Quantum Information Lecture 2

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June 22, 2023

Contents

1 Outline

Here's some questions that these notes shed some light on:

- What is a 2-qubit quantum theory?
- What is entanglement? How do we measure it?
- Which 2-qubit states are entangled?

2 2-Qubit States

In Newtonian physics, a 2-particle state is specified by fixing two one-particle states. In quantum theory, there exist 2-particle states which can not be decomposed nicely as two one-particle states! The problem is that the set of states which are decomposable as two one-particle states is not a vector space; superpositions can generate new states.

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A first approach to modelling composite quantum systems is to work with the kinematical structure of a tensor product [1]. Although we will use a basis-dependent definition of the tensor product, in some sense, the construction does not depend on which bases for the single particle spaces are chosen.

Suppose subsystem A, B is described by $\mathcal{H}_{A,B}$ with the orthonormal basis $|i\rangle_{A,B}$, respectively. For a 2-qubit system, $i \in \{0, 1\}$. Then, the tensor product space is the vector space generated by the basis

$$\mathcal{H}_A \otimes \mathcal{H}_B = \text{span}\left\{(|i\rangle_A, |j\rangle_B) \mid |i\rangle, |j\rangle \in \text{orthonormal bases for } \mathcal{H}_A, \mathcal{H}_B\right\}. \tag{1}$$

Here's some various shorthands which all mean the same thing:

$$\begin{aligned}(|i\rangle_A, |j\rangle_B) &\to |i\rangle_A \otimes |j\rangle_B &(2)\\ &\to |i\rangle \otimes |j\rangle &(3)\\ &\to |ij\rangle. &(4)\end{aligned}$$

For example, the most general element of a 2-qubit Hilbert space is

$$|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \qquad a, b, c, d \in \mathbb{C}. \tag{5}$$

We can consistently define the tensor product of two arbitrary vectors via linearity, $\otimes$: $\mathcal{H}_A \times \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$, which means

$$\begin{aligned}(a|i_1\rangle + b|i_2\rangle) \otimes (|j\rangle) &:= a|i_1\rangle \otimes |j\rangle + b|i_2\rangle \otimes |j\rangle &\forall a, b \in \mathbb{C}, &\quad(6)\\ |i\rangle \otimes (a|j_1\rangle + b|j_2\rangle) &:= a|i\rangle \otimes |j_1\rangle + b|i\rangle \otimes |j_2\rangle &\forall a, b \in \mathbb{C}. &\quad(7)\end{aligned}$$

Formally, the tensor product is a bilinear map.

Exercise: Show that $|\Psi_{++}\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ is not of the form $|\psi\rangle \otimes |\phi\rangle$ for any $|\psi\rangle \in \mathcal{H}_A \simeq \mathbb{C}^2$, $|\phi\rangle \in \mathcal{H}_B \simeq \mathbb{C}$.

Solution: Assume such $|\psi\rangle$ and $|\phi\rangle$ exist. Using the orthonormal basis,

$$\begin{aligned}|\psi\rangle &= \alpha|0\rangle + \beta|1\rangle &(8)\\ |\phi\rangle &= \gamma|0\rangle + \delta|1\rangle, &(9)\end{aligned}$$

and bilinearity of the tensor product,

$$|\psi\rangle \otimes |\phi\rangle = \alpha\gamma|00\rangle + \beta\gamma|10\rangle + \alpha\delta|01\rangle + \beta\delta|11\rangle. \tag{10}$$

If this vector was equal to $|\Psi_{++}\rangle$, the middle two terms would need to vanish, so $\beta\gamma = \alpha\delta = 0$. Thus,

$$\alpha\beta\gamma\delta = 0. \tag{11}$$

But, taking the product of the first and last terms and comparing with the desired expression for $|\Psi_{+}+\rangle$,

$$\alpha\beta\gamma\delta = 1/2, \tag{12}$$

[1] In a certain case, the tensor product model can be derived, see link:Roos 1970 and link:Zanardi 2004 for more details. There are physically important scenarios where the tensor product is insufficient to model local subsystems, such as in fermionic systems link:Bravyi 2001 and in quantum field theory link:Witten 2018.

a contradiction!

Definition: Any 2-particle vector which can be expressed as a tensor product of two 1-particle vectors is called **separable**. Any vector which is not separable is called **entangled**. While there exist entangled vectors, every vector can be expressed as a superposition of separable vectors.

A 2-particle state is a **normalized** 2-particle vector. How is normalization defined? We need an inner product. There are many in-equivalent ways of doing this. The most popular choice is to define the inner product's action on an the orthonormal basis, such as

$$\langle i'j'|ij\rangle := \langle i'|j'\rangle \langle i|j\rangle \,. \tag{13}$$

Then, if you want to calculate inner products of more general states, you can use the sesquilinearity properties of inner products. The reason why this choice of inner product is nice is that if I give you two one-particle normalized states, their tensor product will always be a normalized state. For other options, see chapter 5 of Jacob's thesis, link:Barnett 2023.

3 2-Particle Operators

Given two one-particle operators, $O_A : \mathcal{H}_A \to \mathcal{H}_A$ and $O_B : \mathcal{H}_B \to \mathcal{H}_B$, we would like to define a composite operator derived from these. This composite operator can be defined via its action on the basis elements,

$$(O_A \otimes O_B) |\psi\rangle \otimes |\phi\rangle := (O_A |\psi\rangle) \otimes (O_B |\phi\rangle). \tag{14}$$

If you want to act with the composite operator on an entangled state, you can reduce the problem to determining this operator's action on separable states by using linearity. For example,

$$(O_A \otimes O_B) \left(\sum_i |\psi_i\rangle \otimes |\phi_j\rangle \right) = \sum_i (O_A \otimes O_B)(|\psi_i\rangle \otimes |\phi_j\rangle). \tag{15}$$

We can embed the single-particle operators into the composite setting via the maps $O_A \to O_A \otimes \mathbb{1}$ and $O_B \to \mathbb{1} \otimes O_B$. You would then say that $O_A \otimes \mathbb{1}$ is equivalent to acting with O_A on the $\mathcal{H}_A$ subsystem and acting trivially on the $\mathcal{H}_B$ subsystem.

Exercise: Let

$$|\pm\rangle := \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \in \mathbb{C}^2 \tag{16}$$

and

$$H := |+\rangle \langle 0| + |-\rangle \langle 1| \tag{17}$$

denote the Hadamard gate. Determine

$$(H \otimes \mathbb{1}) |00\rangle \,. \tag{18}$$

Solution:

$$\begin{aligned} (H \otimes \mathbb{1}) |00\rangle &= (H |0\rangle) \otimes (\mathbb{1} |0\rangle) && (19) \\ &= (|+\rangle \langle 0| |0\rangle + |-\rangle \langle 1| |0\rangle) \otimes |0\rangle && (20) \\ &= |+\rangle \otimes |0\rangle \,. && (21) \end{aligned}$$

4 Multi-State Equivalence

Physically, we will say two states are equivalent if any transformation between them is indistinguishable from a local change of measurement basis. The next sentence formalizes this:

A **Local Unitary** is a tensor product of two unitary operators. Two composite states are called (locally unitarily) **equivalent** if there exists a local unitary which maps one into the other.

The following thought experiment clarifies this notion: Suppose Aaron and I separately create qubits in Berkeley and Waterloo, respectively. Suppose we both rotate our lab while leaving the qubit stationary, this is equivalent to changing the local bases $|i\rangle_{\text{Jacob}} \to |i\rangle'_{\text{Jacob}} := U_J\,|i\rangle_{\text{Jacob}}$ and $|i\rangle_{\text{Aaron}} \to |i\rangle'_{\text{Aaron}} := U_A\,|i\rangle_{\text{Aaron}}$. The probability of measuring outcome $|00\rangle'$ is equivalent to the probability of measuring outcome 00 of a modified state,

$$P(00') = |\,\langle 00'|\Psi\rangle\,|^2 = |\,\langle 00|U_A^\dagger \otimes U_J^\dagger \Psi\rangle\,|^2. \tag{22}$$

Thus, any transformation $|\Psi\rangle \to U_A \otimes U_J\,|\Psi\rangle$ is indistinguishable from a local change of measurement bases.

Example: All separable states are equivalent. Furthermore, the only states which are equivalent to a separable state are separable states, which means separable states are an equivalence class.

Example: $a\,|00\rangle + b\,|11\rangle$ is equivalent to $a\,|11\rangle + b\,|00\rangle$, since

$$(\sigma_x \otimes \sigma_x)(a\,|00\rangle + b\,|11\rangle) = a\,|11\rangle + b\,|00\rangle\,, \tag{23}$$

where σ_x is a Pauli matrix, and therefore unitary, $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$.

Theorem 1. *Every 2-qubit state, $|\Psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$, is equivalent to a state of the form $a\,|00\rangle + b\,|11\rangle$ for $a, b \in \mathbb{R}$, $a \geq b \geq 0$, and $a^2 + b^2 = 1$. Equivalently,*

$$|\Psi\rangle = (U_A \otimes U_B)(a\,|00\rangle + b\,|11\rangle) \tag{24}$$

*for some U_A, U_B which could depend on $|\Psi\rangle$. This is called the **Schmidt decomposition** of $|\Psi\rangle$.*

One way to find the Schmidt decomposition of a state is by noticing that composite systems can be re-expressed as matrices, $\mathbb{C}^d \otimes \mathbb{C}^d \simeq \mathfrak{M}_d(\mathbb{C})$, where $\mathfrak{M}_d(\mathbb{C})$ is the algebra of $d \times d$ matrices. Specifically, note

$$\mathbb{C}^d \otimes \mathbb{C}^d \ni \sum_{ij} a_{ij}\,|i\rangle \otimes |j\rangle \to \sum_{ij} a_{ij}\,|i\rangle\langle j| \in \mathfrak{M}_d(\mathbb{C}) \tag{25}$$

is a linear map. Once you've mapped a state into a matrix, you can find the singular value decomposition of this matrix. Since the above map is a bijection, you can go backwards from the SVD matrices to composite states and thereby determine the Schmidt decomposition.

A corollary of the above theorem is that two states are equivalent if and only if they correspond to the same a and b, this is discussed in the exercises. The Schmidt decomposition is unique in the sense that the a and b are unique, but the U_A and U_B may not be unique. An example of the non-uniqueness comes from mapping $U_A \to e^{\mathrm{i}\phi}U_A$, $U_B \to e^{-\mathrm{i}\phi}U_B$ for some $\phi \in [0, 2\pi)$.

5 Measuring Entanglement

Physically, entanglement correlates measurement outcomes performed by spatially distinct observers. These correlations cannot be modelled in any "locally deterministic" theory, such as Newtonian physics.

This section utilizes the Schmidt decomposition of qubit states to measure entanglement.

We want our notion of entanglement to be invariant under equivalence in the sense defined above. Therefore, measuring entanglement can be reduced to measuring it in the state $a\,|00\rangle + b\,|11\rangle$. Furthermore, separable states should have zero entanglement, and the example $|\Psi_{++}\rangle$ from earlier should be entangled.

Example: Let $|\Psi\rangle = a\,|00\rangle + b\,|11\rangle$ be a normalized state, so $|a|^2 + |b|^2 = 1$. If we measure only qubit A and ask what the chance of measuring 0 is, we find (using the Born rule)

$$\begin{aligned} P(0)_A &= P(00) + P(01) && (26)\\ &= |a|^2 + 0. && (27) \end{aligned}$$

Similarly, if we only measure qubit B, we have $P(0)_B = |b|^2$. If the results of measurement outcomes were not correlated, then we'd have

$$P(00)\text{"} = \text{"}|a|^4 \qquad\qquad P(11)\text{"} = \text{"}|b|^4. \tag{28}$$

The probability the measurement outcomes agree would thus be $P(00) + P(11) = |a|^4 + |b|^4$. However, the actual values for these probabilities in the state $|\Psi\rangle$ is

$$P(00) = a^2 \qquad\qquad P(11)b^2. \tag{29}$$

and the probability for agreement is

$$P(00) + P(11) = 1. \tag{30}$$

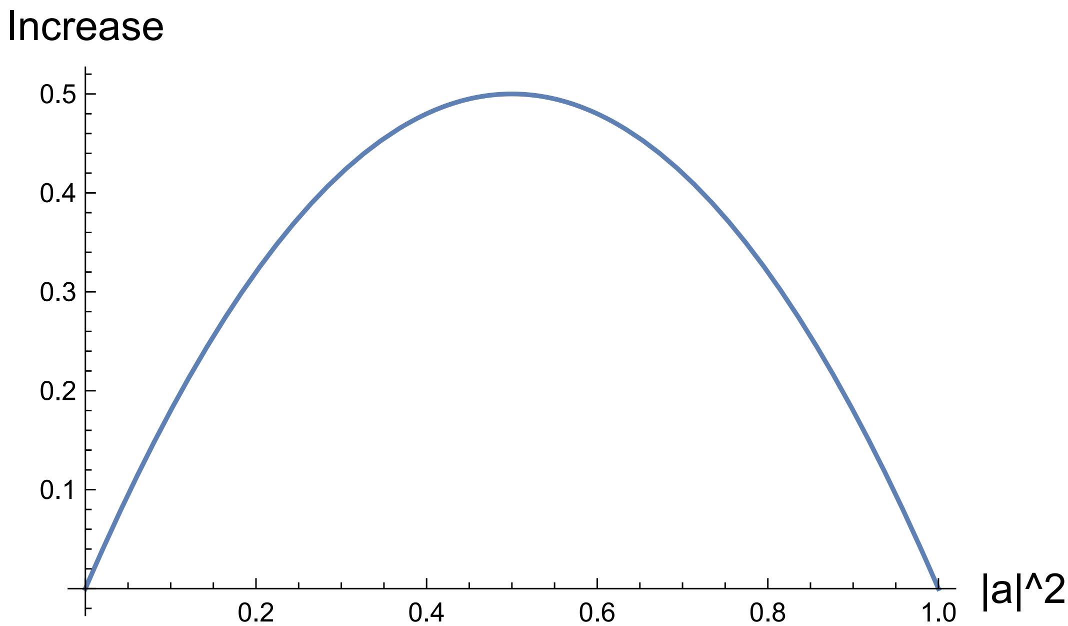
Entanglement increased the probability of agreement! We can measure this increase by taking the difference,

$$\begin{aligned} \text{Increase} &= 1 - (|a|^4 + |b|^4) && (31)\\ &= 1 - (|a|^4 + |b|^4 + 2|a|^2|b|^2 - 2|a|^2|b|^2) && (32)\\ &= 1 - ((|a|^2 + |b|^2)^2 - 2|a|^2|b|^2) && (33)\\ &= 1 - (1 - 2|a|^2|b|^2) && (34)\\ &= 2|a|^2|b|^2 && (35)\\ &= 2|a|^2(1 - |a|^2). && (36) \end{aligned}$$

The increase is plotted below:

The important features of this measure are that it's maximized for $|a|^2 = 1/2$, minimized for $|a|^2 \in \{0, 1\}$, and is monotonic and continuous in the intervals $|a|^2 \in [0, 1/2]$ and $|a|^2 \in [1/2, 1]$.

Something that may be unsatisfactory about the increase in entanglement is that it depends on the measurement basis. One of the exercises is to do the same computation above, but measure the excess correlation in the orthonormal basis corresponding to the $\sigma_x \otimes \sigma_x$ operator. A better measure would be to take the maximum correlation over all measurement bases.



The **von Neumann entanglement entropy** is a ”better” measure of entanglement that also satisfies these properties,

$$S(a, b) = -|a|^2 \log_2(|a|^2) - b^2 \log_2(|b|^2). \tag{37}$$

This measure is genuinely basis independent, even before taking maximums.

Further Reading: A study of what kinds of correlations are possible is given by *nonlocal games* link:Cleve 2004, which generalizes the concept of a Bell inequality. Some foundational papers on the Bell inequality are link:Bell, link:CHSH, and link:Tsirelson.