Lecture Notes

Quantum Mechanics

PHYS 501

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Contents

Chapter 1

Fundamental Concepts

1.1. The Stern–Gerlach Experiment

1.1.1. Description of the Experiment

The **Stern–Gerlach experiment**, conducted in 1922 in Frankfurt, Germany, is a landmark demonstration of the quantization of angular momentum. This experiment provided direct evidence of quantum mechanical principles, showing that certain physical quantities, such as angular momentum, are quantized rather than continuous.

Experimental Setup

In the experiment, a beam of silver (Ag) atoms from an oven was passed through a non-uniform magnetic field created in the gap between the pole faces of a large magnet. The beam was then detected by being deposited on a glass collector plate.

$$\text{Oven} \rightarrow \text{Slit} \rightarrow \text{Magnetic field (non-uniform)} \rightarrow \text{Glass plate}$$

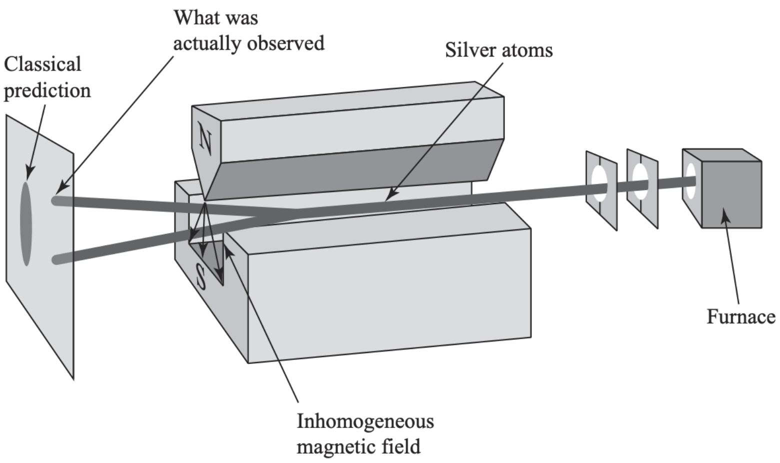

Figure 1.1: Schematic of the Stern–Gerlach experiment. Image taken from Ref. [?]

The classical expectation was that the magnetic moment of the atoms would spread continuously, resulting in a continuous pattern on the glass plate. However, Stern and Gerlach found that the silver atoms struck the plate only in two distinct regions, symmetrically situated about the point of no deflection. This meant that the magnetic moment vector of silver atoms must have only two orientations with respect to the z-axis.

Properties of Silver Atoms

Silver atoms ($Z = 47$) have 47 electrons, with unique properties that make them suitable for this experiment:

- The 46 inner electrons form a closed spherical shell, resulting in no net orbital angular momentum.
- The 47th electron resides in the $5s$ orbital, contributing a nonzero spin angular momentum ($S \neq 0$), which gives rise to the atom's magnetic moment.

Magnetic Moment of Silver Atoms

The magnetic moment of the silver atom is primarily determined by the spin of its 47th electron. It is given by the equation:

$$\vec{\mu} = g_e \left(-\frac{e}{2m}\right) \vec{S}, \tag{1.1}$$

where $g_e = 2$ is the gyromagnetic ratio for an electron, e is the charge of the electron, m is its mass, and $\vec{S}$ is the spin angular momentum of the electron.

The contribution of the nucleus is negligible since its spin is approximately 2×10^5 times smaller than the electron spin. Thus, the magnetic moment of the silver atom is effectively equal to the spin magnetic moment of the 47th electron.

This experiment's results demonstrated that the magnetic moment, and hence the angular momentum, is quantized, aligning with the principles of quantum mechanics.

Magnetic Force on a Current-Currying Wire and Potential Energy

The interaction of magnetic moments with a magnetic field plays a central role in the experiment. For a current-carrying loop, the magnetic moment is defined as:

$$\vec{\mu} = i\vec{A}, \tag{1.2}$$

where i is the current in the loop and $\vec{A}$ is the area vector perpendicular to the plane of the loop.

And the the magnetic force is determined by:

$$\vec{F}_B = i(\vec{L} \times \vec{B}), \tag{1.3}$$

where $\vec{L}$ is the length vector of the wire and $\vec{B}$ is the magnetic field vector.

- The direction of $\vec{F}_B$ depends on the relative orientation of $\vec{L}$ and $\vec{B}$, and it can be determined using the **right-hand rule**.
- This force can induce torque or deflection, depending on the configuration of the magnetic field and the current flow.

The concepts of magnetic force and potential energy provide the theoretical basis for understanding how the Stern–Gerlach apparatus works to separate particles with different magnetic moments.

***Force on a Magnetic* Dipole**

The potential energy of a magnetic dipole in a magnetic field $\vec{B}$ is given by:

$$U = -\vec{\mu} \cdot \vec{B}, \tag{1.4}$$

which shows that the dipole tends to align with the magnetic field to minimize its potential energy.

The force acting on a magnetic dipole in an inhomogeneous magnetic field can be derived from the gradient of the potential energy:

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B}). \tag{1.5}$$

Focusing on the z-component, the force can be expressed as:

$$F_z = \frac{\partial}{\partial z}(\vec{\mu} \cdot \vec{B}) \approx \mu_z \frac{\partial B_z}{\partial z}, \tag{1.6}$$

where μ_z is the z-component of the magnetic moment and $\frac{\partial B_z}{\partial z}$ is the gradient of the z-component of the magnetic field with respect to z.

Observations:

- If $\mu_z > 0$ ($S_z < 0$) $\rightarrow$ **downward deflection**.
- If $\mu_z < 0$ ($S_z > 0$) $\rightarrow$ **upward deflection**.

This differential deflection of the silver atoms due to their magnetic moments is the core observable phenomenon in the Stern–Gerlach experiment.

1.1.2. Sequential Stern–Gerlach Experiments

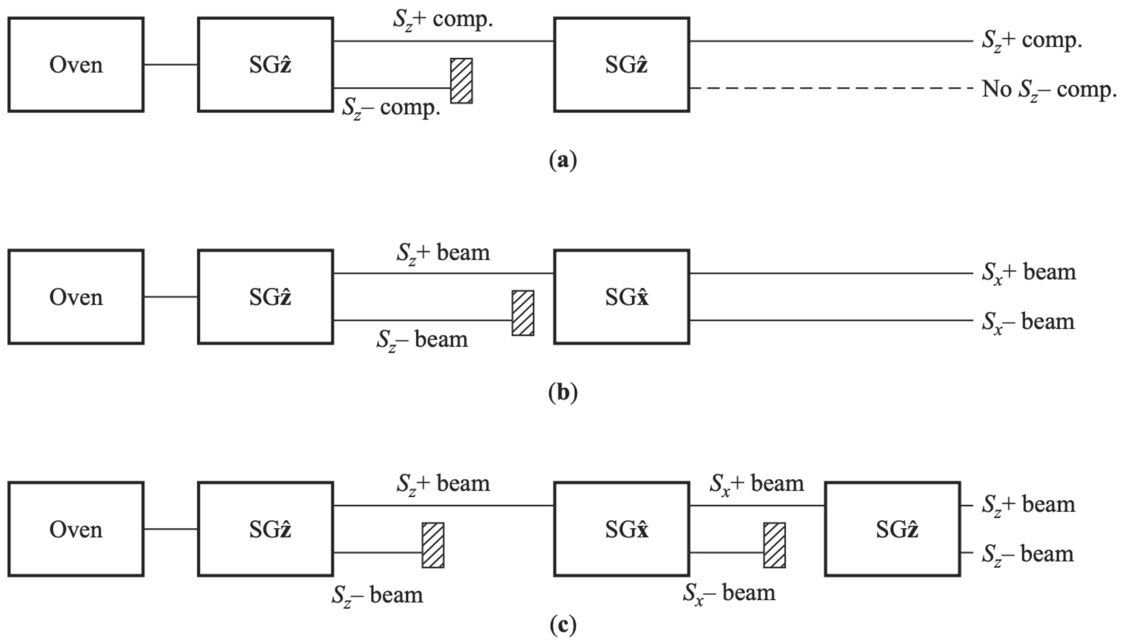

Figure 1.2: Sequential Stern–Gerlach experiments. Image taken from Ref. [?]

First Stern–Gerlach Apparatus ($SG^{\hat{z}}$)

The beam of atoms passes through the first Stern–Gerlach apparatus, which is oriented along the $\hat{z}$-axis. The inhomogeneous magnetic field ($\vec{B} \parallel \hat{z}$) splits the beam into two distinct components:

- The $\vec{S}_z^+$ beam, corresponding to $S_z = +\hbar/2$,
- The $\vec{S}_z^-$ beam, corresponding to $S_z = -\hbar/2$.

Second Stern–Gerlach Apparatus ($SG^{\hat{x}}$)

After the S_z^+ beam is selected using a block, it is passed through a second Stern–Gerlach apparatus oriented along the $\hat{x}$-axis ($SG^{\hat{x}}$). The inhomogeneous magnetic field is now aligned with the $\hat{x}$-axis, splitting the beam into:

- The $\vec{S}_x^+$ beam, corresponding to $S_x = +\hbar/2$,
- The $\vec{S}_x^-$ beam, corresponding to $S_x = -\hbar/2$.

Returning to the $\hat{z}$-Axis

If one of the beams (e.g., S_x^+) is selected and passed through another Stern–Gerlach apparatus oriented along $\hat{z}$, the information about the original spin state along the $\hat{z}$-axis is lost. The final $SG^{\hat{z}}$ apparatus splits the beam once again into S_z^+ and S_z^- components, regardless of the prior measurement along the $\hat{x}$-axis.

Observations from Sequential Stern–Gerlach Experiments

- The outcome of the experiment depends on the chosen axis of orientation ($\hat{z}$, $\hat{x}$, etc.), demonstrating that **spin projections are quantized along any chosen axis**.
- The selection of the S_z^+ beam by the first apparatus destroys any information about S_x, and vice versa.

- These experiments illustrate the principle of the **non-commutativity of quantum measurements**. Spin components along orthogonal axes (S_z and S_x) do not commute, meaning their measurements interfere with one another.

Analogy Between Light Polarization and Spin States

Polarization of Light

Unpolarized Light and Polarizers

Unpolarized light consists of electric field vectors oscillating randomly in directions perpendicular to the propagation direction. A polarizer selects one specific component of the electric field along a defined axis, blocking the others and producing polarized light.

Sequential Polarizers

When polarized light passes through a second polarizer oriented at an angle (e.g., 45°) relative to the first, the electric field splits into components aligned with the new axes:

- $E_{x'}$: Component aligned with the new axis (x'),
- $E_{y'}$: Component perpendicular to x'.

The relationship between the original and new axes is given by:

$$\hat{x}' = \frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}, \quad \hat{y}' = -\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}.$$

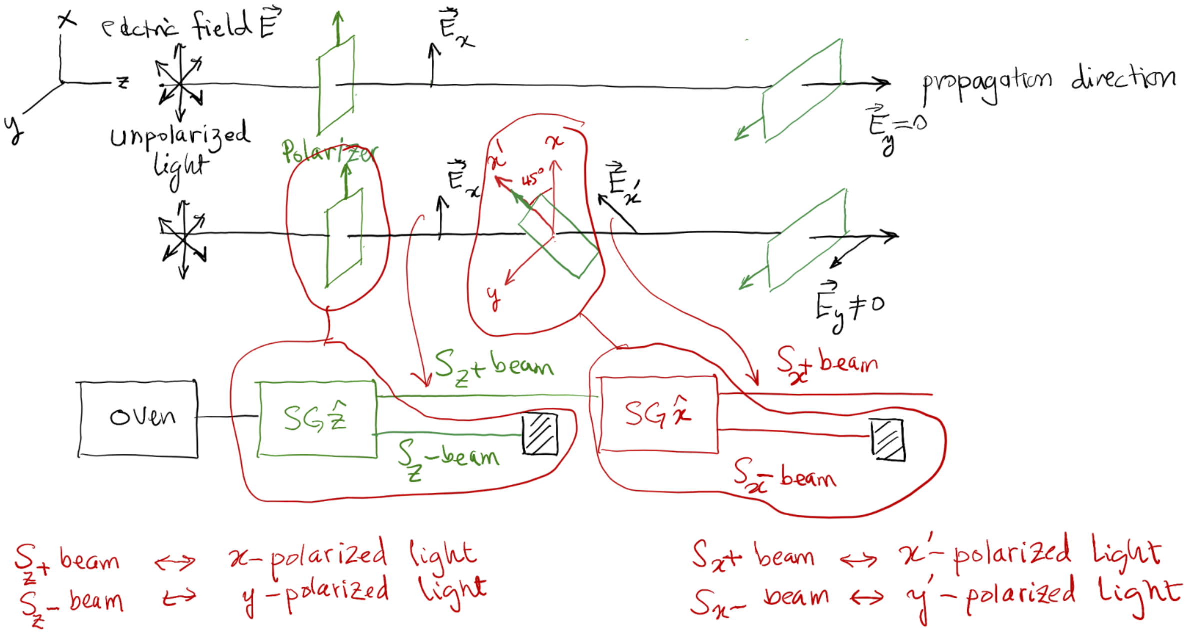

Figure 1.3: Analogy between light polarization and the Stern–Gerlach experiment.

The behavior of light passing through sequential polarizers closely resembles the behavior of spin states in the Stern–Gerlach experiment. Unpolarized light is analogous to an unmeasured spin state, representing a superposition of spin projections. A polarizer projecting light along the x-axis corresponds to a Stern–Gerlach apparatus measuring spin along the $\hat{z}$-axis, which separates the beam into the states $|S_z^+\rangle$ and $|S_z^-\rangle$. Similarly, sequential polarizers oriented at different angles, such as x' at 45°, are analogous to Stern–Gerlach apparatuses aligned along different axes, such as $\hat{x}$, further splitting the beam based on the new spin projections. This analogy highlights the fundamental similarities between light polarization and quantum spin measurements.

The analogy between light polarization and spin states in the Stern–Gerlach experiment can be summarized as follows:

Concept in Light Polarization	Analogous Concept in Stern–Gerlach Experiment
Unpolarized Light	Unmeasured Spin State (Superposition)
First Polarizer (x-polarized)	First Stern–Gerlach Apparatus (S_z^+, S_z^-)
Second Polarizer (e.g., 45°)	Second Stern–Gerlach Apparatus (S_x^+, S_x^-)
x-polarized light	S_z^+ spin state
y-polarized light	S_z^- spin state
x'-polarized light	S_x^+ spin state
y'-polarized light	S_x^- spin state

Table 1.1: Analogy between light polarization and Stern–Gerlach spin states.

Mathematical Representation of Spin States

Spin states in the Stern–Gerlach experiment are described in an abstract two-dimensional quantum state space.

Spin States Along the $\hat{z}$-Axis

The spin operator along the $\hat{z}$-axis has two eigenstates:

$$\left|S_z^+\right\rangle : \text{Spin aligned along } \hat{z},$$
$$\left|S_z^-\right\rangle : \text{Spin anti-aligned along } \hat{z}.$$

These states form the basis for the spin vector space.

Spin States Along the $\hat{x}$-Axis

The eigenstates of the spin operator along the $\hat{x}$-axis are linear combinations of the $\hat{z}$-axis eigenstates:

$$\left|S_x^+\right\rangle = \frac{1}{\sqrt{2}}\left|S_z^+\right\rangle + \frac{1}{\sqrt{2}}\left|S_z^-\right\rangle,$$
$$\left|S_x^-\right\rangle = \frac{1}{\sqrt{2}}\left|S_z^+\right\rangle - \frac{1}{\sqrt{2}}\left|S_z^-\right\rangle.$$

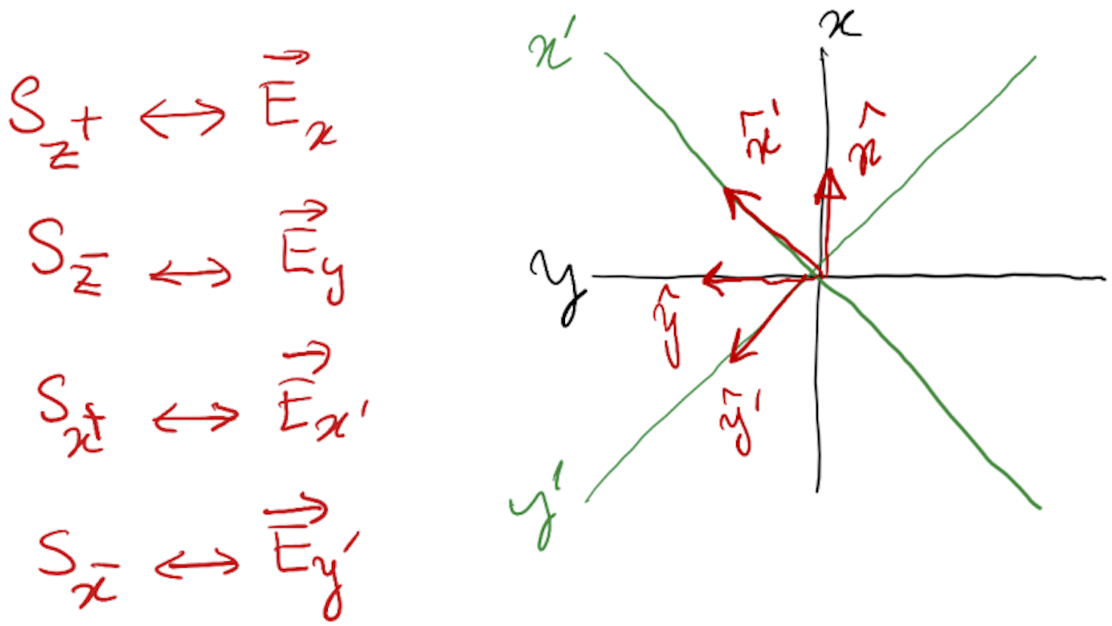

Figure 1.4: Transformation of coordinate axes.

Spin States Along the $\hat{y}$-Axis

Similarly, the eigenstates of the spin operator along the $\hat{y}$-axis are:

$$\left|S_y^+\right\rangle = \frac{1}{\sqrt{2}}\left|S_z^+\right\rangle + \frac{i}{\sqrt{2}}\left|S_z^-\right\rangle,$$
$$\left|S_y^-\right\rangle = \frac{1}{\sqrt{2}}\left|S_z^+\right\rangle - \frac{i}{\sqrt{2}}\left|S_z^-\right\rangle.$$

Coordinate Transformations and Light Polarization

When a second Stern–Gerlach apparatus is oriented at an angle to the first, the new measurement axes (x', y') are linear combinations of the original axes (x, y):

$$\hat{x}' = \frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y},$$
$$\hat{y}' = -\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}.$$

The spin states along these new axes transform similarly:

$$\left|S_{x'}^+\right\rangle = \frac{1}{\sqrt{2}}\left|S_x^+\right\rangle + \frac{1}{\sqrt{2}}\left|S_x^-\right\rangle,$$
$$\left|S_{x'}^-\right\rangle = -\frac{1}{\sqrt{2}}\left|S_x^+\right\rangle + \frac{1}{\sqrt{2}}\left|S_x^-\right\rangle.$$

This transformation mirrors how the electric field components of polarized light transform along new axes:

$$E_0\cos(kz-\omega t)\hat{x}' = E_0\cos(kz-\omega t)\left(\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}\right),$$
$$E_0\cos(kz-\omega t)\hat{y}' = E_0\cos(kz-\omega t)\left(-\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}\right).$$

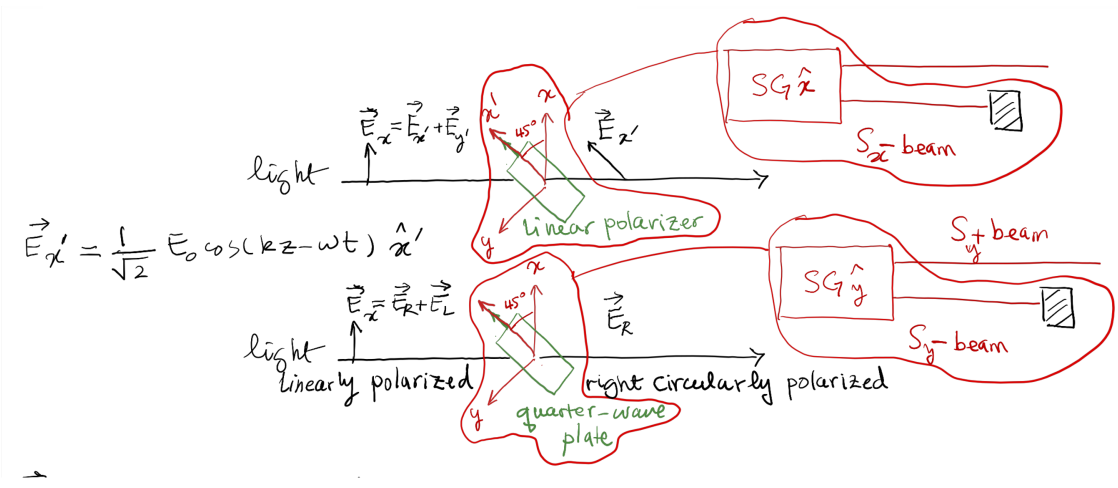

Figure 1.5: Illustration of linear and circular polarization of light and their analogy to spin states.

Circular Polarization and Spin States

Electric Field for Circular Polarization

Circularly polarized light is created when there is a 90° phase difference between the x- and y-components of the electric field:

$$\vec{E}_R = \frac{1}{\sqrt{2}} E_0 \cos(kz - \omega t)\hat{x} + \frac{1}{\sqrt{2}} E_0 \cos\Big(kz - \omega t + \frac{\pi}{2}\Big)\hat{y},$$
$$\vec{E}_L = \frac{1}{\sqrt{2}} E_0 \cos(kz - \omega t)\hat{x} + \frac{1}{\sqrt{2}} E_0 \cos\Big(kz - \omega t - \frac{\pi}{2}\Big)\hat{y}.$$

Using the exponential representation of cosine:

$$e^{i\phi} = \cos\phi + i\sin\phi,$$

these can be rewritten as:

$$\vec{E}_R = \frac{1}{\sqrt{2}} E_0 \,\mathrm{Re}\left[e^{i(kz-\omega t)}(\hat{x} + i\hat{y})\right],$$
$$\vec{E}_L = \frac{1}{\sqrt{2}} E_0 \,\mathrm{Re}\left[e^{i(kz-\omega t)}(\hat{x} - i\hat{y})\right].$$

Spin State Analogy

The spin states $|S_y^+\rangle$ and $|S_y^-\rangle$ are analogous to right and left circularly polarized light:

$$|S_y^+\rangle = \frac{1}{\sqrt{2}} |S_z^+\rangle + \frac{i}{\sqrt{2}} |S_z^-\rangle,$$
$$|S_y^-\rangle = \frac{1}{\sqrt{2}} |S_z^+\rangle - \frac{i}{\sqrt{2}} |S_z^-\rangle.$$

This analogy highlights how phase relationships and complex coefficients play a role in both quantum mechanics and wave optics.

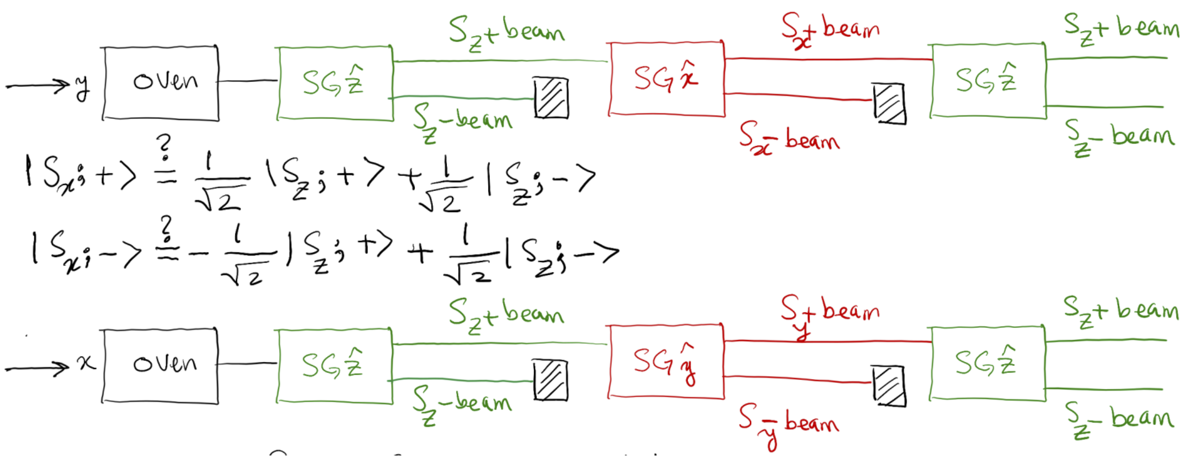

Figure 1.6: Sequential Stern–Gerlach apparatus setup.

1.2. Kets, Bras, and Operators

1.2.1. Ket Space

Definition of a Ket Space

A **ket space** is a vector space consisting of kets ($|\psi\rangle$), which are used to represent quantum states in quantum mechanics. These kets are elements of a larger **Hilbert Space** $\mathcal{H}$, a complete vector space over the field $\mathbb{C}$ (the complex numbers) equipped with an inner product.
The inner product, denoted $\langle\phi|\psi\rangle$, plays a central role in quantum mechanics, allowing us to compute probabilities, normalize states, and describe the principle of superposition. We will discuss the inner product in detail when introducing **bras**, operators, and measurement formalism.
The ket space satisfies the following fundamental properties, inherited from $\mathcal{H}$:

- **Closure under Addition:**
$$|\psi_1\rangle + |\psi_2\rangle \in \mathcal{H}, \quad \forall\, |\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}.$$
- **Closure under Scalar Multiplication:**
$$\lambda\, |\psi\rangle \in \mathcal{H}, \quad \forall\, |\psi\rangle \in \mathcal{H}, \forall \lambda \in \mathbb{C}.$$
- **Associativity of Addition:**
$$(|\psi_1\rangle + |\psi_2\rangle) + |\psi_3\rangle = |\psi_1\rangle + (|\psi_2\rangle + |\psi_3\rangle), \quad \forall\, |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \in \mathcal{H}.$$
- **Commutativity of Addition:**
$$|\psi_1\rangle + |\psi_2\rangle = |\psi_2\rangle + |\psi_1\rangle, \quad \forall\, |\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}.$$
- **Existence of a Null Ket:**
$$\exists\, \mathbf{0} \in \mathcal{H} \quad \text{such that} \quad |\psi\rangle + \mathbf{0} = |\psi\rangle, \quad \forall\, |\psi\rangle \in \mathcal{H}.$$
- **Existence of Additive Inverses:**
$$\exists\, (-\,|\psi\rangle) \in \mathcal{H} \quad \text{such that} \quad |\psi\rangle + (-\,|\psi\rangle) = \mathbf{0}, \quad \forall\, |\psi\rangle \in \mathcal{H}.$$

Example of a Linear Vector Space

Consider the set of all vectors in the xy-plane, denoted $\mathcal{V}$. This set forms a **vector space**, satisfying the following properties:

- **Addition:**
$$\forall \vec{\alpha}, \vec{\beta} \in \mathcal{V}, \quad \vec{\alpha} + \vec{\beta} \in \mathcal{V}.$$
- **Scalar Multiplication:**
$$\forall \vec{\alpha} \in \mathcal{V}, \forall a \in \mathbb{R}, \quad a\vec{\alpha} \in \mathcal{V}.$$
- **Linear Combination:** Any vector $\vec{\alpha} \in \mathcal{V}$ can be expressed as:
$$\vec{\alpha} = \alpha_x \hat{x} + \alpha_y \hat{y}, \quad \alpha_x, \alpha_y \in \mathbb{R},$$
  where $\{\hat{x}, \hat{y}\}$ is a basis for $\mathcal{V}$.

The **dimension** of $\mathcal{V}$ is:
$$\dim(\mathcal{V}) = 2.$$
Alternative bases for $\mathcal{V}$ include:
$$\{\vec{e}_1, \vec{e}_2\},$$
where $\vec{e}_1$ and $\vec{e}_2$ are linearly independent.

Physical States and Dirac Notation

In quantum mechanics, a **physical state** is represented by a **state vector** $|\psi\rangle$ in a Hilbert Space $\mathcal{H}$. We refer to these state vectors as **kets** and denote them using **Dirac notation**. This formalism is particularly convenient for expressing linear operators, inner products, and quantum measurements.

Properties of Kets in a Hilbert Space

The properties of kets in a Hilbert space follow directly from the properties of the Hilbert space. Specifically, for any $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$ and any $c \in \mathbb{C}$:

- **Addition of Kets:**
$$|\alpha\rangle + |\beta\rangle \in \mathcal{H}.$$
- **Scalar Multiplication:**
$$c|\alpha\rangle \in \mathcal{H}.$$
- **Null Ket:** If $c = 0$:
$$c|\alpha\rangle = \mathbf{0},$$
where $\mathbf{0}$ is the null ket.
- **Physical Equivalence:** Two kets $|\psi\rangle$ and $c|\psi\rangle$ are **physically equivalent** if $c \neq 0$:
$$|\psi\rangle \sim c|\psi\rangle.$$

Postulate

The physical state represented by a ket $|\psi\rangle$ is invariant under non-zero scalar multiplication:

$$|\psi\rangle \sim c|\psi\rangle, \quad \forall c \in \mathbb{C},\, c \neq 0.$$

Ket Space: Observables and Operators

Representation of Observables

In quantum mechanics, an **observable** is represented by a **Hermitian operator** $\hat{A}$ acting on a ket space $\mathcal{H}$. Examples of observables include:

- The spin component along the z-axis (S_z).
- Momentum ($\hat{p}$).

Operators Acting on Kets

An operator $\hat{A}$ acts on a ket $|\psi\rangle$ from the left:

$$\hat{A}(|\psi\rangle) = |\phi\rangle,$$

where $|\phi\rangle \in \mathcal{H}$ is another ket in the same vector space.
In simpler terms, operators map elements of the vector space $\mathcal{H}$ to other elements of $\mathcal{H}$. For linear operators, we have:

$$\forall |\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}, \forall c_1, c_2 \in \mathbb{C}, \quad \hat{A}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1\hat{A}|\psi_1\rangle + c_2\hat{A}|\psi_2\rangle.$$

Eigenkets and Eigenvalues

Let $\hat{A}$ be a linear operator acting on $\mathcal{H}$. A ket $|\psi\rangle$ is called an **eigenket** of $\hat{A}$ if it satisfies:

$$\hat{A}|\psi\rangle = a|\psi\rangle,$$

where:

- $a \in \mathbb{C}$ is called the **eigenvalue** corresponding to $|\psi\rangle$.
- $|\psi\rangle$ is the eigenvector (ket) associated with the eigenvalue a.

The eigenket $|\psi\rangle$ remains proportional to itself under the action of $\hat{A}$, and the proportionality constant is the eigenvalue a.

Physical Interpretation

An **eigenstate** is the physical state that corresponds to an eigenket. In quantum mechanics, measuring the observable $\hat{A}$ in the eigenstate $|\psi\rangle$ yields the eigenvalue a with certainty.

Example: Spin Observables

Consider the spin component along the z-axis, represented by the operator S_z. The eigenvalue equation for S_z is given by:

$$S_z|S_z;+\rangle = \frac{\hbar}{2}|S_z;+\rangle,$$
$$S_z|S_z;-\rangle = -\frac{\hbar}{2}|S_z;-\rangle,$$

where:

- S_z is the spin operator.
- $|S_z;+\rangle$ and $|S_z;-\rangle$ are eigenkets of S_z.
- $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are the corresponding eigenvalues.

In a more compact notation:

$$S_z\left|\frac{\hbar}{2}\right\rangle = \frac{\hbar}{2}\left|\frac{\hbar}{2}\right\rangle,$$
$$S_z\left|-\frac{\hbar}{2}\right\rangle = -\frac{\hbar}{2}\left|-\frac{\hbar}{2}\right\rangle.$$

1.2.2. Bra Space and Inner Products

Definition of Bra Space

A **bra space** is the **dual vector space** associated with the **ket space** $\mathcal{H}$, which is a complex vector space. For every ket $|\psi\rangle \in \mathcal{H}$, there exists a corresponding bra $\langle\psi| \in \mathcal{H}^*$, where $\mathcal{H}^*$ is the dual space of $\mathcal{H}$.
This correspondence is known as the **dual correspondence**:

$$|\psi\rangle \longleftrightarrow \langle\psi| \quad \text{(dual correspondence)}.$$

Properties of the Dual Correspondence (DC)

The dual correspondence satisfies the following properties:

- **Addition:** If $|\alpha\rangle$ and $|\beta\rangle$ are kets in $\mathcal{H}$, then:

$$|\alpha\rangle + |\beta\rangle \longleftrightarrow \langle\alpha| + \langle\beta|.$$

- **Scalar Multiplication:** If $c \in \mathbb{C}$ is a complex scalar, then:

$$c|\alpha\rangle \longleftrightarrow c^*\langle\alpha|,$$

where c^* is the **complex conjugate** of c.

- **Linearity:** For any linear combination of kets:

$$c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \longleftrightarrow c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|,$$

where $c_\alpha, c_\beta \in \mathbb{C}$ are complex scalars.

Inner Product of a Bra and a Ket

The **inner product** of a bra $\langle\beta| \in \mathcal{H}^*$ and a ket $|\alpha\rangle \in \mathcal{H}$ is defined as:

$$\langle\beta|\alpha\rangle,$$

where:

- $\langle\beta|$ is the bra on the left.
- $|\alpha\rangle$ is the ket on the right.

The result of the inner product is a **complex number**:

$$\langle\beta|\alpha\rangle \in \mathbb{C}.$$

The inner product satisfies the following key properties:

- **Conjugate Symmetry:**
$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*,$$
where * denotes the complex conjugate.

- **Linearity in the Bra:**
$$\langle c_\alpha\beta_1 + c_\beta\beta_2|\alpha\rangle = c_\alpha^*\langle\beta_1|\alpha\rangle + c_\beta^*\langle\beta_2|\alpha\rangle.$$

- **Linearity in the Ket:**
$$\langle\beta|c_\alpha\alpha_1 + c_\beta\alpha_2\rangle = c_\alpha\langle\beta|\alpha_1\rangle + c_\beta\langle\beta|\alpha_2\rangle.$$

Example

Let us consider an example in a spin-$\frac{1}{2}$ system. For kets $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$, we have:

- The dual correspondence for a linear combination:

$$c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \longleftrightarrow c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|.$$

- The inner product of two states:
$$\langle\beta|\alpha\rangle = (\langle\beta|) \cdot (|\alpha\rangle),$$
which evaluates to a complex number.

This dual structure and the inner product play a fundamental role in the mathematical formulation of quantum mechanics.

Properties and Postulates

The inner product in a Hilbert space $\mathcal{H}$, represented by $\langle\beta|\alpha\rangle$, satisfies the following postulates:

Postulate: Conjugate Symmetry

The inner product of two kets satisfies:

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*,$$

where * denotes the complex conjugate. If $\beta = \alpha$, this implies:

$$\langle\alpha|\alpha\rangle \in \mathbb{R},$$

i.e., the inner product $\langle\alpha|\alpha\rangle$ is a **real number**.

Postulate:

Postulate 2: Positive Semi-Definiteness
The inner product of a ket with itself is non-negative:

$$\langle\alpha|\alpha\rangle \geq 0.$$

Moreover, $\langle\alpha|\alpha\rangle = 0$ if and only if $|\alpha\rangle$ is the **null ket**:

$$|\alpha\rangle = \mathbf{0}.$$

Norm of a Ket

The **norm** of a ket $|\alpha\rangle$ is defined as:

$$\||\alpha\rangle\| = \sqrt{\langle\alpha|\alpha\rangle}.$$

The norm represents the "length" of the ket in the vector space. A ket is considered **normalized** if its norm is equal to 1:

$$\||\alpha\rangle\| = 1 \quad \Leftrightarrow \quad \langle\alpha|\alpha\rangle = 1.$$

Normalization of a Ket

If $|\alpha\rangle$ is not the null ket ($|\alpha\rangle \neq \mathbf{0}$), it can be normalized by dividing it by its norm. The normalized ket $|\tilde{\alpha}\rangle$ is given by:

$$|\tilde{\alpha}\rangle = \frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}|\alpha\rangle,$$

with the property:

$$\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1.$$

Orthogonality of Kets

Two kets $|\alpha\rangle$ and $|\beta\rangle$ are said to be **orthogonal** if their inner product is zero:

$$\langle\alpha|\beta\rangle = 0.$$

This implies that the two kets represent independent directions in the vector space.

1.2.3. Operators

Definition of an Operator

An **operator** $\hat{X}$ is a map that acts on kets in the Hilbert space $\mathcal{H}$. Specifically, an operator $\hat{X}$ takes a ket $|\psi\rangle \in \mathcal{H}$ and produces another ket $|\phi\rangle \in \mathcal{H}$:

$$\hat{X}(|\psi\rangle) = |\phi\rangle, \quad \hat{X}|\psi\rangle = |\phi\rangle.$$

The operator always acts **from the left** on a ket.

Equality of Operators

Two operators $\hat{X}$ and $\hat{Y}$ are equal if and only if:

$$\forall|\psi\rangle \in \mathcal{H}, \quad \hat{X}|\psi\rangle = \hat{Y}|\psi\rangle.$$

Null Operator

An operator $\hat{X}$ is called the **null operator** if:

$$\forall|\psi\rangle \in \mathcal{H}, \quad \hat{X}|\psi\rangle = \mathbf{0},$$

where $\mathbf{0}$ is the null ket.

Addition of Operators

Operators can be added to produce another operator. If $\hat{X}, \hat{Y}, \hat{Z}$ are operators, then:

- **Commutativity:**
$$\hat{X} + \hat{Y} = \hat{Y} + \hat{X}.$$
- **Associativity:**
$$\hat{X} + (\hat{Y} + \hat{Z}) = (\hat{X} + \hat{Y}) + \hat{Z}.$$

Linear Operators

An operator $\hat{X}$ is called a **linear operator** if it satisfies the following two properties:

- **Linearity with respect to addition:**
$$\forall|\psi\rangle, |\phi\rangle \in \mathcal{H}, \quad \hat{X}(|\psi\rangle + |\phi\rangle) = \hat{X}|\psi\rangle + \hat{X}|\phi\rangle.$$
- **Linearity with respect to scalar multiplication:**
$$\forall|\psi\rangle \in \mathcal{H}, \forall c \in \mathbb{C}, \quad \hat{X}(c|\psi\rangle) = c\hat{X}|\psi\rangle.$$

Combining these, for any linear operator $\hat{X}$, the action of $\hat{X}$ on a linear combination of kets satisfies:

$$\hat{X}(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle) = c_\alpha\hat{X}|\alpha\rangle + c_\beta\hat{X}|\beta\rangle, \quad \forall c_\alpha, c_\beta \in \mathbb{C}.$$

Operators and Their Properties

Operators Acting on Bras

Operators $\hat{X}$ act on bras from the **right side**. If $\langle\alpha|$ is a bra and $\hat{X}$ is an operator, then:

$$\langle\alpha| \cdot \hat{X} = \langle\alpha|X,$$

where $\langle\alpha|X$ is another bra in the dual space.

Definition: Adjoint of an Operator

For an operator $\hat{X}$ acting on a ket $|\alpha\rangle$, the **adjoint operator** $\hat{X}^\dagger$ (also called the **Hermitian adjoint**) is defined by the dual correspondence:

$$\hat{X}|\alpha\rangle \longleftrightarrow \langle\alpha|\hat{X}^\dagger.$$

Definition: Hermitian Operator

An operator $\hat{X}$ is said to be **Hermitian** if:

$$\hat{X} = \hat{X}^\dagger.$$

Hermitian operators represent observables in quantum mechanics, as their eigenvalues are guaranteed to be real.

1.2.4. Multiplication

Let $\hat{X}, \hat{Y}, \hat{Z}$ be operators. The following properties govern their multiplication:

- **Non-commutativity:** In general, operator multiplication is not commutative:
$$\hat{X}\hat{Y} \neq \hat{Y}\hat{X}.$$
- **Associativity:** Operator multiplication is associative:
$$\hat{X}(\hat{Y}\hat{Z}) = (\hat{X}\hat{Y})\hat{Z}.$$
- **Action on kets:** If an operator $\hat{X}$ acts on a ket $|\alpha\rangle$, and $\hat{Y}$ acts on $\hat{X}|\alpha\rangle$, then:
$$(\hat{X}\hat{Y})|\alpha\rangle = \hat{X}(\hat{Y}|\alpha\rangle).$$
- **Action on bras:** Similarly, for bras:
$$\langle\beta|(\hat{X}\hat{Y}) = (\langle\beta|\hat{X})\hat{Y} = \langle\beta|(\hat{X}\hat{Y}).$$
- **Hermitian adjoint of a product:** The adjoint of a product of operators satisfies:
$$(\hat{X}\hat{Y})^\dagger = \hat{Y}^\dagger\hat{X}^\dagger.$$

Outer and Inner Products

- **Outer Product:** The outer product of a ket $|\beta\rangle$ and a bra $\langle\alpha|$ is defined as:
$$|\beta\rangle\langle\alpha| = \text{an operator}.$$
This operator acts on a ket $|\psi\rangle$ to produce another ket:
$$\big(|\beta\rangle\langle\alpha|\big)|\psi\rangle = \langle\alpha|\psi\rangle|\beta\rangle.$$
- **Inner Product:** The inner product of a bra $\langle\beta|$ and a ket $|\alpha\rangle$ is a scalar:
$$\langle\beta|\alpha\rangle = \text{a complex number}.$$

Illegal Products

Certain products in quantum mechanics are undefined or **illegal**:

- $|\alpha\rangle\hat{X}$: Operators cannot act on kets from the right.
- $\hat{X}\langle\alpha|$: Operators cannot act on bras from the left.
- $|\alpha\rangle|\beta\rangle$: The product of two kets is undefined unless they belong to different spaces.
- $\langle\beta|\langle\alpha|$: The product of two bras is also undefined unless they belong to different spaces.

Examples of Operator Multiplication and Adjoint

1. Let $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ be operators. Then:
$$(\hat{X}\hat{Y})|\alpha\rangle = \hat{X}(\hat{Y}|\alpha\rangle).$$
2. The adjoint of the product $\hat{X}\hat{Y}$ is given by:
$$(\hat{X}\hat{Y})^\dagger = \hat{Y}^\dagger\hat{X}^\dagger.$$
3. For an outer product $|\beta\rangle\langle\alpha|$, its adjoint is:
$$\big(|\beta\rangle\langle\alpha|\big)^\dagger = |\alpha\rangle\langle\beta|.$$
4. For a Hermitian operator $\hat{X}$:
$$\hat{X} = \hat{X}^\dagger.$$

1.2.5. The Associative Axiom

The **associative property** of multiplication holds for legal combinations of kets, bras, and operators. This property ensures that the grouping of terms does not affect the result.

Associative Property of Outer Product and Inner Product

Let $|\beta\rangle, |\alpha\rangle$ be kets, $\langle\delta|$ a bra, and $\hat{X}$ an operator. The associative property states:

$$(|\beta\rangle\langle\alpha|)|\delta\rangle = |\beta\rangle \cdot (\langle\alpha|\delta\rangle).$$

Here:

- $|\beta\rangle\langle\alpha|$ is an **operator**.
- $\langle\alpha|\delta\rangle$ is the **inner product**, a complex number.
- The operator $|\beta\rangle\langle\alpha|$ rotates $|\delta\rangle$ into the direction of $|\beta\rangle$, scaled by the inner product $\langle\alpha|\delta\rangle$.

Important Note: The dot and brackets must be preserved for proper notation. Removing them would make the expression illegal.

Hermitian Adjoint and Outer Product

If $\hat{X} = |\beta\rangle\langle\alpha|$, then its Hermitian adjoint $\hat{X}^\dagger$ is given by:

$$\hat{X}^\dagger = |\alpha\rangle\langle\beta|.$$

For any ket $|\gamma\rangle$, the action of $\hat{X}$ and $\hat{X}^\dagger$ is:

$$\hat{X}|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle \quad \text{and} \quad \hat{X}^\dagger|\gamma\rangle = |\alpha\rangle\langle\beta|\gamma\rangle^*.$$

Under the dual correspondence:

$$\langle\delta|\hat{X} = \langle\delta|(|\beta\rangle\langle\alpha|) \quad \longleftrightarrow \quad \langle\alpha|\delta\rangle^* = \langle\delta|\hat{X}^\dagger|\beta\rangle.$$

Associative Property of Operators, Bras, and Kets

Let $\hat{X}$ be an operator, $|\alpha\rangle, |\beta\rangle$ kets, and $\langle\delta|$ a bra. The associative property ensures:

$$\langle\beta|(\hat{X}|\alpha\rangle) = (\langle\beta|\hat{X})|\alpha\rangle.$$

Expanding the inner product:

$$\langle\beta|\hat{X}|\alpha\rangle = (\langle\beta|\hat{X})|\alpha\rangle = \langle\beta|(\hat{X}|\alpha\rangle).$$

For any Hermitian operator $\hat{X}$ ($\hat{X} = \hat{X}^\dagger$):

$$\langle\beta|\hat{X}|\alpha\rangle = \langle\alpha|\hat{X}|\beta\rangle^*.$$

Expanded Steps of Associative Action

1. Write the inner product with the operator acting:

$$\langle\beta|\hat{X}|\alpha\rangle = \langle\beta|(\hat{X}|\alpha\rangle).$$

2. By the dual correspondence and the adjoint:

$$\langle\beta|\hat{X}|\alpha\rangle = \langle\alpha|\hat{X}^\dagger|\beta\rangle^*.$$

3. For Hermitian operators ($\hat{X} = \hat{X}^\dagger$):

$$\langle\beta|\hat{X}|\alpha\rangle = \langle\alpha|\hat{X}|\beta\rangle^*.$$

1.3. Base Kets and Matrix Representations

1.3.1. Eigenkets of an Observable

Hermitian Operators Represent Observables

In quantum mechanics, operators representing **physical observables** are **Hermitian operators**. Hermitian operators have the following key properties:

- The **eigenvalues** of a Hermitian operator $\hat{A}$ are **real**.
- The **eigenkets** corresponding to distinct eigenvalues are **orthogonal**.

Theorem

Let $\hat{A}$ be a Hermitian operator, $\hat{A} = \hat{A}^\dagger$. Then:

1. All eigenvalues a of $\hat{A}$ are real.
2. Eigenkets corresponding to distinct eigenvalues are orthogonal.

Proof

Let $\hat{A}$ be Hermitian ($\hat{A} = \hat{A}^\dagger$), and let a' and a'' be eigenvalues of $\hat{A}$, with corresponding eigenkets $|a'\rangle$ and $|a''\rangle$. Then:

$$\hat{A}|a'\rangle = a'|a'\rangle \quad \text{and} \quad \hat{A}|a''\rangle = a''|a''\rangle.$$

Taking the inner product of $|a''\rangle$ with $\hat{A}|a'\rangle$, we have:

$$\langle a''|(\hat{A}|a'\rangle) = \langle a''|a'|a'\rangle = a'\langle a''|a'\rangle.$$

Similarly, using the Hermitian property $\hat{A} = \hat{A}^\dagger$, we consider $\hat{A}^\dagger|a''\rangle = \hat{A}|a''\rangle$, giving:

$$(\langle a''|\hat{A})|a'\rangle = \langle a''|\hat{A}|a'\rangle = a''^*\langle a''|a'\rangle.$$

Since $\hat{A}$ is Hermitian, $a''^* = a''$ (the eigenvalues of Hermitian operators are real), so:

$$\langle a''|\hat{A}|a'\rangle = a''\langle a''|a'\rangle.$$

Equating the two results:

$$a'\langle a''|a'\rangle = a''\langle a''|a'\rangle.$$

Rearranging terms:

$$(a' - a'')\langle a''|a'\rangle = 0.$$

Case 1: $a' = a''$

If $a' = a''$, the eigenvalues are equal, and the equation:

$$(a' - a'')\langle a''|a'\rangle = 0$$

holds trivially because $(a' - a'') = 0$. In this case, there is no restriction on the overlap $\langle a''|a'\rangle$.

Case 2: $a' \neq a''$

If $a' \neq a''$, then:

$$(a' - a'') \neq 0.$$

To satisfy:

$$(a' - a'')\langle a''|a'\rangle = 0,$$

it must be that:

$$\langle a''|a'\rangle = 0.$$

This implies that the eigenkets $|a'\rangle$ and $|a''\rangle$ are **orthogonal**:

$$\langle a''|a'\rangle = 0.$$

From the above cases, we conclude:

1. The eigenvalues of a Hermitian operator $\hat{A}$ are **real**.
2. The eigenkets of $\hat{A}$ corresponding to distinct eigenvalues are **orthogonal**.

Eigenkets of an Observable and Vector Spaces

Orthogonality and Normalization of Eigenkets

Let $\hat{A}$ be a Hermitian operator with eigenkets $|a'\rangle$ and $|a''\rangle$ corresponding to eigenvalues a' and a'', respectively. The following properties hold:

- The eigenkets corresponding to distinct eigenvalues are **orthogonal**:

$$\langle a''|a'\rangle = 0 \quad \text{if} \quad a' \neq a''.$$

- If the eigenkets $|a'\rangle$ and $|a''\rangle$ are normalized, we have:

$$\langle a''|a'\rangle = \delta_{a''a'},$$

  where $\delta_{a''a'}$ is the **Kronecker delta function**, defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Vector Decomposition in a Basis

Consider a three-dimensional vector space $\mathcal{V}$ spanned by the orthonormal basis vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. Any vector $\vec{v} \in \mathcal{V}$ can be expressed as a linear combination of the basis vectors:

$$\vec{v} = v_x\hat{e}_1 + v_y\hat{e}_2 + v_z\hat{e}_3,$$

where $v_x, v_y, v_z \in \mathbb{R}$ are the components of $\vec{v}$ along $\hat{e}_1, \hat{e}_2, \hat{e}_3$, respectively.
Using the inner product, the components v_x, v_y, v_z can be determined as:

$$v_x = (\hat{e}_1 \cdot \vec{v}), \quad v_y = (\hat{e}_2 \cdot \vec{v}), \quad v_z = (\hat{e}_3 \cdot \vec{v}),$$

where $\cdot$ denotes the dot product.
Substituting these values back, the vector can be written as:

$$\vec{v} = (\hat{e}_1 \cdot \vec{v})\hat{e}_1 + (\hat{e}_2 \cdot \vec{v})\hat{e}_2 + (\hat{e}_3 \cdot \vec{v})\hat{e}_3.$$

More generally, in an n-dimensional vector space with an orthonormal basis $\{\hat{e}_i\}_{i=1}^n$, any vector $\vec{v}$ can be expressed as:

$$\vec{v} = \sum_{i=1}^{n} (\hat{e}_i \cdot \vec{v})\hat{e}_i.$$

Properties of the Basis

- The set $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ forms a **basis** of the vector space $\mathcal{V}$, meaning:

$$\mathcal{V} = \text{span}\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}.$$

- The basis vectors satisfy orthonormality:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij},$$

  where δ_{ij} is the Kronecker delta function.

- Any vector $\vec{v} \in \mathcal{V}$ is uniquely represented as:

$$\vec{v} = \sum_{i=1}^{3} v_i\hat{e}_i, \quad v_i = (\hat{e}_i \cdot \vec{v}).$$

1.3.2. Eigenkets as Base Kets

The Completeness of Eigenkets

The set of eigenkets $\{|a^{(1)}\rangle, |a^{(2)}\rangle, \ldots, |a^{(N)}\rangle\}$, corresponding to the eigenvalues of a Hermitian operator $\hat{A}$, forms a **complete basis** for the Hilbert space $\mathcal{H}$. This means:

$$\mathcal{H} = \text{span}\{|a^{(1)}\rangle, |a^{(2)}\rangle, \ldots, |a^{(N)}\rangle\}.$$

Expansion of Kets in Terms of Eigenkets

Any ket $|\psi\rangle \in \mathcal{H}$ can be written as a **linear combination** of the eigenkets of the Hermitian operator $\hat{A}$:

$$|\psi\rangle = c_{a^{(1)}}|a^{(1)}\rangle + c_{a^{(2)}}|a^{(2)}\rangle + \cdots + c_{a^{(N)}}|a^{(N)}\rangle,$$

where $c_{a^{(i)}} \in \mathbb{C}$ are complex coefficients.
In summation notation:

$$|\psi\rangle = \sum_{i=1}^{N} c_{a^{(i)}}|a^{(i)}\rangle.$$

Each coefficient $c_{a^{(i)}}$ can be determined using the inner product:

$$c_{a^{(i)}} = \langle a^{(i)}|\psi\rangle.$$

Properties of the Basis

1. The eigenkets $\{|a^{(i)}\rangle\}_{i=1}^{N}$ are **orthonormal**:

$$\langle a^{(i)}|a^{(j)}\rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker delta function:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

2. The eigenkets form a **complete basis** for $\mathcal{H}$, meaning that any ket in the Hilbert space can be written as a linear combination of these eigenkets.
3. The dimension of the Hilbert space is equal to the number of eigenkets:

$$\dim(\mathcal{H}) = N.$$

Implications of Completeness

The completeness of the eigenkets of a Hermitian operator ensures that:

- Any ket $|\psi\rangle \in \mathcal{H}$ can be **expanded in terms** of the eigenkets of the Hermitian operator:

$$|\psi\rangle = \sum_{i=1}^{N} c_{a^{(i)}}|a^{(i)}\rangle.$$

- The eigenkets of the Hermitian operator $\hat{A}$ can be used as a **basis** for the ket space.

Example

Let $\mathcal{H}$ be a 3-dimensional Hilbert space with a Hermitian operator $\hat{A}$. The eigenkets $\{|a^{(1)}\rangle, |a^{(2)}\rangle, |a^{(3)}\rangle\}$ form a complete orthonormal basis. Any ket $|\psi\rangle \in \mathcal{H}$ can be written as:

$$|\psi\rangle = c_{a^{(1)}}|a^{(1)}\rangle + c_{a^{(2)}}|a^{(2)}\rangle + c_{a^{(3)}}|a^{(3)}\rangle.$$

The coefficients $c_{a^{(1)}}, c_{a^{(2)}}, c_{a^{(3)}}$ are given by:

$$c_{a^{(i)}} = \langle a^{(i)}|\psi\rangle \quad \text{for } i = 1, 2, 3.$$

Expansion of a Ket in Terms of Eigenkets

Let $\{|a^{(i)}\rangle\}_{i=1}^{N}$ be the set of eigenkets of a Hermitian operator $\hat{A}$, forming an orthonormal basis for the Hilbert space $\mathcal{H}$. Any ket $|\alpha\rangle \in \mathcal{H}$ can be expanded as:

$$|\alpha\rangle = \sum_{i=1}^{N} c_{a^{(i)}} |a^{(i)}\rangle,$$

where the coefficients $c_{a^{(i)}}$ are given by:

$$c_{a^{(i)}} = \langle a^{(i)}|\alpha\rangle.$$

Taking the inner product $\langle a^{(j)}|\alpha\rangle$, we have:

$$\langle a^{(j)}|\alpha\rangle = \sum_{i=1}^{N} c_{a^{(i)}} \langle a^{(j)}|a^{(i)}\rangle.$$

Using the orthonormality condition $\langle a^{(j)}|a^{(i)}\rangle = \delta_{ij}$, this simplifies to:

$$\langle a^{(j)}|\alpha\rangle = c_{a^{(j)}}.$$

Thus, the expansion of $|\alpha\rangle$ can be written as:

$$|\alpha\rangle = \sum_{i=1}^{N} \langle a^{(i)}|\alpha\rangle |a^{(i)}\rangle.$$

The Completeness Relation

The completeness relation states that the eigenkets $\{|a^{(i)}\rangle\}_{i=1}^{N}$ form a complete basis for $\mathcal{H}$, and the identity operator can be written as:

$$\hat{I} = \sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|.$$

Inserting the identity operator into the expansion of $|\alpha\rangle$, we have:

$$|\alpha\rangle = \hat{I}|\alpha\rangle = \sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle.$$

Proof of the Completeness Relation

Let $|\alpha\rangle \in \mathcal{H}$. The action of the operator $\hat{I} = \sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|$ on $|\alpha\rangle$ is:

$$\hat{I}|\alpha\rangle = \left(\sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}| \right) |\alpha\rangle.$$

Expanding this:

$$\hat{I}|\alpha\rangle = \sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle.$$

Since $|\alpha\rangle = \sum_{i=1}^{N} c_{a^{(i)}} |a^{(i)}\rangle$ and $c_{a^{(i)}} = \langle a^{(i)}|\alpha\rangle$, the completeness relation is verified:

$$\hat{I}|\alpha\rangle = |\alpha\rangle.$$

Application of the Completeness Relation

The completeness relation $\hat{I} = \sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|$ allows the insertion of the identity operator at any point in a legal sequence of kets, bras, and operators. For example:

$$\langle\alpha|\alpha\rangle = \langle\alpha|\hat{I}|\alpha\rangle = \sum_{i=1}^{N}\langle\alpha|a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle.$$

Expanding further:

$$\langle\alpha|\alpha\rangle = \sum_{i=1}^{N}|\langle a^{(i)}|\alpha\rangle|^2 = \sum_{i=1}^{N}|c_{a^{(i)}}|^2.$$

Normalization Condition

If $|\alpha\rangle$ is normalized, then:

$$\langle\alpha|\alpha\rangle = 1.$$

Using the expansion:

$$1 = \sum_{i=1}^{N}|c_{a^{(i)}}|^2.$$

1.3.3. Matrix Representations

Matrix Representation Using the Completeness Relation

Let $\hat{X}$ be an operator acting on a Hilbert space $\mathcal{H}$ of dimension N. The completeness relation for the eigenkets $\{|a^{(i)}\rangle\}_{i=1}^{N}$ of a Hermitian operator states:

$$\hat{I} = \sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|.$$

Using the identity operator $\hat{I}$, we can express $\hat{X}$ as:

$$\hat{X} = \hat{I} \cdot \hat{X} \cdot \hat{I}.$$

Expanding $\hat{I}$ in terms of the eigenkets:

$$\hat{X} = \left(\sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}| \right) \cdot \hat{X} \cdot \left(\sum_{j=1}^{N} |a^{(j)}\rangle\langle a^{(j)}| \right).$$

Simplifying, we get:

$$\hat{X} = \sum_{i=1}^{N} \sum_{j=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|\hat{X}|a^{(j)}\rangle\langle a^{(j)}|.$$

The scalar quantity $\langle a^{(i)}|\hat{X}|a^{(j)}\rangle$ is called the **matrix element** of $\hat{X}$ in the basis $\{|a^{(i)}\rangle\}$. Thus, we write:

$$\hat{X} = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle a^{(i)}|\hat{X}|a^{(j)}\rangle |a^{(i)}\rangle\langle a^{(j)}|.$$

Matrix Representation

The operator $\hat{X}$ is represented by the $N \times N$ matrix X with elements:

$$X_{ij} = \langle a^{(i)}|\hat{X}|a^{(j)}\rangle,$$

where i corresponds to the row index, and j corresponds to the column index.
The full matrix representation of $\hat{X}$ is:

$$X = \begin{bmatrix} \langle a^{(1)}|\hat{X}|a^{(1)}\rangle & \langle a^{(1)}|\hat{X}|a^{(2)}\rangle & \cdots & \langle a^{(1)}|\hat{X}|a^{(N)}\rangle \\ \langle a^{(2)}|\hat{X}|a^{(1)}\rangle & \langle a^{(2)}|\hat{X}|a^{(2)}\rangle & \cdots & \langle a^{(2)}|\hat{X}|a^{(N)}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a^{(N)}|\hat{X}|a^{(1)}\rangle & \langle a^{(N)}|\hat{X}|a^{(2)}\rangle & \cdots & \langle a^{(N)}|\hat{X}|a^{(N)}\rangle \end{bmatrix}.$$

Properties of the Matrix Representation

1. **Number of Matrix Elements**: There are N^2 elements in the matrix representation of $\hat{X}$. Each element corresponds to $\langle a^{(i)}|\hat{X}|a^{(j)}\rangle$.
2. **Orthonormality of Basis Vectors**: The basis vectors satisfy:

$$\langle a^{(i)}|a^{(j)}\rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.
3. **Action of $\hat{X}$ on a Ket**: If $|\psi\rangle$ is a ket written in the basis $\{|a^{(i)}\rangle\}$:

$$|\psi\rangle = \sum_{i=1}^{N} c_{a^{(i)}} |a^{(i)}\rangle,$$

then the action of $\hat{X}$ on $|\psi\rangle$ is:

$$\hat{X}|\psi\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle a^{(i)}|\hat{X}|a^{(j)}\rangle c_{a^{(j)}} |a^{(i)}\rangle.$$

Matrix Representations: Hermitian Adjoint

Definition of Hermitian Adjoint

For an operator $\hat{X}$ acting on a Hilbert space $\mathcal{H}$, the **Hermitian adjoint** (denoted as $\hat{X}^\dagger$) is defined such that:

$$\forall|\alpha\rangle, |\beta\rangle \in \mathcal{H}, \quad \langle\beta|\hat{X}|\alpha\rangle = \langle\alpha|\hat{X}^\dagger|\beta\rangle^*,$$

where * denotes the complex conjugate.
The Hermitian adjoint of an operator is equivalent to the **complex conjugate transpose** of its matrix representation.

Matrix Representation of $\hat{X}$

Let $\{|a^{(i)}\rangle\}_{i=1}^N$ be an orthonormal basis of the Hilbert space $\mathcal{H}$. The matrix representation of the operator $\hat{X}$ in this basis is given by:

$$X_{ij} = \langle a^{(i)}|\hat{X}|a^{(j)}\rangle, \quad \text{for } i, j \in \{1, 2, \dots, N\}.$$

The full matrix representation of $\hat{X}$ is:

$$X = \begin{bmatrix} \langle a^{(1)}|\hat{X}|a^{(1)}\rangle & \langle a^{(1)}|\hat{X}|a^{(2)}\rangle & \cdots & \langle a^{(1)}|\hat{X}|a^{(N)}\rangle \\ \langle a^{(2)}|\hat{X}|a^{(1)}\rangle & \langle a^{(2)}|\hat{X}|a^{(2)}\rangle & \cdots & \langle a^{(2)}|\hat{X}|a^{(N)}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a^{(N)}|\hat{X}|a^{(1)}\rangle & \langle a^{(N)}|\hat{X}|a^{(2)}\rangle & \cdots & \langle a^{(N)}|\hat{X}|a^{(N)}\rangle \end{bmatrix}.$$

Matrix Representation of $\hat{X}^\dagger$

The Hermitian adjoint $\hat{X}^\dagger$ is represented in the same basis $\{|a^{(i)}\rangle\}$ as the **complex conjugate transpose** of $\hat{X}$. That is:

$$\hat{X}^\dagger \implies X^\dagger = (X^T)^*,$$

where X^T is the transpose of X, and * denotes complex conjugation.
Explicitly, the matrix representation of $\hat{X}^\dagger$ is:

$$X^\dagger = \begin{bmatrix} \langle a^{(1)}|\hat{X}|a^{(1)}\rangle^* & \langle a^{(2)}|\hat{X}|a^{(1)}\rangle^* & \cdots & \langle a^{(N)}|\hat{X}|a^{(1)}\rangle^* \\ \langle a^{(1)}|\hat{X}|a^{(2)}\rangle^* & \langle a^{(2)}|\hat{X}|a^{(2)}\rangle^* & \cdots & \langle a^{(N)}|\hat{X}|a^{(2)}\rangle^* \\ \vdots & \vdots & \ddots & \vdots \\ \langle a^{(1)}|\hat{X}|a^{(N)}\rangle^* & \langle a^{(2)}|\hat{X}|a^{(N)}\rangle^* & \cdots & \langle a^{(N)}|\hat{X}|a^{(N)}\rangle^* \end{bmatrix}.$$

Relation Between X and $X^\dagger$

Each element of $X^\dagger$ is related to the corresponding element of X as:

$$X^\dagger_{ij} = (X_{ji})^*,$$

where $X_{ji} = \langle a^{(j)}|\hat{X}|a^{(i)}\rangle$.

Key Properties

1. **Complex Conjugate Transpose**:

$$X^\dagger = (X^T)^*.$$

2. **Hermitian Operators**: If $\hat{X}$ is Hermitian, then:

$$\hat{X} = \hat{X}^\dagger \implies X = X^\dagger.$$

This implies that X is a real, symmetric matrix:

$$X_{ij} = X^*_{ji}.$$

3. **Diagonal Operators**: If $\hat{X}$ is represented in the basis of its eigenkets, X is diagonal:

$$X = \begin{bmatrix} a^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & a^{(2)} & 0 & \cdots & 0 \\ 0 & 0 & a^{(3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a^{(N)} \end{bmatrix},$$

where $\{a^{(i)}\}$ are the eigenvalues of $\hat{X}$.

Matrix Multiplication and Operator Representations

Matrix Multiplication

Let A and B be two matrices of dimensions $m \times n$ and $n \times p$, respectively. The product $C = AB$ is a matrix of dimension $m \times p$, defined such that the elements c_{ij} of C are given by:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Example with 2×2 Matrices

Consider two 2×2 matrices A and B:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

The product $C = AB$ is given by:

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

where:

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}, \quad c_{12} = a_{11}b_{12} + a_{12}b_{22},$$
$$c_{21} = a_{21}b_{11} + a_{22}b_{21}, \quad c_{22} = a_{21}b_{12} + a_{22}b_{22}.$$

Explicitly:

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

Element Formula

In general, for a matrix product $C = AB$, the element c_{ij} is calculated as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

For example, c_{12} in a 2×2 matrix is given by:

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}.$$

Matrix Multiplication in Terms of Operators

Let $\hat{X}$ and $\hat{Y}$ be two operators on a Hilbert space $\mathcal{H}$. The composite operator $\hat{Z} = \hat{X}\hat{Y}$ acts on a ket $|a^{(i)}\rangle$ as:

$$\hat{Z}|a^{(i)}\rangle = (\hat{X}\hat{Y})|a^{(i)}\rangle.$$

The matrix representation of $\hat{Z}$ in the basis $\{|a^{(i)}\rangle\}$ is:

$$Z_{ij} = \sum_{k=1}^{N} X_{ik} Y_{kj},$$

where $X_{ik} = \langle a^{(i)}|\hat{X}|a^{(k)}\rangle$ and $Y_{kj} = \langle a^{(k)}|\hat{Y}|a^{(j)}\rangle$.

Verification of Matrix Multiplication Rules

The composite operator $\hat{Z} = \hat{X}\hat{Y}$ satisfies:

$$\langle a^{(i)}|\hat{Z}|a^{(j)}\rangle = \langle a^{(i)}|(\hat{X}\hat{Y})|a^{(j)}\rangle.$$

Using the completeness relation $\hat{I} = \sum_{k=1}^{N} |a^{(k)}\rangle\langle a^{(k)}|$, we have:

$$\langle a^{(i)}|\hat{Z}|a^{(j)}\rangle = \langle a^{(i)}|\hat{X}\left(\sum_{k=1}^{N} |a^{(k)}\rangle\langle a^{(k)}|\right)\hat{Y}|a^{(j)}\rangle.$$

Simplifying:

$$\langle a^{(i)}|\hat{Z}|a^{(j)}\rangle = \sum_{k=1}^{N}\langle a^{(i)}|\hat{X}|a^{(k)}\rangle\langle a^{(k)}|\hat{Y}|a^{(j)}\rangle.$$

Thus:

$$Z_{ij} = \sum_{k=1}^{N} X_{ik}Y_{kj}.$$

Key Properties of Matrix Multiplication

1. **Associativity**:
$$(AB)C = A(BC), \quad \forall A, B, C.$$

2. **Non-commutativity** (in general):
$$AB \neq BA.$$

3. **Linearity**:
$$A(B + C) = AB + AC, \quad (B + C)A = BA + CA.$$

4. **Action on Operators**: For operators $\hat{X}, \hat{Y}, \hat{Z}$:
$$\hat{Z} = \hat{X}\hat{Y} \implies Z_{ij} = \sum_{k} X_{ik}Y_{kj}.$$

Matrix Representations of Kets and Operators

Representing a Ket as a Column Matrix

Let $|\alpha\rangle$ be a ket in a Hilbert space $\mathcal{H}$, and let $\{|a^{(i)}\rangle\}_{i=1}^{N}$ be an orthonormal basis of $\mathcal{H}$. The ket $|\alpha\rangle$ can be expressed as:

$$|\alpha\rangle = \sum_{i=1}^{N} c_i|a^{(i)}\rangle,$$

where $c_i = \langle a^{(i)}|\alpha\rangle$ are the expansion coefficients.
The representation of $|\alpha\rangle$ in this basis is given as a column matrix:

$$|\alpha\rangle \sim \begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{bmatrix},$$

which is an $N \times 1$ column vector.

Action of an Operator on a Ket

Let $\hat{X}$ be an operator acting on $\mathcal{H}$. The action of $\hat{X}$ on $|\alpha\rangle$ is represented as:

$$\hat{X}|\alpha\rangle = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} X_{ij} \langle a^{(j)}|\alpha\rangle \right) |a^{(i)}\rangle,$$

where $X_{ij} = \langle a^{(i)}|\hat{X}|a^{(j)}\rangle$ are the matrix elements of $\hat{X}$ in the basis $\{|a^{(i)}\rangle\}$.
In matrix form:

$$\hat{X}|\alpha\rangle \sim \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{21} & X_{22} & \cdots & X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ X_{N1} & X_{N2} & \cdots & X_{NN} \end{bmatrix} \begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{bmatrix}.$$

Matrix Multiplication Interpretation

The product of the operator matrix X and the column matrix $|\alpha\rangle$ results in another column matrix:

$$|\beta\rangle = \hat{X}|\alpha\rangle \sim \begin{bmatrix} \langle a^{(1)}|\beta\rangle \\ \langle a^{(2)}|\beta\rangle \\ \vdots \\ \langle a^{(N)}|\beta\rangle \end{bmatrix}.$$

The element $\langle a^{(i)}|\beta\rangle$ is given by:

$$\langle a^{(i)}|\beta\rangle = \sum_{j=1}^{N} X_{ij} \langle a^{(j)}|\alpha\rangle.$$

This matches the rule for matrix multiplication:

$$b_i = \sum_{j=1}^{N} X_{ij} a_j,$$

where $b_i = \langle a^{(i)}|\beta\rangle$ and $a_j = \langle a^{(j)}|\alpha\rangle$.

Explicit Representation

Let $|\alpha\rangle$ and $|\beta\rangle$ be represented as:

$$|\alpha\rangle \sim \begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{bmatrix}, \quad |\beta\rangle \sim \begin{bmatrix} \langle a^{(1)}|\beta\rangle \\ \langle a^{(2)}|\beta\rangle \\ \vdots \\ \langle a^{(N)}|\beta\rangle \end{bmatrix}.$$

The operator $\hat{X}$ is represented as:

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{21} & X_{22} & \cdots & X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ X_{N1} & X_{N2} & \cdots & X_{NN} \end{bmatrix}.$$

The matrix equation is:

$$|\beta\rangle = X|\alpha\rangle.$$

Key Properties

1. **Column Vector Representation of Kets:** A ket $|\alpha\rangle$ is represented as an $N \times 1$ column vector in the basis $\{|a^{(i)}\rangle\}$.

2. **Matrix Representation of Operators:** An operator $\hat{X}$ is represented as an $N \times N$ matrix X, where $X_{ij} = \langle a^{(i)}|\hat{X}|a^{(j)}\rangle$.

3. **Matrix Multiplication:** The action of $\hat{X}$ on $|\alpha\rangle$ is equivalent to the matrix product $|\beta\rangle = X|\alpha\rangle$.

4. **Completeness Relation:** The completeness relation ensures that the matrix multiplication corresponds to the linear operator acting on the Hilbert space.

Matrix Representations of Bras and Operators

Representing a Bra as a Row Matrix

Let $\langle\gamma|$ be a bra in a Hilbert space $\mathcal{H}$, and let $\{|a^{(i)}\rangle\}_{i=1}^N$ be an orthonormal basis of $\mathcal{H}$. The bra $\langle\gamma|$ can be expressed as:

$$\langle\gamma| = \sum_{i=1}^{N} c_i^* \langle a^{(i)}|,$$

where $c_i = \langle a^{(i)}|\gamma\rangle$. The coefficients c_i^* are the complex conjugates of the expansion coefficients of the corresponding ket $|\gamma\rangle$.
The representation of $\langle\gamma|$ in this basis is a **row matrix**:

$$\langle\gamma| \sim \left[\langle\gamma|a^{(1)}\rangle, \langle\gamma|a^{(2)}\rangle, \dots, \langle\gamma|a^{(N)}\rangle\right],$$

which is a $1 \times N$ matrix.

Action of a Bra on an Operator

When a bra $\langle\gamma|$ acts on an operator $\hat{X}$, the result is another bra:

$$\langle\gamma|\hat{X} = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \langle\gamma|a^{(i)}\rangle X_{ij} \right) \langle a^{(j)}|,$$

where $X_{ij} = \langle a^{(i)}|\hat{X}|a^{(j)}\rangle$ are the matrix elements of $\hat{X}$.
In matrix form:

$$\langle\gamma|\hat{X} \sim \left[\langle\gamma|a^{(1)}\rangle, \langle\gamma|a^{(2)}\rangle, \dots, \langle\gamma|a^{(N)}\rangle\right] \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{21} & X_{22} & \cdots & X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ X_{N1} & X_{N2} & \cdots & X_{NN} \end{bmatrix}.$$

Action of a Bra on a Ket

The action of a bra $\langle\gamma|$ on a ket $|\alpha\rangle$ results in a scalar:

$$\langle\gamma|\alpha\rangle = \sum_{i=1}^{N} \langle\gamma|a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle.$$

Explicitly, in matrix form:

$$\langle\gamma|\alpha\rangle = \left[\langle\gamma|a^{(1)}\rangle, \langle\gamma|a^{(2)}\rangle, \dots, \langle\gamma|a^{(N)}\rangle\right] \begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{bmatrix}.$$

Matrix Representation of Operators

Let $\hat{X}$ be an operator on $\mathcal{H}$. The action of $\langle\gamma|\hat{X}|\alpha\rangle$ is given by:

$$\langle\gamma|\hat{X}|\alpha\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N}\langle\gamma|a^{(i)}\rangle\langle a^{(i)}|\hat{X}|a^{(j)}\rangle\langle a^{(j)}|\alpha\rangle.$$

In matrix notation:

$$\langle\gamma|\hat{X}|\alpha\rangle = \left[\langle\gamma|a^{(1)}\rangle, \langle\gamma|a^{(2)}\rangle, \dots, \langle\gamma|a^{(N)}\rangle\right] \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{21} & X_{22} & \cdots & X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ X_{N1} & X_{N2} & \cdots & X_{NN} \end{bmatrix} \begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{bmatrix}.$$

Key Properties

1. **Row Matrix Representation of Bras:** A bra $\langle\gamma|$ is represented as a $1 \times N$ row vector in the basis $\{|a^{(i)}\rangle\}$.
2. **Matrix Representation of Operators:** An operator $\hat{X}$ is represented as an $N \times N$ matrix X, where $X_{ij} = \langle a^{(i)}|\hat{X}|a^{(j)}\rangle$.
3. **Matrix Multiplication:** The action of $\langle\gamma|\hat{X}$ is equivalent to the product of a $1 \times N$ row vector and an $N \times N$ matrix.
4. **Scalar Product:** The action of $\langle\gamma|\alpha\rangle$ is equivalent to the dot product of a $1 \times N$ row vector and an $N \times 1$ column vector, resulting in a scalar.

Matrix Representations: Examples and Inner Product

Example: Representing Kets and Bras

Consider the ket $|\alpha\rangle$ and its corresponding bra $\langle\alpha|$, expanded in terms of the basis $\{|a^{(1)}\rangle, |a^{(2)}\rangle\}$:

$$|\alpha\rangle = -\frac{3i}{5}|a^{(1)}\rangle + \frac{4i}{5}|a^{(2)}\rangle.$$

The corresponding bra $\langle\alpha|$ is the Hermitian conjugate:

$$\langle\alpha| = \left(-\frac{3i}{5}\right)^{*}\langle a^{(1)}| + \left(\frac{4i}{5}\right)^{*}\langle a^{(2)}|,$$

where * denotes the complex conjugate. Explicitly:

$$\langle\alpha| = \frac{3i}{5}\langle a^{(1)}| - \frac{4i}{5}\langle a^{(2)}|.$$

In matrix representation:

$$|\alpha\rangle \sim \begin{bmatrix} -\frac{3i}{5} \\ \frac{4i}{5} \end{bmatrix}, \quad \langle\alpha| \sim \begin{bmatrix} \frac{3i}{5} & -\frac{4i}{5} \end{bmatrix}.$$

General Representation of Kets and Bras

For a ket $|\alpha\rangle = \sum_{i=1}^{N} c_i|a^{(i)}\rangle$, the bra $\langle\alpha|$ is represented as:

$$\langle\alpha| = \sum_{i=1}^{N} c_i^{*}\langle a^{(i)}|.$$

In matrix form:

$$|\alpha\rangle \sim \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}, \quad \langle\alpha| \sim \begin{bmatrix} c_1^{*} & c_2^{*} & \cdots & c_N^{*} \end{bmatrix}.$$

Example: Inner Product

The inner product $\langle\beta|\alpha\rangle$ is defined as:

$$\langle\beta|\alpha\rangle = \langle\beta| \left(\sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}| \right) |\alpha\rangle,$$

using the completeness relation $\sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}| = \hat{I}$.
Expanding:

$$\langle\beta|\alpha\rangle = \sum_{i=1}^{N} \langle\beta|a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle.$$

In matrix notation:

$$\langle\beta|\alpha\rangle = \begin{bmatrix} \langle\beta|a^{(1)}\rangle & \langle\beta|a^{(2)}\rangle & \cdots & \langle\beta|a^{(N)}\rangle \end{bmatrix} \begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{bmatrix}.$$

The result is a scalar (a 1×1 matrix):

$$\langle\beta|\alpha\rangle = \sum_{i=1}^{N} \langle\beta|a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle.$$

Key Properties of the Inner Product

1. **Linearity:**
$$\langle\beta|(c_1|\alpha_1\rangle + c_2|\alpha_2\rangle) = c_1\langle\beta|\alpha_1\rangle + c_2\langle\beta|\alpha_2\rangle, \quad \forall c_1, c_2 \in \mathbb{C}.$$

2. **Conjugate Symmetry:**
$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*.$$

3. **Positive-Definiteness:**
$$\langle\alpha|\alpha\rangle \geq 0, \quad \langle\alpha|\alpha\rangle = 0 \iff |\alpha\rangle = 0.$$

Matrix Representations: Outer Product

Representing the Outer Product

The **outer product** of a ket $|\beta\rangle$ and a bra $\langle\alpha|$ is denoted by $|\beta\rangle\langle\alpha|$. This is an operator that acts on a Hilbert space $\mathcal{H}$. Let $|\beta\rangle$ and $|\alpha\rangle$ be expanded in terms of an orthonormal basis $\{|a^{(i)}\rangle\}_{i=1}^{N}$:

$$|\beta\rangle = \sum_{i=1}^{N} \langle a^{(i)}|\beta\rangle|a^{(i)}\rangle, \quad \langle\alpha| = \sum_{j=1}^{N} \langle\alpha|a^{(j)}\rangle^*\langle a^{(j)}|.$$

The outer product $|\beta\rangle\langle\alpha|$ is given by:

$$|\beta\rangle\langle\alpha| = \left(\sum_{i=1}^{N} \langle a^{(i)}|\beta\rangle|a^{(i)}\rangle \right) \left(\sum_{j=1}^{N} \langle\alpha|a^{(j)}\rangle^*\langle a^{(j)}| \right).$$

Expanding:

$$|\beta\rangle\langle\alpha| = \sum_{i=1}^{N}\sum_{j=1}^{N} \langle a^{(i)}|\beta\rangle\langle\alpha|a^{(j)}\rangle^*|a^{(i)}\rangle\langle a^{(j)}|.$$

Matrix Representation of the Outer Product

The matrix representation of $|\beta\rangle\langle\alpha|$ in the basis $\{|a^{(i)}\rangle\}$ is:

$$|\beta\rangle\langle\alpha| \sim \begin{bmatrix} \langle a^{(1)}|\beta\rangle \\ \langle a^{(2)}|\beta\rangle \\ \vdots \\ \langle a^{(N)}|\beta\rangle \end{bmatrix} \begin{bmatrix} \langle\alpha|a^{(1)}\rangle^* & \langle\alpha|a^{(2)}\rangle^* & \cdots & \langle\alpha|a^{(N)}\rangle^* \end{bmatrix}.$$

Explicitly, this is an $N \times N$ matrix:

$$|\beta\rangle\langle\alpha| \sim \begin{bmatrix} \langle a^{(1)}|\beta\rangle\langle\alpha|a^{(1)}\rangle^* & \langle a^{(1)}|\beta\rangle\langle\alpha|a^{(2)}\rangle^* & \cdots & \langle a^{(1)}|\beta\rangle\langle\alpha|a^{(N)}\rangle^* \\ \langle a^{(2)}|\beta\rangle\langle\alpha|a^{(1)}\rangle^* & \langle a^{(2)}|\beta\rangle\langle\alpha|a^{(2)}\rangle^* & \cdots & \langle a^{(2)}|\beta\rangle\langle\alpha|a^{(N)}\rangle^* \\ \vdots & \vdots & \ddots & \vdots \\ \langle a^{(N)}|\beta\rangle\langle\alpha|a^{(1)}\rangle^* & \langle a^{(N)}|\beta\rangle\langle\alpha|a^{(2)}\rangle^* & \cdots & \langle a^{(N)}|\beta\rangle\langle\alpha|a^{(N)}\rangle^* \end{bmatrix}.$$

Action of the Outer Product on a Ket

The outer product $|\beta\rangle\langle\alpha|$ acts on a ket $|\gamma\rangle$ as:

$$(|\beta\rangle\langle\alpha|)\,|\gamma\rangle = |\beta\rangle\,(\langle\alpha|\gamma\rangle)\,.$$

This means the operator projects $|\gamma\rangle$ onto $|\alpha\rangle$ and scales $|\beta\rangle$ by the scalar $\langle\alpha|\gamma\rangle$.

Example

Consider a two-dimensional Hilbert space with basis $\{|a^{(1)}\rangle, |a^{(2)}\rangle\}$. Let:

$$|\beta\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \langle\alpha| = \begin{bmatrix} 3 & 4 \end{bmatrix}.$$

The outer product $|\beta\rangle\langle\alpha|$ is:

$$|\beta\rangle\langle\alpha| = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 & 1\cdot 4 \\ 2\cdot 3 & 2\cdot 4 \end{bmatrix}.$$

Simplifying:

$$|\beta\rangle\langle\alpha| = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}.$$

Key Properties of the Outer Product

1. **Rank-One Operator:** The outer product $|\beta\rangle\langle\alpha|$ is a rank-one operator.
2. **Action on Kets:** For any $|\gamma\rangle$:
$$(|\beta\rangle\langle\alpha|)\,|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle.$$
3. **Matrix Representation:** The matrix representation of $|\beta\rangle\langle\alpha|$ is an $N \times N$ matrix constructed from the elements $\langle a^{(i)}|\beta\rangle$ and $\langle\alpha|a^{(j)}\rangle^*$.

1.3.4. Spin $\frac{1}{2}$*Systems*

Spin-$\frac{1}{2}$ Systems

The Base Kets

The spin-$\frac{1}{2}$ system has two orthonormal basis kets, denoted as $|+\rangle$ (spin-up) and $|-\rangle$ (spin-down). These are represented in the standard basis as:

$$|+\rangle \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |-\rangle \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The completeness relation is given by:

$$\hat{I} = |+\rangle\langle+| + |-\rangle\langle-|.$$

Explicitly, in matrix form:

$$\hat{I} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The S_z Operator

The operator S_z measures the spin along the z-axis. It acts on the basis kets as:

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle, \quad S_z|-\rangle = -\frac{\hbar}{2}|-\rangle.$$

The matrix representation of S_z in the $\{|+\rangle, |-\rangle\}$ basis is:

$$S_z = \begin{bmatrix} \langle+|S_z|+\rangle & \langle+|S_z|-\rangle \\ \langle-|S_z|+\rangle & \langle-|S_z|-\rangle \end{bmatrix}.$$

Computing the elements:

$$\langle+|S_z|+\rangle = \frac{\hbar}{2}, \quad \langle-|S_z|-\rangle = -\frac{\hbar}{2}, \quad \langle+|S_z|-\rangle = 0, \quad \langle-|S_z|+\rangle = 0.$$

Thus:

$$S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The S_+ and S_- Operators

The ladder operators S_+ and S_- are defined as:

$$S_+ = \hbar|+\rangle\langle-|, \quad S_- = \hbar|-\rangle\langle+|.$$

Their action on the basis kets is:

$$S_+|+\rangle = 0, \quad S_+|-\rangle = \hbar|+\rangle,$$
$$S_-|-\rangle = 0, \quad S_-|+\rangle = \hbar|-\rangle.$$

The matrix representation of S_+ is:

$$S_+ = \begin{bmatrix} \langle+|S_+|+\rangle & \langle+|S_+|-\rangle \\ \langle-|S_+|+\rangle & \langle-|S_+|-\rangle \end{bmatrix}.$$

Computing the elements:

$$\langle+|S_+|+\rangle = 0, \quad \langle-|S_+|+\rangle = \hbar, \quad \langle+|S_+|-\rangle = 0, \quad \langle-|S_+|-\rangle = 0.$$

Thus:

$$S_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Similarly, the matrix representation of S_- is:

$$S_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Hermiticity of the Operators

The operator S_z is Hermitian:

$$S_z^\dagger = S_z.$$

However, S_+ and S_- are non-Hermitian:

$$S_+^\dagger = S_- \neq S_+.$$

Key Properties of Spin-$\frac{1}{2}$ Operators

1. **Orthonormality of Basis Kets:**

$$\langle +|+\rangle = 1, \quad \langle -|-\rangle = 1, \quad \langle +|-\rangle = 0, \quad \langle -|+\rangle = 0.$$

2. **Completeness Relation:**

$$\hat{I} = |+\rangle\langle +| + |-\rangle\langle -|.$$

3. **Eigenvalues of S_z:**

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle, \quad S_z|-\rangle = -\frac{\hbar}{2}|-\rangle.$$

4. **Ladder Operators:**

$$S_+|+\rangle = 0, \quad S_+|-\rangle = \hbar|+\rangle,$$
$$S_-|-\rangle = 0, \quad S_-|+\rangle = \hbar|-\rangle.$$

1.4. Measurements, Observables, and the Uncertainty Relations

1.4.1. Measurments

In quantum mechanics, a **measurement** of an observable $\hat{A}$ always causes the system to transition into an eigenstate of $\hat{A}$. The result of the measurement is one of the eigenvalues of $\hat{A}$.

1. **Before Measurement:** Let the system initially be in a state $|\psi\rangle$, which can be expressed as a linear combination of the eigenstates $\{|a^{(i)}\rangle\}_{i=1}^N$ of $\hat{A}$:

$$|\psi\rangle = \sum_{i=1}^{N} c_i |a^{(i)}\rangle,$$

2. **After Measurement:** Upon measuring $\hat{A}$, the system is projected into one of the eigenstates $|a^{(i)}\rangle$ of $\hat{A}$. The result of the measurement is the eigenvalue $a^{(i)}$ associated with $|a^{(i)}\rangle$.

3. **Post-Measurement State:** After the measurement, the state of the system becomes:

$$|\psi\rangle \xrightarrow{\text{measurement}} |a^{(i)}\rangle.$$

Postulate: Probability of Measurement

The probability of measuring the eigenvalue $a^{(i)}$ (and the system transitioning into the eigenstate $|a^{(i)}\rangle$) is given by:

$$P(a^{(i)}) = |\langle a^{(i)}|\psi\rangle|^2.$$

This result holds under the condition that $|\psi\rangle$ is normalized:

$$\langle \psi|\psi\rangle = 1.$$

If $|\psi\rangle = \sum_{i=1}^N c_i |a^{(i)}\rangle$, then:

$$P(a^{(i)}) = |c_i|^2.$$

Example: Ensemble Interpretation

Consider an ensemble of 1000 identical quantum systems, all prepared in the same state $|\psi\rangle$. A measurement of $\hat{A}$ is performed on each system. The outcomes are as follows:

1. Each measurement yields one of the eigenvalues $a^{(1)}, a^{(2)}, \ldots, a^{(N)}$.
2. The relative frequency of measuring $a^{(i)}$ is proportional to $|\langle a^{(i)}|\psi\rangle|^2 = |c_i|^2$.
3. For instance, if $|\psi\rangle = c_1|a^{(1)}\rangle + c_2|a^{(2)}\rangle + \cdots + c_N|a^{(N)}\rangle$, the probability of obtaining $a^{(2)}$ is:
$$P(a^{(2)}) = |c_2|^2 = |\langle a^{(2)}|\psi\rangle|^2.$$

Derivation of Probability Formula

Let $\hat{A}$ be an observable with eigenstates $\{|a^{(i)}\rangle\}_{i=1}^N$ and eigenvalues $\{a^{(i)}\}_{i=1}^N$. A general state $|\psi\rangle$ can be expressed as:

$$|\psi\rangle = \sum_{i=1}^{N} c_i|a^{(i)}\rangle,$$

where $c_i = \langle a^{(i)}|\psi\rangle$.

The probability $P(a^{(i)})$ of measuring the eigenvalue $a^{(i)}$ is the square of the projection of $|\psi\rangle$ onto $|a^{(i)}\rangle$:

$$P(a^{(i)}) = |\langle a^{(i)}|\psi\rangle|^2.$$

The matrix representation of the probability calculation involves the coefficients c_i. Let:

$$|\psi\rangle \sim \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}, \quad |a^{(i)}\rangle \sim \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(with 1 in the } i\text{-th position).}$$

The projection $\langle a^{(i)}|\psi\rangle$ is:

$$\langle a^{(i)}|\psi\rangle = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = c_i.$$

Thus:

$$P(a^{(i)}) = |\langle a^{(i)}|\psi\rangle|^2 = |c_i|^2.$$

Postulate: Normalized Probability

The **probability** of measuring the eigenvalue $a^{(i)}$ (and the system jumping into the eigenstate $|a^{(i)}\rangle$) is given by:

$$P(a^{(i)}) = |\langle a^{(i)}|\psi\rangle|^2,$$

provided that the state $|\psi\rangle$ is **normalized**:

$$\langle\psi|\psi\rangle = 1.$$

Properties of the Probability Function

1. **Non-negativity:**
$$P(a^{(i)}) = |\langle a^{(i)}|\psi\rangle|^2 \geq 0, \quad \forall i.$$
2. **Normalization:** The sum of probabilities over all possible outcomes must equal 1:
$$\sum_{i=1}^{N} |\langle a^{(i)}|\psi\rangle|^2 = 1.$$

Proof of Normalization

To show that the probabilities sum to 1, start with the normalized state:

$$|\psi\rangle = \sum_{i=1}^{N} c_i|a^{(i)}\rangle, \quad \text{where } c_i = \langle a^{(i)}|\psi\rangle.$$

The total probability is:

$$\sum_{i=1}^{N} |\langle a^{(i)}|\psi\rangle|^2 = \sum_{i=1}^{N} |c_i|^2.$$

Expanding:

$$\sum_{i=1}^{N} |c_i|^2 = \sum_{i=1}^{N}\langle\psi|a^{(i)}\rangle\langle a^{(i)}|\psi\rangle.$$

Using the completeness relation:

$$\hat{I} = \sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|,$$

we have:

$$\langle\psi|\psi\rangle = \langle\psi| \left(\sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|\right) |\psi\rangle = \sum_{i=1}^{N} |\langle a^{(i)}|\psi\rangle|^2 = 1.$$

Thus, the normalization condition is satisfied.

Expectation Value of an Observable

The **expectation value** of an observable $\hat{A}$, measured with respect to a quantum state $|\psi\rangle$, is denoted by:

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle.$$

Expanding $|\psi\rangle$ in terms of the eigenstates of $\hat{A}$:

$$|\psi\rangle = \sum_{i=1}^{N} c_i|a^{(i)}\rangle, \quad c_i = \langle a^{(i)}|\psi\rangle.$$

Derivation of the Expectation Value

The expectation value is:

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle.$$

Substitute the completeness relation:

$$\langle A\rangle = \langle\psi| \left(\sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|\right) \hat{A} \left(\sum_{j=1}^{N} |a^{(j)}\rangle\langle a^{(j)}|\right) |\psi\rangle.$$

Expanding:

$$\langle A\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N}\langle\psi|a^{(i)}\rangle\langle a^{(i)}|\hat{A}|a^{(j)}\rangle\langle a^{(j)}|\psi\rangle.$$

Since $|a^{(i)}\rangle$ are eigenstates of $\hat{A}$:

$$\hat{A}|a^{(i)}\rangle = a^{(i)}|a^{(i)}\rangle.$$

This reduces to:

$$\langle A\rangle = \sum_{i=1}^{N}\langle\psi|a^{(i)}\rangle a^{(i)}\langle a^{(i)}|\psi\rangle.$$

Simplifying:

$$\langle A\rangle = \sum_{i=1}^{N} |c_i|^2 a^{(i)},$$

where $|c_i|^2 = |\langle a^{(i)}|\psi\rangle|^2$ is the probability of measuring $a^{(i)}$.

Interpretation

The expectation value $\langle A \rangle$ represents the **weighted average** of the eigenvalues $\{a^{(i)}\}$, with weights given by the probabilities $|c_i|^2$.

Matrix Representation

Let the state $|\psi\rangle$ and eigenstates $|a^{(i)}\rangle$ be represented as:

$$|\psi\rangle \sim \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}, \quad \langle\psi| \sim \begin{bmatrix} c_1^* & c_2^* & \cdots & c_N^* \end{bmatrix}.$$

The operator $\hat{A}$ in the $\{|a^{(i)}\rangle\}$ basis is diagonal:

$$\hat{A} \sim \begin{bmatrix} a^{(1)} & 0 & \cdots & 0 \\ 0 & a^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{(N)} \end{bmatrix}.$$

The expectation value is computed as:

$$\langle A \rangle = \langle\psi|\hat{A}|\psi\rangle = \begin{bmatrix} c_1^* & c_2^* & \cdots & c_N^* \end{bmatrix} \begin{bmatrix} a^{(1)} & 0 & \cdots & 0 \\ 0 & a^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{(N)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}.$$

Performing the multiplication:

$$\langle A \rangle = \sum_{i=1}^{N} |c_i|^2 a^{(i)}.$$

Example: Measurement of an Observable

Consider an operator $\hat{A}$ with two eigenstates $\{|0.1\rangle, |0.2\rangle\}$ and eigenvalues $\{0.1, 0.2\}$. The system is initially in the state:

$$|\psi\rangle = c_1|0.1\rangle + c_2|0.2\rangle, \quad \text{where } c_1 = 0.6,\ c_2 = 0.8.$$

The state $|\psi\rangle$ is normalized:

$$|c_1|^2 + |c_2|^2 = 0.6^2 + 0.8^2 = 0.36 + 0.64 = 1.$$

Ensemble Interpretation

An ensemble of 1000 identical systems is prepared in the state $|\psi\rangle$. Measurements of $\hat{A}$ are performed on each system. The possible results are:

1. Eigenvalue 0.1 with eigenstate $|0.1\rangle$,
2. Eigenvalue 0.2 with eigenstate $|0.2\rangle$.

The probabilities of obtaining these results are:

$$P(0.1) = |c_1|^2 = 0.36, \quad P(0.2) = |c_2|^2 = 0.64.$$

The frequencies observed in 1000 measurements are approximately:

$$\text{Frequency of } 0.1 \approx 360, \quad \text{Frequency of } 0.2 \approx 640.$$

Expectation Value of $\hat{A}$

The expectation value of $\hat{A}$ is the weighted average of the eigenvalues:

$$\langle A\rangle = \sum_i P(a^{(i)})a^{(i)}.$$

Substituting the values:

$$\langle A\rangle = P(0.1)(0.1) + P(0.2)(0.2) = 0.36(0.1) + 0.64(0.2).$$

Simplify:

$$\langle A\rangle = 0.036 + 0.128 = 0.164.$$

1.4.2. Spin-$\frac{1}{2}$ Systems, Once Again

Determination of $|S_x;\pm\rangle$, $|S_y;\pm\rangle$, S_x, and S_y in terms of $|S_z;\pm\rangle = |\pm\rangle$ Basis

The spin-$\frac{1}{2}$ system is described by the eigenstates $|S_z;+\rangle$ and $|S_z;-\rangle$ of the operator $\hat{S}_z$:

$$\hat{S}_z|S_z;+\rangle = \frac{\hbar}{2}|S_z;+\rangle, \quad \hat{S}_z|S_z;-\rangle = -\frac{\hbar}{2}|S_z;-\rangle.$$

These states represent the spin components along the z-axis. When a spin-$\frac{1}{2}$ particle passes through a Stern-Gerlach S_z-beam splitter, the beam splits into two components: $|S_z;+\rangle$ and $|S_z;-\rangle$.

Beam Splitting and Probabilities

When the S_z-split beam is passed into another Stern-Gerlach device aligned along S_x, the beams corresponding to $|S_x;+\rangle$ and $|S_x;-\rangle$ emerge. The equal intensities of the split beams indicate equal probabilities, which implies:

$$|c_1|^2 = |c_2|^2 = \frac{1}{2}.$$

The eigenstate $|S_x;+\rangle$ of the operator $\hat{S}_x$ is written as a linear combination of the $\hat{S}_z$-basis states:

$$|S_x;+\rangle = c_1|+\rangle + c_2|-\rangle.$$

Determination of Coefficients

Let the coefficients c_1 and c_2 be complex numbers:

$$c_1 = \frac{1}{\sqrt{2}}e^{i\phi_1}, \quad c_2 = \frac{1}{\sqrt{2}}e^{i\phi_2}.$$

The overall phase factor $e^{i\phi_1}$ can be factored out and ignored without loss of generality. This simplifies the expression for $|S_x;+\rangle$ to:

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i(\phi_2-\phi_1)}|-\rangle.$$

For simplicity, set $\phi_1 = 0$, which gives:

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\phi_2}|-\rangle.$$

Similarly, the eigenstate $|S_x;-\rangle$ is given by:

$$|S_x;-\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}e^{i\phi_2}|-\rangle.$$

Representation of $|S_x;\pm\rangle$ and $|S_y;\pm\rangle$ in Terms of $|S_z;\pm\rangle$

Let $|S_z;+\rangle \equiv |+\rangle$ and $|S_z;-\rangle \equiv |-\rangle$, representing the eigenstates of $\hat{S}_z$. We now express the eigenstates $|S_x;\pm\rangle$ and $|S_y;\pm\rangle$ in terms of the $\hat{S}_z$-basis.

Representation of $|S_x;\pm\rangle$:

The eigenstates of $\hat{S}_x$ can be expressed as:

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right),$$

$$|S_x;-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{i\delta_1}|-\rangle\right),$$

where δ_1 is a relative phase factor.

Orthogonality of $|S_x;\pm\rangle$:

The states $|S_x;+\rangle$ and $|S_x;-\rangle$ must be orthogonal. The inner product:

$$\langle S_x;+|S_x;-\rangle = \frac{1}{2}\left(\langle +| + e^{-i\delta_1}\langle -|\right)\left(|+\rangle - e^{i\delta_1}|-\rangle\right).$$

Simplify using the orthonormality of the $\hat{S}_z$-basis ($\langle +|+\rangle = 1$, $\langle -|-\rangle = 1$, $\langle +|-\rangle = \langle -|+\rangle = 0$):

$$\langle S_x;+|S_x;-\rangle = \frac{1}{2}\left(1 - e^{-i\delta_1}e^{i\delta_1}\right) = \frac{1}{2}(1-1) = 0.$$

This confirms the orthogonality of $|S_x;+\rangle$ and $|S_x;-\rangle$.

Final Representation of $|S_x;\pm\rangle$:

Thus, the eigenstates of $\hat{S}_x$ are:

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right),$$
$$|S_x;-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right).$$

Representation of $|S_y;\pm\rangle$:

The eigenstates of $\hat{S}_y$ involve a phase difference of $\pm i$ between the basis states:

$$|S_y;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right),$$

$$|S_y;-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right).$$

Representation of $|S_y;\pm\rangle$:

The eigenstates $|S_y;\pm\rangle$ of S_y are similarly expressed in the $|S_z;\pm\rangle$ basis:

$$|S_y;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right), \quad |S_y;-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right).$$

Verification of Orthogonality for $|S_y;\pm\rangle$

Compute:

$$\langle S_y;+|S_y;-\rangle = \frac{1}{2}\left(\langle +| + i\langle -|\right)\left(|+\rangle - i|-\rangle\right).$$

Expand:

$$\langle S_y;+|S_y;-\rangle = \frac{1}{2}\left(\langle +|+\rangle - i\langle +|-\rangle + i\langle -|+\rangle - i^2\langle -|-\rangle\right).$$

Using orthonormality:

$$\langle S_y;+|S_y;-\rangle = \frac{1}{2}\left(1-(-1)\right) = 0.$$

Thus, $|S_y;+\rangle$ and $|S_y;-\rangle$ are orthogonal.

Right-Handed Coordinate System

The relative phases of $|S_x\rangle, |S_y\rangle$, and $|S_z\rangle$ are chosen to maintain a right-handed coordinate system. For instance, setting:

$$\delta_1 = 0, \quad \delta_2 = \frac{\pi}{2},$$

ensures consistency with physical conventions.

Matrix Representation of S_x

The operator S_x acts on its eigenstates $|S_x;\pm\rangle$. In the $|+\rangle, |-\rangle$ basis, we use:

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |S_x;-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

The action of S_x on the basis states is given by:

$$S_x|+\rangle = \frac{\hbar}{2}|-\rangle, \quad S_x|-\rangle = \frac{\hbar}{2}|+\rangle.$$

Using the general rule for matrix elements:

$$\langle+|S_x|+\rangle = 0, \quad \langle+|S_x|-\rangle = \frac{\hbar}{2}, \quad \langle-|S_x|+\rangle = \frac{\hbar}{2}, \quad \langle-|S_x|-\rangle = 0.$$

The resulting matrix is:

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Matrix Representation of S_y

The operator S_y acts on its eigenstates $|S_y;\pm\rangle$, which are expressed in the $|+\rangle, |-\rangle$ basis as:

$$|S_y;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle), \quad |S_y;-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle).$$

The action of S_y on the basis states is given by:

$$S_y|+\rangle = \frac{\hbar}{2}i|-\rangle, \quad S_y|-\rangle = -\frac{\hbar}{2}i|+\rangle.$$

Using the general rule for matrix elements:

$$\langle+|S_y|+\rangle = 0, \quad \langle+|S_y|-\rangle = -i\frac{\hbar}{2}, \quad \langle-|S_y|+\rangle = i\frac{\hbar}{2}, \quad \langle-|S_y|-\rangle = 0.$$

The resulting matrix is:

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Summary of Spin Matrices

In the $|+\rangle, |-\rangle$ basis, the spin operators are represented as follows:

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These matrices form the foundation of spin-$\frac{1}{2}$ systems and are essential for understanding quantum mechanics of spin states.

Probabilities and Measurements in Different Bases

Transition Probabilities

When a spin-$\frac{1}{2}$ particle is prepared in an eigenstate of one spin operator (e.g., S_z) and measured along another axis (e.g., S_x), the probability of obtaining a specific result depends on the overlap between the eigenstates of the two operators.

Example: Probability of Measuring $|S_x;+\rangle$ When Prepared in $|S_z;+\rangle$

The eigenstates of S_z are $|S_z;+\rangle \equiv |+\rangle$ and $|S_z;-\rangle \equiv |-\rangle$. The eigenstates of S_x are:

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |S_x;-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

The probability of measuring $|S_x;+\rangle$ when the particle is in $|S_z;+\rangle$ is:

$$P(S_x = +|S_z = +) = |\langle S_x;+|S_z;+\rangle|^2.$$

Compute the inner product:

$$\langle S_x;+|S_z;+\rangle = \frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)|+\rangle = \frac{1}{\sqrt{2}}\langle +|+\rangle = \frac{1}{\sqrt{2}}.$$

Thus:

$$P(S_x = +|S_z = +) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

Similarly:

$$P(S_x = -|S_z = +) = |\langle S_x;-|S_z;+\rangle|^2 = \frac{1}{2}.$$

1.4.3. Compatible Observables

Degeneracy of Eigenvalues:

Given an operator A, its eigenvalue equation is:

$$A|a\rangle = a|a\rangle,$$

where a is the eigenvalue and $|a\rangle$ is the corresponding eigenket.
If there exist two linearly independent eigenkets, $|a_1\rangle$ and $|a_2\rangle$, such that:

$$A|a_1\rangle = a|a_1\rangle, \quad A|a_2\rangle = a|a_2\rangle,$$

then the eigenvalue a is said to be **degenerate**, and $|a_1\rangle$ and $|a_2\rangle$ are **degenerate eigenkets**.

Key Properties of Degeneracy:

1. The eigenvalue alone cannot uniquely label the eigenkets, as multiple eigenkets correspond to the same eigenvalue.

2. Orthogonality cannot be ensured for degenerate eigenkets from the eigenvalue equation alone:

   For degenarate eigenkets, the eigenvalue equation does not impose orthogonality (we cannot use $(a' - a'')\langle a_2|a_1\rangle = 0$). Since $a' = a''$, the factor $(a' - a'')^* = 0$, and:

   $$\langle a_2|a_1\rangle \neq 0 \quad \text{is possible.}$$

3. If the dimensionality of the ket space is larger than the number of distinct eigenvalues of A, there must exist degenerate eigenvalues.

Resolution of Degeneracy:

To label degenerate eigenkets uniquely, we introduce another commuting operator B such that:

$$[A, B] = AB - BA = 0.$$

The eigenkets of A can then be labeled by the eigenvalues of B.

Definition of Compatible Observables:

Two operators A and B are said to **commute** if:

$$[A, B] = AB - BA = 0.$$

Observables A and B are **compatible** if their corresponding operators commute. This means they share a common set of eigenkets.

Theorem:

If A and B are compatible observables and the eigenvalues of A are **non-degenerate**, then the matrix elements $\langle a''|B|a'\rangle$ are all diagonal.

Proof

1. Start with the commutator:
$$[B, A] = BA - AB.$$
For any eigenket $|a'\rangle$ of A:
$$A|a'\rangle = a'|a'\rangle.$$

2. Compute the matrix element:
$$\langle a''|[B, A]|a'\rangle = \langle a''|BA|a'\rangle - \langle a''|AB|a'\rangle.$$
Using $A|a'\rangle = a'|a'\rangle$:
$$\langle a''|AB|a'\rangle = a'\langle a''|B|a'\rangle.$$

3. Similarly:
$$\langle a''|BA|a'\rangle = a''\langle a''|B|a'\rangle.$$

4. Substitute these into the commutator:
$$\langle a''|[B, A]|a'\rangle = a''\langle a''|B|a'\rangle - a'\langle a''|B|a'\rangle.$$
Factorize:
$$\langle a''|[B, A]|a'\rangle = (a'' - a')\langle a''|B|a'\rangle.$$

5. For $a'' \neq a'$, $\langle a''|B|a'\rangle = 0$. Therefore:
$$\langle a''|[B, A]|a'\rangle = 0 \quad \text{if } a'' \neq a'.$$
Hence, $\langle a''|B|a'\rangle$ is diagonal when A has non-degenerate eigenvalues.

Matrix Representation:

Let A and B act on a common eigenbasis $\{|a^{(1)}\rangle, |a^{(2)}\rangle, \ldots, |a^{(N)}\rangle\}$, where A is diagonal:

$$A = \begin{bmatrix} a^{(1)} & 0 & \cdots & 0 \\ 0 & a^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{(N)} \end{bmatrix}.$$

If A and B commute, the matrix B in the same eigenbasis is block-diagonal:

$$B = \begin{bmatrix} b^{(1)} & 0 & \cdots & 0 \\ 0 & b^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^{(N)} \end{bmatrix}.$$

If A has degenerate eigenvalues, B can be used to resolve the degeneracy.

Simultaneous Eigenkets of A and B

The ket $|a'\rangle$ is a simultaneous eigenket of two commuting operators A and B. To demonstrate this, we use the completeness relation of A and express B as follows:

$$B = \mathbb{I}B\mathbb{I} = \left(\sum_{a'} |a'\rangle\langle a'|\right) B \left(\sum_{a''} |a''\rangle\langle a''|\right).$$

Expanding:

$$B = \sum_{a'}\sum_{a''} |a'\rangle\langle a'|B|a''\rangle\langle a''|.$$

Simplifying further:

$$B = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|a'\rangle.$$

Action of B on $|a'\rangle$:

$$B|a'\rangle = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|a'\rangle.$$

Here, the matrix element $\langle a''|B|a'\rangle$ determines the action of B in the basis $\{|a'\rangle\}$.

Resolving Degeneracy:

If the eigenvalue a' of A is **degenerate**, the operator B helps to distinguish the degenerate eigenkets. To account for this, we label the eigenkets using the eigenvalues of both A and B:

$$|a'\rangle \to |a', b'\rangle.$$

Here, b' is the eigenvalue of B, and $|a', b'\rangle$ satisfies:

$$A|a', b'\rangle = a'|a', b'\rangle, \quad B|a', b'\rangle = b'|a', b'\rangle.$$

So, when degeneracy is present, we need both eigenvalues a' and b' to fully label the eigenkets $|a', b'\rangle$. The operator B, which commutes with A, is essential for resolving degeneracy.

Example: Orbital Angular Momentum States

The simultaneous eigenstates of the operators $\hat{L}^2$ (orbital angular momentum squared) and $\hat{L}_z$ (the z-component of angular momentum) are denoted as $|\ell, m_\ell\rangle$, where:

- ℓ: orbital angular momentum quantum number, $\ell = 0, 1, 2, \ldots$,
- m_ℓ: magnetic quantum number, $m_\ell = -\ell, -\ell+1, \ldots, \ell-1, \ell$.

The eigenvalue equations are:

$$\hat{L}^2|\ell, m_\ell\rangle = \hbar^2\ell(\ell+1)|\ell, m_\ell\rangle,$$
$$\hat{L}_z|\ell, m_\ell\rangle = \hbar m_\ell|\ell, m_\ell\rangle.$$

The possible values of ℓ and m_ℓ are summarized below:

ℓ	m_ℓ
0	0
1	-1, 0, 1
2	-2, -1, 0, 1, 2

The table illustrates that for each ℓ, m_ℓ takes $2\ell+1$ possible values ranging symmetrically from $-\ell$ to ℓ.

So:

- $\hat{L}^2$ determines the total orbital angular momentum with eigenvalues proportional to $\ell(\ell+1)$.
- $\hat{L}_z$ determines the projection of the angular momentum along the z-axis, with eigenvalues proportional to m_ℓ.

This example illustrates the concept of simultaneous eigenstates for two commuting operators, $\hat{L}^2$ and $\hat{L}_z$. The quantum numbers ℓ and m_ℓ serve as labels for the eigenstates $|\ell, m_\ell\rangle$, analogous to the labels a' and b' used for general commuting operators A and B. Here, $\hat{L}^2$ resolves the total angular momentum, while $\hat{L}_z$ helps distinguish degenerate states by providing an additional label m_ℓ.

Adding More Observables

If A and B commute, one can introduce additional commuting operators $C, D, \ldots$ to uniquely label the eigenstates:

$$[A, B] = [A, C] = [B, C] = \cdots = 0.$$

The eigenstates can then be labeled as $|a', b', c', \ldots\rangle$, and the completeness relation becomes:

$$1 = \sum_{a', b', c', \ldots} |a', b', c', \ldots\rangle\langle a', b', c', \ldots|.$$

If no more commuting observables can be added, the combination $(a', b', c', \ldots)$ uniquely specifies the eigenstate.

Collective Indices

Orthogonality and Completeness Relations

Let $|a', b', c', \ldots\rangle$ represent the simultaneous eigenkets of a set of compatible observables $A, B, C, \ldots$, labeled collectively by $|K'\rangle$. The orthonormality relation is given by:

$$\langle K''|K'\rangle = \delta_{K''K'} = \delta_{a''a'}\delta_{b''b'}\delta_{c''c'} \cdots ,$$

where $\delta_{K''K'}$ is the Kronecker delta for the collective index.

The completeness relation (closure) is expressed as:

$$\sum_{K'} |K'\rangle\langle K'| = \sum_{a'}\sum_{b'}\sum_{c'} \cdots |a', b', c', \ldots\rangle\langle a', b', c', \ldots| = 1,$$

ensuring that the eigenkets $|a', b', c', \ldots\rangle$ form a complete basis for the Hilbert space.

Physical Interpretation of Degeneracy and Filters

When measurements are performed with two compatible observables A and B, the measurements do not interfere because $[A, B] = 0$. Two cases arise:

1. **No Degeneracy in** A: If A has no degeneracy, the eigenstates $|a'\rangle$ are uniquely labeled by the eigenvalue a'. The measurement process proceeds as follows:

$$|\alpha\rangle \xrightarrow{\text{Filter for } A} |a'\rangle \xrightarrow{\text{Filter for } B} |a', b'\rangle \xrightarrow{\text{Filter for } A} |a', b'\rangle.$$

   Here:

   - The A-filter projects the system onto $|a'\rangle$,
   - The B-filter adds the additional label b', and the system remains in the uniquely labeled state $|a', b'\rangle$.

2. **Degeneracy in** A: If A is degenerate, the eigenvalue a' alone does not uniquely specify the state because multiple eigenstates share the same eigenvalue. The measurement process proceeds as follows:

$$|\alpha\rangle \xrightarrow{\text{Filter for } A} \sum_{i=1}^{n} c_{a'}^{(i)}|a', b^{(i)}\rangle \xrightarrow{\text{Filter for } B} |a', b^{(j)}\rangle \xrightarrow{\text{Filter for } A} |a', b^{(j)}\rangle.$$

   Here:

   - The first A-filter projects the state $|a'\rangle$ into a **linear combination** of degenerate eigenstates $|a', b^{(i)}\rangle$, where:

$$|a'\rangle = \sum_{i=1}^{n} c_{a'}^{(i)}|a', b^{(i)}\rangle,$$

     with n being the degree of degeneracy.
   - The B-filter resolves the degeneracy by collapsing the state into one specific eigenstate $|a', b^{(j)}\rangle$, corresponding to the eigenvalue $b^{(j)}$.
   - The second A-filter leaves the system in the state $|a', b^{(j)}\rangle$, retaining the original eigenvalue a' but now uniquely labeled by $b^{(j)}$.

Summary

Suppose A is degenerate, meaning multiple eigenstates correspond to the same eigenvalue a'. By introducing another compatible observable B, which commutes with A, the degeneracy is resolved as follows:

- $|a', b'\rangle$ is a simultaneous eigenket of A and B, labeled by eigenvalues a' and b'.
- The eigenstates $|a', b'\rangle$ form a complete basis for the system, and the completeness relation holds:

$$\sum_{a',b'} |a', b'\rangle\langle a', b'| = 1.$$

- The measurements of A and B are independent and do not interfere because $[A, B] = 0$.

1.4.4. Incompatible Observables

Definition of Incompatible Observables

Two observables A and B are considered **incompatible** if their operators do not commute, i.e.,

$$[A, B] \neq 0 \quad \text{or equivalently} \quad AB \neq BA.$$

This implies that A and B cannot have a complete set of simultaneous eigenkets. The incompatibility creates fundamental differences in measurement outcomes depending on the order in which the measurements are performed.

Contradiction of Simultaneous Eigenkets

Assume, for contradiction, that A and B do have a complete set of simultaneous eigenkets $|a', b'\rangle$. Then:

$$AB|a', b'\rangle = A(B|a', b'\rangle) = A(b'|a', b'\rangle) = b'(A|a', b'\rangle) = b'(a'|a', b'\rangle),$$
$$BA|a', b'\rangle = B(A|a', b'\rangle) = B(a'|a', b'\rangle) = a'(B|a', b'\rangle) = a'(b'|a', b'\rangle).$$

Thus, we would have:

$$AB|a', b'\rangle = BA|a', b'\rangle \quad \Rightarrow \quad AB = BA,$$

which contradicts the assumption that $[A, B] \neq 0$. Therefore, $|a', b'\rangle$ cannot exist, and incompatible observables do not admit a shared basis of eigenkets.

Measurement Chains with Incompatible Observables

Consider a series of measurements involving incompatible observables A, B, C. The outcomes depend on the specific sequence of measurements. Let the initial state be $|\alpha\rangle$.

1. **First Chain (with B measured)**:

$$|\alpha\rangle \xrightarrow{\text{Filter for } A} |a'\rangle \xrightarrow{\text{Filter for } B} |b'\rangle \xrightarrow{\text{Filter for } C} |c'\rangle.$$

   The total probability for going through all possible b'-routes to reach $|c'\rangle$ is:

$$P_1 = \sum_{b'} |\langle c'|b'\rangle|^2 \, |\langle b'|a'\rangle|^2 .$$

   This can also be expressed using the completeness relation for the intermediate b'-states as:

$$P_1 = \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle.$$

2. **Second Chain (without B measured)**:

$$|\alpha\rangle \xrightarrow{\text{Filter for } A} |a'\rangle = \sum_{b'} |b'\rangle\langle b'|a'\rangle \xrightarrow{\text{Filter for } C} |c'\rangle.$$

   The probability in this case is given directly by:

$$P_2 = |\langle c'|a'\rangle|^2 = \left| \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle \right|^2 .$$

   Expanding this explicitly:

$$P_2 = \sum_{b'} \sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle.$$

3. **Comparison of Probabilities**: It follows that:

$$P_1 \neq P_2,$$

   demonstrating that the outcome of a C-filter depends on whether or not a B-measurement has been made previously.

Impact of Commutation Relations on Measurement Chains

We consider a series of measurements involving observables A, B, and C. The C-filter outcome depends on the commutation relations between the observables. Two scenarios arise:

1. **Case 1:** $[A, B] = 0$

   If A and B commute, the C-filter outcome **does not depend** on whether or not the B-measurement is made. The process is represented as:

$$|\alpha\rangle \xrightarrow{\text{Filter for } A} |a', b'\rangle \xrightarrow{\text{Filter for } B} |a', b'\rangle \xrightarrow{\text{Filter for } C} |c'\rangle,$$
$$|\alpha\rangle \xrightarrow{\text{Filter for } A} |a', b'\rangle \xrightarrow{\text{Filter for } C} |c'\rangle.$$

   - The state $|a', b'\rangle$ is shared by A and B since $[A, B] = 0$. - The C-filter outcome is independent of whether B-filter measurements were performed.

2. **Case 2:** $[B, C] = 0$

   If B and C commute, the C-filter outcome again **does not depend** on whether the B-measurement was made. The process is represented as:

$$|\alpha\rangle \xrightarrow{\text{Filter for } A} |a'\rangle \xrightarrow{\text{Filter for } B} |b', c'\rangle \xrightarrow{\text{Filter for } C} |b', c'\rangle,$$
$$|\alpha\rangle \xrightarrow{\text{Filter for } A} |a'\rangle \xrightarrow{\text{Filter for } C} |b', c'\rangle.$$

   - The B-filter produces $|b', c'\rangle$, and since $[B, C] = 0$, the C-measurement remains unaffected by the B-measurement. - The outcome of the C-filter is consistent in both chains.

We conclude that if $[A, B] = 0$ or $[B, C] = 0$, the C-filter outcome does not depend on whether the intermediate B-measurement has been made. If both commutation relations fail ($[A, B] \neq 0$ and $[B, C] \neq 0$), the C-filter outcome is influenced by the sequence of measurements.

Key Point: C-filter outcome depends on the order of measurements if observables are incompatible.

1.4.5. The Uncertainty Relation

Definition and Variance of an Observable

Let A be an observable. The uncertainty or dispersion in A is defined as:

$$\Delta A = A - \langle A \rangle,$$

where $\langle A \rangle$ is the expectation value of A. The variance, or mean square deviation, is given by:

$$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2.$$

- $\langle A^2 \rangle$: Expectation value of the square of A,
- $\langle A \rangle^2$: Square of the expectation value of A.

Eigenstate Case

If the system is in an eigenstate $|a'\rangle$ of the observable A, then:

$$A|a'\rangle = a'|a'\rangle,$$

where a' is the eigenvalue. In this case:

$$\langle A^2 \rangle = \langle a'|A^2|a'\rangle = (a')^2, \quad \langle A \rangle^2 = \left(\langle a'|A|a'\rangle\right)^2 = (a')^2.$$

Substituting these into the variance formula for A:

$$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2.$$

We find:

$$\langle (\Delta A)^2 \rangle = (a')^2 - (a')^2 = 0.$$

Thus, the uncertainty (variance) in A vanishes when the state is an eigenstate of A.

Application to a Spin-$\frac{1}{2}$ System

Sharp Observable: ΔS_z For a spin-$\frac{1}{2}$ system in the S_z eigenstate $|+\rangle$ (spin-up state), we compute the uncertainty in S_z. Since:

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle,$$

the expectation values are:

$$\langle S_z\rangle = \langle +|S_z|+\rangle = \frac{\hbar}{2}, \quad \langle S_z^2\rangle = \left(\frac{\hbar}{2}\right)^2.$$

Substituting into the variance formula:

$$\langle(\Delta S_z)^2\rangle = \langle S_z^2\rangle - \langle S_z\rangle^2 = \left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2 = 0.$$

Thus, the uncertainty in S_z is zero, and the observable S_z is therefore *sharp* in this state.

Fuzzy Observable: ΔS_x We now compute the uncertainty in S_x for the same state $|+\rangle$. The spin operator S_x is defined as:

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right).$$

Step 1: S_x^2 To calculate S_x^2, square the expression for S_x:

$$S_x^2 = \left(\frac{\hbar}{2}\right)^2 \left(|+\rangle\langle +| + |-\rangle\langle -|\right).$$

Simplifying:

$$S_x^2 = \frac{\hbar^2}{4}I,$$

where I is the identity operator.

Step 2: Expectation Values For the state $|+\rangle$, the expectation value of S_x is:

$$\langle S_x\rangle = \langle +|S_x|+\rangle = 0,$$

since S_x connects $|+\rangle$ and $|-\rangle$ (off-diagonal elements cancel). The expectation value of S_x^2 is:

$$\langle S_x^2\rangle = \frac{\hbar^2}{4}.$$

Step 3: Variance The variance of S_x is given by:

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2.$$

Substitute the values:

$$\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}.$$

Thus, S_x is a *fuzzy* observable in the S_z eigenstate $|+\rangle$, as it has a nonzero uncertainty.

Summary

For a spin-$\frac{1}{2}$ system in the S_z eigenstate $|+\rangle$:

- $\langle(\Delta S_z)^2\rangle = 0$: The observable S_z is *sharp* (zero uncertainty).
- $\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}$: The observable S_x is *fuzzy* (nonzero uncertainty).

This result highlights the uncertainty principle: an observable that is sharp in one state will exhibit nonzero uncertainty for an incompatible observable.

The General Uncertainty Relation

For two observables A and B, the uncertainty relation states:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2,$$

where:

- $\Delta A \equiv A - \langle A \rangle$ is the deviation of A from its expectation value.
- $[A, B] = AB - BA$ is the commutator of A and B.

To Prove:

The uncertainty relation is derived using the following lemmas and the Schwarz inequality:

1. **Lemma 1 (Schwarz Inequality)**: For any states $|\alpha\rangle$ and $|\beta\rangle$,
$$\langle \alpha|\alpha \rangle \langle \beta|\beta \rangle \geq |\langle \alpha|\beta \rangle|^2$$
$$a^2 b^2 \geq |\vec{a} \cdot \vec{b}|^2$$
2. **Lemma 2:** The expectation value of a Hermitian operator is purely real. For a Hermitian operator A ($A = A^\dagger$):
$$\langle A \rangle = \langle \psi | A | \psi \rangle \in \mathbb{R}, \quad \forall \left| \psi \right\rangle.$$
3. **Lemma 3:** The expectation value of an anti-Hermitian operator is purely imaginary. For an anti-Hermitian operator A ($A = -A^\dagger$):
$$\langle A \rangle = \langle \psi | A | \psi \rangle \in i\mathbb{R}, \quad \forall \left| \psi \right\rangle.$$

Proof

Let A and B be two observables. The uncertainties (or dispersions) in A and B are defined as:

$$\Delta A = A - \langle A \rangle, \quad \Delta B = B - \langle B \rangle,$$

where $\langle A \rangle = \langle \psi | A | \psi \rangle$ is the expectation value of A in a given state $|\psi\rangle$.
Define the kets:

$$|\alpha\rangle = \Delta A |\psi\rangle, \quad |\beta\rangle = \Delta B |\psi\rangle.$$

By the Schwarz inequality:

$$\langle \alpha|\alpha \rangle \langle \beta|\beta \rangle \geq |\langle \alpha|\beta \rangle|^2.$$

Substituting the definitions of $|\alpha\rangle$ and $|\beta\rangle$:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq |\langle \Delta A \Delta B \rangle|^2.$$

Decomposition of $\Delta A \Delta B$

The product $\Delta A \Delta B$ can be decomposed into its commutator and anticommutator parts:

$$\Delta A \Delta B = \frac{1}{2} [\Delta A, \Delta B] + \frac{1}{2} \{\Delta A, \Delta B\}.$$

Taking the expectation value in a given state $|\psi\rangle$, we obtain:

$$\langle \Delta A \Delta B \rangle = \frac{1}{2} \langle [\Delta A, \Delta B] \rangle + \frac{1}{2} \langle \{\Delta A, \Delta B\} \rangle.$$

Commutator $[\Delta A, \Delta B]$

The commutator of ΔA and ΔB is defined as:

$$[\Delta A, \Delta B] = (A - \langle A \rangle)(B - \langle B \rangle) - (B - \langle B \rangle)(A - \langle A \rangle).$$

Expanding the terms:

$$[\Delta A, \Delta B] = AB - A\langle B \rangle - \langle A \rangle B + \langle A \rangle \langle B \rangle - (BA - B\langle A \rangle - \langle B \rangle A + \langle A \rangle \langle B \rangle).$$

Combine like terms:

$$[\Delta A, \Delta B] = AB - A\langle B \rangle - \langle A \rangle B + \langle A \rangle \langle B \rangle - BA + B\langle A \rangle + \langle B \rangle A - \langle A \rangle \langle B \rangle.$$

Simplify further by canceling $\langle A \rangle \langle B \rangle$:

$$[\Delta A, \Delta B] = AB - BA - A\langle B \rangle - \langle A \rangle B + B\langle A \rangle + \langle B \rangle A.$$

Reorganize the terms:

$$[\Delta A, \Delta B] = AB - BA = [A, B].$$

Thus, the commutator simplifies to:

$$[\Delta A, \Delta B] = [A, B].$$

Since $[A, B]$ is anti-Hermitian, its expectation value $\langle [A, B] \rangle$ is purely imaginary.

Anticommutator $\{\Delta A, \Delta B\}$

The anticommutator of ΔA and ΔB is defined as:

$$\{\Delta A, \Delta B\} = \Delta A \Delta B + \Delta B \Delta A.$$

Taking the Hermitian conjugate:

$$\{\Delta A, \Delta B\}^\dagger = (\Delta A \Delta B + \Delta B \Delta A)^\dagger.$$

Since ΔA and ΔB are Hermitian ($\Delta A^\dagger = \Delta A$ and $\Delta B^\dagger = \Delta B$):

$$\{\Delta A, \Delta B\}^\dagger = \Delta B \Delta A + \Delta A \Delta B = \{\Delta A, \Delta B\}.$$

Thus, $\{\Delta A, \Delta B\}$ is Hermitian, and its expectation value $\langle \{\Delta A, \Delta B\} \rangle$ is real.

Squaring the Expectation Value of $\Delta A \Delta B$

We compute the square of $\langle \Delta A \Delta B \rangle$:

$$|\langle \Delta A \Delta B \rangle|^2 = \left(\frac{1}{2} \langle [\Delta A, \Delta B] \rangle + \frac{1}{2} \langle \{\Delta A, \Delta B\} \rangle \right)^2.$$

Separating the real and imaginary parts:

$$|\langle \Delta A \Delta B \rangle|^2 = \frac{1}{4} |\langle [\Delta A, \Delta B] \rangle|^2 + \frac{1}{4} |\langle \{\Delta A, \Delta B\} \rangle|^2.$$

Final Bound on the Uncertainty Product

The uncertainty relation states:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq |\langle \Delta A \Delta B \rangle|^2.$$

Using the decomposition of $|\langle \Delta A \Delta B \rangle|^2$, we find:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2 + \frac{1}{4} |\langle \{\Delta A, \Delta B\} \rangle|^2.$$

Since $|\langle \{\Delta A, \Delta B\} \rangle|^2$ is non-negative, we can drop it for a stronger bound:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2.$$

Geometric Interpretation

Given a complex number $c = (ia) + b$, where $a, b \in \mathbb{R}$, its modulus squared is:

$$|c|^2 = c^* c = (-ia + b)(ia + b).$$

Simplifying:

$$|c|^2 = a^2 + b^2.$$

Here:
- a corresponds to the imaginary part $\langle [A, B] \rangle$,
- b corresponds to the real part $\langle \{\Delta A, \Delta B\} \rangle$.

This illustrates the contributions of real and imaginary components to the uncertainty relation.

Lemma 1: The Schwarz Inequality

The Schwarz inequality states:

$$\langle \alpha|\alpha \rangle \langle \beta|\beta \rangle \geq |\langle \alpha|\beta \rangle|^2.$$

Proof: Consider the following general expression:

$$(\langle \alpha| + \lambda^* \langle \beta|)(|\alpha\rangle + \lambda|\beta\rangle) \geq 0,$$

where λ is a complex number.

Expanding the terms:

$$\langle \alpha|\alpha \rangle + |\lambda|^2 \langle \beta|\beta \rangle + \lambda \langle \alpha|\beta \rangle + \lambda^* \langle \beta|\alpha \rangle \geq 0.$$

To simplify the expression, choose:

$$\lambda = -\frac{\langle \beta|\alpha \rangle}{\langle \beta|\beta \rangle}.$$

Substitute λ into the inequality:

$$\langle \alpha|\alpha \rangle + \left| \frac{\langle \beta|\alpha \rangle}{\langle \beta|\beta \rangle} \right|^2 \langle \beta|\beta \rangle - \frac{\langle \beta|\alpha \rangle}{\langle \beta|\beta \rangle} \langle \alpha|\beta \rangle - \frac{\langle \alpha|\beta \rangle \langle \beta|\alpha \rangle}{\langle \beta|\beta \rangle} \geq 0.$$

Simplify each term:

$$\langle \alpha|\alpha \rangle + \frac{|\langle \beta|\alpha \rangle|^2}{\langle \beta|\beta \rangle^2} - \frac{\langle \beta|\alpha \rangle \langle \alpha|\beta \rangle}{\langle \beta|\beta \rangle} - \frac{\langle \alpha|\beta \rangle \langle \beta|\alpha \rangle}{\langle \beta|\beta \rangle} \geq 0.$$

Combining like terms and canceling out the redundant terms:

$$\langle \alpha|\alpha \rangle \langle \beta|\beta \rangle + |\langle \beta|\alpha \rangle|^2 - |\langle \alpha|\beta \rangle|^2 - |\langle \beta|\alpha \rangle|^2 \geq 0.$$

Rearranging:

$$\langle \alpha|\alpha \rangle \langle \beta|\beta \rangle \geq |\langle \alpha|\beta \rangle|^2.$$

This completes the proof of the Schwarz inequality.

Lemma 2: Expectation Value of a Hermitian Operator

The expectation value of a Hermitian operator is purely real.

Proof: Let A be a Hermitian operator. By definition:

$$\langle A \rangle = \langle \alpha|A|\alpha \rangle.$$

Taking the complex conjugate of the expectation value:

$$\langle A \rangle^* = (\langle \alpha|A|\alpha \rangle)^*.$$

Using the property of the Hermitian conjugate, $A^\dagger = A$, we write:

$$\langle A \rangle^* = \langle \alpha|A^\dagger|\alpha \rangle = \langle \alpha|A|\alpha \rangle = \langle A \rangle.$$

Thus, the expectation value $\langle A \rangle$ is equal to its complex conjugate, which implies: $\langle A \rangle$ is real.

Lemma 3: Expectation Value of an Anti-Hermitian Operator

The expectation value of an anti-Hermitian operator is purely imaginary.

Proof: Let A be an anti-Hermitian operator. By definition:

$$A^\dagger = -A.$$

The expectation value is given by:

$$\langle A \rangle = \langle \alpha | A | \alpha \rangle.$$

Taking the complex conjugate of the expectation value:

$$\langle A \rangle^* = \left(\langle \alpha | A | \alpha \rangle \right)^* .$$

Using the property $A^\dagger = -A$, we find:

$$\langle A \rangle^* = \langle \alpha | A^\dagger | \alpha \rangle = \langle \alpha | (-A) | \alpha \rangle = -\langle \alpha | A | \alpha \rangle = -\langle A \rangle.$$

Thus, the expectation value $\langle A \rangle$ satisfies:

$$\langle A \rangle^* = -\langle A \rangle.$$

This condition implies that $\langle A \rangle$ is purely imaginary.

1.5. Change of Basis

1.5.1. Transformation Operator

Definition

Let A and B be two incompatible observables. The ket space can be spanned using either of the two bases:

$$\{|a'\rangle\} \quad \text{or} \quad \{|b'\rangle\}.$$

Here, $\forall\,|\psi\rangle$ in the ket space, it can be expressed as:

$$|\psi\rangle = \sum_i c_i\,|a'_i\rangle = \sum_j d_j\,\left|b'_j\right\rangle,$$

where c_i and d_j are the coefficients in the respective bases.

Example: Spin-$\frac{1}{2}$ System

For a spin-$\frac{1}{2}$ system, the basis states corresponding to S_z and S_x are:

$$\{|S_z;+\rangle, |S_z;-\rangle\} \quad \text{or} \quad \{|S_x;+\rangle, |S_x;-\rangle\}.$$

These bases are related by a transformation, where $\forall\,|\psi\rangle$, we can switch representations.

Change of Basis

Switching from one basis $\{|a'\rangle\}$ to another basis $\{|b'\rangle\}$ can be expressed mathematically as:

$$\left|b'_j\right\rangle = \sum_i T_{ji}\,|a'_i\rangle,$$

where $T_{ji} = \langle b'_j|a'_i\rangle$ are the transformation matrix elements.

Change of Representation

The change of basis corresponds to a change of representation of the observables and states. Specifically:

$$\text{A representation (old)} \quad \longrightarrow \quad \text{B representation (new)}.$$

This transformation can be expressed as a unitary map T that satisfies:

$$T \quad \text{such that} \quad \left|b'_j\right\rangle = T\left|a'_j\right\rangle \quad \text{and} \quad T^\dagger T = I.$$

Here, T preserves the inner product:

$$\langle a'_i|a'_j\rangle = \langle b'_i|b'_j\rangle \quad \forall i, j.$$

Theorem

Let $\{|a'\rangle\}$ and $\{|b'\rangle\}$ be two orthonormal and complete sets of base kets. Then, there exists a unitary operator $\mathcal{U}$ such that:

$$|b^{(1)}\rangle = \mathcal{U}|a^{(1)}\rangle,\ |b^{(2)}\rangle = \mathcal{U}|a^{(2)}\rangle,\ \ldots,\ |b^{(N)}\rangle = \mathcal{U}|a^{(N)}\rangle.$$

Where the unitary operator $\mathcal{U}$ satisfies:

$$\mathcal{U}^\dagger\mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1.$$

Proof

We construct the unitary operator $\mathcal{U}$ explicitly as:

$$\mathcal{U} = \sum_k |b^{(k)}\rangle\langle a^{(k)}|.$$

To show that this operator maps the basis $\{|a'\rangle\}$ to $\{|b'\rangle\}$, consider the action of $\mathcal{U}$ on a basis ket $|a^{(l)}\rangle$:

$$\mathcal{U}|a^{(l)}\rangle = \left(\sum_k |b^{(k)}\rangle\langle a^{(k)}|\right)|a^{(l)}\rangle.$$

Using the orthonormality condition:

$$\langle a^{(k)}|a^{(l)}\rangle = \delta_{kl},$$

we simplify the expression:

$$\mathcal{U}|a^{(l)}\rangle = \sum_k |b^{(k)}\rangle\delta_{kl} = |b^{(l)}\rangle.$$

Thus, the unitary operator $\mathcal{U}$ transforms the basis $|a^{(l)}\rangle$ to $|b^{(l)}\rangle$.

To verify that $\mathcal{U}$ is unitary, we compute $\mathcal{U}^\dagger\mathcal{U}$:

$$\mathcal{U}^\dagger\mathcal{U} = \left(\sum_k |a^{(k)}\rangle\langle b^{(k)}|\right)\left(\sum_l |b^{(l)}\rangle\langle a^{(l)}|\right).$$

Expanding the summation:

$$\mathcal{U}^\dagger\mathcal{U} = \sum_k\sum_l |a^{(k)}\rangle\langle b^{(k)}|b^{(l)}\rangle\langle a^{(l)}|.$$

Using the orthonormality condition of $\{|b^{(k)}\rangle\}$:

$$\langle b^{(k)}|b^{(l)}\rangle = \delta_{kl},$$

we simplify:

$$\mathcal{U}^\dagger\mathcal{U} = \sum_k |a^{(k)}\rangle\langle a^{(k)}|.$$

Since the set $\{|a^{(k)}\rangle\}$ is complete, we have:

$$\sum_k |a^{(k)}\rangle\langle a^{(k)}| = 1.$$

Therefore:

$$\mathcal{U}^\dagger\mathcal{U} = 1.$$

Similarly, we can show that $\mathcal{U}\mathcal{U}^\dagger = 1$, confirming that $\mathcal{U}$ is unitary.

1.5.2. Transformation Matrix

Definition of the Transformation Operator

The transformation matrix $\mathcal{U}$ allows us to switch between two orthonormal bases:
- The **old basis** $\{|a^{(k)}\rangle\}_{\text{old}}$,
- The **new basis** $\{|b^{(k)}\rangle\}_{\text{new}}$.

The operator $\mathcal{U}$ is defined as:

$$\mathcal{U} = \sum_k |b^{(k)}\rangle\langle a^{(k)}|.$$

Here:

- $|b^{(k)}\rangle$: Basis vectors in the new representation,

- $\langle a^{(k)}|$: Basis vectors in the old representation.

To compute the matrix elements of $\mathcal{U}$ in the **old basis** $\{|a^{(\ell)}\rangle\}$, we start with:

$$\langle a^{(k)}|\mathcal{U}|a^{(\ell)}\rangle.$$

Substituting the definition of $\mathcal{U}$:

$$\langle a^{(k)}|\mathcal{U}|a^{(\ell)}\rangle = \langle a^{(k)}| \left(\sum_j |b^{(j)}\rangle\langle a^{(j)}| \right) |a^{(\ell)}\rangle.$$

By linearity, this becomes:

$$\langle a^{(k)}|\mathcal{U}|a^{(\ell)}\rangle = \sum_j \langle a^{(k)}|b^{(j)}\rangle\langle a^{(j)}|a^{(\ell)}\rangle.$$

Using the orthonormality of the old basis:

$$\langle a^{(j)}|a^{(\ell)}\rangle = \delta_{j\ell},$$

we simplify:

$$\langle a^{(k)}|\mathcal{U}|a^{(\ell)}\rangle = \langle a^{(k)}|b^{(\ell)}\rangle.$$

Thus, the matrix elements of $\mathcal{U}$ are the overlaps between the old and new basis vectors:

$$\langle a^{(k)}|\mathcal{U}|a^{(\ell)}\rangle = \langle a^{(k)}|b^{(\ell)}\rangle.$$

Change of Basis for a State $|\alpha\rangle$

Let $|\alpha\rangle$ be a state in the Hilbert space. The state can be expanded in the **old basis** $\{|a^{(\ell)}\rangle\}$ as:

$$|\alpha\rangle = \sum_\ell |a^{(\ell)}\rangle\langle a^{(\ell)}|\alpha\rangle,$$

where $\langle a^{(\ell)}|\alpha\rangle$ are the expansion coefficients in the old basis.
Similarly, in the **new basis** $\{|b^{(k)}\rangle\}$, we have:

$$|\alpha\rangle = \sum_k |b^{(k)}\rangle\langle b^{(k)}|\alpha\rangle,$$

where $\langle b^{(k)}|\alpha\rangle$ are the expansion coefficients in the new basis.

Relating Old and New Basis Coefficients

To relate the coefficients $\langle b^{(k)}|\alpha\rangle$ and $\langle a^{(\ell)}|\alpha\rangle$, we start from the definition of $\mathcal{U}$:

$$|b^{(k)}\rangle = \mathcal{U}|a^{(k)}\rangle \quad \Rightarrow \quad \langle b^{(k)}| = \langle a^{(k)}|\mathcal{U}^\dagger.$$

Taking the inner product with $|\alpha\rangle$, we get:

$$\langle b^{(k)}|\alpha\rangle = \langle a^{(k)}|\mathcal{U}^\dagger|\alpha\rangle.$$

Substitute the expansion of $|\alpha\rangle$ in the old basis:

$$\langle b^{(k)}|\alpha\rangle = \sum_\ell \langle a^{(k)}|\mathcal{U}^\dagger|a^{(\ell)}\rangle\langle a^{(\ell)}|\alpha\rangle.$$

Using the earlier result for $\langle a^{(k)}|\mathcal{U}|a^{(\ell)}\rangle$:

$$\langle a^{(k)}|\mathcal{U}^\dagger|a^{(\ell)}\rangle = \langle b^{(\ell)}|a^{(k)}\rangle^*.$$

Thus:

$$\langle b^{(k)}|\alpha\rangle = \sum_\ell \langle b^{(k)}|a^{(\ell)}\rangle\langle a^{(\ell)}|\alpha\rangle.$$

Matrix Representation of the Coefficients

In matrix notation, let: - $\mathbf{c}_{\text{old}}$: Column matrix of $\langle a^{(\ell)}|\alpha\rangle$, - $\mathbf{c}_{\text{new}}$: Column matrix of $\langle b^{(k)}|\alpha\rangle$.
The relationship between the new and old coefficients is:

$$\mathbf{c}_{\text{new}} = \mathcal{U}^\dagger \mathbf{c}_{\text{old}}.$$

Here: - $\mathcal{U}^\dagger$: Hermitian conjugate (adjoint) of the transformation matrix $\mathcal{U}$.

Transformation of Matrix Elements of an Operator X

The matrix elements of an operator X in the **new basis** $\{|b^{(k)}\rangle\}$ are related to the matrix elements in the **old basis** $\{|a^{(\ell)}\rangle\}$. Specifically:

$$\langle b^{(k)}|X|b^{(\ell)}\rangle = \sum_{m,n} \langle b^{(k)}|a^{(m)}\rangle\langle a^{(m)}|X|a^{(n)}\rangle\langle a^{(n)}|b^{(\ell)}\rangle.$$

Using the transformation operator $\mathcal{U}$, where:

$$\langle b^{(k)}|a^{(m)}\rangle = \langle a^{(m)}|\mathcal{U}^\dagger|a^{(k)}\rangle^*, \quad \text{and} \quad \langle a^{(n)}|b^{(\ell)}\rangle = \langle a^{(n)}|\mathcal{U}|a^{(\ell)}\rangle,$$

we rewrite the expression as:

$$\langle b^{(k)}|X|b^{(\ell)}\rangle = \sum_{m,n} \langle a^{(k)}|\mathcal{U}^\dagger|a^{(m)}\rangle\langle a^{(m)}|X|a^{(n)}\rangle\langle a^{(n)}|\mathcal{U}|a^{(\ell)}\rangle.$$

In matrix notation, this becomes:

$$X_{\text{new}} = \mathcal{U}^\dagger X_{\text{old}} \mathcal{U}.$$

Here: - X_{old}: Matrix representation of X in the old basis, - X_{new}: Matrix representation of X in the new basis, - $\mathcal{U}$: Unitary transformation matrix.

The Trace of an Operator

The trace of an operator X is defined as the sum of its diagonal elements in a given basis:

$$\text{tr}(X) = \sum_{a'} \langle a'|X|a'\rangle.$$

Independence of the Trace from Representation

The trace of X is **independent of the choice of basis**. To show this, start with the trace in the old basis $\{|a'\rangle\}$:

$$\sum_{a'} \langle a'|X|a'\rangle.$$

Insert the completeness relation in the new basis $\{|b'\rangle\}$:

$$\sum_{a'} \langle a'|X|a'\rangle = \sum_{a'} \sum_{b',b''} \langle a'|b'\rangle\langle b'|X|b''\rangle\langle b''|a'\rangle.$$

Using the orthonormality condition $\langle a'|b''\rangle$ and $\langle b'|b''\rangle = \delta_{b'b''}$, the expression simplifies to:

$$\sum_{a'} \langle a'|X|a'\rangle = \sum_{b'} \langle b'|X|b'\rangle.$$

Thus, the trace remains unchanged under a change of basis:

$$\text{tr}(X) = \text{tr}(\mathcal{U}^\dagger X \mathcal{U}).$$

Properties of the Trace

The trace satisfies the following important properties:

1. *Cyclic property:*
$$\mathrm{tr}(XY) = \mathrm{tr}(YX).$$

2. *Invariance under unitary transformations:* For a unitary matrix $\mathcal{U}$, we have:
$$\mathrm{tr}(\mathcal{U}^\dagger X \mathcal{U}) = \mathrm{tr}(X).$$

3. *Trace of a rank-1 operator:* If $X = |a'\rangle\langle a''|$, then:
$$\mathrm{tr}(|a'\rangle\langle a''|) = \delta_{a'a''}.$$

4. *Trace of an outer product in the new basis:* If $X = |b'\rangle\langle a'|$, the trace is:
$$\mathrm{tr}(|b'\rangle\langle a'|) = \langle a'|b'\rangle.$$

1.5.3. Diagonalization of an Operator

Matrix Representation of B

The operator B can be represented in the **old basis** $\{|a^{(k)}\rangle\}$ as a matrix with elements:

$$B = \begin{bmatrix} \langle a^{(1)}|B|a^{(1)}\rangle & \cdots & \langle a^{(1)}|B|a^{(N)}\rangle \\ \vdots & \ddots & \vdots \\ \langle a^{(N)}|B|a^{(1)}\rangle & \cdots & \langle a^{(N)}|B|a^{(N)}\rangle \end{bmatrix}.$$

Here:

- $\langle a^{(k)}|B|a^{(\ell)}\rangle$ are the matrix elements of B in the old basis $\{|a^{(k)}\rangle\}$.

To diagonalize B, we find its eigenvalues and eigenvectors.
—

Eigenvalue Equation

The eigenvalue equation for B is:

$$B|b^{(\ell)}\rangle = b^{(\ell)}|b^{(\ell)}\rangle,$$

where:

- $b^{(\ell)}$: Eigenvalues of B,
- $|b^{(\ell)}\rangle$: Corresponding eigenkets of B.

Expressing the eigenvector $|b^{(\ell)}\rangle$ in terms of the old basis $\{|a^{(k)}\rangle\}$, we have:

$$|b^{(\ell)}\rangle = \sum_k |a^{(k)}\rangle\langle a^{(k)}|b^{(\ell)}\rangle.$$

Substituting this into the eigenvalue equation:

$$B\sum_k |a^{(k)}\rangle\langle a^{(k)}|b^{(\ell)}\rangle = b^{(\ell)}\sum_k |a^{(k)}\rangle\langle a^{(k)}|b^{(\ell)}\rangle.$$

Taking the inner product with $\langle a^{(i)}|$, we get:

$$\sum_k \langle a^{(i)}|B|a^{(k)}\rangle\langle a^{(k)}|b^{(\ell)}\rangle = b^{(\ell)}\langle a^{(i)}|b^{(\ell)}\rangle.$$

—

Unitary Transformation to Diagonalize B

To diagonalize B, we use a **unitary matrix** $\mathcal{U}$ defined as:

$$\mathcal{U} = \sum_j |b^{(j)}\rangle\langle a^{(j)}|.$$

The matrix elements of $\mathcal{U}$ in the old basis are given by:

$$\langle a^{(k)}|\mathcal{U}|a^{(\ell)}\rangle = \sum_j \langle a^{(k)}|b^{(j)}\rangle\langle b^{(j)}|a^{(\ell)}\rangle.$$

Thus, the unitary matrix $\mathcal{U}$ relates the old basis to the eigenbasis of B. The matrix element $\mathcal{U}_{k\ell}$ is:

$$\mathcal{U}_{k\ell} = \langle a^{(k)}|b^{(\ell)}\rangle.$$

—

Diagonal Form of B

In the eigenbasis $\{|b^{(\ell)}\rangle\}$, the operator B becomes diagonal. Its matrix representation is:

$$B_{\text{diag}} = \begin{bmatrix} b^{(1)} & 0 & \cdots & 0 \\ 0 & b^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^{(N)} \end{bmatrix},$$

where $b^{(\ell)}$ are the eigenvalues of B.
The diagonalization process is expressed as:

$$B = \mathcal{U} D \mathcal{U}^{\dagger},$$

where:

- D is the diagonal matrix containing eigenvalues $b^{(\ell)}$,
- $\mathcal{U}$ is the unitary matrix whose columns are the eigenvectors $|b^{(\ell)}\rangle$ expressed in the old basis.

—

Non-Hermitian Operators

If B is **not Hermitian**, it cannot, in general, be diagonalized using a unitary matrix. This is because the eigenvectors of a non-Hermitian operator do not necessarily form an orthonormal basis.

1.5.4. Transformation of the Operator B Between Bases

Matrix Representation of B in the Old Basis

The operator B acts on the eigenvector $|b\rangle_{\text{old}}$ in the **old basis** as:

$$B_{\text{old}}|b\rangle_{\text{old}} = b|b\rangle_{\text{old}},$$

where:

- B_{old}: The matrix representation of B in the old basis,
- $|b\rangle_{\text{old}}$: The eigenvector in the old basis,
- b: The eigenvalue corresponding to $|b\rangle_{\text{old}}$.

—

Transformation of B to the New Basis

To express B in the new basis, we use a unitary matrix $\mathcal{U}$, which relates the old and new bases.
Start with the eigenvalue equation:

$$B_{\text{old}}|b\rangle_{\text{old}} = b|b\rangle_{\text{old}}.$$

Multiply both sides from the left by $\mathcal{U}^{\dagger}$, the Hermitian conjugate of $\mathcal{U}$:

$$\mathcal{U}^{\dagger} B_{\text{old}}|b\rangle_{\text{old}} = b\mathcal{U}^{\dagger}|b\rangle_{\text{old}}.$$

The eigenvector $|b\rangle_{\text{old}}$ can be transformed to the new basis using:

$$|b\rangle_{\text{new}} = \mathcal{U}^{\dagger}|b\rangle_{\text{old}}.$$

Thus, substituting $|b\rangle_{\text{new}}$ into the equation:

$$\mathcal{U}^{\dagger} B_{\text{old}} \mathcal{U}|b\rangle_{\text{new}} = b|b\rangle_{\text{new}}.$$

This shows that B in the **new basis** is given by:

$$B_{\text{new}} = \mathcal{U}^{\dagger} B_{\text{old}} \mathcal{U}.$$

—

Interpretation of $\mathcal{U}$

The unitary matrix $\mathcal{U}$ diagonalizes B and contains the eigenvectors $|b^{(k)}\rangle$ of B expressed in the old basis. Its structure is:

$$\mathcal{U} = \begin{bmatrix} |b^{(1)}\rangle & |b^{(2)}\rangle & \cdots & |b^{(N)}\rangle \end{bmatrix}.$$

Each column corresponds to an eigenvector of B.

Action of B in the New Basis

In the new basis, the operator B_{new} acts as:

$$B_{\text{new}}|b\rangle_{\text{new}} = b|b\rangle_{\text{new}},$$

where:

- B_{new}: Matrix representation of B in the new basis,
- $|b\rangle_{\text{new}}$: Eigenvector in the new basis.

1.5.5. Finding Eigenvalues and Eigenvectors

Eigenvalue Equation

To find the eigenvalues and eigenvectors of an operator A, we solve the eigenvalue equation:

$$A|\alpha\rangle = \lambda|\alpha\rangle,$$

where:

- A: Linear operator,
- $|\alpha\rangle$: Eigenvector (non-null ket),
- λ: Eigenvalue.

Rewriting the equation, we have:

$$(A - \lambda I)|\alpha\rangle = 0.$$

Here, I is the identity operator.

—

Condition for Eigenvalues

The equation $(A - \lambda I)|\alpha\rangle = 0$ implies that $(A - \lambda I)$ has no inverse. This happens if and only if the determinant of $(A - \lambda I)$ vanishes:

$$\det(A - \lambda I) = 0.$$

This equation is called the **characteristic equation**. If A is an $N \times N$ matrix, the characteristic equation is of order N and has N roots.

—

Steps to Find Eigenvalues and Eigenvectors

1. **Find the Eigenvalues:** Solve the characteristic equation:

   $$\det(A - \lambda I) = 0.$$

   This yields N roots, which are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$.

2. **Find the Eigenvectors:** For each eigenvalue λ, solve the equation:

   $$(A - \lambda I)|\alpha\rangle = 0.$$

   This equation determines the eigenvector $|\alpha\rangle$ up to an overall constant.

—

Choice of Normalization

The eigenvector $|\alpha\rangle$ can be normalized in one of the following ways:

1. By choosing one of its components (e.g., the first component) to be 1, or
2. By using the normalization condition:
$$\langle\alpha|\alpha\rangle = 1.$$

1.5.6. Example: Eigenvalues and Eigenvectors of B

Given Matrix B

The matrix B is:
$$B = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$$
We aim to determine the eigenvalues and eigenvectors of B.
—

Step 1: Characteristic Equation

The eigenvalues are obtained by solving the characteristic equation:
$$\det(B - \lambda I) = 0.$$
Substitute $B - \lambda I$:
$$B - \lambda I = \begin{bmatrix} 1-\lambda & -i \\ i & 1-\lambda \end{bmatrix}.$$
The determinant is:
$$\det(B - \lambda I) = (1-\lambda)(1-\lambda) + i^2 = (1-\lambda)^2 - 1 = 0.$$
Simplify:
$$(1-\lambda)^2 + i^2 = 0 \implies \lambda = 0 \, \text{or} \, \lambda = 2.$$

—

Step 2: Eigenvectors for $\lambda = 0$

For $\lambda = 0$, solve:
$$(B - 0I)\begin{bmatrix} a \\ b \end{bmatrix} = 0.$$
Substitute B:
$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
This gives the equations:
$$a - ib = 0 \quad \text{and} \quad ia + b = 0.$$
From $a - ib = 0$, we get:
$$a = ib.$$
Choose $b = 1$, then $a = i$. The corresponding eigenvector is:
$$|0\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

—

Step 3: Eigenvectors for $\lambda = 2$

For $\lambda = 2$, solve:

$$(B - 2I)\begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

Substitute $B - 2I$:

$$\begin{bmatrix} 1-2 & -i \\ i & 1-2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Simplify the equations:

$$-a - ib = 0 \quad \text{and} \quad ia - b = 0.$$

From $-a - ib = 0$, we find:

$$a = -ib.$$

Choose $b = 1$, then $a = -i$. The corresponding eigenvector is:

$$|2\rangle = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

—

Step 4: Verification

1. **Eigenvalues:** Both eigenvalues $\lambda = 0$ and $\lambda = 2$ are real, as expected.
2. **Orthogonality:** Verify the orthogonality of eigenvectors $|0\rangle$ and $|2\rangle$:

$$\langle 0|2\rangle = \begin{bmatrix} -i & 1 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = (-i)(i) + (1)(1) = 0.$$

3. **Action of B on $|0\rangle$:** Verify that $B|0\rangle = 0|0\rangle$:

$$B|0\rangle = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

4. **Action of B on $|2\rangle$:** Verify that $B|2\rangle = 2|2\rangle$:

$$B|2\rangle = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = 2\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Given Matrix and Bases

The matrix B_{old} in the old basis is:

$$B_{\text{old}} = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$$

The **old basis** vectors are:

$$|a_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |a_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The **new basis** vectors, corresponding to the eigenvectors of B, are:

$$|0\rangle_{\text{old}} = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}, \quad |2\rangle_{\text{old}} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

—

Transformation Matrix $\mathcal{U}$

The unitary matrix $\mathcal{U}$, which transforms from the old basis to the new basis, is constructed as:

$$\mathcal{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}.$$

Here: - The columns of $\mathcal{U}$ are the eigenvectors $|0\rangle_{\text{old}}$ and $|2\rangle_{\text{old}}$.
—

Diagonalization of B

The matrix B in the new basis, B_{new}, is obtained using the relation:

$$B_{\text{new}} = \mathcal{U}^\dagger B_{\text{old}} \mathcal{U}.$$

We compute each term step by step:

1. Compute $\mathcal{U}^\dagger$:

$$\mathcal{U}^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}.$$

2. Perform the multiplication:

$$B_{\text{new}} = \mathcal{U}^\dagger B_{\text{old}} \mathcal{U}.$$

Substitute the matrices:

$$B_{\text{new}} = \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}.$$

Step-by-step multiplication gives:

$$B_{\text{new}} = \frac{1}{2} \begin{bmatrix} 0 & -2i \\ 0 & 2 \end{bmatrix}.$$

Simplifying:

$$B_{\text{new}} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

—

Verification of Eigenvectors in the New Basis

To confirm the diagonal form, we verify the action of B_{new} on the eigenvectors $|0\rangle$ and $|2\rangle$ in the new basis:

- For $|0\rangle$:

$$B_{\text{new}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

  Thus, $B_{\text{new}}|0\rangle = 0|0\rangle$.

- For $|2\rangle$:

$$B_{\text{new}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

  Thus, $B_{\text{new}}|2\rangle = 2|2\rangle$.

1.5.7. Unitary Equivalent Observables

Theorem

If $\{|a'\rangle\}$ and $\{|b'\rangle\}$ are two sets of orthonormal bases connected by a unitary matrix $\mathcal{U}$ such that:

$$\mathcal{U} = \sum_k |b^{(k)}\rangle\langle a^{(k)}|,$$

then the operators A and $\mathcal{U}A\mathcal{U}^{-1}$ have the **same eigenvalues**.

—

Proof

Start with the eigenvalue equation for A in the old basis:

$$A|a^{(\ell)}\rangle = a^{(\ell)}|a^{(\ell)}\rangle.$$

1. Multiply both sides of the equation by $\mathcal{U}$ from the left:

$$\mathcal{U}A|a^{(\ell)}\rangle = a^{(\ell)}\mathcal{U}|a^{(\ell)}\rangle.$$

   Define the new basis vectors as:

$$|b^{(\ell)}\rangle = \mathcal{U}|a^{(\ell)}\rangle.$$

   Substituting into the equation:

$$\mathcal{U}A|a^{(\ell)}\rangle = a^{(\ell)}|b^{(\ell)}\rangle.$$

2. Observe that:

$$(\mathcal{U}A\mathcal{U}^{-1})(\mathcal{U}|a^{(\ell)}\rangle) = a^{(\ell)}(\mathcal{U}|a^{(\ell)}\rangle).$$

   Since $\mathcal{U}|a^{(\ell)}\rangle = |b^{(\ell)}\rangle$, this becomes:

$$(\mathcal{U}A\mathcal{U}^{-1})|b^{(\ell)}\rangle = a^{(\ell)}|b^{(\ell)}\rangle.$$

Thus, $\mathcal{U}A\mathcal{U}^{-1}$ has the same eigenvalues $a^{(\ell)}$ as A, but its eigenvectors are expressed in the new basis $\{|b^{(\ell)}\rangle\}$.

1.5.8. Position, Momentum, and Translation: Continuous Spectra

Observables with Discrete Eigenvalues

For observables with **discrete eigenvalues**, such as S_z, the eigenvalues are finite and countable. For example:

$$S_z \in \left\{+\frac{\hbar}{2}, -\frac{\hbar}{2}\right\}.$$

Here:

- The vector space is spanned by a **finite set of eigenkets** $\{|a'\rangle\}$,
- The eigenvalue equation is:

$$A|a'\rangle = a'|a'\rangle,$$

  where A is the observable.

For discrete eigenvalues, the orthonormality condition of the eigenkets is expressed using the **Kronecker delta** δ_{ij}:

$$\langle a^{(i)}|a^{(j)}\rangle = \delta_{ij}.$$

The completeness relation for the finite set of eigenkets is:

$$\sum_{a'} |a'\rangle\langle a'| = I,$$

where I is the identity operator.

—

Observables with Continuous Eigenvalues

For observables with **continuous eigenvalues**, such as position x, the eigenvalues form a continuous spectrum over an interval:

$$x \in (-\infty, +\infty).$$

Here:

- The vector space is spanned by an **infinite set of eigenkets** $\{|\xi\rangle\}$,
- The eigenvalue equation becomes:

  $$\Xi|\xi'\rangle = \xi'|\xi'\rangle,$$

  where Ξ is the observable and ξ' is a continuous eigenvalue.

For continuous eigenvalues, the orthonormality condition of the eigenkets is expressed using **Dirac's delta function** $\delta(\xi - \xi')$:

$$\langle\xi|\xi'\rangle = \delta(\xi - \xi').$$

The completeness relation for the infinite set of eigenkets becomes:

$$\int d\xi' \, |\xi'\rangle\langle\xi'| = I,$$

where I is the identity operator.
—

Comparison: Discrete vs. Continuous Spectra

- For **discrete eigenvalues**:

  $$\langle a^{(i)}|a^{(j)}\rangle = \delta_{ij}, \quad \sum_{a'} |a'\rangle\langle a'| = I.$$

- For **continuous eigenvalues**:

  $$\langle\xi|\xi'\rangle = \delta(\xi - \xi'), \quad \int d\xi' \, |\xi'\rangle\langle\xi'| = I.$$

—

Summary

1. **Discrete Eigenvalues**: The vector space is spanned by a finite set of eigenkets $\{|a'\rangle\}$, with orthonormality expressed via the Kronecker delta δ_{ij}. 2. **Continuous Eigenvalues**: The vector space is spanned by an infinite set of eigenkets $\{|\xi'\rangle\}$, with orthonormality expressed via Dirac's delta function $\delta(\xi - \xi')$. 3. **Completeness Relations**:

- Discrete: $\sum_{a'} |a'\rangle\langle a'| = I$,
- Continuous: $\int d\xi' \, |\xi'\rangle\langle\xi'| = I$.

1.5.9. Position, Momentum, and Translation: Continuous Spectra

Observables with Discrete Eigenvalues

For observables with **discrete eigenvalues**, the eigenvectors $|a'\rangle$ satisfy the orthonormality condition:

$$\langle a'|a''\rangle = \delta_{a'a''},$$

where $\delta_{a'a''}$ is the Kronecker delta.
The completeness relation for the discrete eigenkets is expressed as:

$$\sum_{a'} |a'\rangle\langle a'| = I.$$

Any arbitrary state $|\alpha\rangle$ can be expanded in terms of the discrete eigenkets as:

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle.$$

The coefficients $\langle a'|\alpha\rangle$ represent the projections of $|\alpha\rangle$ onto the eigenbasis $\{|a'\rangle\}$. The normalization condition for $|\alpha\rangle$ is:

$$\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1.$$

For two states $|\alpha\rangle$ and $|\beta\rangle$, their inner product is given by:

$$\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle.$$

If A is an operator with discrete eigenvalues a', the matrix elements of A in this basis are:

$$\langle a''|A|a'\rangle = a'\delta_{a'a''}.$$

Observables with Continuous Eigenvalues

For observables with **continuous eigenvalues**, the eigenvectors $|\xi\rangle$ satisfy the orthonormality condition:

$$\langle\xi|\xi'\rangle = \delta(\xi - \xi'),$$

where $\delta(\xi - \xi')$ is the Dirac delta function.
The completeness relation for the continuous eigenkets is:

$$\int d\xi'\, |\xi'\rangle\langle\xi'| = I.$$

Any arbitrary state $|\alpha\rangle$ can be expanded in terms of the continuous eigenkets as:

$$|\alpha\rangle = \int d\xi'\, |\xi'\rangle\langle\xi'|\alpha\rangle.$$

The coefficients $\langle\xi'|\alpha\rangle$ represent the projections of $|\alpha\rangle$ onto the continuous eigenbasis $\{|\xi'\rangle\}$. The normalization condition for $|\alpha\rangle$ is:

$$\int d\xi'\, |\langle\xi'|\alpha\rangle|^2 = 1.$$

For two states $|\alpha\rangle$ and $|\beta\rangle$, their inner product is:

$$\langle\beta|\alpha\rangle = \int d\xi'\, \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle.$$

If Ξ is an operator with continuous eigenvalues ξ', the matrix elements of Ξ in this basis are:

$$\langle\xi''|\Xi|\xi'\rangle = \xi'\delta(\xi - \xi').$$

1.5.10. Position Eigenkets and Position Measurements

Position Operator and Eigenkets

The position operator $\mathcal{X}$ acts on its eigenkets $|x'\rangle$ as:

$$\mathcal{X}|x'\rangle = x'|x'\rangle,$$

where x' is the eigenvalue of the position operator, representing a specific position in space. The eigenkets $\{|x'\rangle\}$ form a complete basis for the Hilbert space.

Completeness of Position Eigenkets

The completeness relation for the position eigenkets is:

$$\int_{-\infty}^{\infty} dx'\, |x'\rangle\langle x'| = I,$$

where I is the identity operator. This ensures that any state $|\alpha\rangle$ can be expanded in terms of the position eigenkets:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'\, |x'\rangle\langle x'|\alpha\rangle.$$

The expansion coefficients $\langle x'|\alpha\rangle$ represent the projection of the state $|\alpha\rangle$ onto the position eigenkets $|x'\rangle$.

Normalization of the State

The state $|\alpha\rangle$ is normalized if:

$$\langle\alpha|\alpha\rangle = 1.$$

In terms of the position eigenbasis, this normalization condition becomes:

$$\int_{-\infty}^{\infty} dx'\, |\langle x'|\alpha\rangle|^2 = 1.$$

Probability of Finding the Particle

The probability of finding a particle in the region between $x' - \frac{\Delta}{2}$ and $x' + \frac{\Delta}{2}$ is given by:

$$P(x') = |\langle x'|\alpha\rangle|^2\Delta,$$

where Δ is the width of the position interval.
As $\Delta \to dx'$ (infinitesimally small), the probability of finding the particle at position x' within the interval dx' becomes:

$$P(x') = |\langle x'|\alpha\rangle|^2 dx'.$$

Here:

- $|\langle x'|\alpha\rangle|^2$: Probability density function, with dimensions of $\frac{1}{\text{length}}$,
- dx': Infinitesimal position interval.

Measurement of Position

A position measurement corresponds to projecting the state $|\alpha\rangle$ onto the position eigenket $|x'\rangle$, which yields the probability amplitude $\langle x'|\alpha\rangle$. Mathematically:

$$\langle x'|\alpha\rangle = \text{amplitude for the particle to be at position } x'.$$

The measurement process depends on the detector's resolution. If the detector has finite resolution Δ, the position measurement integrates the probability density over the interval $[x' - \frac{\Delta}{2}, x' + \frac{\Delta}{2}]$.
—

Summary of Key Results

- Position eigenvalue equation:

$$\mathcal{X}|x'\rangle = x'|x'\rangle.$$

- Completeness relation for position eigenkets:

$$\int_{-\infty}^{\infty} dx'\, |x'\rangle\langle x'| = I.$$

- State expansion in the position eigenbasis:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'|\alpha\rangle.$$

- Probability density for finding the particle at x':

$$P(x') = |\langle x'|\alpha\rangle|^2 dx'.$$

- Normalization condition for the state $|\alpha\rangle$:

$$\int_{-\infty}^{\infty} dx' \, |\langle x'|\alpha\rangle|^2 = 1.$$

1.5.11. The Canonical Commutation Relations

The Canonical Commutators

The canonical commutation relations form the foundation of quantum mechanics, describing the algebra of position and momentum operators. They are given as:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}.$$

Here:

- x_i and x_j: Position operators in the i-th and j-th directions.
- p_i and p_j: Momentum operators in the i-th and j-th directions.
- δ_{ij}: Kronecker delta, which ensures that the commutator of x_i and p_j is non-zero only when $i = j$.
- $\hbar$: Reduced Planck's constant.

The commutation relations indicate:

1. Position operators commute:

$$[x_i, x_j] = x_i x_j - x_j x_i = 0.$$

2. Momentum operators commute:

$$[p_i, p_j] = p_i p_j - p_j p_i = 0.$$

3. Position and momentum operators do not commute, and their commutator is proportional to $\hbar$:

$$[x_i, p_j] = x_i p_j - p_j x_i = i\hbar\delta_{ij}.$$

These relations are referred to as the **fundamental commutation relations** in quantum mechanics.

Justification of the Commutation Relations

There are two primary ways to justify the canonical commutation relations:

1. **From Properties of Translations:** The commutation relations arise naturally when considering the effect of translations in space. Specifically, the position operator x and momentum operator p generate translations through their exponential forms:

$$T(a) = e^{-\frac{i}{\hbar}ap}, \quad T(a)xT(a)^\dagger = x + a,$$

where $T(a)$ represents a translation by a. This leads to the commutator $[x, p] \propto i\hbar$.

2. **Dirac's Analogy Between Poisson Brackets and Commutators:** In classical mechanics, the Poisson bracket of position and momentum is given by:

$$\{x, p\}_{\text{Poisson}} = 1.$$

Dirac proposed replacing Poisson brackets with quantum commutators using the rule:

$$\{A, B\}_{\text{Poisson}} \to \frac{1}{i\hbar}[A, B].$$

Applying this rule to x and p gives:

$$[x, p] = i\hbar.$$

Physical Significance

The canonical commutation relation $[x_i, p_j] = i\hbar\delta_{ij}$ implies the following:

- The position and momentum operators cannot simultaneously have well-defined eigenvalues. This is a statement of the Heisenberg uncertainty principle.
- The non-commutativity of x and p underlies the probabilistic nature of quantum measurements.
- The commutation relations ensure that quantum mechanical observables form an algebraic structure consistent with the principles of quantum theory.

1.5.12. Translation Operator

Definition of the Translation Operator

The translation operator $\mathcal{T}(\vec{a})$ acts on position eigenkets $|\vec{x}'\rangle$ to displace them by a vector $\vec{a}$:

$$\mathcal{T}(\vec{a})|\vec{x}'\rangle = |\vec{x}' + \vec{a}\rangle,$$

where:

- $\mathcal{T}(\vec{a})$: Translation operator,
- $|\vec{x}'\rangle$: Position eigenket,
- $\vec{a}$: Displacement vector.

The position eigenkets $|\vec{x}'\rangle$ are not eigenkets of the translation operator $\mathcal{T}(\vec{a})$ because the operator shifts the position by $\vec{a}$.

—

Infinitesimal Translation

For an infinitesimal displacement $d\vec{x}'$, the translation operator is given by:

$$\mathcal{T}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle.$$

—

Action of the Translation Operator on a State

A general state $|\alpha\rangle$ in the position basis can be expanded as:

$$|\alpha\rangle = \int d^3x'\, |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle,$$

where $\langle\vec{x}'|\alpha\rangle \equiv \psi(\vec{x}')$ is the wavefunction of the state $|\alpha\rangle$ in the position representation.
The action of the translation operator $\mathcal{T}(d\vec{x}')$ on $|\alpha\rangle$ is:

$$\mathcal{T}(d\vec{x}')|\alpha\rangle = \int d^3x'\, \mathcal{T}(d\vec{x}')|\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle.$$

Substituting $\mathcal{T}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle$, we get:

$$\mathcal{T}(d\vec{x}')|\alpha\rangle = \int d^3x'\, |\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle.$$

—

Transformation of the Wavefunction

The position wavefunction of the translated state $\mathcal{T}(d\vec{x}')|\alpha\rangle$ is given by:

$$\langle\vec{x}'|\mathcal{T}(d\vec{x}')|\alpha\rangle = \langle\vec{x}' - d\vec{x}'|\alpha\rangle.$$

Therefore, the wavefunction of the translated state becomes:

$$\psi(\vec{x}') \to \psi(\vec{x}' - d\vec{x}'),$$

where $\psi(\vec{x}') = \langle\vec{x}'|\alpha\rangle$ is the original wavefunction.

1.5.13. Translation: Infinitesimal Translation Operator

Infinitesimal Translation Operator

The infinitesimal translation operator $\mathcal{T}(d\vec{x}')$ is defined as:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}',$$

where:

- $\vec{K}$: The generator of translations (momentum operator in position space),
- $d\vec{x}'$: Infinitesimal displacement vector,
- 1: The identity operator.

The operator acts on position eigenkets $|\vec{x}'\rangle$ as:

$$\mathcal{T}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle.$$

—

Properties of the Infinitesimal Translation Operator

1. **Unitarity of $\mathcal{T}(d\vec{x}')$:** If $|\alpha\rangle$ is a normalized state, then $\mathcal{T}(d\vec{x}')|\alpha\rangle$ remains normalized. Given:

   $$\langle\alpha|\alpha\rangle = 1,$$

   the condition of unitarity requires:

   $$\langle\alpha|\mathcal{T}^\dagger(d\vec{x}')\mathcal{T}(d\vec{x}')|\alpha\rangle = 1.$$

   Substituting $\mathcal{T}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}'$, we compute:

   $$\mathcal{T}^\dagger(d\vec{x}') = (1 - i\vec{K}\cdot d\vec{x}')^\dagger = 1 + i\vec{K}^\dagger\cdot d\vec{x}'.$$

   Since $\vec{K}$ is Hermitian ($\vec{K}^\dagger = \vec{K}$), this simplifies to:

   $$\mathcal{T}^\dagger(d\vec{x}') = 1 + i\vec{K}\cdot d\vec{x}'.$$

   Now, calculate $\mathcal{T}^\dagger(d\vec{x}')\mathcal{T}(d\vec{x}')$:

   $$\mathcal{T}^\dagger(d\vec{x}')\mathcal{T}(d\vec{x}') = (1 + i\vec{K}\cdot d\vec{x}')(1 - i\vec{K}\cdot d\vec{x}').$$

   Expanding to first order in $d\vec{x}'$, we get:

   $$\mathcal{T}^\dagger(d\vec{x}')\mathcal{T}(d\vec{x}') = 1 + i\vec{K}\cdot d\vec{x}' - i\vec{K}\cdot d\vec{x}' + \mathcal{O}((d\vec{x}')^2).$$

   The terms involving $i\vec{K}\cdot d\vec{x}'$ cancel out, leaving:

   $$\mathcal{T}^\dagger(d\vec{x}')\mathcal{T}(d\vec{x}') \simeq 1.$$

   Therefore, $\mathcal{T}(d\vec{x}')$ is **unitary** to first order in $d\vec{x}'$.

2. **Hermitian Property of $\vec{K}$:** The generator $\vec{K}$ of the translation operator is Hermitian:

   $$\vec{K}^\dagger = \vec{K}.$$

   This property ensures that the translation operator $\mathcal{T}(d\vec{x}')$ is unitary.

1.5.14. Properties of the Infinitesimal Translation Operator

Two Successive Infinitesimal Translations

Consider two successive infinitesimal translations by $d\vec{x}'$ and $d\vec{x}''$. The combined effect of these two translations is equivalent to a single translation by $d\vec{x}' + d\vec{x}''$.
The translation operators satisfy:

$$\mathcal{T}(d\vec{x}'')\mathcal{T}(d\vec{x}') = \mathcal{T}(d\vec{x}'' + d\vec{x}').$$

To verify, recall the definition of the infinitesimal translation operator:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}'.$$

For two translations, we compute:

$$\mathcal{T}(d\vec{x}'')\mathcal{T}(d\vec{x}') = (1 - i\vec{K} \cdot d\vec{x}'')(1 - i\vec{K} \cdot d\vec{x}').$$

Expanding to first order in $d\vec{x}'$ and $d\vec{x}''$, and ignoring higher-order terms $\mathcal{O}((d\vec{x}')^2)$, we get:

$$\mathcal{T}(d\vec{x}'')\mathcal{T}(d\vec{x}') \simeq 1 - i\vec{K} \cdot (d\vec{x}'' + d\vec{x}').$$

This is equivalent to a single translation by $d\vec{x}'' + d\vec{x}'$:

$$\mathcal{T}(d\vec{x}'' + d\vec{x}') = 1 - i\vec{K} \cdot (d\vec{x}'' + d\vec{x}').$$

—

Inverse of the Translation Operator

The translation operator for a displacement $-d\vec{x}'$ is the inverse of the translation operator for $d\vec{x}'$:

$$\mathcal{T}(-d\vec{x}') = \mathcal{T}^{-1}(d\vec{x}').$$

To show this, consider the definition of the translation operator:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}'.$$

The inverse operator is:

$$\mathcal{T}^{-1}(d\vec{x}') = \frac{1}{1 - i\vec{K} \cdot d\vec{x}'}.$$

Expanding the denominator to first order in $d\vec{x}'$ using the geometric series:

$$\mathcal{T}^{-1}(d\vec{x}') \simeq 1 + i\vec{K} \cdot d\vec{x}'.$$

Thus, for a displacement $-d\vec{x}'$, the operator becomes:

$$\mathcal{T}(-d\vec{x}') \simeq 1 + i\vec{K} \cdot d\vec{x}' \simeq \mathcal{T}^{-1}(d\vec{x}').$$

—

Limit of the Translation Operator as $d\vec{x}' \to 0$

As the displacement $d\vec{x}' \to 0$, the translation operator $\mathcal{T}(d\vec{x}')$ approaches the identity operator:

$$\lim_{d\vec{x}' \to 0} \mathcal{T}(d\vec{x}') = 1.$$

Explicitly, from the definition:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}',$$

taking the limit $d\vec{x}' \to 0$ gives:

$$\mathcal{T}(0) = 1.$$

1.5.15. Translation and Commutator with Position Operator

Definition of the Infinitesimal Translation Operator

The infinitesimal translation operator is defined as:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}',$$

where $\vec{K}$ is the generator of translations (momentum operator), and $d\vec{x}'$ is the infinitesimal displacement vector.

Action on Position Eigenkets

The action of $\mathcal{T}(d\vec{x}')$ on a position eigenket $|\vec{x}'\rangle$ shifts the position by $d\vec{x}'$:

$$\mathcal{T}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle.$$

The position operator $\vec{x}$ acting on this translated state gives:

$$\vec{x}\mathcal{T}(d\vec{x}')|\vec{x}'\rangle = \vec{x}|\vec{x}' + d\vec{x}'\rangle = (\vec{x}' + d\vec{x}')|\vec{x}' + d\vec{x}'\rangle.$$

Commutator Between $\vec{x}$ and $\mathcal{T}(d\vec{x}')$

The commutator $[\vec{x}, \mathcal{T}(d\vec{x}')]$ acting on any position ket $|\vec{x}'\rangle$ can be explicitly computed. Start from:

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = \vec{x}\mathcal{T}(d\vec{x}') - \mathcal{T}(d\vec{x}')\vec{x}.$$

Applying this to $|\vec{x}'\rangle$:

$$\vec{x}\mathcal{T}(d\vec{x}')|\vec{x}'\rangle = (\vec{x}' + d\vec{x}')|\vec{x}' + d\vec{x}'\rangle,$$
$$\mathcal{T}(d\vec{x}')\vec{x}|\vec{x}'\rangle = \vec{x}'|\vec{x}' + d\vec{x}'\rangle.$$

Thus, the commutator becomes:

$$[\vec{x}, \mathcal{T}(d\vec{x}')]|\vec{x}'\rangle = d\vec{x}'|\vec{x}' + d\vec{x}'\rangle.$$

To first order in $d\vec{x}'$, this simplifies to:

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = d\vec{x}'.$$

Operator Identity Verification

We verify the commutator identity using the definition $\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}'$. Compute:

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = [\vec{x}, 1 - i\vec{K} \cdot d\vec{x}'].$$

The commutator of $\vec{x}$ with the identity is zero:

$$[\vec{x}, 1] = 0.$$

Thus:

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = -i[\vec{x}, \vec{K} \cdot d\vec{x}'].$$

Using the property $[\vec{x}, \vec{K} \cdot d\vec{x}'] = id\vec{x}'$, we substitute:

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = -i(id\vec{x}') = d\vec{x}'.$$

Conclusion

The commutator between the position operator $\vec{x}$ and the infinitesimal translation operator $\mathcal{T}(d\vec{x}')$ satisfies:

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = d\vec{x}'.$$

1.5.16. Translation and Commutators with Momentum Generators

Infinitesimal Translation Operator and Position Shifts

The translation operator acting on position kets $|\vec{x}'\rangle$ translates the position by an infinitesimal vector $d\vec{x}'$. Recall:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}',$$

where $\vec{K}$ is the generator of translations.
If we take the position operator $\vec{x}$ and act on the eigenket $|\vec{x}'\rangle$, the position operator satisfies:

$$\vec{x}\mathcal{T}(d\vec{x}')|\vec{x}'\rangle = d\vec{x}' + \mathcal{T}(d\vec{x}')\vec{x}|\vec{x}'\rangle.$$

The final commutator simplifies, leading to:

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = d\vec{x}'.$$

Coordinate Representation and Position Components

We write the position vector $\vec{x}$ in terms of its components:

$$\vec{x} = x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3,$$

where x_1, x_2, x_3 are the Cartesian coordinates, and $\hat{x}_1, \hat{x}_2, \hat{x}_3$ are the corresponding unit vectors. For an infinitesimal displacement along $\hat{x}_1$, we have:

$$d\vec{x}' = dx'\hat{x}_1 \implies \vec{K} \cdot d\vec{x}' = K_1 dx'.$$

This leads to:

$$-i\vec{x}K_1 dx' + iK_1 dx'\vec{x} = dx'\hat{x}_1.$$

The commutator between x_1 and K_1 can now be computed:

$$[\vec{x}, K_1] = i\hat{x}_1 \implies [x_1, K_1] = i.$$

Commutators Between Position and Momentum Generators

For general components x_i and K_j, we write the commutators explicitly:

$$[x_1, K_1] = i, \quad [x_2, K_1] = 0, \quad [x_3, K_1] = 0.$$

This arises from the orthogonality of the components. Similarly, for translations along $\hat{x}_2$:

$$d\vec{x}' = dx'\hat{x}_2 \implies \vec{K} \cdot d\vec{x}' = K_2 dx'.$$

The commutators become:

$$[x_2, K_2] = i, \quad [x_1, K_2] = 0, \quad [x_3, K_2] = 0.$$

Finally, for translations along $\hat{x}_3$:

$$d\vec{x}' = dx'\hat{x}_3 \implies \vec{K} \cdot d\vec{x}' = K_3 dx',$$

yielding:

$$[x_3, K_3] = i, \quad [x_1, K_3] = 0, \quad [x_2, K_3] = 0.$$

General Commutator Relation

Combining all results, the commutator between the position components x_i and the translation generators K_j satisfies:

$$[x_i, K_j] = i\delta_{ij},$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

1.5.17. Momentum as a Generator of Translations

Definition of the Translation Operator

The infinitesimal translation operator is defined as:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}',$$

where $\vec{K}$ is the generator of translations (momentum operator).

Free Particle Wavefunction

The wavefunction for a free particle can be expressed as:

$$\psi(\vec{x}, t) = e^{i(\vec{K}\cdot\vec{x}-\omega t)},$$

where $\vec{K}$ corresponds to the wavevector.

Translation of the Wavefunction

Under an infinitesimal translation $d\vec{x}'$, the wavefunction $\psi(\vec{x}, t)$ is translated as:

$$\psi(\vec{x} - d\vec{x}', t).$$

Substituting into the exponential form of ψ, we find:

$$\psi(\vec{x} - d\vec{x}', t) = e^{i(\vec{K}\cdot(\vec{x}-d\vec{x}')-\omega t)}.$$

Simplifying the exponent:

$$\vec{K} \cdot (\vec{x} - d\vec{x}') = \vec{K} \cdot \vec{x} - \vec{K} \cdot d\vec{x}',$$

so:

$$\psi(\vec{x} - d\vec{x}', t) = e^{-i\vec{K}\cdot d\vec{x}'} e^{i(\vec{K}\cdot\vec{x}-\omega t)}.$$

Thus:

$$\psi(\vec{x} - d\vec{x}', t) = e^{-i\vec{K}\cdot d\vec{x}'} \psi(\vec{x}, t).$$

Expanding $e^{-i\vec{K}\cdot d\vec{x}'}$ to first order in $d\vec{x}'$, we get:

$$e^{-i\vec{K}\cdot d\vec{x}'} \simeq 1 - i\vec{K} \cdot d\vec{x}'.$$

Therefore, the translated wavefunction is:

$$\psi(\vec{x} - d\vec{x}', t) \simeq \mathcal{T}(d\vec{x}')\psi(\vec{x}, t),$$

where:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K} \cdot d\vec{x}'.$$

Momentum Operator and De Broglie Wavelength

The generator $\vec{K}$ is related to the momentum operator $\vec{p}$ by:

$$\vec{K} = \frac{\vec{p}}{\hbar}.$$

From De Broglie's hypothesis, the wavelength λ of a free particle is related to its momentum:

$$\lambda = \frac{h}{p}, \quad \vec{k} = \frac{2\pi}{\lambda} = \frac{\vec{p}}{\hbar}.$$

Commutator Relation

The position operator x_i and the momentum operator p_j satisfy the commutation relation:

$$[x_i, p_j] = i\hbar\delta_{ij},$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Momentum as a Generator of Translations

Combining the results, the translation operator in terms of the momentum operator becomes:

$$\mathcal{T}(d\vec{x}') = 1 - i\frac{\vec{p} \cdot d\vec{x}'}{\hbar}.$$

This explicitly shows that momentum generates spatial translations.

1.6. Momentum as a Generator of Translation

Position-Momentum Commutation Relation

The fundamental commutator between the position operator x_i and the momentum operator p_j is given by:

$$[x_i, p_j] = i\hbar\delta_{ij},$$

where δ_{ij} is the Kronecker delta, defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The other commutators involving position and momentum operators are:

$$[x_i, p_x] = i\hbar, \quad [x_i, p_y] = 0, \quad [x_i, p_z] = 0.$$

Position-Momentum Uncertainty Relation

The uncertainty principle for two non-commuting observables A and B states:

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}\left|\langle[A, B]\rangle\right|^2.$$

For the position x and momentum p_x, this becomes:

$$\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \frac{\hbar^2}{4}.$$

Finite Translation as Successive Infinitesimal Translations

A finite translation can be considered as the result of successive infinitesimal translations. The finite translation operator in the x-direction by an amount $\Delta x'$ is given by:

$$\mathcal{T}(\Delta x'\hat{x})|\vec{x}'\rangle = |\vec{x}' + \Delta x'\hat{x}\rangle.$$

The finite translation operator can be expressed as the limit of successive infinitesimal translations:

$$\mathcal{T}(\Delta x'\hat{x}) = \lim_{N\to\infty}\left(1 - i\frac{p_x(\Delta x'/N)}{\hbar}\right)^N.$$

This limit results in the exponential form:

$$\mathcal{T}(\Delta x'\hat{x}) = \exp\left(-\frac{ip_x\Delta x'}{\hbar}\right).$$

Action of the Translation Operator on Position Kets

The translation operator $\mathcal{T}(\Delta x'\hat{x})$ acts on the position eigenket $|\vec{x}'\rangle$ as follows:

$$\mathcal{T}(\Delta x'\hat{x})|\vec{x}'\rangle = e^{-\frac{ip_x\Delta x'}{\hbar}}|\vec{x}'\rangle.$$

The translated position ket corresponds to:

$$\mathcal{T}(\Delta x'\hat{x})|\vec{x}'\rangle = |\vec{x}' + \Delta x'\hat{x}\rangle.$$

1.6.1. Momentum as a Generator of Translation

Definition of the Finite Translation Operator

The finite translation operator in the x-direction by a displacement $\Delta x'$ is defined as:

$$\mathcal{T}(\Delta x'\hat{x}) = \exp\left(-\frac{iP_x\Delta x'}{\hbar}\right),$$

where P_x is the momentum operator in the x-direction, and $\hbar$ is the reduced Planck's constant. This operator translates the position eigenstates $|\vec{x}\rangle$ as follows:

$$\mathcal{T}(\Delta x'\hat{x})|\vec{x}\rangle = |\vec{x} + \Delta x'\hat{x}\rangle.$$

Successive Translations in Different Directions

Consider two successive translations:

- First, a translation in the y-direction by $\Delta y'\hat{y}$,
- Second, a translation in the x-direction by $\Delta x'\hat{x}$.

The corresponding operators are:

$$\mathcal{T}(\Delta y'\hat{y})\mathcal{T}(\Delta x'\hat{x}) \quad \text{and} \quad \mathcal{T}(\Delta x'\hat{x})\mathcal{T}(\Delta y'\hat{y}).$$

If the translations commute, then the total displacement is independent of the order:

$$\mathcal{T}(\Delta y'\hat{y})\mathcal{T}(\Delta x'\hat{x}) = \mathcal{T}(\Delta x'\hat{x} + \Delta y'\hat{y}).$$

Commutator of Translation Operators

To test the commutativity of successive translations, consider the commutator:

$$\delta = [\mathcal{T}(\Delta y'\hat{y}), \mathcal{T}(\Delta x'\hat{x})].$$

Substitute the exponential forms of the translation operators:

$$\mathcal{T}(\Delta x'\hat{x}) = \exp\left(-\frac{iP_x\Delta x'}{\hbar}\right), \quad \mathcal{T}(\Delta y'\hat{y}) = \exp\left(-\frac{iP_y\Delta y'}{\hbar}\right).$$

The commutator becomes:

$$\delta = \left[\exp\left(-\frac{iP_y\Delta y'}{\hbar}\right), \exp\left(-\frac{iP_x\Delta x'}{\hbar}\right)\right].$$

Using the Baker-Campbell-Hausdorff formula, expand the exponentials to first order:

$$\exp(X)\exp(Y) \simeq \exp\bigg(X + Y + \frac{1}{2}[X, Y] + \dots\bigg).$$

Here $X = -\frac{iP_y\Delta y'}{\hbar}$ and $Y = -\frac{iP_x\Delta x'}{\hbar}$. Substituting into the commutator:

$$\delta \simeq -\frac{\Delta x'\Delta y'}{\hbar^2}[P_x, P_y].$$

Commutation Relation Between Momentum Components

The momentum components P_x and P_y commute, so:

$$[P_x, P_y] = 0.$$

Thus, the commutator of the translation operators vanishes:

$$\delta = 0 \implies \mathcal{T}(\Delta y'\hat{y})\mathcal{T}(\Delta x'\hat{x}) = \mathcal{T}(\Delta x'\hat{x})\mathcal{T}(\Delta y'\hat{y}).$$

Conclusion

The translation operators in different directions commute:

$$[P_i, P_j] = 0.$$

Here P_i and P_j represent the momentum operators in orthogonal directions x, y, z.

1.6.2. Momentum as a Generator of Translation

Commutation of Momentum Operators

The momentum components P_x, P_y, P_z satisfy the commutation relation:

$$[P_i, P_j] = 0 \quad \forall\, i, j \in \{x, y, z\}.$$

Since the commutator is zero, the momentum operators are said to commute. When the generators of a transformation group commute, the group is Abelian. Thus:

The translation group in three dimensions is Abelian.

Compatibility of Momentum Operators

The operators P_x, P_y, P_z are compatible, meaning they share a common set of eigenstates. The simultaneous eigenstates of P_x, P_y, P_z are denoted as $|\vec{P}'\rangle$, where:

$$|\vec{P}'\rangle \equiv |P'_x, P'_y, P'_z\rangle.$$

The eigenvalue equations for the momentum operators acting on the eigenket $|\vec{P}'\rangle$ are:

$$P_x|\vec{P}'\rangle = P'_x|\vec{P}'\rangle, \quad P_y|\vec{P}'\rangle = P'_y|\vec{P}'\rangle, \quad P_z|\vec{P}'\rangle = P'_z|\vec{P}'\rangle.$$

Translation Operator Acting on Momentum Eigenstates

The translation operator $\mathcal{T}(d\vec{x}')$, defined as:

$$\mathcal{T}(d\vec{x}') = 1 - i\frac{\vec{P} \cdot d\vec{x}'}{\hbar},$$

acts on a momentum eigenstate $|\vec{P}'\rangle$. The action is given by:

$$\mathcal{T}(d\vec{x}')|\vec{P}'\rangle = \left(1 - i\frac{\vec{P} \cdot d\vec{x}'}{\hbar}\right)|\vec{P}'\rangle.$$

Substituting $\vec{P}|\vec{P}'\rangle = \vec{P}'|\vec{P}'\rangle$, we get:

$$\mathcal{T}(d\vec{x}')|\vec{P}'\rangle = \left(1 - i\frac{\vec{P}' \cdot d\vec{x}'}{\hbar}\right)|\vec{P}'\rangle.$$

Complex Eigenvalues and Hermiticity

The eigenvalue $1 - i\frac{\vec{P}' \cdot d\vec{x}'}{\hbar}$ is **complex** because of the imaginary unit i. This implies that the operator $\mathcal{T}(d\vec{x}')$ is not Hermitian:

$$\mathcal{T}(d\vec{x}') \neq \mathcal{T}^\dagger(d\vec{x}').$$

The operator is still unitary to first order because it preserves the normalization of states.

Conclusion

The momentum eigenstates $|\vec{P}'\rangle$ remain eigenstates of the translation operator $\mathcal{T}(d\vec{x}')$, but with complex eigenvalues. The momentum components P_x, P_y, P_z commute, ensuring that the translation group in three dimensions is Abelian.

1.7. The Canonical Commutation Relations

Dirac's Correspondence

In quantum mechanics, the canonical commutation relations can be obtained from the corresponding classical relations by replacing classical Poisson brackets with commutators. This procedure is known as **Dirac's quantization rule**.
The substitution is given as:

$$\{A, B\}_{\text{classical}} \longrightarrow \frac{1}{i\hbar}[A, B]_{\text{quantum}},$$

where:

- $\{\,,\}$ represents the classical Poisson bracket,
- $[\,,]$ represents the quantum commutator,
- $\hbar$ is the reduced Planck's constant.

Classical Poisson Bracket

The Poisson bracket of two classical observables $A(q, p)$ and $B(q, p)$, where q and p are canonical coordinates and momenta, is defined as:

$$\{A, B\}_{\text{classical}} \equiv \sum_s \left(\frac{\partial A}{\partial q_s} \frac{\partial B}{\partial p_s} - \frac{\partial A}{\partial p_s} \frac{\partial B}{\partial q_s} \right).$$

For the position x_i and momentum p_j, the classical Poisson bracket is:

$$\{x_i, p_j\}_{\text{classical}} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Transition to Quantum Commutation Relations

Using Dirac's prescription, the classical Poisson bracket $\{x_i, p_j\}_{\text{classical}}$ is replaced by the commutator $[x_i, p_j]$ in quantum mechanics:

$$\{x_i, p_j\}_{\text{classical}} \longrightarrow \frac{1}{i\hbar}[x_i, p_j].$$

Rearranging this gives the quantum commutation relation:

$$[x_i, p_j] = i\hbar\delta_{ij}.$$

Generalization to Multiple Degrees of Freedom

For systems with multiple degrees of freedom, the commutation relations extend to:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}.$$

Canonical Quantization

The process of replacing classical observables and Poisson brackets with quantum operators and commutators is called **canonical quantization**. In this framework:

- x_i represents the position operator,
- p_j represents the momentum operator.

The fundamental commutation relations form the basis of quantum mechanics and ensure the correct transition from classical to quantum theory.

Chapter 2

Quantum Dynamics

Chapter 3

Theory of Angular Momentum