Path Integrals Day 6

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1 Direct and exchange paths

Identical particles in three spatial dimensions[1] obey either Bose-Einstein or Fermi-Dirac statistics. Today we will use path integrals to argue that these two cases are the only possibilities in three spatial dimensions. In the exercises, you will show that other possibilities exist in two spatial dimensions.

The amplitude A for two particles to go from some intitial positions $\mathbf{q}_{1i}$ and $\mathbf{q}_{2i}$ to some final positions $\mathbf{q}_{1f}$ and $\mathbf{q}_{2f}$ is

$$A(\mathbf{q}_{1f}, \mathbf{q}_{2f}, T, \mathbf{q}_{1i}, \mathbf{q}_{2i}, 0) = \int \mathcal{D}\mathbf{q}_1(t)\mathbf{q}_2(t) e^{iS[\mathbf{q}_1(t), \mathbf{q}_2(t)]/\hbar} \tag{1}$$

where T is the final time and 0 is the initial time. If the particles are distinguishable, the paths $\mathbf{q}_1$ and $\mathbf{q}_2$ satisfy the boundary conditions

$$\begin{aligned} \mathbf{q}_1(0) &= \mathbf{q}_{1i} \\ \mathbf{q}_2(0) &= \mathbf{q}_{2i} \\ \mathbf{q}_1(T) &= \mathbf{q}_{1f} \\ \mathbf{q}_2(T) &= \mathbf{q}_{2f}\,. \end{aligned} \tag{2}$$

If the particles are indistinguishable, there is another possibility. The particles can take a "direct" path and obey the boundary conditions (2), or they can "exchange" places and obey the boundary conditions

$$\begin{aligned} \mathbf{q}_1(0) &= \mathbf{q}_{1i} \\ \mathbf{q}_2(0) &= \mathbf{q}_{2i} \\ \mathbf{q}_1(T) &= \mathbf{q}_{2f} \\ \mathbf{q}_2(T) &= \mathbf{q}_{1f}\,. \end{aligned} \tag{3}$$

It is convenient to change variables from $\mathbf{q}_{1,2}$ to a center of mass position variable

$$\mathbf{Q} = \frac{\mathbf{q}_1 + \mathbf{q}_2}{2} \tag{4}$$

[1] We will work in real time for all of today's class.

and a relative position variable

$$\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2 \, . \tag{5}$$

The center of mass position is not relevant to particle statistics and we will not discuss it further.

If the initial and final relative positions of the two particles are the same $\mathbf{q}_i = \mathbf{q}_f$, and we also assume that the particles cannot be at the same location $\mathbf{q} \neq 0$, then direct paths are closed loops in three-space minus the origin $\mathbb{R}^3 - \{0\}$. The exchange paths are not closed - they go from $\mathbf{q}_i$ to $-\mathbf{q}_i$ in the space $\mathbb{R}^3 - \{0\}$. Two exchange paths taken in sequence give a direct path.

The space $\mathbb{R}^3 - \{0\}$ is not the space of all physical relative positions. The space $\mathbb{R}^3 - \{0\}$ is known as the covering space. The relative positions $\mathbf{q}$ and $-\mathbf{q}$ are physically equivalent, so the configuration space is the covering space with opposite points $\mathbf{q}$ and $-\mathbf{q}$ identified or $(\mathbb{R}^3 - \{0\})/\mathbb{Z}_2$.

Exchange paths are closed in the configuration space. They start and end at the same point (since $\mathbf{q}$ and $-\mathbf{q}$ are identified). In configuration space, we can distinguish a direct from an exchange paths by determining if the path is contractible. A path is contractible if it can be continuously deformed in to the trivial path (with no relative motion or $\mathbf{q}$ constant). Direct paths are contractible and exchange paths are not. In other words, we can classify paths as direct or exchange based on their topolopgy. In three spatial dimensions, all paths are either direct or exchange paths.

2 Amplitude

Assume for simplicity that action S only depends on the relative position of the two particles. Then the amplitude (1) for indistinguishable particles is the sum of two contributions

$$A_{\mathbf{q},\mathbf{q}} \equiv A(\mathbf{q}, T, \mathbf{q}, 0) = \bar{A}_{\mathbf{q},\mathbf{q}} + \bar{A}_{-\mathbf{q},\mathbf{q}} \tag{6}$$

where

$$\bar{A}_{\mathbf{q},\mathbf{q}} = \int_{\text{direct}} \mathcal{D}\mathbf{q}(t) e^{iS[\mathbf{q}(t)]/\hbar} \tag{7}$$

and

$$\bar{A}_{-\mathbf{q},\mathbf{q}} = \int_{\text{exchange}} \mathcal{D}\mathbf{q}(t) e^{iS[\mathbf{q}(t)]/\hbar} \, . \tag{8}$$

The amplitude $A_{\mathbf{q},\mathbf{q}}$ is a path integral in the configuration space. Both $\bar{A}_{\mathbf{q},\mathbf{q}}$ and $\bar{A}_{\mathbf{q},\mathbf{q}}$ are path integrals in the covering space. For distinhuishable particles the amplitude is given by either (7) or (8) depending on the boundary conditions. As a consequence, these amplitudes obey the Schrödinger equation. Due to the linearity of the Schrödinger equation, the configuration space amplitude $A_{\mathbf{q},\mathbf{q}}$ also satisfies the Schrödinger equation.

Adding together the contributions of the direct and exchange paths as in (6) is not the most general way to construct amplitudes in the configuration space. If we add a relative phase ϕ between the contributions of the direct and the exchange paths

$$A^{\phi}_{\mathbf{q},\mathbf{q}} = \bar{A}_{\mathbf{q},\mathbf{q}} + e^{i\phi}\bar{A}_{-\mathbf{q},\mathbf{q}} \tag{9}$$

the amplitude $A^{\phi}_{\mathbf{q},\mathbf{q}}$ also obeys the Schrödinger equation.

We will see that the defining the configuration space amplitude using (9) is inconsistent for some choices of the phase ϕ.

To see which phases are consistent, let's drop the restriction that the initial and final relative position are the same. Then the amplitude is given by

$$A^{\phi}_{\mathbf{q}_f,\mathbf{q}_i} = \bar{A}_{\mathbf{q}_f,\mathbf{q}_i} + e^{i\phi}\bar{A}_{-\mathbf{q}_f,\mathbf{q}_i}\,. \tag{10}$$

Since we identify opposite points in the configuration space, we can also write the amplitude as

$$A^{\phi}_{-\mathbf{q}_f,\mathbf{q}_i} = \bar{A}_{-\mathbf{q}_f,\mathbf{q}_i} + e^{i\phi}\bar{A}_{\mathbf{q}_f,\mathbf{q}_i}\,. \tag{11}$$

Since (10) and (11) describe the same physical process, they can differ by at most a phase α

$$A^{\phi}_{\mathbf{q}_f,\mathbf{q}_i} = e^{i\alpha}A^{\phi}_{-\mathbf{q}_f,\mathbf{q}_i}\,. \tag{12}$$

Combining (10), (11), and (12) gives

$$e^{i\phi} = e^{i\alpha} \tag{13}$$

and

$$e^{2i\phi} = 1\,. \tag{14}$$

There are two physically distinct solutions to this equation. If $e^{i\phi} = 1$, the direct and exchange paths add so the particles obey Bose-Einstein statistics. If $e^{i\phi} = -1$, exchanging the particles produces a minus sign so the particles obey Fermi-Dirac statistics.

3 Exercise: Anyons

In this exercise, we will repeat the above analysis except we will work in two spatial dimensions instead of three. We will find that particles in two dimensions do not have to obey Bose-Einstein or Fermi-Dirac statistics!

(a) By considering paths with $\mathbf{q}_i = \mathbf{q}_f$, argue that there are countably infinitely many topological classes of paths in two spatial dimensions (unlike in three spatial dimensions, where we found two topological classes of paths).[2]

Argue further that the paths can be classified into topological classes based on the change in the polar angle θ of the relative coordinate. In the nth class, the polar angle changes by $n\pi$.

[2] It is essential that we exclude the origin $\mathbf{q} \neq 0$!

(b) As before, let us relax the condition $\mathbf{q}_i = \mathbf{q}_f$. The path integral in configuration space can be written in polar coordinates (q, θ) as

$$A(q,\theta) = \sum_{n=-\infty}^{\infty} C_n \bar{A}_n(q,\theta) \tag{15}$$

where the C_n are phases. Here A represents the configuration space amplitude and $\bar{A}$ represents the covering space amplitude as before. Here (q, θ) represents the final relative position of the particles. We are suppressing the dependence on the final time, and initial time and relative position.

By comparing the amplitudes $A(q, \theta)$ and $A(q, \theta + \pi)$, show that the phases C_n obey the recursion relation

$$C_{n+1} = e^{i\phi} C_n \tag{16}$$

for some phase ϕ.

(c) What does your result in part (c) imply about particle statistics in two spatial dimensions?