Proof of Schmidt decomposition (lecture notes only)

Let $|\Psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$

Organize the coeffs into a 2x2 matrix, $M = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}$

This matrix has a "singular value decomposition" or SVD,

$$M = UDV^\dagger = \begin{pmatrix} | & | \\ u_0 & u_1 \\ | & | \end{pmatrix} \begin{pmatrix} d_0 & 0 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} - v_0^* - \\ - v_1^* - \end{pmatrix}, \quad d_0 \geq d_1 \geq 0$$

(The columns $\begin{pmatrix} | \\ u_0 \\ | \end{pmatrix}$ and $\begin{pmatrix} | \\ u_1 \\ | \end{pmatrix}$ are orthonormal. Likewise for V.)

$$M = d_0 \cdot \begin{pmatrix} | \\ u_0 \\ | \end{pmatrix} (- v_0^* -) + d_1 \cdot \begin{pmatrix} | \\ u_1 \\ | \end{pmatrix} (- v_1^* -) = \sum_k d_k \cdot \begin{pmatrix} | \\ u_k \\ | \end{pmatrix} (- v_k^* -)$$

Let $\begin{pmatrix} | \\ u_0 \\ | \end{pmatrix} = \begin{pmatrix} u_{00} \\ u_{10} \end{pmatrix}$ etc, and $U_A = u_{00}|0\rangle\langle 0| + u_{01}|0\rangle\langle 1| + \ldots$

$U_B = v_{00}^*|0\rangle\langle 0| + v_{01}^*|0\rangle\langle 1| + \ldots$

Then $(U_A \otimes U_B)(d_0|00\rangle + d_1|11\rangle) = d_0(u_{00}|0\rangle + u_{10}|1\rangle)(v_{00}^*|0\rangle + v_{10}^*|1\rangle)$

$+ d_1(u_{01}|0\rangle + u_{11}|1\rangle)(v_{01}^*|0\rangle + v_{11}^*|1\rangle)$

$= d_0 \sum_i u_{i0}|i\rangle \sum_j v_{j0}^*|j\rangle + d_1 \sum_i u_{i1}|i\rangle \sum_j v_{j1}^*|j\rangle$

$= \sum_{ij} \left(\sum_k d_k u_{ik} v_{jk}^* \right) |ij\rangle$

$= \sum_{ij} \left(\sum_k d_k u_{ik} (V^\dagger)_{kj} \right) |ij\rangle = \sum_{ij} c_{ij} |ij\rangle$ ∎