HW #4:

$\boxed{2.3.3}$

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Temporarily ignore the nonzero I.V., and separate variables

general sln.: $u(x,t) = \phi(x)\, G(t)$

$$\phi(x)\frac{dG(t)}{dt} = kG(t)\,\frac{d^2\phi(x)}{dx^2}$$

$$\Rightarrow \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

We get 2 ODE's

$G' + \lambda k G = 0$ ---- (1)

$\phi'' + \lambda\phi = 0$ ---- (2)

Applying B.C.'s (nontrivial solutions)

$u(0,t) = 0 \Rightarrow \phi(0)\,G(t) = 0 \Rightarrow \phi(0) = 0$

$u(L,t) = 0 \Rightarrow \phi(L)\,G(t) = 0 \Rightarrow \phi(L) = 0$

solving for equ. (1)

1st order lin. O.E. w/ constant coefficients $\Rightarrow$ general sln.: $\boxed{G = ce^{-\lambda kt}}$

solving for equ. (2) (eigenvalue problem)

Try $\phi(x) = e^{rx}$ $\Rightarrow r^2 + \lambda = 0$ $\quad r = \pm\sqrt{-\lambda}$

Case 1. $\lambda < 0$ $\quad r_{1,2} = \pm\sqrt{\lambda}$

$\phi(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

B.C.'s

$$\left.\begin{aligned}
&\phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \\
&\phi(L) = 0 \Rightarrow c_1 e^{\sqrt{\lambda}L} - c_1 e^{-\sqrt{\lambda}L} = 0 \\
&\qquad \Rightarrow c_1\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) = 0 \Rightarrow c_1 = 0
\end{aligned}\right\} \text{trivial sln.}$$

Case 2. $\lambda = 0$ $\quad r_{1,2} = 0$

$\phi(x) = c_1 + c_2 x$

$$\left.\begin{aligned}
&\text{B.C.'s} \\
&\phi(0) = 0 \Rightarrow c_1 = 0 \\
&\phi(L) = 0 \Rightarrow c_2 L = 0 \Rightarrow c_2 = 0
\end{aligned}\right\} \text{also trivial}$$

Case 3. $\lambda < 0$ $\quad r_{1,2} = \pm i\sqrt{\lambda}$

$\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

B.C.'s

$\phi(0) = 0 \Rightarrow c_1 = 0$

$\phi(L) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}x = 0$

$\sin\sqrt{\lambda}x = 0 \Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \ldots$

$\Rightarrow \phi(x) = c_n \sin\frac{n\pi x}{L}$

Applying superposition principal

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda t} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

Now, consider I.V. to solve for $u(x,t)$

a) $u(x,0) = 6\sin\frac{9\pi x}{L}$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = 6\sin\frac{9\pi x}{L}$$

By incpection, $C_n = \begin{cases} 6 & \text{at } n=9 \\ 0 & \text{at } n \neq 9 \end{cases}$

$$\Rightarrow u(x,t) = 6\, e^{-\left(\frac{9\pi}{L}\right)^2 kt} \sin\left(\frac{9\pi x}{L}\right)$$

b) $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$$

$C_n \begin{cases} 3 & \text{at } n=1 \\ -1 & \text{at } n=3 \\ 0 & \text{otherwise} \end{cases}$

2.3.5

Evaluate $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$ $\quad n>0$ & $m>0$

Use $\sin a\sin b = \frac{1}{2}[\cos(a-b)-\cos(a+b)]$

$$\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \frac{1}{2}\left[\cos\left(\frac{(n-m)\pi x}{L}\right)-\cos\left(\frac{(m+n)x\pi}{L}\right)\right]$$

Case 1. $m=n\neq 0$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L\left[1-\cos\left(\frac{2nx\pi}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[\int_0^L dx-\int_0^L\cos\frac{2n\pi x}{L}\,dx\right]$$

$$= \frac{1}{2}\left(L-\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\Big|_0^L\right)$$

$$= \frac{1}{2}(L-0) = \frac{L}{2}$$

Case 2. $m\neq n$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L\left[\cos\frac{(n-m)\pi x}{L}-\cos\frac{(n+m)\pi x}{L}\right]dx$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\Big|_0^L-\frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\Big|_0^L\right]$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}(\underbrace{\sin(n-m)\pi}_{=0}-\underbrace{\sin 0}_{=0})-\frac{L}{(n+m)\pi}(\underbrace{\sin(n+m)\pi}_{=0}-\underbrace{\sin 0}_{=0})\right]$$

$$= 0$$

$$\therefore \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} L/2 & m=n \\ 0 & m\neq n\end{cases}$$

2.3.6

Evaluate $\displaystyle\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$ $\qquad m \geq 0$ & $n \geq 0$

Use $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$

$$\Rightarrow \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \frac{1}{2}\left[\cos\left[(n-m)\frac{\pi x}{L}\right] + \cos\left[(n+m)\frac{\pi x}{L}\right]\right]$$

Case 1. $\quad m = n \neq 0$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L \left(\cos\frac{2n\pi x}{L} + 1\right)dx$$

$$= \frac{1}{2}\int_0^L \cos\frac{2n\pi x}{L}\,dx + \frac{1}{2}\int_0^L dx$$

$$= \frac{1}{2}\left[\cancelto{0}{\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\Big|_0^L} + L\right]$$

$$= \frac{1}{2}(0 + L) = \frac{L}{2}$$

Case 2. $\quad m \neq n$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\left(\int_0^L \left[\cos(n+m)\frac{\pi x}{L} + \cos(n-m)\frac{\pi x}{L}\right]dx\right)$$

$$= \frac{1}{2}\left[\left(\frac{L}{\pi(n+m)}\right)\sin(n+m)\frac{\pi x}{L}\Big|_0^L + \left(\frac{L}{\pi(n-m)}\right)\sin(m-n)\frac{\pi x}{L}\Big|_0^L\right]$$

$$= \frac{1}{2}\left[\frac{L}{\pi(n+m)}\left(\cancelto{0}{\sin(n+m)\pi} - \cancelto{0}{\sin 0}\right) + \frac{L}{\pi(n-m)}\left(\cancelto{0}{\sin(m-n)\pi} - \cancelto{0}{\sin 0}\right)\right]$$

$$= 0$$

Case 3. $\quad m = n = 0$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\left[\int_0^L (\cos 0 + \cos 0)\,dx\right] = \frac{1}{2}\left[\int_0^L dx + \int_0^L dx\right] = L$$

$$\therefore \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} L & m=n=0 \\ 0 & m\neq n \\ L/2 & m=n\neq 0 \end{cases}$$

2.5.1

Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ inside $0 \le y \le H$, $0 \le x \le L$

a.

Sol: $u(x,y) = \phi(x) h(y)$

Plug into Laplace's equation,

$$h(y) \frac{d^2\phi}{dx^2} + \phi(x) \frac{d^2 h}{dy^2} = 0 \qquad \div \phi h$$

$$\Rightarrow \frac{1}{\phi}\frac{d^2\phi}{dx^2} + \frac{1}{h}\frac{d^2 h}{dy^2} = 0$$

$$\Rightarrow \frac{h''}{h} = -\frac{\phi''}{\phi} = -\lambda$$

This results in 2 ODE's:

$$h'' = -\lambda h \quad \text{---- (1)}$$

$$\phi'' = \lambda\phi \quad \text{---- (2)}$$

BC's

$u'(0,y) = 0 \Rightarrow \phi'(0) + h'(y) = 0 \Rightarrow \phi'(0) = 0$

$u'(L,y) = 0 \Rightarrow \phi'(L) + h'(y) = 0 \Rightarrow \phi'(L) = 0$

$u(x,0) = 0 \Rightarrow \phi(x) h(0) = 0 \Rightarrow h(0) = 0$

Eigenvalue problem

$$\begin{cases} \frac{\phi''}{\phi} + \lambda = 0 \\ \phi'(0) = \phi'(L) = 0 \end{cases}$$

$\phi(x) = e^{rx} \Rightarrow r^2 + \lambda = 0$

1. $\lambda < 0$ $\qquad r_{1,2} = \pm\sqrt{-\lambda}$

general sol.: $\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$\phi'(x) = \sqrt{-\lambda}c_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda}c_2 e^{-\sqrt{-\lambda}x}$

BC:

$\phi'(0) = 0 \Rightarrow \sqrt{-\lambda}c_1 - \sqrt{-\lambda}c_2 = 0 \Rightarrow \sqrt{-\lambda}(c_1 - c_2) = 0 \Rightarrow c_1 = c_2$

$\phi'(L) = 0 \Rightarrow \sqrt{-\lambda}c_1 e^{\sqrt{-\lambda}L} - \sqrt{-\lambda}c_2 e^{-\sqrt{-\lambda}L} = 0 \Rightarrow c_1\sqrt{-\lambda}(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}) = 0$

$\Big\}$ trivial

2. $\lambda = 0$ $\qquad r_{1,2} = 0$

general sol.: $\phi(x) = c_1 x + c_2$

$\phi'(x) = c_1$

BC:

$\phi'(0) = \phi'(L) = c_1 = 0$

3. $\lambda > 0$ $\qquad r_{1,2} = \pm i\sqrt{\lambda}$

general sol.: $\phi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\phi'(x) = \sqrt{\lambda}(c_2\cos\sqrt{\lambda}x - c_1\sin\sqrt{\lambda}x)$

BC:

$\phi'(0) = 0 \Rightarrow \sqrt{\lambda}c_2\cos\sqrt{\lambda}x = 0 \Rightarrow c_2 = 0$

$\phi'(L) = 0 \Rightarrow -\sqrt{\lambda}c_1\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$

$\Rightarrow \phi_n(x) = \cos\frac{n\pi}{L}x$

Now, solve equ. (1) with condition,

$$\begin{cases} h'' - \lambda h = 0 \\ h(0) = 0 \end{cases}$$

equ. (1) becomes $h'' - \left(\frac{n\pi}{L}\right)^2 h = 0$

• $n=0 \Rightarrow h''=0 \Rightarrow \begin{cases} h(y) = C_1 + C_2 y \Rightarrow h(y) = C_2 y \\ h(0) = 0 \Rightarrow C_1 = 0 \end{cases}$

• $n \neq 0 \Rightarrow h'' - \left(\frac{n\pi}{L}\right)^2 h = 0, \quad h(y) = e^{ry} \Rightarrow r - \left(\frac{n\pi}{L}\right)^2 = 0 \Rightarrow r = \pm \frac{n\pi}{L}$

$\Rightarrow h_n(y) = C_{n1} e^{\frac{n\pi}{L}y} + C_{n2} e^{-\frac{n\pi}{L}y}$

$h(0) = 0 \Rightarrow C_{n1} + C_{n2} = 0 \Rightarrow C_{n1} = -C_{n2}$

$\Rightarrow h_n(y) = C_{n1}\left(e^{\frac{n\pi}{L}y} - e^{-\frac{n\pi}{L}y}\right) \Rightarrow h_n(y) = C_n \sinh \frac{n\pi}{L} y$

From superposition

$$u(x,y) = C_0 y + \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{L} y \cos \frac{n\pi}{L} x$$

Now, consider the condition

$u(x,H) = f(x) \Rightarrow C_0 H + \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{L} H \cos \frac{n\pi}{L} x = f(x)$

$\int_0^L f(x)\,dx = C_0 H \int_0^L dx + \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi H}{L} \underbrace{\int_0^L \cos \frac{n\pi x}{L}\,dx}_{\to 0} \Rightarrow C_0 = \frac{1}{HL}\int_0^1 f(x_0)\,dx_0$

$$\Rightarrow C_n = \frac{2\int_0^L f(x_1) \cos \frac{n\pi x_1}{L}\,dx_1}{L \sinh \frac{n\pi H}{L}}$$

$$\Rightarrow u(x,y) = \frac{1}{HL}\int_0^L f(x_1)\,dx_1\, y + \sum_{n=1}^{\infty} \frac{2}{L\sinh \frac{n\pi H}{L}} \int_0^L f(x_1) \cos\frac{n\pi x_1}{L}\,dx_1 \sinh \frac{n\pi y}{L} \cos \frac{n\pi x}{L}$$

c.

let $u(x,y) = h(x)\,\phi(y)$, plug into PDE: $\phi(y)\, h'' + h(x)\, \phi'' = 0 \quad (\div \phi h)$

$$\frac{h''}{h} + \frac{\phi''}{\phi} = 0$$

$$\frac{h''}{h} = -\frac{\phi''}{\phi} = -\lambda$$

2 ODE's :

$h'' = \lambda h$ ---- (1)

$\phi'' = -\lambda \phi$ ---- (2)

BC

$u(x,0)=0 \Rightarrow \phi(0)=0$

$u(x,H)=0 \Rightarrow \phi(H)=0$

Eigenvalue Problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = \phi(H) = 0 \end{cases}$$

$\phi(y) = e^{ry} \quad \Rightarrow r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$

1. $\lambda < 0 \Rightarrow \phi(y) = c_1 e^{\sqrt{-\lambda} y} + c_2 e^{-\sqrt{-\lambda} y}$

$\phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = c_2$

$\phi(H) = 0 \Rightarrow c_1 e^{\sqrt{-\lambda} H} - c_1 e^{-\sqrt{-\lambda} H} = 0 \quad \Rightarrow c_1\left(e^{\sqrt{-\lambda} H} - e^{-\sqrt{-\lambda} H}\right) = 0$ if $\lambda = 0 \Rightarrow$ trivial

2. $\lambda = 0 \Rightarrow \phi(y) = c_1 + c_2 y$

$\phi(0) = 0 \Rightarrow c_1 = 0$

$\phi(H) = 0 \Rightarrow c_2 H = 0 \Rightarrow c_2 = 0 \qquad$ trivial

3. $\lambda > 0 \Rightarrow \phi(y) = c_1 \sin\sqrt{\lambda}\, y + c_2 \cos\sqrt{\lambda}\, y$

$\left.\begin{array}{l} \phi(0) = 0 \Rightarrow c_2 = 0 \\ \phi(H) = 0 \Rightarrow c_1 \sin\sqrt{\lambda} H = 0 \Rightarrow c_1 = 0 \end{array}\right\}$ trivial

or $\Rightarrow \sqrt{\lambda} H = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{H}\right)^2$

$\Rightarrow \phi_n(y) = c_1 \sin \frac{n\pi}{H} y \qquad \lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad n = 1, 2, \ldots$

Now, we solve for equ. (1)

$$\begin{cases} h'' - \lambda h = 0 \\ h'(0) = 0 \end{cases} \qquad \lambda = \left(\frac{n\pi}{H}\right)^2 \qquad n = 1, 2, \ldots$$

$h = e^{rx} \quad \Rightarrow r^2 - \lambda = 0 \qquad ; \lambda > 0$

$\Rightarrow h(x) = C_3 e^{\sqrt{\lambda} x} + C_4 e^{\sqrt{\lambda} x} = C_3 \cosh\sqrt{\lambda} x + C_4 \sinh\sqrt{\lambda} x$

$\Rightarrow h'(x) = \sqrt{\lambda}\left(C_3 \sinh\sqrt{\lambda} x + C_4 \cosh\sqrt{\lambda} x\right)$

$h'(0)=0 \Rightarrow \sqrt{\lambda}\, C_4 = 0 \Rightarrow C_4 = 0$

$\Rightarrow h(x) = C_3 \cosh\sqrt{\lambda}\, x = C_3 \cosh \frac{n\pi}{H} x$

Superposition

$$u(x,y) = \sum_{n=1}^{\infty} C_n \cosh \frac{n\pi x}{H} \sin \frac{n\pi y}{H}$$

Now, consider

$$u(L,u) = g(y) \Rightarrow \sum_{n=1}^{\infty} C_n \cosh \frac{n\pi L}{H} \sin \frac{n\pi y}{H} = g(y)$$

$$\Rightarrow C_n = \frac{2\int_0^H g(y_0) \sin \frac{n\pi y_0}{H} dy_0}{H \cosh \frac{n\pi L}{H}}$$

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} \frac{2\int_0^H g(y_0) \sin \frac{n\pi y_0}{H} dy_0}{H \cosh \frac{n\pi L}{H}} \cosh \frac{n\pi x}{L} \sin \frac{n\pi y}{H} \quad ; n = 1, 2, \cdots$$

$\boxed{2.5.9}$ a.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{s.o.v } u(r,\theta) = G(r)\Phi(\theta)$$

It is known that if $u(r,\theta) = \Phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\theta^2}$

$$\Rightarrow \frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\theta^2} = \lambda$$; λ is the separation constant

This results in 2 ODEs:

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda = 0 \text{ ---- } (1)$$

$$\frac{1}{\Phi}\frac{d^2\Phi(\theta)}{d\theta^2} + \lambda = 0 \text{ ..... } (2)$$

Applying B.C.'s:

$u(r,0) = 0 \Rightarrow \Phi(0)G(r) = 0 \Rightarrow \Phi(0) = 0$

$u(r,\frac{\pi}{2}) = 0 \Rightarrow \Phi(\frac{\pi}{2})G(r) = 0 \Rightarrow \Phi(\pi/2) = 0$

So, we have the eigenvalue problem

$$\begin{cases} \frac{d^2\Phi}{d\theta^2} + \Phi\lambda = 0 \\ \Phi(0) = \Phi(\frac{\pi}{2}) = 0 \end{cases}$$

let $\Phi = e^{rx} \Rightarrow r^2 + \lambda = 0$

1. $\lambda < 0$ (two distinct real roots) $r_{1,2} = \pm\sqrt{-\lambda}$

general sol.: $\phi(\theta) = c_1 e^{\sqrt{-\lambda}\,\theta} + c_2 e^{-\sqrt{-\lambda}\,\theta}$

B.C.'s:

$\phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$\phi(\frac{\pi}{2}) = 0 \Rightarrow c_1 e^{\sqrt{-\lambda}\frac{\pi}{2}} + c_2 e^{-\sqrt{-\lambda}\frac{\pi}{2}} = 0$

$\Rightarrow c_1 \left(e^{\sqrt{-\lambda}\frac{\pi}{2}} - e^{-\sqrt{-\lambda}\frac{\pi}{2}}\right) = 0 \quad \Rightarrow c_1 = c_2 = 0$ trivial sol.

2. $\lambda > 0$ (complex conjugate roots) $r_{1,2} = \pm i\sqrt{\lambda}$

general sol.: $\phi(\theta) = c_1 \cos\sqrt{\lambda}\,\theta + c_2 \sin\sqrt{\lambda}\,\theta$

B.C.'s: $\phi(0) = 0 \Rightarrow c_1 = 0$

$\phi(\frac{\pi}{2}) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}\,(\frac{\pi}{2}) = 0 \quad \Rightarrow \sqrt{\lambda}\,(\frac{\pi}{2}) = n\pi$

$\Rightarrow \sqrt{\lambda} = 2n \quad \Rightarrow \lambda = (2n)^2$

$\phi_n = C_n \sin 2n\theta \qquad \lambda_n = 4n^2 \qquad ; n = 1, 2, \ldots$

3. $\lambda = 0$ (identical roots) $r_{1,2} = 0 \quad e^{0\theta} = 1$

general sol.: $\phi(\theta) = c_1\theta + c_2$

B.C.'s:

$$\left.\begin{aligned} &\phi(0) = 0 \Rightarrow c_2 = 0 \\ &\phi(\tfrac{\pi}{2}) = 0 \Rightarrow c_1\left(\tfrac{\pi}{2}\right) = 0 \Rightarrow c_1 = 0 \end{aligned}\right\} \text{trivial}$$

For equ. (1)

$r \frac{d}{dr}\left(r \frac{dG}{dr}\right) = \lambda G \quad \Rightarrow r\left(r \frac{d^2G}{dr^2} + \frac{dG}{dr}\right) = \lambda G$

$r^2 G'' + r G' - \lambda G = 0 \qquad \phi(\theta) G(a) = 0 \Rightarrow G(a) = 0$

Cachy-Euler

let $r = e^t$, $t = \ln r$, $\bar{D} = \frac{d}{dt}$ $\Rightarrow$ $rDG = \bar{D}G$, $r^2D^2G = \bar{D}(\bar{D}-1)G$

$$\bar{D}(\bar{D}-1)G + \bar{D}G - \lambda = 0$$

$$\bar{D}^2 - \lambda = 0 \Rightarrow \bar{D} = \pm\sqrt{\lambda} = \pm 2n$$

$$G(r) = C_1 r^{2n} + C_2 r^{-2n}$$

$$G(a) = 0 \Rightarrow C_1 a^{2n} + C_2 a^{-2n} = 0 \Rightarrow C_1 a^{2n} = -C_2 a^{-2n}$$

$$a^{-2n} G(r) = C_1 r^{2n} a^{-2n} + C_2 a^{-2n} r^{-2n}$$

$$= C_1 r^{2n} a^{-2n} - C_1 r^{-2n} a^{2n}$$

$$= C_1 \left[\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right]$$

From the principle of superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n \left[\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right] \sin(2n\theta)$$

Now, consider the condition

$$u(b,\theta) = f(\theta)$$

$$\Rightarrow f(\theta) = \sum_{n=1}^{\infty} C_n \left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right] \sin(2n\theta)$$

$$C_n \left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right] \sin(2n\theta) = \frac{4}{\pi} \int_0^{2\pi} f(\theta) \sin(2n\theta)\, d\theta$$

From arthaganalit of sines

$$C_n = \frac{4 \int_0^{2\pi} f(\theta) \sin 2n\theta \, d\theta}{\pi \left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right]}$$

$\boxed{2.5.9}$ b.

$$u(r,\theta) = G(r)\,\phi(\theta)$$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = -\lambda$$

This results in 2 ODEs:

$$\begin{cases} r^2G'' + rG' + \lambda G = 0 \text{ ------ } (1) \\ \phi'' - \lambda\phi = 0 \text{ ---- } (2) \end{cases}$$

B.C.'s:

$u(a,\theta) = 0 \Rightarrow G(a)\phi(\theta) = 0 \Rightarrow G(a) = 0$

$u(b,\theta) = 0 \Rightarrow G(b)\phi(\theta) = 0 \Rightarrow G(b) = 0$

Eigenvalue problem

$$\begin{cases} r^2G'' + rG' + \lambda G = 0 \\ G(a) = G(b) = 0 \end{cases}$$

let $r = e^t$, $t = \ln r$, $\bar{D} = \frac{d}{dt} \Rightarrow rDG = \bar{D}G$, $r^2D^2G = \bar{D}(\bar{D}-1)G$

$$\bar{D}(\bar{D}-1)G + \bar{D}G + \lambda = 0$$

$$\bar{D}^2 + \lambda = 0$$

1. $\lambda < 0$ $\quad \bar{D}_{1,2} = \pm\sqrt{-\lambda}$

general sol.: $G(r) = c_1 r^{\sqrt{-\lambda}} + c_2 r^{\sqrt{-\lambda}}$

BCs:

$$\left.\begin{aligned} G(a) = 0 \Rightarrow c_1 a^{\sqrt{-\lambda}} + c_2 a^{-\sqrt{-\lambda}} = 0 \\ G(b) = 0 \Rightarrow c_1 b^{\sqrt{-\lambda}} + c_2 b^{-\sqrt{-\lambda}} = 0 \end{aligned}\right\} \text{ trivial}$$

2. $\lambda > 0$ $\quad\quad \bar{D}_{1,2} = \pm i\sqrt{\lambda}$

general sol.: $G(r) = C_1 \cos\sqrt{\lambda}\,Ln r + C_2 \sin\sqrt{\lambda}\,Ln r$

if $G(r) = r^{\bar{D}}$ satisfies equ. (1), then $G(\frac{r}{a}) = (\frac{r}{a})^{\bar{D}}$ also satisfies the equ.

$$G(\tfrac{r}{a}) = C_1 \cos\left(\sqrt{\lambda}\,\ln(\tfrac{r}{a})\right) + C_2 \sin\left(\sqrt{\lambda}\,Ln(\tfrac{r}{a})\right)$$

$$\boxed{\begin{aligned} G &= r^{\bar{D}} \\ G' &= \bar{D}\, r^{\bar{D}-1} \\ G'' &= \bar{D}(\bar{D}-1)\, r^{\bar{D}-2} \end{aligned}}$$

BC

$G(a) = 0 \Rightarrow \quad C_1 = 0$

$G(b) = 0 \Rightarrow \quad C_2 \sin(\sqrt{\lambda}\,Ln(\frac{b}{a})) = 0$

$\Rightarrow \sin(\sqrt{\lambda}\,Ln(\frac{b}{a})) = 0$

$\Rightarrow \sqrt{\lambda}\,Ln(\frac{b}{a}) = n\pi \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{Ln(\frac{b}{a})}\right)^2 \quad\quad n = 1, 2, \cdots$

$\Rightarrow G_n(r) = C_n \sin\left(n\pi \frac{Ln(r/a)}{Ln(b/a)}\right)$

3. $\lambda = 0$ $\quad\quad \bar{D}_{1,2} = 0$

general sol.: $G(r) = C_1 + C_2 r$

BC

$$\left.\begin{aligned} G(a) = 0 \quad &\Rightarrow \quad C_1 + C_2 Ln a = 0 \\ G(b) = 0 \quad &\Rightarrow \quad C_1 + C_2 Ln b = 0 \end{aligned}\right\} \text{trivial}$$

The second problem

$\Phi'' - \lambda\Phi = 0 \quad\quad \Phi(0) = 0$

$r^2 - \lambda = 0 \quad\quad r_{1,2} = \pm\sqrt{\lambda}$

$\Phi(\theta) = C_1 e^{\sqrt{\lambda}\,\theta} + C_2 e^{-\sqrt{\lambda}\,\theta}$

BC

$\Phi(0) = 0 \Rightarrow \quad C_1 = -C_2$

$\Rightarrow C_1\left[e^{\sqrt{\lambda}\theta} - e^{-\sqrt{\lambda}\theta}\right]$

$\Phi_n(\theta) = C_n\left[e^{\sqrt{\lambda}\theta} - e^{-\sqrt{\lambda}\theta}\right] = \sinh\left(\frac{n\pi\theta}{Ln(b/a)}\right)$

$$u(r,\theta)=\sum_{n=1}^{\infty} C_n \sin\left[n\pi \frac{Ln(r/a)}{Ln(b/a)}\right] \sinh\left(\frac{n\pi\theta}{Ln(b/a)}\right)$$

Now, consider

$$u(r,\tfrac{\pi}{2}) = f(r)$$

$$f(r)=\sum_{n=1}^{\infty} C_n \sin\left[n\pi \frac{Ln(r/a)}{Ln(r/b)}\right] \sinh\left[\frac{n\pi^2}{2Ln(b/a)}\right]$$

let $z=\dfrac{Ln(r/a)}{Ln(b/a)}$ if $a<r<b$ $\quad 0<z<1$

$$dz=\frac{1}{rLn(b/a)}\,dr$$

$$f(r)=\sum_{n=1}^{\infty} C_n \sin(n\pi z) \sinh\left(\frac{n\pi z}{2Ln(b/a)}\right) \qquad \times \sin mz\pi$$

$$\int_0^1 f(r) \sin mz\pi \, dz = \frac{1}{2} C_n \sinh\left(\frac{n\pi^2}{2Ln(b/a)}\right)$$

$$C_n = \frac{2\int_0^1 f(r) \sin mz\pi \, dz}{\sinh\left(\frac{n\pi^2}{2Ln(b/a)}\right)}$$

$$= \frac{2\int_a^b f(r) \sin\left(m\pi \frac{Ln(r/a)}{Ln(b/a)}\right)\left(\frac{dr}{rLn(b/a)}\right)}{\sinh\left(\frac{n\pi^2}{2Ln(b/a)}\right)}$$

$$= \frac{2\int_a^b \frac{f(r)}{r} \sin\left(m\pi \frac{Ln(r/a)}{Ln(b/a)}\right) dr}{Ln\left(\frac{b}{a}\right) \sinh\left(\frac{n\pi^2}{2Ln(b/a)}\right)}$$