Problem Set 4:

Part I

Problem Set 3.5

2.

C(A) is the line thru $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ (the 2 other columns are linear combinations of this one)

basis

$\Rightarrow$ $\dim(C(A)) = 1$

C(AT) is the line thru $\left\{\begin{bmatrix}1\\2\\4\end{bmatrix}\right\}$ (same reasoning)

basis

$\Rightarrow$ $\dim(C(A^T)) = 1$

reduced A: $\begin{bmatrix}1&2&4\\0&0&0\end{bmatrix}$; R_2-2R_1

It is clear that pivot column is C1 $\Rightarrow$ free var.'s are: x_2, x_3

let $x_2=1$ & $x_3=0$ $\Rightarrow$ $x_1+2=0$ $\Rightarrow$ $x_{s1} = \begin{bmatrix}-2\\1\\0\end{bmatrix}$

let $x_2=0$ & $x_3=1$ $\Rightarrow$ $x_1+4=0$ $\Rightarrow$ $x_{s2} = \begin{bmatrix}-4\\0\\1\end{bmatrix}$

Thus, basis for $N(A) = \left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\end{bmatrix}\right\}$

$\Rightarrow$ $\dim(N(A)) = 2$

Since $R_2 - R_1$ produces a row of zeros; $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ which is the basis of $N(A^T)$

$\dim(N(A^T)) = 1$

Since $C1$ & $C2$ are lin. ind., the basis for $C(B)$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ \& \ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

$\dim(C(B)) = 2$

Since R_1 & R_2 are lin. ind., the basis for $C(B^T)$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \ \& \ \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$

$\dim(C(B^T)) = 2$

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

pivot

x_3 is the free var.

let $x_3 = 1$

$x_1 + 2x_2 + 4 = 0$

$x_2 = 0$

$\Rightarrow x_s = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \longrightarrow$ the basis for $N(B)$

$\dim(N(B)) = 1$

Since R_1 & R_2 are lin. ind., the basis & dim. of $N(B^T)$ is empty.

3.

From R.H.S., we can see that pivot columns are 2 & 4

$\Rightarrow$ basis for $C(A)$ is $\begin{bmatrix}1\\1\\0\end{bmatrix}$ & $\begin{bmatrix}3\\4\\1\end{bmatrix}$

Since R_1 & R_2 (in RHS) are ind. $\Rightarrow$ $C(A^T)$ has the basis

$\begin{bmatrix}0\\1\\2\\3\\4\end{bmatrix}$ & $\begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}$

The free var.'s are x_1, x_3, x_5 (in echelon form)

$$\boxed{\begin{aligned} x_2 + 2x_3 + 3x_4 + 4x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}}$$

① let

$x_1 = 1, x_3 = x_5 = 0$

$x_2 + 3x_4 = 0$

$x_4 = 0$

$x_2 = 0$

$\Rightarrow x_{s1} = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$

② let

$x_3 = 1, x_1 = x_5 = 0$

$x_2 + 2 + 3x_4 = 0$

$x_4 = 0$

$\therefore x_2 = -2$

$\Rightarrow x_{s2} = \begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix}$

③ let

$x_5 = 1, x_1 = x_3 = 0$

$x_2 + 3x_4 + 4 = 0$

$x_4 + 2 = 0$

$\Rightarrow x_4 = -2$

$\Rightarrow x_2 = 2$

$\Rightarrow x_{s3} = \begin{bmatrix}0\\2\\0\\-2\\1\end{bmatrix}$

$\Rightarrow$ Basis for $N(A)$ is $\{x_{s1}, x_{s2}, x_{s3}\}$

Assume that $E=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$ $\qquad |E|=1$

last row of E^{-1} is $\begin{bmatrix}1\\-1\\1\end{bmatrix}$; since last row of echelon form is 0

$a_{13}=1-0=1 \qquad a_{23}=-(1-0)=-1 \quad a_{33}=1-0=1$

$\downarrow$

basis for $N(A^T)$

8.

$$A=\begin{array}{c}\begin{matrix}*\;\;*\;\;*\qquad\quad\end{matrix}\\ \left[\begin{matrix}\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}&\underbrace{\begin{matrix}0&0\\0&0\\0&0\end{matrix}}_{0}\end{matrix}\right]\end{array}$$

$A_{3\times5}$, $\underline{\text{3 lin. ind. Cs}} \Rightarrow \dim(C(A))=3$
$=\dim(C(A^T))$

Recall: $\dim(C(A))=\dim(C(A^T))$

$\underline{\text{rank of } A=3}$

$\Rightarrow \dim(N(A))=n-r=5-3=2$

$\Rightarrow \dim(N(A^T))=m-r=3-3=0$

$$B=\begin{array}{c}\begin{bmatrix}1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\\ \begin{matrix}*\;\;*\;\;*\qquad\qquad\quad\end{matrix}\end{array}$$

$B_{5\times6}$, $\underline{\text{3 lin. ind. Cs}}$ $(*) \Rightarrow \dim(C(B))=3$
$=\dim(C(B^T))$

$\underline{\text{rank of } B=3}$

$\Rightarrow \dim(N(B))=n-r=6-3=3$

$\Rightarrow \dim(N(B^T))=m-r=5-3=2$

$C=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}$ $\qquad C_{3\times2}$, $\underline{\text{all columns}=0} \Rightarrow \dim(C(C))=0=\dim(C(C^T))$

$\underline{\text{rank of } C=0}$

$\Rightarrow \dim(N(C))=2-0=2 \qquad \Rightarrow \dim(N(C^T))=3-0=3$

20.

$\boxed{a}$ If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a sol. to $A\vec{x} = \vec{0}$. Then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4x_1 + 2x_2 + x_4 \\ x_3 + 3x_4 \\ 0 \end{bmatrix} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 4x_1 + 2x_2 + x_4 \\ 8x_1 + 4x_2 + 2x_4 + x_3 + 3x_4 \\ 12x_1 + 6x_2 + 3x_4 + 4x_3 + 12x_4 \end{bmatrix} = \begin{bmatrix} 4x_1 + 2x_2 + x_4 \\ 8x_1 + 4x_2 + x_3 + 5x_4 \\ 12x_1 + 6x_2 + 4x_3 + 15x_4 \end{bmatrix} = \vec{0}$$

From the pattern we can see that there're 2 solutions:

$\vec{b}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ & $\vec{b}_2 = \begin{bmatrix} -1/4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$

Now, evaluate $\vec{r}_1 \cdot \vec{b}_1$ & $\vec{r}_2 \cdot \vec{b}_1^T$ & $\vec{r}_3 \cdot \vec{b}_1^T$

$\begin{bmatrix} 4 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$ & $\begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

& $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$ ✓ perp.

Now, $\vec{r}_1 \cdot \vec{b}_2^T$ & $\vec{r}_2 \cdot \vec{b}_2^T$ & $\vec{r}_3 \cdot \vec{b}_3^T$

$\begin{bmatrix} 4 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/4 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \vec{0}$ & $\begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1/4 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \vec{0}$ & $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/4 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \vec{0}$

✓ also perp.

b)

from (a), $A = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 8 & 2 & 1 & 5 \\ 12 & 6 & 4 & 15 \end{bmatrix}$ $\Rightarrow A^T = \begin{bmatrix} 4 & 8 & 12 \\ 2 & 2 & 6 \\ 0 & 1 & 4 \\ 1 & 5 & 15 \end{bmatrix}$

As we did in (a), suppose that $\vec{y}$ is a sln. to $A^T\vec{y} = \vec{0}$

Notice that 3rd colmn is lin. dep. (lin. comb. of C1 & C2)

$\Rightarrow$ rank of $A^T = 2$

$N(A^T) = 3 - 2 = 1$ $\Rightarrow$ only one ind. sol.

what is it?

$$A^T\vec{y} = \begin{bmatrix} 4y_1 + 8y_2 + 12y_3 \\ 2y_1 + 4y_2 + 6y_3 \\ y_2 + 4y_3 \\ y_1 + 5y_2 + 15y_3 \end{bmatrix} = \vec{0}$$

sln. of this system of equations is $\vec{y}^T = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}^T = [5 \quad -4 \quad 1]$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$$ y^T = 3rd row of E^{-1}

Problem Set 3.6

6.

The 2 ind. clm's are 2 & 3

$\Rightarrow$ Basis for $C(A)$ is $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \text{ \& } \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow \operatorname{Dim}(C(A)) = 2$

$\Rightarrow$ Basis for $C(A^T)$ is $\left\{ \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \end{bmatrix} \text{ \& } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \operatorname{Dim}(C(A^T)) = 2$

$N(A)$: $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 0 & 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \longleftrightarrow R_1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 0 \end{array}\right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right]$$

pivot

free

$x_2 + x_4 = 0$

$3x_3 = 0$

① let $x_1 = 1$ & $x_4 = 0$

$\Rightarrow x_3 = 0 \Rightarrow x_2 = 0$

$x_{s1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

② let $x_1 = 0$ & $x_4 = 1$

$\Rightarrow x_2 = -1 \Rightarrow x_3 = 0$

$x_{s2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ Basis for $N(A)$: $\{x_{s1}, x_{s2}\}$

$\Rightarrow \dim(N(A)) = 2$

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only R_2 of A gives zero rows

$\Rightarrow$ Basis for $N(A^T)$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \dim(N(A^T)) = 1$

Matrix B has only one column; ind.

$\Rightarrow$ Basis of $C(B) = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

$\Rightarrow \text{Dim}(C(B)) = 1$

R1 is ind. $\Rightarrow$ Basis for $C(B^T)$ is 1

$\Rightarrow \text{Dim}(C(B^T)) = 1$

No pivot var $\Rightarrow$ no free var.'s

$\Rightarrow$ Basis for $N(B)$ is $\vec{0}$

$\Rightarrow \dim(N(B)) = 0$ (basis is empty)

$B^T = [1 \quad 4 \quad 5]$ only 1 pivot position (x_2 & x_3 are free var.)

$x_1 + 4x_2 + 5x_3 = 0$

let $x_2 = 1$ & $x_3 = 0 \Rightarrow x_1 = -4$

$\Rightarrow x_{s1} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$

let $x_2 = 0$ & $x_2 = 1 \Rightarrow x_1 = -5$

$\Rightarrow x_{s2} = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ Basis for $N(B^T)$ is $\{x_{s1}, x_{s2}\}$

$\Rightarrow$ Dim. of $N(B^T) = 2$

$\boxed{11.}$

a. $\boxed{r \leq m \quad r \leq n}$

If a matrix has full row rank, then $r = m \Rightarrow A\vec{x} = \vec{b}$ always has a sln.

If $A\vec{x} = \vec{b}$ has no solo., then $r \neq m$

Thus $\boxed{\begin{matrix} r < m \\ r \leq n \end{matrix}}$

b.

$r < m \Rightarrow m - r > 0$

$\Rightarrow$ dim. of left null space is > 0

$\Rightarrow$ there is a nonzero vector in the left null space

$\Rightarrow A^T\vec{y} = \vec{0}$ has sln.'s other than $\vec{y} = \vec{0}$

$\boxed{24.}$

Since the row space of $A = C(A^T)$, $A^T\vec{y} = \vec{d}$ is solvable when $\vec{d} \in$ row space of A

when rank = # of columns & no free var.'s

nullspace of A^T contains $\vec{0}$

since the null space of A^T is the same as $N(A^T)$
left null...

$\Rightarrow$ the sln $\vec{y}$ is unique when the left null space of A contains only $\vec{0}$

28.)

ind. rows are R1 & R2 $\Rightarrow$ rank(B) = 2

$\Rightarrow$ Basis for the row space

(1,0,1,0,1,0,1,0), (0,1,0,1,0,1,0,1)

For B^T, only R1 & R2 ind. $\Rightarrow$ all other R's = 0

$\Rightarrow$ Free var.'s are $\overset{\text{from}}{x_3} - \overset{\text{to}}{x_8}$

① let $x_3 = 1$ and the rest = 0

$$x_1 + x_3 + x_5 + x_7 = 0 \qquad \& \qquad x_2 + x_4 + x_6 + x_8 = 0$$

$$x_1 + 1 + 0 + 0 = 0 \qquad\qquad x_2 = 0$$

$$x_1 = -1$$

$$\Rightarrow \vec{x}_{s1} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $x_4 = 1$ and the rest $= 0$

$x_1 + x_3 + x_5 + x_7 = 0$ & $x_2 + x_4 + x_6 + x_8 = 0$

$x_1 = 0$ $\qquad$ $x_2 = -1$

$$\Rightarrow \vec{x}_{s2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $x_5 = 1$ all others $= 0$

$x_1 + x_3 + x_5 + x_7 = 0$ & $x_2 + x_4 + x_6 + x_8 = 0$

$x_1 = -1$ $\qquad$ $x_2 = 0$

$$\Rightarrow \vec{x}_{s3} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, let $x_6 = 0 \Rightarrow \vec{x}_{s4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

let $x_7 = 0 \Rightarrow \vec{x}_{s5} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $x_8 = 0 \Rightarrow \vec{x}_{s6} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ Basis for left null space of B $\{\vec{x}_{s1}, \vec{x}_{s2}, \vec{x}_{s3}, \vec{x}_{s4}, \vec{x}_{s5}, \vec{x}_{s6}\}$

If $p \neq 0$, we have C

only 2 ind. rows, R1 & R2 $\Rightarrow$ rank$(C) = 2$

$$C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & - & - & - & - & - & - & p \\ & & & \text{4 zeros rows} & & & & \\ p & - & - & - & - & - & - & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

These 2 rows are the basis of the row space

For C^T; the left null space of C is

$(-1,0,0,0,0,0,0,1), (0,-1,0,0,0,0,1,0)$ and columns from 3-6 of I

Basis for null space

$(-1,0,0,0,0,0,0,1), (0,-1,0,0,0,0,1,0), (0,0,-1,0,0,1,0,0)$

The rest are lin. comb. of these

$(k-q,0,0,r-k,q-r,0,0,0), (0,k-q,0,n-k,q-n,0,0,0)$ and

$(0,0,k-q,b-k,q-b,0,0,0)$

If $p = 0 \Rightarrow$ rank$(C) = 1 \Rightarrow$ R1 is the basis for row space

& left null space has $(-1,0,0,0,0,0,0,1)$ & from C2-C7 of I

The other sln's

$(-\frac{n}{r},1,0,0,0,0,0,0), (-\frac{b}{r},0,1,0,0,0,0,0), (-\frac{q}{r},0,0,1,0,0,0,0)$

and $(-\frac{k}{r},0,0,0,1,0,0,0)$

30.

The matrix is 2x2 and rank = 1 $\Rightarrow$ dim of row space & column space = 1

Dim of Null space $= n - r$
$= 2 - 1$
$= 1$

Dim of left null space $= m - r$
$= 2 - 1$
$= 1$

since $r = 1 \Rightarrow$ col. space the set of lin. comb. of the ind. column u, and the row space is the set of lin. comb. of the ind. row v^T

Null space $\perp$ row space

$\Rightarrow$ null space is the plane $\perp v$

left null space $\perp$ column space

$\Rightarrow$ left null space $\perp u$

If rank(B) = 1 $\Rightarrow B = u'v'^T \Rightarrow$ produces the same 4 subspaces

$\Rightarrow u'$ is a multiple of u

& v' is ———— of v

$\Rightarrow B$ is a multiple of A

$\boxed{4.1}$

$\boxed{6.}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & 9 \end{array}\right] \xrightarrow{R_1+R_2-R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 2 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \text{no sol.}$$

$\Rightarrow \vec{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ in the left null space

$$\Rightarrow \vec{y}^T\vec{b} = [1 \quad 1 \quad -1] \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix} = \underline{1}$$

$$\& \ \vec{y}^T A = \underset{1\times 3}{[1 \quad 1 \quad -1]} \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}}$$

$$= [0 \quad 0 \quad 0]$$

$\Rightarrow (\vec{y}^T A)\vec{x} = \underline{0}$ $\qquad 0 \neq 1$ $\qquad$ no sol. $\vec{x}$

$0 = (\vec{y}^T A)\vec{x} = \vec{y}^T\vec{b} = 1$ is not possible

$\boxed{7.}$

$x_1 - x_2 = 1 \ \& \ x_2 - x_3 = 1 \ \& \ x_1 - x_3 = 1$

$$\Rightarrow A\vec{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \vec{x} = \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$y_1 = y_2 = 1 \ \& \ y_3 = -1 \Rightarrow$ for equations to add up to $0 = 1$

$$\Rightarrow \vec{y}^T A\vec{x} = [1 \quad 1 \quad -1] \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \vec{x}$$

$$= [1 \quad 1 \quad -1] \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix} = x_1 - x_2 - x_3 - x_1 + x_3 = 0$$

and

$\vec{y}^T\vec{b} = [1 \quad 1 \quad -1]\begin{bmatrix}1\\1\\1\end{bmatrix} = 1+1-1 = 1$

So, $\vec{y}^T A\vec{x} = \vec{y}^T\vec{b}$ reduces to $0 = 1 \Rightarrow$ no sol.

Since $\vec{y}^T A\vec{x} = \vec{0}$, $\vec{y}$ is $\perp$ to $C(A)$

$N(A^T)$ is the orthogonal complement of $C(A)$

$\Rightarrow \vec{y} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$ in $N(A)$

$\boxed{9.}$

Assume that $A\vec{x} = \vec{b} \Rightarrow \vec{b} \in C(A)$

Then, $A^T\vec{b} = \vec{0} \Rightarrow \vec{b} \in N(A^T)$

Since $C(A) \perp N(A^T) \Rightarrow \vec{y}$ is perp. to itself. That is

$\vec{y}\cdot\vec{y} = 0 \Rightarrow \|\vec{y}\| = 0 \Rightarrow \vec{y} = \vec{0}$

$A\vec{x} = \vec{0}$

$\boxed{11.}$

a.

$C(A)$ is the span $\left\{c\begin{bmatrix}1\\3\end{bmatrix}\right\}$; $c \in \mathbb{R}$

$C(A^T)$ is the span of $\left\{c\begin{bmatrix}1\\2\end{bmatrix}, c \in \mathbb{R}\right\}$; $\quad [3, 6]^T = 3[1, 2]^2$

$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$

$= \left\{c\begin{bmatrix}-2\\1\end{bmatrix}, c \in \mathbb{R}\right\}$

free, set $x_2 = 1$

$A = \left[\begin{array}{cc|c}1 & 2 & 0\\3 & 6 & 0\end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c}1 & 2 & 0\\0 & 0 & 0\end{array}\right]$

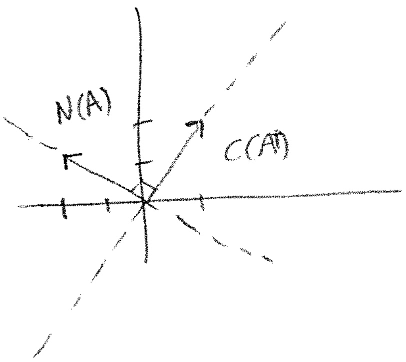
$N(A^T) = \{\vec{x} \mid \begin{bmatrix}1&3\\2&6\end{bmatrix}\vec{x} = \vec{0}\}$

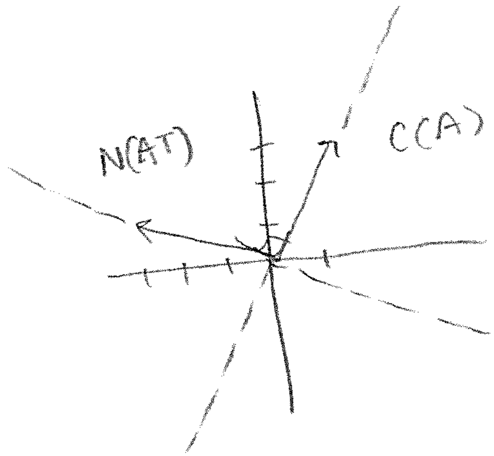
$= \{c\begin{bmatrix}-3\\1\end{bmatrix}, c \in \mathbb{R}\}$

$\left[\begin{array}{cc|c}1&3&0\\2&6&0\end{array}\right] \rightarrow [$

$\dim(C(A^T)) = \dim C(A)$

$\dim(N(A^T) = \dim(N(A))$





b.

$C(B^T) = \text{span}\{\begin{bmatrix}1\\0\end{bmatrix}c, c \in \mathbb{R}\}$

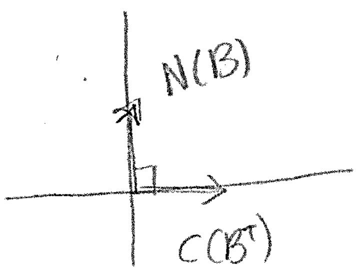
$B\vec{x} = \vec{0} = \left[\begin{array}{cc|c}1&0&0\\3&0&0\end{array}\right] \rightarrow \begin{array}{cc} & \text{free} \\ \end{array}\left[\begin{array}{cc|c}1&0&0\\0&0&0\end{array}\right]$

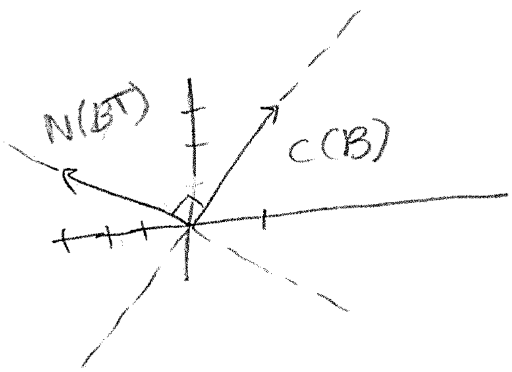
$C(B) = \text{span}\{c\begin{bmatrix}1\\3\end{bmatrix}, c \in \mathbb{R}\}$

$N(B) = \text{span}\{c\begin{bmatrix}0\\1\end{bmatrix}, c \in \mathbb{R}\}$

$B^T\vec{x} = \vec{0} = \begin{array}{cc} & \text{free} \\ \end{array}\left[\begin{array}{cc|c}1&3&0\\0&0&0\end{array}\right]$

$N(B^T) = \text{span}\{c\begin{bmatrix}-3\\1\end{bmatrix}, c \in \mathbb{R}\}$





$\dim(C(B^T)) = \dim C(B)$

$\dim(N(B^T) = \dim(N(B))$

14.

We want a vector $\in C(B)$ & $C(A)$

$A\vec{x} = B\hat{\vec{x}} \Rightarrow A\vec{x} - B\hat{\vec{x}} = \vec{0} \Rightarrow [A \quad B]\begin{bmatrix} \vec{x} \\ -\hat{\vec{x}} \end{bmatrix} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}\vec{x} = \vec{0}$$

free

① let $x_3 = 1$

$x_1 + 2x_2 - 5\cancel{x_3} - 4x_4 = 0$

$x_2 - 1 - x_4 = 0$

$x_4 = 0$

$\Rightarrow x_2 = +1 \quad x_1 = 3 \quad \Rightarrow \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \hat{\vec{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$ & $B\hat{\vec{x}} = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$

The vector is $\begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$

$\boxed{2b.}$

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \Rightarrow N(A) = S^{\perp}$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & \underbrace{1 \quad -1}_{\text{free}} \end{bmatrix} \Rightarrow \begin{array}{l} x_1 + 2x_2 + 2x_3 + 3x_4 = 0 \\ x_2 + x_3 - x_4 = 0 \end{array}$$

① let $x_3 = 1$ & $x_4 = 0$

$x_2 = -1$

$x_1 = 0$

$$\vec{x}_{s1} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

② let $x_3 = 0$ & $x_4 = 1$

$x_2 = 1$ $\quad x_1 = 5$

$$\vec{x}_{s2} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

S^{+} is spanned by $\vec{x}_{s1}$ & x_{s2}

$\boxed{32.}$

a.

Assume that $\vec{r}$ & $\vec{n}$ are bases for $C(A^T)$ & $N(A)$. then $\vec{r} \cdot \vec{n} = \vec{0}$, as $C(A^T) \perp N(A)$. The same applies for $\vec{l}$ & $\vec{c}$

b.

A could be $b\vec{c}\vec{r}^T$, $b \neq 0$

columns of A are multiples of $\vec{c}$, so we have correct $C(A)$

& rows will be multiples of $\vec{r}$, we have correct $C(A^T)$

$N(A)$ will also be correct; $C(A^T) \perp N(A)$

$N(A^T)$ $\quad C(A) \perp N(A^T)$

33.

a) $A_{4\times 4}$

$C(A^T) \perp N(A)$ ✓ & $C(A) \perp N(A^T)$

$\Rightarrow \{\vec{r}_1, \vec{r}_2\}, \{\vec{n}_1, \vec{n}_2\}, \{\vec{c}_1, \vec{c}_2\}, \{\vec{l}_1, \vec{l}_2\}$ should be lin. ind.

and $\vec{r}_i \cdot \vec{n}_j = \vec{0}$ as well as $\vec{c}_i \cdot \vec{l}_j = \vec{0}$ ✓

b) use the vectors to construct a 4 x 4 matrix A, all satisfies conditions in (a). Then,

$$A = [\underbrace{\overset{2\times 2}{\vec{c}_1 \quad \vec{c}_2}}_{\text{form basis for } C(A)} \quad \underbrace{\overset{2\times 2}{\vec{n}_1 \quad \vec{n}_2}}_{\text{form basis for } N(A)}]$$

same applies for $\vec{r}$ & $\vec{l}$

If conditions in (a) are satisfied, then we will have the correct 4 fundamental subspaces

Part II

1.

a)

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_1-R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

3 pivot columns → rank = 3 = # of unknowns

They're lin. ind. ✓

b)

# of vectors > # of unknowns

4 > 3

⇒ some vectors of the given set are lin. dep.

c) The set is the basis of $\mathbb{R}^n$, which are always lin. ind.

2.

a) $\begin{bmatrix} 3 & 0 & -6 & 0 \\ 1 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{-3R_2+R_1} \begin{bmatrix} 3 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

* pivot column

A 2x4 ⇒ $C(A) \in \mathbb{R}^2$

C(A) is the span of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ↳ basis of C(A)

Since we have one pivot c ⇒ dim C(A) = 1

$$AT = \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -6 & -2 \\ 0 & 0 \end{bmatrix}$$

C2 is multiple of C1

$\Rightarrow$ only one pivot column $\Rightarrow$ dim $C(AT) = 1$

$\Rightarrow$ basis of $C(AT) = \begin{bmatrix} 3 \\ 0 \\ -6 \\ 0 \end{bmatrix} \in \mathbb{R}^4$

b) Third row = first row, so last row will be 0 0 0 0

$\Rightarrow$ 2 pivot columns (1st & 2nd)

$\Rightarrow$ dim $(C(A)) = 2$

$\Rightarrow$ basis of $C(A) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$

If we consider AT, last column will be 0's

$\Rightarrow$ 2 pivot columns (1st & 2nd) $(R_4 + 2R_1)$

$\Rightarrow$ dim $(C(AT)) = 2$

$\Rightarrow$ basis of $C(AT)$ is $\begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^4$

c)

3 ind. columns $\Rightarrow$ 3 pivot clms $\Rightarrow$ dim $C(A) = 3$

$\Rightarrow$ basis $C(A)$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$C(A) \in \mathbb{R}^3$

Similarly for $C(AT)$, 3 ind. rows $\Rightarrow$ dim $C(AT) = 3$

$\Rightarrow$ basis $C(AT)$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$C(AT) \in \mathbb{R}^4$

d) $C(A) \in \mathbb{R}^4$ $C(A^T) \in \mathbb{R}^3$

same reasoning as previous

$\dim C(A) = 3$

basis of $C(A)$ és $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim C(A^T) = 3$

basis of $C(A^T)$ és $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$