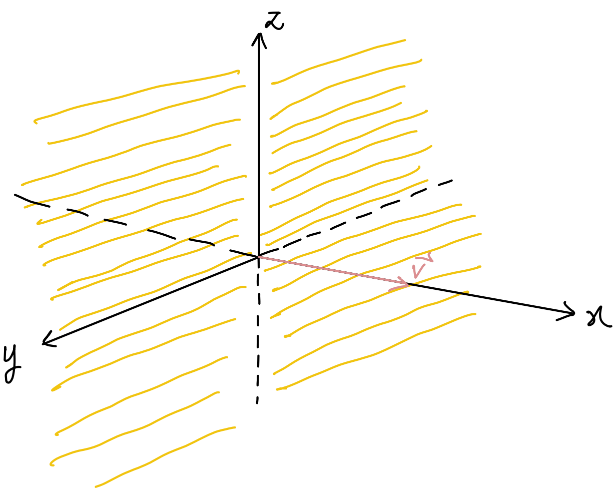
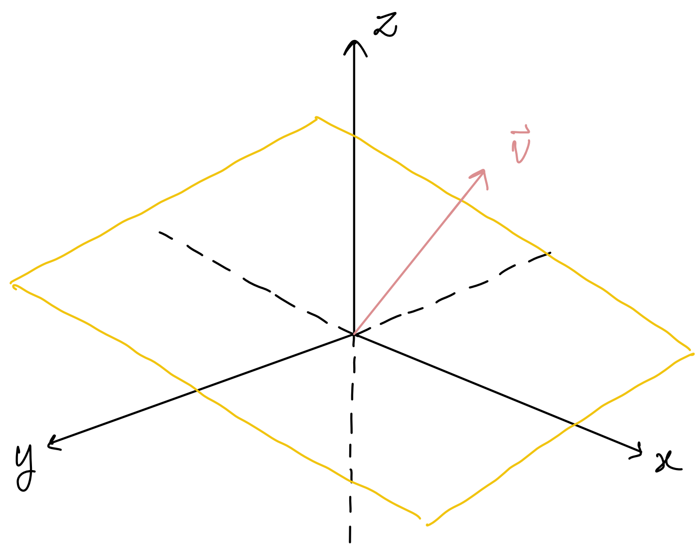
Solutions. Fatimah H. Alhazmi

S1) The dot product is given by $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$, $|\vec{v}|$ and $|\vec{w}|$ are magnitudes (cannot have negative values). This implies that $\cos\theta < 0$, which means that the angle between $\vec{v}$ and $\vec{w}$ is an obtuse angle; $90^\circ < \theta < 270^\circ$.



or



The plane splits the 3D space into 2 regions. One contains the $\vec{v}$, the other region contains the points that give negative dot product.

* The $\vec{w}$'s fill half of the 3-dimensional space. θ must be between 90° & 270°.

2) Yes. Three vectors in a plane could make angles between 90° and 270° with each other. Example: $\vec{u} = (2,0)$, $\vec{v} = (-1,3)$, and $\vec{w} = (-3,-5)$.

3) $\vec{w}_2 = \dfrac{(\vec{w}_1 + \vec{w}_3)}{2} \Rightarrow \vec{x} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix} \Rightarrow \frac{1}{2}\vec{w}_1 - \vec{w}_2 + \frac{1}{2}\vec{w}_3 = \vec{0}$

$$\Rightarrow \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 3.5 \\ 4 \\ 4.5 \end{bmatrix} = \vec{0}$$

- The vectors are <u>dependent</u> and lie on a <u>plane</u>.

4) The five centered difference equations lead to

$b_1 + b_3 + b_5 = 0;$

$$\begin{aligned} x_2 &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ - x_4 &= b_5 \end{aligned}$$

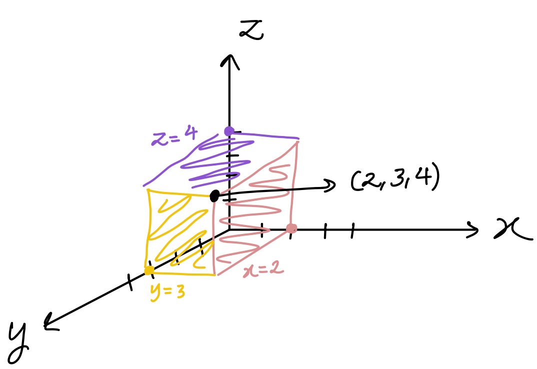
* There cannot be a solution unless

$b_1 + b_3 + b_5 = 0$

5) • row pic.:

$1x + 0y + 0z = 2$

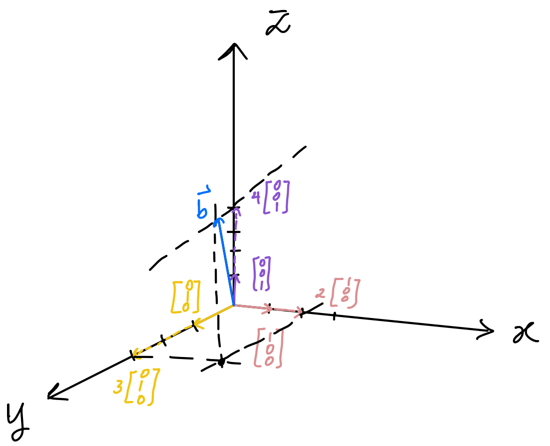
$0x + 1y + 0z = 3$



$0x + 0y + 1z = 4$

• Caulmn pic.:

$$2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix} = \vec{b} = \begin{bmatrix}2\\3\\4\end{bmatrix}$$



• The linear combination of the caulmns of A that results in $\vec{b}$ is:

$$2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix} = \vec{b} = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

6)

a) $AB = \begin{bmatrix}1\end{bmatrix}$ $\quad BA = \begin{bmatrix}-2 & -6\\1 & 3\end{bmatrix}$ $\quad AB \neq BA$

b) $AB = \begin{bmatrix} 1 & 4 & 1 & 2 \\ -4 & 5 & 3 & -1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$ BA (not possible)

7)

a) example of $A^2 = -I$: $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$; $A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$, where $i^2 = -1$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

b) $B^2 = 0$, $B \neq 0$ example: $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

c) $CD = -DC$, $CD \neq 0$ example: $C = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow CD = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix} \neq 0$ and $DC = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow CD = -DC$

d) $EF = 0$; no zero components

example: $E = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$ and $F = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$

$\Rightarrow EF = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8)

a) True;

<u>Def.</u>:

Suppose that the kth caulmn of B is $\vec{b}_k$. Then AB is the matrix of which the kth column is $A\vec{b}_k$.

Let A be an $m \times n$ matrix and B is an $n \times l$ matrix.

Then, AB is an $m \times l$ matrix, say $C_{m \times l}$.

Then, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ $\qquad n \geq 5$

- $b_{i2} = b_{i5}$ for $i = 1, 2, \ldots n$

$\Rightarrow c_{i2} = \sum_{k=1}^{n} a_{ik} b_{k2}$ $\qquad i = 1, 2, \ldots, m$

$$= \sum_{k=1}^{n} a_{ik} b_{k5}$$

$$= c_{i5}$$

$\therefore$ 2nd & 5th columns of AB are equal.

b) False;

Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times l}$ $\qquad n \geq 5$

$\Rightarrow AB = C_{m \times l}$; $C = \sum_{k=1}^{n} a_{ik} b_{kj}$

$\Rightarrow b_{2j} = b_{5j}$ $\qquad \Rightarrow c_{2j} = \sum_{k=1}^{n} a_{2k} b_{kj}$

a_{2k} not necessairly $= a_{5k}$

$\Rightarrow C_{2j} \neq C_{5j}$

c) True;

Consider the same previous A, B, and C.

Since 2nd and 5th rows of A are equal.

$\Rightarrow a_{2j} = a_{5j}$

$$\Rightarrow C_{2j} = \sum_{k=1}^{n} a_{2k} b_{kj}$$

$$= \sum_{k=1}^{n} a_{5k} b_{kj}$$

$$= C_{5j}$$

d) False

$(AB)^2 = (AB)(AB)$

we can see from (6) that $AB \neq BA$

Thus, $(AB)^2 \neq A^2B^2$