$\boxed{5.1}$

a) P15

$$\begin{pmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 103 & 203 & 303 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 101 & 201 & 301 \\ 02 & 202 & 302 \\ 103 & 203 & 303 \end{vmatrix} = 0 \text{ ; since we have 2 identical rows}$$

$$\begin{pmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & t & t^2 \\ t-1 & 1-t & t-t^2 \\ t^2 & t & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & t & t^2 \\ t-1 & 1-t & t-t^2 \\ t^2-1 & 0 & 1-t^2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & t & t^2 \\ t-1 & -(t-1) & -t(t-1) \\ t^2-1 & 0 & -(t^2-1) \end{pmatrix} \Rightarrow (t^2-1)(t-1)\begin{pmatrix} 1 & t & t^2 \\ t-1 & -(t-1) & -t(t-1) \\ t^2-1 & 0 & -(t^2-1) \end{pmatrix}$$

$$\begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{vmatrix} = (t^2-1)(t-1)\begin{vmatrix} 1 & t & t^2 \\ t-1 & -(t-1) & -t(t-1) \\ t^2-1 & 0 & -(t^2-1) \end{vmatrix} = (t^2-1)(t-1)\left[1 - t(-1+t) + t^2(0+1)\right]$$

$$= (t^2-1)(t-1)(1+t-t^2+t^2)$$

$= (t^2-1)(t-1)(t+1)$

$= (t^2-1)(t^2-1) = t^4 - 2t^2 + 1$

b) P16

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 2 & -8 & 10 \\ 3 & -12 & 15 \end{bmatrix} \xrightarrow{C_2 + 4C_1} \begin{bmatrix} 1 & 0 & 5 \\ 2 & 0 & 10 \\ 3 & 0 & 15 \end{bmatrix} \Rightarrow |A| = 0$$

$$|K| = \begin{vmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{vmatrix} = 0 + 1(-12-0) + 3(4-0) = 12 - 12 = 0$$

We could also say: $|K^T| = |-K|$ and $|K^T| = |K|$ for any square matrix of order n

and $|-K| = (-1)^{3}|K|$ (3 → order)

$\Rightarrow |K| = (-1)^n |K| \quad \Rightarrow |K| = -|K| \Rightarrow |K| + |K| = 0 \Rightarrow 2|K| = 0$

$\Rightarrow |K| = 0$

c) P17

$$K = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

taking (-1) from 3^{rd} row: $K = (-1)\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{bmatrix} \xrightarrow{\frac{R_3}{b}} (-1)\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ 1 & \frac{c}{b} & 0 \end{bmatrix}$

$$\xrightarrow{\frac{R_2}{a}} (-1)\begin{bmatrix} 0 & a & b \\ -1 & 0 & \frac{c}{a} \\ 1 & \frac{c}{b} & 0 \end{bmatrix} \xrightarrow{R_2 + R_3} (-1)\begin{bmatrix} 0 & a & b \\ -1 & 0 & \frac{c}{a} \\ 0 & \frac{c}{b} & \frac{c}{a} \end{bmatrix}$$

$\rightarrow$ Expanding along 1^{st} column: $(-1)(-(-1)(c-c)) = 0$

- Example of a skew-symmetric matrix w/ det. $= 1$

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \Rightarrow |A| = 1$$

d) P19

a)

$$U = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$|U| = 1 \times 2 \times 3 = 6$ $\qquad 1(2\times3-0)+4(0)+6(0)=6$

product of diagonal entries

$|U^{-1}||U| = 1 \Rightarrow |U^{-1}| = \frac{1}{|U|} = \frac{1}{6}$ $\qquad |U^2| = |U||U| = 6\times6 = 36$

b)

$$U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$|U| = ad$ $\qquad$ product of diagonal entries

$|U^{-1}| = \frac{1}{|U|} = \frac{1}{ad}$ $\qquad$ -

$|U^2| = |U||U| = ad \times ad - a^2d^2$

e) P24

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 8 & -5 \end{bmatrix} \xrightarrow{2R_3-8R_2} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

Then, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$

$|L| = 1$ $\quad |U| = 3 \times 2 \times -2 = -12$ $\quad 3(2x-2-0)+3(0)+4(0) = -12$

$|A| = |LU| = |L||U| = 1 \times -12 = -12$

$|U^{-1}L^{-1}| = |U^{-1}||L^{-1}| = \frac{1}{|U|} \cdot \frac{1}{|L|} = \frac{1}{1} \cdot \frac{1}{-12} = \frac{1}{-12}$

$|U^{-1}L^{-1}A| = |U^{-1}L^{-1}||A| = -\frac{1}{12} \cdot -12 = 1$

f) P 25

consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad$ (ij entry $= i \times j$)

$\Rightarrow |A| = 4 - 4 = 0$

This is not the case if $A = [1]$.

g) P27

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 0 & b \\ 0 & a & 0 \\ c & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \Rightarrow |A| = abc$$

$$B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \Rightarrow |B| = dabc$$

$$C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix} \Rightarrow |C| = a[(b-a)(c-a) - (b-a)(b-a)] + a(0) + a(0)$$

$= a(b-a)\left[(c-a)-(b-a)\right]$

$c-\cancel{a}-b+\cancel{a}$

$= a(b-a)(c-b)$

$\boxed{5.2}$

a) P3

$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$ Using Big formula: $|A| = \sum \det(P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$; $P = (\alpha, \beta, \dots, \omega)$

$= x0x + xx0 + x00 - xx0 - x0x - x00 = 0$

$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

- Cofactors along r1:

$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} = xx0 - x0x = 0$

$a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} = x0x - xx0 = 0$

$a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} = x00 - x00 = 0$

- rank: since $|A| = 0 \rightarrow$ row are lin. dep. $\rightarrow r(A) \leq 2$

- The 6 terms are:

$a_{11}a_{22}a_{33} = x0x = 0$ $\qquad a_{12}a_{21}a_{33} = x0x = 0$

$a_{11}a_{23}a_{32} = xx0 = 0$ $\qquad a_{13}a_{21}a_{32} = x00 = 0$

$a_{12}a_{23}a_{31} = xx0 = 0$ $\qquad a_{13}a_{22}a_{31} = x00 = 0$

b) P4

Consider A, choose A_{23} (the only nonzero entry). In C2, we are left w/ one possible choice; A_{32}, we choose it. In C1, 2 possible choices; A_{11} & A_{41}. If we choose A_{11}, then we're left w/ A_{44} in C4, and if A_{41}, we're left w/ A_{14} in C4.

$\Rightarrow$ The 2 possible ways: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$$|A| = (1)(1)(1)(1)\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} + (1)(1)(1)(1)\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1+1=0$$
$$= 1-1 = 0$$

Consider B, for C3 we choose A_{23}. C2; A_{32}. C1, A_{11} & A_{41}.

$A_{11} \downarrow$ A_{44} in C4 $\quad$ $A_{41} \hookrightarrow$ A_{14} in C4

$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$ & $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$|B| = (1)(4)(4)(1)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + (2)(4)(4)(2)\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -16+64 = 48$$

c) P13

$C_1 = |0| = 0$

$C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

crossing out 1^{st} r & 1^{st} c

cofactor $C_{11} = (-1)^{1+1} \det M_{11} = (-1)^2 |0| = 0$

$C_{12} = (-1)^3 |1| = -1$

$\Rightarrow C_2 = a_{11}C_{11} + a_{12}C_{12} = 0 + (1)(-1) = -1$

C_3 :

$C_{11} = (-1)^2 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$C_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$C_{13} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\Rightarrow C_3 = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= 0(-1) + (1)(0) + (0)(1) = 0$

C_4 :

$C_{11} = (-1)^2 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 1(0) = 0$

$C_{14} = (-1)^5 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(1) = -1$

$C_{12} = (-1)^3 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)(-1) = 1$

$\Rightarrow C_4 = (0)(0) + (1)(1) + (0)(0) + (0)(-1) = 1$

$C_{13} = (-1)^4 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$

The relation: $C_1 = 0$, $C_2 = -1$, $C_3 = 0$, $C_4 = 1$

Then, $C_3 = 0$ and $C_4 = 1$ By cofactors of r1 then of C1

$= -0$ $\quad$ $= -(-1)$ $\Rightarrow$ The relation is $C_n = -C_{n-2}$

$= -C_1$ $\quad$ $= -C_2$

$= -C_{3-2}$ $\quad$ $= -C_{4-2}$

- C_{10} $\quad n = 10$

$\Rightarrow C_{10} = -C_8$

$= -(-C_6) = C_6$

$= -C_4$

$= -(-C_2) = C_2 = -1$ $\Rightarrow$ $C_{10} = -1$

d) P23

i) $|X| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$; X is 2x2 matrix

$\Rightarrow \begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| - \underbrace{|0||B|}_{=0} = |A||D|$ True

Now, take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\Rightarrow \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} \xrightarrow{C_4 - C_1} \begin{vmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0$

all 0

$|A||D| - |C||B| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (1-1)(0-1) - (1-0)(2-1) = 0 - 1 = -1 \Rightarrow \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$

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ii)

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0$$

and $|A||D| - |C||B| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$ Thus $\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$ fails for the entry of C

iii)

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$AD = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $CB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $AD - CB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$|AD - CB| = 0$

from (ii) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0$, So, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |AD - CB|$ is wrong

$\boxed{5.3}$

a) P8

$C^T = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

cofactor:

$C_{11} = \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 6$ $C_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$ $C_{22} = \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} = 1$

$C_{12} = \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -3$ $C_{21} = -\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = 3$ $C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$C_{31} = \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} = -6$ $C_{32} = -\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 2$ $C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$

$Adj(A) = C^T$

$\Rightarrow$ Matrix of cofactor A : $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ $\quad Adj(A) = \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

$AC^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$Adj(A) = C^T \Rightarrow AC^T = |A| I_{3\times3}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= (1)(6) + (1)(-3) + (4)(0) = 3$

$C_{13} = 0 \Rightarrow$ Multiplying by 4 or 100 $\rightarrow$ no change

b) P14

i) cofactors of L: $C_{ij} = (-1)^{i+j} M_{ij}$

$C_{11} = \begin{vmatrix} c & 0 \\ e & f \end{vmatrix} = cf$ $\quad C_{12} = -\begin{vmatrix} b & 0 \\ d & f \end{vmatrix} = -bf$ $\quad C_{13} = \begin{vmatrix} b & c \\ d & e \end{vmatrix} = be - cd$ $\quad C_{21} = -\begin{vmatrix} 0 & 0 \\ e & f \end{vmatrix} = 0$ $\quad C_{22} = \begin{vmatrix} a & 0 \\ d & f \end{vmatrix} = af$

$C_{23} = -\begin{vmatrix} a & 0 \\ d & e \end{vmatrix} = -ae$ $\quad C_{31} = \begin{vmatrix} 0 & 0 \\ c & 0 \end{vmatrix} = 0$ $\quad C_{32} = -\begin{vmatrix} a & 0 \\ b & 0 \end{vmatrix} = 0$ $\quad C_{33} = \begin{vmatrix} a & 0 \\ b & c \end{vmatrix} = ac$ $\quad C = \begin{bmatrix} cf & -bf & be-dc \\ 0 & af & -ae \\ 0 & 0 & ac \end{bmatrix}$

ii) Cofactors of S:

$C_{11} = \begin{vmatrix} c & e \\ e & f \end{vmatrix} = cf - ce$ $\quad C_{12} = -\begin{vmatrix} b & e \\ d & f \end{vmatrix} = de - bf$ $\quad C_{13} = \begin{vmatrix} b & c \\ d & e \end{vmatrix} = be - cd$ $\quad C_{21} = -\begin{vmatrix} b & d \\ e & f \end{vmatrix} = de - bf$

$C_{22} = \begin{vmatrix} a & b \\ d & f \end{vmatrix} = af - d^2$ $\quad C_{23} = -\begin{vmatrix} a & b \\ d & e \end{vmatrix} = bd - ae$ $\quad C_{31} = \begin{vmatrix} b & d \\ c & e \end{vmatrix} = be - cd$ $\quad C_{32} = -\begin{vmatrix} a & d \\ b & e \end{vmatrix} = bd - ae$

$C_{33} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$ $\qquad C = \begin{bmatrix} cf-ce & de-bf & be-cd \\ de-bf & af-d^2 & bd-ae \\ be-cd & bd-ae & ac-b^2 \end{bmatrix}$

Notice that $C_{12} = C_{21}$ $\quad C_{31} = C_{13}$ $\quad C_{32} = C_{23}$ $\quad \Rightarrow S^{-1}$ is symmetric

iii) $|Q| = 1$ or -1 $\qquad$ Q has cofactor matrix $C = |Q||Q^{-1}| = \pm Q$ also orthogonal

c) P18

Area of the triangle: $\frac{\text{det.}}{2}$

$$A = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = \frac{1}{2}\left[2\begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix} - (1)\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + (1)\begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix}\right]$$

$$= \frac{1}{2}[2(4-5) - 3 + 15] = \frac{1}{2}[-2-3+15] = 5$$

Area of the new region = 5 + area of the new triangle formed

$$= 5 + \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 5 + \frac{1}{2}\left[2\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} - (1)\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 5 \\ -1 & 0 \end{vmatrix}\right]$$

$$= 5 + \frac{1}{2}[2\cdot5 - 5] = 5 + 7 = 12$$