Projet Monte Carlo

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1 Exercice 8 : Loi normale tronquée

Notre but dans cette exercice est de générer une loi normale tronquée de support [b ,+ ∞ [qui est définie par la densité f proportionnelle à:

$$f1(x) = exp(-\frac{(x-u)^2}{2\ \sigma^2})\ \mathbb{1}_{x \geq\ b}$$

Or on a f est la densité de la loi normale tronquée tel que $f = \alpha f$. Tout d'abord on doit chercher α.

On sait que :

$$\int \alpha\ exp(-\frac{(x-u)^2}{2\ \sigma^2})\ \mathbb{1}_{x \geq\ b}\, dx = 1$$

On pose $y = \frac{x-u}{\sigma} \Rightarrow \begin{cases} dy & = & \frac{dx}{\sigma} \\ si\ y & = & b \Rightarrow x = \frac{b-u}{\sigma} \\ y & \to + & \infty \Rightarrow x \to \infty \end{cases}$

$$\Rightarrow \int_{\frac{b-u}{\sigma}}^{+\infty} \alpha\ \sigma\ exp(\frac{-y^2}{2})\, dy = 1$$

$$\Leftrightarrow \int_{\frac{b-u}{\sigma}}^{+\infty} exp(\frac{-y^2}{2})\, dy = \frac{1}{\alpha\ \sigma}$$

$$\Leftrightarrow \int_{\frac{b-u}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\, dy = \frac{1}{\sqrt{2\ \pi}}\ \frac{1}{\alpha\ \sigma}$$

Et on sait que

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\, dy = \int_{-\infty}^{\frac{b-u}{\sigma}} \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\, dy + \int_{\frac{b-u}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\, dy = 1$$

$$d'où\ \ \frac{1}{\sqrt{2\ \pi}}\ \frac{1}{\alpha\ \sigma} = \int_{\frac{b-u}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\, dy = 1 - \int_{-\infty}^{\frac{b-u}{\sigma}} \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\, dy$$

On sait que la loi normale $\mathcal{N}(0,\, 1)$ est une loi symétrique donc :

$$F(x) + F(-x) = 1\ \ d'\text{où}\ F(x) = 1 - F(-x)$$

Donc

$$\frac{1}{\sqrt{2\ \pi}}\ \frac{1}{\alpha\ \sigma} = 1 - F(\frac{b-u}{\sigma}) = F(\frac{u-b}{\sigma})$$

$$\Rightarrow \alpha = \frac{1}{\sqrt{2\ \pi\ \sigma^2}} * \frac{1}{F(\frac{u-b}{\sigma})}$$

1.1 Question 1

On a

$$f = \frac{1}{F(\frac{u-b}{\sigma})} \frac{1}{\sqrt{2\ \pi\ \sigma^2}}\ exp(-\frac{(x-u)^2}{2\ \sigma^2})\ \mathbb{1}_{x \geq\ b} \leq \frac{1}{F(\frac{u-b}{\sigma})} \frac{1}{\sqrt{2\ \pi\ \sigma^2}}\ exp(-\frac{(x-u)^2}{2\ \sigma^2})\ \mathbb{1}_{\mathbb{R}}(x)$$

avec une probabilité d'acceptation p tel que :

$$p = \frac{1}{M} = F(\frac{u-b}{\sigma})$$

1.1.1 trunc.norm

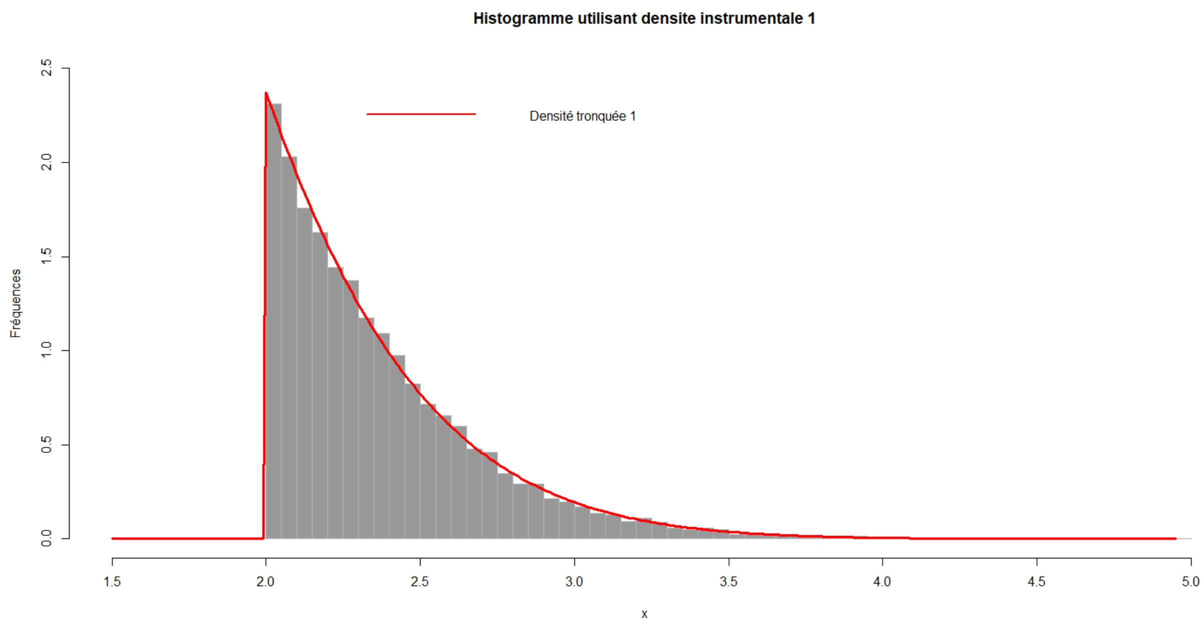
```
f=function(x,b,mean=0,sigma=1){
  x=(1/pnorm((mean-b)/sigma))*dnorm(x,mean,sigma)*(x>=b)
  return(x)}

sim_rejet1=function(n,b,mean=0,sigma=1){
  res=c()
  m=n
  while(m>0){

    u=runif(m*pnorm(mean-b/sigma)%/%1+1)
    y=rnorm(m*pnorm(mean-b/sigma)%/%1+1,mean,sigma)
    res=append(res,y[which(y>b)])
    m=n-length(res)
 }
  return(res[1:n])
}
```

1.2 Question 2

```
n <- 10000
b <- 2
x <- sim_rejet1(n, b)
hist(x, freq = FALSE, main = "Histogramme de x", ylab = "Fréquences",
col = "grey60", border = "grey70", breaks = 40, ylim = c(0,
2.5), xlim = c(1.5, max(x)))
y <- seq(1.5, max(x), 0.01)
lines(y, f(y, b), col = "olivedrab3", lwd = 3)
legend("topright", "Densité tronquée", box.lty = 0, bg = "gray95",inset = 0.05, col = "yellow", lty = 1
```



1.3 Question 3

La probabilité d'acceptation est

$$p = \phi(-50) = pnorm(-50) \approx 0$$

```r
pnorm(-50)
```

```
## [1] 0
```

1.4 Question 4

$$g_\lambda(x) = \lambda\ exp(-\lambda\ (x-b))\ \mathbb{1}_{x \geq\ b}$$

on cherche M tel que

$$\frac{f}{g} \leq M \Leftrightarrow \frac{\alpha\ f}{g} \leq M$$
$$\Leftrightarrow \frac{f_1}{g} \leq \frac{M}{\alpha} = M'$$

Soit

$$h(x) = \frac{f_1}{g}(x) = \frac{1}{\lambda}\ exp(\lambda\ (x-b) - \frac{(x-u)^2}{2\ \sigma^2})$$

$$h'(x) = \frac{1}{\lambda}\ (\lambda - \frac{2\ (x-u)}{2\ \sigma^2})\ exp(\lambda\ (x-b) - \frac{(x-u)^2}{2\ \sigma^2})$$

$$h'(x) = 0 \Leftrightarrow \lambda = \frac{x-u}{\sigma^2} \Leftrightarrow x = u + \lambda\ \sigma^2$$

x	$-\infty$	b	$\lambda\ \sigma^2 + u$	b	$+\infty$
$\lambda - \frac{x-u}{\sigma^2}$		$+$	0	$-$	
$h'(x)$		$+$	0	$-$	
$h(x)$		$\nearrow$		$\searrow$	

Remarque : discussion suivant (b)

$$si\ b < \lambda\ \sigma^2 + u \quad h(x)\ atteint\ son\ max\ en \quad x = u + \lambda\ \sigma^2$$

\ Alors

$$h(x) = \frac{f_1}{g}(x) \leq h(u + \lambda\ \sigma^2) = \frac{1}{\lambda}\ exp(\lambda\ (u - b) + \frac{\lambda\ \sigma^2}{2})$$

$$si\ b \geq u + \lambda\ \sigma^2$$

son max est atteint en h(b)

$$\frac{f_1}{h}(x) \leq h(b) = \frac{1}{\lambda} exp(\frac{-(b-u)^2}{2\ \sigma^2})$$

et on a

$$\forall\ b \in\ \mathbb{R}$$

on a

$$\frac{f_1}{g}(x) \leq \frac{1}{\lambda}\ exp(\lambda\ (u - b) + \frac{\lambda\ \sigma^2}{2}) = M' = \frac{M}{\alpha}$$

Or on sait que la probabilité d'acceptation est $p = \frac{1}{M}$

Donc on va chercher λ^* pour laquelle le temps moyenne de calcul T soit optimale

si p $\nearrow$ alors on a plus de chance d'accepter donc T $\searrow$
Donc minimiser T $\Leftrightarrow$ maximiser $p = \frac{1}{M}$ $\Leftrightarrow$ minimiser $M = \alpha\ M'$
Cherchons à minimiser M' par rapport à λ

$$\lambda \longmapsto M'(\lambda) = \frac{1}{\lambda}\ exp(\lambda\ (u - b) + \frac{\lambda^2\ \sigma^2}{2})$$

$$\frac{\partial M'}{\partial \lambda} = \frac{-1}{\lambda^2}\ exp(\lambda\ (u - b) + \frac{\lambda^2\ \sigma^2}{2}) + ((u - b) + \lambda\ \sigma^2)\ \frac{1}{\lambda}\ exp(\lambda\ (u - b) + \frac{\lambda^2\ \sigma^2}{2})$$

$$= exp(\lambda\ (u - b) + \frac{\lambda^2\ \sigma^2}{2})(\frac{-1}{\lambda^2} + (u - b)\ \frac{1}{\lambda} + \sigma^2)$$

$$\frac{\partial M'}{\partial \lambda} = 0 \Leftrightarrow \frac{-1}{\lambda^2} + (u - b)\ \frac{1}{\lambda} + \sigma^2 = 0$$

$$\Leftrightarrow -1 + (u - b)\ \lambda + \sigma^2\ \lambda^2 = 0$$

$$\Delta\ = (u - b)^2 + 4\sigma^2\ >\ 0$$

$$\lambda_1 = \frac{-(u - b) - \sqrt{\Delta}}{2\ \sigma^2}\ <\ 0$$

Si $\lambda > 0$ on rejette

$$\lambda_2 = \frac{-(u - b) + \sqrt{\Delta}}{2\ \sigma^2}\ >\ 0$$

$$\frac{\partial^2 M'}{\partial^2 \lambda}(\lambda_2) < 0$$

Donc λ_2 est un minimum de M'(λ) par suite λ^* optimale égale

$$\lambda^* = \frac{(b-u) + \sqrt{(u-b)^2 + 4\ \sigma^2}}{2\ \sigma^2}$$

Conclusion

$$\frac{f_1}{g_\lambda}(x) \leq \begin{cases} \frac{1}{\lambda} exp(\lambda\ (u-b) + \frac{(\lambda\ \sigma)^2}{2}) & si\ b < u + \lambda\ \sigma^2 \\ \frac{1}{\lambda} exp(-\frac{(b-u)^2}{2\ \sigma^2}) & sinon \end{cases}$$

$$f = \alpha\ f_1 = \frac{1}{\sqrt{2\ \pi\sigma^2}}\ \frac{1}{F(\frac{u-b}{\sigma})}\ f_1$$

$$\Rightarrow \frac{f}{g_\lambda}(x) \leq \begin{cases} \frac{1}{\sqrt{2\ \pi\sigma^2}}\ \frac{1}{F(\frac{u-b}{\sigma})}\ \frac{1}{\lambda}\ exp(\lambda\ (u-b) + \frac{(\lambda\ \sigma)^2}{2}) & si\ b < u + \lambda\ \sigma^2 \\ \frac{1}{\sqrt{2\ \pi\sigma^2}}\ \frac{1}{F(\frac{u-b}{\sigma})}\ \frac{1}{\lambda}\ exp(-\frac{(b-u)^2}{2\ \sigma^2}) & sinon \end{cases}$$

$$M = \begin{cases} \frac{1}{\sqrt{2\ \pi\sigma^2}}\ \frac{1}{F(\frac{u-b}{\sigma})}\ \frac{1}{\lambda}\ exp(\lambda\ (u-b) + \frac{(\lambda\ \sigma)^2}{2}) & si\ b < u + \lambda\ \sigma^2 \\ \frac{1}{\sqrt{2\ \pi\sigma^2}}\ \frac{1}{F(\frac{u-b}{\sigma})}\ \frac{1}{\lambda}\ exp(-\frac{(b-u)^2}{2\ \sigma^2}) & sinon \end{cases}$$

$$U \rightarrow \mathbb{U}[0,1]$$

$$y \rightarrow g_\lambda$$

$$U \leq \frac{f(y)}{M\ g_\lambda(y)} = \frac{f(y)}{g_\lambda(y)}\ p$$

1.4.1 Densite instrumentale

```
g <- function(x, lambda, b) {
  return(lambda * exp(lambda * (b - x)) * (x >= b))
}
```

1.4.2 lambda optimale d'apres l'exercice 3 TD 2

```
lambda_opti=function(b,mean,sigma){
  x=(b-mean+sqrt((mean-b)^2+4*sigma^2))/(2*sigma^2)
  return(x)
}
```

1.4.3 probabilité d'acceptation p=1/m discussion suivant la valeur de b

```
prob_accp=function(b,lambda,mean,sigma){
  ans=lambda*sqrt(2*pi*sigma^2)*pnorm((mean-b)/sigma)
  if (b<mean+lambda*sigma^2){
    p=ans/exp(lambda*(mean-b)+((lambda*sigma)^2)/2)
  }
  else
```

```
    p=ans/exp(-(((b-u)/sigma)^2)/2)

  return(p)
}
```

1.4.4 trunc norm 2

```
sim_rejet2=function(n,b,mean,sigma){
  res=c()
  m=n
  lambda=lambda_opti(b,mean,sigma)
  p=prob_accp(b,lambda,mean,sigma)
  while (m>0){
    u=runif(m%/%p+1)
    y=rexp(m%/%p+1,lambda)+b
    res=append(res,y[which(u <= p*f(y,b,mean,sigma)/g(y,lambda,b))])
    m=n-length(res)

  }
  return(res[1:n])
}
```

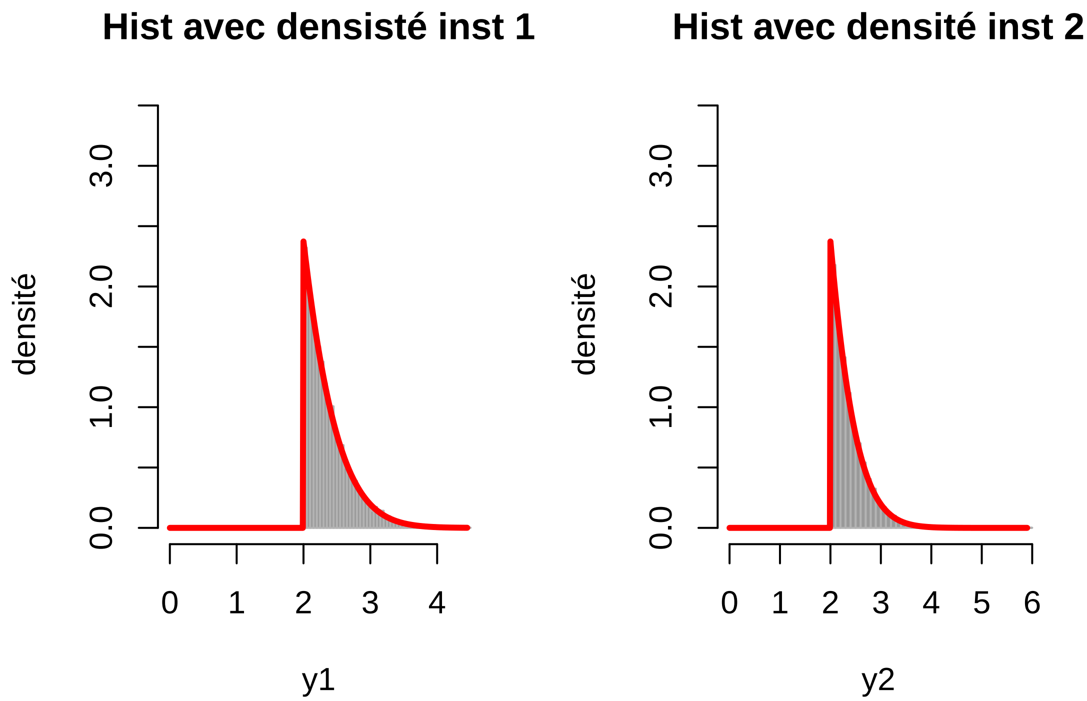
1.4.5 application numerique

```
n=10000
b=2
mean=0
sigma=1


par(mfrow = c(1, 2))

y1=sim_rejet1(n,b,mean,sigma)
x1=seq(b-2, max(y1), 0.01)
densite1=f(x1,b,mean,sigma)
hist(y1,freq = FALSE ,xlim=c(b-2,max(y1)),ylim=c(0,max(densite1)+1),main = "Hist avec densité inst 1")
lines(x1,densite1,lwd=3 , col = "red" )

y2=sim_rejet2(n,b,mean,sigma)
x2=seq(b-2, max(y2), 0.01)
densite2=f(x2,b,mean,sigma)
hist(y2,freq = FALSE ,xlim=c(b-2,max(y2)),ylim=c(0,max(densite2)+1),main = "Hist avec densité inst 2")
lines(x2,densite2,lwd=3,col = "red" )
```



1.5 Comparaison de deux methodes

1.5.1 probabilté d'acceptation

```
p1=pnorm(mean-b/sigma)
p2=prob_accp(b,lambda_opti(b,mean,sigma),mean,sigma)
cat("probabilité d'acceptation de Densité instrumentale n°1 = ",p1)
```

```
## probabilité d'acceptation de Densité instrumentale n°1 =  0.02275013
```

```
cat("probabilité d'acceptation de Densité instrumentale n°2=",p2)
```

```
## probabilité d'acceptation de Densité instrumentale n°2= 0.9336453
```

```
cat("on utisant la methode 2 on  a p2/p1=",p2/p1,"de chance qu'une variable soit accepté en utilisant la
```

```
## on utisant la methode 2 on  a p2/p1= 41.03912 de chance qu'une variable soit accepté en utilisant la
```

1.6 le rapport

```
p2/p1
```

```
## [1] 41.03912
```

la deuxieme methode est plus rapide

```
cat("on utisant la methode 2 on  a p2/p1=",p2/p1,"de chance qu'une variable soit accepté en utilisant la
```

```
## on utisant la methode 2 on  a p2/p1= 41.03912 de chance qu'une variable soit accepté en utilisant la
```

2 Exercice 13

Soit X une variable aléatoire de loi N (0, 1) et K une constante réelle. On s'intéresse au calcul de

$$p = \mathbb{P}(X \geq K)$$

Pour les applications numériques, on pourra prendre K = 2. Les intervalles de confiance seront donnés au niveau 0.95.($\alpha = 0.05$)

2.1 Question 1

$$X \sim \mathbb{N}(0,1) \quad et\ k \in \mathbb{R} p = \mathbb{P}(X \geq K)$$

$$p = \mathbb{P}(X \geq K) = 1 - \mathbb{P}(X < K) = 1 - F_X(k) = F_X(-k) : pnorm(-k)$$

```
Sn=function(y){
  m=length(y)
  return((sum(y^2)-m*(mean(y)^2))/(n-1))
}

mc.estim=function(y,level=0.95){
  m=length(y)
  I=mean(y)
  delta=sqrt(Sn(y)/n)*qnorm(0.5*(1+level))
  binf=I-delta
  bsup=I+delta
  return(data.frame(value=I,var=Sn(y),binf=binf,bsup=bsup,level=level))
}

mc.estim.evol=function(y,level=0.95){
  m=length(y)
  I=cumsum(y)/(1:n)
  sn=c(0)
  for (i in 2:m){
    sn=c(sn, unctionTokSn(y[1:i]))
  }
  delta=qnorm(0.5*(1+level))
  binf=I-sqrt(sn/(1:m))
  bsup=I+sqrt(sn/(1:m))
  return(data.frame(value=I,var=sn,binf=binf,bsup=bsup,level=level))
}
```

```
k=2
p=pnorm(-k)
```

2.2 Question 2

$$p = \int_K^{+\infty} f(x)\,dx = \int_{\mathbb{R}} \mathbb{I}_{[K,+\infty[}(x)\ f(x)\,dx = \mathbb{E}(\mathbb{I}_{[K,+\infty[}(X)) \qquad X \sim \mathbb{N}(0,1)$$

LFGN

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{[K,+\infty[}(x_i) \xrightarrow{ps} \mathbb{E}(\mathbb{I}_{[K,+\infty[}(X_1))$$

```r
 n <- 10000
x <- rnorm(m)
mc.estim((x >= k))
```

```
##    value        var       binf       bsup level
## 1 0.0232 0.02266403 0.02024936 0.02615064  0.95
```

2.3 Question 3

loi d'importance $\mathbb{N}(m, 1)$

$$g(x) = \frac{1}{\sqrt{2\ \pi}}\ exp(-\frac{(x-m)^2}{2})$$

$$p = \int_K^{+\infty} \frac{1}{\sqrt{2\ \pi}}\ exp(-\frac{x^2}{2})$$

$$p = \int_{\mathbb{R}} \mathbb{I}_{[K,+\infty[} \frac{1}{\sqrt{2\ \pi}}\ exp(-\frac{(x-m)^2}{2})\ exp(\frac{m^2}{2} - mx)\, dx$$

$$posons \quad y = x - m \Rightarrow \begin{cases} dy & = & dx \\ x & = & y + m \\ si\ x & = & K \to y = k - m \\ x & \to + & \infty \Rightarrow y \to +\infty \end{cases}$$

$$p = \int_{\mathbb{R}} \mathbb{I}_{[K-m,+\infty[}(y)\ \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\ exp(-my - \frac{m^2}{2})\, dy \quad \mathbb{E}(H'(X))$$

LFGN

$$\frac{1}{n} \sum_{i=1}^{n} H'(y_i) \quad \xrightarrow{ps} \quad \mathbb{E}(H'(y_1)) \qquad Y \sim \mathbb{N}(0, 1)$$

$$\mathbb{P}(X \geq K - m)) = 1 - \alpha$$

$$\alpha = \mathbb{P}(X \leq K - m)) = F(K - m)$$

$$q_\alpha^{\mathbb{N}(0,1)} = k - m$$

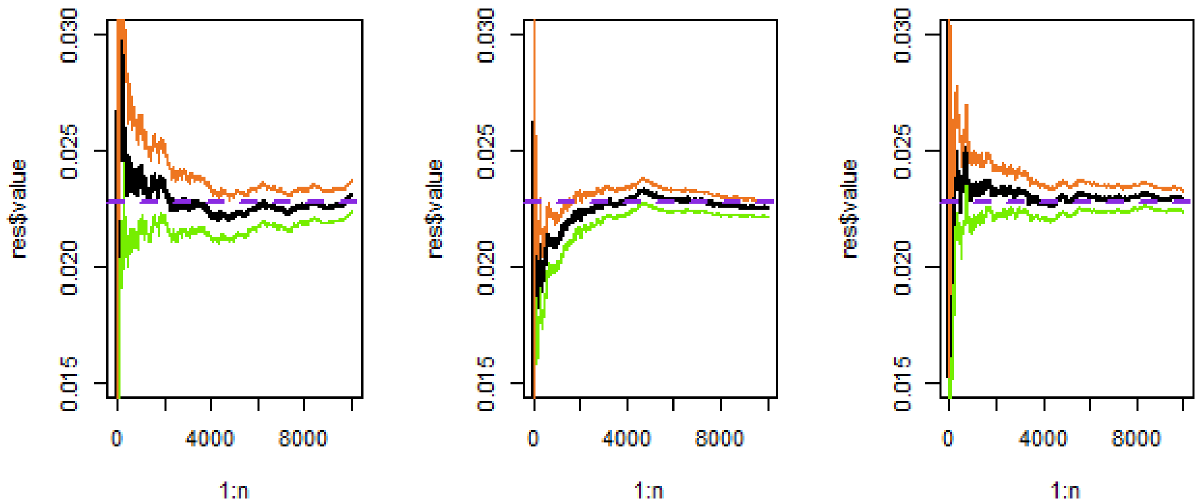
```r
n <- 10000
x <- rnorm(n)

h <- function(x, K , m ) {
  return(exp(-0.5 * m * m - m * x) * (x >= (K - m)))
}

compa1=c()

graph=function(x,k=2,m,h){
  y=h(x,k,m)
  m=length(y)
  res=mc.estim.evol(y)
  plot(1:n,res$value,type = "l", lwd = 2,ylim = c(0.02, 0.025))
  lines(1:n,res$binf,col="charteuse2",lwd = 1)
  lines(1:n,res$bsup,col="chocolate2",lwd = 1)
  abline(h=p, lwd = 2, lty = 2,col="blueviolet")
  return(c(m,res$value[n],res$var[n]))
}
```

```
level <- c(0.05, 0.5, 0.75)
m <- k - qnorm(level)
par(mfrow=c(1,3))
for (i in m){
  compa1=cbind(compa1,graph(x,k,i,h))
}
```




```
compa1=array(compa1,c(3,3),dimnames=list(c("m","value","variance")))
print(compa1)
```

```
##                 [,1]        [,2]        [,3]
## m         3.64485363 2.000000000 1.325510250
## value     0.02312393 0.022212556 0.022911906
## variance  0.00458483 0.001200609 0.002034903
```

α 0.5 est la valeur optimale car elle admet la variance minimale

2.4 Question 4

on a

$$g(x) = \lambda\ exp(-\lambda\ (x-k))\mathbb{I}_{x \geq K}$$

$$p = \int_K^{+\infty} \frac{1}{\sqrt{2\ \pi}} exp(\frac{-y^2}{2})\, dy$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\ \pi}}\ exp(-\frac{x^2}{2} + \lambda\ (x-k))\ \lambda exp(-\lambda\ (x-k))\ \mathbb{I}_{x \geq K}\, dx$$

$$= \mathbb{E}(\frac{1}{\sqrt{2\ \pi}\lambda} exp(-\frac{x^2}{2} + \lambda\ (x-k))) \quad X \sim \epsilon_{T_K}(\lambda)$$

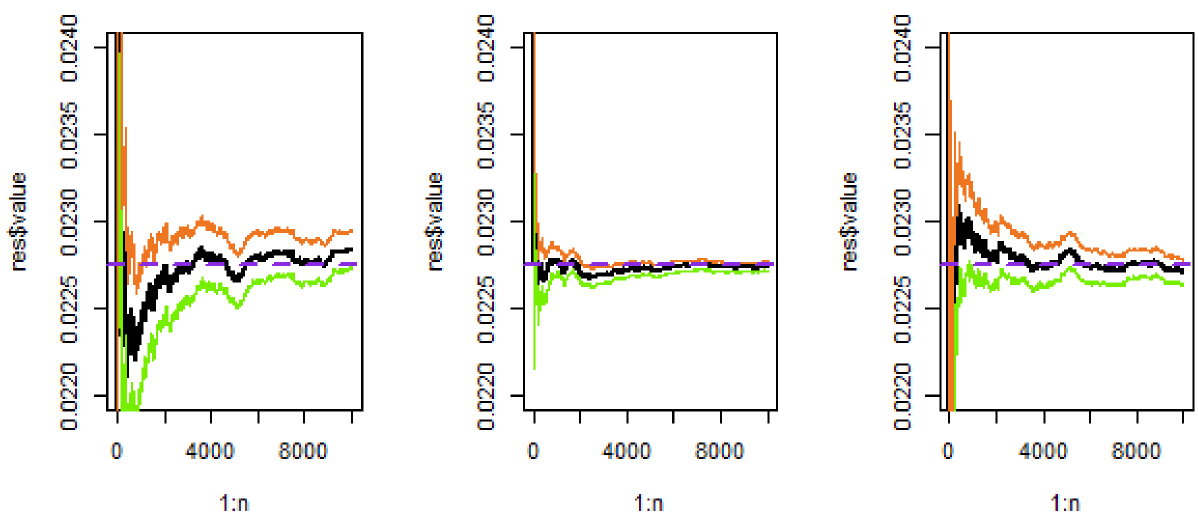
$$si\ on\ pose \quad y = \lambda\ (x-k) \Rightarrow \begin{cases} dy & = & \lambda\ dx \Rightarrow dx = \frac{dy}{\lambda} \\ si\ x & = & K \rightarrow y = 0 \\ x & \rightarrow + & \infty \Rightarrow y \rightarrow +\infty \\ \lambda & > & 0 \end{cases}$$

$$p = \int_{mathbbR} \frac{1}{\sqrt{2\ \pi}\lambda} exp(\frac{-1}{2}\ (\frac{y}{\lambda}+k)^2+y)\ exp(-y)\mathbb{I}_{\mathbb{R}+}(y)\,dy$$

$$\mathbb{E}(\frac{1}{\sqrt{2\ \pi}\lambda} exp(\frac{-1}{2}\ (\frac{y}{\lambda}+k)^2+y) \quad ouy\ \sim \epsilon(1)$$

```
h.exp <- function(x, K = 2, r = 1) {
  return(1/(r * sqrt(2 * pi)) * exp(x - 0.5 * (x/r + K)^2))}

x <- rexp(n)
lambda=c(1.5,2.5,3.5)
compa2=c()
for (i in lambda){
  compa2=cbind(compa2,graph(x,k,i,h.exp))
}
```




```
compa2=array(compa2,c(3,3),dimnames=list(c("lambda","value","variance")))
print(compa2)
```

```
##                 [,1]         [,2]         [,3]
## lambda   1.5000000000 2.500000e+00 3.500000e+00
## value    0.0228249149 2.270361e-02 2.274122e-02
## variance 0.0001164727 6.874682e-06 5.230287e-05
```

λ_2 **est la valeur optimale car elle admet la variance minimale**