Lecture 4 Exercises

Quantum Information, PSI START Summer 2023

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Exercise #1: CNOT and entanglement

(a) Compute $\text{CNOT}(a|00\rangle + b|11\rangle)$. Check that the result is normalized if the input state is, and that the result is not entangled.

(b) Show that $\text{CNOT}^2 = \text{Id}$.

(c) Explain why (a) and (b) together imply that CNOT can also create entanglement.

Exercise #2: CNOT in X basis

Rewrite CNOT in the (X,X) basis using the following steps:

(a) Write out the four (X,X) basis states in terms of the (Z,Z) basis states, e.g. $(|++\rangle = |00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$.

(b) Apply CNOT to each of the four basis states. You should find that the operator just permutes the four states.

(c) Write CNOT in the new basis.

What happened to the roles of the first and second qubits in the new basis?

Exercise #3: CNOT and entanglement (Part 2)

For each of the following states, after applying CNOT, does the entanglement increase, decrease, or stay the same? [Hint: each one requires only reasoning, no calculation!]

(a) $(|00\rangle + |11\rangle)/\sqrt{2}$

(b) $\left(\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle\right) \otimes |0\rangle$

(c) $(|0\rangle \otimes |-\rangle + |1\rangle \otimes |+\rangle)/\sqrt{2}$

Exercise #4: Bell basis

For each of the four Bell basis states, namely

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

Find a unitary operator U such that the state can be written as

$$(\text{Id} \otimes U)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$$

Exercise #5: Bell measurement

Find the possible outcomes and corresponding probabilities for measurement in the Bell basis of the following states:

(a) $(|00\rangle + |11\rangle)/\sqrt{2}$

(b) $|00\rangle$

(c) $|++\rangle$

Exercise #6: Quantum teleportation

We begin with

$$|\Psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

(a) Write out the full state of $|\Psi\rangle$ in the (Z,Z,Z) basis, $|000\rangle, \cdots, |111\rangle$.

(b) Re-write $|\Psi\rangle$ in the (Bell,Z) basis: $|\Phi^+\rangle \otimes |0\rangle$, $|\Phi^+\rangle \otimes |1\rangle$, $|\Phi^-\rangle \otimes |0\rangle$, $\cdots$, $|\Psi^-\rangle \otimes |1\rangle$

(c) For each possible outcome of a Bell measurement on qubits A and B, what is the post-collapse state of the full ABC system?

(d) For each post-collapse state from (c), find a unitary operator on C, $\mathrm{Id}_A \otimes \mathrm{Id}_B \otimes U_C$, that converts the state to $|\Psi_{AB}\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)_C$

(e) Argue that if the lab at PI with qubits A and B classically sends the result of the Bell measurement to the lab at IQC with qubit C, then the state of C can deterministically be converted to $\alpha|0\rangle + \beta|1\rangle$.

(f) Without this classical communication, can C know whether their state is the same as the original $\alpha|0\rangle + \beta|1\rangle$? What does this imply about causality in quantum teleportation?