Path Integrals Day 4

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1 Imaginary time and statistical physics

We have seen that the propagator can be expressed as a path integral

$$K = \langle q_f | e^{iHT/\hbar} | q_i \rangle \tag{1}$$
$$= \int \mathcal{D}q(t) e^{iS[q(t)]/\hbar} \,. \tag{2}$$

The i in the exponent causes oscillations. It can be challenging to determine if an oscillatory integral is well defined.

The Euclidean[1] propagator

$$K_E = \langle q_f | e^{-\mathbf{H}T_E/\hbar} | q_i \rangle \tag{3}$$

with $T_E = iT$ real has no such oscillations. The Euclidean propagator can be obtained from the real time propagator by "Wick rotating" real time t to imaginary time[2] τ

$$t = -i\tau \,. \tag{4}$$

Under this change of variables the action becomes

$$iS[q(t)] = i \int_0^T dt \left(\frac{m}{2} \left(\frac{dq}{dt} \right)^2 - V(q) \right) \tag{5}$$
$$= + \int_0^{T_E} d\tau \left(-\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 - V(q) \right) \tag{6}$$
$$= -S_E[q(\tau)] \,. \tag{7}$$

Note the change in the relative sign between the kinetic and potential terms! The Euclidean propagator can then be expressed as

$$K_E = \int \mathcal{D}q(\tau) e^{-S_E[q(\tau)]/\hbar} \,. \tag{8}$$

[1]The name Euclidean is used because after a Wick rotation $t \to -i\tau$, the Minkowski metric $ds^2 = -dt^2 + d\mathbf{x}^2$ becomes the Euclidean metric $ds^2 = d\tau^2 + d\mathbf{x}^2$.

[2]Note that the name can be misleading. We are often interested in the case that t is an imaginary number and τ is a real number!

The Euclidean propagator has no annoying oscillations. In many cases, we can compute a quantity of interest in imaginary time τ and analytically continue our answer back to real time t.

The Euclidean propagator may remind you of statistical physics. The partition function can be written as a sum over the energy eigenvalues E_j

$$Z(\beta) = \sum_j e^{-\beta E_j} \tag{9}$$

where $\beta = \frac{1}{K_B(\text{temp})}$ is the inverse temperature. Alternatively we can write the partition function as

$$Z(\beta) = \text{tr} e^{-\beta \mathbf{H}} \tag{10}$$

where the trace can be over any complete set of states. This expression looks very similar to the expression (3) if we make the identification $\beta = \frac{T_E}{\hbar}$. In one of the exercises, you will show that by writing the partition function in a position basis the partition function is given by the Euclidean path integral

$$\int_{q(0)=q(\beta\hbar)} \mathcal{D}q(\tau) e^{-S_E[q(\tau)]/\hbar} \tag{11}$$

with periodic boundary conditions $q(0) = q(\beta\hbar)$.

Statistical physics is quantum mechanics in imaginary time!

2 Other quantities of interest

We can use path integrals to compute other quantities of interest besides the propagator. Let's look at an example to see how. In the exercises we will look at a generalization.

Suppose we want to compute the expectation value of some operator $\mathbf{A}(\tau)$ at finite temperature

$$\langle \mathbf{A}(\tau) \rangle_\beta = \frac{\text{tr} \mathbf{A} e^{-\beta \mathbf{H}}}{\text{tr} e^{-\beta H}} \,. \tag{12}$$

The expecation value is time independent since thermal states are stationary states. We already know how to write the denominator as a path integral. We can evaluate the trace in the numerator in the position basis

$$\text{tr} \mathbf{A} e^{-\beta \mathbf{H}} = \int dq \langle q | \mathbf{A}(0) e^{-\beta \mathbf{H}} | q \rangle \,. \tag{13}$$

For simplicity assume that $|q\rangle$ is an eigenstate of $\mathbf{A}$ so that

$$\text{tr} \mathbf{A} e^{-\beta \mathbf{H}} = \int dq \mathbf{A}(q(0)) \langle q | e^{-\beta \mathbf{H}} | q \rangle \,. \tag{14}$$

The factor $\langle q|e^{-\beta\mathbf{H}}|q\rangle$ is the Euclidean propagator, which can be written as a path integral

$$\mathrm{tr}\mathbf{A}e^{-\beta\mathbf{H}} = \int_{q(0)=q(\beta\hbar)} \mathcal{D}q(\tau)A(q(0))e^{-S_E[q(\tau)]/\hbar}. \tag{15}$$

Note that $A(q(0))$ is a number, not an operator in path integral.

3 Exercise 1: Conceptual review

Which quantities in quantum mechanics correspond to which quantities in statistical physics?

4 Exercise 2: Statistical physics and imaginary time

Show that the partition function

$$Z(\beta) = \mathrm{tr}e^{-\beta\mathbf{H}} \tag{16}$$

can be expressed as a sum over imaginary time propagators

$$Z[\beta] = \int dqK_E(q, \beta\hbar, q, 0). \tag{17}$$

This demonstrates that the partition function can be written as a path integral:

$$K_E = \int_{q(0)=q(\beta\hbar)} \mathcal{D}q(\tau)e^{-S_E[q(\tau)]/\hbar}. \tag{18}$$

5 Exercise 3: Operators and correlation functions in real time

(a) Show that (in real time) inserting a position operator $\mathbf{Q}(t_1)$ is equavalent to inserting a function $q(t_1)$:

$$\langle q_f, t_f|Q(t_1)|q_i, t_i\rangle = \int \mathcal{D}q(t)e^{iS[q(t)]/\hbar}q(t_1) \tag{19}$$

(b) Show that if we insert two position functions at different times, the corresponding operators are naturally time-ordered

$$\int \mathcal{D}q(t)e^{iS[q(t)]/\hbar}q(t_1)q(t_2) = \langle q_f, t_f|T\mathbf{Q}(t_1)\mathbf{Q}(t_2)|q_i, t_i\rangle\,, \tag{20}$$

where

$$T\mathbf{Q}(t_1)\mathbf{Q}(t_2) = \begin{cases} \mathbf{Q}(t_1)\mathbf{Q}(t_2) \text{ if } t_1 > t_2 \\ \mathbf{Q}(t_2)\mathbf{Q}(t_1) \text{ if } t_2 > t_1\,. \end{cases} \tag{21}$$

Hint: break up the paths into segments.