Path Integrals Day 3

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1 Classical limit

We have seen that the propagator can be expressed as a path integral

$$K = \int \mathcal{D}q(t) e^{iS[q(t)]/\hbar} . \tag{1}$$

Based on our experience with classical mechanics, where it makes sense to talk about the path that a particle took to go from one point to another, we expect that classical paths are the most important contributions to the path integral (1) in the classical limit $\hbar \to 0$.

But the classical paths contribute the same amount as any other path in (1)! Since the action $S[q(t)]$ is real, $e^{iS[q(t)]/\hbar}$ is a complex number with magnitude one for every path, classical or not. Every path is equally important.

However, not every set of nearby paths is equally important. We can understand many of the features of how this works by considering a toy model for the propagator (in zero dimensions)

$$I = \int dx e^{if(x)/\hbar} . \tag{2}$$

Here I is analogous to the propagator, x is analogous to the path q (but is a real number instead of a function), and f is analogous to the action S.

Let's consider a path $q'(t)$ that is close to some path $q(t)$:

$$q'(t) = q(t) + \eta(t) \tag{3}$$

where $\eta(t)$ is small for all time. In the toy model, we would instead look at a point x' near some other point x

$$x' = x + \epsilon \tag{4}$$

with ϵ a small parameter.

The action of the nearby point (or path) is to leading order in ϵ (or η)

$$f(x') = f(x) + \epsilon f'(x) + \mathcal{O}(\epsilon^2) \tag{5}$$

or

$$S[q'(t)] = S[q(t)] + \int \eta(t) \frac{\delta S[q(t)]}{\delta q(t)} dt + \mathcal{O}(\eta^2)\,. \tag{6}$$

The functional derivative $\frac{\delta S[q(t)]}{\delta q(t)}$ is a generalization of the partial derivative. If we discretized $q(t)$, the action would be a function of a finite number of variables (the q_i) instead of being a functional. In that case, we would sum over the first derivative of S in each direction times the change in q in that direction. In the continuum limit the partial derivative becomes a functional derivative in (6).

Neglecting the higher-order terms, the contributions of these two nearby points (or paths) is

$$e^{if(x)/\hbar} + e^{if(x')/\hbar} \approx e^{if(x)/\hbar}\left(1 + e^{i\epsilon f'(x)/\hbar}\right) \tag{7}$$

or

$$e^{iS[q(t)]/\hbar} + e^{iS[q'(t)]/\hbar} \approx e^{iS[q(t)]/\hbar}\left(1 + e^{i\int \eta(t)\frac{\delta S[q(t)]}{\delta q(t)} dt/\hbar}\right)\,. \tag{8}$$

In the classical limit $\hbar \to 0$, the phase difference between the two points (or paths) becomes large, leading to destructive interference.

That is, unless the derivative vanishes $f'(x) = 0$ (or $\frac{\delta S[q(t)]}{\delta q(t)} = 0$). But points (or paths) satisfying these equations are exactly the classical paths. The phase changes slowly in the vicinity of a classical path, so a path that is close to a classical path will interfere constructively with the classical path.

2 Perturbation theory

Few path integrals can be computed exactly. Fortunately we have powerful techniques to approximate path integrals. In exercise 3, we will develop one such technique (and see a surprising result!).

We will focus on the toy model above to describe perturbation theory for path integrals. Or rather, a slight variation of the toy model we looked at above

$$I_E = \int dx e^{-f(x)/\hbar} \tag{9}$$

where the E stands for Euclidean for reasons we will discuss briefly next time. The difference between this toy model and the previous version is the presence of a minus sign instead of an i in the exponent. This change will simplify the mathematical discussion, although perturbation theory works in essentially the same way in both cases. There are some interesting physics applications where the path integral has a minus sign instead of an i that we will discuss in future classes.

For most $f(x)$, we cannot evaluate I_E exactly. We can approximate I_E by noting that the integrand is largest where $f(x)$ is smallest. If we expand $f(x)$ around its minimum at x_c we get

$$f(x) = f(x_c) + \frac{1}{2} f''(x_c)(x - x_c)^2 + \frac{1}{3!} f'''(x_c)(x - x_c)^3 + \cdots \tag{10}$$

since the first derivative vanishes at a minimum. If we stop at second order, we can evaluate the resulting integral

$$I_E \approx \int dx e^{-[f(x_c) + \frac{1}{2} f''(x_c)(x - x_c)^2]/\hbar} \tag{11}$$

as it is Gaussian. If we go to higher order, we seem to be in trouble as

$$I_E \approx \int dx e^{-[f(x_c) + \frac{1}{2} f''(x_c)(x - x_c)^2 + \frac{1}{3!} f'''(x_c)(x - x_c)^3]/\hbar} \tag{12}$$

is not Gaussian. However, as you will see in the exercise, integrals of the form

$$\int dx x^N \exp[-ax^2 + bx + c] \tag{13}$$

can also be evaluated exactly. We can expand the troublesome exponential

$$I_E \approx \int dx e^{-[f(x_c) + \frac{1}{2} f''(x_c)(x - x_c)^2]/\hbar} \left(1 + \frac{1}{3!} f'''(x_c)(x - x_c)^3]/\hbar + \cdots\right) \tag{14}$$

to put our approximation to I_E in the desired form. Each term can be computed.

3 Exercise 1: Conceptual review

(a) In what sense are classical paths the most important contributions to the propagator in the classical limit?

(b) Fixing the Lagrangian (or action) and initial and final times and positions, is there a necessarily unique classical path? If so, why? If not, give a counter example. What are the implications of the uniqueness or non-uniqueness of the classical solution for perturbation theory?

4 Exercise 2: Semi-classical limit

(a) Let $f(x)$ be a real valued function. By expanding $f(x)$ around its minimal values x_c show that we can approximate

$$I_E = \int_{-\infty}^{\infty} dx e^{-f(x)/\hbar} \tag{15}$$

by

$$I_E \approx \sum_{x_c} \sqrt{\frac{2\pi\hbar}{f''(x_c)}} e^{-f(x_c)/\hbar} (1 + \mathcal{O}(\hbar^a)) \,. \tag{16}$$

(b) Express the most important corrections to equation (16) in the form

$$\int_{-\infty}^{\infty} dx C_1 x^n e^{-C_2 x^2} \tag{17}$$

where C_1 and C_2 are constants.

(c) Determine the value of a in equation (16) by making a change of variables.

5 Exercise 3: Perturbation theory

(a) Show that

$$\exp\left[-\lambda\left(\frac{d}{dJ}\right)^4\right]\int_{-\infty}^{\infty} dx \exp[-x^2 + xJ] = \int_{-\infty}^{\infty} dx \exp[-x^2 - \lambda x^4 + xJ]\,. \tag{18}$$

Path integrals with interactions can be computed by taking derivatives of free path integrals with a source J using generalizations of this formula.

(b) Use induction to show that

$$\int_{-\infty}^{\infty} dx x^{2n} \exp\left[-\frac{1}{2}ax^2\right] = \left(\frac{2\pi}{a}\right)^{1/2}\frac{1}{a^n}(2n-1)!! \tag{19}$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$. Hint: differentiate with respect to a.

(c) Show that

$$\int_{-\infty}^{\infty} dx \exp[-x^2 - \lambda x^4] = \sqrt{\pi}\sum_{j=0}^{\infty}\frac{(-\lambda)^j(4j-1)!!}{2^{2j}j!}\,. \tag{20}$$

(d) Argue that the integral in part (c) converges for $\lambda \geq 0$.

(e) Show that the series in part (c) diverges.

(f) Which step in the derivation was not justified?