Space discretization of velocity terms

Francesco Pesce

October 8, 2024

1 Space discretization

1.1 Original term

The non-discretized term in vectorial form is:

$$-\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}+\frac{1}{\mathrm{Re}}\nabla^2\vec{u}$$

corresponding to the components:

$$-\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$-\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$
$$-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

1.2 Trival terms

The terms u, v, w on their respective grids need no approximation. Similarly, the single and double derivatives that need to be computed can be approximated with their usual finite difference approximations, i.e.

$$\frac{\partial u}{\partial x}|_{i+0.5,j,k}\approx\frac{u|_{i+1.5,j,k}-u|_{i-0.5,j,k}}{2\Delta x}$$
$$\frac{\partial u}{\partial y}|_{i+0.5,j,k}\approx\frac{u|_{i+0.5,j+1,k}-u|_{i+0.5,j-1,k}}{2\Delta y}$$
$$\frac{\partial u}{\partial z}|_{i+0.5,j,k}\approx\frac{u|_{i+0.5,j,k+1}-u|_{i+0.5,j,k-1}}{2\Delta z}$$
$$\frac{\partial^2 u}{\partial x^2}|_{i+0.5,j,k}\approx\frac{u|_{i+1.5,j,k}-2u|_{i+0.5,j,k}+u|_{i-0.5,j,k}}{\Delta x^2}$$
$$\frac{\partial^2 u}{\partial y^2}|_{i+0.5,j,k}\approx\frac{u|_{i+0.5,j+1,k}-2u|_{i+0.5,j,k}+u|_{i+0.5,j-1,k}}{\Delta y^2}$$
$$\frac{\partial^2 u}{\partial z^2}|_{i+0.5,j,k}\approx\frac{u|_{i+0.5,j,k+1}-2u|_{i+0.5,j,k}+u|_{i+0.5,j,k-1}}{\Delta z^2}$$

with similar results for v, w. All approximations are of order 2.

1.3 Interpolation of u, v, w

The terms u, v, w need to be approximated in the other grids as well. This is done taking the average of the values in the 4 closest points where the values are defined, i.e.

$$v|_{i+0.5,j,k}\approx\frac{v|_{i,j+0.5,k}+v|_{i,j-0.5,k}+v|_{i+1,j+0.5,k}+v|_{i+1,j-0.5,k}}{4}$$
$$w|_{i+0.5,j,k}\approx\frac{w|_{i,j,k+0.5}+w|_{i,j,k-0.5}+w|_{i+1,j,k+0.5}+w|_{i+1,j,k-0.5}}{4}$$

with similar results for for the y, z grids. All approximations are of order 2.

1.4 Overall discretization

The sum and product of order 2 approximations remains of order 2. The first component of the original term in $i+0.5, j, k$ is approximated as

$$
\begin{aligned}
-\Bigg(\quad & u|_{i+0.5,j,k} \frac{u|_{i+1.5,j,k} - u|_{i-0.5,j,k}}{2\Delta x} + \\
& \frac{v|_{i,j+0.5,k} + v|_{i,j-0.5,k} + v|_{i+1,j+0.5,k} + v|_{i+1,j-0.5,k}}{4} \frac{u|_{i+0.5,j+1,k} - u|_{i+0.5,j-1,k}}{2\Delta y} + \\
& \frac{w|_{i,j,k+0.5} + w|_{i,j,k-0.5} + w|_{i+1,j,k+0.5} + w|_{i+1,j,k-0.5}}{4} \frac{u|_{i+0.5,j,k+1} - u|_{i+0.5,j,k-1}}{2\Delta z} \Bigg) + \\
\frac{1}{\mathrm{Re}} \Bigg(\quad & \frac{u|_{i+1.5,j,k} - 2u|_{i+0.5,j,k} + u|_{i-0.5,j,k}}{\Delta x^2} + \\
& \frac{u|_{i+0.5,j+1,k} - 2u|_{i+0.5,j,k} + u|_{i+0.5,j-1,k}}{\Delta y^2} + \\
& \frac{u|_{i+0.5,j,k+1} - 2u|_{i+0.5,j,k} + u|_{i+0.5,j,k-1}}{\Delta z^2} \Bigg)
\end{aligned}
$$

with similar results for the other two components.

1.5 Coding

Mapping the index $i+0.5, j, k$ to (i, j, k) on the x grid, the index $i, j+0.5, k$ to (i, j, k) on the y grid and the index $i, j, k+0.5$ to (i, j, k) on the z grid, the first component of the term becomes:

$$
\begin{aligned}
-\Bigg(\quad & u(i,j,k) \frac{u(i+1,j,k) - u(i-1,j,k)}{2\Delta x} + \\
& \frac{v(i,j,k) + v(i,j-1,k) + v(i+1,j,k) + v(i+1,j-1,k)}{4} \frac{u(i,j+1,k) - u(i,j-1,k)}{2\Delta y} + \\
& \frac{w(i,j,k) + w(i,j,k-1) + w(i+1,j,k) + w(i+1,j,k-1)}{4} \frac{u(i,j,k+1) - u(i,j,k-1)}{2\Delta z} \Bigg) + \\
\frac{1}{\mathrm{Re}} \Bigg(\quad & \frac{u(i+1,j,k) - 2u(i,j,k) + u(i-1,j,k)}{\Delta x^2} + \\
& \frac{u(i,j+1,k) - 2u(i,j,k) + u(i,j-1,k)}{\Delta y^2} + \\
& \frac{u(i,j,k+1) - 2u(i,j,k) + u(i,j,k-1)}{\Delta z^2} \Bigg)
\end{aligned}
\tag{1}
$$

The second component becomes:

$$
\begin{aligned}
-\Bigg(\quad & \frac{u(i,j,k) + u(i-1,j,k) + u(i,j+1,k) + u(i-1,j+1,k)}{4} \frac{v(i+1,j,k) - v(i-1,j,k)}{2\Delta x} + \\
& v(i,j,k) \frac{v(i,j+1,k) - v(i,j-1,k)}{2\Delta y} + \\
& \frac{w(i,j,k) + w(i,j,k-1) + w(i,j+1,k) + w(i,j+1,k-1)}{4} \frac{v(i,j,k+1) - v(i,j,k-1)}{2\Delta z} \Bigg) + \\
\frac{1}{\mathrm{Re}} \Bigg(\quad & \frac{v(i+1,j,k) - 2v(i,j,k) + v(i-1,j,k)}{\Delta x^2} + \\
& \frac{v(i,j+1,k) - 2v(i,j,k) + v(i,j-1,k)}{\Delta y^2} + \\
& \frac{v(i,j,k+1) - 2v(i,j,k) + v(i,j,k-1)}{\Delta z^2} \Bigg)
\end{aligned}
\tag{2}
$$

The third component becomes:

$$
\begin{aligned}
-\Bigg(\quad & \frac{u(i,j,k)+u(i-1,j,k)+u(i,j,k+1)+u(i-1,j,k+1)}{4}\frac{w(i+1,j,k)-w(i-1,j,k)}{2\Delta x}+ \\
& \frac{v(i,j,k)+v(i,j-1,k)+v(i,j,k+1)+v(i,j-1,k+1)}{4}\frac{w(i,j+1,k)-w(i,j-1,k)}{2\Delta y}+ \\
& w(i,j,k)\frac{w(i,j,k+1)-w(i,j,k-1)}{2\Delta z}\Bigg)+ \\
\frac{1}{\mathrm{Re}}\Bigg(\quad & \frac{w(i+1,j,k)-2w(i,j,k)+w(i-1,j,k)}{\Delta x^2}+ \\
& \frac{w(i,j+1,k)-2w(i,j,k)+w(i,j-1,k)}{\Delta y^2}+ \\
& \frac{w(i,j,k+1)-2w(i,j,k)+w(i,j,k-1)}{\Delta z^2}\Bigg)
\end{aligned}
\tag{3}
$$