Section 4.2

13. Suppose A is $I_{4\times4}$ w/ last column removed. A is 4x3.

Project $\vec{b} = (1,2,3,4)$ onto C(A)....

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = \vec{p}$$

- $A^TA = \underset{3\times4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \underset{4\times3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \Rightarrow (A^TA)^{-1} = I$

- $A(I)A^T\vec{b} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underbrace{\underset{3\times4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \vec{p}$$

16. what lin. comb. of $(1,2,-1)$ & $(1,0,1)$ is closest to $(2,1,1) = \vec{b}$?

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad 2x_1 = 1 \Rightarrow \boxed{x_1 = \frac{1}{2}}$$

$$\frac{1}{2} + x_2 = 2 \Rightarrow \boxed{x_2 = \frac{3}{2}}, \Rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

17. If $P^2 = P$, show $(I-P)^2 = I-P$. When P projects onto the column space of A, $I-P$ projects onto $\underline{(N(A^T))}$

Proof

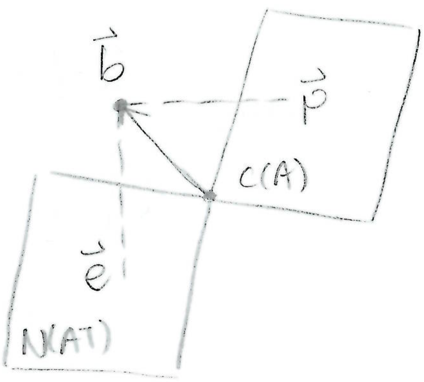
$$(I-P)^2 = I^2 - IP - PI + P^2$$
$$= I - P - \cancel{P} + \cancel{P}$$
$$= I - P$$

suppose that $\vec{v} \in C((I-P)\vec{b}$; $\vec{b} \in \mathbb{R}^m$

then $\vec{v} = (I-P)\vec{b}$
$= \vec{b} - P\vec{b}$
$= \vec{b} - \vec{p} \rightarrow$ the error vector when $\vec{b}$ projected onto $C(A)$

$\Rightarrow \vec{v} = \vec{e}$ is orthogonal to $C(A) \Rightarrow \vec{v} \in N(A^T)$ $(C(A) \perp N(A^T))$

if $\vec{b} \in N(A^T) \Rightarrow \underset{\substack{\downarrow \\ \text{projects onto } C(A)}}{P}\vec{b} = \vec{0}$



$\Rightarrow (I-P)\vec{b} = \vec{b} - P\vec{b}$
$= \vec{b} - \vec{0}$
$= \vec{b}$

and $(I-P)\vec{b} \in C(I-P)$
$\vec{b} \in C(I-P) \Rightarrow N(A^T) \subset C(I-P)$

30.

a. 2nd & 3rd Cs are lin. dep. $\Rightarrow$ C1 is the only ind. col.

$\Rightarrow C(A)$ is the line thru $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\vec{a}\vec{a}^T = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \end{bmatrix}$$

$$\Rightarrow P_c = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$\hookrightarrow \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 9 + 16 = 25$

b.

$$\begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix} \xrightarrow{\frac{1}{4}R_2 - \frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$C(A^T)$ is the line throu $\vec{r} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$P_R = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 9$$

$$\vec{a}\vec{a}^T = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

$B = P_C A P_R$

$$= \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

$$= \frac{1}{225}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\begin{bmatrix} 27 & 54 & 54 \\ 36 & 72 & 72 \end{bmatrix}$$

$$= \frac{1}{225}\begin{bmatrix} 675 & 1350 & 1350 \\ 900 & 1800 & 1800 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}}_{= A\,!}$$

surprising indeed

why?
because $AP_R = A$ & $P_C A = A \Rightarrow P_C A P_R = A$

34.

P_1P_2 is a projection matrix $\Leftrightarrow P_1P_2 = P_2P_1$

We know that $P_i^2 = P_i \Rightarrow (P_1P_2)^2 = P_1P_2$

So,

$(\Rightarrow)$ $(P_1P_2)^2 = (P_1P_2)(P_1P_2) = P_1(P_2P_1)P_2$ From assumption

$= P_1(P_1P_2)P_2$ $P_1P_2 = P_2P_1$

$= P_1^2 P_2^2 = P_1P_2$

$(\Leftarrow)$ Assume for contradiction that $P_1P_2 \neq P_2P_1$, where P_1P_2 is a projection matrix. Then,

$(P_1P_2)^2 = P_1P_2 = P_1^2P_2^2 = (P_1P_2)(P_1P_2)$

$= P_1(P_2P_1)P_2 = P_1(P_1P_2)P_2$ since P_1P_2 is a projection matrix

However, that contradicts the assumption $P_1P_2 \neq P_2P_1$

so, we have $P_1P_2 = P_2P_1$

Proved

Section 4.3

4.

$E = \|A\vec{x} - \vec{b}\|^2$

$= (C-0)^2 + (C+D-8)^2 + (C+3D-8)^2 + (C+4D-20)^2$

$\frac{\partial E}{\partial C} = 2C + 2(C+D-8) + 2(C+3D-8) + 2(C+4D-20) = 0$ ---(1)

$\frac{\partial E}{\partial D} = 2(C+D-8) + 6(C+3D-8) + 8(C+4D-20) = 0$ ---(2)

Divide (1) & (2) by 2

$$(1) \rightarrow C + (C+D-8) + (C+3D-8) + (C+4D-20) = 0$$

$$4C + 8D = 36$$

$$(2) \rightarrow (C+D-8) + 3(C+3D-8) + 4(C+4D-20) = 0$$

$$8C + 26D = 112$$

$\Rightarrow$ we have the system

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

7.

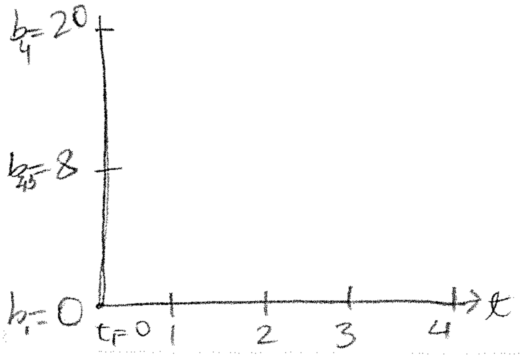
$$A\hat{x} = \vec{b}$$

$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \hat{x} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$A^T A \hat{x} = A^T \vec{b}$$

$$\begin{bmatrix} 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \hat{x} = \begin{bmatrix} 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$26\hat{x} = 112 \Rightarrow \hat{x} = \frac{112}{26} = \frac{56}{13}$$



$$\leftarrow b = \frac{56}{13} t$$

9.

$$\left.\begin{aligned} C &= 0 \\ C+D+E &= 8 \\ C+3D+9E &= 8 \\ C+4D+16E &= 20 \end{aligned}\right\} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \\ E \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}}_{\vec{b}}$$

$$A^TA\hat{\vec{x}} = A^T\vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \hat{\vec{x}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

what's happening in the figure is that we're computing the best fit to the span of 3 vectors.

measured in the least squared

26.

$$\left.\begin{aligned} C+D &= 0 \\ C+E &= 1 \\ C-D &= 3 \\ C-E &= 4 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$A^T A \hat{x} = A^T \vec{b}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix}$$

$\Rightarrow 4C = 8 \Rightarrow C = 2$ $\quad 2D = -3 \Rightarrow D = -\frac{3}{2}$ $\quad 2E = -3 \Rightarrow E = -\frac{3}{2}$

$\Rightarrow$ Equ. of a plane

$C + Dx + Ey = 2 - \frac{3}{2}x - \frac{3}{2}y$

at $(0,0)$:

$2 - 0 - 0 = 2$

average of b's:

$\frac{0+1+3+4}{4} = 2$

$\therefore C + Dx + Ey =$ average of the b's

Section 4.4   10, 11, 18, 36

10.

[a] When the $\vec{q}$'s are orthogonormal & $c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3 = \vec{0}$

Then,

$\vec{q}_1 \cdot (c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3) = \vec{q}_1 \cdot \vec{0}$

$\Rightarrow c_1 \vec{q}_1^T \vec{q}_1 = 0 \quad \Rightarrow c_1 = 0 \quad \Rightarrow$ dot product w/ $\vec{q}_1$ leals to $c_1 = 0$

and the same for the dot product w/ $\vec{q}_2$ & $\vec{q}_3$.

$\vec{q}_3 \rightarrow$ leads to $c_3 = 0$

$\vec{q}_2 \rightarrow$ leads to $c_2 = 0$

[b]

$Q\vec{x} = \vec{0} \Longrightarrow \vec{x} = \vec{0}$

$Q^TQ = I \quad \Rightarrow \quad Q^TQ\vec{x} = Q^T\vec{0}$

$I\vec{x} = \vec{0}$

$\vec{x} = \vec{0}$

11.