Mid-course Problem Set

Quantum Information, PSI START Summer 2023

Instructor: Aaron Szasz

TA: Jake Barnett

Instructions: You should attempt all the problems, but if you get stuck it's ok. I highly encourage you to use the Slack channel for our course to coordinate study/collaboration groups for working on this. Submit your solutions using the Dropbox link posted on Slack.

Problem #1:

Suppose we make a change of basis by mapping $|0\rangle$ to $|0\rangle'$ and $|1\rangle$ to $|1\rangle'$, for

$$|0\rangle' = \alpha|0\rangle + \beta|1\rangle, \quad |1\rangle' = \gamma|0\rangle + \delta|1\rangle \tag{1}$$

where the coefficients α, β, γ, and δ are complex numbers with $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$ so that the new basis vectors $|0\rangle'$ and $|1\rangle'$ are normalized.

Consider a general normalized state $|\psi\rangle = a|0\rangle + b|1\rangle$ with $a^2 + b^2 = 1$.

(a) Transform $|\psi\rangle$ to $a|0\rangle' + b|1\rangle'$ by substitution. The resulting state must be normalized. For each of the following choices of (a, b), what conditions on α, β, γ, and δ must be satisfied to ensure normalization of $|\psi\rangle$: $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$? Show that these conditions precisely correspond to orthogonality of $|0'\rangle$ and $|1'\rangle$.

(b) If we write the matrix form of the basis transformation,

$$U_M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix},$$

show that the conditions you found in (a), together with normalization of $|0\rangle'$ and $|1\rangle'$, are equivalent to the statement $U_M^\dagger U_M = \mathrm{Id}_M$.

(c) Argue (for example using the matrix determinant), that a square matrix M satisfying $M^\dagger M = \mathrm{Id}$ must also satisfy $MM^\dagger = \mathrm{Id}$. Conclude that U_M is unitary.

(d) (Optional) Using the definition of adjoint from linear algebra, for an operator

$$U = \sum_{ij} U_{ij}|i\rangle\langle j|,$$

find $U^\dagger$. [Hint: use $\langle v|w\rangle = \langle w|v\rangle^*$.]

Then show, using your result for $U^\dagger$, that $U^\dagger U = \mathrm{Id}$ precisely when $U_M^\dagger U_M = \mathrm{Id}_M$.

(e) Conclude that norm-preserving transformations of one qubit are unitary.

Problem #2:

(a) Consider $|\psi\rangle = a|0\rangle + b|1\rangle$. If we measure in the z-basis $\{|0\rangle, |1\rangle\}$, what outcomes can we get, and with what probabilities?

(b) We will now discuss measurement in the "x-basis," which is the basis of σ^x eigenvectors. Show that $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle \equiv (|0\rangle - |1\rangle)/\sqrt{2}$ are eigenvectors of σ^x.

(c) Next, we invert the transformation. Find $|0\rangle$ and $|1\rangle$ in terms of $|+\rangle$ and $|-\rangle$.

(d) We can measure $|\psi\rangle$ in the new basis by computing the probabilities of measuring $|+\rangle$ and $|-\rangle$; these are $P(+) = |\langle + |\psi\rangle|^2$ and $P(-) = |\langle - |\psi\rangle|^2$. Compute $P(+)$ and $P(-)$.

(e) Alternatively, we can transform $|\psi\rangle$ and leave the basis invariant. In this method, we make the opposite transformation of $|\psi\rangle$, namely we replace $|+\rangle$ by $|0\rangle$ and $|-\rangle$ by $|1\rangle$. One way of doing this is to first substitute the results of (c) into $|\psi\rangle$, so we write it entirely in terms of $|+\rangle$ and $|-\rangle$. Then we just make the replacements. Finally, we measure the resulting state in the basis $\{|0\rangle, |1\rangle\}$. Compute $P(0)$ and $P(1)$, and show that they match the results of (d).

Problem #3:

One way of understanding the dimensionality of a space or set is the number of parameters required to describe it, minus the number of constraints on those parameters. For example, a circle is defined by $a^2 + b^2 = 1$, with 2 parameters and 1 constraint, and is thus 1-dimensional. Show the following:

(a) A normalized single-qubit state is described by 4 real numbers with one constraint, hence the space of such states is 3-dimensional.

(b) A normalized product state $|\psi_1\rangle_A \otimes |\psi_2\rangle_B$ is described by 7 real numbers with 2 constraints, hence the space of such states is 5-dimensional.

(c) A fully general 2-qubit state is described by 8 real numbers with 1 constraint, hence the space of such states is 7-dimensional.

Thus product states are a lower-dimensional space within the space of all 2-qubit states. This means that almost all two-qubit states are entangled.

[Note: I have neglected the fact that overall phase of a wavefunction is not measurable. You may choose to further reduce the number of parameters in each case as a result, but your overall conclusion about dimensionality of the space of all two-qubit states vs the space of product states will be the same.]

Problem #4:

For a state $|\Psi\rangle = (U_A \otimes U_B)(a|00\rangle + b|11\rangle)$, we have established that entanglement is 0 when $(a, b) = (1, 0)$, maximized when $a = b$, and varies continuously in between. There are many functions of a and b that behave in this manner, including a class of functions called Rényi entanglement entropies. Specifically, the Rényi entropy of order α is

$$S^{(\alpha)}(a, b) = \frac{1}{1-\alpha} \log\left(a^{2\alpha} + b^{2\alpha}\right).$$

(a) Using the normalization $a^2 + b^2 = 1$, plot $S^{(\alpha)}$ as a function of a^2 for $\alpha = 2$ and $\alpha = 1/2$.

(b) Draw what the function looks like in the limit $\alpha \to 0^+$. You can try plotting numerically with a software package of your choice for various small values of α, and the shape of the limiting function should become clear. What about the limit $\alpha \to \infty$? The limit isn't as clear in this case, but you can plot it for large values of α to get a sense of what it looks like. Is it smooth at $a^2 \approx 1/2$?

(c) Show that in the limit $\alpha \to 1$, the Rényi entropy reduces to the Von Neumann entanglement entropy,

$$S(a, b) = -a^2 \log_2(a^2) - b^2 \log_2(b^2)$$

Problem #5: (Optional but highly recommended

Again, consider the two-qubit state $|\Psi\rangle = a|00\rangle + b|11\rangle$. Another intuitive way of getting at entanglement is to ask "how far is it from a product state?" Specifically, we want to know:

$$\text{"Product-ness"} = \max_{|\psi_A\rangle,|\psi_B\rangle} |\langle\psi_A \otimes \psi_B \mid |\Psi\rangle|^2 \tag{2}$$

In other words, we compute the largest possible overlap between our state and a product state. If this is small, then our state must be quite different from the most similar product state, thus it is probably "more entangled."

Argue that the "Product-ness" of $|\Psi\rangle$ is exactly a^2. Does this make sense?

[Note: in general this is quite challenging! If you would like, you can solve an easier version. Assume that both U_A and U_B are rotations by θ, so that their matrix versions are both

$$U_M = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Then show that the optimal θ is 0.]