Path Integrals Day 7

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1 Path integrals and relativity

So far we have been working in the non-relativistic limit. If we want to combine quantum mechanics and special relativity several challenges are apparent:

- In relativistic quantum mechanics, particles can be created and destroyed. In the non-relativistic path integral, paths have fixed boundary conditions. They do not start or end at intermediate times.
- It is not immediately clear how the path integral formulation of quantum mechanics is compatible with causality. Should our integration over all paths include paths outside of the lightcone?
- We treated time and space differently in the development of the path integral. Time is a parameter and position is an integration variable. If we want a relativistic theory, we may want to put time and space on an equal footing.

All of these potential issues can be resolved using quantum field theory. We comment briefly on each of the above issues.

1.1 Particle creation and annihilation

Experimentally we have observed the creation and annihilation of particles. For example, a neutron can decay into a proton, electron, and anti-neutrino. The creation or annihilation of a massive particle is inherently a relativistic process. In the non-relativistic limit $c \to \infty$ it requires an infinite amount of energy to change the mass of a system of particles.

If a particle is created or destroyed, the path of that particle starts or ends. A relativistic path integral must be able to accommodate particle creation and annihilation. It can be difficult mathematically to include paths that start or end. Relativistic path integrals can avoid this difficulty by integrating over possible histories of something other than paths (such as fields).

1.2 Causality

In the path integral expression for the propagator

$$K = \int \mathcal{D}q(t) e^{iS[q(t)]/\hbar} \tag{1}$$

we integrate over all paths that satisfy the boundary conditions, regardless of the choice of action S. For a relativistic action, these paths include paths that go outside of the lightcone. A "typical" path in the path integral is wild (nowhere differentiable) and has infinitely many spacelike segments.

It gets worse - the propagator $K(q, T, 0, 0)$ for a single free relativistic particle is non-zero for spacelike separation $q > cT$ between the initial and final points. In single-particle relativistic quantum mechanics, there is a non-zero probability for a particle to propagate outside of the lightcone!

You will examine this problem further in the exercise.

1.3 Time versus space

In the path integral formulation of quantum mechanics, time is a parameter and space is an integration variable. If we want to treat time and space in the same way, we can treat both as integration variables or both as parameters.

If we treat both as integration variables, we still need a parameter to describe evolution. One possibility is proper time τ. It is possible to develop a relativistic quantum theory along these lines.

An easier path is to treat both time and space as parameters. In other words, we will integrate over a function of both x and t. Our path integral becomes a functional integral, or integral over functions $\phi(x, t)$. Here ϕ is a field, or function that takes values at each point in spacetime. The functional integral analog of the propagator is

$$\int \mathcal{D}\phi(x, t) e^{iS[\phi(x,t)]/\hbar} . \tag{2}$$

Instead of integrating over paths, we integrate over field configurations.

Both the path integral and functional integral can be constructed from a continuum limit, and both can be thought of as infinite dimensional integrals. As a result, all of the techniques we developed to study path integrals can be applied to functional integrals.

2 Exercise: Single-Particle Relativistic Propagator

In single-particle relativistic quantum mechanics, there is a non-zero amplitude for a particle to propagate outside of its lightcone. By single-particle relativistic quantum mechanics (in one spatial dimension) we mean that:

- $\mathbf{1} = \int \frac{dp}{2\pi} |p\rangle\langle p|$
- $E_p = \sqrt{p^2c^2 + m^2c^4}$

- $\hat{P}|p\rangle = p|p\rangle$
- $\hat{Q}|q\rangle = q|q\rangle$
- $\hat{H}|p\rangle = E_p|p\rangle$
- $\langle q|p\rangle = e^{ipq/\hbar}$

(a) Which of the above equations tells us that we are dealing with single-particle quantum mechanics? Which of the above equations tells us that we are dealing with relativistic quantum mechanics?

(b) Show that the propagator is

$$\langle q|e^{-iHT/\hbar}|q=0\rangle = \int \frac{dp}{2\pi} e^{ipq/\hbar} e^{-iE_pT/\hbar} \,. \tag{3}$$

(c) It can be shown that the propagator you computed in part (b) is non-zero for $q > ct$. Why is this problematic?

(d) To compute the single-particle relativistic propagator using path integrals our starting point is the action of a relativistic particle

$$S = -\int mc^2 \sqrt{1 - \frac{\dot{q}^2}{c^2}} dt \,. \tag{4}$$

The equation of motion is

$$\frac{m\ddot{q}}{\left(1 - \frac{\dot{q}^2}{c^2}\right)^{3/2}} = 0 \,. \tag{5}$$

Find the classical paths that contribute to the spacelike propagator. Does this quantum theory of a free relativistic particle have the expected classical limit?

We can fix all of the problems you found using quantum field theory. In quantum field theory, particles are excitations of fields and can be created and annihilated.[1]

[1]Some quantum field theories do not allow for particle creation or annihilation. They avoid the problems here because it is not possible to construct position operators in a relativistic theory that satisfy the assumptions at the beginning of this exercise.