Nanophotonics

EM Waves

- Models of light
- Motivation
- Maxwell’s equations
- Unpacking equations & sources
- Helmholtz (wave equations)
- Plane wave solutions
- Other solutions

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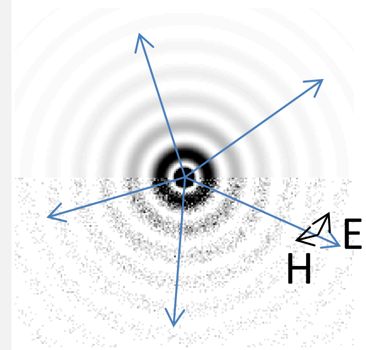
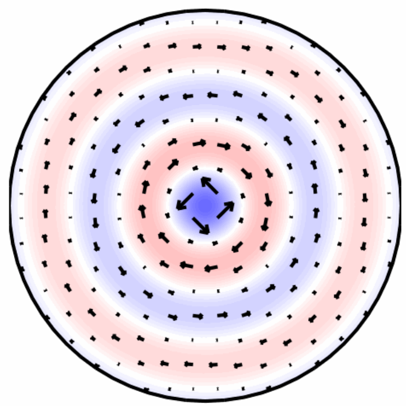
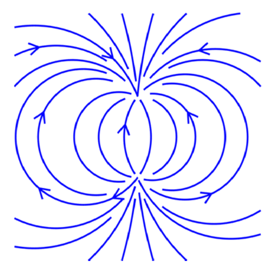
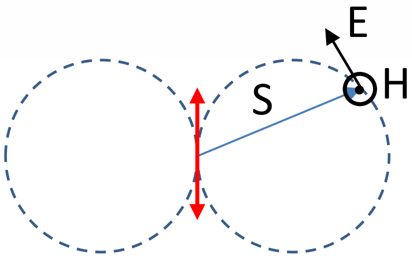








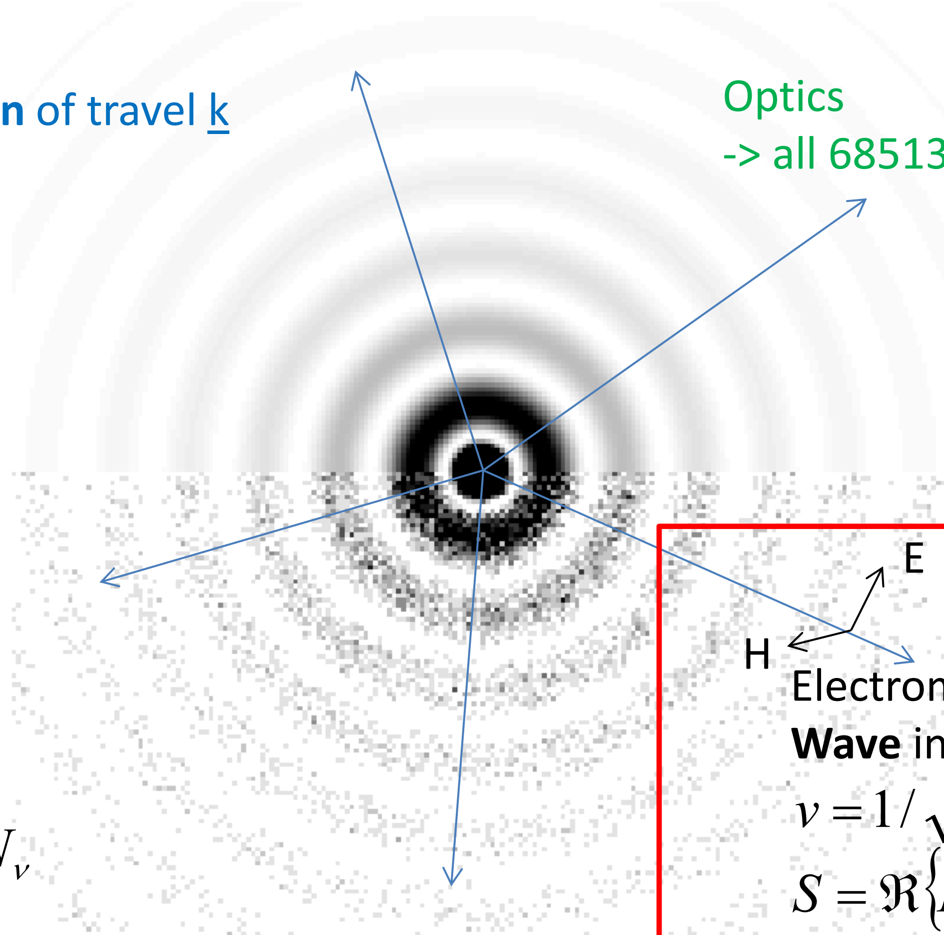
Models of light: different views of same reality

Ray: **direction** of travel <u>k</u>

Optics
-> all 68513

Wave

$$\exp[i(\vec{k}.\vec{r} - \omega t)]$$
$$k = 2\pi / \lambda$$
$$n = c / v$$



First half

Electromagnetism:
Wave in both E & M

$$v = 1 / \sqrt{\varepsilon\mu}$$
$$S = \Re\left\{\vec{E} \times \vec{H}^*\right\} / 2$$

EM an *extension* of wave model distinguishes light from other waves

Quantum
-> final weeks

Particle

$$u = \hbar\omega$$
$$p = \hbar k$$
$$S = u v N_v$$

Why Maxwell's equations?

"The work of James Clerk Maxwell changed the world forever" A Einstein

- Fundamental to technology
  - Electronic devices (resistors, capacitors, inductors)
  - Electricity transformation (motors, generators, transformers, microwave)
  - Comms & compute (radio/TV, wifi, processors, phone, internet, storage)
- Deeper understanding of light explaining the wave model
  - Specialising the wave model to light (this week)
  - Interaction with matter: macro & nanoscale (week 2)
  - Transformation at interfaces (week 3)
  - Polarization (week 4)
- Possibilities for future nanoscale tech
  - Resonators & Waveguides (week 5)
  - Photonic crystals, waveguides, resonators (week 6)
  - Plasmonic materials, waveguides, resonators (week 7)

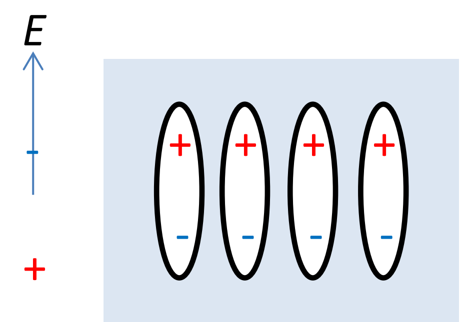
EM waves summary

- **Maxwell's equations are the fundamental "rules" for EM**

We will *not* cover everything related to M.E.: only the basics needed for waves

This week

- **Unpack** M.E. with *some* important examples
- M.E. + properties of space → **Helmholtz** (wave equations)
- W.E. **plane wave** solutions
  - orthogonality of fields
  - Poynting vector (how energy is transported)



Next week

- **Constitutive equations** (EM properties of materials)
- **Models** for properties of materials

Then

- M.E. + properties of materials → **boundary conditions**
- B.C. + kE conservation → **Fresnel equations** (ref & trans)

If you need more background on electromag there are many free resources: e.g. [Wikipedia]

Electromagnetic **wave** roadmap

- Maxwell's equations

$$\nabla \bullet \mathbf{D} = \rho_{free}$$
$$\nabla \bullet \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$$
$$\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$$
$$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$$

Constitutive equations

- Helmholtz (**wave**) equations

$$\nabla^2 \mathbf{E} = \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{H} = \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- **Harmonic** waves

$$E = E_0 \exp[i\{kr - \omega t\}]$$
$$H = H_0 \exp[i\{kr - \omega t\}]$$

- Harmonic Maxwell

$$i\mathbf{k}.\mathbf{D} = \rho_{free}$$
$$\mathbf{k}.\mathbf{B} = 0$$
$$\mathbf{k} \times \mathbf{H} = -i\mathbf{J} - \omega\mathbf{D}$$
$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}$$

$$S_{av} = \mathrm{Re}[\mathbf{E} \times \mathbf{H}^*]/2$$

- **Properties** of time-harmonic EM wave in simple media

$$\mathbf{k} \perp \mathbf{E} \perp \mathbf{H} \qquad \boxed{n \equiv \frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}} \qquad \boxed{\eta \equiv \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = 1/(v\varepsilon)} \qquad v = \frac{\omega}{k} \qquad \boxed{S_{av} \propto \mathrm{Re}[n]|E|^2}$$

Saleh & Teich Ch5

Yes these look challenging – but they're so important we will work through them

Maxwell's equations

"Macroscopic" formulation

- Differential

$$\nabla \cdot \boldsymbol{D} = \rho_{free}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Gauss & Stokes theorems

- Integral

$$\oint \boldsymbol{D} \cdot dA = \int \rho_{free} dV$$

$$\oint \boldsymbol{B} \cdot dA = 0$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \int \boldsymbol{B} \cdot dA$$

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \int \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot dA$$

E electric field [V/m=N/C]
H magnetic field [A/m]
D electric flux [C/m^2=N/(Vm)]
B magnetic flux [N/m/A]
J electric current density [A/m^2]
ρ electric charge [A s]
t time [s]

Definitions of quantities

Lots of ways to define fields.

One that may be familiar is the Lorentz force $\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

e.g. electric field produces a collinear force on a charged particle

Simplest linear definition of relationships between fields (*not* always true):

D displacement/electric flux	$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$	ε (electric) permittivity
B magnetic flux density	$\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$	μ (magnetic) permeability
J electric current density	$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$	σ (electric) conductivity

properties (More next week)

The last one is Ohm's law!

$$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E} \rightarrow \mathbf{J}A = (\boldsymbol{\sigma}A/d)(\mathbf{E}d) \rightarrow \mathbf{I} = R^{-1}\mathbf{V}$$

$E=10^2 V/m$

$J=10^5 A/m^2$

$I=10^{-1} A$

$R=10\Omega$

ME: Diff & Int Operators

"Macroscopic" formulation

- Differential

$$\nabla \cdot \boldsymbol{D} = \rho_{free}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Gauss & Stokes theorems

- Integral

$$\oint \boldsymbol{D} \cdot dA = \int \rho_{free} dV$$

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$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \int \boldsymbol{B} \cdot dA$$

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \int \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot dA$$

$\nabla \cdot$ "divergence"

$\nabla \times$ "rotation", "curl"

l distance along path

A area of surface

V volume in volume

Suppose we have 1cm long carbon resistor ($\sigma=10^3/(\Omega m)$).
What is the current *density* J if we apply 1V across it?
If the *area* is 1(mm)^2, what is the total current?

$$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E} \rightarrow \mathbf{J}A = (\boldsymbol{\sigma}A/d)(\mathbf{E}d) \rightarrow \mathbf{I} = R^{-1}\mathbf{V}$$

Maxwell's equations:

Differential Form

Rules at any point in space

$$\nabla \cdot \boldsymbol{D} = \rho_{free}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

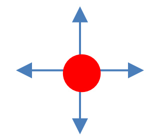
$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Reminder of maths of vector fields & operators

Dealing with **fields**: we'll use rectangular **components** here

Scalar "divergence" of field (2D)

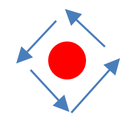
$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y}$$



+ve div

Vector "rotation"/"curl" (2D)

$$\nabla \times \mathbf{E} = \left\{ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right\} \hat{\mathbf{z}}$$



Curl
out of plane

$$\mathbf{r} = r\hat{\mathbf{r}}$$

vector = magnitude x direction

IMPORTANT:
Div/Curl non-zero @ source
Zero *everywhere else*

Vector fields & *differential* operators

Cylindrical coordinates **2D in plane (no z):** may be useful below

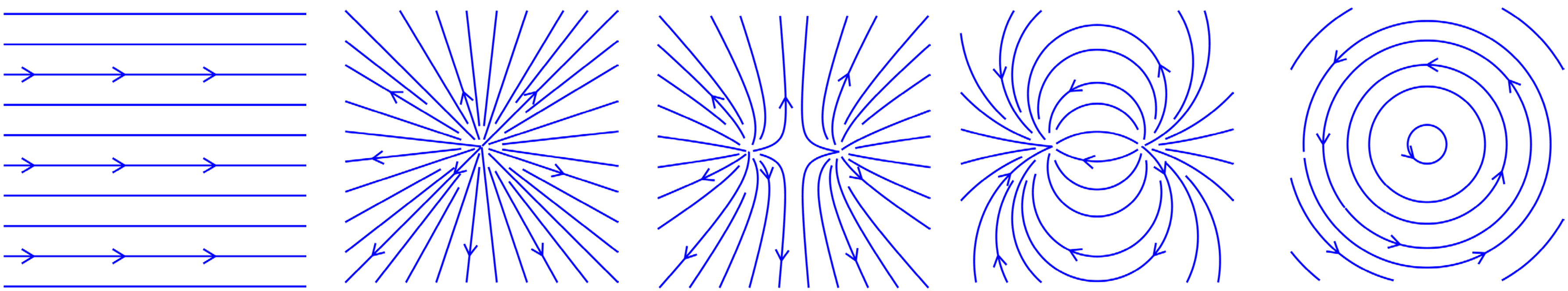
Scalar "divergence" of field

$$\nabla \cdot \mathbf{D} = \frac{1}{r}\left\{\frac{\partial(rD_r)}{\partial r} + \frac{\partial D_\theta}{\partial \theta}\right\}$$

Vector "rotation"/"curl" of field

$$\nabla \times \mathbf{D} = \frac{1}{r}\left\{\frac{\partial(rD_\theta)}{\partial r} - \frac{\partial D_r}{\partial \theta}\right\}\hat{\mathbf{z}}$$

Q: (ignoring z) find the sources of div & curl below and their sign/direction

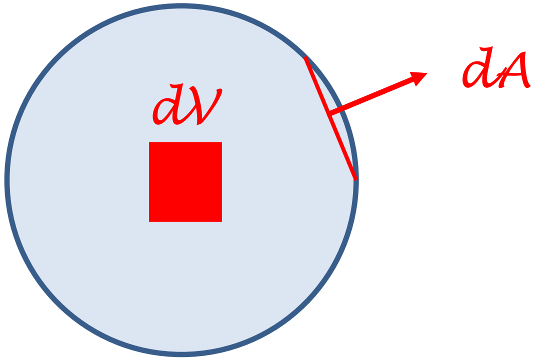


Come to tutorial to check your answers

Maxwell's equations:

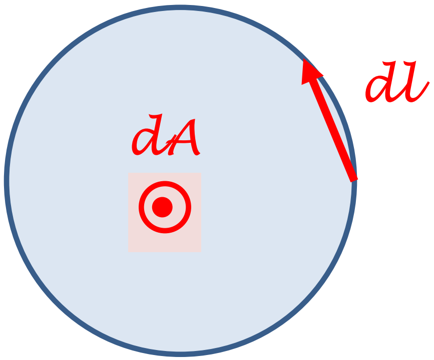
Integral Form

Closed vector surface & enclosed volume (2D)



dA **normal to surface**

Closed path & enclosed vector area (2D)



(dA out of plane here)

Rules on *closed* paths, surfaces & volumes

$$\oint \boldsymbol{D} \cdot d\boldsymbol{A} = \int \rho_{free} dV$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \int \boldsymbol{B} \cdot d\boldsymbol{A}$$

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \int \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{A}$$

l distance along path

A area of surface

V volume in volume

Vector fields & *integral* operators

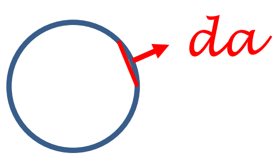
Reminder of maths of vector fields & operators

Dealing with **fields**: we'll use rectangular **components** here

Surface integral of field

$$\mathbf{D.da} = D_x da_x + D_y da_y + D_z da_z$$

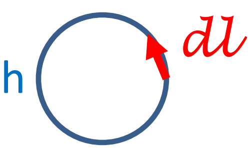
Closed (2D) surface



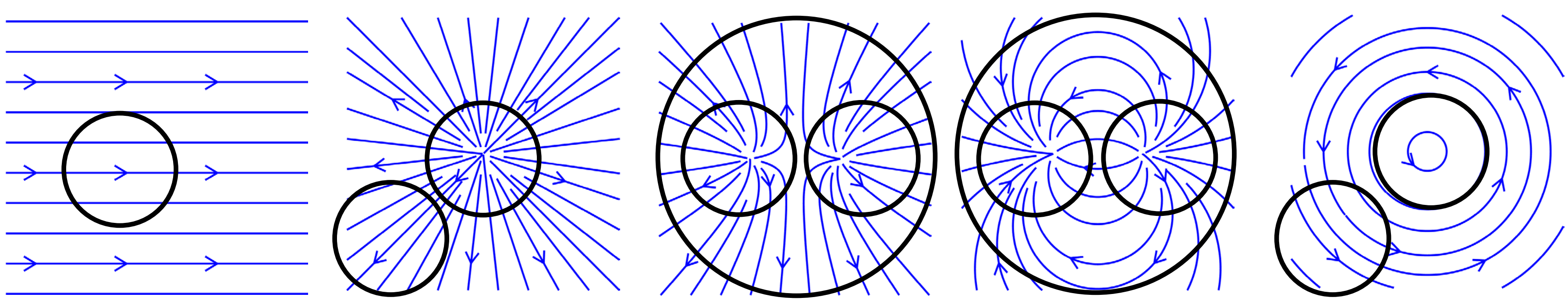
Line integral of field

$$\mathbf{E.dl} = E_x dx + E_y dy + E_z dz$$

Closed path



Q: determine the sign of the (in-plane) surface & line integral for each contour:



Unpacking M.E.

- Two time-independent equations "statics"

$$\nabla \cdot \boldsymbol{D} = \rho_{free} \qquad \nabla \cdot \boldsymbol{B} = 0$$

e.g. they tell us about the *shape* of fields that are allowed

- Two time-dependent equations "dynamics"

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

e.g. they tell us about how field circulation & oscillation in time couple → waves

Some things to consider:

- M.E. do not directly give us field from sources, only validity of field
- Most of space is not "on" a source, so the integral form is probably more insightful
- Given source distribution, we can work out field by symmetry and/or trying many surfaces
- Much easier to use known solutions (e.g. starting from a point charge or line current)

(1) Gauss's (electric flux) law

Divergence of field is due to enclosed charge

	Differential	Integral
Macroscopic/material $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$	$\nabla \cdot \boldsymbol{D} = \rho_{free}$	$\oint \boldsymbol{D} \cdot d\boldsymbol{A} = \int \rho_{free} dV$
Microscopic/**vacuum** $\mathbf{D} = \varepsilon_0 \mathbf{E}$	$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0$	$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \varepsilon_0^{-1} \int \rho dV$

Macro formulation incorporates permittivity: affected by bound charges in materials
Notice that the **only source in the *macro* formulation is *free* charges**
Bound charges are paired with an opposite charge so they appear to cancel
In between electron & proton we would use the *microscopic* formulation

Let's look at some static examples (which will also help us understand magnetic fields)
We will spend some time on this, because they also tell us about the *shape* of waves
We will compare how easy it is to predict fields with source equations vs M.E.

Two views of the same reality: Coulomb & Gauss

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} q \frac{\hat{\mathbf{r}}}{r^2}$$

$$\oint \varepsilon \boldsymbol{E} \cdot d\boldsymbol{A} = \int \rho_{free} dV$$

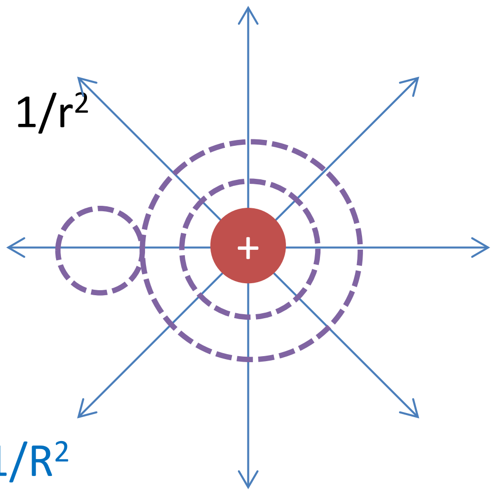
Point sources, Direct

Test surfaces, Indirect

"Monopole" charge, E field sketched from Coloumb's law

Draw Gaussian surfaces; calculate net flux to check

1. Any surface enclosing charge has non-zero net flux
2. Larger surface = larger area & same charge = smaller flux, $1/r^2$
3. Any surface without enclosed charge have <u>zero</u> net flux



We can **guess** the form of the field by **symmetry**:

This charge has **spherical** symmetry

Spherical test surfaces centered at charge always contain it

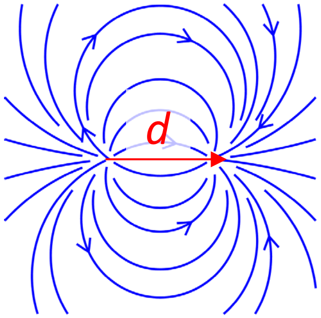
Test **area** increases as R^2, so **field normal to surface** must $\rightarrow 1/R^2$

Symmetry of these surfaces implies **radial field**

Another example: Dipole + & -

At *large distance* from dipole size *d* along *x*:

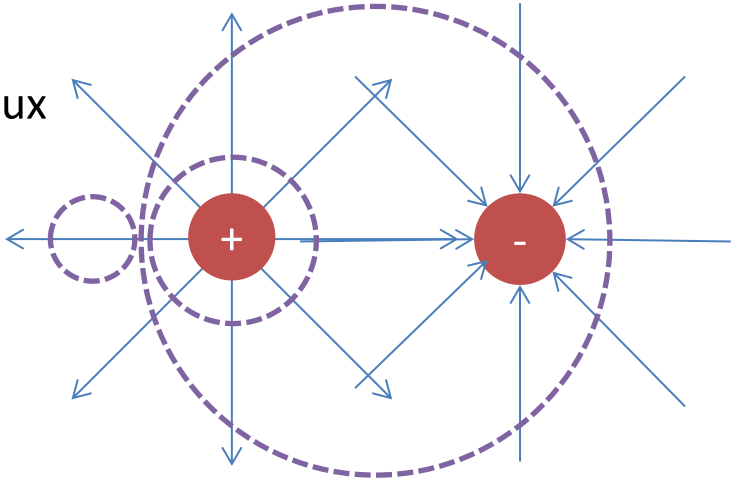
$$E_x = \frac{3x^2 - r^2}{r^5} \sim \frac{1}{r^3}$$



“Dipole” charges, E field sketched from Coulomb’s law

Draw Gaussian surfaces; calculate net flux to check

1. Any surface enclosing one charge has net flux
2. Any surface enclosing both charges cancel = zero net flux
3. Any surface without enclosed charge = zero net flux



Note: *Oscillating* dipoles also important in electromagnetic waves e.g. atomic/molecular absorption & excitation, antennae

Note that *radiating* dipoles have extra terms with different scaling

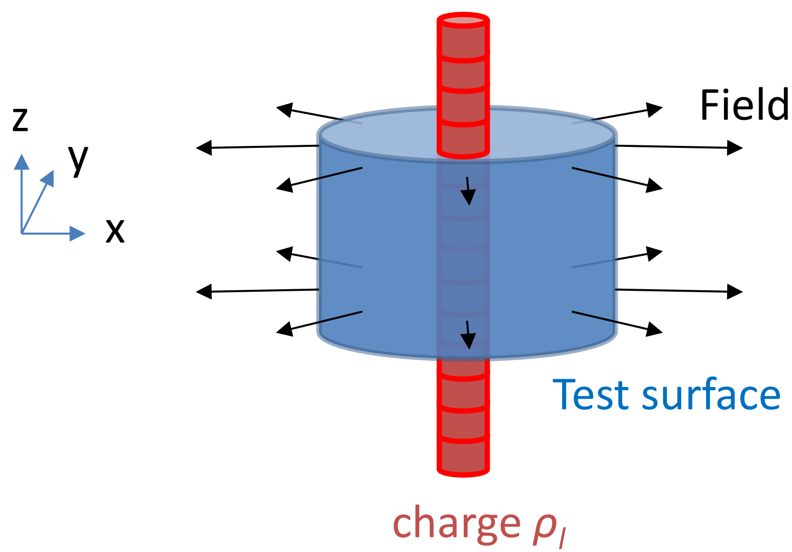
Uniform line & sheet

Determine the field at any point by integrating charge distribution

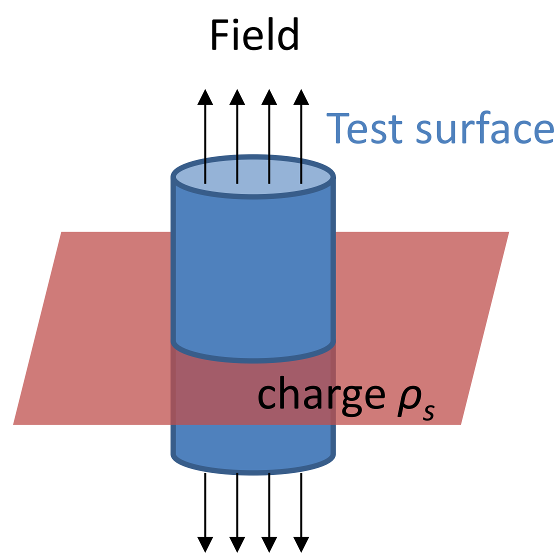
We can **guess** the form of the fields by **symmetry:**

- Pick test surfaces that have similar symmetry
- Must contain a fixed amount of charge as surface is moved away from charge
- Conservation of flux as the surface changes will give the scaling of the field with distance
- Normals of well-chosen flux-capturing surface gives the symmetry of the field

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\rho_l \frac{2\hat{\mathbf{r}}_c}{r_c}$$



$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\rho_s 2\pi\hat{\mathbf{z}}$$



Unpacking M.E.

- ✓ Gauss's law

$$\nabla \cdot \boldsymbol{D} = \rho_{free}$$

- ➢ Magnetic divergence

$$\nabla \cdot \boldsymbol{B} = 0$$

- ➢ Two time-dependent equations "dynamics"

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

e.g. they tell us about how field circulation & oscillation in time couple → waves

(2) No magnetic monopoles

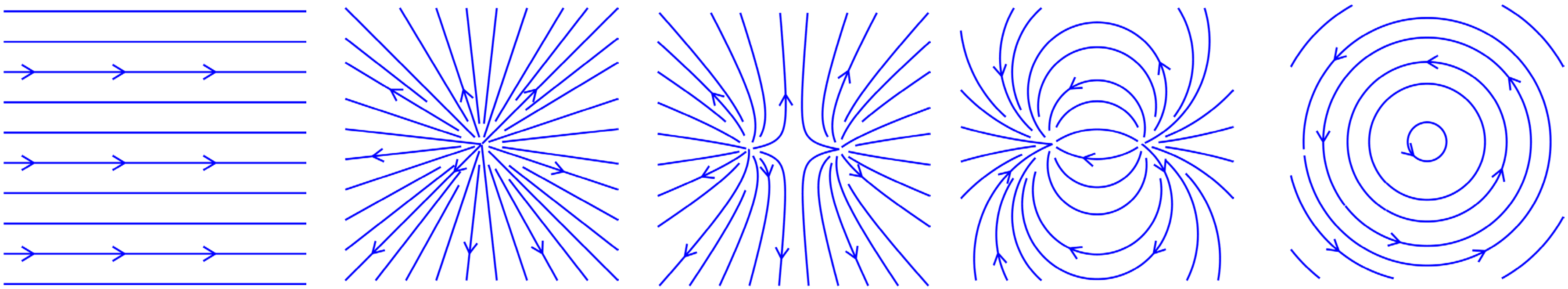
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \oint \boldsymbol{B} \cdot dA = 0$$

Analog of Gauss' law: but no "magnetic charges" observed

○ No monopole fields

✓ Rotational fields allowed

Which of the following B fields are allowed?
(Hint: this does require some thought)



Come to tutorial to check your answers

(2) Biot-Savart law

$$B = \frac{\mu_0}{4\pi}\int \frac{\mathbf{Idl}\times\hat{\mathbf{r}}}{r^2}$$

Again, Gauss' law is somewhat inconvenient

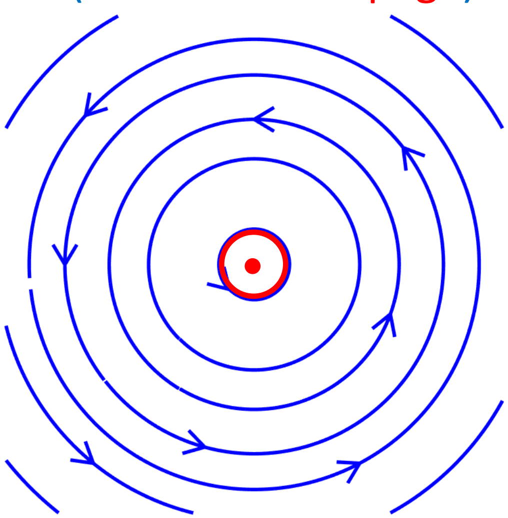
An important source of magnetic field is a current element $I\,d\mathbf{l}$

Biot-Savart is the magnetic equivalent of Coulomb's law (both @ low-frequency)

This law explains makes it easy to calculate fields of electromagnetics

B field around a wire

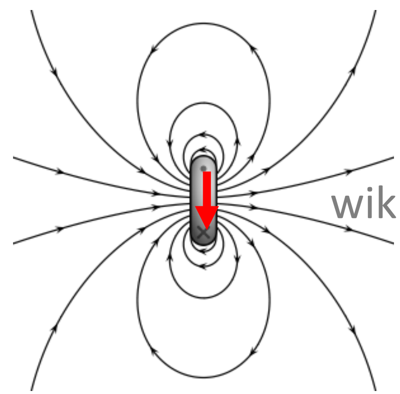
(current out of page)



$$B = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}}$$

Field near a current loop

(magnetic "dipole": far-field $1/r^3$)

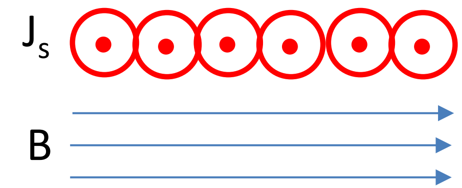


Notice: differs from E dipole shape

(no point singularities)

Sheet current

Constant field



$B=\mu_0 J_s/2$

Applies to electric generators & transformers

(3)Lenz/Faraday law of induction

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = \int (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{A}$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \int \boldsymbol{B} \cdot d\boldsymbol{A}$$

A **time-changing** magnetic field will induce a voltage **around a loop**
(the induced current would then oppose this change in magnetic field)

Q: suppose we could turn off the Earth's field 10^{-4} T in 10^{-2}s
What's the maximum voltage we would see around a 1m long loop of wire?

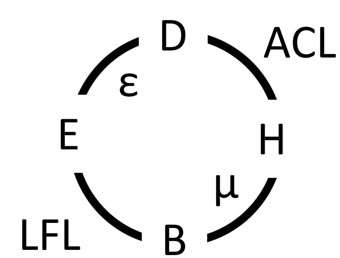
dB/dt=10^{-2}T/s; L=1m; r=1/(2π); circle A=πr^2=0.08m^2; V=ELA dB/dt=0.8mV

(4) Ampere's circuital law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

A **current** *OR* a **time-changing electric flux** will induce a **circulating magnetic field**

Together with Faraday, allows **oscillating EM field**



Electromagnetic **wave** roadmap

✓ Maxwell's equations

$$\nabla \bullet \mathbf{D} = \rho_{free}$$
$$\nabla \bullet \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$$
$$\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$$
$$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$$

Constitutive equations

➢ Helmholtz (**wave**) equations

$$\nabla^2 \mathbf{E} = \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{H} = \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

❖ **Harmonic** waves

$$E = E_0 \exp[i\{kr - \omega t\}]$$
$$H = H_0 \exp[i\{kr - \omega t\}]$$

❖ Harmonic Maxwell

$$i\mathbf{k}.\mathbf{D} = \rho_{free}$$
$$\mathbf{k}.\mathbf{B} = 0$$
$$\mathbf{k} \times \mathbf{H} = -i\mathbf{J} - \omega\mathbf{D}$$
$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}$$

$$S_{av} = \mathrm{Re}[\mathbf{E} \times \mathbf{H}^*]/2$$

❖ **Properties** of time-harmonic EM wave in simple media

Saleh & Teich Ch5

$$\mathbf{k} \perp \mathbf{E} \perp \mathbf{H}$$

$$\boxed{n \equiv \frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}}$$

$$\boxed{\eta \equiv \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = 1/(v\varepsilon)}$$

$$v = \frac{\omega}{k}$$

$$\boxed{S_{av} \propto \mathrm{Re}[n]|E|^2}$$

We will assume…

some restrictions on constitutive properties

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E} \quad \varepsilon \ \text{(electric) permittivity}$$

$$\mathbf{B} = \boldsymbol{\mu}\mathbf{H} \quad \mu \ \text{(magnetic) permeability}$$

$$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E} \quad \sigma \ \text{(electric) conductivity}$$

[In SI, convenient to use $\varepsilon=\varepsilon_r\varepsilon_0$ etc, i.e. relative (r) to free space (0).]

Typical assumption is "simple media":

- Most materials have μ_r=1 (non-magnetic)
- Transparent materials have $\sigma\approx0$ and no currents (relax this later)
- These equations are *linear* approximations – not valid for very strong fields
- These equations hide directional dependence (anisotropy) – we will ignore
- Space (and time) dependence are interesting but difficult – assume none

More next week

Derivation of Helmholtz

Using ME + simple media + vector calc $\nabla\times\nabla\times\equiv\nabla(\nabla\cdot)-\nabla^2$

$$\nabla\bullet\mathbf{D}=\rho_{free}$$

$\nabla\bullet(\boldsymbol{\varepsilon}\mathbf{E})=\rho_{free}$ Substituting constitutive equation

$\boldsymbol{\varepsilon}\nabla\bullet\mathbf{E}=\rho_{free}$ Assuming permittivity is constant in space

$\nabla\bullet\mathbf{E}=0$ (1) Assuming no free charges (simple medium)

$\nabla\times\nabla\times\mathbf{E}=\nabla(\nabla\bullet\mathbf{E})-\nabla^2\mathbf{E}$ Vector identity

$\nabla\times\nabla\times\mathbf{E}=\boxed{-\nabla^2\mathbf{E}}$ (2) using eq (1)

$\nabla\times\nabla\times\mathbf{E}=-\nabla\times\dfrac{\partial\mathbf{B}}{\partial t}$ Curl of Faraday equation

$\nabla\times\nabla\times\mathbf{E}=-\nabla\times\dfrac{\partial}{\partial t}(\mu H)$ Substituting constitutive equation

$\nabla\times\nabla\times\mathbf{E}=\boxed{-\mu\dfrac{\partial}{\partial t}}\nabla\times H$ (3) Assume permeability constant space/time

---

$\nabla\times H=J+\dfrac{\partial(\varepsilon E)}{\partial t}$ ACL ← constitutive equation

$\nabla\times H=\boxed{\varepsilon\dfrac{\partial E}{\partial t}}$ (4) Constant permittivity + no source J

(4) Into (3),
RHS (3) = (2)

$$\nabla^2\boldsymbol{E}=\mu\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2}$$

Be like Maxwell & Helmholtz: see if you can derive the other field!

Helmholtz (wave) equations

Using ME + simple media + vector calc:

$$\nabla^2 \boldsymbol{H} = \mu\varepsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2} \qquad \nabla^2 \boldsymbol{E} = \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

Helmholtz equations

Compare general wave equation $$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

Q: What is the speed of this EM wave in free space, and why is this result significant?

(use ε_0 = 8.854 x 10^{-12} F·m^{-1}, μ_0 = 4π x 10^{-7} H·m^{-1})

Electromagnetic **wave** roadmap

✓ Maxwell's equations

$$\nabla \bullet \mathbf{D} = \rho_{free}$$
$$\nabla \bullet \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$$
$$\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$$
$$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$$

Constitutive equations

✓ Helmholtz (**wave**) equations

$$\nabla^2 \mathbf{E} = \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{H} = \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

➢ **Harmonic** waves

$$E = E_0 \exp[i\{kr - \omega t\}]$$
$$H = H_0 \exp[i\{kr - \omega t\}]$$

➢ Harmonic Maxwell

$$i\mathbf{k}.\mathbf{D} = \rho_{free}$$
$$\mathbf{k}.\mathbf{B} = 0$$
$$\mathbf{k} \times \mathbf{H} = -i\mathbf{J} - \omega\mathbf{D}$$
$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}$$

$$S_{av} = \mathrm{Re}[\mathbf{E} \times \mathbf{H}^*]/2$$

➢ **Properties** of time-harmonic EM wave in simple media

Saleh & Teich Ch5

$$\mathbf{k} \perp \mathbf{E} \perp \mathbf{H}$$

$$\boxed{n \equiv \frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}}$$

$$\boxed{\eta \equiv \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = 1/(v\varepsilon)}$$

$$v = \frac{\omega}{k}$$

$$\boxed{S_{av} \propto \mathrm{Re}[n]|E|^2}$$

“Harmonic” plane-wave solutions to WE

Maths is convenient and we can use superposition

Assume

$$E = E_0 \exp[i\{kr - \omega t\}]$$

$$H = H_0 \exp[i\{kr - \omega t\}]$$

$$\nabla \rightarrow i\mathbf{k}$$ (think δ/δr if it helps)

then

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$

so ME become

$$\nabla \bullet \mathbf{D} = \rho_{free}$$

$$\nabla \bullet \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\longrightarrow$$

$$i\mathbf{k.D} = \rho_{free}$$

$$\mathbf{k.B} = 0$$

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$$

$$\mathbf{k} \times \mathbf{H} = -i\mathbf{J} - \omega \mathbf{D}$$

Incorporating induced currents

For convenience we assumed no explicit charges & **currents**
This is not actually necessary if we assume time-harmonic fields
And it hides some important physics (*induced* current):

$$\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$$

An electric field will *induce* a current to flow

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$$

The displacement field is related to the electric field

Substituting into harmonic M.E. $\mathbf{k} \times \mathbf{H} = -i\mathbf{J} - \omega\mathbf{D} \rightarrow -\omega\boldsymbol{\varepsilon}_{\mathbf{c}}\mathbf{E}$

$$i\mathbf{J} + \omega\mathbf{D} = (i\boldsymbol{\sigma} + \varepsilon\omega)\mathbf{E} = \omega\boldsymbol{\varepsilon}_{\mathbf{c}}\mathbf{E}$$

$$\boldsymbol{\varepsilon}_{\mathbf{c}} = \boldsymbol{\varepsilon} + i\frac{\boldsymbol{\sigma}}{\omega}$$

Complex permittivity incorporating induced harmonic current
Notice that this is frequency dependent
only accurate for low frequencies: Drude model better (next week)

Notice:
At radio frequencies electrodynamics is often strongly affected by σ
At higher optical frequencies, electrodynamics is often strongly affected by ε
Ironically, *some* optical properties can be predicted with partially electrostatic arguments

EM plane wave properties

Wave Equation $\longrightarrow$ $\boxed{n \equiv \frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}}$ $\quad v = \frac{\omega}{k}$

Harmonic M.E.

See if you can explain how these follow:

$$i\mathbf{k}.\mathbf{D} = \rho_{free}$$

$$\mathbf{k}.\mathbf{B} = 0$$

$$\mathbf{k} \times \mathbf{H} = \mathbf{J} - \omega\mathbf{D}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}$$

$\Rightarrow$ $\mathbf{k} \perp \mathbf{E} \perp \mathbf{H}$ **Field directions**

$$\boxed{\eta \equiv \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = 1/(v\varepsilon) = v\mu}$$

Field ratio/impedance (to eliminate H)

$$\boxed{S_{av} \equiv \mathrm{Re}[\mathbf{E} \times \mathbf{H}^*]/2 = \mathrm{Re}[n]|E|^2/(2\eta_0)}$$

Power flow / intensity

*x is cross product, Re is real part, * is complex conjugate*

Useful values in SI

$\boxed{c \approx 2.998 \times 10^8 m\, s^{-1}}$

$\boxed{\mu_0 = 4\pi 10^{-7} \approx 1.257 \times 10^{-6}\,\mathrm{H}/m}$

$\boxed{\varepsilon_0 = 1/(c^2\mu_0) \approx 8.854 \times 10^{-12}\ \mathrm{F/m}}$

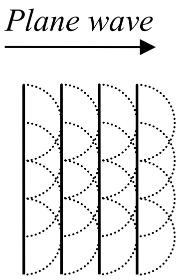
$\boxed{\eta_0 = \sqrt{\mu_0/\varepsilon_0} \approx 376.7\Omega}$

EM plane wave questions

The integrated intensity of sunlight reaching the earth's surface is approximately 1kW/m^2 (ignoring the incoherence of this light) what it the equivalent electric field amplitude in SI?

What about D, H, B? (compare B to the Earth's static field 10^{-4} T)

Cylindrical & spherical waves



Multiple spherical wavelets propagate the wavefront

So far we have just considered *plane waves* for convenience
Also useful to consider **spherical waves** (e.g. Huygens principle).

Main difference is **energy spreads out**

Since we can look at a tiny part of the wavefront as nearly planewave,
many (but not all) of the previous equations hold at a given point in space

Cylindrical/spherical waves are useful for:

- General understanding of light (plane waves are special case)
- Point/line sources, e.g. atoms, dipole antennae
- Diffraction from points, slits, and edges

Q: for each of spherical & cylindrical waves,

- how should the intensity change with dist from the source?
- what does this imply about the electric field?
- what types of source is this consistent with?

	Sph	Cyl
I		
E		
source		

EM sources & waves

At high frequency M.E. are still true,
but need to modify source equations to account for finite speed of wave

Current sheet

Plane wave

$H_0 = J_s/2$ along sheet perp to J
$E_0 = \eta_0 J_s/2$ parallel to J
k normal to sheet

$$E = E_0 \exp[i\{kr - \omega t\}]$$

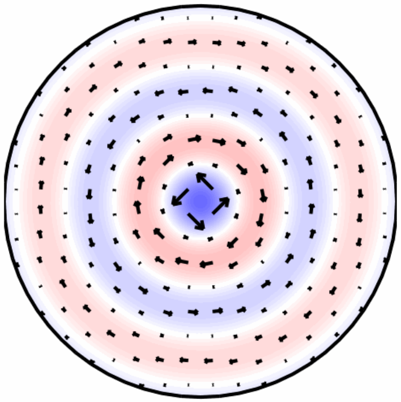
$$H = H_0 \exp[i\{kr - \omega t\}]$$

Current loop

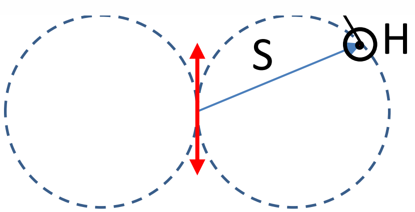
Complicated…
Also relatively weak

Small dipole

Spherical dipole wave
[Equations](), [Pictures]()
Take E static dipole + oscillation
S is outwards (donut – **nothing along dipole**)



Top view (middle slice)
Source in/out
E colour in/out
H arrows circulating
S in-plane outwards



sketch
Source up/down
E colour in/out
H arrows circulating in/out
S outwards

Additional info on waves

Sign convention:

$$E = E_0 \exp[i\{kr - \omega t\}]$$
$$H = H_0 \exp[i\{kr - \omega t\}]$$

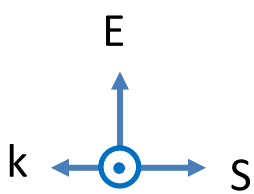
"Physics" $\boldsymbol{\varepsilon}_{\mathbf{c}} = \boldsymbol{\varepsilon} + i\dfrac{\boldsymbol{\sigma}}{\omega}$

$$E = E_0 \exp[j\{-kr + \omega t\}]$$
$$H = H_0 \exp[j\{kr + \omega t\}]$$

"Engineering" $\boldsymbol{\varepsilon}_{\mathbf{c}} = \boldsymbol{\varepsilon} - j\dfrac{\boldsymbol{\sigma}}{\omega}$

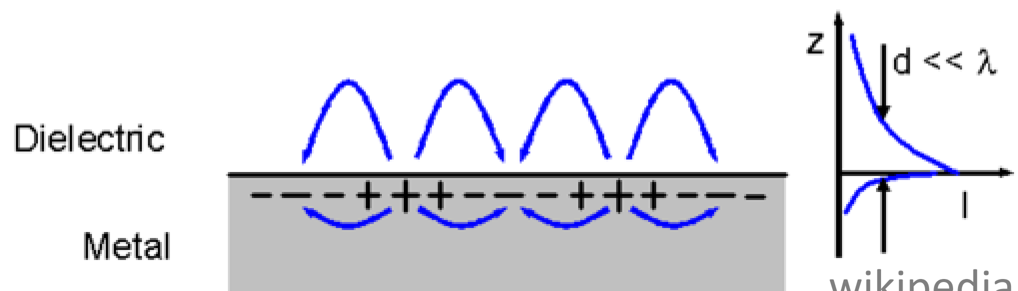
Two important examples of "plane-wave" variations to $\mathbf{S} \| \mathbf{k} \perp \mathbf{E} \perp \mathbf{H}$

Negative refraction (n<0):



Note:
opposite energy & phase flow

Evanescent/surface wave (*k imaginary*):

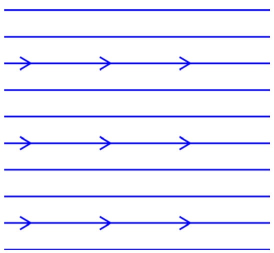


wikipedia

Additional background on electromagnetism and EM waves

Summary of M.E. → EM waves

- Electromagnetic model is more complete model of waves
- Maxwell's equations = "law" + constitutive "material properties"
- Manipulate equations to find allowed solutions
- Easiest to assume harmonic plane waves



Check that you know how & when these apply

Simple wave equation $\nabla^2 \boldsymbol{E} = \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$

+ Material properties $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$, $\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$

Plane wave properties in simple (i.e. constant) media

$$\boxed{n \equiv \frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}} \quad \boxed{\eta \equiv \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = 1/(v\varepsilon) = v\mu} \quad \mathbf{k} \perp \mathbf{E} \perp \mathbf{H} \quad \boldsymbol{\varepsilon}_\mathbf{c} = \boldsymbol{\varepsilon} + i\frac{\boldsymbol{\sigma}}{\omega}$$

$$\boxed{S_{av} \equiv \mathrm{Re}[\mathbf{E} \times \mathbf{H}^*]/2 = \mathrm{Re}[n]|E|^2/(2\eta_0)} \quad E = E_0 \exp[i\{kr - \omega t\}]$$

$$\boxed{c \approx 2.998 \times 10^8 m\, s^{-1}} \quad \boxed{\mu_0 = 4\pi 10^{-7} \approx 1.257 \times 10^{-6} \mathrm{H}/m} \quad \boxed{\varepsilon_0 = 1/(c^2\mu_0) \approx 8.854 \times 10^{-12}\ \mathrm{F/m}} \quad \boxed{\eta_0 = \sqrt{\mu_0/\varepsilon_0} \approx 376.7\Omega}$$