Navier-Stokes discretization

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1 Time discretization

Time discretization is performed using the Heun method (explicit and second order) and a fractional step method.

1.1 Step 1

$$\vec{u}^{**} = \vec{u}^n - \Delta t \left(\left(\vec{u}^n \cdot \vec{\nabla} \right) \vec{u}^n - \frac{1}{\mathrm{Re}} \nabla^2 \vec{u}^n + \vec{\nabla} p^n \right)$$

This step computes a first approximation of $\vec{u}^*$ and is introduced by the usage of the Heun method. It consists in 3 independent scalar equations that can be solved directly.

1.2 Step 2

$$\vec{u}^* = \vec{u}^n - \Delta t \left(\frac{1}{2} \left(\left(\vec{u}^n \cdot \vec{\nabla} \right) \vec{u}^n - \frac{1}{\mathrm{Re}} \nabla^2 \vec{u}^n \right) + \frac{1}{2} \left(\left(\vec{u}^{**} \cdot \vec{\nabla} \right) \vec{u}^{**} - \frac{1}{\mathrm{Re}} \nabla^2 \vec{u}^{**} \right) + \vec{\nabla} p^n \right)$$

This step is the first step of the fractional step method, and uses the approximation of $\vec{u}^*$, i.e. $\vec{u}^{**}$, computed at the previous step, to compute $\vec{u}^*$. It consists in 3 independent scalar equations that can be solved directly.

1.3 Step 3

$$\nabla^2 \left(p^{n+1} - p^n \right) = \frac{1}{\Delta t} \vec{\nabla} \cdot \vec{u}^*$$

This step consists in the calculation of p^{n+1} from the value of $\vec{u}^*$ calculated in the previous step, and is the first part of the second step of the fractional step method. It consists in a scalar Poisson problem, so a linear system needs to be solved.

1.4 Step 4

$$\vec{u}^{n+1} = \vec{u}^n - \Delta t \vec{\nabla} \left(p^{n+1} - p^n \right)$$

This step concludes the final step of the fractional step method by calculating $\vec{u}^{n+1}$ from p^{n+1}. It consists in 3 independent scalar equations that can be solved directly.