So we want to use a staggered grid to recover the coupling between even and odd nodes.

For ease of notation I will refer to $\mathbf{u}$ as the velocity in vector components $\vec{u} = (u, v, w)^T$ to avoid useless repetition i will call $u$ the $u_x$ component, $v$ the $u_y$, and $w$ the $u_z$ component.

Also another usefull tipical notation is to indicate $\frac{\partial \Phi}{\partial \psi} = \partial_\psi \Phi$
Starting from Navier-Stockes equations :

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}$$

We rewrite them in such a way to have a 2 step second order method in the form:

- I step:

$$\mathbf{u}^* = \mathbf{u}^n + \Delta t \cdot (-\mathbf{u}^{**} \cdot \nabla \mathbf{u}^{**} + \frac{1}{Re} \cdot \nabla^2 \mathbf{u}^{**} - \nabla p^n)$$

NOTE:
We need to calculate all the terms above for $u^*|_{i+0.5,j,k}$, $w^*|_{i,j,k+0.5}$ and $v^*|_{i,j+0.5,k}$
$\mathbf{u}^{**}$ has to be provided as function of $\mathbf{u}^n$, the prof didn't give use the formula yet !!

- II step:

$$\nabla^2 (p^{n+1} - p^n) = \frac{\nabla \cdot \mathbf{u}^*}{\delta t}$$

That comes with the necessary BCs to have a well defined system $\partial_n (p^{n+1} - p^n) = 0$ on $\partial\Omega$

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \nabla (p^{n+1} - p^n)$$

OK so I start as in the photo with the discretization of $\nabla p$ needed in the momentum equations. We want to evaluate it in all point where we evaluate $\mathbf{u}$

$$\partial_x p|_{i+0.5,j,k} \simeq \frac{p_{i+1,j,k} - p_{i,j,k}}{\Delta x}$$
$$\partial_y p|_{i,j+0.5,k} \simeq \frac{p_{i,j+1,k} - p_{i,j,k}}{\Delta y}$$
$$\partial_z p|_{i,j,k+0.5} \simeq \frac{p_{i,j,k+1} - p_{i,j,k}}{\Delta z}$$

At this point the biggest problem of the system, the non linear part $\mathbf{u} \cdot \nabla \mathbf{u}$ in Einstein's notation becomes : $u_i \partial_{x_i} u_j$ and in explicit form is a vector with components:

$$\mathbf{u}^{**} \cdot \nabla \mathbf{u}^{**} = \begin{bmatrix} u^{**} \cdot \partial_x u^{**} + v^{**} \cdot \partial_y u^{**} + w^{**} \cdot \partial_z u^{**} \\ u^{**} \cdot \partial_x v^{**} + v^{**} \cdot \partial_y v^{**} + w^{**} \cdot \partial_z v^{**} \end{bmatrix} \tag{1}$$

The discretized form should be like this: (I'm a noob in LaTeX, I can't make this fit in the page in another way than this)

$$
\mathbf{u}^{**} \cdot \nabla \mathbf{u}^{**}|_{i,j,k} = \begin{bmatrix} u^{**}|_{i,j,k} \cdot \frac{u^{**}|_{i+0.5,j,k} - u^{**}|_{i-0.5,j,k}}{\Delta x} + v^{**}|_{i,j,k} \cdot \frac{u^{**}|_{i,j+0.5,k} - u^{**}|_{i,j-0.5,k}}{\Delta y} + \\ + w^{**}|_{i,j,k} \cdot \frac{u^{**}|_{i,j,k+0.5} - u^{**}|_{i,j,k-0.5}}{\Delta z} \\ u^{**}|_{i,j,k} \cdot \frac{v^{**}|_{i+0.5,j,k} - v^{**}|_{i-0.5,j,k}}{\Delta x} + v^{**}|_{i,j,k} \cdot \frac{v^{**}|_{i,j+0.5,k} - v^{**}|_{i,j-0.5,k}}{\Delta y} + \\ + w^{**}|_{i,j,k} \cdot \frac{v^{**}|_{i,j,k+0.5} - v^{**}|_{i,j,k-0.5}}{\Delta z} \\ u^{**}|_{i,j,k} \cdot \frac{w^{**}|_{i+0.5,j,k} - w^{**}|_{i-0.5,j,k}}{\Delta x} + v^{**}|_{i,j,k} \cdot \frac{w^{**}|_{i,j+0.5,k} - w^{**}|_{i,j-0.5,k}}{\Delta y} + \\ + w^{**}|_{i,j,k} \cdot \frac{w^{**}|_{i,j,k+0.5} - w^{**}|_{i,j,k-0.5}}{\Delta z} \end{bmatrix}
$$

We need to evaluate it in the points where we look for the velocity (In the first step)

The discretization of the laplacian was not provided, but it would assume it can be done as we did in the second lecture: Here is the discretizzation made for the 3 components:

$$
(\frac{1}{Re} \cdot \nabla^2 u^{**})|_{i,j,k} \simeq \frac{1}{Re} \cdot (\frac{u^{**}_{i+0.5,j,k} - 2 \cdot u^{**}_{i,j,k} + u^{**}_{i-0.5,j,k}}{\Delta x^2} + \frac{u^{**}_{i,j+0.5,k} - 2 \cdot u^{**}_{i,j,k} + u^{**}_{i,j+0.5,k}}{\Delta y^2} + \frac{u^{**}_{i,j,k+0.5} - 2 \cdot u}{\Delta}
$$

$$
(\frac{1}{Re} \cdot \nabla^2 v^{**})|_{i,j,k} \simeq \frac{1}{Re} \cdot (\frac{v^{**}_{i+0.5,j,k} - 2 \cdot v^{**}_{i,j,k} + v^{**}_{i-0.5,j,k}}{\Delta x^2} + \frac{v^{**}_{i,j+0.5,k} - 2 \cdot v^{**}_{i,j,k} + v^{**}_{i,j+0.5,k}}{\Delta y^2} + \frac{v^{**}_{i,j,k+0.5} - 2 \cdot v^{**}_{i,}}{\Delta z^2}
$$

$$
(\frac{1}{Re} \cdot \nabla^2 w^{**})|_{i,j,k} \simeq \frac{1}{Re} \cdot (\frac{w^{**}_{i+0.5,j,k} - 2 \cdot w^{**}_{i,j,k} + w^{**}_{i-0.5,j,k}}{\Delta x^2} + \frac{w^{**}_{i,j+0.5,k} - 2 \cdot w^{**}_{i,j,k} + w^{**}_{i,j+0.5,k}}{\Delta y^2} + \frac{w^{**}_{i,j,k+0.5} - 2}{}
$$

AT THIS POINT THE FIRST EQ IS TOTALLY DISCRETIZED, LET'S GO FOR THE SECOND STEP.

$$
\nabla^2 (p^{n+1} - p^n) = \frac{\nabla \cdot \mathbf{u}^*}{\Delta t} \text{ in } \Omega
$$

$$
\partial_n (p^{n+1} - p^n) = 0 \text{ on } \partial\Omega
$$

$$
\mathbf{u}^{n+1} = \mathbf{u}^* - \nabla (p^{n+1} - p^n) \text{ in } \Omega
$$

Starting with (2), playing a bit with the equations (You'll see the image I made on paper or later I try to include them in this file) we can end up with the system below where A in general is a point on $\partial\Omega$ and C is the first point along the normal direction (But opposite verse since n point outwards). Let's suppose we're on the face of the inlet and A is the point (0, 1, 1) then C is (1, 1, 1) (We skip the 0,5 since is just for the velocity!)

$$
\frac{p_C^{n+1} - p_A^{n+1}}{2 * \Delta x} = \frac{p_C^n - p_A^n}{2 * \Delta x}
$$

Simplifying the denominators since they both are equal and remembering that we have $p_A^{n+1}$ from the BCs we can then recover the value for $p_C^{n+1} = p_A^{n+1} + p_C^n - p_A^n$

At this point we have initialized the value for the internal points of the domain ( $\Omega$ / $\partial\Omega$ to be heavy)

$$\nabla^2 p^{n+1} = \underbrace{\nabla^2 p^n + \frac{1}{\Delta t} \cdot \nabla \cdot \mathbf{u}^*}_{\text{This whole part is know, I will just call it } f \text{from now on}}$$

$$f|_{i,j,k} = \frac{p^n_{i+1,j,k} - 2p^n_{i,j,k} + p^n_{i-1,j,k}}{\Delta x} + \frac{p^n_{i,j,k+1} - 2p^n_{i,j,k} + p^n_{i,j-1,k}}{\Delta y} + \frac{p^n_{i,j,k+1} - 2p^n_{i,j,k} + p^n_{i,j,k-1}}{\Delta z} + \\ + \frac{1}{\Delta t} \cdot \left( \frac{u^*_{i-0.5,j,k} - u^*_{i+0.5,j,k}}{\Delta x} + \frac{v^*_{i,j-0.5,k} - v^*_{i,j+0.5,k}}{\Delta y} + \frac{w^*_{i,j,k-0.5} - w^*_{i,j,k+0.5}}{\Delta z} \right)$$

Now the not so much fun part, cause here we may encounter a problem since we have 3 unknowns ($p^{n+1}_{i+1,j,k}$ , $p^{n+1}_{i,j+1,k}$ and $p^{n+1}_{i,j,k+1}$) I have no clue on how to avoid this problem but to use a decentered scheme (Obv we need to keep it still second order) or just solve the linear system. Blindly applying the second order scheme studied earlier we obtain:

$$\frac{p^{n+1}_{i+1,j,k} - 2p^{n+1}_{i,j,k} + p^{n+1}_{i-1,j,k}}{\Delta x} + \frac{p^{n+1}_{i,j,k+1} - 2p^{n+1}_{i,j,k} + p^{n+1}_{i,j-1,k}}{\Delta y} + \frac{p^{n+1}_{i,j,k+1} - 2p^{n+1}_{i,j,k} + p^{n+1}_{i,j,k-1}}{\Delta z} = f|_{i,j,k}$$

Once we get how to sort out this problem the next step is trivial.

$$u^{n+1}_{i+0.5,j,k} = u^*_{i+0.5,j,k} - \Delta t \cdot \frac{p^{n+1}_{i,j,k} - p^{n+1}_{i-1,j,k}}{\Delta x}$$

$$v^{n+1}_{i,j+0.5,k} = v^*_{i,j+0.5,k} - \Delta t \cdot \frac{p^{n+1}_{i,j,k} - p^{n+1}_{i,j-1,k}}{\Delta y}$$

$$w^{n+1}_{i+0.5,j,k} = w^*_{i,j,k+0.5} - \Delta t \cdot \frac{p^{n+1}_{i,j,k} - p^{n+1}_{i,j,k-1}}{\Delta z}$$