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9.3.1)

Take the Fourier sine series of

$$G(x,x_0) = \begin{cases} \frac{-x(L-x_0)}{L} & x < x_0 \\ \frac{-x_0(L-x)}{L} & x > x_0 \end{cases} \qquad (\text{Let } L=1)$$

We first find Fourier coefficients b_n (3.3.2), which is

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

For $G(x,x_0)$ w/ $L=1$

$$b_n = 2\left(\int_0^{x_0} G(x,x_0) \sin n\pi x \, dx + \int_{x_0}^1 G(x,x_0) \sin n\pi x\, dx\right)$$

constant

$$= 2\left(\underbrace{\int_0^{x_0} -x(1-x_0) \sin n\pi x\, dx}_{1} + \int_{x_0}^1 -x_0(1-x) \sin n\pi x\, dx\right)$$

$$= 2\left[(1-x_0)\left(\frac{x_0 \cos n\pi x_0}{n\pi} - \int \frac{\cos n\pi x_0}{n\pi}\, dx\right) + \underbrace{\left(x_0\int_{x_0}^1 x \sin(n\pi x)\, dx - x_0\int_{x_0}^1 \sin(n\pi x)\, dx\right)}_{2}\right]$$

① $\int u\,dv = uv - \int v\,du$

$u = x \qquad dv = \sin \pi x\, dx$

$du = dx \qquad v = \frac{\cos n\pi x}{\pi n}$

② same as ①

$$= 2\left[(1-x_0)\left(-\frac{x_0 \cos n\pi x_0}{n\pi} - \int \frac{\cos n\pi x_0}{n\pi}\, dx\right) + \frac{x_0(-\sin n\pi x_0 + \pi n(x_0-1)\cos \pi n x_0 + \sin n\pi}{(n\pi)^2}\right]$$

$$\Rightarrow b_n = \frac{2}{(n\pi)^2}\Big[(1-x_0)(n\pi x_0 \cos n\pi x_0 - \sin n\pi x_0) - x_0 \sin n\pi x_0 - x_0 n\pi$$
$$(1-x_0)\cos n\pi x_0\Big]$$

$$= \frac{2}{(n\pi)^2}(-\sin n\pi x_0)$$

$\Rightarrow$ Forier series for $G(x, x_0)$

$$G(x, x_0) = \sum_{n=1}^{\infty} 2\left[\frac{-\sin n\pi x_0}{(n\pi)^2}\right] \sin n\pi x$$

$$= -\sum_{n=1}^{\infty} 2\,\frac{\sin n\pi x_0 \cdot \sin n\pi x}{(n\pi)^2}$$

9.3.9)

a) Variation of parameters

① u_c: $u'' + u = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m_1 = m_2 = 0 \pm i$

$\Rightarrow u_1(x) = \sin x \qquad u_2(x) = \cos(x)$

$\Rightarrow u_c(x) = c_1 \sin x + c_2 \cos x$

② $u_p(x)$: $u_p(x) = V_1(x) u_1(x) + V_2(x) u_2(x)$

first, find w; $w = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$

$$V_1 = \int \frac{-u_1(x_0) f(x_0)}{w(x_0)} dx_0 = -\int_a^x \frac{\cos x_0 f(x_0)}{-1}$$

$$V_2 = \int \frac{u_2(x_0) f(x_0)}{w} dx_0 = \int_a^x \frac{\sin(x_0) f(x_0)}{-1} dx_0$$

$$\Rightarrow u_p(x) = \sin x \int_a^x \cos x_0 f(x_0) dx_0 - \cos x \int_a^x \sin x_0 f(x_0) dx_0$$

③ $u(x)$:

general sol:

$$u(x) = c_1 \sin x + c_2 \cos x + \sin x \int_a^x \cos x_0 f(x_0) dx_0 - \cos x \int_a^x \sin f(x_0) dx_0$$

Applying B.C. $u(0)=0$ $u(L)=0$

$$u(0)=0 \Rightarrow 0 = C_2 - \int_a^0 \sin x_0 f(x_0)\,dx_0 \Rightarrow C_2 = \int_a^0 \sin x_0 f(x_0)\,dx_0$$

$$u(L)=0 \Rightarrow 0 = C_1 \sin L + \left(\int_a^0 \sin x_0 f(x_0)\,dx_0\right)\cos L + \sin L \int_a^L \cos x_0 f(x_0)\,dx_0$$
$$- \cos L \int_a^L \sin x_0 f(x_0)\,dx_0$$

$$\Rightarrow C_1 = \frac{1}{\sin L}\left(-\cos L \int_a^0 \sin x_0 f(x_0)\,dx_0 - \sin L \int_a^L \cos x_0 f(x_0)\,dx_0 \right.$$
$$\left. + \cos L \int_a^L \sin x_0 f(x_0)\,dx_0\right)$$

$$\Rightarrow C_1 = \frac{1}{\sin L}\left(\cos L\left(\int_a^L \sin x_0 f(x_0)\,dx_0 - \int_a^0 \sin x_0 f(x_0)\,dx_0\right) - \sin L \right.$$
$$\left. \int_a^L \cos x_0 f(x_0)\,dx_0\right)$$

$$= \frac{1}{\sin L}\left(\cos L\left(\int_0^L \sin x_0 f(x_0)\,dx_0\right) - \sin L \int_a^L \cos x_0 f(x_0)\,dx_0\right)$$

$$u = \frac{\sin x}{\sin L}\left[\cos L\left(\int_0^L \sin x_0 f(x_0)\,dx_0\right) - \sin L\int_a^L \cos x_0 f(x_0)\,dx_0\right] + \left(\int_a^0 \sin x_0 f(x_0)\,dx_0\right)$$

③

$$\cos x + \sin x\int_a^x \cos x_0 f(x_0)\,dx_0 - \cos x\int_a^x \sin x_0 f(x_0)\,dx_0$$

⑤

③+⑤

$$= \left(-\cos x\int_0^x \sin x_0 f(x_0)\,dx_0\right) + \frac{\sin x\cos L}{\sin L}\int_0^x \sin x_0 f(x_0)\,dx_0 + \frac{\sin x\cos L}{\sin L}$$

$$\int_x^L \cos x_0 f(x_0)\,dx_0 - \sin x\int_x^L \cos x_0 f(x_0)\,dx_0$$

$$= \int_0^x \left(\frac{\sin x\cos L}{\sin L} - \cos x\right)\sin x_0 f(x_0)\,dx_0 + \int_x^L \left(\frac{\sin x\cos L}{\sin L} - \sin x\right)\cos x_0 f(x_0)\,dx_0$$

b)

$$\Rightarrow u(x) = \int_0^L G(x,x_0) f(x_0)\,dx_0$$

$$G(x,x_0) = \begin{cases} \left(\frac{\sin x\cos L}{\sin L} - \cos x\right)\sin x_0 & x > x_0 \\ \left(\frac{\sin x\cos L}{\sin L} - \sin x\right)\cos x_0 & x < x_0 \end{cases}$$

9.3.10)

The associated eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \phi = -\lambda\phi \quad ; \ \phi(0) = 0 \quad \phi(L) = 0$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \phi + \lambda\phi = 0$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \phi(1+\lambda) = 0$$

$$\Rightarrow \phi(x) = c_1 \cos(\sqrt{1+\lambda})x + c_2 \sin(\sqrt{1+\lambda})x$$

Applying B.C.

$$\phi(0) = 0 \Rightarrow 0 = c_1 \cos(0) + c_2 \sin(0)$$

$$\Rightarrow c_1 = 0 \quad \Rightarrow \phi(x) = c_2 \sin(\sqrt{1+\lambda})L \quad \text{---} \ (*)$$

$$\phi(L) = 0 \Rightarrow 0 = c_2 \sin(\sqrt{1+\lambda})L$$

Substitute in $(*)$

$$0 = c_2 \sin(\sqrt{1+\lambda})L$$

$$\sin(\sqrt{1+\lambda})L = n\pi \Rightarrow (\sqrt{1+\lambda}) = \frac{n\pi}{L}$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 - 1$$

$$\therefore \phi(x) = c_2 \sin\left(\frac{n\pi}{L}\right)x$$

Hence, eigenvalues are

$\lambda = \left(\frac{n\pi}{L}\right)^2 - 1$, the eigenfun. $\Phi_n(x) = \sin\frac{n\pi x}{L}$, $\sigma = 1$

using $u(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x)$, we get

$$u(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

Now, substituting in the given DE, we get

$$\frac{d^2}{dx^2}\left(\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x)$$

$$\sum_{n=1}^{\infty} a_n \frac{d^2}{dx^2}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x)$$

$$-\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x)$$

$$\sum_{n=1}^{\infty} a_n \left(-\left(\frac{n\pi}{L}\right)^2 + 1\right) \sin\frac{n\pi x}{L} = f(x)$$

Using orthagonality, we get

$$a_n = \frac{\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}{\left(-\left(\frac{n\pi}{L}\right)^2 + 1\right)\int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx}$$

$$= \int_0^L f(x_0) \sum_{n=1}^{\infty} \frac{2}{L} \frac{\sin\frac{n\pi x}{L}\sin\frac{n\pi x}{L}}{\left(-\left(\frac{n\pi}{L}\right)^2 + 1\right)}\,dx_0$$

$$= \int_0^L f(x_0)\,G(x, x_0)\,dx_0 \quad \Rightarrow G(x, x_0) = \sum_{n=1}^{\infty} \frac{2}{L} \frac{\sin\frac{n\pi x}{L}\sin\frac{n\pi x}{L}}{\left(-\left(\frac{n\pi}{L}\right)^2 + 1\right)}$$

9.3.11)

a) $G'' + G = 0 \Rightarrow m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1}$

$m = \pm i$

$\Rightarrow G = A\cos x + B\sin x$

generally, $G(x, x_0) = \begin{cases} A_1 \cos x + B_1 \sin x & 0 \le x < x_0 \\ A_2 \cos x + B_2 \sin x & x_0 < x \le L \end{cases}$

G should be (i) continuous at x_0 (ii) $\left.\frac{dG}{dx}\right|_{x=x_0^+} - \left.\frac{du}{dx}\right|_{x=x_0^-} = -1$

(iii) $G(0, x_0) = 0$ & $G(L, x_0) = 0$

(iii) (Applying B.C.)

$G(0, x_0) = 0 \Rightarrow A_1 = 0$

$G(L, x_0) = 0 \Rightarrow A_2 \cos L + B_2 \sin L = 0$

(ii)

$-A_2 \sin x_0 + B_2 \cos x_0 + A_1 \sin x_0 - B \cos x_0 = -1$

$\Rightarrow -A_2 \sin x_0 + B_2 \cos x_0 - B_1 \cos x_0 = -1$

(i) $\cancel{A_1 \cos x_0} + B_1 \sin x_0 = A_2 \cos x_0 + B_2 \sin x_0$

$B_1 \sin x_0 = A_2 \cos x_0 + B_2 \sin x_0$

$$\Rightarrow A_2 \cos x_0 + B_2 \sin x_0 = \sin x_0 \left(\frac{-A_2 \sin x_0 + B_2 \cos x_0 + 1}{\cos x_0} \right)$$

$$\Rightarrow A_2 \cos x_0 + B_2 \sin x_0 = -A_2 \frac{\sin^2 x_0}{\cos x_0} + B_2 \sin x_0 + \frac{\sin x_0}{\cos x_0}$$

$$\Rightarrow A_2 \cos^2 x_0 + A_2 \sin^2 x_0 = \sin x_0$$

$$\Rightarrow B_2 = -\frac{\sin x_0 \cos L}{\sin L} \qquad ; L \neq n\pi$$

$$\Rightarrow B_1 \cos x_0 = -A_2 \sin x_0 + B_2 \cos x_0 + 1$$

$$= \frac{\cos x_0 \sin x_0 \cos L}{\sin L} + \cos^2 x_0$$

$$\Rightarrow B_1 = \frac{\sin x_0 \sin L + \sin L \cos x_0}{\sin L}$$

$$= \frac{\sin (x_0 + L)}{\sin L}$$

$$G(x, x_0) = \begin{cases} \dfrac{\sin(L+x_0)\sin x}{\sin L} & x < x_0 \\[2ex] \dfrac{\sin x_0 \sin(L+x)}{\sin L} & x > x_0 \end{cases}$$

- why $L \neq n\pi$? because the obtained system of linear equations is consistent iff its determinant $\neq 0$. This holds only if $L \neq n\pi$.

b) $G(x, x_0) = \sin x \dfrac{\sin(L+x_0)}{\sin L} \qquad x < x_0$

$\sin x_0 \dfrac{\sin(L+x)}{\sin L} \qquad x_0 < x$

$\Rightarrow G(x_0, x) = G(x, x_0)$