AMCS 201 HW #4 Due on October 18, 2023

Haberman's book

2.3.3 (a) (b)

2.3.5

2.3.6

2.5.1 (a) (c)

2.5.9 (a) (b)

ERCISES 2.3

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

* (a) $\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

(b) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$

* (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$

* (e) $\frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4}$

* (f) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

2.3.2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

***(b)** $\phi(0) = 0$ and $\phi(1) = 0$

(c) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$ (If necessary, see Section 2.4.1.)

***(d)** $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(e) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

***(f)** $\phi(a) = 0$ and $\phi(b) = 0$ (You may assume that $\lambda > 0$.)

(g) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + \phi(L) = 0$ (If necessary, see Section 5.8.)

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$ (b) $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

*(c) $u(x,0) = 2\cos\frac{3\pi x}{L}$ (d) $u(x,0) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$

(e) $u(x,0) = f(x)$

[Your answer in part (c) may involve certain integrals that do not need to be evaluated.]

2.3.4. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = 0, u(L,t) = 0$, and $u(x,0) = f(x)$.

*(a) What is the total heat energy in the rod as a function of time?

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

*(c) What relationship should exist between parts (a) and (b)?

2.3.5. Evaluate (be careful if $n = m$)

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \qquad \text{for } n > 0, m > 0.$$

Use the trigonometric identity

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right].$$

***2.3.6.** Evaluate

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \qquad \text{for } n \ge 0, m \ge 0.$$

Use the trigonometric identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right].$$

(Be careful if $a - b = 0$ or $a + b = 0$.)

2.3.7. Consider the following boundary value problem (if necessary, see Section 2.4.1):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ with } \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, \text{ and } u(x,0) = f(x).$$

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint:* The answer is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t}\cos\frac{n\pi x}{L}.\Big]$$

What is λ_n?

EXERCISES 2.5

2.5.1. Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions [*Hint*: Separate variables. If there are two homogeneous boundary conditions in y, let $u(x,y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in x, let $u(x,y) = \phi(x)h(y)$.]:

*(**a**) $\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = f(x)$

(**b**) $\frac{\partial u}{\partial x}(0,y) = g(y)$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = 0$

*(**c**) $\frac{\partial u}{\partial x}(0,y) = 0$, $u(L,y) = g(y)$, $u(x,0) = 0$, $u(x,H) = 0$

(**d**) $u(0,y) = g(y)$, $u(L,y) = 0$, $\frac{\partial u}{\partial y}(x,0) = 0$, $u(x,H) = 0$

*(**e**) $u(0,y) = 0$, $u(L,y) = 0$, $u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0$, $u(x,H) = f(x)$

(**f**) $u(0,y) = f(y)$, $u(L,y) = 0$, $\frac{\partial u}{\partial y}(x,0) = 0$, $\frac{\partial u}{\partial y}(x,H) = 0$

(**g**) $\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$, $\frac{\partial u}{\partial y}(x,H) = 0$

(**h**) $u(0,y) = 0$, $u(L,y) = g(y)$, $u(x,0) = 0$, $u(x,H) = 0$

2.5.6. Solve Laplace's equation inside a semicircle of radius $a(0 < r < a,\ 0 < \theta < \pi)$ subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

*(a) $u = 0$ on the diameter and $u(a,\theta) = g(\theta)$

(b) the diameter is insulated and $u(a,\theta) = g(\theta)$

2.5.7. Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u(r,0) = 0, \quad u\left(r,\frac{\pi}{3}\right) = 0, \quad u(a,\theta) = f(\theta)$

*(b) $\frac{\partial u}{\partial \theta}(r,0) = 0, \quad \frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{3}\right) = 0, \quad u(a,\theta) = f(\theta)$

2.5.8. Solve Laplace's equation inside a circular annulus $(a < r < b)$ subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

*(a) $u(a,\theta) = f(\theta), \quad u(b,\theta) = g(\theta)$

(b) $\frac{\partial u}{\partial r}(a,\theta) = 0, \quad u(b,\theta) = g(\theta)$

(c) $\frac{\partial u}{\partial r}(a,\theta) = f(\theta), \quad \frac{\partial u}{\partial r}(b,\theta) = g(\theta)$

If there is a solvability condition, state it and explain it physically.

***2.5.9.** Solve Laplace's equation inside a 90° sector of a circular annulus $(a < r < b,\ 0 < \theta < \pi/2)$ subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u(r,0) = 0, \quad u(r,\pi/2) = 0, \quad u(a,\theta) = 0, \quad u(b,\theta) = f(\theta)$

(b) $u(r,0) = 0, \quad u(r,\pi/2) = f(r), \quad u(a,\theta) = 0, \quad u(b,\theta) = 0$

2.5.10. Using the maximum principles for Laplace's equation, prove that the solution of Poisson's equation, $\nabla^2 u = g(\boldsymbol{x})$, subject to $u = f(\boldsymbol{x})$ on the boundary, is unique.

2.5.11. Do Exercise 1.5.8.

2.5.12. (a) Using the divergence theorem, determine an alternative expression for $\iint u\, \nabla^2 u\, dx\, dy\, dz$.

EXERCISES 9.5

9.5.1. Consider (9.5.10), the eigenfunction expansion for $G(\boldsymbol{x}, \boldsymbol{x}_0)$. Assume that $\nabla^2 G$ has some eigenfunction expansion. Using Green's formula, verify that $\nabla^2 G$ may be obtained by term-by-term differentiation of (9.5.10).

9.5.2. (a) Solve

$$\nabla^2 u = f(x, y)$$

on a rectangle $(0 < x < L, 0 < y < H)$ with $u = 0$ on the boundary using the method of eigenfunction expansion.

(b) Write the solution in the form

$$u(\boldsymbol{x}) = \int_0^L \int_0^H f(\boldsymbol{x}_0) G(\boldsymbol{x}, \boldsymbol{x}_0)\, dx_0\, dy_0.$$

Show that this $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function obtained previously.