Path Integrals Day 1

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1 References

Two good introductory references on path integrals are:

- Feynman and Hibbs, *Quantum Mechanics and Path Integrals*
- Mackenzie, arXiv:quant-ph/0004090, ”Path Integral Methods and Applications”

Parts of these notes are also based on unpublished lecture notes by Francois David.

2 Many screens with many slits

In the double slit experiment, a screen with two small slits is placed in between a particle emitter and a detector. The amplitude K for a particle to reach a given point on the detector is the sum of the amplitudes A_1, A_2 for the particle to take either of the two possible paths through the slits

$$K = A_1 + A_2 \,. \tag{1}$$

The amplitudes for the two paths can interfere constructively or destructively. If the experiment is repeated many times this interference will be visible to the detector. At some locations the amplitudes for the two paths will interfere destructively and the probability of detecting the particle will vanish. At other locations the amplitudes will interfere constructively.

Feynman, building on ideas from Dirac and Weiner, developed the path integral by generalizing the double slit experiment. What if we add more screens? Or more slits? Or better yet, both?

The amplitude for the particle to go from the emitter to the detector should be the sum over all possible paths γ of the amplitude for a given path $A[\gamma]$

$$K = \sum_{\text{paths } \gamma} A[\gamma] \,. \tag{2}$$

In the limit that the density of screens and slits approaches infinity, the screens disappear! A particle moving through empty space can be viewed as a particle going through a series of infinitesimally separated “screens” with infinitesimally separated “slits.” In this limit the sum over paths becomes an integral over paths, or path integral.

3 Path integrals from discretized paths

It will simplify our discussion if we specify the time t_i that the particle is emitted and the time t_f that the particle reaches the detector. The amplitude K is a function of the initial and final positions q_i and q_f in addition to the initial and final times. In the Schrödinger picture the amplitude K is

$$K(q_f, t_f, q_i, t_i) = \langle q_f, t_f | q_i, t_i \rangle \tag{3}$$

and in the Heisenberg picture it is

$$K(q_f, t_f, q_i, t_i) = \langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle \,. \tag{4}$$

The amplitude K is known as the propagator.

We can discretize the paths appearing in (2) in more than one way. A convenient discretization is to divide the time interval $T = t_f - t_i$ into N time intervals each of duration Δt. Any given path can be discretized by specifying the location q_n of the particle at each time $n\Delta t$.

The sum over paths becomes an integral of the locations of the particle at intermediate times

$$\sum_{\text{paths } \gamma} = \int dq_1 dq_2 \cdots dq_{N-1}. \tag{5}$$

The key question in determining a path integral expression for the propagator is: What is $A[\gamma]$? The discretization we chose above is the easiest way to determine $A[\gamma]$.

Let's start by breaking the path into two pieces. If particle number is conserved and a particle propagates from position q_i at time t_i to position q_f at time t_f, then it must be somewhere at an intermediate time t_{int}

$$K(q_f, t_f, q_i, t_i) = \int dy K(q_f, t_f, y, t_{\text{int}}) K(y, t_{\text{int}}, q_i, t_i) \,. \tag{6}$$

We can derive this formula by inserting the resolution of the identity as an integral over position eigenstates $\mathbf{1} = \int dy |y\rangle\langle y|$ into the Heisenberg picture expression for the propagator

$$\begin{aligned}
K(q_f, t_f, q_i, t_i) &= \langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle & (7)\\
&= \langle q_f | e^{-iH(t_f - t_{\text{int}})} e^{-iH(t_{\text{int}} - t_i)} | q_i \rangle & (8)\\
&= \langle q_f | e^{-iH(t_f - t_{\text{int}})} \int dy |y\rangle \langle y | e^{-iH(t_{\text{int}} - t_i)} | q_i \rangle & (9)\\
&= \int dy K(q_f, t_f, y, t_{\text{int}}) K(y, t_{\text{int}}, q_i, t_i) \,. & (10)
\end{aligned}$$

Let's repeat the above procedure N times. First we can rewrite the Heisenberg picture expression for the propagator as

$$K(q_f, t_f, q_i, t_i) = \langle q_f | \left(e^{iH\Delta t} \right)^N | q_i \rangle \,. \tag{11}$$

We can then insert the identity $\int dq_n|q_n\rangle\langle q_n|$ for $n = 1, ..., N-1$ in between each factor of $e^{iH\delta t}$

$$K(q_f, t_f, q_i, t_i) = \langle q_f|e^{-iH\Delta t}\int dq_{N-1}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\Delta t}\int dq_{N-2}|q_{N-2}\rangle \cdots e^{-iH\Delta t}|q_i\rangle \tag{12}$$

$$= \int dq_1 dq_2 \cdots dq_{N-1} A[\gamma]\,. \tag{13}$$

We see that $A[\gamma]$ is a product of propagators

$$A[\gamma] = K_{q_N,q_{N-1}} \cdots K_{q_1,q_0} \tag{14}$$

where $q_N = q_f$, $q_0 = q_i$, and

$$K_{q_{n-1},q_n} = K(q_n, n\Delta t, q_{n-1}, (n-1)\Delta t)\,. \tag{15}$$

4 Expectations

We will sketch a derivation of a more useful form for $A[\gamma]$ next time. We can almost guess what form $A[\gamma]$ must take based on the expectation that the propagator should satisfy a composition property, and that classical paths should be the most important[1] in the classical limit.

4.1 Composition

We have seen that (6) implies the composition property

$$A[\gamma_{12}] = A[\gamma_1]A[\gamma_2] \tag{16}$$

In one of the exercieses, you will show the converse (i.e. that (16) implies (6)).

4.2 Classical limit

In the classical limit $\hbar \to 0$[2], we expect that the quantum effects are unimportant and we should recover classical mechanics. The first-order variation of the action vanishes[3] for classical paths:

$$\delta S = 0 \tag{17}$$

In the classical limit we expect that paths that satisfy (17) are the most important contributions to the propagator.

[1] In a sense that we will describe next time.
[2] Or if you prefer in the limit $S \gg \hbar$.
[3] Or alternatively the functional derivative of the action vanishes $\frac{\delta S}{\delta q(t)} = 0$.

5 Exercise 1: Composition

You will solve these exercises in small groups during class.

(a) In lecture we argued that the propagator K can be written as a sum (or integral) over paths γ of a functional $A[\gamma]$

$$K = \sum_{\gamma} A[\gamma] \,. \tag{18}$$

Show that the propagator obeys the composition property

$$K(q_f, t_f, q_i, t_i) = \int dy K(q_f, t_f, y, t_{\text{int}}) K(y, t_{\text{int}}, q_i, t_i) \tag{19}$$

if A has the factorization property

$$A[\gamma_{12}] = A[\gamma_1] A[\gamma_2] \,. \tag{20}$$

(b) Explain why the factorization property

$$A[\gamma_{12}] = A[\gamma_1] A[\gamma_2] \tag{21}$$

holds if $A[\gamma] = e^{iS[\gamma]/\hbar}$ where $S[\gamma]$ is the classical action of the trajectory γ.

(c) Suppose $A[\gamma] = C_1 e^{C_2 S[\gamma]}$ for some constants C_1 and C_2. Which values of C_1 and C_2 give rise to a composition property?

6 Exercise 2: Classical limit

(a) Suppose $f(x)$ is a real function. Which values of x provide the most important contribution to

- $\int e^{-f(x)} dx$? Why?
- $\int e^{if(x)} dx$? Why?

(b) The classical equation of motion for a particle with classical action S is that the first-order variation vanishes

$$\delta S = 0 \,. \tag{22}$$

A trajectory with $\delta S = 0$ is a classical trajectory. No extra conditions on the second-order variation of the action are required.[4] Why does $A[\gamma] = e^{iS[\gamma]/\hbar}$ give the correct classical limit?

(c) What other choices $A[\gamma]$ give rise to the correct classical limit? Do they have a factorization property?

(d) Do any of the other possibilities you identified in 1 (c) have the correct classical limit?

[4] Can you come up with an example of a classical trajectory that is not a local minimum of the action?