A way not using Gaussian integral to obtain Eq.(24) in Path Integral Day 2's notes

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1 Motivation

(Let $\hbar = 1 = c$)
Eq.(20) in Path Integral Day 2

$$K = \int \mathcal{D}[q]\mathcal{D}[p]exp\{i\int_0^T dt(p_j\dot{q}^j - \mathcal{H})\} \tag{1}$$

$p_j\dot{q}^j - \mathcal{H}$ looks the Lagrangian density $\mathcal{L}$, where p can be written as $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$
Thus

$$\int_0^T dt(p_j\dot{q}^j - \mathcal{H}) = \int_0^T dt\mathcal{L} = S \tag{2}$$

This may implies we can obtain Eq.(24) without using Gaussian integral.

2 Gauge

Let's rewritten K

$$K = \frac{1}{Vol}\int \mathcal{D}[q]\mathcal{D}[p]e^{iS}; \quad Vol = 1 \tag{3}$$

where the Volume "Vol" means we shouldn't be integrating over all field configurations. Since the path integral, as written, sums over all fields, the "Vol" term means that we need to divide out by the volume of the gauge action on field space. But what gauge symmetry here? I will talk it later.
Notice the coordinate transform in Dirac-delta function

$$\int d^n x\delta^n(y(x)) = \sum_{y=0}\frac{1}{det(\frac{\partial y^\mu}{\partial x^\nu})} \tag{4}$$

And in path integral, we face the infinity dimension, so we need use the def. of Faddeev-Popov determinant

$$\int \mathcal{D}[\xi]\delta(p - p^\xi) = \Delta_{FP}^{-1}(p) \tag{5}$$

And the ξ here is related with gauge symmetry and satisfies

$$\int \mathcal{D}[\xi] = Vol \tag{6}$$

insert $\Delta_{FP}(p) \int \mathcal{D}[\xi]\delta(p - p^{\xi}) = 1$ into K

$$K = \frac{1}{Vol} \int \mathcal{D}[\xi]\mathcal{D}[q]\mathcal{D}[p]\Delta_{FP}(p)\delta(p - p^{\xi})e^{iS} = \int \mathcal{D}[q]\mathcal{D}[p]\delta(p - p^{\xi})\Delta_{FP}(p)e^{iS} \tag{7}$$

Use the property of Dirac delta function. Thus

$$K = \int \mathcal{D}[q]\Delta_{FP}(p^{\xi})e^{iS} \tag{8}$$

So now let's check what the gauge symmetry here. It is not hard to see the only way can choose ξ is

$$\frac{\partial p^{\xi}}{\partial p} = 1 \tag{9}$$

thus we can see

$$\Delta_{FP} = \Delta_{FP}^{-1} = 1 \tag{10}$$

Again, it can be seen ξ is the most trivial choice but the unique choice.
Thus we can get Eq.(24)

$$K = \int \mathcal{D}[q]e^{iS} \tag{11}$$

The good news is the trivial choice of ξ here will not cause the grassmann algebra in FP, which means avoid adding the ghost field.