Part I:

1.

3.1, 18)

(a) True;

consider A, B

$A^T = A$, $B^T = B$

$\Rightarrow (A+B)^T = A^T + B^T$

$= A + B$

$\Rightarrow$ sum of symm. matrices is also a symmetric matrix $\Rightarrow$ the symm. matrices form a subspace of M

(b) True;

$A^T = -A$ $\quad B^T = -B$

$(A+B)^T = A^T + B^T$

$= -A + (-B)$

$= -(A+B)$

(c) False; consider e.g., $\underbrace{A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \text{ \& } B = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}}_{\text{unsymmetric}}$

$A + B = \begin{bmatrix} 4 & 5 \\ 5 & 8 \end{bmatrix}$

$(A+B)^T = \begin{bmatrix} 4 & 5 \\ 5 & 8 \end{bmatrix}$ their sum is symm.

(a)

reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ -1 & -4 & -2 & b_3 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3+b_1 \end{array}\right]$$

$\Rightarrow b_2 - 2b_1 = 0$

$\boxed{b_2 = 2b_1}$

and

$b_3 + b_1 = 0$

$\boxed{b_3 = -b_1}$

$$\Rightarrow \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(b)
$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2-2b_1 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2-2b_1 \\ 0 & 0 & b_3+b_1 \end{array}\right]$$

$\Rightarrow b_3 = -b_1$

$$\begin{bmatrix} b_1 \\ b_2 \\ -b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3.1, 23)

unless the vector b is in the column space of A

Now, add an extra column so that $[A \ \vec{b}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\vec{b} \notin C(A) \Rightarrow$ no sol.

Add a different column

$$[A \ b] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$C(A)$ hasn't changed

the new column is already

in $C(A)$

Solvability:

$A\vec{x} = \vec{b}$ is solvable (exactly) when $C(A)$ doesn't get larger

2. from 3.2

P2.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[3R_3 - R_2]{R_2 - R_1} \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}2\\0\\0\end{matrix} & \boxed{\begin{matrix}2\\1\\0\end{matrix}} & \begin{matrix}4\\2\\0\end{matrix} & \begin{matrix}6\\3\\0\end{matrix} \end{bmatrix} = U$$

free

associated to pivot columns

$$x_1 + 2x_2 + 2x_3 + 4x_4 + 6x_5 = 0$$

$$x_3 + 2x_4 + 3x_5 = 0$$

① $x_2 = 1, \ x_4 = x_5 = 0$

$x_3 + 0 + 0 = 0 \Rightarrow x_3 = 0$

& $x_1 + 2 = 0 \Rightarrow x_1 = -2$

$$S_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

② $x_2 = x_5 = 0$ $\quad x_4 = 1$

$x_3 = -2$

& $x_1 \cancel{-4+4} = 0$

$$S_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

③ $x_2 = x_4 = 0 \quad x_5 = 1$

$$x_3 + 3 = 0$$
$$x_3 = -3$$

$\& \ x_1 = 0$

$$S_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$\underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}}_{\text{pivot columns}} \xrightarrow{2R_3 - R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

pivots   free

$$2x_1 + 4x_2 + 2x_3 = 0$$
$$4x_2 + 4x_3 = 0$$

$x_3 = 1$

$$4x_2 + 4 = 0$$
$$4x_2 = -4$$
$$x_2 = -1$$

$$\Rightarrow 2x_1 - 4 + 2 = 0$$
$$2x_1 = 2$$
$$x_1 = 1$$

$$S = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

P.16.

$\boxed{\text{five pivots}}$ in which case, we have no free variables and we only have the trivial sol..

$\boxed{\text{5 pivots}}$ (dim of $C(A)$ = # of pivots)

18.

$x-3y-z=12$

$\Rightarrow x=12+3y+z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12+3y+z \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

24.

Since we have 2 vectors in the nullspace, and the matrix is 3×3, the # of pivots : $3-2=1$ at most

the column space contains 2 vectors $\Rightarrow$ # of pivots $=2$

$\Rightarrow$ constructing such a matrix is impossible

$\boxed{3.3}$

19 20

38 50 56

7.

special solutions of $R\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

free

$x_1 + 2x_3 + 3x_4 = 0$

$x_2 + 4x_3 + 5x_4 = 0$

let $x_3 = c_1$ & $x_4 = c_2$

$x_1 = -(2c_1 + 3c_2)$

$x_2 = -(4k_1 + 5k_2)$

$$\Rightarrow \vec{x} = \begin{bmatrix} -(2c_1 + 3c_2) \\ -(4c_1 + 5c_2) \\ c_1 \\ c_2 \end{bmatrix} \Rightarrow \text{special sol.'s} \left\{ \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\vec{y}^T R = \vec{0}$

$$[y_1 \; y_2 \; y_3] \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \vec{0}$$

$y_1 = 0$

$y_2 = 0$

$2y_1 + 4y_2 = 0$

$3y_1 + 5y_2 = 0$

free variable is y_3, $y_3 = C$

$\Rightarrow$ the special sol. of $\vec{y}^T R = \vec{0}$ is $(0 \ 0 \ 1]$

Now, $R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- solutions of $R\vec{x} = 0$

x_1 & $x_3 \rightarrow$ free

$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_2 + 2x_3 = 0$

let $x_1 = C_1$ & $x_3 = C_2$

$x_2 + 2x_3 = 0$

$x_2 = -2C_3$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ -2C_2 \\ C_2 \end{bmatrix} \Rightarrow$ special sol.'s $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

$= C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

- $\vec{y}^T R = \vec{0}$

$[y_1 \ y_2 \ y_3] \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \vec{0} \Rightarrow \begin{matrix} y_1 = 0 \\ 2y_1 = 0 \end{matrix}$

$\Rightarrow$ special sol. $\left\{ \begin{matrix} (0,1,0), \\ (0,0,1) \end{matrix} \right\}$

free var.

y_2 & y_3

$y_2 = C_1 \quad y_3 = C_2$

$(y_1, y_2, y_3)^T = (0, C_1, C_2)$

19.

$A \& B$
$n \times n$

since $AB = I$

$r(AB) = r(I)$
$= n$

$r(AB) \leq r(A)$

$\Rightarrow r(A) \geq n$

However, an $n \times n$ matrix cannot have rank $> n$

$\Rightarrow r(A) = n$ which means A has no free variables
$\Rightarrow A$ is invertible (A^{-1} exists)

$\Rightarrow A^{-1}AB = IA^{-1}$

$B = A^{-1}$ B is the two-sided inverse of A

Multiplying by A

$BA = I$

25.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 - R_2} \underbrace{\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rank} = 2}$$

pivot columns

$$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$\boxed{3.4}$

13.

(a)

linear comb. of x_p & x_s = the complete sol.

$x = x_p + x_s \rightarrow$ complete sol.

x_p always multiplied by 1

$\Rightarrow x$ cannot be any lin. comb. of x_p & x_s

(b)

If N(A) has the sol. x_s and the particular x_p

$\Rightarrow$ Another sol. is $x_p + x_s$

The # of x_p can be more than 1

$\Rightarrow$ Statement is false

(c)

consider $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $\Rightarrow \left[\begin{array}{cc|c} 2 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$

x_2 is the free variable

let $x = 0$

$2x_1 + 2x_2 = 4$

$x_1 = 2$

$\Rightarrow x_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ it's length : $\sqrt{2^2+0} = 2$

Now, let $x_2 = 1$

$2x_1 + 2x_2 = 4$

$2x_1 + 2 = 4$ $\qquad x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and has length $= \sqrt{2}$

$x_1 = 1$

$\Rightarrow$ shortest length is not of particular sol.

(d)

If A is invertable, the solution ~~to~~ N(A) is $x_s = 0$

$\Rightarrow$ statement is false

18. we are solving $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 6 & 10 \end{bmatrix} \xrightarrow[R_3 - 2R_2]{R_1 - \frac{4}{3}R_2} \begin{bmatrix} 1 & 0 & -20/3 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 2$ $\qquad r(A^T) = 2$ $\qquad$ (T doesn't alter r)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$$

$q - 2 = 0$

$q = 2$

last row: 0 0 0

$\therefore$ $r(A) = 2$

$r(A^T) = 2$

when $q \neq 2$

$r(A) = 3$ & $r(A^T) = 3$

25.

(a) $b \notin C(A)$

$\Rightarrow r < m$

$r \leq n$

(b) $N(A)$ must contain $\vec{0}$s

$\Rightarrow r < n$

$r = m$

(c)

if it has one sol. exactly for one $\vec{b}$

$\Rightarrow$ A must have full column rank

if it has no sol. for other $\vec{b}$

$\Rightarrow r$ must be $< m$

$\Rightarrow r < m$

$r = n$

d)

A has full row & column rank

$\Rightarrow r = m$

$r = n$

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Part II

1.

(I) $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

clearly, $\underline{C(A_1)}$ consists of the $\vec{0}$ only; $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, <u>which is the origin in $\mathbb{R}^2$</u>

$A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A_2) = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

$\Rightarrow$ <u>$C(A_2)$ is the xy-plane in $\mathbb{R}^3$</u>

(II) Null space

$N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$

$N(A_1) = \left\{ \vec{x} ; x_1, x_2, x_3 \in \mathbb{R} \mid \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \right\}$

$= \underline{\mathbb{R}^3}$

$N(A_2) = \{ \vec{x} ; x_1, x_2, x_3 \in \mathbb{R} \mid \underbrace{2x_1 + x_2 + x_3 = 0, \ x_3 = 0}_{} \}$

<u>the line defined by $(0,0,0)$ and $(1,-2,0)$</u>

2.

$$A \begin{bmatrix} 2a \\ 0 \\ a \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{2\cancel{1}} & a_{2\cancel{2}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2a \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2a a_{11} + 0 + a a_{13} = 0$$

$$2a a_{22} + 0 + a a_{23} = 0$$

$$2a a_{31} + 0 + a a_{23} = 0$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \\ 3 & 0 & -6 \end{bmatrix}$$

3. $$\begin{bmatrix} 3 & 0 & -6 & 0 \\ 1 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{3}R_1} \begin{bmatrix} 3 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 1$ free variables: $4 - 1 = 3$

$\Rightarrow A\vec{x} = \vec{0}$ has ∞ many solutions

$A\vec{x} = \vec{b}$ if $b \in \operatorname{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} \Rightarrow \infty$ many

and no solutions otherwise

b)

$$A = \begin{bmatrix} -1 & 3 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ -1 & 3 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} -1 & 3 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 2 < 4$

$\Rightarrow A\vec{x}$ has infinitly many sol.'s

$\Rightarrow A\vec{x} = \vec{b}$ has ∞ many sol.'s as well

if $\vec{b} \in \left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$ otherwise is has 0 solutions

c) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $r(A) = 3 < 4$

$\Rightarrow A\vec{x} = 0$ has ∞ many sol.'s

$A\vec{x} = \vec{b}$ has ∞ many sol. $\forall \vec{b}$

d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $r(A) = 3 =$ # of columns

$\Rightarrow A\vec{x} = \vec{0}$ has a unique sol.

$A\vec{x} = \vec{b}$ has a unique sol. if $\vec{b} \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$