Quantum Information Lecture 3, PSI Start

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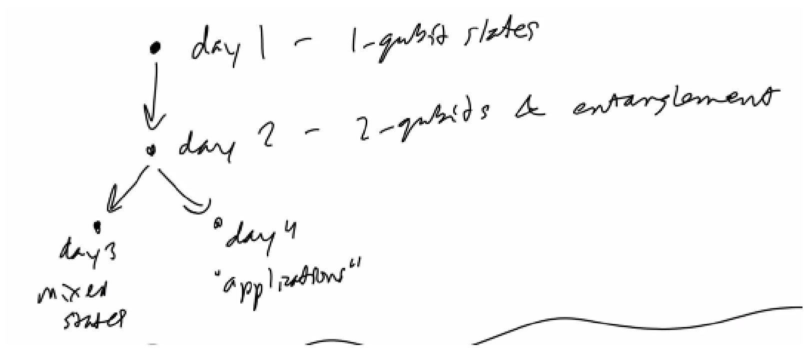
June 27, 2023

Contents

1 Outline

Today: Mixed Quantum States and Reduced Density Matrices

Overall Course outline:



2 Motivating mixed states

Note: I got a bit carried away and ended up writing an intro to mixed states that's a bit different than Aaron's. Aaron's takes the same physics and thought experiment discussed in these notes, but uses qubits to flesh out the details of how things are calculated. Hopefully the contrast is helpful.

Here's a fact we discussed previously: when a two qubit system, AB, is entangled, you can't describe the state of subsystem A alone using a pure state. Thus, one could be lead to

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the "conclusion" that when a state is entangled, you can't describe subsystem A without also describing subsystem B. But this is incorrect! The following thought experiment explains this.

When there's no entanglement, so that the state is of the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ it's straightforward to show that the results of subsystem A can be described using only the mathematics associated to subsystem A, in particular, using the pure state $|\psi_A\rangle$. We'll now argue that even if there is entanglement, you can also describe subsystem A by only modelling subsystem A instead of modelling the composite system AB. However, when there's entanglement, we can't do this using a pure state, we have to use a mixed state.

Suppose A and B share an entangled qubit. Furthermore, suppose A and B are spatially separated (e.g. the A stands for Aaron, who is not in the same town as B as in Barnett). Unless quantum theory allows faster-than-light information transfer, when performing a local measurement, no measurement A can perform can distinguish between two cases:

- case 1: B has performed a measurement but hasn't told A the results.
- case 2: B has not performed a measurement.

In case 1, if subsystem B performs a measurement of some (non-degenerate) observable with outcomes $\{j\}$, entanglement is eliminated after the measurement! More explicitly, consider a general initial state of the form

$$\Psi = \sum_{ij} \psi_{ij} |i\rangle \otimes |j\rangle . \tag{1}$$

The probability of B getting measurement outcome j is

$$\mathbf{Pr}(j) = \sum_i |\psi_{ij}|^2. \tag{2}$$

After measuring j, observer B models the state as

$$\begin{aligned} \Psi'_j &= \left(\frac{1}{\mathbf{Pr}(j)} \sum_i \psi_{ij} |i\rangle \right) \otimes |j\rangle && (3) \\ &=: |\psi'_j\rangle \otimes |j\rangle , && (4) \end{aligned}$$

which is a pure state (the prefactor of $1/\mathbf{Pr}(j)$ is a normalization factor). Since A doesn't immediately know the result of B's measurement, they'd model the system after measurement as being in an ensemble of the states Ψ'_j occurring with probability $\mathbf{Pr}(j)$. Therefore, in case 1, if A wants to compute the expectation value of some local observable, O_A, they'd write down

$$\langle O_A \rangle = \sum_j \mathbf{Pr}(j) \langle \Psi'_j | O_A \otimes \mathrm{Id} | \Psi'_j \rangle . \tag{5}$$

To summarize, the idea that A can't physically distinguish between a case with entanglement (case 2) and a case they'd describe as a probabilistic distribution over cases with no entanglement (case 1), can be verified by showing that the above derivation of expectation values equals what you'd get without performing a measurement in B, which is $\langle \Psi | O_A \otimes \mathrm{Id} | \Psi \rangle$. I'd recommend convincing yourself of this fact mathematically.

So what is a mixed state? It's effectively a nice rebranding of eq. (5). You can write it as

$$\langle O_A \rangle = \mathrm{Tr}\left[\left(\sum_j \mathbf{Pr}(j) \left|\Psi'_j\right\rangle \left\langle\Psi'_j\right|\right)(O_A \otimes \mathrm{Id})\right] \tag{6}$$

$$= \mathrm{Tr}\left[\sum_j \left(\mathbf{Pr}(j) \left|\psi'_j\right\rangle \left\langle\psi'_j\right|\right) O_A\right] \tag{7}$$

Defining the mixed state ρ_A, a.k.a. the reduced density operator,

$$\rho_A := \sum_j \left(\mathbf{Pr}(j) \left|\psi'_j\right\rangle \left\langle\psi'_j\right|\right), \tag{8}$$

which is an operator acting only on $\mathcal{H}_A$, we have

$$\langle O_A \rangle = \mathrm{Tr}\rho_A O_A. \tag{9}$$

It turns out that if you know ρ_A, you know A is entangled to some subsystem B, and you know that the state AB is pure (even if you don't know what it is), you can measure entanglement using only ρ_A. The Renyi entropy turns out to be

$$S^{(\alpha)}(\rho_A) = \frac{1}{1-\alpha}\log_2\left(\mathrm{Tr}\left(\rho_A^\alpha\right)\right) \tag{10}$$

and the von neumann entropy is

$$S^{(1)}(\rho_A) = -\mathrm{Tr}\left(\rho \log_2 \rho\right). \tag{11}$$

(Yes, you can define the logarithm of a matrix. The trick is to use the eigenvalue (spectral) decomposition).

Here's another nice thing about reduced density matrices: If two pure states are locally equivalent, say by $|\Phi\rangle = (U_A \otimes U_B)|\Psi\rangle$, then their reduced density matrices are unitarily similar,

$$\rho_A(\Phi) = U_A \rho_A(\Psi) U^\dagger. \tag{12}$$

Importantly, the eigenvalues of ρ_A, and consequently the entanglement measures mentioned above, do not change if we apply a local unitary, $U_A \otimes U_B$.