Path Integrals Day 5

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1 Instantons

Often we are unable to compute observables exactly, so we use perturbation theory. The result of a perturbative computation frequently takes the form

$$F(g) \sim \sum_n a_n g^n \tag{1}$$

where $F(g)$ is the quantity we are computing and g is a small parameter (such as $\hbar$). In many cases the sum on the right hand side of (1) diverges![1]

Contributions to $F(g)$ that are exponentially small (such as $e^{-1/g}$) are invisible to perturbation theory. All terms in the Taylor series of $e^{-1/g}$ around $g = 0$ are zero. If we approximate a Euclidean path integral using the techniques from day 3[2], and look for contributions of paths near the most important classical path (the one with the smallest Euclidean action), we will generally obtain a divergent series approximation of the form (1). There may be other classical paths whose contributions are exponentially smaller. These other classical paths are known as instantons.

One example of an exponentially small effect is quantum tunneling. If a particle is incident on a potential barrier, but does not have enough energy to classically go to the other side of the barrier, it can tunnel to the other side of the barrier quantum mechanically. The transmission coefficient T is exponentially small in $\hbar$

$$T \propto \exp\left[-\frac{2}{\hbar}\int \sqrt{2m(V-E)}dx\right] . \tag{2}$$

We can understand tunneling by studying instantons.

2 Decay of a metastable state

Consider a potential $V(q)$ with a local minimum at q_0 with $\lim\limits_{q\to\infty} = -\infty$ and $\lim\limits_{q\to-\infty} V(q) = \infty$. Suppose further that $V(q_0) = 0$. By continuity, there must be another point q_* with $V(q_*) = 0$.

[1] See exercise 3 from day 3 for an example.

[2] The techniques from day 3 and their generalizations to complex integrals are often refered to as the saddle point approximation.

A particle that is initially localized near q_0 will eventually tunnel through the barrier. This potential has no ground state, but we can compute the decay rate Γ by computing the imaginary part of the "ground state" energy E_0[3]

$$\frac{\Gamma}{2} = |\mathrm{Im} E_0| \, . \tag{3}$$

The ground state energy can be computed from the partition function $Z = \sum_j e^{-\beta E_j}$ from last time

$$E_0 = - \lim_{\beta \to \infty} \frac{1}{\beta} \log Z \tag{4}$$

since the ground state energy goes to zero the most slowly. We know how to compute the partition function using path integrals.

We can also compute the ground state energy directly from the Euclidean propagator

$$\begin{aligned} K_E\left(q, \frac{\beta\hbar}{2}, q, -\frac{\beta\hbar}{2}\right) &= \langle q | e^{-\beta H} | q \rangle & (5)\\ &= \langle q | e^{-\beta H} \sum_j |j\rangle\langle j|q\rangle & (6)\\ &= \sum_j e^{-\beta E_j} \phi_j(q) \phi_j^*(q) & (7) \end{aligned}$$

where we inserted the resolution of the identity $1 = \sum_j |j\rangle\langle j|$ in the second line and used $\langle q|j\rangle = \phi_j(q)$. The "extra" terms involving the wavefunction $\phi_j(q)$ are independent of β so we also have

$$E_0 = - \lim_{\beta \to \infty} \frac{1}{\beta} \log K_E\left(q, \frac{\beta\hbar}{2}, q, -\frac{\beta\hbar}{2}\right) \tag{8}$$

3 Euclidean propagator and decay rate

The Euclidean propagator can be expressed as a path integral

$$K_E\left(q_0, \frac{\beta\hbar}{2}, q_0, -\frac{\beta\hbar}{2}\right) = \int \mathcal{D}q(\tau) e^{-S_E[q(\tau)]/\hbar} \tag{9}$$

where the paths must satisfy the boundary conditions $q\left(\frac{\beta\hbar}{2}\right) = q\left(-\frac{\beta\hbar}{2}\right) = q_0$. We have chosen to set the initial and final position to be the local minimum of the potential q_0. Using the techniques of day 3, we can approximate this path integral as

$$K_E\left(q_0, \frac{\beta\hbar}{2}, q_0, -\frac{\beta\hbar}{2}\right) \approx \sum_{\bar{q}(\tau)} e^{-S_E[\bar{q}(\tau)]\hbar} * C(\bar{q}) \tag{10}$$

[3] More precisely this is a resonant energy (a quantized complex energy computed with Gamow-Siegert boundary conditions [boundary conditions where the incoming wavefunction vanishes]).

where $\bar{q}$ are the "classical" paths (in Euclidean time) and $C(\bar{q})$ accounts for the fluctuations around those paths.

The classical paths satisfy $\frac{\delta S_E[\bar{q}]}{\delta \bar{q}(\tau)} = 0$ or

$$-m\frac{d^2\bar{q}}{d\tau^2} + V'(\bar{q}) = 0 \tag{11}$$

with boundary conditions $q\left(\frac{\beta\hbar}{2}\right) = q\left(-\frac{\beta\hbar}{2}\right) = q_0$.

In the exercises, you will show that such solutions exist, and determine their Euclidean actions. To get the tunneling rate, we need to sum the contributions of each of the saddles, then divide by β and take the log. The result is

$$\Gamma = \hbar|C_0|e^{-S_0/\hbar}. \tag{12}$$

The tunneling rate is exponentially small in $\hbar$. Here S_0 is the smallest non-zero Euclidean action of a classical solution, and C_0 is the purely imaginary factor that accounts for fluctuations around that solution.

4 Exercise 1: Instantons

(a) Let's set the mass $m = 1$ to simply the algebra. We want to solve the equation

$$-\frac{d^2\bar{q}}{d\tau^2} + V'(\bar{q}) = 0 \tag{13}$$

with boundary conditions

$$\bar{q}\left(\frac{\pm\beta\hbar}{2}\right) = q_0 \tag{14}$$

to find the instantons that determine the decay rate. These equations are the equations of motion of a classical particle with mass $m = 1$ moving in a potential. What is the potential (be careful of minus signs!)?

(b) There is a trivial solution to equations of motion and boundary conditions. What is it? What is the Euclidean action of this trivial solution?

(c) Using your knowledge of a particle moving in a potential, sketch a plot of a non-trivial zero-energy solution to the equations of motion and boundary conditions. (Assume the potential has the shape described at the beginning of section 2 and in lecture.)

(d) Using conservation of energy, show that a solution to the equation of motion with zero energy has

$$\bar{q}(\tau) - q_0 \approx e^{-\omega|\tau|} \tag{15}$$

for large $|\tau|$ (that is $|\tau| \approx \frac{\beta\hbar}{2}$).

(e) Refine your sketch if necessary.

(f) We are interested in the $\beta \to \infty$ limit. Is the Euclidean action of the solution you found finite and non-zero in this limit?

(g) Change integration variables and use conservation of energy to show that the Euclidean action

$$S_0 = \lim_{\beta \to \infty} \int_{-\beta\hbar/2}^{\beta\hbar/2} d\tau \left(\frac{1}{2} \left(\frac{d\bar{q}}{d\tau} \right)^2 + V(\bar{q}) \right) \tag{16}$$

of the solution you found can be expressed as

$$S_0 = 2 \int_{q_0}^{q_*} d\bar{q} \sqrt{2V(\bar{q})} \tag{17}$$

where q_* is the other point where the potential vanishes $V(q_*) = 0$.

(h) Are there any solutions with non-zero energy with finite Euclidean action in the $\beta \to \infty$ limit?

(i) Are there any other solutions with zero energy and finite Euclidean action in the $\beta \to \infty$ limit?

Summing over all of the solutions you found in parts (h) and (i) gives the saddle-point approximation to the Euclidean propagator. Plugging this result into equations (3) and (8) gives the decay rate in equation (12). For more details ask Dan during office hours or see *Aspects of Symmetry* by Sydney Coleman.