Lecture 1 : Vectors and Linear Combinations

$\mathbb{R}$ → the set of all real numbers

(V_1, V_2) → A pair of real #s

$\mathbb{R}^2$ → the set of all pairs of real #s

$\mathbb{R}^3$ → the set of all triplets (, ,) of real #s

$\mathbb{R}^n$ → the set of all vectors with n components or entries

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Vectors:

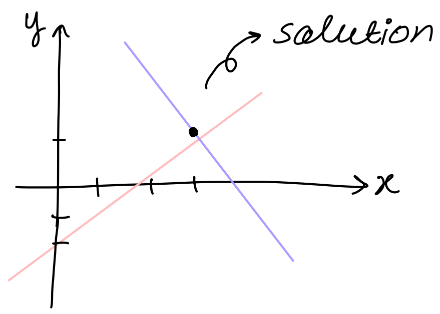
III The geometry of linear equations :

The fundamental problem of LA is to solve n linear equations in n unkowns;

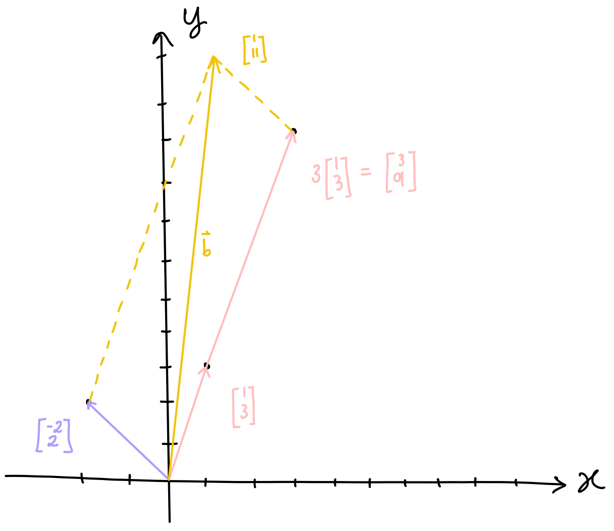
examples:

1) Two equations $x - 2y = 1$
Two unknowns $3x + 2y = 11$

• Row Pic.



• Coulmn Pic.



$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

• Matrix Pic.

$$\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

row pic. corresponds to dot products
(row of A) $\cdot \vec{x}$

$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

column pic. corresponds to $A\vec{x}$ represented as a linear combinations of the coulmns of A

2) Three equations
Three unknowns

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$

- **Row Pic.**

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$

each equation represents a plane

intersection of the 2 planes is a line

Intersection of a plane and a line is a point

- **Coulmn Pic.**

$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\underbrace{\begin{bmatrix}3\\2\\1\end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix}6\\4\\2\end{bmatrix}}_{\vec{b}}$$

$$\vec{b} = 2 \cdot \vec{V}$$

$$0\begin{bmatrix}1\\2\\6\end{bmatrix} + 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

Matrix form $A\vec{x} = \vec{b}$

$$\begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -1 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\9\end{bmatrix}$$

$\vec{b}$ is a linear comb. of the coulmns of A and $\vec{x}$ tells us how to combine the caulmns of A

Linear Independence

$$A\vec{x} = \vec{b} \Rightarrow A^{-1}\vec{b} = \vec{x}$$

* Not every matrix A has an inverse A^{-1}; not every problem or system of linear equations is invertable.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

The coulmns of A <u>are independent,</u> which means that ONLY the zero lin. comb. produces the zero vector, i.e.,

$$\underbrace{x_1}_{\text{zero}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \underbrace{x_2}_{\text{zero}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \underbrace{x_3}_{\text{zero}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ A has an inverse, and so it's possible to find an $\vec{x}$ for every $\vec{b}$ so that $A\vec{x} = \vec{b}$

The space of all lin. comb. of the coulmns of A describe the entire 3D space $\mathbb{R}^3$.

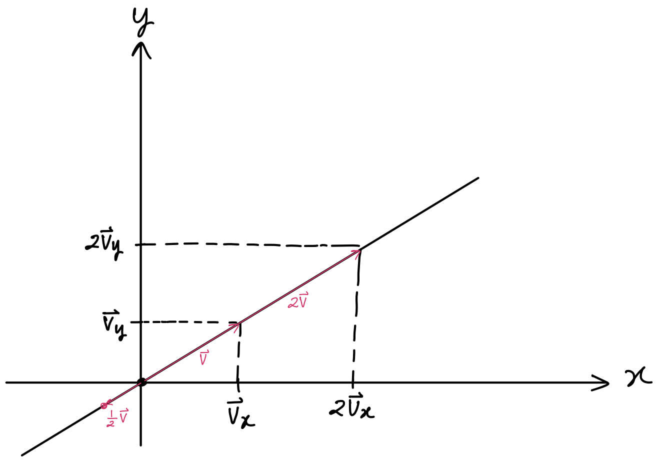
* If the linear combinations of the coulmn vectors fill the xy-plane (or space, in the 3d case), then A (in $A\vec{x} = \vec{b}$) is non-singular. Otherwise, A is a singular matrix; which means its coulmn vectors are linearly dependent. All lin. combs. of these vectors lie on a point or line (in 2d) or on a point, line, or plane (in 3d).

III Vector operations:

① Scalar Multiplication of a vector

$\vec{v} \in \mathbb{R}^2$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\lambda \in \mathbb{R}$

$$\Rightarrow \lambda\vec{v} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \in \mathbb{R}^2$$

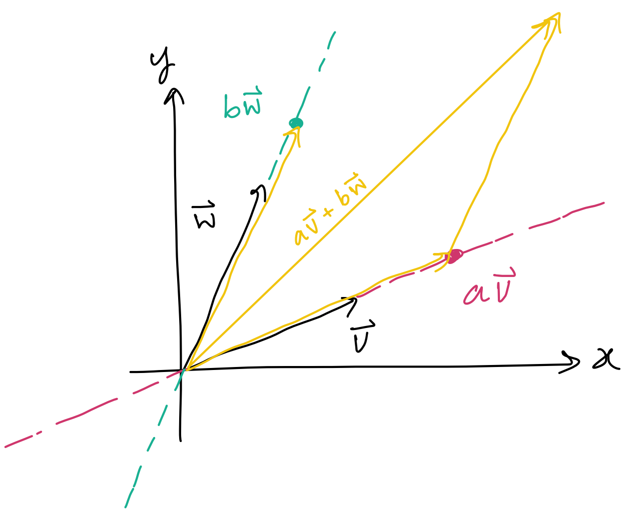


$\lambda\vec{v}$ represents a line through the origin with the same orientation of $\vec{v}$

② Addition of vectors

2 vectors: $\vec{v}, \vec{w} \in \mathbb{R}^n$

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$



$a\vec{v} + b\vec{w}$ is a lin. comb. of $\vec{v}$ with $\vec{w}$ with scalars a & b

Q: in $\mathbb{R}^2$, w/ any $\vec{v}, \vec{w} \in \mathbb{R}^2$, can I always generate $\mathbb{R}^2$?

No ;

① $\vec{v} = \vec{w} = \vec{0}$ $\qquad a\vec{v} + b\vec{w} = \vec{0} \quad \neq \mathbb{R}^2$

② $\vec{v} = \vec{0},\ \vec{w} \neq \vec{0}$ $\qquad a\vec{v} + b\vec{w}$ is the line in $\mathbb{R}^2$ $b\vec{w}$

③ $\vec{v} = \lambda\vec{w}$ $\qquad a\vec{v} + b\vec{w} = a\lambda\vec{w} + b\vec{w} = (a\lambda + b)\,\vec{w}$ line in $\mathbb{R}^2$

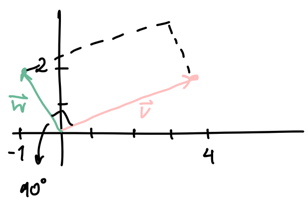
- **Dot product (inner product)**

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$\vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_2$

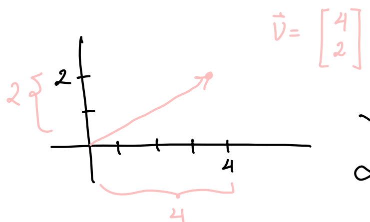
e.g. $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\qquad \vec{v} \cdot \vec{w} = -4 + 4 = 0$

$\Rightarrow$ $\vec{v}$ and $\vec{w}$ are perpendicular, i.e., $\theta = 90°$

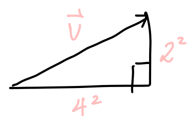


- **Length of a vector**

$\vec{v} \cdot \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1^2 + v_2^2$



Pythagoras theorem gives the length of $\vec{v}$



length of $\vec{v}$ is given by

$$\boxed{\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}}$$

For higher dimentions :

Given 2 vectors $\vec{v}$ & $\vec{w}$ in $\mathbb{R}^n$, their dot product is given by

$\vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + \dots + v_n w_n$

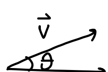
The length of $\vec{v}$ is given by

$\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{v \cdot v}$

- Connecting length of a vector & dot product

Cosine formula:

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

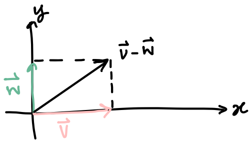


$\vec{w}$

Two vectors are perpendicular (orthogonal; $\theta = 90°$) if and only if their dot product is 0.

Proof

Suppose that $\vec{v}$ & $\vec{w}$ are perpendicular



The Pythagoras Law $a^2 + b^2 = c^2$

$\Rightarrow \|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} - \vec{w}\|^2$

In 2D:

$$(\cancel{v_1^2} + \cancel{v_2^2}) + (\cancel{w_1^2} + \cancel{w_2^2}) = (v_1 - w_1)^2 + (v_2 - w_2)^2$$

$$= \cancel{v_1^2} - 2v_1w_1 + \cancel{w_1^2} + \cancel{v_2^2} - 2v_2w_2 + \cancel{w_2^2}$$

$$0 = (-2v_1w_1 - 2v_2w_2) \,/ -2$$

$$\Rightarrow 0 = v_1w_1 + v_2w_2$$

* Note: The zero vector $\vec{0}$ is perpendicular to every vector $\vec{w}$, because $\vec{0} \cdot \vec{w}$ is always zero.

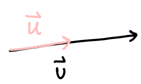
- Unit Vectors

Def.: A unit vector $\vec{u}$ is a vector whose length equals 1.

Equivalently, we have $\boxed{\vec{u} \cdot \vec{u} = 1}$

e.g. $\vec{u} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \Rightarrow \vec{u} \cdot \vec{u} = (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = 1$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$



$\vec{u} = \frac{1}{2}\vec{v} \Rightarrow \|\vec{u}\| = |\frac{1}{2}| \cdot \|\vec{v}\| = \frac{1}{2}\|\vec{v}\| \Rightarrow \|\vec{v}\| = 2$

Given a vector $\vec{v}$, we can find a unit vector $\vec{u}$ in the same direction as $\vec{v}$, as follows,

$$u = \frac{1}{\|\vec{v}\|} \cdot \vec{v} \quad \text{or} \quad \vec{u} = \frac{-1}{\|\vec{v}\|} \cdot \vec{v}$$

- SCHWARZ INEQUALITY

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

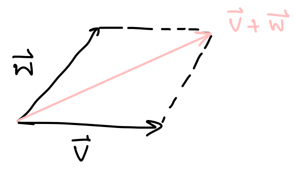
Remember, $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

$|\cos\theta| \leq 1$ —Always—

$\Rightarrow |\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\| \cdot 1$

- TRIANGLE INEQUALITY

$$\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$$



The longest side of the tringle is shorter than the sum of the 2 other lengths

Exercises on the geometry of linear equations

Problem 1.1: (1.3 #4. *Introduction to Linear Algebra:* Strang) Find a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent)(dependent).

The three vectors lie in a __________. The matrix W with those columns is *not invertible*.

Problem 1.2: Multiply: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Problem 1.3: True or false: A 3 by 2 matrix A times a 2 by 3 matrix B equals a 3 by 3 matrix AB. If this is false, write a similar sentence which is correct.

$A_{3\times2} \cdot B_{2\times3} = AB_{3\times3}$

1.1:

sol. we might observe that $\vec{w}_1 + \vec{w}_3 - 2\vec{w}_2 = \vec{0}$, or we can solve

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow 0x_3 = 0 \Rightarrow \infty$ many slns.

let $x_3 = 2 \Rightarrow -3x_2 - 6(2) = 0$

$-3x_2 = 12$

$x_2 = -4$

$\Rightarrow x_1 + 4(-4) + 7(2) = 0 \quad \Rightarrow x_1 = 2$

- The coulmn vectors of W are dependent.
- The 3 vectors lie in a plane.

1.2:

sln. $3\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 - 4 + 0 \\ 6 - 0 + 3 \\ 12 - 2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$

1.3:

sln. True. $A_{m \times n} \cdot B_{n \times p} = AB_{m \times p}$

Lecture 2: Multiplication of Matrices & Inverses

Problem 1) Suppose $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

a) what space do u & v belong to?

$\mathbb{R}^2$ because they have 2 components

b) Are u and v perpendicular?

$u^T v = [4 \quad 2] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0$ Yes, they are.

c) what is the magnitude of v?

$\|v\| = \sqrt{v^T v} = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

d) what is the unit vector along the direction of v?

$$e_v = \frac{v}{\|v\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}$$

Problem 2: Express $u = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ as a lin. comb. of

$v = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$xv + yw = u \Rightarrow x\begin{bmatrix} 6 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

system of lin. equations:

$6x + 2y = 4$

$2x + 4y = 4$

in the form of $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \xrightarrow{\text{Augmented matrix}} \left[\begin{array}{cc|c} 6 & 2 & 4 \\ 2 & 4 & 4 \end{array}\right]$$

$R_2 - R_1/3 \longrightarrow R_3$

$$\left[\begin{array}{cc|c} 6 & 2 & 4 \\ 0 & \frac{10}{3} & \frac{8}{3} \end{array}\right] \Rightarrow \frac{10}{3}y = \frac{8}{3} \Rightarrow y = \frac{8}{\cancel{3}} \cdot \frac{\cancel{3}}{10} = \frac{4}{5}$$

$$\Rightarrow 6x + 2\left(\frac{4}{5}\right) = 4 \Rightarrow 6x = 4 - \frac{8}{5} = \frac{12}{5} \Rightarrow x = \frac{2}{5}$$

Rules for Matrix operations :

Size

$\underbrace{m}_{\text{rows}} \times \underbrace{n}_{\text{caulmns}}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]$$

a_{ij} → entry of A in the i^{th} raw and j^{th} caulmn

MATRIX ADDITION

- we can add matrices provided they have the same size.
- addition is term by term
- $A + B = B + A$ (commutative law)
- $c(A+B) = cA + cB$ (distributive law)
- $A + (B + C) = (A+B) + C$ (assosiative law)

MATRIX MULTIPLICATION

The product AB is defined if A has n caulmns and B has n raws.

- $AB \neq BA$

  $(m \times n)(n \times k) = m \times k$

  $(n \times k)(m \times n$ = possible only if $k = m$

- $A(B + C) = AB + AC$ from the left
- $(B + C)A = BA + CA$ from the right

} distributive law

- $A(BC) = (AB)C$

This multiplication can be done in 4 ways:

1) Standard (rows times columns) -dot product-

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

2) Columns

$$\underbrace{\begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}_{m \times n}}_{A} \underbrace{\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}_{n \times l}}_{B} = \underbrace{\begin{bmatrix} | \\ c_j \\ | \end{bmatrix}_{m \times l}}_{C}$$

$$\begin{bmatrix} c_j \end{bmatrix} = b_{1j}\begin{bmatrix} c_1 \end{bmatrix} + b_{2j}\begin{bmatrix} c_2 \end{bmatrix} + \cdots + b_{nj}\begin{bmatrix} c_n \end{bmatrix}$$

3) Rows

This multiplication can be

$$\underbrace{\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix}_{m \times n}}_{A} \underbrace{\begin{bmatrix} — b_1 — \\ — b_2 — \\ \vdots \\ — b_n — \end{bmatrix}_{n \times l}}_{B} = \begin{bmatrix} — c_i — \end{bmatrix}_{m \times l}$$

4) Column times row

$$AB = \sum_{k=1}^{n} \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1} & \cdots & b_{kn} \end{bmatrix}$$

* Solving $A\vec{x} = \vec{b}$ → Gauss Elimination → Back Substitution

$$[A|b] \to [\;\;|c]$$

Inverses:

• defenition and properties

$\boxed{\text{Def.}}$ A matrix A is invertable if & only if there exists a matrix A^{-1} given that $A \cdot A^{-1} = I$ and $A^{-1}A = I$.

Properties:

1) The inverse exists if and only if elimination produces n pivots (A is an $n \times n$ matrix).

2) If an inverse exists, it is unique.

$\boxed{\text{Proof}}$ (By contradiction)

Assume that a matrix A has 2 inverses, B and C.

$$BAC = B\underbrace{(AC)}_{I} = B$$
$$\parallel$$
$$\underbrace{(BA)}_{I}C = C$$

then $B = C$

3) $A\vec{x} = \vec{b}$

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b} \Rightarrow x = A^{-1}\vec{b}$$

4) Suppose there's a non-zero vector $\vec{x}$, such that $A\vec{x} = \vec{0}$. Then A cannot have an inverse.

No matrix can bring zero back to x.

If A is invertable, then $A\vec{x} = \vec{0}$ can only have the zero solution $\vec{x} = A^{-1}\vec{0} = \vec{0}$.

• The inverse of a product AB

Assume that 2 matrecis A, B wich are invertable. Then the matrix $C = AB$ is also invertable. Its inverse

$$C^{-1} = B^{-1}A^{-1} = (AB)^{-1}$$

$\boxed{\text{Proof}}$

$$C \cdot C^{-1} = I \quad \text{and} \quad C^{-1}C = I$$

$$A\underbrace{BB^{-1}}_{I}A^{-1} = AA^{-1} = I$$

$$B^{-1}\underbrace{A^{-1}A}_{I}B = B^{-1}B = I$$

5) A 2 by 2 matrix is invertable if and only if the determinant $\neq 0$.

6) A diagonal matrix has an inverse provided no diagonal entries are zero.

If $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$

Lecture 3: Elimination

Elimination: $A\vec{x}=\vec{b} \xrightarrow{\text{Gauss elimination}} U\vec{x}=\vec{c} \rightarrow$ back substit.

The technique most commonly used by computer software to solve systems of lin. equations. It finds a soln. $\vec{x}$ to $A\vec{x}=\vec{b}$ whenever A is invertable.

We start w/ an invertable matrix A and we should end up with an upper triangular matrix U. Thus, we transform $A\vec{x}=\vec{b}$ into $U\vec{x}=\vec{c}$. The last equation is easy to solve by back substitution.

* pivots may not be 0. If there's a 0 in the pivot position, we must exchange that row w/ one below to get a non-zero value in the pivot position.

* If there's a 0 in the pivot position and no non-zero value below it, then <u>the matrix A is not invertable</u>. Elimination cannot be used to find a unique solution to the system of equations - it doesn't exist.

In other words, elimination is a systematic way to solve lin. equ..

$$\left.\begin{aligned} x-2y&=1 \\ 3x+2y&=11 \end{aligned}\right\} \text{ Augumented matrix } [A|\vec{b}]: \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right]$$

$$\underbrace{\begin{matrix}1 & -2\\ 3 & 2\end{matrix}}_{A} \qquad \begin{matrix}1\\11\end{matrix} \rightarrow \vec{b}$$

Elimination: produce an upper triangular system, so that the non-zero elements are within the triangular and everything else is zero pivot (cannot be zero)

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right] \Rightarrow \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

U $\quad \vec{c}$

"To solve n equations, we want n pivot"

$$A\vec{x} = \vec{b} \xrightarrow{R_2 - 3R_1 \to R_2} U\vec{x} = \vec{c}$$

Now, we solve $U\vec{x} = \vec{c}$ using back substitution

$y = 1 \Rightarrow x - 2(1) = 1 \Rightarrow x = 3$

Break Down Elimination

1) Permanent failure with no sol.

$x - 2y = 1$

$3x - 6y = 11$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 11 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 8 \end{array}\right] \to 0 \cdot x + 0 \cdot y = 8$$

impossible

- the system has no solution.
- the system is inconsistent

2) Failure with infinitely many solutions

$x - 2y = 1$

$3x - 6y = 3$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 3 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \end{array}\right] \to 0 = 0$$

for any choise of y, we can find an x.

The solutions to $x \cdot 2y = 1$ give a line in $\mathbb{R}^2$.

3) Temporary failure (zero in pivot). A row exchange provide 2 pivots.

$0x + 2y = 4$

$3x - 2y = 5$

$$\left[\begin{array}{cc|c} 0 & 2 & 4 \\ 3 & -2 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 3 & -2 & 5 \\ 0 & 2 & 4 \end{array}\right]$$

$$\begin{aligned} 3x - 2y &= 5 \\ 2y &= 4 \end{aligned} \Rightarrow y = 2 \Rightarrow 3x = 5 + 4 = 9 \Rightarrow x = 3$$

* If we start with a matrix A (square), if elimination produces pivots in every row and every column, then A is invertable.

Otherwise, A is NOT invertable (A is singular) and the columns of A are linearly dependent.

Elimination Using Matrices

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow[R_3 + R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right] \end{array}$$

Elementary row-operations can be represented as matrix multiplication using elementary matrices.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

The elimination matrix used to eliminate the entry in row m column n is denoted E_{mn}. The 3 elimination steps leading to U: $E_{32}(E_{31}(E_{21}A)) = U$.

- Permutation matrix exchanges 2 rows of a matrix; for example: $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The 1st & 2nd rows of PA are the 2nd & 1st rows of A.

* Note: $PA \neq AP$

Lecture 4: Gauss-Jordan Elimination

Gauss-Jordan Algorithm: to calculate A^{-1}

$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, $\quad A^{-1}$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad a, b, c, d \text{ ?}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\underbrace{\begin{matrix} 1 & 3 \\ 2 & 7 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}\right] \longrightarrow \left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \middle| \; A^{-1} \right]$$

$$\left[\begin{matrix} 1 & 3 \\ 2 & 7 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{matrix} 1 & 3 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ -2 & 1 \end{matrix}\right] \xrightarrow{R_1 - 3R_2} \left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} 7 & -3 \\ -2 & 1 \end{matrix}}_{A^{-1}}\right]$$

- Why does Gauss-Jordan work?

$$[A \mid I] \xrightarrow{GE} [I \mid ?]$$

$(E_{12}E_{21})A = I \Rightarrow E_{12}E_{21} = A^{-1}$

• <u>Properties:</u> A, B of the same size

A^{-1}, B^{-1} exist

1. $(AB)^{-1} = A^{-1}B^{-1}$

$$(AB)(B^{-1}A^{-1}) = A\underbrace{BB^{-1}}_{I}A^{-1}$$

$$= AIA^{-1}$$
$$= AA^{-1} = I$$

2. $(A^T)^{-1} = (A^{-1})^T$ inversion & transportation are commutative operations

(a) $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

(b) $(AB)^T = B^T A^T$

(c) $(A^{-1}A)^T = I^T = I$

$$(A^{-1}A)^T \overset{(b)}{=} A^T(A^{-1})^T = I \Rightarrow (A^T)^{-1} = (A^{-1})^T$$

$\boxed{A_{3\times3}}$:

$$E_{32}E_{31}E_{21}A = U$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(r_2)_{new} = (r_2)_{old} - m_{21}r_1$

$(r_2)_{old} = (r_2)_{new} + m_{21}r_1$

$$\underbrace{(E_{32}E_{31}E_{21})^{-1}(E_{32}E_{31}E_{21})}_{I} A = \underbrace{(E_{32}E_{31}E_{21})^{-1}}_{L} U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

Factorization into $A = LU$

- Inverse of a product

$(AB)^{-1} = B^{-1}A^{-1}$

- Transpose of a product

- $(AB)^T = B^T A^T$

- For any invertable matrix A, the inverse of A^T is $(A^{-1})^T$

- $A = LU$

A is an arbitrary $m \times n$ matrix

$$A \xrightarrow{E_1} \xrightarrow{E_2} \cdots \xrightarrow{E_3} U$$

$$\underbrace{(E_k^{-1}(E_k)}_{I} \cdots E_2 E_1) A = E_k^{-1} U$$

$$E_k^{-1} \cdots E_2 E_1 A = E_k^{-1} U$$

$$A = (E_1^{-1} E_2^{-1} \cdots E_k^{-1})\, U = LU$$

lower - triangular

invertible
(takes us from U to A)

- Using $A = LU$ to solve $A\vec{x} = \vec{b}$

$$A\vec{x} = \vec{b}$$

$$(L\underbrace{U)\vec{x}}_{\vec{y}} = \vec{b}$$

1) $L\vec{y} = \vec{b}$

2) $U\vec{x} = \vec{y}$

Lecture 5:

Permutations (P matrices)

Multiplication by a P swaps the rows of a matrix. We use Ps when applying elimination to move zeros out of pivot positions. $A = LU$ then becomes $PA = LU$.

Recall that $P^{-1} = P^T$, i.e. that $P^TP = I$

Transposes

- $A^T_{ij} = A_{ji}$
- A matrix A is <u>symmetric</u> if $A^T = A$
- Given any matrix R, the product R^TR is always symmetric;

$(R^TR)^T = R^T(R^T)^T = R^TR$

* Note $(R^T)^T = R$

Vector spaces

<u>Def.</u>: A <u>vector space</u> V is a set on which there are defined 2 operations; addition and scalar multiplication, such that the following properties hold:

1. $\vec{v} + \vec{w} = \vec{w} + \vec{v} \quad \forall \; \vec{v}, \vec{w} \in V;$ commutativity

2. $\vec{v} + (\vec{w} + \vec{u}) = (\vec{v} + \vec{w}) + \vec{u} \quad \forall \; \vec{v}, \vec{w}, \vec{u} \in V;$ associativity

3. $\lambda, \mu \in F \; , \; \vec{v} \in V \Rightarrow (\lambda\mu)\vec{v} = \lambda(\mu\vec{v});$ --- --- ---

4. $\exists \; \vec{0} \in V$ s.t. $\forall \; \vec{v} \in V, \; \vec{v} + \vec{0} = \vec{v};$ additive identity

5. $\exists \; -\vec{v} \in V$ s.t. $\forall \; \vec{v} \in V, \; \vec{v} + (-\vec{v}) = \vec{0};$ additive inverse

6. $1\vec{v} = \vec{v} \quad \forall \in V;$ multiplicative identity

7. $\lambda \in F \; , \; \vec{v}, \vec{w} \in V \Rightarrow \lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ disturbeivety

8. $\lambda \in F, \; \vec{v} \in V \Rightarrow \lambda\vec{v} \in V.$

- Real Vector Space: when scalars are real #s.
- Complex Vector Space: When scalars are all complex #s.
- A "field" F is – informally – a number system in which every non-zero vector element has a multiplicative inverse. We say that V is a vector space over a field F. It should be noted, however, that F is either the field $\mathbb{R}$ of real #s or the field $\mathbb{C}$ of complex #s.

Examples:

1. is $\mathbb{R}^2$ a vector space?

$\vec{v}, \vec{u} \in \mathbb{R}^2 \qquad a, b \in \mathbb{R}$

$$a\vec{v}+b\vec{u} = a\begin{bmatrix}v_1\\v_2\end{bmatrix} + b\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}av_1+bu_1\\av_2+bv_2\end{bmatrix} \in \mathbb{R}^2$$

$\Rightarrow$ $\mathbb{R}^2$ is a vector space in $\mathbb{R}^2$

(2) is $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ a vector space in $\mathbb{R}^2$?

$a, b \in \mathbb{R} \Rightarrow a\begin{bmatrix}0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \in \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

$\Rightarrow$ $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is a vector space in $\mathbb{R}^2$, and it's <u>the smallest</u> one.

* a vector space contained in another vector space. e.g., $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is a subset of $\mathbb{R}^2$.

(3) is $\mathbb{R}^2 \setminus \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ a vector space in $\mathbb{R}^2$?

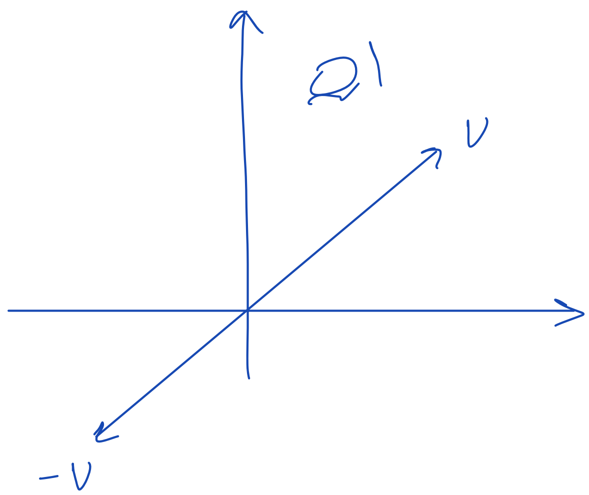
$\vec{v} \in \mathbb{R}^2 / \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ such that $v_1 \neq 0$
$v_2 \neq 0$

$0v = \begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \notin \mathbb{R}^2 / \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

$\Rightarrow \mathbb{R}^2 / \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is not a vector space

$\Rightarrow$ the zero of $\mathbb{R}^n$ is always included in a vector space in $\mathbb{R}^n$.

(4) is the 1st quadrant of $\mathbb{R}^2$, i.e., $Q_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0 \cap y \geq 0 \right\}$ a vector space?



$-v$ isn't in the 1st quadrant

$\Rightarrow$ Q1 is not a vector space

(5) is the set of real numbers $\mathbb{R}$ a vector space?

$\alpha, \beta \in \mathbb{R} \qquad a, b \in \mathbb{R} \quad \Rightarrow \alpha a + \beta b \in \mathbb{R}$

$\Rightarrow \mathbb{R}$ is a vector space.

(6) is $[-1, 1]$ a vector space in $\mathbb{R}$?

$\frac{1}{2} \in [-1,1]$, $10 \cdot \frac{1}{2} = 5 \notin [-1,1]$

$\Rightarrow [-1,1]$ not a vector space

⑦ is the line $y = x$ a vector space in $\mathbb{R}^2$?

is $Z = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x \right\}$ a vector space?

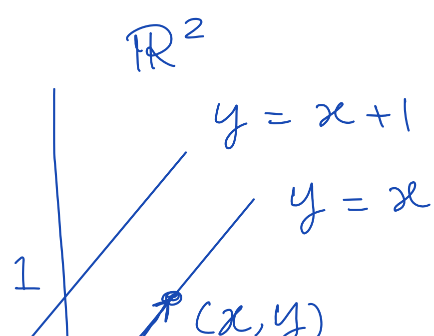
$u, v \in Z$; $a, b \in \mathbb{R}$

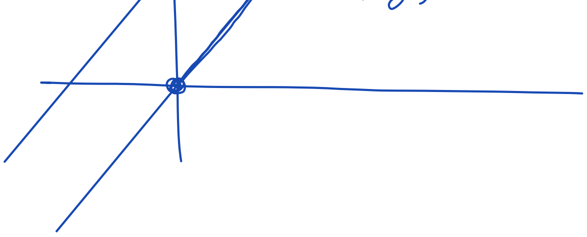
$$au + bv = a\begin{bmatrix} u_1 \\ u_1 \end{bmatrix} + b\begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} au_1 + bv_1 \\ au_1 + bv_1 \end{bmatrix} \in Z$$

$\Rightarrow Z$ is a vector space in $\mathbb{R}^2$ (a subspace of $\mathbb{R}^2$).

(7b) is $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x+1 \right\}$ a vector space?

No, because $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$ why? $0 \cdot \begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

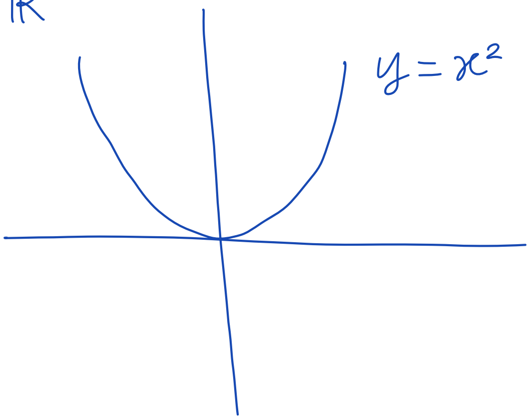




⑧ is $M = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x^2 \right\}$ a vector space?

NO, it's not

$\mathbb{R}^2$



$f, g : \mathbb{R} \longrightarrow \mathbb{R}$

f, g are continous

$a, b \in \mathbb{R}$

$af + bg$ is also a continous fcn.

or:

$a \in \mathbb{R}$, $\begin{bmatrix} x \\ x^2 \end{bmatrix} \in S$ $\Rightarrow a \begin{bmatrix} x \\ x^2 \end{bmatrix} = \begin{bmatrix} ax \\ ax^2 \end{bmatrix} \to \neq x^2 \notin S$

$\Rightarrow S$ is not a vector space.

⑨ W in $\mathbb{R}^3$, $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + 2y + 3z = 0 \right\}$. Is W a vector space?

$a, b \in \mathbb{R}$ $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ & $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in W$

$a \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} ay_1 + by_2 \\ az_1 + bz_2 \end{bmatrix}$$

$$ax_1 + bx_2 + 2(ay_1 + by_2) + 3(az_1 + bz_2)$$

$$= a\underbrace{(x_1 + 2y_1 + 3z_1)}_{0} + b\underbrace{(x_2 + 2y_2 + 3z_2)}_{0} = 0 \quad \checkmark \text{ Yes}$$

Subspaces

<u>Def.</u>: Let V be a vector space over a field F. A subspace is a non-empty set W of V that is closed under the operations of V, in the sense that

(1) if $\vec{x}, \vec{y} \in W$, then $\vec{x} + \vec{y} \in W$;

(2) if $\vec{x} \in W$ and $\lambda \in F$, then $\lambda\vec{x} \in W$.

Examples:

① Every vector space (trivially) is a subspace of itself. V itself is therefore the <u>biggest subspace of V.</u>

② The singleton subset $\{\vec{0}\}$ is a subspace of V. This is then the <u>smallest subspace of V</u> since we have that $\vec{0} \in W$ for every subspace W of V.

③ The subset $X = \{(x, 0), x \in \mathbb{R}\}$ is a subspace of the vector space $\mathbb{R}^2$. It's closed under addition and multiplication by scalars; $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$;

$$\lambda(x, 0) = (\lambda x, 0).$$

This subspace is simply the 'x-axis' in the cartesian plane $\mathbb{R}^2$. Similarly, the 'y-axis':

$Y = \{(0, y);\ y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

④ In $\mathbb{R}^3$, every plane through the origin can be described as: $P = \{(x, y, z);\ \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z} = \vec{0}\}$.

If $(x_1, y_1, z_1) \in P$ and $(x_2, y_2, z_2) \in P \Rightarrow (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in P$,

and if $(x_1, y_1, z_1) \in P$, then for every $\lambda \in \mathbb{R}$,

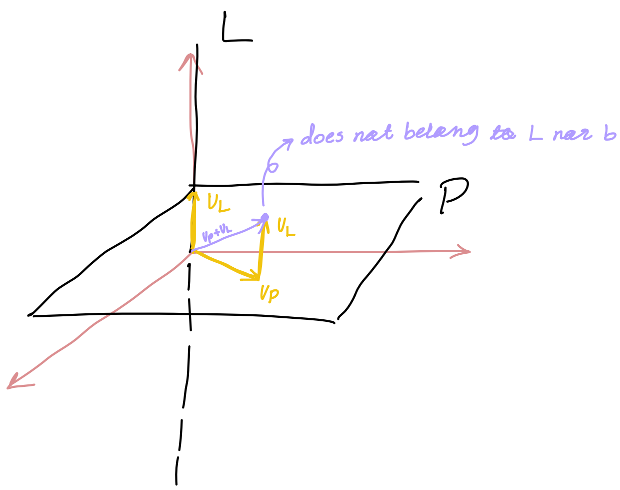
$\lambda(x_1, y_1, z_1) = (\lambda x_1, \lambda y_1, \lambda z_1) \in P$.

Thus, every plane through the origin is a subspace of $\mathbb{R}^3$.

* Apart from $\{(0, 0, 0)\}$ and $\mathbb{R}^3$ itself, the only subspaces of $\mathbb{R}^3$ are lines through the origin.

⑤ Consider two subspaces L & P in $\mathbb{R}^3$, where L is a line through the origin and P is a plane through the origin

(a) is $P \cup L$ a subspace of $\mathbb{R}^3$? $V_p + V_L \notin P \cup L$ isn't a vector space in $\mathbb{R}^3$. Thus, $P \cup L$ isn't a subspace of $\mathbb{R}^3$



(b) is $P \cap L$ a subspace of $\mathbb{R}^3$?

- $P \cap L = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is subspace of $\mathbb{R}^3$

- $L \in P$, $P \cap L = L$ is a subspace of $\mathbb{R}^3$

⑥ An $n \times n$ matrix over a field F is said to be lower triangular if it's of the form $\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$

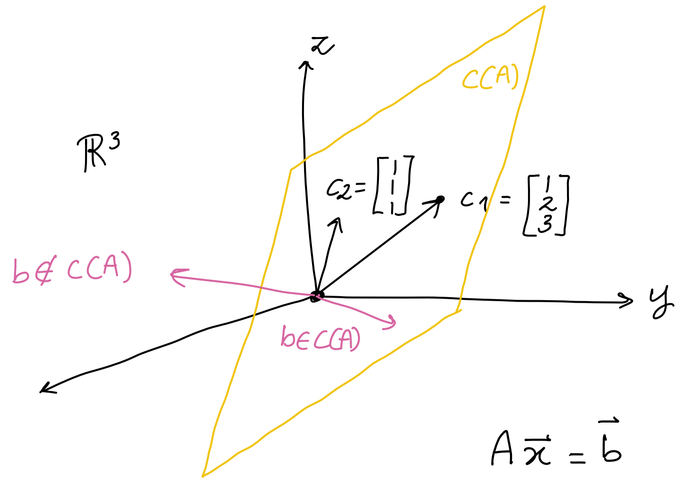
i.e., if $a_{ij} = 0$ whenever $i < j$.

The set of lower triangular matrices is a subspace of the vector space $\mathrm{Mat}_{n \times n} F$; If A & B are lower triangular matrices then so is $A + B$ and so is λA.

Lecture 6:

Caulmn Space

Def.: A caulmn space $C(A)$ of a matrix A is the vector space made up of all lin. combs. of the caulmns of A.



- Solving $A\vec{x} = \vec{b}$

Given a matrix A, for what vectors $\vec{b}$ does $A\vec{x} = \vec{b}$ have a sol. $\vec{x}$?

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$. The 3^{rd} column is the sum of the first 2 caulmns, so it doesn't add anything to the subspace. The caulmn space of A is a 2d subspace of $\mathbb{R}^4$.

→ what are the b's such that $Ax = b$ has a sln.?
(can I find a lin. comb. of the caulmns of A equal to b?)

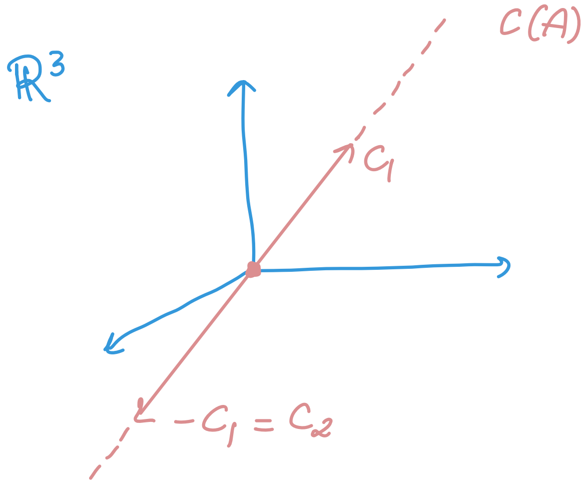
→ $b \in C(A)$, we have at least one sln.

→ $b \notin C(A)$, $Ax = b$ has no sln.

If $b \in C(A)$, then $Ax = b$ has at least 1 sln.

$\rightarrow A = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}$ c_1 and c_2 are not collinear, i.e., they have different directions - they span different lines; so there's only 1 sln. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$, such that $x_1 c_1 + x_2 c_2 = b$

$\rightarrow A = \begin{bmatrix} | & | \\ c_1 & -c_1 \\ | & | \end{bmatrix}$

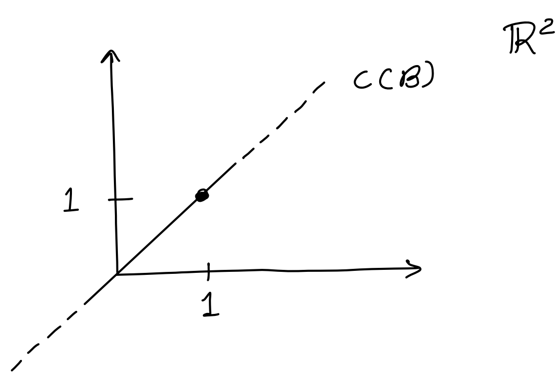


$\Rightarrow Ax = b$ has infinitly many slns

$c_2 = a c_1$, $a \in \mathbb{R}$

Examples:

① $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $C(B) \subseteq \mathbb{R}^2$



② $C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad C(C) = \mathbb{R}^2$

③ $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ $\quad C(A) \subseteq \mathbb{R}^4$

$C_1 \quad C_2 \quad C_3 = C_1 + C_2$

- $C(A)$ is the plane in $\mathbb{R}^4$ containing 0, C_1, and C_2.

④ $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad C(A) \subseteq \mathbb{R}^3$

$A\vec{x} = \vec{b}$ has a solution;

$\vec{b} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in C(A)$

Nullspace of A

<u>Def.</u>: A nullspace of a matrix A, $N(A)$, is the collection of all solns. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equation $A\vec{x} = \vec{0}$.

Note that:

- The column space of a matrix $A_{m \times n}$ is a subspace of $\mathbb{R}^m$.
- The nullspace of a matrix $A_{m \times n}$ is a subspace of $\mathbb{R}^n$.

$C(A) \subseteq \mathbb{R}^m$

&

$N(A) \subseteq \mathbb{R}^n$

Examples:

(1)

$$A \quad \vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(2) $$A = \begin{array}{ccc} \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 3 & -1 & 2 \\ 4 & -1 & 3 \end{bmatrix}_{4\times 3} \\ c_1 \quad c_2 \quad c_3 = c_1 + c_2 \end{array}$$

$C(A) \subseteq \mathbb{R}^4$

$N(A) \subseteq \mathbb{R}^3$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in N(A) \; ; \quad \vec{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\substack{c_1 + c_2 - c_3 = 0 \\ A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0}} \in N(A) \; ; \quad \vec{x} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \in N(A) \; ;$$

$$\vec{x} = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad c \in \mathbb{R}$$

Lecture 7:

Computing the nullspace

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

1st pivot

2nd pivot

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$A\vec{x} = \vec{0} \xrightarrow{GE} U\vec{x} = \vec{0}$$

$$\Rightarrow N(A) = N(U)$$

Special solutions

$N(U) = ?$

free columns

$$U\vec{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \boxed{x_2} \\ x_3 \\ \boxed{x_4} \end{bmatrix}$$

pivot variables

pivot columns

free variables

we can assign any value to x_2 & x_4.

① Suppose $x_2 = 1$ and $x_4 = 0$

$$2x_3 + 4\cancel{x_4} = 0 \Rightarrow x_3 = 0$$

and $x_1 + 2x_2 + 2\cancel{x_3} + 2\cancel{x_4} = 0 \Rightarrow x_1 = -2$

So, $\vec{x}_s^1 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A\vec{x}_s^1 = \vec{0}$ 1st special sln.

$$\begin{bmatrix} | & | & | & | \\ c_1 & c_2 & c_3 & c_4 \\ | & | & | & | \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \Rightarrow -2c_1 + c_2 = \vec{0}$$

Now, suppose $x_2 = 0$ and $x_4 = 1$

$$2x_3 + 4x_4 = 0 \Rightarrow x_3 = -2$$

$$x_1 + 2\cancel{x_2} + 2x_3 + 2x_4 = 0$$

$$x_1 - 4 + 2 = 0 \Rightarrow x_1 = 2$$

So, $\vec{x}_s^2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow c_4 = 2c_3 - 2c_1$ 2nd special sln.

* The nullspace of A is the collection of all linear combs. of these "special slns." vectors.

• The rank r of A is the # of pivot columns, so the # of free columns is n−r. This equals the # of special sln. vectors and the dimension of the nullspace.

Reduced row echelon form

By continuing to use the method of elimination we can convert U to a matrix R in *reduced row echelon form* (rref form), with pivots equal to 1 and zeros above and below the pivots.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

By exchanging some columns, R can be rewritten with a copy of the identity matrix in the upper left corner, possibly followed by some free columns on the right. If some rows of A are linearly dependent, the lower rows of the matrix R will be filled with zeros:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}. = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Here I is an r by r square matrix.)

If N is the *nullspace matrix* $N = \begin{bmatrix} -F \\ I \end{bmatrix}$ then $RN = 0$. (Here I is an $n - r$ by $n - r$ square matrix and 0 is an m by $n - r$ matrix.) The columns of N are the special solutions.

Lecture 8 :

Solving $A\mathbf{x} = \mathbf{b}$: row reduced form R

When does $A\mathbf{x} = \mathbf{b}$ have solutions $\mathbf{x}$, and how can we describe those solutions?

Solvability conditions on b

We again use the example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

The third row of A is the sum of its first and second rows, so we know that if $A\mathbf{x} = \mathbf{b}$ the third component of $\mathbf{b}$ equals the sum of its first and second components. If $\mathbf{b}$ does not satisfy $b_3 = b_1 + b_2$ the system has no solution. If a combination of the rows of A gives the zero row, then the same combination of the entries of $\mathbf{b}$ must equal zero.

One way to find out whether $A\mathbf{x} = \mathbf{b}$ is solvable is to use elimination on the augmented matrix. If a row of A is completely eliminated, so is the corresponding entry in $\mathbf{b}$. In our example, row 3 of A is completely eliminated:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}.$$

If $A\mathbf{x} = \mathbf{b}$ has a solution, then $b_3 - b_2 - b_1 = 0$. For example, we could choose $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$.

From an earlier lecture, we know that $A\mathbf{x} = \mathbf{b}$ is solvable exactly when $\mathbf{b}$ is in the column space $C(A)$. We have these two conditions on $\mathbf{b}$; in fact they are equivalent.

Complete solution

1

In order to find all solutions to $A\mathbf{x} = \mathbf{b}$ we first check that the equation is solvable, then 2 find a particular solution. We get the complete solution of the equation by adding the particular solution to all the vectors in the nullspace.

A particular solution

One way to find a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ is to set all free variables to zero, then solve for the pivot variables.

For our example matrix A, we let $x_2 = x_4 = 0$ to get the system of equations:

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \end{aligned}$$

which has the solution $x_3 = 3/2$, $x_1 = -2$. Our particular solution is:

$$\mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}.$$

Combined with the nullspace

The general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_{\text{complete}} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n$ is a generic vector in the nullspace. To see this, we add $A\mathbf{x}_p = \mathbf{b}$ to $A\mathbf{x}_n = \mathbf{0}$ and get $A\left(\mathbf{x}_p + \mathbf{x}_n\right) = \mathbf{b}$ for every vector $\mathbf{x}_n$ in the nullspace.

Last lecture we learned that the nullspace of A is the collection of all combinations of the special solutions $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So the complete solution to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is:

$$\mathbf{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where c_1 and c_2 are real numbers.

The nullspace of A is a two dimensional subspace of $\mathbb{R}^4$, and the solutions to the equation $A\mathbf{x} = \mathbf{b}$ form a plane parallel to that through $x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

Rank

The rank of a matrix equals the number of pivots of that matrix. If A is an m by n matrix of rank r, we know $r \leq m$ and $r \leq n$.

Full column rank

If $r = n$, then from the previous lecture we know that the nullspace has dimension $n - r = 0$ and contains only the zero vector. There are no free variables or special solutions.

If $A\mathbf{x} = \mathbf{b}$ has a solution, it is unique; there is either 0 or 1 solution. Examples like this, in which the columns are independent, are common in applications.

We know $r \leq m$, so if $r = n$ the number of columns of the matrix is less than or equal to the number of rows. The row reduced echelon form of the matrix will look like $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$. For any vector $\mathbf{b}$ in $\mathbb{R}^m$ that's not a linear combination of the columns of A, there is no solution to $A\mathbf{x} = \mathbf{b}$.

Full row rank

If $r = m$, then the reduced matrix $R = \begin{bmatrix} I & F \end{bmatrix}$ has no rows of zeros and so there are no requirements for the entries of $\mathbf{b}$ to satisfy. The equation $A\mathbf{x} = \mathbf{b}$ is solvable for every $\mathbf{b}$. There are $n - r = n - m$ free variables, so there are $n - m$ special solutions to $A\mathbf{x} = \mathbf{0}$.

Full row and column rank

If $r = m = n$ is the number of pivots of A, then A is an invertible square matrix and R is the identity matrix. The nullspace has dimension zero, and $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Summary

If R is in row reduced form with pivot columns first (rref), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
# solutions to $A\mathbf{x} = \mathbf{b}$	1	0 or 1	infinitely many	0 or infinitely many

Exercises on solving $A\mathbf{x} = \mathbf{b}$ and row reduced form R

Problem 8.1: (3.4 #13.(a,b,d) *Introduction to Linear Algebra:* Strang) Explain why these are all false:

a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

b) The system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.

c) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Problem 8.2: (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices $[U \;\; 0]$ and $[U \;\; \mathbf{c}]$ to $[R \;\; 0]$ and $[R \;\; \mathbf{d}]$. Solve $R\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{d}$.

Check your work by plugging your values into the equations $U\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{c}$.

Problem 8.3: (3.4 #36.) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

18.06SC Linear Algebra
Fall 2011

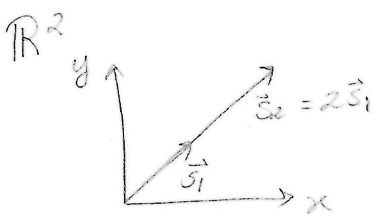
Linear Independence (in lec. 11)

Def.: Let S be a non-empty subset of a vector space V over a field F. Then S is said to be linearly independent if the only way of expressing $\vec{O}_V$ as a linear combination of elements of S is the trivial way (in which all scalars are O_F).

Equivalently, S is said to be linearly independent for any given $\vec{s}_1, \ldots, \vec{s}_n \in S$, we have

$$\lambda_1 \vec{s}_1 + \ldots + \lambda_n \vec{s}_n = \vec{O}_V \Rightarrow \lambda_1 = \ldots = \lambda_n = O_F.$$

Ex. #1



$\{\vec{s}_1, \vec{s}_2\}$ is lin. dep.

why? $\vec{s}_2 = 2\vec{s}_1$

$\Rightarrow 2\vec{s}_1 - \vec{s}_2 = 0$

$\lambda_1 = 2 \neq 0 \quad \lambda_2 = -1 \neq 0$

There is a nonzero pair (λ_1, λ_2), such that the equ. is verified,

$$\lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2 = 0.$$

Hence, $\{\vec{s}_1, \vec{s}_2\}$ is lin. dep..

Ex. #2

The subset $\{(1,0),(0,1)\} \in \mathbb{R}^2$ is lin. indep. If $\lambda_1(1,0) + \lambda_2(0,1) = (0,0)$ then $(\lambda_1, \lambda_2) = (0,0)$ and hence $\lambda_1 = \lambda_2 = O_F$.

Ex. #3

Every singleton subset $\{s\}$, where $s \neq 0$, is lin. indep..

Thus, every dependent subset, other than $\{\vec{O}_V\}$ must contain at least 2 elements

Theorem #1

No lin. indep. subset of a vector space V can contain $\vec{0}$.

Ex. #4

$\{\vec{s}_1, \vec{0}\}$, $\vec{s}_1 \neq \vec{0} \in \mathbb{R}^2$ lin. indep.?

$\lambda_1 \vec{s}_1 + \lambda_2 \vec{0} = \vec{0}$

$\lambda_1 = 0, \quad \lambda_2 = 2/3 \neq 0$

$\Rightarrow$ The set is lin. dep. because $\lambda_1 = 0$, and $\lambda_2 = 2/3$ verify the equation.

Linear Independence from nullspace perspective:

A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, $\vec{v}_i \in \mathbb{R}^m$, is linearly indep. if and only if

$$N\left(\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}\right) = \{\vec{0}\}.$$

Problem #1) Show that the set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ $\vec{v}_i \in \mathbb{R}^m$, $n > m$, is lin. dep.

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

* Everytime we have more vectors than components, the set is lin. dep.

sol.

- if $N(A) = \{\vec{0}\}$, then the set is lin. indep.
- if $N(A) \neq \{\vec{0}\}$, then the set is lin. dep..

- $\text{rank}(A) \leq m$ (I can have more pivots than rows).
- $n > m$ (the assumption).
- then $n > r$, then I have free coulmn and therefore $N(A) \neq \{\vec{0}\}$.

$\therefore$ The set is lin. dep. :)

• Linearly dependent sets can be characterised as follows:

Theorem #2

Let V be a vector space over a field F. If S is a subset of V that contains at least two elements, then the following statements are equivalent:

$\boxed{1}$ S is lin. dep..

$\boxed{2}$ at least one element of S can be expressed as a lin. comb. of the other elements of S.

Ex. #5

$S \in \mathbb{R}^3$, $S = \{(1,1,0), (2,5,3), (0,1,1)\}$ is lin. dep.;

$$(2,5,3) = 2(1,1,0) + 3(0,1,1).$$

Let U and W be subspaces of V.

Def.: The intersection of $U \cap W$ is the set of elements that are both elements of U and W.

$$U \cap W = \{x \mid x \in U \text{ and } x \in W\}$$

Theorem #3

The intersection of any set of subspaces of a vector space V is a subspace of V.

Proof

Let T be the intersection of U and W. Then $T \neq \emptyset$, since every subspace of V contains the zero vector, whence so also does T.

Suppose that $x, y \in T$. Since x and y belong to U and to W, so does $x+y$, and hence $x+y \in T$.

Now, let $x \in T$ and $\lambda \in \mathbb{R}$. As $x \in T$ then x belongs to every subspace in the set T, thus $\lambda x \in T$ as well. Thus, we see that T is a subspace of V.

In contrast, the union of a set of subspaces of a vector space V need not be a subspace of V.

Proof

Since $\mathcal{U}$ is not contained in V, there exist a vector $u \in \mathcal{U}$ but $u \notin V$. So the subset $\mathcal{U} \cup V$ is not closed under addition or multip. and therefore cannot be a subspace.

Def: Let V be a vector space over a field F and let S be a non-empty subset of V. We say that $\vec{v} \in V$ is a linear combination of elements of S if there exist $\vec{s}_1, \dots, \vec{s}_n \in S$ and $\lambda_1, \dots, \lambda_n \in F$, s.t.

$$\vec{v} = \lambda_1 \vec{s}_1 + \dots + \lambda_n \vec{s}_n = \sum_{i=1}^{n} \lambda_i \vec{s}_i.$$

If $\vec{v} = \sum_{i=1}^{n} \lambda_i \vec{s}_i$ and $\vec{w} = \sum_{i=1}^{n} \mu_i \vec{u}_i$ are linear combinations of elements of S then so is $\vec{v} + \vec{w}$, so is $\lambda \vec{v}$ for every $\lambda \in F$. Thus, the linear combs. of elements of S is a subspace of V. We call this the <u>subspace spanned by S</u> and denote it by <u>Span S</u>.

Ex. #6

Consider the subset $S = \{(1,0), (0,1)\}$ of the cartesian plane $\mathbb{R}^2$. For every $(x, y) \in \mathbb{R}^2$ we have

$$(x, y) = (x, 0) + (0, y) = x(1,0) + y(0,1)$$

so that every element of $\mathbb{R}^2$ is a linear combination of elements of S. Thus S is a spanning set of $\mathbb{R}^2$.

Ex. #7 If the n-tuple $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ has the 1 in the i-th position then for every $(x_1, \dots, x_n) \in \mathbb{R}^n$ we have

$$(x_1, \dots, x_n) = x_1 e_1 + \dots + x_n e_n.$$

Consequently, $\{e_1, \dots, e_n\}$ spans $\mathbb{R}^n$.

Ex. #8 In $\mathbb{R}^3$ we have $\text{span}\{(1,0,0)\} = \{\lambda(1,0,0);\ \lambda \in \mathbb{R}\}$
$= \{(\lambda,0,0);\ \lambda \in \mathbb{R}\}$;

i.e. the subspace of $\mathbb{R}^3$ spanned by the singleton $\{(1,0,0)\}$ is the x-axis.

Ex. #9 In $\mathbb{R}^3$ we have $\text{span}\{(1,0,0),(0,0,1)\} = \{x(1,0,0) + z(0,0,1);\ x, z \in \mathbb{R}\} = \{(x,0,z);\ x,z \in \mathbb{R}\}$

i.e. the subspace of $\mathbb{R}^3$ spanned by the subset $\{(1,0,0),(0,0,1)\}$ is the 'x, z-plane'.

The combination of the notions of linearly independent set and spanning set gives an important concept.

Bases

Def.: A basis of a vector space V is a linearly independent subset of V that spans V.

* A basis of a vector space isn't unique, but all bases share the same # of vectors, this # is the dim of the vector space.

Ex. #10

The subset $\{(1,0),(0,1)\}$ is a basis of the cartesian plane $\mathbb{R}^2$. Likewise the subset $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$. Generally, $\{e_1,\dots,e_n\}$ is a basis for $\mathbb{R}^n$, where

$$e_i = (0,0,\dots,0,1,0,\dots,0,0),$$

the 1 being in the ith position.

They are called the natural (or canonical) bases.

Ex. #11

The subset $\{(1,1),(1,-1)\}$ is a basis in $\mathbb{R}^2$;

For every $(x,y)\in\mathbb{R}^2$, we have $(x,y) = \lambda_1(1,1)+\lambda_2(1,-1)$, where $\lambda_1 = \frac{1}{2}(x+y)$ and $\lambda_2 = \frac{1}{2}(x-y)$. Thus, $\{(1,1),(1,-1)\}$ spans $\mathbb{R}^2$; and if $\alpha(1,1)+\beta(1,-1)=(0,0)$. Then $\alpha+\beta=0$ and $\alpha-\beta=0 \Rightarrow \alpha=\beta=0$, so $\{(1,1),(1,-1)\}$ is also lin. indep..

The <u>span</u> of a set of vectors is the subspace consisting of all lin. combs. of the vectors in the set.

Given a subspace S, we say that <u>a set S of vectors spans the subspace</u> if the span of the set S is the subspace.

A <u>basis</u> of a subspace is a set of vectors that spans the subspace and is lin. ind..

The <u>nullspace</u> of a matrix A is the subspace consisting of slns. of $A\vec{x}=\vec{0}$.
It has a basis consisting of the "fundamental slns." of $A\vec{x}=\vec{0}$.

The span of a set of vectors is a subspace. when we put these vectors in a matrix, the subspace is called <u>column space</u> of that matrix. To find a basis of the span, put the vectors in a matrix A. The columns of A that wind up with leading in Gaussian Elimination form a basis of that subspace.

The <u>dimension</u> of a subspace U is the number of vectors in a basis of U. There are many choices of a basis, each with diff. features, but the # of vectors is always the same.

Summary

Lecture 12:

Problem #1 Show that a set of $\{\vec{v}_1, \dots, \vec{v}_n\}$, $\vec{v}_i \in \mathbb{R}^n$, is a basis for $\mathbb{R}^n$ if and only if A^{-1} exists, with $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

① Assume that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$. we show that A^{-1} exists.

- if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis, then the $\vec{v}_i$'s are lin. indep.;

- $\boxed{GJ}$ $[A \mid I] \longrightarrow [I \mid A^{-1}]$ $r = n$, therefore A^{-1} exists.

② if A^{-1} exists, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$

- the set has the right # of vectors because a basis of $\mathbb{R}^n$ has n vectors.
- now we only need to prove the $\vec{v}_i$'s are lin. indep.;
  - if A^{-1} exists, then $N(A) = \{\vec{0}\}$
  - if $N(A) = \{\vec{0}\}$, then the columns are lin. indep.

---

if A^{-1} exists, then $N(A) = \{\vec{0}\}$

$A\vec{x} = \vec{0}$, $\vec{x} \neq 0$

$A^{-1}A\vec{x} = \vec{0}$

$\vec{x} = \vec{0}$

Problem #2

Suppose $\underset{m \quad n}{A_{3\times5}}$ and rank$(A) = 3$.

$\boxed{a}$ Is it true that $A\vec{x} = \vec{b}$ always has a solution?

There're 2 ways to answer;

[1] Row approach / condition of solvability; there're no rows of zeros, so the last equ. $a\vec{x}_i = \vec{b}_3$

$a \neq 0$

and it can always be solved.

[2] Column approach: $b \in C(A)$?

- $rank(A) = 3 \Rightarrow$ there're 3 lin. indep. columns
- $C(A) \subseteq \mathbb{R}^3$
- then $C(A) = \mathbb{R}^3$
- $A\vec{x} = \vec{b} \in \mathbb{R}^3$

$\Rightarrow A\vec{x} = \vec{b}$ always has a solution

[b] How many solutions for $A\vec{x} = \vec{b}$?

Theorem: $n - r = \dim N(A)$

$5 - 3 = 2 = \dim N(A) > 0$

$x_c = x_p + N(A) \Rightarrow \infty$-many solutions

$$\#\text{ of slns. of } A\vec{x}=\vec{b} = \begin{cases} \text{finite} & \text{if F is finite} \\ \infty & \text{if F is } \infty \end{cases}$$

The Fundamental Subspaces of a Matrix

4 :)

$A_{m \times n}$ $rank(A) = r$

[1]
- $C(A)$ is the span of the columns of A;
- $C(A)$ is a subset of $\mathbb{R}^m$;
- $\dim C(A) = r$
- basis of $C(A)$ is the set of the pivot columns in A.

2 • N(A) is the set of all $\vec{x} \in \mathbb{R}^n$ that verify $A\vec{x} = \vec{0}$;

• $N(A) \subseteq \mathbb{R}^n$;

• $\dim N(A) = n - r$;

• a basis for N(A) is the set of all special slns..

3 • C(AT) – row space, it's the set of all lin. comb. of the rows of A.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

• $C(A^T) \subseteq \mathbb{R}^n$

• $\dim C(A^T) = r$

• a basis for C(AT): check next example

4 • N(AT) – left nullspace, it's the set of all $\vec{y}$ s.t. $A^T\vec{y} = \vec{0}$

• $N(A^T) \subseteq \mathbb{R}^m$ $\qquad A_{m\times n} \quad \boxed{A^T_{n\times m}}$

• $\dim N(A^T) = m - r$

• bases for N(AT): check exercise

$A^T\vec{y} = \vec{0}$

$(A^T\vec{y})^T = \vec{0}^T$

$\vec{y}^T(A^T)^T = 0^T$

$y^T A = 0^T$

$$[1\ 2\ \dots\ m] \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix} = [0\ \dots\ 0]$$

$\Rightarrow$ Any $y \in N(A^T) \perp$ to any element in $C(A)$

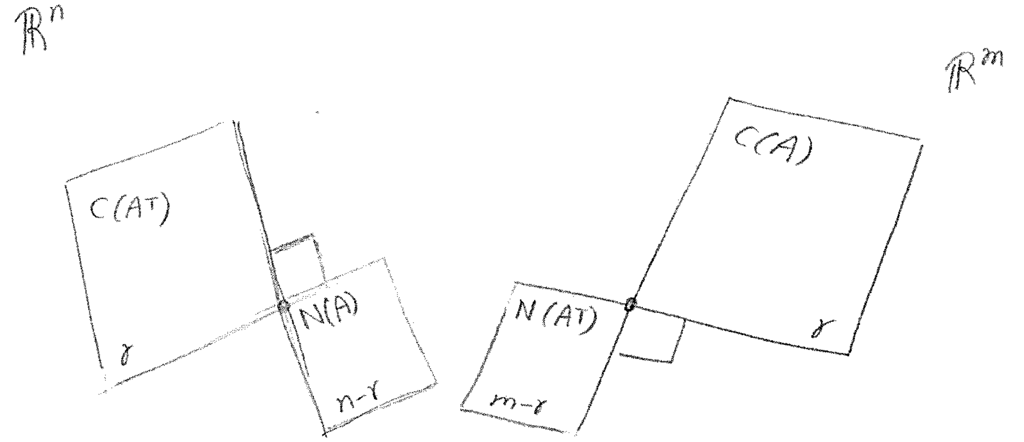
$A\vec{x} = \vec{0}$

$$\begin{bmatrix} - & r_1^T & - \\ & \vdots & \\ - & r_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

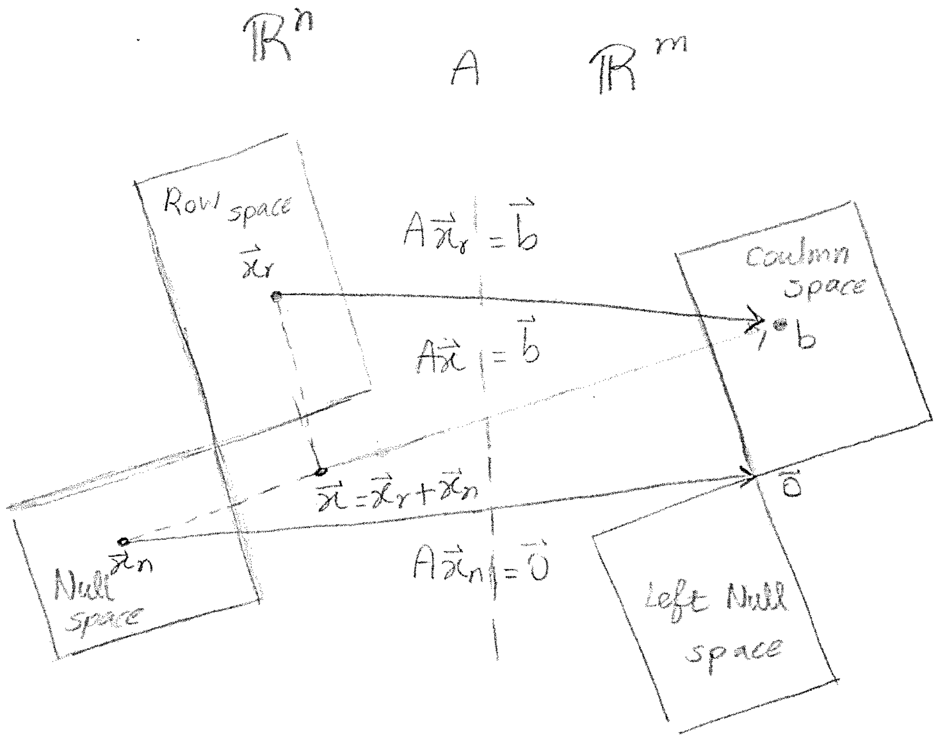
$r_1^T \vec{x} = \vec{0} \iff r_1 \perp x$

$r_i^T \vec{x} = \vec{0} \iff r_i \perp x$

$\Rightarrow$ all vectors in $C(A^T)$ are $\perp$ to all the elements in $N(A)$



Lecture 13:



- $\vec{x} \in \mathbb{R}^n$
- $\vec{x}_r \in C(A^T)$
- $\vec{x}_n \in N(A)$

$\vec{x} = \vec{x}_r + \vec{x}_n$ because $C(A^T)$ and $N(A)$ are orthogonal complements in $\mathbb{R}^n$.

- $\vec{y} \in \mathbb{R}^m$
- $\vec{y}_c \in C(A)$
- $\vec{y}_{ln} \in N(A^T)$

$\vec{y} = \vec{y}_c + \vec{y}_{ln}$ because $C(A)$ and $N(A^T)$ are orthogonal complements in $\mathbb{R}^m$.

- Finding bases for the four fundamental subspaces

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{array}{cccc} \downarrow & \downarrow & & \\ \begin{array}{c}\text{pivot col.}\end{array} & & & \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot col. (columns 1, 2); free col. (columns 3, 4)

$\longrightarrow$ Basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

- Dim. $= 2$

$\rightarrow$ Basis for $C(A^T)$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$

- Dim. = 2

$\rightarrow$ Basis for $N(A)$: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$

$$x_1 + 2x_2 + 3x_3 + x_4 = 0$$
$$-x_2 - x_3 = 0$$

- choose $\left.\begin{matrix} x_3 = 0 \\ x_4 = 1 \end{matrix}\right\}$ $\begin{matrix} x_1 = -1 \\ x_2 = 0 \end{matrix}$ $\Rightarrow x_{s1} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

- choose $\left.\begin{matrix} x_3 = 1 \\ x_4 = 0 \end{matrix}\right\}$ $\begin{matrix} x_1 = -1 \\ x_2 = -1 \end{matrix}$ $\Rightarrow x_{s2} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

- Dim. = 4 - 2 = 2

$\rightarrow$ Basis for $N(A^T)$: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_1 \end{bmatrix} \boxed{= 0} \Rightarrow -b_1 + 0 + b_3 = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Dim. = 3 - 2 = 1

Show that $C(A^T) \perp N(A)$

$$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\} \Rightarrow A\vec{x} = \begin{bmatrix} \text{— row}_1 \text{—} \\ \text{— row}_2 \text{—} \\ \vdots \\ \text{— row}_m \text{—} \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} \text{row}_1 \cdot x_1 \\ \text{row}_2 \cdot x_2 \\ \vdots \\ \text{row}_3 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ Any vector $\vec{x} \in N(A)$ is orthogonal to every row of A

Now, take a vector $\vec{V} \in C(A^T)$,

$\vec{V} = c_1 \cdot R_1 + c_2 \cdot R_2 + \cdots + c_m R_m \quad , c \in \mathbb{R}$

$\Rightarrow \vec{x}^T \vec{V} = \vec{x}^T (c_1 R_1 + c_2 \cdot R_2 + \cdots + c_m R_m) \quad , \vec{x} \in N(A)$

$= c_1 (\vec{x}^T \cdot R_1) + c_2 (\vec{x}^T \cdot R_2) + \cdots + c_3 (\vec{x}^T \cdot R_3)$

$= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_m \cdot 0$

$= 0$

$\therefore C(A^T) \perp N(A)$

Show that $C(A) \perp N(A^T)$

Let $B = A^T$, then $B^T = {A^T}^T = A$

$\Rightarrow C(B^T) \perp N(B)$

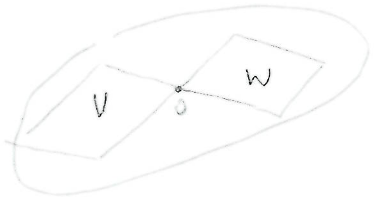
$\Rightarrow C(A) \perp N(A^T)$

Orthaganality of the Four Subspaces

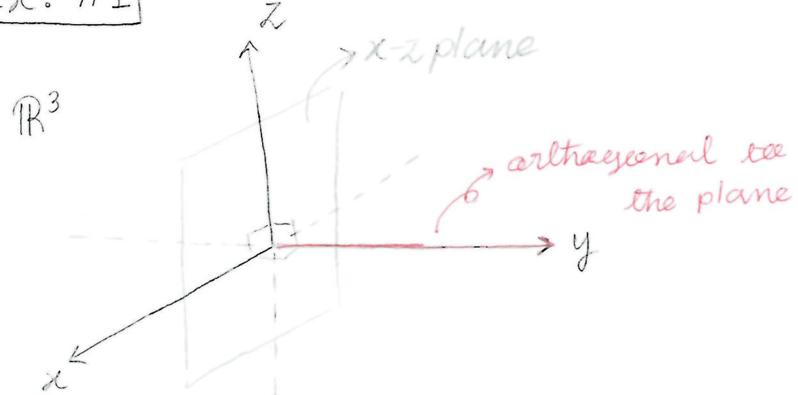
Def. :

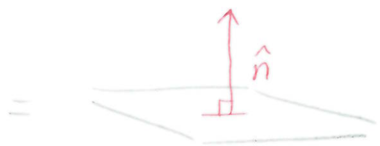
Let V and W be subspaces of same space $\mathcal{U}$. We say that V is <u>orthaganal</u> to W if for every $\vec{v} \in V$ and for every $\vec{w} \in W$, we have

$\vec{v}^T \vec{w} = \vec{0}$

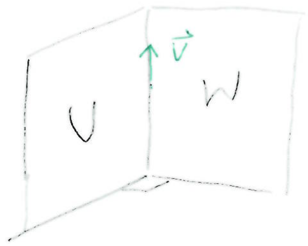


Ex. #1





* Two planes in $\mathbb{R}^3$ cannot be orthaganal to each other.



$\vec{v}^T\vec{v} = 0$, then $\vec{v} = 0$

($\vec{v}$ is not orthagonal to itself)

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\quad \vec{v}^T\vec{v} = v_1^2 + v_2^2 + v_3^2$

Consider the matrix A of size $m \times n$

→ The rows of A have n components.

→ The columns of A have m components.

$C(A) \subseteq \mathbb{R}^m$, $N(A^T) \subseteq \mathbb{R}^m$

$\boxed{C(A^T) \subseteq \mathbb{R}^n, \quad N(A) \subseteq \mathbb{R}^n}$ See the following

Recall: $N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$

$A\vec{x} = \vec{0}$ for $\vec{x} \in N(A)$

$$A\vec{x} = \begin{bmatrix} -\text{row}_1- \\ \vdots \\ -\text{row}m- \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} \text{row}1 \cdot x \\ \vdots \\ \text{row}m \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{x}$ is orthagonal to the rows of A.

→ Every $\vec{x} \in N(A)$ is orthagonal to the rows of A.

* To show that 2 vector spaces are orthagonal, it's enough to check that (given a basis for one space and given a basis for the other) each vector in the one base is orthagonal to every element in the other base.

⇒ Take $\vec{V} = c_1 R_1 + c_2 R_2 + \cdots + c_m R_m$

$$\vec{x}^T \vec{V} = \vec{x}^T (c_1 R_1 + c_2 R_2 + \cdots + c_m R_m)$$
$$= c_1 (\vec{x}^T R_1) + c_2 (\vec{x}^T R_2) + \cdots + c_m (\vec{x}^T R_m)$$
$$= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_3 \cdot 0$$
$$= 0$$

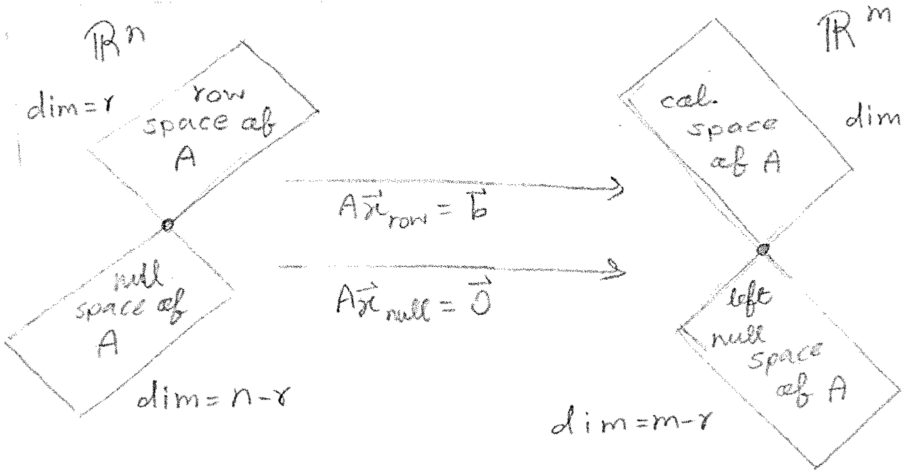
$\therefore C(A^T) \perp N(A)$

From here, we also have $C(A) \perp N(A^T)$ How?

Let $B = A^T$, then $B^T = A^{TT} = A$

$C(B^T) \perp N(B)$

$C(A) \perp N(A^T)$



The Four Fundamental subspaces

Given a matrix A of size $m \times n$ (rows are in $\mathbb{R}^n$ and columns are in $\mathbb{R}^m$.) The 4 fundamental subspaces of A are:

$\boxed{1}$ The column space $C(A)$; a subspace of $\mathbb{R}^m$

It is the space that is spanned by the columns of A.

$\rightarrow$ Dim. $C(A) = r$ $\qquad$ $r = \text{rank}(A)$

$\rightarrow$ Basis: the set of the pivot columns in A.

$\boxed{2}$ The null space $N(A)$; a subspace of $\mathbb{R}^n$

$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0},\ \vec{x} \in \mathbb{R}^n\}$

$\rightarrow$ Dim. $N(A) = n - r$

$\rightarrow$ Basis: the set of all special slns.

$\boxed{3}$ The row space $C(A^T)$; a subspace of $\mathbb{R}^n$

It is the set of all lin. combs. of the rows of A.

$\rightarrow$ Dim. $C(A^T) : r$

$\rightarrow$ Basis:

$\boxed{4}$ The left null space $N(A^T)$; a subspace of $\mathbb{R}^m$

$N(A^T) = \{ \vec{y} \mid A\vec{y} = \vec{0} , \vec{y} \in \mathbb{R}^m \}$

$\rightarrow$ Dim. $N(A^T) = m - r$

$\rightarrow$ Bases :

why it is called the "left" null space ?

$A^T \vec{y} = \vec{0}$

$(A^T \vec{y})^T = (\vec{0})^T$

$\Rightarrow \vec{y}^T A = \vec{0}^T$

Ex. #1

Find a basis for each fundamental subspace of A.

$$A = \begin{bmatrix} \boxed{1} & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

pivot caulms

$$A \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A)$$

free caulms.

- **Basis for C(A):** $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

Dim. = 2

- **Basis for C(AT):** $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^5$

* Pivot raws

* It's also a basis for $C(R^T)$.

Dim. = 2

- **Basis for N(A):** $\left\{ \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^5$

- Remember: $N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\} \Leftrightarrow \{\vec{x} \mid R\vec{x} = \vec{0}\}$
$= N(R)$

$\left.\begin{array}{l} x_5 = 1 \\ x_2 = 0 \\ x_3 = 0 \end{array}\right\}$ $\begin{array}{l} x_1 = -7 \\ x_4 = -2 \end{array}$ $\Rightarrow x_{s_1} = \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$\left.\begin{array}{l} x_2 = 1 \\ x_3 = 0 \\ x_5 = 0 \end{array}\right\}$ $\begin{array}{l} x_1 = -3 \\ x_4 = 0 \end{array}$ $\Rightarrow x_{s_2} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\left.\begin{array}{l} x_3 = 1 \\ x_2 = 0 \\ x_5 = 0 \end{array}\right\}$ $\begin{array}{l} x_1 = -5 \\ x_4 = 0 \end{array}$ $x_{s_3} = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Dim. = 5 − 2 = 3

• Basis for $N(A^T)$: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

Remember

$N(A^T) \neq N(R^T)$; $N(R^T) = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ which means: $\underbrace{0\,\text{row}1 + 0\,\text{row}2 + 1\,\text{row}3}_{\text{rows of } R} = 0$

$\Rightarrow$ To find a basis for $N(A^T)$, we can use conditions on the components of so that $A\vec{x} = \vec{b}$ has a solution.

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 - b_1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} b_1 \\ b_2 \\ \boxed{b_3 - b_1 - b_2} \end{bmatrix} = 0$$

$\Rightarrow$ $b_3 - b_1 - b_2 = 0$

$-b_1 - b_2 + b_3 = 0$

$\begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0$

basis for $N(A^T) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

• Dim: $3 - 2 = 1$

Projections onto subspaces : (Lecture 14)

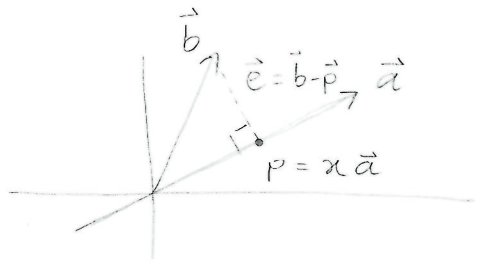


Fig. : The projection of $\vec{b}$ onto $\vec{a}$

suppose that we have a vector $\vec{b}$ and a line determined by the vector $\vec{a}$, the closest point to $\vec{b}$ on the line is $\vec{p}$ (the intersection formed by the line passing through $\vec{b}$ that is orthogonal to $\vec{a}$)

Clearly, $\vec{p}$ is some multiple of $\vec{a}$; $\vec{p} = x\vec{a}$. x is what we would like to find.

orthogonality here is the key fact; $\vec{a}^T e = 0$

$$\vec{a}^T(\vec{b} - \vec{p}) = 0$$

$$\vec{a}^T(\vec{b} - x\vec{a}) = 0$$

$$\Rightarrow x\vec{a}^T\vec{a} = \vec{a}^T\vec{b}$$

$$\Rightarrow x = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$$ $\vec{a}^T\vec{a} \longrightarrow$ a number $= \vec{a}^2$

$$\Rightarrow \vec{p} = x\vec{a} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\vec{a}$$ Doubling $\vec{b}$ doubles $\vec{p}$, but doubling $\vec{a}$ does nothing

Projection Matrix

we could express projection as a matrix P acting on $\vec{b}$;

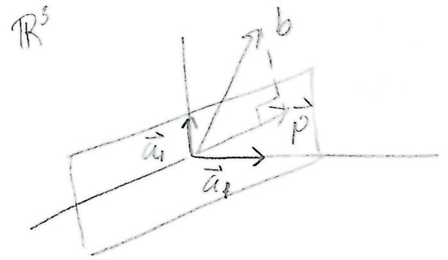
$$\vec{p} = \vec{a}x = \underbrace{\left(\vec{a}\,\frac{\vec{a}^T}{\vec{a}^T\vec{a}}\right)}_{P}\vec{b} \Rightarrow \text{projection} = P\vec{b}, \quad P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}$$ $\longrightarrow$ an $n \times n$ matrix

- $C(P)$ is spanned by $\vec{a}$, because for any $\vec{b}$, $P\vec{b}$ lies on the line determined by $\vec{a}$.
- $C(P)$ = line through $\vec{a}$, rank $(P) = 1$.
- P is symmetric; $P^T = P$; $(\vec{a}\vec{a}^T)^T = \vec{a}\vec{a}^T$, and $\vec{a}^T\vec{a}$ is a #.
- $P^2\vec{b} = P\vec{b}$ because the projection of a vector already on the line through $\vec{a}$ is just that vector.

(II) Why project?

$A\vec{x}=\vec{b}$ may have no solution; so we solve the closest problem instead. Which is $A\vec{y}=\vec{p}$, where $\vec{p}$ is the projection of $\vec{b}$ onto the column space.

(III) Projection in higher dimensions:



The bases for the plane are $\vec{a}_1$ and $\vec{a}_2$.

$\Rightarrow$ The plane is the column space of $A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

$\Rightarrow \vec{p} = y_1\vec{a}_1 + y_2\vec{a}_2 = A\vec{y}$. $\vec{y}$ is what we'd like to find here.

$\Rightarrow \vec{e} = \vec{b} - \vec{p} = \vec{b} - A\vec{y}$; $\vec{e}$ is $\perp$ to $\vec{a}_1$ & $\vec{a}_2$

- Note that $\vec{e}$ is $\perp$ to the plane.

$\Rightarrow \vec{a}_1^T(\vec{b}-A\vec{y}) = 0$ and $\vec{a}_2^T(\vec{b}-A\vec{y}) = 0 \Rightarrow \underbrace{A^T(\vec{b}-A\vec{y}) = 0}$

from here, we can see that $e \in N(A^T)$ and we know that $N(A^T) \perp C(A)$, so we see that $e \perp C(A)$, which is true!

$A^T(\vec{b}-A\vec{y}) = 0$ can be rewritten as $A^TA\vec{y} = A^T\vec{b}$

$\Rightarrow \vec{y} = (A^TA)^{-1}A^T\vec{b}$

$\Rightarrow \vec{p} = A\vec{y} = (\underbrace{A(A^TA)^{-1}A^T}_{P})\vec{b}$

$\Rightarrow P = A(A^TA)^{-1}A^T$ ------ $*$

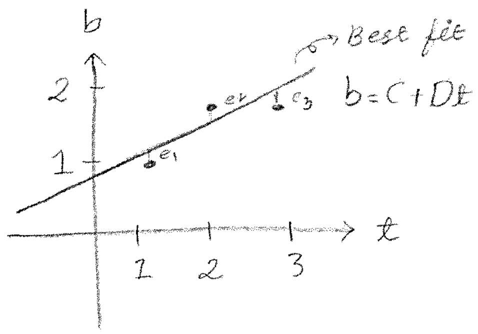
- If A is a square matrix, we could rewrite $*$ as, $P = AA^{-1}(A^T)^{-1}A^T = I$, A should be inverteable too.

$\rightarrow C(A)$ in this case is the whole of $\mathbb{R}^n$. Thus, the projection matrix is I.

$\rightarrow P^T = P$; $(AA^{-1}A^{T^{-1}}A^T)^T = AA^{-1}A^{T^{-1}}A^T$

$\rightarrow P^2 = P$; $A(A^TA^{-1})\underbrace{\cancel{A^T \cdot A(A^TA)^{-1}}}_{A^TAA^{-1}A^{T^{-1}}}A^T = P$:)

Least Square Fitting by a Line



Suppose that we're given a collection of data points (t, b): $\{(1,1), (2,2), (3,2)\}$

The line $b = C + Dt$ is the closest to the collect.

If the line went through all three points, we'd have:

$$\left.\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}\right\} \text{ This is equivalent to: } \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

But the line doesn't go through the 3 points! so this equ. isn't solvable. Instead, we can solve: $A^T A \vec{x} = A^T \vec{b}$.

Projection onto Subspaces (Recap) Lec. 15) :

Suppose that we have vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{R}^m$. Assume that our vectors are linearly independent; so they form bases for $\mathbb{R}^m$.

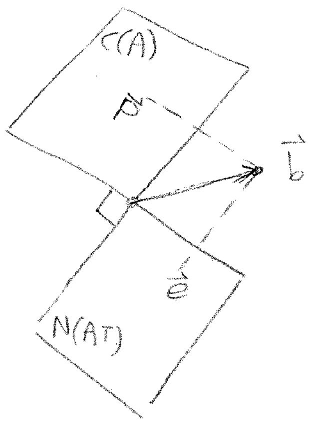
The span $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ is a subspace of $\mathbb{R}^m$ $(n<m)$.

Now, if we have the vector $\vec{b} \in \mathbb{R}^m$, to find the projection $\vec{p}$ onto span $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$:

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}_{m \times n}$$

→ if $\vec{b} \perp C(A)$, then it's in $N(A^T)$ & $P\vec{b} = \vec{0}$.

→ if $\vec{b}$ is in $C(A)$, then $\vec{b} = A\vec{y}$ for some $\vec{y}$, and $P\vec{b} = \vec{b}$. see geometrically:



$\vec{b} = \vec{p} + \vec{e}$, $\vec{e} = \vec{b} - \vec{p}$

$\vec{e} \perp \vec{p}$

$\vec{e} \perp C(A)$

* The matrix projecting $\vec{b}$ onto $N(A^T)$ is $I-P$:

$$\vec{e} = \vec{b} - \vec{p} = \vec{b} - P\vec{b} = \underbrace{(I-P)}_{\text{has all the properties of } P}\vec{b}$$

Ex. #1

If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. Find $\vec{y}$, $\vec{p}$ and P

sol.: $\vec{y} = (A^T A)^{-1} A^T \vec{b}$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\Rightarrow A\vec{y} = \vec{b} \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 0 & 2 & -6 \end{array}\right] \Rightarrow \begin{array}{l} 3y_1 + 3y_2 = 6 \\ 2y_2 = -6 \end{array} \Rightarrow y_2 = -3$$

$$\Rightarrow 3y_1 + 3(-3) = 6$$

$$3y_1 = 6 + 9 = 15$$

$$y_1 = 5$$

$$\therefore \vec{y} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$\vec{p} = A\vec{y} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T$$

$$A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \left[\begin{array}{cc|cc} 3 & 3 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|cc} 3 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{2R_1 - 3R_2} \left[\begin{array}{cc|cc} 6 & 0 & 5 & -3 \\ 0 & 2 & -1 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{\frac{1}{6}R_1} \left[\begin{array}{cc|cc} 1 & 0 & 5/6 & -3/6 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right]$$

$$\underbrace{\begin{matrix} 5/6 & -3/6 \\ -1/2 & 1/2 \end{matrix}}_{(A^TA)^{-1}}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ -3 & 0 & 3 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

- A^TA is invertable if and only if A has linearly independent cols.

$\boxed{\text{Proof}}$ A $m \times n$ matrix

A^TA is an $n \times n$ matrix. $\Rightarrow N(A^TA) = \{\vec{0}\}$ iff $N(A) = \{\vec{0}\}$ ($\Longleftrightarrow$)

1) If $N(A) = \{\vec{0}\}$, we need to show that $N(A^TA) = \{\vec{0}\}$.

Take $\vec{x} \in N(A^TA)$, i.e., $\boxed{A^TA\vec{x} = \vec{0}} \longrightarrow (\vec{x}^T A^T A \vec{x}) = \vec{x}^T \vec{0} = \vec{0}$

$(A\vec{x})^T (A\vec{x}) = \vec{0}$

$\|A\vec{x}\|^2 = \vec{0}$

$\|A\vec{x}\| = \vec{0} \Rightarrow$

the only way for a vector to have a 0 length is for that vector to be 0

Remember: $a^Ta = \|a\|^2$

$\Rightarrow A\vec{x} = \vec{0} \Rightarrow \vec{x} \in N(A)$

$\vec{x} = \vec{0}$

$\Rightarrow N(A^TA) = \{\vec{0}\}$

* we can also see from here that $N(A^TA)$ is a subset of $N(A)$ ----- *

2) If $N(A^TA) = \{\vec{0}\}$, we'll show that $N(A) = \{\vec{0}\}$.

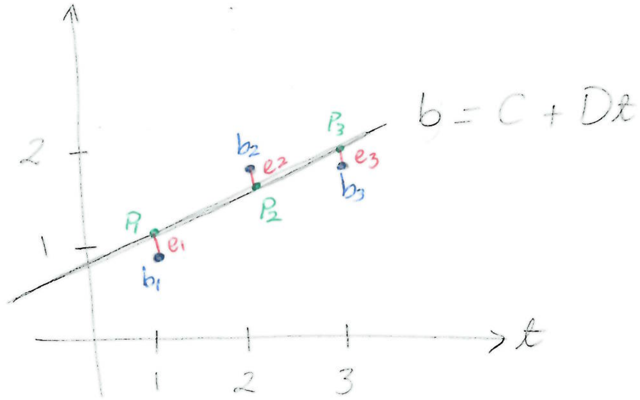
$\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0}$

$(A^TA)\vec{x} = \vec{0} \Rightarrow \vec{x} \in N(A^TA) = \{\vec{0}\}$

* we can see from here that $N(A)$ is a subset of $N(A^TA)$ ---- **

These 2 facts; * & ** show us that $N(A) = N(A^TA)$; if we have for example $\left.\begin{matrix} S \subseteq T \\ T \subseteq S \end{matrix}\right\}$ then we say that $T = S$

① Lease Squares



$$\left.\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}\right\} \text{ they don't have a sol.}$$

$$\|$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \quad \text{-------} \; ***$$

we want to find the <u>closest</u> line b = C + Dt to the points (1,1), (2,2), (3,2). closest means a line that minimizes the error represented by the distance from the points to the line. We measure these errors by adding up the squares of these distances; we want to minimize

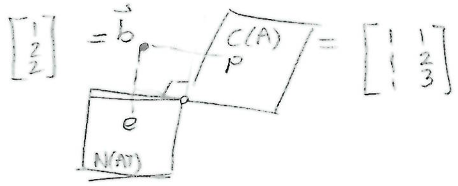
$$\|A\vec{x} - \vec{b}\|^2 = \|e\|^2 = e_1^2 + e_2^2 + e_3^2$$

The used method here is called <u>linear regression</u>, It is most useful if non of the data points are outliers.

There are two ways to look at $***$:

1) In the space of the line we're trying to find, e_1, e_2, and e_3 are the vertical distances from the data points to the line. The components p_1, p_2, and p_3 are the values of C + Dt near each data point; $\vec{p} \approx \vec{b}$.

2) The vector $\vec{b}$ is in $\mathbb{R}^3$, it's projection $\vec{p}$ onto the C(A), and its projection $\vec{e}$ onto N(AT);



Now, find $\vec{y} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$A^TA\vec{y} = A^T\vec{b}$

$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$\left.\begin{matrix} 3\hat{C} + 6\hat{D} = 5 \\ 6\hat{C} + 14\hat{D} = 11 \end{matrix}\right\}$ normal equations

solving the system will show that $\hat{C} = \frac{2}{3}$, $\hat{D} = \frac{1}{2}$

$\Rightarrow$ the best line: $\frac{2}{3} + \frac{1}{2}t$

We could also have used calculus to find the minimum of the following function of 2 variables: $e_1^2 + e_2^2 + e_3^2 = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$.

Either way, we end up solving a system of linear eqns. to find that the closest line to our points is $b = \frac{2}{3} + \frac{1}{2}t$.

$\Rightarrow P_1 = \frac{2}{3} + \frac{1}{2} = 7/6$ $\qquad e_1 = 1 - 7/6 = -1/6$

$P_2 = \frac{2}{3} + \frac{1}{2} \cdot 2 = 5/3$ $\qquad e_2 = 2 - 5/3 = 1/3$

$P_3 = \frac{2}{3} + \frac{1}{2} \cdot 3 = 13/6$ $\qquad e_3 = 2 - 13/6 = -1/6$

Remember that $\vec{p} + \vec{e} = \vec{b}$

$\Rightarrow \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 1/3 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

* $\vec{e}$ & $\vec{p}$ are orthogonal.
* $\vec{e} \perp$ columns of A

Orthogonal Matrices and Gram-Schmidt : (Lecture 16)

- Gram Schmidt : A process starts with any basis and produces an ortho[normal]

[b]asis that spans the same space as the original basis.

Orthonormal Vectors

The vectors $\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_n$ are <u>orthonormal</u> if :

$$\vec{q}_i^T \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

i.e., they have (normal) length 1, and perpendicular (ortho) [to] each other. Orthonormal vectors are always independent.

Orthonormal Matrix

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad Q^T Q = \begin{bmatrix} — q_1^T — \\ \vdots \\ — q_n^T — \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

* If Q is square, then $Q^T Q = I$ tells us $Q^T = Q^{-1}$

$\boxed{\text{Ex. \#1}}$

$$\text{perm } Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow Q Q^T = I$$

$\boxed{\text{Ex. \#2}}$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ ✓ orthogonal

Ex. #3

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad Q^TQ = \sqrt{2} \neq I$$

not orthogonal

How shall we fix it to be orthogonal?

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Ex. #4

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Ex. #5

$$Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$ rectangular matrix w/ orthonormal coulmns

- Orthonormal coulmns are good

Q has orthonormal coulmns. The matrix that projects onto the coulmn space of Q is: $P = Q^T(Q^TQ)^{-1}Q^T$

$= QQ^T = I$ if Q square

because the coulmns of Q span the entire space.

* Many equations become trivial when using a matrix w/ orthon.
coulmns;

$A^T A \hat{x} = A^T \vec{b}$

Now, A is Q

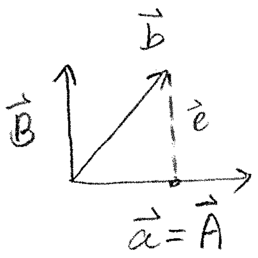
$\underbrace{Q^T Q}_{I} \hat{x} = Q^T \vec{b} \Rightarrow \hat{x}_i = q_i^T \vec{b}$ ---- Important

---

Gam-Schmidt

- Elimination $\longrightarrow$ make the matrix triangular
- GS $\longrightarrow$ make the matrix orthonormal.

consider two independent vectors $\vec{a}, \vec{b}$. Find orthonormal vectors $\vec{q}_1$ and $\vec{q}_2$ that span the same plane:



$\boxed{1}$ Find orthogonal vectors $\vec{A}$ and $\vec{B}$ that span the same plane as $\vec{a}$ and $\vec{b}$.

$\vec{a} = \vec{A}$ $\quad$ $\vec{B} = \vec{b} - \vec{p}$

$$= \vec{b} - \frac{\vec{A}^T \vec{b}}{\vec{A}^T \vec{A}} \vec{A}.$$

$* \vec{A}^T \vec{B} = 0.$

$\boxed{2}$ The unit vectors $\vec{q}_1 = \frac{\vec{A}}{\|\vec{A}\|}$ and $\vec{q}_2 = \frac{\vec{B}}{\|\vec{B}\|}$

form the desired orthonormal bases.

what if we had started with 3 independent vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$?

Then we'd find a vector $\vec{C}$ orthogonal to $\vec{A}$ & $\vec{B}$;

$$\vec{C} = \vec{c} - \frac{\vec{A}^T\vec{b}}{\vec{A}^T\vec{A}}\vec{A} - \frac{\vec{B}^T\vec{c}}{\vec{B}^T\vec{B}}\vec{B}.$$

$\boxed{\text{Ex. \#6}}$

$A = QR$

Properties of Determinants (Lec. 17)

- The determinant is a number associated with any square matrix, it's written like det A or |A|. The determinant encodes a lot of info. about the matrix; the matrix is invertable exactly when the determinant is non-zero.

Def. :

consider the matrix $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]^{n\times n}$, where $\vec{a}_i$ is the i-th coulmn of A. A mapping $D : Mat_{n\times n} F \longrightarrow F$ is determinantal if it is :

① 1-preserving in the sense that

$D(I_n) = 1_F$;

② Alternating in the sense that

$D[\ldots, \vec{a}_i, \ldots, \vec{a}_j, \ldots] = - D[\ldots, \vec{a}_j, \ldots, \vec{a}_i, \ldots]$;

③ Multilinear (a linear function of each coulmn) in the sense that

a- $D[\ldots, \lambda\vec{a}_i, \ldots] = \lambda D[\ldots, \vec{a}_i, \ldots]$;

b- $D[\ldots, \vec{b}_i + \vec{c}_i, \ldots] = D[\ldots, \vec{b}_i, \ldots] + D[\ldots, \vec{c}_i, \ldots]$.

Theorem #1 :

If D satisfies (3b), then D satisfies ② if and only if it satisfies the property

②' $D(A) = 0$ whenever A has 2 identical coulmns

Determinants (Lecture 17 & 18)

3 defining properties :

① $\det I = 1$;

② if 2 rows are exchanged, then the sign of $\det A$ changes;

Ex. :

$P \longrightarrow$ permutation matrix $\begin{cases} \det = +1 \\ \text{if even \# of row exchanges} \\ \det = -1 \\ \text{if odd \# --- --- ---} \end{cases}$

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \underset{②}{=} -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \underset{①}{=} -1$$

③ Linearity row-by-row

a) $k \in \mathbb{R}$: $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix}$

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

$$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k \underset{3a}{=} \begin{vmatrix} ka & kb \\ c & d \end{vmatrix}$$

$$\overset{3a}{=} kk \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$\Rightarrow k^n \det [A_{n \times n}] = \det [kA_{n \times n}]$

---

b) $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

* $\det[A+B] \neq \det[A] + \det[B]$

From these 3 properties we can deduce many others

(4) If 2 rows of A are equal, then $\det A = 0$.

$\det A = -\det A \Rightarrow \det A = 0$

(5) If there's a row of zeros, then $\det A = 0$

(6) $\det A = \pm \det U$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{GE} \begin{bmatrix} a & b \\ c-la & d-lb \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cancelto{0}{- l \begin{vmatrix} a & b \\ a & b \end{vmatrix}}$$

(7) $U = \begin{bmatrix} d_1 & * & * & * \\ & d_2 & * & * \\ & & \ddots & * \\ 0 & & & d_n \end{bmatrix}$, $\det U = d_1 d_2 \cdots d_n$

(8) $\det A = 0 \Leftrightarrow$ A is singular (i.e. A^{-1} does not exist)

$\Rightarrow$: If $\det A = 0 \Rightarrow A^{-1}$ doesn't exist

A^{-1} exists $\Rightarrow \det A \neq 0$

$\Rightarrow$ A is full rank

$\Rightarrow$ A has n pivots

$\Rightarrow \det A = \pm d_1 d_2 \cdots d_n \neq 0$

$\Leftarrow$: If A is singular $\Rightarrow \det A = 0$

A is singular $\Rightarrow$ U has at least one row of zeros

$\Rightarrow \det U = 0$

$\Rightarrow \det A = \pm \det U$

⑨ $\det[AB] = \det A \cdot \det B$

$\rightarrow \det(AA^{-1}) = \det(I) \overset{①}{=} 1$

$= \det A \cdot \det A^{-1}$

$\Rightarrow \det A^{-1} = \dfrac{1}{\det A}$

$\rightarrow \det(Q^TQ) = \det Q^T \cdot \det Q = \det I = 1$

$= (\det Q)^2 = 1 \longrightarrow$ ⑩

$\Rightarrow \det Q = \pm 1$

⑩ $\det A = \det A^T$

All properties written for rows can be stated in terms of columns.

$\det A \overset{?}{=} \det A^T$; $A = LU$

$\det(LU) = \det(U^TL^T)$

$\underbrace{\det L}_{=1} \cdot \underbrace{\det U}_{= d_1 d_2 \dots d_n} = \underbrace{\det U^T}_{= d_1 d_2 \dots d_n} \underbrace{\det L^T}_{= 1}$

⑪ Formula for the determinant:

- 2x2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{=0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{=0}$$

$$= ad \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=1} + bc \underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{-1}$$

$$= ad - bc$$

- 3x3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

BIG FORMULA

$$\det A = \sum_{n!\text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

$(\alpha, \beta, \gamma, \ldots, \omega)$ = Permutation of $(1, 2, 3, \ldots, n)$

$\boxed{\text{Ex. \# 1}}$

Compute the determinant of : $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

The permutation corresponding to the diagonal running from a_{14} to a_{41} is

$(4, 3, 2, 1) \Rightarrow +1$

Another non-zero term of $\sum \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$ comes from the permutation

$(3, 2, 1, 4) \Rightarrow -1$

$\Rightarrow$ the determinant is zero

Cofactor Formula

A formula that rewrites the big formula for the determinant of an $n \times n$ matrix in terms of the determinants of smaller matrices.

- 3×3 matrix :

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$ cofactors

- cofactor of a_{ij} $= \underset{\text{depends on } i+j}{\pm} \det(n-1$ matrix with row i, col. j erased$)$
  $= C_{ij}$

$\Rightarrow \det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$, along row 1 $\qquad C_{ij} = (-1)^{i+j}$

Tridiagonal Matrix

A **tridiagonal matrix** is one for which the only non-diagonal entries lie on or adjacent to the diagonal.

$\boxed{\text{Ex. \#2}}$

- $|T_1| = [1] = 1$

- $|T_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1\cdot 1 - 1\cdot 1 = 0$

- $|T_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(1\cdot 1 - 1\cdot 1) + 1(0 - 1\cdot 1) + 0 = -1$

- $|T_4| = \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1\underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}}_{A_3} - 1\underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}}_{0 = A_2} + 0 + 0$

  $= 1|T_3| - 1|T_2| = -1$

$\Rightarrow |T_n| = |T_{n-1}| - |T_{n-2}|$

• We get a sequence which repeats every 6 terms.

$|T_1| = 1, |T_2| = 0, |T_3| = -1, |T_4| = -1, |T_5| = 0, |T_6| = 1, |T_7| = 1.$

ⓘⓘⓘ How the determinant can be used?

• Formula for A^{-1} assuming it exists

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T$$ → product of $n-1$ entries

$\det A$ → product of n entries

Check $AC^T = (\det A) I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

The entry in the 1st row & 1st caulmn of the product matrix is:

$$\sum_{j=1}^{n} a_{1j} C_{j1} = \det A$$ → cofactor formula

Now, check that the off diagonal entries of AC^T are zero. In the 2x2 case, entries of row 1 times entries of caulmn 2 = 0; $a.-b + b.a = 0$. This is the det. of $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$. This happens with all off-diagonal entries, which confirms that $AC^T = \frac{1}{\det A} C^T$.

* This formula helps us answer questions about how the inverse changes when the matrix changes.

• Cramer's Rule for $A\vec{x} = \vec{b}$

We know that if $A\vec{x} = \vec{b}$ and A is non-singular, then $\vec{x} = A^{-1}\vec{b}$, $A^{-1} = \frac{C^T}{\det A}$

$$\Rightarrow \vec{x} = \frac{C^T \vec{b}}{\det A}$$ Cramer's rule

To derive this rule, break $\vec{x}$ to it's components;

$x_1 = \frac{\det B_1}{\det A}$ → any time I multiply cofactor by a number ($C^T.b$ in this case), I get a determinant

$x_j = \frac{\det B_j}{\det A}$

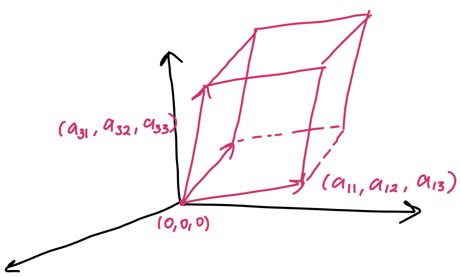
$B_1 = \begin{bmatrix} | & \text{last } n-1 \\ b & \text{columns} \\ | & \text{of } A \end{bmatrix}$, $B_j =$ A with column j replaced by b.

* computing inverses using Cramer's rule is usually less efficient than using elimenation.

• |det A| = Volume of box

<u>claim</u>: |det A| is the volume of a box (parellepiped) whose edges are the columns of A.

(using row vectors is valid too, it forms different box but with same volume.)



→ If $A = I$, then the box is a unit volume cube and its volume = 1, because it agrees with the claim.

⇒ The volume obeys determinant property;

1. If $A = Q$ is an orthogonal matrix, then the box is a unit cube in a different orientation with volume $= 1 = |\det Q|$.

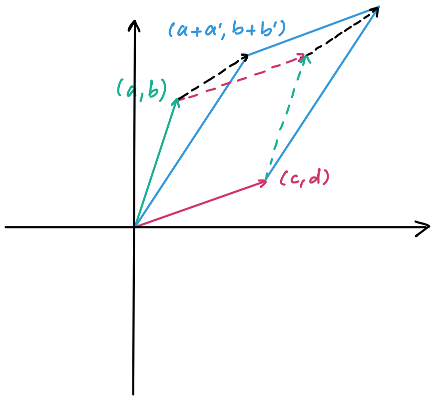
$\det Q = \pm 1$, why? from property 9 of determinats, $Q^TQ = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$

2. Swapping 2 columns of A doesn't change the volume of the box (remembering that $|A| = |A^T|$) the absolute volume of the determinent (property 2). If we show that the volume of the box also obeys property 3, we will have proven $|A| =$ the volume of the box.

3. a) If we double the length of one coulmn of A, we double the volume of the box formed by its coulmns.

b) The determinant is linear in the rows of the matrix.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$



4. If 2 edges are equal, the box flatterns out and has no volume.

Note: If you know the coordinates for the corners of a box, then computing the volume is as easy as calculating the determinant. In particular, the area of a parallelogram with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ = $ad - cb$.

The area of a triangle with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is half the area of that parallelogram; $\frac{1}{2}(ad - cb)$.

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is: $\frac{1}{2}\begin{vmatrix} x_1 & x_2 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Eigenvalues and Eigenvectors : (Lec. 19)

consider an $n \times n$ matrix A

Def. :

Eigenvectors are non-zero vectors for which $A\vec{x}$ is parallel to $\vec{x}$; $A\vec{x} = \lambda\vec{x}$. The scalar λ is the eigenvalue.

- By definition, eigenvalues are non-zero.
- If $\lambda = 0$, then $A\vec{x} = \vec{0}$. vectors with $\lambda = 0$ make up the nullspace of A; if A is singular, then $\lambda = 0$ is an eigenv. of A.

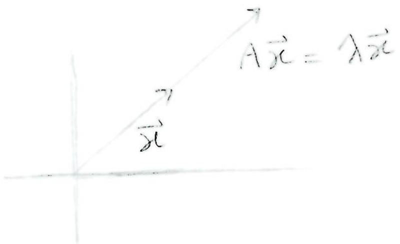
$A\vec{x} \neq k\vec{x} \qquad k \in \mathbb{R}$

I'm looking for all $\vec{x} \neq \vec{0}$ s.t.

$A\vec{x} = \lambda\vec{x}$

$\vec{x} \rightarrow$ eigenvector of A

$\lambda \rightarrow$ eigenvalue of A



① $\vec{x} \neq \vec{0}$

② $A\vec{x} = \lambda\vec{x} \rightarrow 2$ unkowns: λ, x

$A\vec{x} = \vec{b} \rightarrow 1$ unknown

③ $\underbrace{A\vec{x} = \lambda\vec{x}}$, $\quad \vec{x} \neq \vec{0}$

$c \in \mathbb{R}$

$$A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x})$$

$\rightarrow$ if $\vec{x}$ is evector of A w/ λ evalue, then $c\vec{x}$ is also an evector w/ the same evalue

---

Ex. #1

I $n \times n$; $\quad I\vec{x} = \vec{x} \quad \boxed{\lambda = 1}$

* Any $\vec{x} \neq 0, \vec{x} \in \mathbb{R}^n$ is an evector with $\lambda = 1$. I can choose n orthogonal $\vec{x} \neq 0$ as n eigenvectors.

Ex. #2

$A\vec{x} = \lambda\vec{x}, \quad \vec{x} \neq 0$

$\lambda > 0$ $\quad \vec{x}, A\vec{x}$ point towards the same direction.

$\lambda < 0$ $\quad \vec{x}, A\vec{x}$ point in opposite directions.

$\lambda = 0, \quad A\vec{x} = 0$

Ex. #3

The permutation matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has an evector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w/ $\lambda = 1$ and another evector $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ w/ $\lambda = -1$;

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \Rightarrow B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the evector of B and $\lambda = 1$

$\Rightarrow B\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the evector of B and $\lambda = -1$

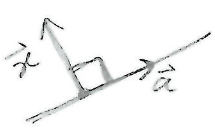
* These 2 evectors span the space.
They are $\perp$ because $B = B^T$ (as will be proven).

Ex. #4

Suppose P is the matrix of a projection onto a plane.

a) $\vec{x} \in C(P)$

$P\vec{x} = 1\vec{x} \Rightarrow$ all $\vec{x} \neq \vec{0}$ in $C(P)$ are evector of P w/ $\lambda = 1$.



b) $\vec{x} \perp C(P), \; \vec{x} \neq \vec{0} \iff \vec{x} \in N(P^T)$

$P\vec{x} = 0 = 0\vec{x}$ so this is an evector w/ $\lambda = 0$.

* The evectors of P span the whole space (but this's not true for every matrix).

Theorem #1

A scalar λ is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0$$

This is called the characteristic equation of A.

Proof

$A\vec{x} = \lambda\vec{x}$ can be rewritten as $(A - \lambda I_n)\vec{x} = 0$

$\rightarrow$ λ is an evalue of A if and only if the homogeneous system of equations $(A - \lambda I_n)\vec{x} = \vec{0}$ has a non-zero solution. This is the case if and only if the matrix $A - \lambda I_n$ is singular, and this is equivalent to $\det(A - \lambda I)$ being zero.

* We can use the characteristic equation for λ to get n solutions. Once we're found a λ, we can use elimination to find the nullspace of $A - \lambda I$. The vectors in that nullspace are evectors of A w/ λ.

* Note that with $A = [a_{ij}]_{n \times n}$ we have

$$\det(A - \lambda I_n) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

We see that this is a polynomial of degree n in λ (recalling that the product of the diagonal elements is a term in the $\sum_\sigma$-expansion; $\det(A - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$

Ex. #5

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ Find λ and $\vec{x}$:

---

* $A^T = A$: • all the n evalues are real
• there're n lin. ind. evectors

I) λ:

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)^2 - 1 = 0$$

$$9 - 6\lambda + \lambda^2 - 1 = 0$$

$$8 - 6\lambda + \lambda^2 = 0$$

$$\Rightarrow \lambda = \frac{6 \pm \sqrt{36-32}}{2} \quad \lambda_1 = 4, \quad \lambda_2 = 2$$

II) • $\lambda_1 = 4$ in the nullspace of $A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

• $\lambda_2 = 2$ in the nullspace of $A - \lambda_2 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The nullspace is an entire line, which could be any vector on that line. Natural choise is $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

or $A^T = A$, so $\vec{x}_2 \perp \vec{x}_1$, let $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Note that these eigenvectors are the same as those of $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$A = B + 3I$, Adding $3I$ to B added 3 to each of its evalues and did not change the evectors.

Because $A\vec{x} = (B+3I)\vec{x} = \lambda\vec{x} + 3\vec{x} = (\lambda + 3)\vec{x}$

---

similarly, if $A\vec{x} = \lambda\vec{x}$ and $B\vec{x} = \alpha\vec{x} \Rightarrow (A+B)\vec{x} = (\lambda+\alpha)\vec{x}$

or $B = C + kI \quad , k \in \mathbb{R}$

$C\vec{x} = \lambda\vec{x}$

$$\Rightarrow B\vec{x} = (C + kI)\vec{x}$$
$$= C\vec{x} + kI\vec{x}$$
$$= \lambda\vec{x} + k\vec{x}$$
$$= (\lambda + k)\vec{x}$$

* This is only true if A & B have the same eigenvectors.

* The eigenvalues of $AB \neq \lambda(A)\lambda(B)$

$\boxed{\text{Ex. \#6}}$

Diagonlization:

if

$A_{n\times n}$ has n lin. ind. eigenvectors

→ then we can do a multiplication to transform it to a diagonal matrix (easy to compute with).

- 2×2 matrix:

$\lambda_1, \vec{x}_1 = \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix}$ $\lambda_2, \vec{x}_2 = \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix}$

$A_{2\times 2}$: $\left.\begin{array}{l} A\vec{x}_1 = \lambda_1\vec{x}_1 \\ A\vec{x}_2 = \lambda_2\vec{x}_2 \end{array}\right\}$ $A\underbrace{\begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}}_{S} = \begin{bmatrix} \lambda x_1 & \lambda_2 x_2 \end{bmatrix}$ --- *

* can be decomposed: $\begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

⇒ $AS = S\Lambda$

Λ ↳ our diagonal matrix

S ↓ if its caulmns are lin. ind. → it's invertable

⇒ $A = S\Lambda S^{-1}$ ⟶ factorization

⇒ $\Lambda = S^{-1}AS$ ⟶ diagonalization

Diagonlization (Lecture 21)

The point of this is to make calculations easier, the same with LU & QR.

Consider n lin. ind. evectors of A, then put these coulmns in a square and invertible matrix S. Then

$$AS = A\begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix}\underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & \lambda_2 & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{diagonal evalue matrix}} = S\Lambda$$

Because the coulmns of S are ind., S^{-1} exists.
We can multiply both sides of $AS = S\Lambda$ by S^{-1}; $S^{-1}AS = \Lambda$ or

$A = S\Lambda S^{-1}$

Powers of A

what are the evectors and evalues of A^2?
If $A\vec{x} = \lambda\vec{x}$, then $A^2\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}$. The evalues of A^2 are the squares of the evalues of A. If we write $A = S\Lambda S^{-1}$, then:
$A^2 = S\Lambda \cancel{S^{-1}S}\Lambda S^{-1} = S\Lambda^2 S^{-1}$. Similarly, $A^K = S\Lambda^K S^{-1}$ tells us that raising the evalues of A to the kth power gives us the eigenvalues of A^K, and that the evectors of A^K are the same as those of A.

$\boxed{\text{Theorem \#1}}$

If A has n ind. evectors w/ evalues λ_i, then $A^K \to 0$ as $K \to \infty$ if and only if all $|\lambda_i| < 1$.

* A is guaranteed to have n ind. evectors (and be diagonaliza-
ble) if all it's evalues are different.

Repeated Eigenvalues

If A has repeated eigenvalues, it may or may not have n ind. evectors. Take the id matrix for example. All its evalues are 1, but the matrix still has n ind. evectors.

If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ its evalues are 2 & 2. its evectors are in the nullspace of $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is spanned by $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} c \\ 0 \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This particular A doesn't have 2 ind. evectors.

Difference Equations $\vec{u}_{k+1} = A\vec{u}_k$

start with $\vec{u}_0$ (given). We can creat a sequence of vectors in which each new vector is A times the previous vector: $\vec{u}_{k+1} = A\vec{u}_k$

which is <u>a first order difference equation</u>, and $\vec{u}_k = A^k \vec{u}_0$ is a solution to this system. We can also write $\vec{u}_0$ as a comb. of evectors of A:

$$\vec{u}_0 = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = S\vec{c}$$

Then: $A\vec{u}_0 = c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \cdots + c_n\lambda_n\vec{x}_n$

and: $\vec{u}_k = A^k\vec{u}_0 = c_1\lambda_1^k\vec{x}_1 + c_2\lambda_2^k\vec{x}_2 + \cdots + c_n\lambda_n^k\vec{x}_n = \Lambda^k S\vec{c}$.

Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, ... In general, $F_{k+2} = F_{k+1} + F_k$. To know how fast the numbers in the sequence are increasing, we could use evalues and evectors. $\vec{u}_{k+1} = A\vec{u}_k$ was a first order system. $F_{k+2} = F_{k+1} + F_k$ is a 2nd order scalar equation, but we can convert it to 1st order linear system by using a ~~trick~~. If

$\vec{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, then:

$$F_{k+2} = F_{k+1} + F_k \text{ ------ } (1)$$

$$F_{k+1} = F_{k+1} \text{ ------ } (2)$$

is equivalent to the 1st order system $\vec{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}_k$.

Now, find evectors of A.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = -\lambda + \lambda^2 - 1 = 0$$

$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$; i.e. $\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ & $\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -1.618$. The growth rate of the F_k is controlled by λ_1, the only $|\lambda| > 1$. This tells us for large k, $F_k \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^k$ for some constant c_1.

Find evectors of A.

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ when } \vec{x} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

Finally, $\vec{u}_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\vec{x}_1 + c_2\vec{x}_2$ tells us that $c_1 = -c_2 = \frac{1}{\sqrt{5}}$.

Because $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$, we get:

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k.$$

using eigenvalues and eigenvectors, we have found a closed form expression for the Fibonacci #s.

What is the 10001 element in Fibonacci sequence? (Lec. 22)

$\vec{u}_{k+1} = A\vec{u}_k \qquad k \geq 0$

$k=0$: $\vec{u}_1 = A\vec{u}_0$

$k=1$: $\vec{u}_2 = A\vec{u}_1 = A(A\vec{u}_0) = A^2\vec{u}_0$

$\vdots$

$k \geq 0 \quad \vec{u}_k = A^k\vec{u}_0 = \underbrace{(A \cdots A)}_{k\ \text{times}}\vec{u}_0$

$\vec{u}_k = \begin{bmatrix} F_{k+1} \\ f_k \end{bmatrix}$

Remember:

- $\vec{u}_0 = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$

$\rightarrow \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_3\}$ is a basis for $\mathbb{R}^n$

- $\vec{u}_k = A^k\vec{u}_0 \qquad , A^k = S\Lambda^k S^{-1}$

$= A^k(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n)$

$= c_1\lambda_1^k\vec{x}_1 + \cdots + c_n\lambda_n^k\vec{x}_n$

$$\begin{cases} \vec{u}_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A}\vec{u}_k \\ \vec{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

$\text{trace}A = 1 + 0 = \sum a_{ii}$

1st step: solve for evalues & evectors of A.

$|A - \lambda I| = \vec{0}$

$\lambda^2 - (\text{trace}A)\lambda + \det A = 0$

$$\begin{cases} \lambda_1 \lambda_2 = \det \\ \lambda_1 + \lambda_2 = \text{trace } A = 1 \end{cases}$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \text{ "golden \#"}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$$

- $\lambda_1, \vec{x}_1$?

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}}_{\vec{x}_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-\lambda_1)\lambda + 1 = 0$$

$$\lambda_1 - \lambda_1^2 + 1 = 0$$

- $\lambda_2, \vec{x}_2$?

$$A^T = A \Rightarrow \vec{x}_2 \perp \vec{x}_1 \Rightarrow \vec{x}_2 = \begin{bmatrix} -1 \\ \lambda_1 \end{bmatrix}$$

<u>2nd step</u>: $\vec{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{u}_0 = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} \lambda_1 \\ 1 \end{matrix}}_{\vec{x}_1} & \underbrace{\begin{matrix} -1 \\ \lambda_1 \end{matrix}}_{\vec{x}_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\lambda_1^2 + 1} \begin{bmatrix} \lambda_1 & 1 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\lambda_1^2 + 1} \begin{bmatrix} \lambda_1 \\ -1 \end{bmatrix}$$

3rd step:

$$\vec{u}_k = c_1 \lambda_1^k \vec{x}_1 + c_2 \lambda_2^k \vec{x}_2$$

$$\vec{u}_k = \frac{\lambda_1}{\lambda_1^2+1} \lambda_1^k \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{\lambda_2^k}{\lambda_1^2+1} \begin{bmatrix} -1 \\ \lambda_1 \end{bmatrix}$$

f_{10001} ? $\quad u_{10000} = \begin{bmatrix} f_{10001} \\ f_{10000} \end{bmatrix} \qquad \vec{u}_k = \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$

$$f_{10001} = \frac{\lambda_1^{10002}}{\lambda_1^2+1} + \frac{\lambda_2^{10000}}{\lambda_1^2+1}$$

$$\rightarrow \vec{u}_k = \begin{bmatrix} \dfrac{\lambda_1^{k+2}}{\lambda_1^2+1} + \dfrac{\lambda_2^k}{\lambda_1^2+1} \\ \dfrac{\lambda^{k+1}}{\lambda_1^2+1} - \dfrac{\lambda_1 \lambda_2^k}{\lambda_1^2+1} \end{bmatrix}$$

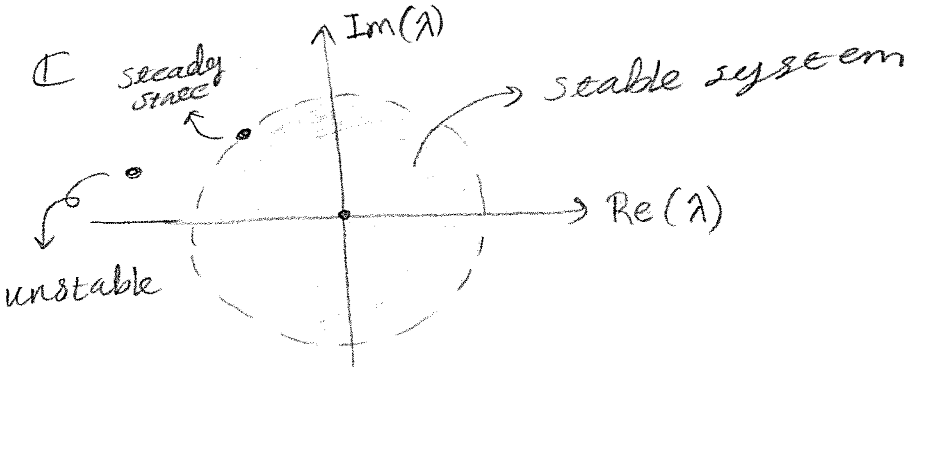
Stability

The eigenvalues of A will tell us what sort of solutions to expect;

① stable: if $|\lambda_i| < 1$, then $\lim_{t \to +\infty} u_t = 0$

② steady state: One evalue is zero and all other evalues have ⊖ real part.

③ Blow up: if $Re(\lambda) > 0$ for any evalue λ.



$\lim_{k \to +\infty} \vec{u}_k$?

$\lambda_1 \simeq 1.618$

$\lambda_2 \simeq -0.618$

$$\vec{u}_k = c_1 \overset{\nearrow +\infty}{\lambda_1^k} \vec{x}_1 + c_2 \overset{\nearrow 0}{\lambda_2^k} \vec{x}_2$$

c_1, $\vec{x}_1$, c_2, $\vec{x}_2$ ↓ All constants

(Blow up, unstable system)

System of 1^{st} order linear ODE

$$\begin{cases} \dfrac{du_1}{dt} = -u_1 + 2u_2. \\ \dfrac{du_2}{dt} = u_1 - 2u_2. \end{cases}$$

This system describes how the values of variables u_1 and u_2 affect each other over time.

We can use LA to solve this differential equation, just as we applied it to solve a difference equation.

For example, the initial condition can be written as $\vec{u}_0 = \vec{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\frac{d\vec{u}}{dt} = \begin{bmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\vec{u}} \Rightarrow \dot{\vec{u}} = A\vec{u}$$

1^{st} step to solve the system is to find λ's & $\vec{x}$'s:

I) λ_1, λ_2

Because A is singular and its trace is -3, we know that $\lambda_1 = 0$ & $\lambda_2 = -3$

$\det A = \lambda_1 \lambda_2$

$\det A = 0 \Rightarrow \lambda_1 = 0$

$\text{trace } A = \lambda_1 + \lambda_2$

$\Rightarrow \lambda_2 = -3$

The solution will turn out to include e^{-3t} and e^{0t}. As t increase e^{-3t} vanishes and $e^{0t}=1$ remains constant. evalues $=0$ have evectors that are (steady state) solutions.

#) $\lambda_1 = 0$, $\vec{x}_1$?

$(A-\lambda_1 I)\vec{x}_1 = \vec{0}$

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{x}_1} = \vec{0}$$

$\lambda_2 = -3$, $\vec{x}_2 = ?$

$(A+3I)\vec{x}_2 = \vec{0}$

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{x}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2nd step: the general saultion

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{x}_1 + c_2 e^{\lambda_2 t}\vec{x}_2$$

Plug in $\vec{u} = e^{\lambda_1 t}\vec{x}_1$ to see if it's really a solu of $\frac{d\vec{u}}{dt}$:

$\frac{d\vec{u}}{dt} = \lambda_1 e^{\lambda_1 t}\vec{x}_1$, which really agrees w/ $A\vec{u} = e^{\lambda_1 t}A\vec{x}_1 = \lambda_1 e^{\lambda_1 t}\vec{x}_1$.

Plugging in the values of the evectors:

$$\vec{u}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

3rd step: initial condition

at $t=0$, $\vec{u}(t) = \vec{u}(0) = c_1\vec{x}_1 + c_2\vec{x}_2$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-2-1}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$$\Rightarrow \vec{u}(t) = \tfrac{1}{3}\cancel{e^{0t}}_{1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \tfrac{1}{3}e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lim_{t \to +\infty} \vec{u}(t) = 1/3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = u_{ss}$$

$\hookrightarrow$ steady state

⑪ Similar Matrices

<u>Def.</u>: $A \sim B$ if there's some P so

$$A = PBP^{-1} \iff P^{-1}AP = B$$

A and B are equivalent on a long list of different properties: example $r(A) = r(B)$

- $\det(A - \lambda I) = \det(PBP^{-1} - \lambda PIP^{-1})$

$$= \det(P[B - \lambda I]P^{-1})$$

$$= \underbrace{\cancel{\det(P)}}_{\#} \det(B - \lambda I) \underbrace{\cancel{\det(P^{-1})}}_{1/\#}$$

$$= \det(B - \lambda I)$$

* similar matrices have the <u>same</u> eigenvalues

Positive Definite Matrices

A square matrix A is +ve definite and anyone of the following is true:

1. All evalues are +ve
2. All its pivots (w/o row exchange) +ve.
3. All upper left determinant of order 1, 2, ..., n of an $n \times n$ matrix A are +ve

- Example: A^TA +ve definite? $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$