Heat Equation

Finite Difference Methods for ODE & PDE

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1 Solving the heat equations using Finite Difference Methods (FDMs):

In the last week, finite difference methods were briefly introduced. Now, I am going to use these methods to approximate the heat equation and then try to do some implementations with Python.

First, recall that:

- **Forward Differences**:

$$\frac{df}{dt} \simeq \frac{f(t+\Delta t) - f(t)}{\Delta t}. \tag{1}$$

  Error: $O(\Delta t)$

- **Backward Differences**:

$$\frac{df}{dt} \simeq \frac{f(t) - f(t+\Delta t)}{\Delta t}. \tag{2}$$

  Error: $O(\Delta t)$

- **Central Differences**:

$$\frac{df}{dt} \simeq \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t}. \tag{3}$$

  Error: $O(\Delta t^2)$

- **Second Derivative**:

$$f''(t) = \frac{f(t+\Delta t) + f(t-\Delta t) - 2f(t)}{\Delta t^2}. \tag{4}$$

  Error: $O(\Delta t^2)$

1.1 Partial Differential Equations and Heat Equation

The heat equation is a Partial Differential Equation (PDE), e.i., it has more than one independent variable. Second order PDEs have a general form of:

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = \Phi(x,y,u,u_x,u_y), \tag{5}$$

where u_{xx} is equivalent to $\frac{\partial^2 u}{\partial x^2}$, and u_{xy} equivalent to $\frac{\partial^2 u}{\partial x \partial y}$, ... etc. I am going to use both notations interchangebly.

Hyperbolic, parabolic, and elliptic PDEs are the three main categories of PDEs. The wave propagation, the time-dependent diffusion processes, and the steady state or equilibrium processes are each represented respectively by one of these PDEs from a physical perspective. The transfer of a physical quantity, such as fluids or waves, is thus modeled by hyperbolic equations. Evolutionary processes that result in a stable state defined by an elliptic equation are described by parabolic issues. Elliptic equations are connected to a certain system state that, in theory, corresponds to the energy minimum.

- if $B^2 - 4AC > 0$, then the PDE is hyperbolic.
- if $B^2 - 4AC = 0$, then the PDE is parabolic.
- if $B^2 - 4AC < 0$, then the PDE is elliptic.

The heat equation, which represents the main concern here, is a parabolic PDE.

1.1.1 Solving 1D heat equation:

Consider an isolated heat-conducting rod. The heat equation that governs the tempurature distribution at any time along that rod is:

$$\frac{\partial u}{\partial t} - C\frac{\partial^2 u}{\partial x^2} = 0. \tag{6}$$

Initial Conditions:
$u(x, 0) = u^0(x)$.

Boundary Conditions:
$u(a, t) = \alpha(t)$.
$u(b, t) = \beta(t)$.

Note that since the boundary conditions are functions of time, this means they will not be chnaging with time.

I am in a position now to rewrite the equation in a "decrtized" form by using FDMs, so $x_i = i\Delta x$, and $t_k = k\Delta t$. Therefore, $u(x, t)$ can be written as u_i^k. Accordingly, the approximation of 1D heat equation will have this form,

$$u_t = \frac{u_i^{k+1} - u_i^k}{\Delta t}. \tag{7}$$

And,

$$u_{xx} = \frac{u_{i+1}^k + u_{i-1}^k - 2u_i^k}{\Delta x^2}. \tag{8}$$

Hence,

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} - C[\frac{u_{i+1}^k + u_{i-1}^k - 2u_i^k}{\Delta x^2}] = 0. \tag{9}$$

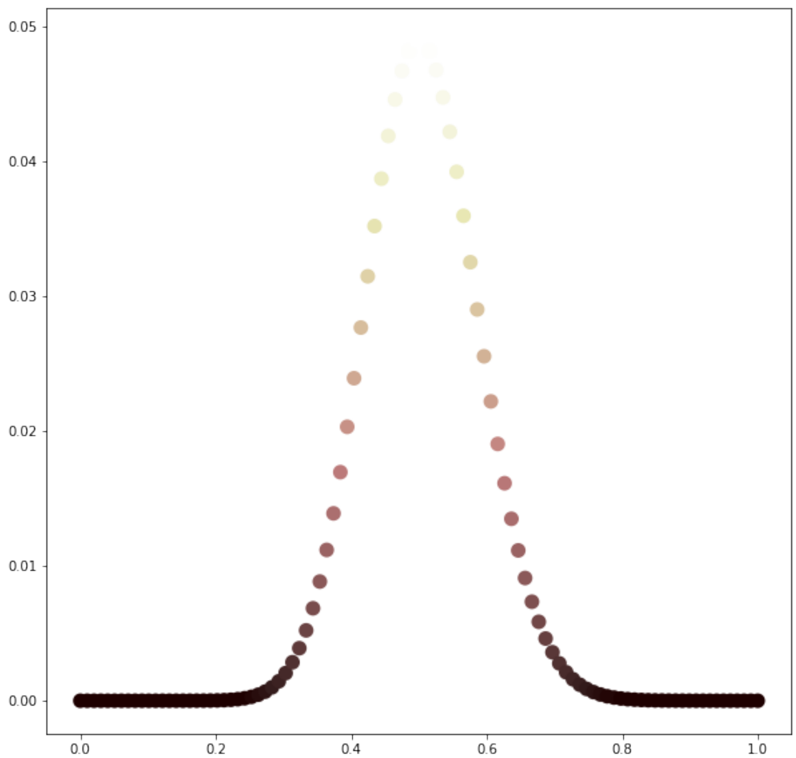
The total error $= O(\Delta t + \Delta x^2)$.

The following equation results from arranging the previous one,

$$u_i^{k+1} = \gamma(u_{i+1}^k + u_{i-1}^k - 2u_i^k) + u_i^k, \tag{10}$$

where $\gamma = C\frac{\Delta t}{\Delta x^2}$. For simplicity, assume that C = 1. Further, assume that $\frac{\Delta t}{\Delta x^2} = \frac{1}{4}$.

Now, by converting these things into codes we get the following figure,



The exact same applies for 2D and 3D, where we can get the following approximations, respectivly,

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} - C[\frac{u_{i+1,j}^k + u_{i-1,j}^k - 2u_{i,j}^k}{\Delta x^2} + \frac{u_{i,j+1}^k + u_{i,j-1}^k - 2u_{i,j}^k}{\Delta y^2}] = 0. \tag{11}$$

The total error $= O(\Delta t + \Delta x^2 + \Delta y^2)$.

And finally,

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} - C[\frac{u_{i+1,j,f}^k + u_{i-1,j,f}^k - 2u_{i,j,f}^k}{\Delta x^2} + \frac{u_{i,j+1,f}^k + u_{i,j-1,f}^k - 2u_{i,j,f}^k}{\Delta y^2} + \frac{u_{i,j,f+1}^k + u_{i,j,f-1}^k - 2u_{i,j,f}^k}{\Delta z^2}] = 0. \tag{12}$$

The total error $= O(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2)$.