KAUST

CEMSE151 - Linear Algebra

Problem Set 1

To be returned by September 9, 2023, 5:00pm

August 31, 2023

The first 5 problems are taken from the book of Strang, Gilbert. Introduction to Linear Algebra. 4th ed. Wellesley, MA: Wellesley-Cambridge Press, February 2009. ISBN: 9780980232714.

1. If $\mathbf{v} \cdot \mathbf{w} < 0$, what does this say about the angle between $\mathbf{v}$ and $\mathbf{w}$? Draw a 3-D vector $\mathbf{v}$ and show where to find all $\mathbf{w}$'s with $\mathbf{v} \cdot \mathbf{w} < 0$.

2. Can three vectors in the xy plane have $\mathbf{u} \cdot \mathbf{v} < 0$ and $\mathbf{v} \cdot \mathbf{w} < 0$ and $\mathbf{u} \cdot \mathbf{w} < 0$?

3. Find a (non-trivial) combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector, for
$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$
Those vectors are (independent)(dependent). The three vectors lie in a .... The matrix W with those columns is not invertible.

4. The very last words in Chapter 1.3 say that the 5 by 5 centered difference matrix is not invertible. Write down the 5 equations $C\mathbf{x} = \mathbf{b}$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero? Note that the 5 columns lie on a 4D hyperplane in 5D space.

5. With $A = I$, in which I is the 3×3 identity matrix (products of an arbitrary identity matrix I, either to the right or the left, result in the original arbitrary matrix), i.e.,
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

draw the planes in the row picture:

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Verify that the three sides of the box meet at the solution $(2, 3, 4)$. Draw the vectors in the column picture and find the linear combination of the columns of A that result in $\mathbf{b}$.

6. Determine AB and BA if possible:

   a)
   $$A = \begin{bmatrix} 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

   b)
   $$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

7. Let A be a square matrix. A^k is the product of A by itself k times:

   $$A^k = \underbrace{A \ldots A}_{k \text{ times}}$$

   Give examples of 2×2 matrices with the following properties:

   a) $A^2 = -I$, in which I is the 2×2 identity matrix, i.e.,
   $$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

   b) $B^2 = 0, B \neq 0$;

   c) $CD = -DC, CD \neq 0$;

   d) $EF = 0$, with all components of both E and F nonzero.

8. True or false?

   a) if the second and fifth columns of B are equal, then the second and fifth columns of AB are equal;

   b) if the second and fifth rows of B are equal, then the second and fifth rows of AB are equal;

   c) if the second and fifth rows of A are equal, then then second and fifth rows of AB are equal;

   d) $(AB)^2 = A^2B^2$.