KAUST

CEMSE151 - Linear Algebra

Problem Set 2

To be returned by September, 21st, 2023, 5:00pm

September 7, 2023

The first 11 problems are taken from the book of Strang, Gilbert. Introduction to Linear Algebra. 4th ed. Wellesley, MA: Wellesley-Cambridge Press, February 2009. ISBN: 9780980232714.

1. Start with the vector $\mathbf{u}_0 = (1, 0)$. Multiply again and again by the same ?Markov matrix? $A = [.8 \quad .3; .2 \quad .7]$. What are the next three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$? What property do you notice for all four vectors $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

2. For which 3 values of k does elimination break down? Which can be fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞?

$$\begin{bmatrix} k & 3 \\ 3 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

3. Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a linear combination of the first two rows. Find a third equation that cannot be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

4. Which three matrices E_{21}, E_{31} and E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \qquad E_{32}E_{31}E_{21}A = U.$$

Multiply the E's to get the one matrix M that does elimination $MA = U$.

5. The entries of A and x are a_{ij} and x_j. So the first component of Ax is $\sum a_{1j}x_j = a_{11}x_1 + \ldots + a_{1n}x_n$. If E_{21} subtracts row 1 from row 2, write a formula for
   a) the 3rd component of Ax;
   b) the $(2,1)$ entry of $E_{21}A$;
   c) the $(2,1)$ entry of $E_{21}(E_{21}A)$;
   d) the 1st component of $E_{21}Ax$.
6. What rows or columns or matrices do you multiply to find
   a) the 3rd column of AB?
   b) the first row of AB?
   c) the entry in row 3, column 4 of AB?
   d) the entry in row 1, column 1 of CDE?
7. Suppose you solve $Ax = b$ for three special right b:
$$Ax_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad Ax_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad Ax_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$
   If the 3 solutions x_1, x_2 and x_3 are the columns of a matrix X, what is AX?
8. If A has row 1+row 2=row 3, show that A is not invertible:
   a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.
   b) Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?
   c) What happens to row 3 in elimination?
9. from section 2.5: problems 24 and 31 (in the 5th edition, these are problems 24 and 30 (only the part with matrix A));
10. from section 2.6: problems 13, 18 and 23 (in the 5th edition, these are problems 13, 18 and 23 as well);
11. from section 2.7: problems 13 and 36 (in the 5th edition, these are problems 13 and 35).
12. Consider
$$E_{21} = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$
    and explain what happens to a 3×3 matrix A when multiplied on the left and on the right by E_{21}.
13. If A and B are $n \times n$ such that all the components of A are 1 and all the components of B are 2, find the values of all the components of AB.

14. Solve the following systems of equations using Gauss elimination on the augmented matrix and backwards substitution:

a)

$$\begin{cases} 2x + y + 4z &= 2 \\ 6x + y &= -10 \\ -x + 2y - 10z &= -4 \end{cases}$$

b)

$$\begin{cases} y + z &= 3 \\ x + 2y - z &= 1 \\ x + y + z &= 4 \end{cases}$$

c)

$$\begin{cases} x + 2y + 3z + 4w &= 0 \\ 5z + 6w &= 0 \\ az + 6w &= 0 \\ y + 7z + 8w &= 1 \end{cases}$$

for $a \in \mathbb{R}$.

15. Problem 14a) asked to solve a system of linear equations using Gauss elimination. Suppose A is the matrix of the system:

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 6 & 1 & 0 \\ -1 & 2 & 10 \end{bmatrix}.$$

a) Write this Gauss elimination in terms of the product of elementary matrices.

b) Write the LU decomposition of A and explain how you determined L.

c) Use the row approach to multiplication to write A as a product of L, a diagonal matrix D whose elements in the diagonal are the pivots of Gauss elimination and an upper triangular matrix U^* with 1 in the diagonal:

$$A = LDU^*.$$

16. Let $A = LU$ with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solve $Ax = b$, $b = [1 \quad 2 \quad 4]^T$, using two systems of linear equations with triangular matrices.

17. Consider again

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 6 & 1 & 0 \\ -1 & 2 & 10 \end{bmatrix}.$$

a) Explain how to reduce A to a diagonal matrix D using only products of elementary matrices. Show those elementary matrices.

b) What are the diagonal components of D?

c) Explain how to determine A^{-1} using elementary and diagonal matrices.

18. Consider the following 2 matrices:

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}.$$

For each of these matrices A,

a) determine, if possible, its LU decomposition; if $A = LU$ is not possible, find a permutation matrix P such that $PA = LU$;

b) determine conditions on the components of b such that $Ax = b$ has a solution (look for the rows of zeros in U).