Lecture Notes

Classical Mechanics

PHYS 571

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Chapter 1

Survey of the Elementary Principles

Elementary classical mechanics deals with the motion of particles and rigid bodies which are **classical** in their nature: observable in size, with low speeds compared to the speed of light $c = 3 \times 10^8$ m/s, and hence low energies compared to their rest mass energy.

According to prevailing views, a classical system moves along a predictable trajectory with **position** and **velocity** known to an indefinite degree of accuracy. In cases where there is a large number of particles, it does not make sense to follow individual particles; rather, we are satisfied with understanding the average or statistical properties of a dynamical system. All these nice properties of classical systems are due to the linearity of their **dynamical equations of motion**, primarily **Newton's second law**.

However, more than a century ago, mathematicians discovered that some apparently simple mechanical systems can exhibit very complicated and random motion. Their behavior is very sensitive to initial conditions. Such **chaotic systems** owe their complicated behavior to the nonlinearity of their dynamical equations of motion. These complicated systems are introduced in Chapter 11 of Ref. [**?**], which unfortunately is not part of the syllabus of this course. In this chapter, we will review most of the important concepts needed to investigate the dynamics of particles.

1.1. Mechanics of a Single Particle

A **particle**, being a physical object with mass m and infinitesimally small in size, has at any instant of time t:

- A **position vector** $\vec{r}$ relative to a given inertial frame.
- A **velocity** $\vec{v}$ defined by

$$\vec{v} = \frac{d\vec{r}}{dt}, \tag{1.1}$$

Its **linear momentum** is defined by

$$\vec{p} = m\vec{v}.$$

Under the action of external forces, whose resultant is denoted by $\vec{F}$, the trajectory of the particle is described by **Newton's Second Law** in an inertial frame:

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}, \tag{1.2}$$

where

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}. \tag{1.3}$$

This represents a differential equation of second order in time. If we imagine that we have a fixed star in our space, then Newton's law is valid in any reference frame moving with constant velocity with respect to this star. This means that reference frames attached to the surface of the Earth are not strictly

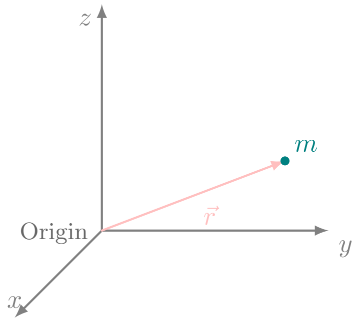


Figure 1.1: Illustration of a particle in an inertial frame with position vector $\vec{r}$ and mass m.

inertial. However, if we are interested in motions occurring over short periods of time, say $\Delta t = 10\,\mathrm{s}$, we can approximate this frame as inertial.

Oscillations of a pendulum over one year on the surface of the Earth, however, cannot be considered as moving in an inertial frame, since $\Delta t = 1\,\text{year} \gg T = 24\,\mathrm{h}$, where T is the period of rotation of the Earth about its axis.

1.2. Conservation Laws

Conservation laws play an important role in classical mechanics; any physical quantity whose derivative vanishes will lead to a conserved quantity. For example, if

$$\frac{d\mathcal{H}}{dt} = 0 \Rightarrow \mathcal{H} = \text{constant or is conserved in time.} \tag{1.4}$$

However, not all conservation laws are of equal importance.

• Conservation of Linear Momentum

Since $\vec{F} = \frac{d\vec{p}}{dt}$, if $\vec{F} = 0$ then

$$\frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = \text{constant.} \tag{1.5}$$

The total momentum $\vec{p}$ of a system is conserved if the net force acting on it vanishes.

• Conservation of Angular Momentum

The **angular momentum** $\vec{L}$ is defined as

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}. \tag{1.6}$$

The rate of change of **angular momentum** $\vec{L}$ for a particle in an external force field is given by:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \vec{v} \times \vec{p} = \vec{r} \times \vec{F} = \vec{\tau}_{\text{ext}}. \tag{1.7}$$

If $\vec{\tau}_{\text{ext}} = 0$, then

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant and conserved.} \tag{1.8}$$

Thus, if the net external torque vanishes, the net angular momentum of the system is conserved.

• Conservation of Energy

The work done by an external force on an object moving from point 1 to point 2 is defined by

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}, \tag{1.9}$$

where $d\vec{s}$ represents an infinitesimal displacement along the trajectory of the object. If the object's velocity is $\vec{v}$, then

$$d\vec{s} = \vec{v}(t)\,dt \quad \text{and} \quad \vec{F} = m\frac{d\vec{v}}{dt}.$$

So that

$$\begin{aligned} W_{12} &= m\int_1^2 \frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = \frac{m}{2}\int_1^2 \frac{d}{dt}\left(\vec{v}\cdot\vec{v}\right)dt \\ &= \frac{m}{2}\left(v_2^2 - v_1^2\right) = \Delta K = T_2 - T_1, \end{aligned} \tag{1.10}$$

where $T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v}$ is the kinetic energy.

If the work done by a force is independent of the path taken between 1 and 2, then the force is said to be **conservative**. Hence, for conservative forces, the work done on any closed path is zero:

$$\oint \vec{F}\cdot d\vec{s} = 0. \tag{1.11}$$

This implies that during a trip along the closed path, the quantity $\vec{F}\cdot d\vec{s}$ must keep changing its sign so as to sum up to zero. However, forces such as friction, which keep

$$\oint \vec{f_k}\cdot d\vec{s} < 0, \tag{1.12}$$

have the negative sign at all times and hence will never satisfy this requirement 1.11. These are **non-conservative or dissipative forces**.

Using **Stokes' Theorem**

$$\oint \vec{F}\cdot d\vec{s} = 0 = \int_{surface}\left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{a}, \tag{1.13}$$

for any closed surface, we have

$$\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{F} = -\vec{\nabla}V,$$

where V is a scalar function. The minus sign is a matter of convention. Thus,

$$\vec{F} = -\vec{\nabla}V(\vec{r}), \tag{1.14}$$

for any **conservative force**.

See here an example of using conservation laws in orbital dymanics

1.3. System of Particles

Consider a system of N particles labeled by $i = 1, 2, \ldots, N$, each with a constant mass m_i. Newton's second law for the i-th particle is given by:

$$m_i\ddot{\vec{r}}_i = \vec{F}_i, \tag{1.15}$$

where $\vec{F}_i$ is the total force acting on the i-th particle.

The force $\vec{F}_i$ can be decomposed into two components:

$$\vec{F}_i = \vec{F}_i^{\text{ext}} + \sum_{j\neq i}\vec{F}_{ij}, \tag{1.16}$$

where:

- $\vec{F}_i^{\text{ext}}$: external force acting on the i-th particle,
- $\vec{F}_{ij}$: internal force due to the interaction between the i-th and j-th particles.

If the internal forces satisfy the weak form of Newton's third law:

$$\vec{F}_{ij} = -\vec{F}_{ji},$$

then the total contribution of internal forces vanishes:

$$\sum_i \sum_{j \neq i} \vec{F}_{ij} = 0.$$

Summing Newton's second law over all particles, we find:

$$\sum_i m_i \ddot{\vec{r}}_i = \sum_i \vec{F}_i = \sum_i \vec{F}_i^{\text{ext}} = \vec{F}^{\text{ext}}, \tag{1.17}$$

where $\vec{F}^{\text{ext}}$ is the total external force acting on the system.

Center of Mass Motion

The total mass of the system is:

$$M = \sum_i m_i.$$

The position of the center of mass (C.M.) is defined as:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M}. \tag{1.18}$$

Differentiating twice with respect to time, we find:

$$M\ddot{\vec{R}} = \sum_i m_i \ddot{\vec{r}}_i = \vec{F}^{\text{ext}}.$$

Thus, the motion of the center of mass is governed by the equation:

$$M\ddot{\vec{R}} = \vec{F}^{\text{ext}}. \tag{1.19}$$

This result implies that the center of mass moves as if all the mass of the system were concentrated at $\vec{R}$ and acted upon by the net external force $\vec{F}^{\text{ext}}$.

If $\vec{F}^{\text{ext}} = 0$, the center of mass moves with constant velocity:

$$\ddot{\vec{R}} = 0 \quad \Rightarrow \quad \vec{V} = \text{constant},$$

where $\vec{V} = \frac{d\vec{R}}{dt}$ is the velocity of the center of mass.

Total Linear Momentum

The total linear momentum of the system is defined as:

$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i. \tag{1.20}$$

Using the definition of the center of mass velocity $\vec{V} = \frac{d\vec{R}}{dt}$, the total momentum can also be written as:

$$\vec{P} = M\vec{V}.$$

Differentiating with respect to time:

$$\dot{\vec{P}} = M\ddot{\vec{R}}.$$

Substituting from (1.19), we find:

$$\dot{\vec{P}} = \vec{F}^{\text{ext}}. \tag{1.21}$$

Thus, the rate of change of the total linear momentum is equal to the net external force acting on the system. If $\vec{F}^{\text{ext}} = 0$, then:

$$\dot{\vec{P}} = 0 \quad \Rightarrow \quad \vec{P} = \text{constant}.$$

This implies that the total linear momentum of the system is conserved when no external forces act.

Moments of Force and Momentum

We now consider the angular momentum equivalents. Starting with Newton's second law:

$$\dot{\vec{p}}_i = \vec{F}_i,$$

we take the cross product with $\vec{r}_i$ and sum over all particles:

$$\sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \vec{F}_i.$$

The left-hand side (LHS) of this equation expands as:

$$\sum_i \vec{r}_i \times \dot{\vec{p}}_i = \frac{d}{dt} \sum_i (\vec{r}_i \times \vec{p}_i) - \sum_i \dot{\vec{r}}_i \times \vec{p}_i.$$

Since $\vec{p}_i = m_i \dot{\vec{r}}_i$, the second term vanishes, leaving:

$$\sum_i \vec{r}_i \times \dot{\vec{p}}_i = \frac{d}{dt} \sum_i (\vec{r}_i \times \vec{p}_i).$$

Defining the total angular momentum of the system as:

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i, \tag{1.22}$$

the LHS becomes:

$$\frac{d\vec{L}}{dt}. \tag{1.23}$$

For the right-hand side (RHS), substituting $\vec{F}_i = \vec{F}_i^{\text{ext}} + \sum_{j \neq i} \vec{F}_{ij}$:

$$\sum_i \vec{r}_i \times \vec{F}_i = \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} + \sum_i \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij}.$$

The second term simplifies using symmetry. Since $\vec{F}_{ij} = -\vec{F}_{ji}$, we can write:

$$\sum_i \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2} \sum_{i,j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}.$$

If the forces satisfy the strongest form of Newton's third law, such that $\vec{F}_{ij}$ lies along $\vec{r}_i - \vec{r}_j$, this term vanishes:

$$\sum_i \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij} = 0.$$

Thus, we have:

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}}.$$

Defining the total external torque:

$$\vec{\tau}^{\text{ext}} = \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}}, \tag{1.24}$$

the equation becomes:

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{\text{ext}}. \tag{1.25}$$

If $\vec{\tau}^{\text{ext}} = 0$, the angular momentum is conserved:

$$\vec{L} = \text{constant}.$$

Energy of the System

The work-energy theorem states:

$$\vec{F} \cdot d\vec{r} = d\left(\frac{1}{2}mv^2\right) = dT, \tag{1.26}$$

where T is the kinetic energy. For a system of N particles, summing over all particles gives:

$$dT = \sum_i \vec{F}_i \cdot d\vec{r}_i.$$

Substituting $\vec{F}_i = \vec{F}_i^{\text{ext}} + \sum_{j \neq i} \vec{F}_{ij}$:

$$dT = \sum_i \vec{F}_i^{\text{ext}} \cdot d\vec{r}_i + \sum_i \sum_{j \neq i} \vec{F}_{ij} \cdot d\vec{r}_i.$$

If the external forces are conservative, there exists a potential function $V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ such that:

$$\vec{F}_i^{\text{ext}} = -\nabla_i V. \tag{1.27}$$

Similarly, if the internal forces are conservative, there exists a potential energy function $U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ such that:

$$\vec{F}_{ij} = -\nabla_i U. \tag{1.28}$$

The total change in energy can then be written as:

$$dT = -dV - dU.$$

Thus, the total energy is conserved:

$$T + V + U = \text{constant}. \tag{1.29}$$

Energy and Conservation Laws

The total energy of a system includes contributions from kinetic and potential energy. For N particles, the total kinetic energy is:

$$T = \frac{1}{2}\sum_i m_i v_i^2.$$

If both external and internal forces are conservative, they can be expressed as gradients of scalar potential energy functions. The external forces are derived from a potential V:

$$\vec{F}_i^{\text{ext}} = -\nabla_i V,$$

and the internal forces are derived from a potential U:

$$\vec{F}_{ij} = -\nabla_i U.$$

The total work done by external and internal forces over a displacement is:

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{r}_i.$$

Substituting $\vec{F}_i = \vec{F}_i^{\text{ext}} + \sum_{j \neq i} \vec{F}_{ij}$, the work becomes:

$$W_{12} = \sum_i \int_1^2 \vec{F}_i^{\text{ext}} \cdot d\vec{r}_i + \sum_i \int_1^2 \sum_{j \neq i} \vec{F}_{ij} \cdot d\vec{r}_i. \tag{1.30}$$

If the external forces are conservative, the work done is:

$$\sum_i \int_1^2 \vec{F}_i^{\text{ext}} \cdot d\vec{r}_i = -\Delta V, \tag{1.31}$$

where V is the total potential energy due to external forces.
Similarly, the work done by the internal forces is:

$$\sum_i \int_1^2 \sum_{j\neq i} \vec{F}_{ij} \cdot d\vec{r}_i = -\frac{1}{2} \sum_{ij} \Delta U_{ij}.$$

Combining these results, the total work becomes:

$$W_{12} = -\Delta V - \Delta U.$$

Using the work-energy theorem:

$$W_{12} = T_2 - T_1,$$

we find the conservation of energy:

$$T + V + U = \text{constant}. \tag{1.32}$$

This expresses the conservation of mechanical energy, where:

- T: total kinetic energy,
- V: potential energy due to external forces,
- U: potential energy due to internal forces.

Energy of Rigid Bodies

For a rigid body, the relative positions of particles are fixed, so the internal energy U remains constant. The total energy simplifies to:

$$E = T + V.$$

In rotational motion, the kinetic energy is expressed in terms of the moment of inertia I and angular velocity ω:

$$T = \frac{1}{2} I\omega^2.$$

Galilean Transformations Between Frames

A Galilean transformation describes the relationship between an inertial frame S and another frame S' moving with a constant velocity $\vec{V}$ relative to S. If $\vec{r}$ is the position vector in S, the corresponding position vector in S' is:

$$\vec{r}' = \vec{r} - \vec{V}t + \vec{b},$$

where $\vec{b}$ is the displacement of the origins of S and S' at $t = 0$.
The velocity and momentum in the two frames are related by:

$$\vec{v} = \vec{v}' + \vec{V}, \quad \vec{p} = \vec{p}' + m\vec{V}.$$

Thus, Newton's equations remain invariant under Galilean transformations:

$$\dot{\vec{p}} = \vec{F} \quad \Rightarrow \quad \dot{\vec{p}}' = \vec{F}.$$

For a system of particles, the total momentum and its rate of change transform as:

$$\vec{P} = \vec{P}' + M\vec{V}, \quad \dot{\vec{P}} = \dot{\vec{P}}'.$$

The Center of Momentum Frame

The center of momentum frame is defined as the inertial frame in which the total momentum of the system vanishes instantaneously:

$$\vec{P}' = \sum_a \vec{p}'_a = 0.$$

In this frame, the origin is usually chosen at the instantaneous center of mass. The relationship between the center of mass $\vec{R}$ in the fixed frame and the center of mass $\vec{R}'$ in the moving frame is:

$$M\vec{R} = M\vec{R}' + M\vec{V}t + M\vec{b}.$$

Differentiating twice with respect to time:

$$M\ddot{\vec{R}} = M\ddot{\vec{R}}' = \vec{F}^{\text{ext}}.$$

Thus, in the center of momentum frame, Newton's second law applies as usual, with the center of mass motion described by:

$$\vec{R}' = \vec{0}.$$

Intrinsic Angular Momentum

The total angular momentum $\vec{L}$ and the external torque $\vec{\tau}^{\text{ext}}$ are generally origin-dependent. To explore this, consider a shift in the origin of the coordinate system from O to O', such that:

$$\vec{r} = \vec{r}' + \vec{b}, \quad \dot{\vec{r}} = \dot{\vec{r}}', \quad \vec{p} = \vec{p}'.$$

The angular momentum of a single particle about the origin O is:

$$\vec{L}_a = \vec{r}_a \times \vec{p}_a = \vec{r}'_a \times \vec{p}_a + \vec{b} \times \vec{p}_a = \vec{L}'_a + \vec{b} \times \vec{p}_a.$$

Summing over all particles gives:

$$\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a \vec{r}'_a \times \vec{p}_a + \vec{b} \times \sum_a \vec{p}_a.$$

Using the definitions:

$$\vec{L}' = \sum_a \vec{r}'_a \times \vec{p}_a, \quad \vec{P} = \sum_a \vec{p}_a,$$

we can write:

$$\vec{L} = \vec{L}' + \vec{b} \times \vec{P}. \tag{1.33}$$

If O' is chosen as the center of momentum frame, then $\vec{P} = 0$, and $\vec{L} = \vec{L}'$. This decomposition shows that $\vec{L}$ consists of two components:

- $\vec{L}'$: the intrinsic angular momentum,
- $\vec{b} \times \vec{P}$: the angular momentum due to the motion of the center of mass.

The torque is similarly affected:

$$\vec{\tau}^{\text{ext}} = \sum_a \vec{r}_a \times \vec{F}_a = \sum_a \vec{r}'_a \times \vec{F}_a + \vec{b} \times \sum_a \vec{F}_a.$$

If $\vec{b} = \vec{R}$, where $\vec{R}$ is the center of mass, then:

$$\vec{\tau}^{\text{ext}} = \vec{\tau}'^{\text{ext}} + \vec{R} \times \vec{F}^{\text{ext}}.$$

Transformation Law for Kinetic Energy

The kinetic energy T of a system in a reference frame S is given by:

$$T = \sum_a \frac{1}{2} m_a v_a^2.$$

Expanding $\vec{v}_a = \vec{v}'_a + \vec{V}$, where $\vec{v}'_a$ is the velocity in a frame S' moving with velocity $\vec{V}$ relative to S:

$$T = \sum_a \frac{1}{2} m_a (\vec{v}'_a + \vec{V}) \cdot (\vec{v}'_a + \vec{V}).$$

Expanding the terms:

$$T = \sum_a \frac{1}{2} m_a {v'_a}^2 + \sum_a m_a \vec{v}'_a \cdot \vec{V} + \sum_a \frac{1}{2} m_a V^2.$$

The cross term vanishes if S' is the center of momentum frame ($\sum_a m_a \vec{v}'_a = 0$):

$$T = T' + \frac{1}{2} M V^2,$$

where:

- $T' = \sum_a \frac{1}{2} m_a {v'_a}^2$: the kinetic energy relative to the center of mass,
- $\frac{1}{2} M V^2$: the kinetic energy of the center of mass.

For a rigid body with moment of inertia I rotating about an axis with angular velocity ω, the kinetic energy simplifies to:

$$T = \frac{1}{2} I \omega^2 + \frac{1}{2} M V^2.$$

Rotating Frames and Non-Inertial Dynamics

Non-inertial frames are reference frames that accelerate relative to an inertial frame. A key example is a rotating frame R, which rotates with a constant angular velocity $\vec{\omega}$ relative to an inertial frame S. For a particle with position vector $\vec{r}$ in S, its velocity in R is related by:

$$[\vec{v}]_S = [\vec{v}]_R + \vec{\omega} \times \vec{r},$$

where $[\vec{v}]_S$ and $[\vec{v}]_R$ are the velocities of the particle as seen from S and R, respectively.
The acceleration in S is given by:

$$[\vec{a}]_S = [\vec{a}]_R + 2\vec{\omega} \times [\vec{v}]_R + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}).$$

Equation of Motion in Rotating Frames

From Newton's second law:

$$\vec{F} = m[\vec{a}]_S,$$

the equation of motion in R becomes:

$$m[\vec{a}]_R = \vec{F} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times [\vec{v}]_R - m\dot{\vec{\omega}} \times \vec{r}.$$

Rearranging, we identify the terms:

- $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$: centrifugal force,
- $-2m\vec{\omega} \times [\vec{v}]_R$: Coriolis force,
- $-m\dot{\vec{\omega}} \times \vec{r}$: Euler force.

The equation of motion in R becomes:

$$m[\vec{a}]_R = \vec{F} + \vec{F}_{\text{centrifugal}} + \vec{F}_{\text{Coriolis}} + \vec{F}_{\text{Euler}}, \tag{1.34}$$

where the fictitious forces are defined as:

$$\vec{F}_{\text{centrifugal}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}), \quad \vec{F}_{\text{Coriolis}} = -2m\vec{\omega} \times [\vec{v}]_R, \quad \vec{F}_{\text{Euler}} = -m\dot{\vec{\omega}} \times \vec{r}.$$

Energy in Rotating Frames

In a rotating frame, the kinetic energy of a particle is:

$$T = \frac{1}{2}m[\vec{v}]_R^2 + \frac{1}{2}m(\vec{\omega} \times \vec{r})^2.$$

The potential energy in a rotating frame includes the effects of the centrifugal force:

$$V_{\text{eff}} = V - \frac{1}{2}m\omega^2 r^2.$$

The total energy in a rotating frame is:

$$E_{\text{rot}} = T + V_{\text{eff}}, \tag{1.35}$$

where V_{eff} represents the effective potential energy including rotational effects.

Summary

- **Newton's Second Law**: In an inertial frame, the motion of each particle in a system is governed by:
$$m_a \ddot{\vec{r}}_a = \vec{F}_a, \quad a = 1, 2, \ldots, N.$$
This leads to a system of $3N$ second-order differential equations, which can, in principle, be solved given the initial positions, velocities, and forces acting on all particles. The forces depend on the positions, velocities, and potentially time. For instance, for two particles connected by a spring, the force depends on the relative positions and any external driving forces.

- **Conservation Laws**:
  - **Linear Momentum**: If the weak form of Newton's third law holds ($\vec{F}_{ab} = -\vec{F}_{ba}$), the total linear momentum of the system is conserved in the absence of external forces:
$$\dot{\vec{P}} = \vec{F}^{\text{ext}}.$$
  - **Angular Momentum**: If the strong form of Newton's third law holds ($\vec{F}_{ab} = -\vec{F}_{ba} \parallel \vec{r}_{ab}$), the total angular momentum is conserved in the absence of external torques:
$$\dot{\vec{L}} = \vec{\tau}^{\text{ext}}.$$
  - **Energy**: If all forces (external and internal) are conservative, the total energy of the system is conserved:
$$E = T + V + U.$$

- **Non-Inertial Frames**: Newton's second law can be extended to non-inertial frames by introducing fictitious forces, such as centrifugal and Coriolis forces, which depend on the motion of the frame.

Limitations of Newtonian Mechanics

Despite its success, the Newtonian formulation has several limitations, which are addressed by more advanced approaches:

- **Vectorial Formulation and Coordinates**: Newtonian mechanics relies heavily on vector analysis, which becomes cumbersome in non-Cartesian coordinate systems. The Lagrangian formulation avoids these issues by using scalar quantities and generalized coordinates.

- **Constraints**: Systems often involve constraints (e.g., rigid bodies, inextensible strings) that impose restrictions on particle motion. While Newtonian mechanics requires explicit calculation of the forces enforcing these constraints, Lagrangian methods incorporate them naturally at the outset through generalized coordinates and Lagrange multipliers.

- **Conservation Laws**: In the Newtonian framework, conservation of energy, linear momentum, and angular momentum are derived consequences of Newton's laws, rather than fundamental principles. This derivation often depends on specific assumptions, such as the weak or strong forms of Newton's third law. The Lagrangian approach, by contrast, reveals the deeper connection between conservation laws and symmetries (via Noether's theorem).

- **Relation to Modern Physics**: Newtonian mechanics is a limiting case of more general theories:
  - **Special Relativity**: At high velocities ($v \sim c$), Newtonian mechanics is replaced by relativistic mechanics, which accounts for the invariance of the speed of light and the equivalence of mass and energy.
  - **Quantum Mechanics**: At microscopic scales, Newtonian mechanics is replaced by quantum mechanics. The transition is governed by the correspondence principle, connecting classical and quantum descriptions.

1.4. Degrees of Freedom and Constraints

1.4.1. Degrees of Freedom

The **degrees of freedom (DoF)** of a system are the minimum number of independent quantities required to completely specify the state of the system. For a system of N particles, each particle typically has three spatial coordinates, resulting in $3N$ degrees of freedom. However, constraints reduce the number of independent coordinates.

Example 1.1 – Pendulum

A simple pendulum consists of a mass m suspended from a fixed point by a rigid, massless rod of length L. The pendulum's motion is constrained to a plane, reducing its degrees of freedom to 1. The position of the pendulum is fully described by the angle θ made with the vertical.
The Cartesian coordinates of the mass can be expressed in terms of θ:

$$x = L\sin\theta, \quad y = -L\cos\theta.$$

The generalized coordinate is θ, and the corresponding generalized velocity is:

$$\dot{\theta} = \frac{d\theta}{dt}.$$

Thus, the pendulum's motion is entirely described by a single degree of freedom, simplifying the analysis of its dynamics.

Example 1.2 – Rigid Body

A rigid body, which consists of multiple particles with fixed relative positions, has six degrees of freedom in three-dimensional space. These include:

1. **Three translational degrees of freedom**: The position of the center of mass, described by (x, y, z).
2. **Three rotational degrees of freedom**: The orientation of the rigid body, typically described using Euler angles (ϕ, θ, ψ).

The translational motion is represented by the center of mass coordinates:

$$\vec{r}_{\text{cm}} = (x, y, z).$$

The rotational motion can be described using the Euler angles:

$$\phi \text{ (rotation about the } z\text{-axis)}, \quad \theta \text{ (inclination angle)}, \quad \psi \text{ (rotation about the new } z'\text{-axis)}.$$

Together, these six degrees of freedom fully describe the position and orientation of the rigid body in space.

If constraints are applied, they reduce the degrees of freedom by restricting the possible motions of the system.

1.4.2. Constraints

Constraints are conditions or restrictions imposed on a system that reduce its degrees of freedom. Our primary focus is on **holonomic constraints**, which provide explicit relationships between the coordinates of the system and simplify its mathematical description by reducing the number of independent variables. For comparison, we briefly note the existence of **non-holonomic constraints**, which involve inequalities or non-integrable relationships but are not the subject of this discussion.

Consider a particle with position vector $\vec{r}$, constrained to move on a fixed plane. The motion of the particle is restricted by a constraint force $\vec{f^C}$, which acts perpendicular to the plane. This ensures the particle remains confined to the plane. Constraint forces, by their nature, are designed to enforce these restrictions. Importantly, for displacements that are consistent with the constraints, such forces do no work because their direction is always orthogonal to the allowed displacements.
To formalize this idea, we define an *instantaneous displacement* as an infinitesimal change in the position of the system at a given instant, fully described by the system's degrees of freedom. The following principle then holds:

Constraint forces do no work in any instantaneous displacement consistent with the constraints.

This principle, however, does not mean that constraint forces can never perform work during the actual motion of a system. For instance, if the plane itself is in motion, the particle's velocity may include a component in the direction of the constraint force, resulting in work being done. We will return to such cases in more detail later.

The notion of instantaneous displacement is closely related to what is called a **virtual displacement**, a key concept in analytical mechanics that will be introduced and used in deriving D'Alembert's principle.

Holonomic Constraints

A constraint is called **holonomic** if it can be expressed as an algebraic equation involving the coordinates of the system and possibly time:

$$f(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t) = 0. \tag{1.36}$$

Holonomic constraints explicitly reduce the number of independent coordinates and simplify the description of the system.

Example 1.3 – Rolling Without Slipping

A cylinder of radius a rolls without slipping on a flat surface, see Fig. (1.2). The holonomic constraint is:

$$x' = a\theta,$$

where x' is the translational displacement of the cylinder's center, and θ is the angle of rotation. This constraint relates the translational and rotational motion of the cylinder, reducing the system's degrees of freedom. The velocity of the cylinder's center of mass is:

$$v = \dot{x}' = a\dot{\theta}.$$

This relationship ensures that the point of contact between the cylinder and the surface has zero velocity relative to the surface, satisfying the rolling without slipping condition. The generalized coordinates for this system are:

$$x' \quad \text{and} \quad \theta.$$

The generalized velocities are:

$$\dot{x}' = \frac{dx'}{dt}, \quad \dot{\theta} = \frac{d\theta}{dt}.$$

The constraint simplifies the dynamics by coupling the translational and rotational motions of the cylinder.

If k holonomic constraints are applied to a system of N particles, the number of degrees of freedom is reduced from $3N$ to $3N - k$.

Non-Holonomic Constraints

A constraint is called **non-holonomic** if it cannot be expressed as a simple algebraic equation like (1.36). Non-holonomic constraints often involve inequalities or differential equations. These constraints are not

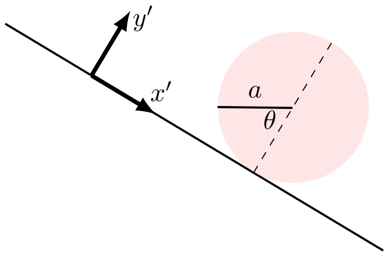


Figure 1.2: A cylinder of radius a rolls without slipping on an inclined plane. The angle θ denotes the inclination of the plane.

our primary focus but are discussed here for clarity.

Example 1.4 – Inequality Constraints

Consider a particle confined to move within a rectangular box of width a and height b as in Fig. (1.3). The particle's motion is restricted by the following inequality constraints:

$$0 \leq x \leq a, \quad 0 \leq y \leq b,$$

where:

- x is the horizontal position of the particle,
- y is the vertical position of the particle,
- a and b define the dimensions of the rectangular box.

These constraints ensure that the particle's position remains inside the box. Since the constraints do not explicitly depend on the velocities or accelerations, they are classified as inequality constraints. However, they impose a restriction on the permissible region of motion, meaning the particle cannot exit the boundaries defined by the box.

Non-holonomic constraints generally do not reduce the degrees of freedom in the same explicit way as holonomic constraints, making them more complex to handle mathematically.

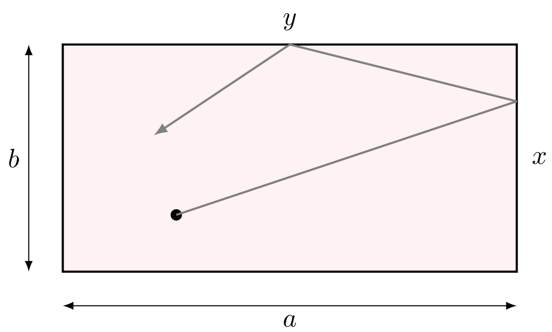


Figure 1.3: A particle confined within a rectangular box of width a and height b.

1.5. Generalized Quantities

1.5.1. Generalized Coordinates

The minimum number of independent coordinates required to describe a system's state are called **generalized coordinates**. These coordinates inherently account for all constraints in the system, reducing the complexity of the mathematical description.

For a system with $3N$ initial degrees of freedom and k holonomic constraints, the number of generalized coordinates is $q_1, q_2, \ldots, q_{3N-k}$. The positions of all particles can then be expressed as functions of these coordinates:

$$\begin{aligned} \vec{r}_1 &= \vec{r}_1(q_1, q_2, \ldots, q_{3N-k}, t), \\ &\vdots \\ \vec{r}_i &= \vec{r}_i(q_1, q_2, \ldots, q_{3N-k}, t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{1.37}$$

Generalized coordinates can represent:

- Distances, such as the length of a pendulum.
- Angles, such as the orientation of a rigid body.
- Combinations of distances and angles, as in cylindrical or spherical coordinates.

By using generalized coordinates, we incorporate the constraints into the mathematical formulation and simplify the description of the system.

1.5.2. Generalized Velocities

The **generalized velocities** are the time derivatives of the generalized coordinates. For each generalized coordinate q_i, the corresponding generalized velocity is defined as:

$$\dot{q}_i = \frac{dq_i}{dt}, \quad i = 1, \ldots, 3N - k.$$

These velocities describe the rate of change of the generalized coordinates with respect to time. They may represent physical velocities, angular rates, or other quantities depending on the nature of the coordinates. For example:

- In Cartesian coordinates, generalized velocities are the standard velocity components v_x, v_y, v_z.
- For angular coordinates, such as the angle θ of a pendulum, the generalized velocity is the angular rate $\dot{\theta}$.

Generalized velocities are fundamental in describing the dynamics of the system, as they are essential for defining kinetic energy and formulating equations of motion.

1.5.3. Generalized Forces

The **generalized forces** are quantities associated with the generalized coordinates that account for the effects of all forces in the system. They are defined through the **virtual work** principle:

$$\delta W = \sum_i Q_i \delta q_i,$$

where δW is the virtual work done by the forces during a **virtual displacement**, δq_i are the virtual displacements of the generalized coordinates, and Q_i are the generalized forces.
For each generalized coordinate q_i, the corresponding generalized force Q_i is given by:

$$Q_i = \sum_j F_j \frac{\partial x_j}{\partial q_i},$$

where F_j are the physical forces acting on the system, and x_j are the original Cartesian coordinates.
Generalized forces retain the dimensional consistency of the forces they represent and can include contributions from various physical effects, such as:

- Gravitational force, expressed in terms of the coordinates.
- Constraint forces, which do no work but influence the motion indirectly.

By expressing forces in terms of generalized coordinates and velocities, the equations of motion can be derived more naturally and compactly.

Example 1.5 – Particle on a Circular Track

Consider a particle of mass m moving under the influence of gravity on a smooth circular track of radius a. The motion of the particle is subject to a holonomic constraint:

$$x^2 + z^2 = a^2.$$

This constraint reduces the system to a single degree of freedom. A natural choice for the **generalized coordinate** is the angle θ, which specifies the position of the particle along the circular path. The Cartesian coordinates can then be expressed as functions of θ:

$$x = a\cos\theta, \quad z = a\sin\theta.$$

Generalized coordinates inherently incorporate the system's constraints and reduce the complexity of its description. In this case, the single generalized coordinate θ is sufficient to fully describe the particle's position.

The **generalized velocity** is the time derivative of the generalized coordinate. For this system:

$$\dot{\theta} = \frac{d\theta}{dt}.$$

The velocity components in Cartesian coordinates are obtained by differentiating x and z with respect to time:

$$\dot{x} = \frac{dx}{dt} = -a\sin\theta\,\dot{\theta}, \quad \dot{z} = \frac{dz}{dt} = a\cos\theta\,\dot{\theta}.$$

The magnitude of the velocity is:

$$v = \sqrt{\dot{x}^2 + \dot{z}^2} = a\dot{\theta}.$$

This illustrates how the generalized velocity $\dot{\theta}$ relates to the Cartesian velocity components and provides a compact description of the particle's motion.

The forces acting on the particle include:

- A **constraint force** exerted by the track, which ensures the particle remains on the circular path. This force does no work since it acts perpendicular to the displacement.
- The **gravitational force**, which has components $F_x = 0$ and $F_z = -mg$.

To compute the **generalized force**, we use the virtual work principle. For a virtual displacement $\delta\theta$, the virtual work done by the non-constraining force is:

$$\delta W = \sum_j F_j \delta x_j = \left[F_x \frac{\partial x}{\partial \theta} + F_z \frac{\partial z}{\partial \theta}\right] \delta\theta.$$

Substituting $F_x = 0$, $F_z = -mg$, and the derivatives of x and z with respect to θ:

$$\frac{\partial x}{\partial \theta} = -a\sin\theta, \quad \frac{\partial z}{\partial \theta} = a\cos\theta,$$

we find:

$$\delta W = [0 \cdot (-a\sin\theta) + (-mg) \cdot a\cos\theta]\,\delta\theta.$$

Simplifying:

$$\delta W = -mga\sin\theta\,\delta\theta.$$

Thus, the **generalized force** corresponding to the coordinate θ is:

$$Q_\theta = -mga\sin\theta.$$

1.5.4. Functions of $\{q\}, \{\dot{q}\}, t$

Physical quantities, such as the kinetic energy, can be expressed in terms of the generalized coordinates $\{q\}$, generalized velocities $\{\dot{q}\}$, and time t. For example, the kinetic energy of a particle is:

$$T = \frac{1}{2}\sum_i m_i \dot{x}_i^2.$$

Using the relationship between Cartesian and generalized coordinates, $x_i = x_i(q, t)$, and their time derivatives:

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t},$$

the kinetic energy becomes:

$$T = \frac{1}{2}\sum_i m_i \left(\sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}\right)^2.$$

This formulation aligns with the treatment of generalized velocities and forces, as it expresses dynamics in terms of the independent generalized quantities. By expressing kinetic energy, and later other quantities like potential energy, in terms of generalized coordinates, we simplify the equations of motion while fully incorporating the effects of constraints.

1.6. D'Alembert's Principle and Lagrange's Equations

It is important to mention from the outset that Lagrange's equations of motion were derived by Lagrange and Hamilton more than 45 years before its formulation by D'Alembert. The principle of virtual work provides a basis for a rigorous derivation of the Lagrangian formalism. A virtual displacement $\delta\vec{r}_i$ of a system refers to changes in the coordinates of a system consistent with the forces of constraint imposed on the system at a given instant, contrary to a real displacement that would occur in a time interval dt. D'Alembert postulated that the virtual work should vanish:

$$\sum_{i=1}^{N}\left(\vec{F}_i - \dot{\vec{P}}_i\right)\cdot\delta\vec{r}_i = 0, \quad i = 1, 2, \ldots, N \tag{1.38}$$

where $\vec{F}_i = \vec{F}_i^A + \vec{f}_i^C$; $\vec{F}_i^A$ is the applied force and $\vec{f}_i^C$ the constraint force. However, due to the constraints, the variables $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ are not independent.

Eliminating the constraints will result in n-independent generalized coordinates q_j, $j = 1, 2, \ldots, n$, so that

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n, t), \quad i = (1, 2, \ldots, N) \tag{1.39}$$

whose virtual displacement can be written as

$$\delta\vec{r}_i = \sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \tag{1.40}$$

Note that virtual displacements do not involve any time variation dt by definition. The velocities of the particles are defined by

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial\vec{r}_i}{\partial t} \tag{1.41}$$

So that

$$\frac{\partial\vec{v}_i}{\partial\dot{q}_j} = \frac{\partial\vec{r}_i}{\partial q_j}$$

Returning to D'Alembert's virtual work equation

$$\sum_{i=1}^{N}\left(\vec{F}_i^A + \vec{f}_i^C - \dot{\vec{P}}_i\right)\cdot\delta\vec{r}_i = 0 \tag{1.42}$$

Then we make the transformation to the generalized coordinates in each term to get

$$\sum_{i=1}^{N} \vec{F}_i^A \cdot \delta\vec{r}_i = \sum_{i=1}^{N} \vec{F}_i^A \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} Q_j \delta q_j \tag{1.43}$$

where

$$Q_j = \sum_{i=1}^{N} \vec{F}_i^A \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Q_j is a generalized force associated with the generalized coordinate q_j. However, it is important to note that the units of Q_j and q_j are not necessarily N and m, but the product of $Q_j \delta q_j$ is necessarily Joules.

The other term in D'Alembert's equation is

$$\sum_{i=1}^{N} \dot{\vec{P}}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{N} \dot{\vec{P}}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} \delta q_j \left(\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) \tag{1.44}$$

Using the identity

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} \right]$$

Using the fact that

$$\frac{\partial \vec{v}_i}{\partial q_j} = \frac{\partial \vec{r}_i}{\partial q_j} \quad \text{and} \quad \frac{\partial \dot{\vec{v}}_i}{\partial q_j} = \frac{\partial}{\partial q_j} \dot{\vec{r}}_i \tag{1.45}$$

Then our above equation becomes:

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right]$$

which simplifies to:

$$= \frac{d}{dt} \frac{\partial}{\dot{q}_j} \left(\sum_{i=1}^{N} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \right) - \frac{\partial}{\partial q_j} \left(\sum_{i=1}^{N} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \right)$$

Thus, we arrive at:

$$= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \tag{1.46}$$

Now, due to the presence of the constraints, the displacement $\delta\vec{r}_i$ in Eq. (1.42) is not independent since holonomic constraints

$$g_k(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t) = 0, \quad k = 1, 2, \ldots, m$$

will reduce the number of degrees of freedom to $n = N - m$, represented by the generalized coordinates q_j for $i = 1, 2, \ldots, n$, which on the other hand are independent. Hence, the constraint term in Eq. (1.42) becomes:

$$\sum_{i=1}^{N} \vec{f}_i^C \cdot \delta\vec{r}_i = \sum_{i=1}^{N} \vec{f}_i^C \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} Q_j^C \delta q_j, \tag{1.47}$$

where

$$Q_j^C = \sum_{i=1}^{N} \vec{f}_i^C \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

is the force of constraint corresponding to q_j.

The generalized coordinates are independent and obey all constraints, so that their virtual work vanishes:

$$\sum_{j=1}^{n} Q_j^C \delta q_j = 0. \tag{1.48}$$

Substituting the results from Eqs. (1.43), (1.46), and (1.48) into the virtual work equation, Eq. (1.42), yields:

$$\sum_{j=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}\right)\delta q_j=\sum_{j=1}^{n} Q_j\delta q_j \tag{1.49}$$

Since q_j degrees of freedom are independent, then

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=Q_j, \quad j=1,2,\ldots,n \tag{1.50}$$

If the applied forces are conservative, then

$$\vec{F}_i=-\vec{\vec{\nabla}}_i V, \quad V=\text{scalar potential function}$$

$$Q_j=\sum_{i=1}^{N}\vec{F}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}=-\sum_{i=1}^{N}\vec{\vec{\nabla}}_i V\cdot\frac{\partial \vec{r}_i}{\partial q_j}$$

So that

$$Q_j=-\frac{\partial V}{\partial q_j} \quad \text{since} \quad V=V(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,t) \tag{1.51}$$

The equation of motion becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=-\frac{\partial V}{\partial q_j} \tag{1.52}$$

Since in general $T=T(q_i,\dot{q}_i)$ while $V=V(q_i)$, then the above equation reduces to the famous Lagrange's equation,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j}-\frac{\partial \mathcal{L}}{\partial q_j}=0, \quad j=1,2,\ldots,n \quad ; \quad T-V=\mathcal{L} \tag{1.53}$$

$\mathcal{L}$ is called the Lagrangian of the conservative system. However, it is important to note that there is no unique choice of Lagrangian for a given Lagrange equation of motion, as indicated in Eq. (1.53). In particular, we note that:

- Any addition of a total derivative to $\mathcal{L}$ will not affect the equations of motion.

$$\mathcal{L}'(q_i,\dot{q}_i,t)=\mathcal{L}(q_i,\dot{q}_i,t)+\frac{d}{dt}f(q_i,t) \tag{1.54}$$

  $\mathcal{L}$ and $\mathcal{L}'$ obey the same Lagrange equations.

- Scaling the Lagrangian by a constant will not affect the equations of motion: $\mathcal{L}\to\alpha\mathcal{L}$.

1.7. Velocity-Dependent Potentials and Dissipation Function

Looking back at the original equation of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=Q_j, \quad j=1,2,\ldots,n \tag{1.55}$$

we can still cast this equation in a Lagrangian formalism,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j}-\frac{\partial \mathcal{L}}{\partial q_j}=0, \quad j=1,2,\ldots,n \tag{1.56}$$

if we define a new velocity-dependent potential $V(q,\dot{q})$ so that

$$Q_j=-\frac{\partial V}{\partial q_j}+\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right) \tag{1.57}$$

and hence define

$$\mathcal{L}=T-V$$

This is a simple generalization of our Lagrangian formalism for a single particle of mass m under the influence of a potential $V(\vec{x})$ where

$$\mathcal{L}(\vec{x},\vec{v},t) = \frac{1}{2}mv^2 - V(\vec{x}) = \frac{1}{2}m\vec{v}\cdot\vec{v} - V(\vec{x})$$

so that Lagrange's equation gives

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\vec{v}} - \frac{\partial\mathcal{L}}{\partial\vec{x}} = 0 \Rightarrow m\frac{d^2\vec{x}}{dt^2} = -\frac{\partial V}{\partial\vec{x}} = -\vec{\nabla}V = \vec{F}$$

which is just Newton's law.

Consider now a situation where a charged particle q of mass m is subject to an electric and magnetic field, then its Lorentz equation of motion is

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{1.58}$$

To obtain this equation from a Lagrangian formalism, we need to consider a velocity-dependent potential and generalize V to depend on $(\vec{x},\vec{v},t)$ since $\vec{E}=\vec{E}(\vec{x},t)$ and $\vec{B}=\vec{B}(\vec{x},t)$. Then, the Euler-Lagrange equations become

$$\frac{\partial\mathcal{L}}{\partial\vec{x}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\vec{v}} = -\frac{\partial V}{\partial\vec{x}} - \frac{d}{dt}(m\vec{v}) + \frac{d}{dt}\frac{\partial V}{\partial\vec{v}} = 0$$

which implies

$$m\frac{d\vec{v}}{dt} = -\frac{\partial V}{\partial\vec{x}} + \frac{d}{dt}\frac{\partial V}{\partial\vec{v}} \tag{1.59}$$

This expresses the generalized force mentioned in Eq. (1.57). To determine the explicit form of $V(\vec{x},\vec{v},t)$ in the context of the Lorentz force, as given in Eq. (1.58), it is necessary to rewrite this equation in terms of the vector and scalar potentials $(\vec{A},\Phi)$:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Eq. (1.58) becomes

$$m\frac{d\vec{v}}{dt} = q\left\{-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right\} \tag{1.60}$$

Now, we manipulate the above equation to bring it closer to Eq. (1.59) and thereby identify $V(\vec{x},\vec{v},t)$. To achieve this, we apply the following transformation to the RHS of Eq. (1.60):

$$\vec{v}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A} = \frac{\partial}{\partial\vec{x}}(\vec{v}\cdot\vec{A}) - \vec{v}\cdot\frac{\partial\vec{A}}{\partial\vec{x}} \tag{1.61}$$

Thus, Eq. (1.60) becomes

$$m\frac{d\vec{v}}{dt} = q\left\{-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} + \frac{\partial}{\partial\vec{x}}(\vec{v}\cdot\vec{A}) - \vec{v}\cdot\frac{\partial\vec{A}}{\partial\vec{x}}\right\} \tag{1.62}$$

Since $\Phi=\Phi(\vec{x},t)$, we have $\vec{A} = \frac{\partial}{\partial\vec{v}}(\vec{A}\cdot\vec{v}) = \frac{\partial}{\partial\vec{v}}(\vec{A}\cdot\vec{v}-\Phi)$. Thus, Eq. (1.62) becomes:

$$m\frac{d\vec{v}}{dt} = q\left\{-\frac{\partial}{\partial\vec{x}}(\Phi - \vec{v}\cdot\vec{A}) - \frac{d}{dt}\frac{\partial}{\partial\vec{v}}(\vec{A}\cdot\vec{v}-\Phi)\right\} = -\frac{-\partial V}{\partial\vec{x}} + \frac{d}{dt}\frac{\partial V}{\partial\vec{v}} \tag{1.63}$$

where

$$V = (\Phi - \vec{v}\cdot\vec{A})q$$

We then identify the electromagnetic potential in the Lagrangian formalism as

$$V(\vec{x},\vec{v},t) = q\left(\Phi - \vec{v}\cdot\vec{A}\right) \tag{1.64}$$

So that our single particle Lagrangian in an $\vec{E}$ and $\vec{B}$ field becomes

$$\mathcal{L}(\vec{x},\vec{v},t) = \frac{1}{2}m\vec{v}\cdot\vec{v} - q\Phi(\vec{x},t) + q\,\vec{A}(\vec{x},t)\cdot\vec{v} \tag{1.65}$$

Whose Lagrange equation of motion is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\vec{v}} - \frac{\partial\mathcal{L}}{\partial\vec{x}} = 0$$

Expanding, we have

$$\frac{d}{dt}\left(m\vec{v} + q\vec{A}(\vec{x},t)\right) + q\frac{\partial\Phi}{\partial\vec{x}} - q\frac{\partial}{\partial\vec{x}}(\vec{A}(\vec{x},t)\cdot\vec{v}) = 0$$

which gives

$$m\frac{d^2\vec{x}}{dt^2} = q\left(-\frac{\partial A}{\partial t} - V\frac{\partial\vec{A}}{\partial\vec{x}} - q\frac{\partial\Phi}{\partial\vec{x}} + q\vec{\nabla}\left(\vec{A}(\vec{x},t)\cdot\vec{v}\right)\right)$$

Now simplifying, we get:

$$\begin{aligned} m\frac{d^2\vec{x}}{dt^2} &= q\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} + q\left(\vec{\nabla}(\vec{A}\cdot\vec{v}) - (\vec{v}\cdot\vec{\nabla})\vec{A}\right)\right. \\ &= q\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} + q(\vec{v}\times(\vec{\nabla}\times\vec{A}))\right) \\ &= q\left(\vec{E} + \vec{v}\times\vec{B}\right) \end{aligned} \tag{1.66}$$

We can also show that under a gauge transformation

$$\vec{A}\to\vec{A}+\vec{\nabla}\chi, \quad \Phi\to\Phi-\frac{\partial\chi}{\partial t} \tag{1.67}$$

where $\chi = \chi(\vec{x},t)$ is a scalar gauge function, Maxwell's equations remain invariant, as does Lagrange's equation. In fact, the transformed Lagrangian becomes

$$\mathcal{L}'(\vec{x},\vec{v},t) = \frac{1}{2}m\vec{v}\cdot\vec{v} - q\left(\Phi(\vec{x},t) - \frac{\partial\chi(\vec{x},t)}{\partial t}\right) + q\,\vec{v}\cdot\left(\vec{A}(\vec{x},t) + \vec{\nabla}\chi(\vec{x},t)\right)$$

Expanding this further, we get

$$\begin{aligned} \mathcal{L}'(\vec{x},\vec{v},t) &= \mathcal{L}(\vec{x},\vec{v},t) + q\left(\frac{\partial\chi(\vec{x},t)}{\partial t} + \frac{d\vec{x}}{dt}\cdot\chi(\vec{x},t)\right) \\ &= \mathcal{L}(\vec{x},\vec{v},t) + q\frac{d}{dt}\left(\chi(\vec{x},t)\right) \end{aligned} \tag{1.68}$$

which indeed leaves Lagrange's equations invariant.

From the Lagrangian, we can also define the canonical momentum:

$$\vec{P} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{x}}} = m\vec{v} + q\vec{A} \tag{1.69}$$

The Hamiltonian is then defined through the Legendre transformation.

Transformation:

$$H(\vec{x},\vec{P}) = \vec{P}\cdot\vec{V} - \mathcal{L} = \left(\frac{\vec{P}-q\vec{A}}{m}\right)\cdot\vec{P} - \frac{1}{2}m\left(\frac{\vec{P}-q\vec{A}}{m}\right)^2 + q\Phi + q\left(\frac{\vec{P}-q\vec{A}}{m}\right)\cdot\vec{A}$$

where we used $\mathcal{L}$ as

$$\mathcal{L}(\vec{x},\vec{v},t) = \frac{1}{2}m\vec{v}\cdot\vec{v} - q\Phi(\vec{x},t) + q\,\vec{A}(\vec{x},t)\cdot\vec{v},$$

and replaced $\vec{v}$ all over with

$$\vec{v} = \frac{\vec{P}-q\vec{A}}{m},$$

from (71) to get

$$H(\vec{x}, \vec{P}) = \frac{(\vec{P} - q\vec{A})^2}{2m} + q\Phi(\vec{x}, t). \tag{1.70}$$

This clearly shows the minimum substitution principle in going from the free particle Hamiltonian to the one in the presence of $\vec{E}$ and $\vec{B}$ fields:

$$\vec{P} \to \vec{P} - q\vec{A} \tag{1.71}$$

plus the scalar potential $q\Phi(\vec{x}, t)$.

1.8. Simple Applications of the Lagrangian Formulation

To write down the Lagrangian for a single particle,

$$\mathcal{L} = \frac{1}{2} m\vec{v} \cdot \vec{v} - V(\vec{x}, t) \tag{1.72}$$

we need to express T in different coordinate systems.

$$\vec{v} \cdot \vec{v} = v^2 = \left(\frac{d\vec{s}}{dt}\right)^2 \tag{1.73}$$

$d\vec{s}$ is an infinitesimal displacement which depends on the coordinates selected:

$$\begin{aligned}
d\vec{s} &= dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \quad \text{(Cartesian coordinates)} \\
d\vec{s} &= dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \quad \text{(Spherical coordinates)} \\
d\vec{s} &= dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z} \quad \text{(Cylindrical coordinates)}
\end{aligned}$$

Thus, in Cartesian, Spherical, and Cylindrical coordinates, respectively, we have:

$$T = \frac{1}{2} mv^2 = \begin{cases} \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) & \text{in Cartesian coordinates} \\ \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2) & \text{in Spherical coordinates} \\ \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) & \text{in Cylindrical coordinates} \end{cases} \tag{1.74}$$

Examples of Lagrangian Formulation

Write down the Lagrangian for the following systems:

Example 1.6 – Single Particle in Cartesian Coordinates

A particle of mass m moving in a potential $V(x)$. In Cartesian coordinates, the particle's position is described by (x_1, x_2, x_3), and its velocity is $\vec{v} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$. The Lagrangian for the system is given by:

$$\mathcal{L} = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 - V(\{x_i\}).$$

The Euler-Lagrange equation for each coordinate x_j is:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\right) - \frac{\partial \mathcal{L}}{\partial x_j} = 0.$$

For this system:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_j} = m\dot{x}_j, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\right) = m\ddot{x}_j, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial V}{\partial x_j}.$$

Substituting these into the Euler-Lagrange equation, we obtain:

$$m\ddot{x}_j = -\frac{\partial V}{\partial x_j}.$$

This is the familiar Newtonian equation of motion for a particle in a potential field.

Example 1.7 – Single Particle in Polar Coordinates

A particle of mass m moving in a plane, described by polar coordinates (r, θ). The radial and angular components of the velocity are $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$, respectively. The kinetic energy is:

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$$

The Lagrangian for the system is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r, \theta).$$

For $q_1 = r$, the Euler-Lagrange equation becomes:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0.$$

Here:

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r}, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 + \frac{\partial V}{\partial r}.$$

Substituting, we find:

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial r}.$$

For $q_2 = \theta$, the Euler-Lagrange equation becomes:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0.$$

Here:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}), \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial V}{\partial \theta}.$$

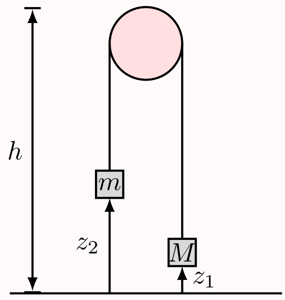
Substituting, we find:

$$m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = -\frac{\partial V}{\partial \theta}.$$

These two equations describe the radial and angular motion of the particle in polar coordinates.

Example 1.8 – Atwood's Machine

Atwood's machine consists of a smooth pulley, a light inextensible string, and two masses M and m. The pulley is fixed to a support, and the string moves frictionlessly over the pulley. The masses are free to move vertically under the influence of gravity. The system is illustrated in the following Fig.



The system has one constraint: the length of the string is fixed. Define the following symbols for clarity:

- z_1: Position of mass M relative to the ground.
- z_2: Position of mass m relative to the ground.
- l: Fixed string length.
- h: Distance from the pulley to the ground.

The constraint is given by:

$$z_1 + z_2 + l = 2h,$$

where l is the fixed string length, and h is the distance from the pulley to the ground. Defining $q = z_1$, the generalized coordinate, we have:

$$z_2 = 2h - l - z_1 \quad \Rightarrow \quad \dot{z}_2 = -\dot{z}_1.$$

The kinetic and potential energies are:

$$T = \frac{1}{2}M\dot{z}_1^2 + \frac{1}{2}m\dot{z}_2^2 = \frac{1}{2}(M+m)\dot{q}^2,$$

$$V = Mgz_1 + mgz_2 = (M-m)gq + \text{const}.$$

The Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{q}^2 - (M-m)gq.$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0,$$

we get:

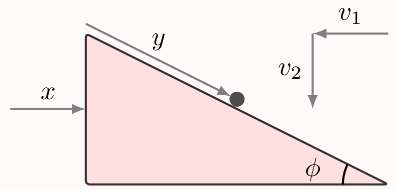
$$(M+m)\ddot{q} = -(M-m)g.$$

Thus, the acceleration of the system is:

$$\ddot{q} = -\frac{(M-m)g}{M+m}.$$

Example 1.9 – Particle and Wedge

A particle of mass m sliding without friction down a wedge of mass M, which is free to slide on a frictionless horizontal surface. The wedge makes an angle ϕ with the horizontal. The system is illustrated in Fig.



To describe the motion of the system, we choose generalized coordinates x for the horizontal displacement of the wedge and y for the position of the particle along the inclined surface. The kinetic energy of the system has two components:

$$T_{\text{wedge}} = \frac{1}{2}M\dot{x}^2, \quad T_{\text{particle}} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\phi\right).$$

The total kinetic energy is:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\phi\right).$$

The potential energy of the system is given by:

$$V = -mgy\sin\phi + \text{const.}$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\phi\right) + mgy\sin\phi.$$

Applying the Euler-Lagrange equations to x and y, we obtain:

$$\frac{d}{dt}\left((M+m)\dot{x} + m\dot{y}\cos\phi\right) = 0,$$
$$\frac{d}{dt}\left(m(\dot{y} + \dot{x}\cos\phi)\right) = mg\sin\phi.$$

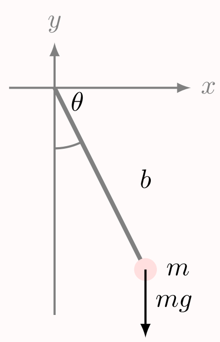
By solving these equations simultaneously, we find the acceleration of the wedge:

$$\ddot{x} = \frac{g\sin\phi}{\cos\phi - \frac{(M+m)}{m\cos\phi}},$$

and the motion of the particle along the wedge.

Example 1.10 – Simple Pendulum

Consider a simple pendulum of length b and mass m, where the bob is displaced by an angle θ from the vertical. The system is illustrated in the following figure.



The Cartesian coordinates of the mass are:

$$x = b\sin\theta, \quad y = -b\cos\theta.$$

The velocities in terms of $\dot{\theta}$ are:

$$\dot{x} = b\dot{\theta}\cos\theta, \quad \dot{y} = b\dot{\theta}\sin\theta.$$

The kinetic energy T is:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mb^2\dot{\theta}^2.$$

The potential energy U is:

$$U = mgy = -mgb\cos\theta.$$

Therefore, the Lagrangian $\mathcal{L} = T - U$ is:

$$\mathcal{L} = \frac{1}{2}mb^2\dot{\theta}^2 + mgb\cos\theta.$$

To obtain the equation of motion, we use the Euler-Lagrange equation for θ:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0.$$

Calculating:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mb^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = mb^2\ddot{\theta}, \quad \frac{\partial\mathcal{L}}{\partial\theta} = -mgb\sin\theta.$$

Substituting these into the Euler-Lagrange equation:

$$mb^2\ddot{\theta} + mgb\sin\theta = 0.$$

Simplifying, we get:

$$\ddot{\theta} + \frac{g}{b}\sin\theta = 0.$$

For small angles θ ($\theta \ll 1$), we use the approximation $\sin\theta \approx \theta$, which simplifies the equation to:

$$\ddot{\theta} + \frac{g}{b}\theta = 0.$$

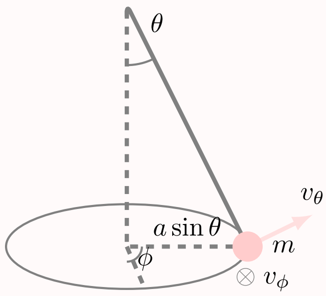
This is the equation for simple harmonic motion with angular frequency $\omega = \sqrt{\frac{g}{b}}$. The solution for small oscillations is:

$$\theta(t) = \theta_0\cos(\omega t + \phi),$$

where θ_0 and ϕ are constants determined by initial conditions.

Example 1.11 – Spherical Pendulum

A spherical pendulum consists of a mass m swinging at a fixed length a from the origin. The motion is described by two angles: θ, the angle from the vertical axis, and ϕ, the rotation angle in the horizontal plane. The system is illustrated in Fig.



The kinetic energy of the system is given by:

$$T = \frac{1}{2}m\left(v_\theta^2 + v_\phi^2\right),$$

where $v_\theta = a\dot{\theta}$ and $v_\phi = a\sin\theta\,\dot{\phi}$. Substituting these, we get:

$$T = \frac{1}{2}ma^2\sin^2\theta\,\dot{\phi}^2 + \frac{1}{2}ma^2\dot{\theta}^2.$$

The potential energy is:

$$V = -mga\cos\theta + \text{constant}.$$

The Lagrangian $\mathcal{L}$ is:

$$\mathcal{L} = T - V = \frac{1}{2}ma^2\sin^2\theta\,\dot{\phi}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta.$$

To find the equations of motion, we apply the Euler-Lagrange equations.
For θ:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0,$$

leading to:

$$ma^2\ddot{\theta} = ma^2\sin\theta\cos\theta\,\dot{\phi}^2 - mga\sin\theta.$$

For ϕ:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0,$$

which simplifies to:

$$\frac{d}{dt}\left(ma^2\sin^2\theta\,\dot{\phi}\right) = 0.$$

This implies conservation of angular momentum about the vertical axis:

$$ma^2\sin^2\theta\,\dot{\phi} = L_z = \text{constant}.$$

Substituting $\dot{\phi}$ into the equation for θ, we find:

$$ma^2\ddot{\theta} = ma^2\sin\theta\cos\theta\left(\frac{L_z^2}{m^2a^4\sin^4\theta}\right) - mga\sin\theta.$$

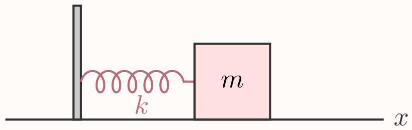
This equation, along with the energy conservation:

$$E = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{L_z^2}{2ma^2\sin^2\theta} - mga\cos\theta,$$

fully describes the motion of the spherical pendulum.

Example 1.12 – Harmonic Oscillator

A mass m attached to a spring with spring constant k, resting on a flat surface. The spring is fixed to a vertical wall, and the mass can oscillate along the horizontal x-axis. The interaction between the spring's restoring force and the inertia of the mass results in simple harmonic motion.



The Lagrangian for the system is given by:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

To obtain the equation of motion, we apply the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

- $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$
- $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x}$
- $\frac{\partial \mathcal{L}}{\partial x} = -kx$

Substituting these results:

$$m\ddot{x} + kx = 0$$

This is the differential equation for a simple harmonic oscillator with angular frequency $\omega = \sqrt{\frac{k}{m}}$. The general solution is:

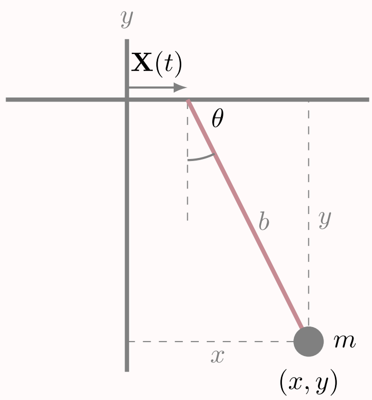
$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

where A and B are constants determined by initial conditions.

Example 1.13 – Driven Simple Pendulum

Consider a driven simple pendulum where the pivot point moves horizontally according to $X(t)$. Let us assume:

$$X(t) = A\sin(\omega_0 t),$$



where A is the amplitude and ω_0 is the driving angular frequency. The coordinates of the mass

m are then given by:

$$x = X(t) + b\sin\theta,$$
$$y = -b\cos\theta.$$

The velocities in terms of $\dot{\theta}$ and $\dot{X}$ are:

$$\dot{x} = \dot{X} + b\dot{\theta}\cos\theta,$$
$$\dot{y} = b\dot{\theta}\sin\theta.$$

The kinetic energy T is:

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m\left[\dot{X}^2 + 2b\dot{X}\dot{\theta}\cos\theta + (b\dot{\theta})^2\right]. \end{aligned}$$

The potential energy U is:

$$U = mgy = -mgb\cos\theta.$$

Thus, the Lagrangian $\mathcal{L} = T - U$ becomes:

$$\mathcal{L} = \frac{1}{2}m\left[\dot{X}^2 + 2b\dot{X}\dot{\theta}\cos\theta + (b\dot{\theta})^2\right] + mgb\cos\theta.$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0,$$

we find:

- Calculate $\frac{\partial\mathcal{L}}{\partial\dot{\theta}}$:
$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m\left(b\dot{X}\cos\theta + b^2\dot{\theta}\right).$$
- Differentiate with respect to time:
$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = m\left(b\ddot{X}\cos\theta - b\dot{X}\dot{\theta}\sin\theta + b^2\ddot{\theta}\right).$$
- Calculate $\frac{\partial\mathcal{L}}{\partial\theta}$:
$$\frac{\partial\mathcal{L}}{\partial\theta} = -mb\dot{X}\dot{\theta}\sin\theta - mgb\sin\theta.$$

Thus, the equation of motion is:

$$b\ddot{\theta} + g\sin\theta = -\ddot{X}\cos\theta.$$

Chapter 2

Variational Principle and Lagrange's Equations

We have seen that Lagrange's equations are equivalent to Newton's equations of motion but provide a better formulation that works independently of the coordinate system used and avoids the concept of force, which can be very complicated. Lagrange's equations have been derived in Chapter 1 using D'Alembert's virtual work principle; however, we will see in this chapter that there is a more general and elegant way to derive Lagrange's equations as the variational principle in mathematics. This mathematical principle is even considered to be a "Universal Fundamental Principle," called the least-action principle by physicists. This principle enables us to treat all physical phenomena on equal footing and in a unified way. For background on this subject, see Ref. [?].

Lagrange's equations are derived from Hamilton's principle of least-action, which states that the motion of a given system between time t_i and t_f is such that the time integral, or action,

$$I = \int_{t_i}^{t_f} \mathcal{L}(\vec{x}, \dot{\vec{x}}, t)\, dt \tag{2.1}$$

where $\mathcal{L} = T - V$ is the Lagrangian, has a stationary value for the actual path of motion.

2.1. Variational Calculus

The variational calculus involves finding the extremum of a quantity that is expressible in integral form. To be more explicit, we look for a function $y(x)$ connecting two fixed points $(x_1, y_1 = y(x_1))$ and $(x_2, y_2 = y(x_2))$, so that the following functional

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx \quad ; \quad y' = \frac{dy}{dx} \tag{2.2}$$

is extremum (minimum or maximum). In contrast to our usual question: Find the extremum of a function $F(x)$, which requires

$$\frac{dF}{dx} = 0 \Rightarrow x_{\max} \quad (\text{or } x_{\min})$$

Here, we look for a function $y(x)$ that makes I extremum. I is called a *functional* since $I = I(y(x))$ is a function of $y(x)$.

Examples:

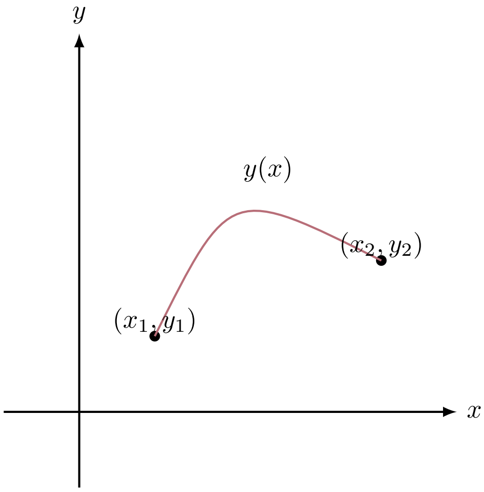
- Fermat's principle in optics states that light travels through a path that takes the least amount of time between any two fixed points in space:

$$I = \int_{t_1}^{t_f} dt = \int_{r_1}^{r_f} \frac{ds}{v} = \frac{1}{c} \int_i^f n(\vec{r})\, ds$$

  where $n(\vec{r})$ is the refractive index of the medium, and c is the speed of light in vacuum.

- Find the curve $y(x)$ connecting two fixed points (x_1, y_1) and (x_2, y_2) so that its length is minimum:

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx$$



In order to find $y(x)$, we consider a neighboring curve, which we parametrize by α:

$$y(\alpha, x) = y(x) + \alpha\eta(x)$$

so that $\eta(x_1) = \eta(x_2) = 0$ by construction. $y(0, x) = y(x)$ is the solution we seek. Then, our original functional (2.2) becomes

$$I(\alpha) = \int_{x_1}^{x_2} F\left(y(x, \alpha), y'(x, \alpha), x\right)\, dx \tag{2.3}$$

The algebraic condition for $I(\alpha)$ to have an extremum at $\alpha = 0$ is that

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0 \tag{2.4}$$

Using the fact that

$$\begin{aligned} \frac{dI}{d\alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\frac{d\eta}{dx}\right) dx \end{aligned}$$

using the fact that $\frac{\partial y}{\partial \alpha} = \eta(x)$ and using integration by parts in the second

$$\begin{aligned} \frac{dI}{d\alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta(x) + \eta(x)\frac{d}{dx}\frac{\partial F}{\partial y'}\right) dx \\ &\quad + \eta(x)\left.\frac{\partial F}{\partial y'}\right|_{x_1}^{x_2} \\ 0 &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta(x)\, dx \quad \forall\, \eta(x) \end{aligned} \tag{2.5}$$

Since $\eta(x_1) = \eta(x_2) = 0$, the above equality being valid for an arbitrary $\eta(x)$ requires that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \tag{2.6}$$

This is the so-called Euler's equation. So, the solution of the variational principle/problem is a solution to Euler's equation.

Example:
For the shortest distance between two fixed points, the curve is such that it gives an extremum for

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} dx\, \sqrt{1 + y'(x)^2}$$

Thus, $F(y, y', x) = \sqrt{1 + y'^2}$, with $\frac{\partial F}{\partial y} = 0$.

$$\Rightarrow \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \Rightarrow \frac{\partial F}{\partial y'} = \text{constant} = C = \frac{y'}{\sqrt{1 + y'^2}}$$

$$\Rightarrow y' = a = \text{constant} \Rightarrow y(x) = ax + b$$

The expected straight line with $y_1 = ax_1 + b$, $y_2 = ax_2 + b$ to define the constants a and b.

2.2. Lagrange's Equations from Hamilton's Principle

Euler's equations closely resemble Lagrange's equations derived in Chapter 1:

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0, \quad \mathcal{L} = T - V \tag{2.7}$$

This similarity suggests that Lagrange's equations can be obtained from a variational principle. The resulting variational principle for mechanics is known as *Hamilton's Principle*. Hamilton's principle for conservative systems states that mechanical systems move so as to extremize the action defined by

$$S = \int_{t_i}^{t_f} \mathcal{L}(\vec{x}, \dot{\vec{x}}, t)\, dt \tag{2.8}$$

which directly gives rise to the Lagrange-Euler equation (**??**).

Generalizing this principle to many degrees of freedom with generalized coordinates $q_1, q_2, \ldots, q_n$, then

$$S = \int_{t_i}^{t_f} \mathcal{L}(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)\, dt \tag{2.9}$$

The variation $\delta S = 0$ is given by

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta \dot{q}_i \right) dt \\ &= \int_{t_i}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i \right) dt\end{aligned}$$

Integration by parts of the second term gives, using $\delta q_i\big|_{t_i}^{t_f} = 0$,

$$\delta S = \int_{t_i}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \delta q_i\, dt \tag{2.10}$$

The above equation is valid for an arbitrary variation δq_i, which requires that (knowing that δq_i are independent)

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0, \quad i = 1, 2, \ldots, n \tag{2.11}$$

which form the n-Lagrange equations of motion.

2.3. Lagrange's Equations in Presence of Constraints

In the presence of holonomic constraints, we can use the concept of constraint functions f_α, where

$$f_\alpha(q_1, q_2, \ldots, q_n, t) = 0, \quad \alpha = 1, 2, \ldots, k$$

$$\implies \delta f_\alpha = \sum_{i=1}^{M} \frac{\partial f_\alpha}{\partial q_i}\delta q_i = 0' \quad \alpha = 1, 2, \ldots, k \tag{2.12}$$

resulting in k constraints on the system. These constraints imply that the virtual displacements δq_i are no longer independent. To account for these k-constraint relations between δq_i, we apply the method of undetermined multipliers, introducing λ_α as follows:

$$\mathcal{L}' = \mathcal{L} + \sum_{\alpha=1}^{k} \lambda_\alpha f_\alpha \tag{2.13}$$

Thus, our action becomes

$$S' = \int_{t_i}^{t_f} \left[\mathcal{L}(q_i, \dot{q}_i, t) + \sum_{\alpha=1}^{k} \lambda_\alpha f_\alpha(q_i, t)\right] dt \tag{2.14}$$

For $\delta S' = 0$, we obtain

$$\delta S' = \int_{t_i}^{t_f} \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_{\alpha=1}^{k} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i}\right) \delta q_i \, dt = 0$$

This leads to the system of equations:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_{\alpha=1}^{k} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n$$

$$f_\alpha(q_1, q_2, \ldots, q_n, t) = C_\alpha, \quad \alpha = 1, 2, \ldots, k \tag{2.15}$$

which represents $n + k$ equations for $n + k$ unknown, δq_i and λ_α.

Physical Meaning of Lagrange Parameters

Using $\mathcal{L} = T - V$ for conservative systems, we can write:

$$-\frac{\partial V}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \sum_{\alpha=1}^{k} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} = 0$$

This leads to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = m_i \ddot{q}_i = -\frac{\partial V}{\partial q_i} + \sum_{\alpha=1}^{k} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} \tag{2.16}$$

which is the generalized Newton's law with $F = -\frac{\partial V}{\partial q_i}$, where V is the conservative potential. The term $\lambda_\alpha \frac{\partial f_\alpha}{\partial q_i}$ can be interpreted as the generalized force required by the constraint to maintain the restriction.

The purpose of the Lagrangian formalism is to avoid directly considering forces; however, if a physical interpretation is needed, we can rely on the guiding principles above with physical intuition. Note that if we rewrite Lagrange's equations for the $n - k$ independent coordinates, constraint forces will not appear in the equations.

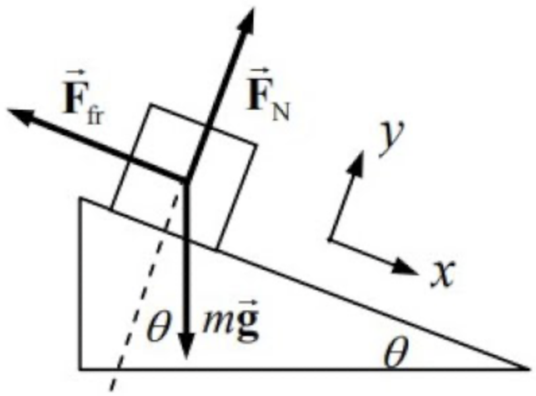


Figure 2.1: Diagram of a box sliding on an incline, showing forces and constraint relations.

Example

Consider a box of mass m sliding along an incline with angle θ.

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

The constraint is given by

$$\frac{y}{x} = \tan\theta \quad \Rightarrow \quad y - x\tan\theta = 0 \quad \Rightarrow \quad f = y - x\tan\theta$$

Thus, the modified Lagrangian becomes

$$\mathcal{L}' = \mathcal{L} + \lambda f = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\theta)$$

The equations of motion are:

$$\frac{\partial \mathcal{L}'}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{x}} = 0 \quad \Rightarrow \quad -\lambda\tan\theta - m\ddot{x} = 0$$
$$\frac{\partial \mathcal{L}'}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{y}} = 0 \quad \Rightarrow \quad \lambda - mg - m\ddot{y} = 0$$

This yields:

$$\begin{cases} m\ddot{x} = -\lambda\tan\theta \\ m\ddot{y} = \lambda - mg \end{cases}$$

Comparing this to Newton's equations:

$$\begin{cases} m\ddot{x} = -N_x \\ m\ddot{y} = N_y - mg \end{cases}$$

we clearly find that

$$N_x = \lambda\tan\theta \quad \text{and} \quad N_y = \lambda$$

Thus,

$$N = \sqrt{N_x^2 + N_y^2} = \frac{\lambda^2}{\cos^2\theta} \quad \Rightarrow \quad \lambda = N\cos\theta$$

2.4. Advantages of Variational Formulation

Since the Lagrangian is constructed from scalar quantities, it is automatically invariant with respect to coordinate transformations. The Lagrangian formalism is based on the correct degree of freedom, meaning it includes all constraints and reduces the number of independent variables to its optimum value, thereby including the constraints automatically address themselves by excluding the presence of constraints from the problem. The variational approach is not limited to classical mechanics.

LC-Circuit

Consider an LC circuit with a capacitor initially charged to $Q_0 = CV_0$. The loop equation is given by

$$L\frac{dI}{dt} + \frac{q}{C} = 0; \quad I = \frac{dq}{dt}$$

This implies

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

This equation is analogous to $m\frac{d^2x}{dt^2} + kx = 0$ for a mass-spring system. We observe the following correspondences:

$$L \leftrightarrow m; \quad \frac{1}{C} \leftrightarrow k$$

The Lagrangian for the circuit can be written as

$$\mathcal{L} = T - V = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C}$$

A similar analysis can be done for other circuits.

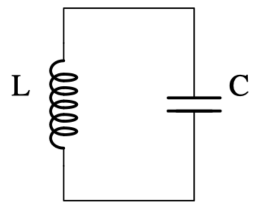


Figure 2.2: Diagram of an LC circuit showing inductor L and capacitor C in series.

RC-Circuit

For an RC circuit, the loop equation becomes

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

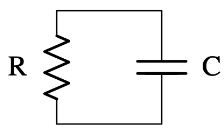


Figure 2.3: Diagram of an RC circuit.

RL-Circuit

For an RL circuit, the loop equation is

$$L\frac{dI}{dt} + RI = 0$$

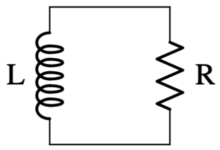


Figure 2.4: Diagram of an RL circuit.

RLC-Circuit

For an RLC circuit, the equation takes the form

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

This equation is analogous to the damped mass-spring system

$$m\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + kx = 0$$

where α represents the coefficient of friction.

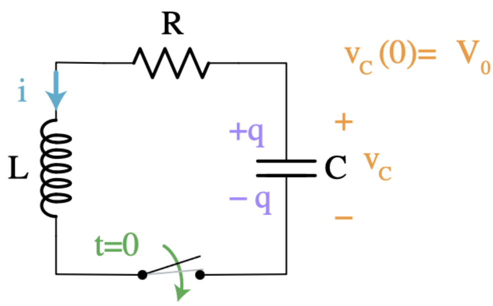


Figure 2.5: Diagram of an RLC circuit showing inductor L, resistor R, and capacitor C in series.

2.5. Symmetry and Conservation Laws

During the motion of a mechanical system, the $2N$ quantities $q_i, \dot{q}_i$ $(i = 1, 2, \ldots, N)$, which specify the state of the system, vary with time. There exist, however, certain quantities whose values remain constant during the motion and depend only on the initial conditions. Such functions are called *integrals of motion*.

There are many problems for which the full solution of the Lagrange's equations of motion is not integrable. However, even when complete solutions cannot be obtained, it is often possible to extract significant information about the physical nature of the system.

Noether's Theorem

Noether's Theorem states that whenever there is a continuous symmetry of the Lagrangian, an associated conservation law will exist. By symmetry, we mean a generalized coordinate transformation q_i, generalized velocities $\dot{q}_i$, and possibly the time variable t that leaves the value of the Lagrangian invariant (unaffected).

The set of all symmetry transformations forms a group of continuous transformations called a *Lie group*. These transformations are continuous if they depend on a parameter ϵ that takes continuous values in a given interval and can be represented by analytic functions so that for $\epsilon = 0$, they coincide with the identity transformation. Thus, the main features of these transformations can be obtained by considering infinitesimal transformations.

Theorem 2.1 – Symmetry

Consider a Lagrangian with generalized coordinates $q_i, \dot{q}_i, \ldots, q_n$, for a certain function $\xi_i(t)$. The infinitesimal transformation for $\epsilon \ll 1$ is given by:

$$q_i(t) \to q_i(t) + \epsilon\xi_i(t)$$

$$\dot{q}_i(t) \to \dot{q}_i(t) + \epsilon\dot{\xi}_i(t) \tag{2.17}$$

If this is a symmetry transformation, then the quantity

$$J = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\xi_i$$

is a constant of motion, i.e.,

$$\frac{dJ}{dt} = 0 \quad \text{or} \quad J = \text{Constant}.$$

In fact:

$$\begin{aligned}\delta\mathcal{L} &= \mathcal{L}\left(q_i + \epsilon\xi_i, \dot{q}_i + \epsilon\dot{\xi}_i, t\right) - \mathcal{L}(q_i, \dot{q}_i, t) \\ &= \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\epsilon\xi_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\epsilon\dot{\xi}_i\right)\end{aligned}$$

using $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$, we have:

$$\delta\mathcal{L} = \epsilon \sum_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\xi_i\right)$$

$$\delta\mathcal{L} = 0,\ \forall\epsilon \Rightarrow \frac{d}{dt}\left(\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\xi_i\right) = 0$$

$$\Rightarrow J = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\xi_i = \text{Constant}. \tag{2.18}$$

Examples:

1. Cyclic Coordinates

If a generalized coordinate q_s does not appear explicitly in $\mathcal{L}$, it is called a cyclic coordinate. Obviously, the Lagrangian is invariant under the infinitesimal transformation

$$q_s \to q_s + \epsilon; \quad q_i \to q_i \quad \text{for} \quad i \neq s \tag{2.19}$$

Thus, $\xi_i = \delta_{is}$ in 2.17, so that

$$J = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta_{is} = \frac{\partial \mathcal{L}}{\partial \dot{q}_s} = p_s = \text{Constant} \tag{2.20}$$

This means that the momentum conjugate of a cyclic coordinate is conserved. Of course, this is easily obtained from the corresponding Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_s} = \frac{\partial \mathcal{L}}{\partial q_s} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_s} = p_s = \text{Constant} \tag{2.21}$$

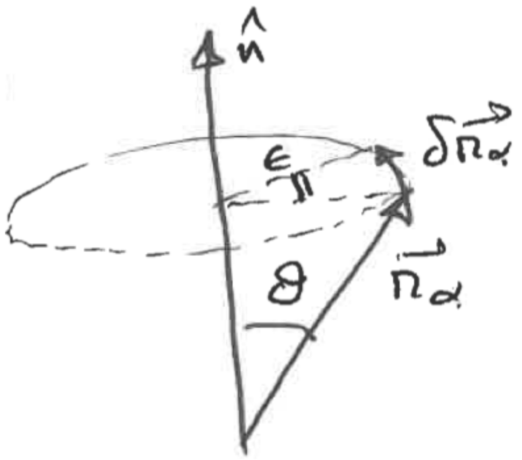


Figure 2.6: Isotopy of Space

2. Homogeneity of Space

The system is translationally invariant under

$$\vec{r}_i \to \vec{r}_i + \epsilon\,\hat{n} \tag{2.22}$$

where $\hat{n}$ is an arbitrary direction in space, so that in component form:

$$x_{i\alpha} \to x_{i\alpha} + \epsilon\, n_\alpha, \quad ; \quad \alpha = 1,2,3 \quad \text{or} \quad x,y,z$$

The conserved current corresponding to this translational invariance is then
The conserved current is then

$$J = \sum_{i=1}^{N}\sum_{\alpha=1}^{3} \frac{\partial \mathcal{L}}{\partial \dot{x}_{\alpha i}} n_\alpha = \sum_{\alpha=1}^{3} n_\alpha \left(\sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_{\alpha i}}\right)$$

$$J = \sum_{\alpha=1}^{3} n_\alpha \sum_{i=1}^{N} P_{i\alpha} = \sum_{\alpha=1}^{3} n_\alpha P_\alpha = \hat{n}\cdot\vec{P} \tag{2.23}$$

where

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i$$

is the total momentum of the system. Since $\hat{n}$ is an arbitrary direction, this implies that

$$J = \hat{n}\cdot\vec{P} = \text{Constant} \quad \forall\,\hat{n} \quad \text{(arbitrary direction)}$$

Therefore, we conclude that

$$\vec{P} = \text{Constant} \tag{2.24}$$

indicating that the total momentum $\vec{P}$ of the system is conserved.

3. Isotropy of Space

Isotropy of space means that the mechanical system does not vary its properties when it is rotated as a whole about an arbitrary axis $\hat{n}$. Since an infinitesimal rotation by an angle ϵ about an axis $\hat{n}$ is given by

$$\delta\vec{r}_\alpha = \epsilon\,\hat{n}\times\vec{r}_\alpha \tag{2.25}$$

we have

$$|\delta\vec{r}_\alpha| = r_\alpha \sin\theta\,\epsilon$$

where $\delta\phi = \epsilon$ is an infinitesimal rotation . By construction, $\delta\vec{r}_\alpha \perp (\hat{n}, \vec{r}_\alpha)$.

Thus,

$$
\begin{aligned}
J &= \sum_{\alpha=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{r}_\alpha} \cdot \xi_\alpha = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial \dot{x}_{i\alpha}} \left(\hat{n} \times \vec{r}_\alpha\right) \\
&= \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \sum_{j,k=1}^{3} p_{\alpha i}\, \epsilon_{ijk} n_j x_{\alpha k} \\
&= \sum_{\alpha=1}^{N} \sum_{j=1}^{3} n_j \sum_{i,k=1}^{3} \epsilon_{ijk} x_{\alpha k} p_{\alpha i} \\
&= \sum_{\alpha=1}^{N} n_j \left(\vec{r}_\alpha \times \vec{p}_\alpha\right)_j \\
&= \hat{n} \cdot \vec{L} = \text{Constant} \quad \forall\, \hat{n}
\end{aligned}
$$

where

$$
\vec{L} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha = \text{Constant} \tag{2.26}
$$

which expresses the conservation of angular momentum (total).

4. Homogeneity in Time

In this case, the properties of our system are invariant under the time translation

$$
t \to t + \epsilon \tag{2.27}
$$

Thus, we require

$$
\delta \mathcal{L} = \mathcal{L}(q, \dot{q}, t + \epsilon) - \mathcal{L}(q, \dot{q}, t) = \frac{\partial \mathcal{L}}{\partial t} \epsilon + \mathcal{O}(\epsilon^2) \tag{2.28}
$$

Since this must hold for any arbitrary time translation, we have

$$
\delta \mathcal{L} = 0, \quad \forall\, \epsilon \implies \frac{\partial \mathcal{L}}{\partial t} = 0 \tag{2.29}
$$

Using this result in the total time derivative of the Lagrangian:

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \frac{\partial \mathcal{L}}{\partial t} + \sum_\alpha \frac{\partial \mathcal{L}}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \ddot{q}_\alpha \\
&= \sum_\alpha \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \right) \dot{q}_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \ddot{q}_\alpha \\
&= \frac{d}{dt} \left(\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \dot{q}_\alpha \right)
\end{aligned} \tag{2.30}
$$

Therefore,

$$
\frac{d}{dt} \left(\sum_\alpha \dot{q}_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} - \mathcal{L} \right) = 0
$$

which implies that

$$
H = \sum_\alpha \dot{q}_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} - \mathcal{L} = \text{Constant}. \tag{2.31}
$$

This homogeneity in time gives rise to the conservation of the Hamiltonian of the system, which represents the total mechanical energy for conservative systems. Since

$$
\mathcal{L} = T(q, \dot{q}) - U(q)
$$

we have

$$\sum_\alpha \dot{q}_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \sum_\alpha \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T \tag{2.32}$$

since T is a homogeneous function of degree 2 in $\dot{q}_\alpha$. Thus,

$$\begin{aligned} H &= \sum_\alpha \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} - (T - V) \\ &= 2T - T + V = T + V \end{aligned} \tag{2.33}$$

Example 2.1 – Example 1

For a particle moving in three-dimensional space with Lagrangian

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - V(z)$$

since $\frac{\partial \mathcal{L}}{\partial x} = 0$, we conclude that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = p_x = \text{Constant}$$

implying that the momentum p_x is conserved.

Example 2.2 – Example 2

For a particle in polar coordinates with Lagrangian

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r) \quad \text{in 2D}$$

since $\frac{\partial \mathcal{L}}{\partial \phi} = 0$, we conclude that

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2 \dot{\phi} = \text{Constant} = L_\phi$$

which implies the system is invariant about the z-axis rotation, and thus the corresponding angular momentum L_ϕ is conserved. If $V = V(r)$, the system is unaffected by rotation in ϕ, leading to the conservation of $L_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{Constant}$.

Chapter 3

The Central Force Problem

In this chapter, we consider two interacting particles through a central potential, a potential that depends only on the distance between these two particles. The forces acting between the two particles obey Newton's Third Law, either in its weak form:

$$\vec{F}_{ij} = -\vec{F}_{ji},$$

or in its strong form, which requires that the action-reaction forces act along the line joining the two particles.

3.1. Reduction to an Equivalent One-Body Central Problem

Consider two point particles with masses m_1 and m_2, interacting through a potential U that depends only on the relative position $\vec{r}_2 - \vec{r}_1$, the relative velocity $\dot{\vec{r}}_2 - \dot{\vec{r}}_1$, or any higher-order derivatives of these quantities. Such a system has six degrees of freedom. We choose to describe the system using its center-of-mass coordinate $\vec{R}$ and the relative coordinate $\vec{r}$, defined as:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \quad \vec{r} = \vec{r}_2 - \vec{r}_1. \tag{3.1}$$

Substituting these coordinates into the system's Lagrangian, we obtain:

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}}, \ldots), \tag{3.2}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of the system.

3.1.1. Center of Mass

The derivation of the general form for the motion of the C.M. requires the use of Newton's Second and Third Laws. For each particle, the equation of motion is:

$$m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \vec{f}_\alpha^{\text{ext}} + \sum_\beta \vec{f}_{\alpha\beta}^{\text{int}}, \tag{3.3}$$

where $\vec{f}_\alpha^{\text{ext}}$ is the external force acting on m_α, and $\vec{f}_{\alpha\beta}^{\text{int}}$ is the internal force exerted on particle α by particle β.
Summing over all the particles in the system, we have:

$$\frac{d^2}{dt^2}\left(\sum_\alpha m_\alpha \vec{r}_\alpha\right) = \sum_\alpha \vec{f}_\alpha^{\text{ext}} + \sum_{\alpha,\beta} \vec{f}_{\alpha\beta}^{\text{int}} = \vec{F}_{\text{ext}}. \tag{3.4}$$

Using Newton's Third Law, which states that the internal forces cancel each other ($\vec{f}_{\alpha\beta}^{\text{int}} = -\vec{f}_{\beta\alpha}^{\text{int}}$), we find:

$$\sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \vec{f}_{\alpha\beta}^{\text{int}} = 0. \tag{3.5}$$

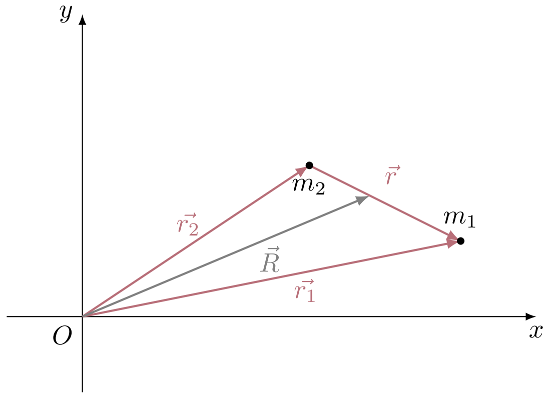


Figure 3.1: Diagram illustrating the relative and center-of-mass coordinates in a two-body system.

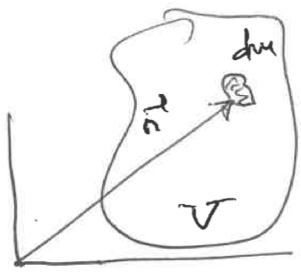


Figure 3.2: Center of mass for a continuous mass distribution.

Then we can rewrite (3.4) as follows:

$$M = \sum_\alpha m_\alpha, \quad M\frac{d^2\vec{R}}{dt^2} = \vec{F}_{\text{ext}}, \quad \vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha. \tag{3.6}$$

Here, $\vec{F}_{\text{ext}} = \sum_\alpha \vec{f}_\alpha^{\text{ext}}$ is the net external force on the system. This shows the well-known result that the dynamics of the center of mass are affected solely by external forces.

3.1.2. Continuous Mass Distribution

For a continuous mass distribution:

$$\vec{R} = \frac{1}{M}\int_V \vec{r}\,dm, \tag{3.7}$$

where the integration is taken over the volume V.

For a system that is subdivided into many subsystems M_i with volume V_i, and centers of mass $\vec{R}_i$, we can write:

$$\vec{R} = \frac{1}{M}\left[\sum_i \int_{V_i} \vec{r}\,dm\right] = \frac{\sum_i M_i \vec{R}_i}{M}, \quad \vec{R}_i = \frac{1}{M_i}\int_{V_i} \vec{r}\,dm. \tag{3.8}$$

Theorem 3.1 – Center of Mass and Symmetry

If a mass distribution $\rho(\vec{r})$ in three-dimensional space has an axis of symmetry (e.g., the z-axis), then the center of mass $\vec{R}$, defined as:

$$\vec{R} = \frac{\int_V \vec{r}\,\rho(\vec{r})\,dV}{\int_V \rho(\vec{r})\,dV},$$

lies on the axis of symmetry. Specifically, if $\rho(\vec{r})$ is invariant under rotations about the z-axis,

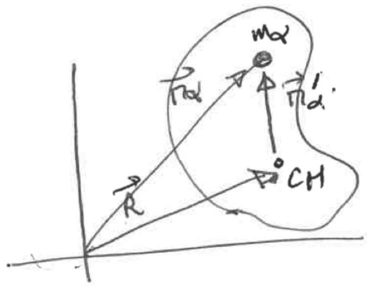


Figure 3.3: Kinetic energy decomposition with respect to the center of mass.

then:

$$\int_V x\,\rho(\vec{r})\,dV = \int_V y\,\rho(\vec{r})\,dV = 0,$$

and:

$$\vec{R} = (0, 0, R_z).$$

3.1.3. Kinetic Energy

The total kinetic energy of the system is:

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha.$$

Using:

$$\dot{\vec{r}}_\alpha = \dot{\vec{R}} + \dot{\vec{r}}'_\alpha,$$

where $\dot{\vec{r}}'_\alpha$ is the velocity of m_α relative to the C.M., we can write:

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left(\dot{\vec{R}} + \dot{\vec{r}}'_\alpha\right)\cdot\left(\dot{\vec{R}} + \dot{\vec{r}}'_\alpha\right),$$

which expands to:

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left[\dot{\vec{R}}^2 + 2\dot{\vec{R}}\cdot\dot{\vec{r}}'_\alpha + \dot{\vec{r}}'^2_\alpha\right].$$

Since:

$$\sum_\alpha m_\alpha \dot{\vec{r}}'_\alpha = \frac{d}{dt}\sum_\alpha m_\alpha \vec{r}'_\alpha = M\frac{d\vec{R}'}{dt} = 0,$$

where $\vec{R}'$ is the position of the C.M. with respect to itself, the second term vanishes. Thus:

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}'^2_\alpha = T_{\text{C.M.}} + T_{\text{particles/C.M.}}, \tag{3.9}$$

where:

$$T_{\text{C.M.}} = \frac{1}{2}M\dot{\vec{R}}^2 \quad \text{and} \quad T'_{\text{particles/C.M.}} = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}'^2_\alpha.$$

In our case of two particles:

$$\vec{r}_1 = \vec{R} + \vec{r}'_1, \quad \vec{r}_2 = \vec{R} + \vec{r}'_2, \quad \vec{r} = \vec{r}_2 - \vec{r}_1, \quad \vec{r}' = \vec{r}'_2 - \vec{r}'_1,$$

so that:

$$\dot{\vec{r}}'_1 = -\frac{m_2}{m_1+m_2}\dot{\vec{r}}, \quad \dot{\vec{r}}'_2 = \frac{m_1}{m_1+m_2}\dot{\vec{r}}.$$

Substituting:

$$m_1\dot{\vec{r}}'^2_1 + m_2\dot{\vec{r}}'^2_2 = \left[m_1\left(\frac{m_2}{m_1+m_2}\right)^2 + m_2\left(\frac{m_1}{m_1+m_2}\right)^2\right]\dot{\vec{r}}^2.$$

Simplifying:

$$m_1\dot{\vec{r}}_1'^2 + m_2\dot{\vec{r}}_2'^2 = \frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2. \tag{3.10}$$

Defining the effective mass of the two particles by:

$$\mu = \frac{m_1 m_2}{m_1+m_2}.$$

Putting these results back in (3.9) gives:

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2, \tag{3.11}$$

and

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}}, \dots). \tag{3.12}$$

Of course, the formula could be derived directly as follows:

$$\dot{\vec{r}}_1 = \dot{\vec{R}} - \frac{m_2}{m_1+m_2}\dot{\vec{r}}, \quad \dot{\vec{r}}_2 = \dot{\vec{R}} + \frac{m_1}{m_1+m_2}\dot{\vec{r}}.$$

Then:

$$T = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2,$$

$$T = \frac{1}{2}m_1\left(\dot{\vec{R}} - \frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} + \frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2.$$

Expanding the terms:

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2, \tag{3.13}$$

where $M = m_1 + m_2$ is the total mass and $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

3.2. Equations of Motion and First Integrals

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}}, \dots). \tag{3.14}$$

For conservative central forces, let $U = V(r)$. Thus, the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r). \tag{3.15}$$

Since $\vec{R}$ is cyclic (it does not appear explicitly in the Lagrangian), the Euler-Lagrange equation gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}} = M\dot{\vec{R}} = \text{const} = \vec{P}, \tag{3.16}$$

where $\vec{P}$ is the total momentum of the center of mass. Hence:

$$\dot{\vec{R}} = \vec{V}_{\text{CM}} \implies \vec{R}(t) = \vec{V}_{\text{CM}}t + \vec{R}_0. \tag{3.17}$$

This eliminates the motion of the center of mass, allowing us to focus entirely on the relative motion.

In the center of mass reference frame, the Lagrangian reduces to:

$$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r). \tag{3.18}$$

For $V = V(r)$, the system is spherically symmetric, which implies that angular momentum is conserved:

$$\frac{d\vec{\ell}}{dt} = \vec{r}\times\vec{F} = \vec{r}\times(-\nabla V) = \vec{r}\times\left(-\frac{dV}{dr}\hat{r}\right) = 0. \tag{3.19}$$

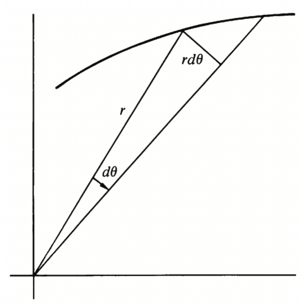


Figure 3.4: The area swept out by the radius vector in a time dt. Image take from Ref. [?]

Thus:

$$\vec{\ell} = \vec{r} \times \vec{p} = \text{const}.$$

This implies that $\vec{r}$ and $\vec{p}$ always lie in a plane normal to $\vec{\ell}$. Therefore, the motion of $\vec{r}(t)$ is planar. Let us take the xy-plane as the plane of motion, so that:

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}. \tag{3.20}$$

The Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \tag{3.21}$$

Since θ is a cyclic coordinate, it follows that:

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{const} = \ell, \tag{3.22}$$

where ℓ is the conserved angular momentum.

In fact:

$$\vec{\ell} = \vec{r} \times \vec{p} = \mu \vec{r} \times \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) = \mu r^2 \dot{\theta}\hat{z}.$$

The conservation of angular momentum can be interpreted geometrically: the area swept by $\vec{r}$ in time dt is:

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}r^2 d\theta.$$

Hence:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2\mu} = \text{const}. \tag{3.23}$$

Thus, the conservation of angular momentum is equivalent to a constant areal speed.

The radial Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0, \tag{3.24}$$

gives:

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{dV}{dr} = 0. \tag{3.25}$$

Using conservation of angular momentum $\dot{\theta} = \frac{\ell}{\mu r^2}$, we get:

$$\mu\ddot{r} - \frac{\ell^2}{\mu r^3} + \frac{dV}{dr} = f(r), \tag{3.26}$$

where $f(r)$ is the centripetal force.

Since $\mathcal{L}$ in (3.21) does not depend explicitly on time, the total energy is conserved:

$$E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r). \tag{3.27}$$

The term:

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2\mu r^2}, \tag{3.28}$$

is called the ***effective potential***. The centripetal term arises from the conservation of angular momentum:

$$\frac{1}{2}\mu r^2 \dot{\theta}^2 = \frac{1}{2}\mu r^2 \left(\frac{\ell}{\mu r^2}\right)^2 = \frac{\ell^2}{2\mu r^2}.$$

We are left with a single degree of freedom, $r(t)$, since:

$$\dot{\theta}(t) = \frac{\ell}{\mu r^2}, \tag{3.29}$$

and so:

$$\theta(t) = \int \frac{\ell}{\mu r^2}\, dt. \tag{3.30}$$

The function $\theta(t)$ will be known once $r(t)$ is solved. We can then integrate (3.27) to get:

$$\dot{r}^2 = \frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right) \implies dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right)}}. \tag{3.31}$$

Or equivalently:

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right)}}. \tag{3.32}$$

Recall that this problem has been reduced to an effective 1D problem. The knowledge of the shape of the effective potential $V_{\text{eff}}(r)$ dictates the type of orbit (periodic or open) and can be studied using phase space diagrams. See Fig 3.5

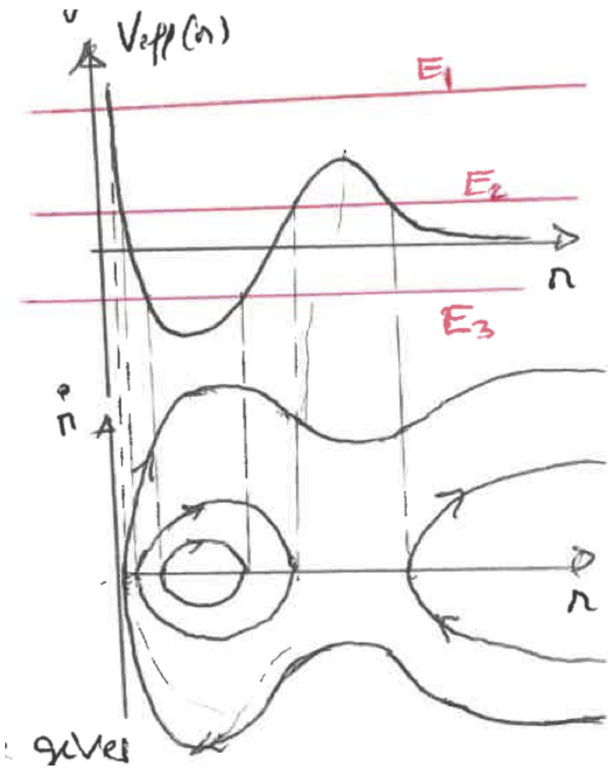


Figure 3.5: Phase space analysis for the effective potential.

Full integration of (3.20) is possible for $V(r) \propto r^n$, where $n = 1, 2, -1, \ldots$. The shape of the fictitious or effective potential:

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2\mu r^2}, \tag{3.33}$$

dictates the type of possible orbits: open, closed, periodic, etc.
For the Coulomb potential, see Fig. ??:

$$V(r) = -\frac{k}{r}, \quad k > 0, \tag{3.34}$$

we have:

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}. \tag{3.35}$$

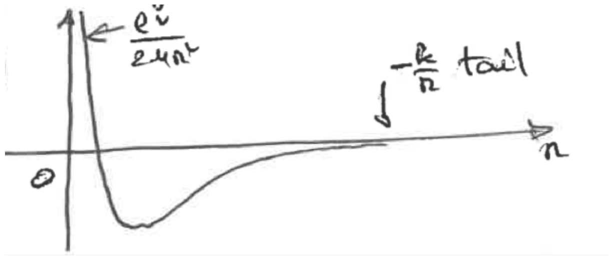


Figure 3.6: Effective potential $V_{\text{eff}}(r)$ for the Coulomb potential with angular momentum contributions.

3.3. Isotropic Harmonic Oscillator

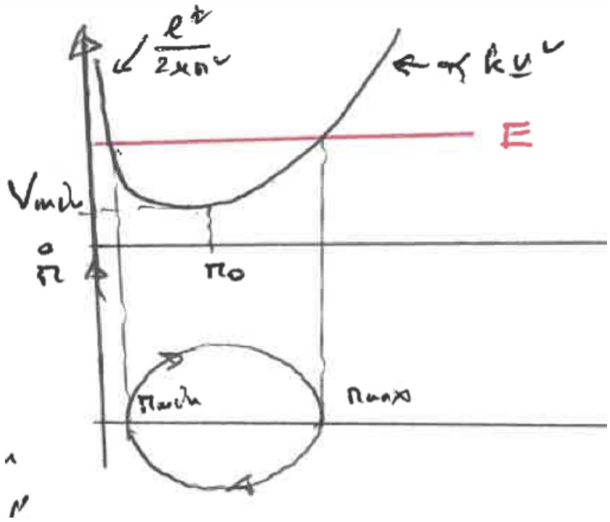


Figure 3.7: Effective potential with turning points and energy level E.

For the isotropic harmonic oscillator, the force and potential are given as:

$$f = -kr, \quad V(r) = \frac{1}{2}kr^2.$$

The effective potential becomes:

$$V_{\text{eff}}(r) = \frac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2}. \tag{3.36}$$

To find the turning points of the motion, solve:

$$\frac{dV_{\text{eff}}}{dr} = 0 \implies r = r_0. \tag{3.37}$$

The solution of $E = V_{\text{eff}}(r)$ at $r = r_{\min}, r_{\max}$ gives the turning points at which $\dot{r} = 0$. The period of the motion can then be computed either analytically or numerically:

$$T = 2\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right)}}. \tag{3.38}$$

Close to $E = V_{\min}$, we can expand $V_{\text{eff}}(r)$ around r_0 to get:

$$V_{\text{eff}}(r) = V_{\min} + (r - r_0)\frac{dV_{\text{eff}}}{dr}\bigg|_{r_0} + \frac{1}{2}(r - r_0)^2\frac{d^2V_{\text{eff}}}{dr^2}\bigg|_{r_0}. \tag{3.39}$$

Since $\frac{dV_{\text{eff}}}{dr}\big|_{r_0} = 0$, this reduces to:

$$V_{\text{eff}}(r) = V_{\min} + \frac{1}{2}K(r - r_0)^2, \quad K = \frac{d^2V_{\text{eff}}}{dr^2}\bigg|_{r_0}. \tag{3.40}$$

The careful computation of (3.38) gives the classical result for the period of oscillation:

$$T = 2\pi\sqrt{\frac{\mu}{K}}, \tag{3.41}$$

for $E = V_{\min} + \epsilon$, where $\epsilon \ll 1$.

3.4. Conditions for Closed Orbits and Stability Analysis

Classification of Orbits in a Central Potential

From our previous equations:

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r); \quad V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2\mu r^2}. \tag{3.42}$$

However, in solving such problems, we should not forget that the particle is moving in two dimensions (xy) and thus:

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}. \tag{3.43}$$

If $\dot{r} = 0$, there is no radial velocity, and the velocity is purely tangential ($r\dot{\theta}$).

In general, depending on the shape of $V_{\text{eff}}(r)$, we can have either:

- Closed (periodic) orbits
- Open orbits.

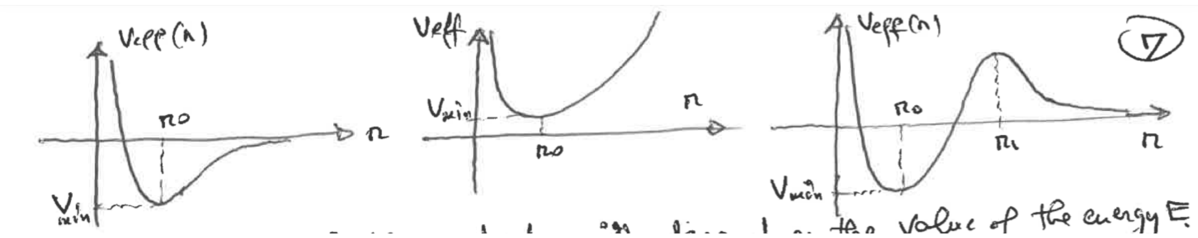


Figure 3.8: Effective potential $V_{\text{eff}}(r)$ showing different energy levels and corresponding trajectories, determined by the conservation of angular momentum.

The trajectory of the particle will depend on the value of the energy E. From the conservation of angular momentum expressed by (3.30):

$$\dot{\theta} = \frac{\ell}{\mu r^2} > 0, \tag{3.44}$$

which shows that θ varies monotonically with time since $\dot{\theta}$ cannot change sign.

From (3.42), we have:

$$\dot{r}^2 = \frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right), \tag{3.45}$$

which requires $E \geq V_{\text{eff}}(r)$ for allowed motion. The points at which $E = V_{\text{eff}}(r)$, such that $\dot{r} = 0$, are called the turning points. These are the radial positions where the radial part of the velocity vanishes:

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}. \tag{3.46}$$

At the turning points, the radial velocity vanishes ($\dot{r} = 0$), but the tangential velocity $r\dot{\theta}$ remains. Depending on the roots of the equation, there may be two turning points, r_{min} and r_{max}. If the motion is confined within these limits, the particle will oscillate radially between r_{min} and r_{max}, forming an annulus bounded by circles of radii r_{min} and r_{max}.

To analyze the motion, we evaluate the angular deviation during one complete revolution around the center of the circles:

$$\Delta\theta = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\ell\, dr/r^2}{\sqrt{2\mu\,(E - V(r)) - \frac{\ell^2}{r^2}}}. \tag{3.47}$$

From the conservation laws:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\,(E - V(r)) - \frac{\ell^2}{\mu^2 r^2}}, \tag{3.48}$$

and:

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{\ell}{\mu r^2}. \tag{3.49}$$

Using the chain rule:

$$\frac{\dot{\theta}}{\dot{r}} = \frac{d\theta}{dr} = \frac{\ell}{\mu r^2\sqrt{\frac{2}{\mu}\,(E - V(r)) - \frac{\ell^2}{\mu^2 r^2}}}. \tag{3.50}$$

The total angular deviation during one complete oscillation between r_{min} and r_{max} is:

$$\Delta\theta = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\ell\, dr/r^2}{\sqrt{\frac{2}{\mu}\,(E - V(r)) - \frac{\ell^2}{\mu^2 r^2}}}. \tag{3.51}$$

The condition for the path to be closed is that:

$$\Delta\theta = 2\pi\frac{m}{n}; \quad m, n \in \mathbb{Z}^+. \tag{3.52}$$

If $\Delta\theta$ is not a rational multiple of 2π, then after a long time, the path of the particle will cover all space, as the path will not close.
The only types of central potentials for which the path closes are:

- Coulomb-like potential $\frac{1}{r}$,
- Oscillator potential r^2.

The presence of the centrifugal term $\frac{\ell^2}{2\mu r^2}$ in the potential energy, which becomes infinite as $r \to 0$, generally makes it impossible for the effective (or fictitious) particle to reach the center unless:

$$V(r) = -\frac{\alpha}{r^n}; \quad \alpha = \text{constant},\ n \geq 2. \tag{3.53}$$

Circular Orbits

Circular orbits are possible if $V_{\text{eff}}(r)$ has an extremum at $r = r_0$, so that:

$$\left.\frac{dV_{\text{eff}}}{dr}\right|_{r=r_0} = \left.\frac{dV}{dr}\right|_{r=r_0} - \frac{\ell^2}{\mu r_0^3} = 0, \tag{3.54}$$

which implies:

$$f(r_0) = -\left.\frac{dV}{dr}\right|_{r=r_0} = -\frac{\ell^2}{\mu r_0^3} < 0. \tag{3.55}$$

This means that the force $f(r)$ (centripetal force) must be attractive for the circular orbit to be possible.

Among these orbits, only those for which $V_{\text{eff}}(r)$ has a minimum at $r = r_0$ will be effectively stable. For stability, $V_{\text{eff}}(r)$ must satisfy:

$$\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_0} = \left.\frac{d^2V}{dr^2}\right|_{r=r_0} + \frac{3\ell^2}{\mu r_0^4} > 0. \tag{3.56}$$

Substituting $f(r) = -\frac{dV}{dr}$, this becomes:

$$-\left.\frac{df}{dr}\right|_{r=r_0} + \frac{3\ell^2}{\mu r_0^4} > 0. \tag{3.57}$$

Using $f(r_0) = -\frac{\ell^2}{\mu r_0^3}$, we get:

$$-\left.\frac{df}{dr}\right|_{r=r_0} - \frac{3}{r_0} f(r_0) > 0. \tag{3.58}$$

This leads to:

$$\left.\frac{d\ln f}{d\ln r}\right|_{r=r_0} > -3. \tag{3.59}$$

Applying the condition in (3.59) for a central force model:

$$f(r) = -Kr^n, \quad K > 0, \tag{3.60}$$

gives:

$$\left.\frac{\partial f}{\partial r}\right|_{r=r_0} = -nKr_0^{n-1} < -\frac{3}{r_0} f(r_0) = 3Kr_0^{n-1}. \tag{3.61}$$

This leads to:

$$n > -3. \tag{3.62}$$

3.5. Stability of Circular Orbits

Circular orbits occur when $E = V_{\text{eff}}(r_0)$, and $V_{\text{eff}}(r)$ has a minimum. Then a small increase in $E' = E + \epsilon$, $\epsilon \ll 1$, should result in small oscillatory motion about the circular orbit $r = r_0$.
Using the radial equation of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0, \tag{3.63}$$

we get:

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = -\frac{\partial V}{\partial r} = f(r), \tag{3.64}$$

and substituting $\mu r^2\dot{\theta} = \ell$, this becomes:

$$\mu\ddot{r} - \frac{\ell^2}{\mu r^3} = f(r). \tag{3.65}$$

Let:

$$r = r_0 + \epsilon, \quad \epsilon \ll r_0, \tag{3.66}$$

so that we can expand in a Taylor series:

$$\frac{1}{r^3} = \frac{1}{(r_0+\epsilon)^3} = \frac{1}{r_0^3}\frac{1}{\left(1+\frac{\epsilon}{r_0}\right)^3}. \tag{3.67}$$

Using the binomial expansion for $(1+x)^{-3}$, we write:

$$\frac{1}{\left(1+\frac{\epsilon}{r_0}\right)^3} \approx 1 - 3\frac{\epsilon}{r_0}, \tag{3.68}$$

which gives:

$$\frac{1}{r^3} \approx \frac{1}{r_0^3}\left(1 - 3\frac{\epsilon}{r_0}\right) = \frac{1}{r_0^3} - \frac{3\epsilon}{r_0^4}. \tag{3.69}$$

Expanding $f(r)$ around r_0 using a Taylor series gives:

$$f(r) = f(r_0 + \epsilon) = f(r_0) + f'(r_0)\epsilon + \mathcal{O}(\epsilon^2). \tag{3.70}$$

The radial equation of motion:

$$\mu\ddot{r} - \frac{\ell^2}{\mu r^3} = f(r), \tag{3.71}$$

becomes:

$$\mu\ddot{\epsilon} - \frac{\ell^2}{\mu}\left(\frac{1}{r_0^3} - \frac{3\epsilon}{r_0^4}\right) = f(r_0) + f'(r_0)\epsilon. \tag{3.72}$$

At $r = r_0$, we have:

$$f(r_0) = -\frac{\ell^2}{\mu r_0^3}. \tag{3.73}$$

Substituting $f(r_0)$ into the equation of motion and simplifying gives:

$$\mu\ddot{\epsilon} + \frac{3\ell^2\epsilon}{\mu r_0^4} = f'(r_0)\epsilon. \tag{3.74}$$

Dividing through by μ, we find:

$$\ddot{\epsilon} + \epsilon\left(\frac{3\ell^2}{\mu^2 r_0^4} - \frac{f'(r_0)}{\mu}\right) = 0. \tag{3.75}$$

Thus, ϵ satisfies the following ordinary differential equation (ODE):

$$\ddot{\epsilon} + \Omega^2\epsilon = 0; \quad \Omega^2 = \frac{3\ell^2}{\mu^2 r_0^4} - \frac{f'(r_0)}{\mu} > 0. \tag{3.76}$$

For an attractive force:

$$f(r) = -\frac{K}{r^n}, \tag{3.77}$$

then:

$$f'(r) = n\frac{K}{r^{n+1}}. \tag{3.78}$$

Substituting $f'(r_0) = n\frac{K}{r_0^{n+1}}$ into (3.76), we have:

$$\Omega^2 = \frac{3\ell^2}{\mu^2 r_0^4} + n\frac{K}{\mu r_0^{n+1}} > 0. \tag{3.79}$$

From the equilibrium condition:

$$f(r_0) = -\frac{K}{r_0^n} = -\frac{\ell^2}{\mu r_0^3}, \tag{3.80}$$

we solve for K:

$$K = \frac{\ell^2 r_0^{n-3}}{\mu}. \tag{3.81}$$

Substituting this expression for K into (3.79), we get:

$$\Omega^2 = \frac{3\ell^2}{\mu^2 r_0^4} - n\frac{\ell^2 r_0^{n-3}}{\mu^2 r_0^{n+1}}. \tag{3.82}$$

Simplifying:

$$\Omega^2 = \frac{\ell^2}{\mu^2 r_0^4}(3-n). \tag{3.83}$$

For stability ($\Omega^2 > 0$):

$$n < 3. \tag{3.84}$$

This is the stability requirement for circular orbits. The limiting case $n = 3$ should be treated separately.

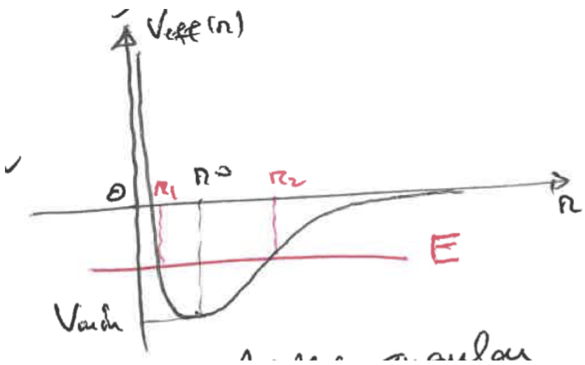


Figure 3.9

3.6. The Kepler Problem: Inverse Square Law of Force

Historically, finding the orbits under a central force problem:

$$f = -\frac{k}{r^2}, \quad V(r) = -\frac{k}{r}, \tag{3.85}$$

was one of the most important problems and deserved special attention because Kepler's famous three laws of planetary motion are based on this problem. Actually, the success of Newton's laws of motion in explaining Kepler's assertions validated Newton's theory of planetary motion, which is based on the inverse square law.
For the effective potential:

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}, \tag{3.86}$$

the path of the particle can be obtained by quadrature using conservation laws of angular momentum and energy:

$$\ell = \mu r^2 \dot{\theta} \implies \theta(t) = \frac{\ell}{\mu} \int \frac{dt}{r^2(t)}. \tag{3.87}$$

Using $r = r(\theta)$, we get:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{\ell}{\mu r^2}\frac{dr}{d\theta}, \tag{3.88}$$

and:

$$\frac{d\theta}{dr} = \frac{\mu r^2}{\ell} \frac{1}{\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right)}}. \tag{3.89}$$

Thus:

$$\theta(r) = \int \frac{\ell\, dr/r^2}{\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right)}}. \tag{3.90}$$

Using the change of variable $u = \frac{1}{r}$, we have:

$$du = -\frac{1}{r^2}dr, \quad dr = -\frac{du}{u^2}. \tag{3.91}$$

Substituting this, we get:

$$\theta(u) = \int \frac{\ell\left(-du/u^2\right)}{\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(1/u)\right)}}. \tag{3.92}$$

$$\Theta(n) = -\int \sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu k}{\ell^2}u - u^2}\, du. \tag{3.93}$$

Using the result from the table of integrals:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right) + \text{const},$$

gives in our case:

$$\Theta(n) = \sin^{-1}\left(\frac{2\mu k}{\ell^2}u + \frac{2\mu E}{\ell^2}\right) + \text{const.} \tag{3.94}$$

Thus:

$$\Theta(n) = \sin^{-1}\left(\sqrt{\frac{2\mu k}{\ell^2}u + \frac{8\mu E}{\ell^2}}\right) + \text{const.} \tag{3.95}$$

Solving for u gives:

$$u = \frac{1}{r} = \frac{\mu k}{\ell^2}\left(1 + \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\sin(\Theta(n) + \Theta_0)\right), \tag{3.96}$$

where Θ_0 is a constant.
To bring our formula closer to that of a conic section, we define:

$$e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}, \quad \alpha = \frac{\mu k}{\ell^2}. \tag{3.97}$$

Select the origin of $\Theta(n)$ so that $\Theta_0 = -\frac{\pi}{2}$. Then:

$$\frac{\alpha}{r} = 1 + e\cos\Theta. \tag{3.98}$$

The trajectory described by this curve depends on the magnitude and sign of e and E, as shown in the following table:

$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$0 < e < 1$	$V_{\text{min}} < E < 0$	Ellipse
$e = 0$	$E = V_{\text{min}} = -\frac{\mu k^2}{2\ell^2}$	Circle

If instead of looking for the trajectory $n(\Theta)$, we would like to find $n(t)$ and $\Theta(t)$, we can proceed as follows:

$$\dot{\Theta} = \frac{\ell}{\mu n^2}, \quad dt = \frac{\mu n^2}{\ell}d\Theta. \tag{3.99}$$

Thus:

$$t = \frac{\mu}{\ell}\int n^2(\Theta)d\Theta, \tag{3.100}$$

which gives $t(\Theta)$ to be evaluated. Substituting this result back, $n(t) = n(\Theta(t)) = n(t)$.

3.9 The Laplace-Runge-Lenz Vector

The Kepler problem is also characterized by the existence of an additional conserved vector quantity, besides the angular momentum. Consider the quantity (recall $\vec{\ell}$ is conserved):

$$\frac{d}{dt}\left(\vec{p} \times \vec{\ell}\right) = \frac{d\vec{p}}{dt} \times \vec{\ell} = m\ddot{\vec{r}} \times \left(\vec{r} \times \dot{\vec{r}}\right). \tag{3.101}$$

Using $\ddot{\vec{r}} = -\frac{K}{r^2}\hat{n}$, we get:

$$\begin{aligned} m\ddot{\vec{r}} \times \left(\vec{r} \times \dot{\vec{r}}\right) &= -\frac{K}{r^2}\hat{n} \times \left(\vec{r} \times \dot{\vec{r}}\right) \\ &= -\frac{K}{r^2}\left[\hat{n}\left(\vec{r} \cdot \vec{r}\right) - (\hat{n} \cdot \vec{r})\vec{r}\right]. \end{aligned}$$

Simplifying:

$$\frac{d}{dt}\left(\vec{p} \times \vec{\ell}\right) = +\frac{d}{dt}\left(\frac{K}{r}\hat{n}\right). \tag{3.102}$$

Thus:

$$\frac{d}{dt}\left(\vec{p} \times \vec{\ell} - \frac{K}{r}\hat{n}\right) = 0. \tag{3.103}$$

Define:

$$\vec{A} = \vec{p} \times \vec{\ell} - \frac{K}{r}\hat{n}, \qquad \frac{d\vec{A}}{dt} = 0, \tag{3.104}$$

so $\vec{A}$ is a conserved vector.
Since $\vec{p} \times \vec{\ell} \perp \vec{\ell}$ and $\hat{n} \cdot \vec{\ell} = 0$, it follows that $\vec{A}$ must lie in the plane of the orbit. If we denote by Θ the angle between $\hat{n}$ and $\vec{A}$, then:

$$\vec{A} \cdot \hat{n} = A\cos\Theta = \hat{n} \cdot \left(\vec{p} \times \vec{\ell}\right) - Kmr. \tag{3.105}$$

Using:

$$\hat{n} \cdot \left(\vec{p} \times \vec{\ell}\right) = \vec{\ell} \cdot \vec{\ell} = \ell^2, \tag{3.106}$$

we get:

$$Ar = \ell^2 - Kmr. \tag{3.107}$$

Simplifying:

$$\frac{1}{r} = \frac{mK}{\ell^2}\left(1 + \frac{A}{mK}\cos\Theta\right). \tag{3.108}$$

Comparing this equation to the conic equation (3.98), we deduce that:

$$e = \frac{A}{mK}. \tag{3.109}$$