Lecture 3 Exercises

Quantum Information, PSI START Summer 2023

Instructor: Aaron Szasz

TA: Jake Barnett

Exercise #1:

For each of the following two states:

1. What is the probability, when qubit B is measured in the z-basis, of getting each outcome, $|0\rangle$ or $|1\rangle$?
2. Suppose qubit A is *not* measured. After B is measured, for each possible outcome $|0\rangle$ or $|1\rangle$, what is the "collapsed" post-measurement state? [This one is conceptually challenging for the second state—try to come up with a thought experiment to find/justify the answer.]

States:

(a) $(|00\rangle + |11\rangle)/\sqrt{2}$

(b) $(|00\rangle + |10\rangle + |11\rangle)/\sqrt{3}$

Exercise #2:

Consider the state $|\Psi\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_B$, with $|\psi_1\rangle = a|0\rangle + b|1\rangle$, $|\psi_2\rangle = c|0\rangle + d|1\rangle$.

(a) Find $\langle \sigma^z \otimes Id \rangle$ directly, using the full state.

(b) Find ρ_A

(c) Find $\mathrm{Tr}[\rho_A \sigma^z]$

Then repeat all three parts for $|\Psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

Exercise #3:

Consider the two states

$$|\Phi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\Psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

Show that both give

$$\rho_A = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$$

With this example in mind, suppose you know ρ_A. Is it possible to reconstruct the full original $|\Psi\rangle$ that it came from?

Exercise #4:

Write a general $|\Psi\rangle$ in its Schmidt-decomposed form,

$$|\Psi\rangle = (U_A \otimes U_B)(a|00\rangle + b|11\rangle)$$

Show that the reduced density matrix is

$$\rho_A = U_A(a^2|0\rangle\langle 0| + b^2|1\rangle\langle 1|)U_A^\dagger$$

[Hint: First try with $U_A = U_B = \mathrm{Id}$, then just with $U_B = \mathrm{Id}$. Finally, if we do include U_B, in what basis should you measure B?]

Conclude that the eigenvalues of ρ are a^2 and b^2. What does this tell you about the uniqueness of the Schmidt coefficients a and b?

Exercise #5:

Show that the reduced density operator satisfies (1) $\mathrm{Tr}[\rho_A] = 1$ and (2) $\rho_A^\dagger = \rho_A$. For the latter problem, recall that $(a|i\rangle\langle j|)^\dagger = a^*|j\rangle\langle i|$.

Exercise #6:

Consider a generic reduced density matrix

$$\rho_A = \begin{pmatrix} a & ce^{i\theta} \\ ce^{-i\theta} & b \end{pmatrix}$$

where a, b, c, and θ are real numbers.

Compute $\mathrm{Tr}[\rho_A]$, $\mathrm{Tr}[\rho_A\sigma^z]$, $\mathrm{Tr}[\rho_A\sigma^x]$, $\mathrm{Tr}[\rho_A\sigma^y]$.

Then show that

$$\rho_A = \frac{1}{2}\left[\mathrm{Tr}[\rho_A]\cdot\mathrm{Id} + \mathrm{Tr}[\rho_A\sigma^z]\cdot\sigma^z + \mathrm{Tr}[\rho_A\sigma^x]\cdot\sigma^x + \mathrm{Tr}[\rho_A\sigma^y]\cdot\sigma^y\right]$$

Recalling that $\mathrm{Tr}[\rho_A\mathcal{O}_A]$ is the expectation value of $\mathcal{O}_A$, explain how you can use the above equation to experimentally find ρ_A if you have many copies of it in a lab. This process is called "tomography."