

# 1 Notation

$$\text{Position in body frame is } \mathbf{x}_B = [x_B \ y_B \ z_B]^T \quad (1)$$

$$\text{Position in inertial frame is } \mathbf{x}_I = [x_I \ y_I \ z_I]^T \quad (2)$$

$$\text{Attitude is } \Theta = [\phi \ \theta \ \psi]^T = [\text{roll pitch yaw}]^T \quad (3)$$

$$\text{Rotation from frame 1 to frame 2 is } H_1^2 \quad (4)$$

$$\text{Rotation from inertial to body frame is } H_I^B \quad (5)$$

$$\text{Velocity in body frame is } \mathbf{v}_B = [u \ v \ w]^T \quad (6)$$

$$\text{Angular rate in body frame is } \omega_B = [p \ q \ r]^T \quad (7)$$

$$\text{Total velocity in body frame is } V = \sqrt{u^2 + v^2 + w^2} \quad (8)$$

$$\text{Angle of attack in body frame is } \alpha = \tan^{-1}(w/u) \quad (9)$$

$$\text{Flight path angle is } \gamma = \theta - \alpha \quad (10)$$

$$\text{Sideslip angle in body frame is } \beta = \sin^{-1}(v/V) \quad (11)$$

$$\text{Flight path heading is } \xi = \psi + \beta \quad (12)$$

$$\text{Bank angle is } \beta \quad (13)$$

$$\text{Magnitude of the thrust vector is } T \quad (14)$$

If the thrust of vehicle is aligned with the centerline ( $x_B$  axis), then

$$[T_x \ T_y \ T_z]^T = \begin{bmatrix} T \cos \alpha \\ 0 \\ -T \sin \alpha \end{bmatrix} \quad (15)$$

$$\text{Aerodynamic forces are } \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B = H_I^B \begin{bmatrix} -D \\ SF \\ -L \end{bmatrix} \quad (16)$$

where  $L$  is lift,  $D$  is drag, and  $SF$  is side force.

$$(17)$$

$$\text{Aerodynamic moments are } \begin{bmatrix} L \\ M \\ N \end{bmatrix}_B = H_I^B \begin{bmatrix} L \\ M \\ N \end{bmatrix}_I \quad (18)$$

Yes, the roll moment and lift force are both denoted by  $L$ ...

$$\text{Mass of vehicle is } m \quad (19)$$

$$\text{Reference area (e.g., wing area) is } S \quad (20)$$

$$\text{Wing span is } b \quad (21)$$

$$\text{Mean aerodynamic chord } \bar{c} \quad (22)$$

$$\text{Elevator deflection is } \delta E \quad (23)$$

$$\text{Aileron deflection is } \delta A \quad (24)$$

$$\text{Rudder deflection is } \delta R \quad (25)$$

$$\text{Dynamic pressure is } \bar{q} = \frac{1}{2} \rho V^2 \quad (26)$$

## 2 Longitudinal Model:

Longitudinal equations of motion are

$$\dot{x}_I = u \cos \theta + w \sin \theta \quad (27)$$

$$\dot{z}_I = -u \sin \theta + w \cos \theta \quad (28)$$

$$\dot{\theta} = q \quad (29)$$

$$\dot{u} = F_X/m - qw \quad (30)$$

$$\dot{w} = F_Z/m + qu \quad (31)$$

$$\dot{q} = M_m/I_{yy} \quad (32)$$

To get forces and moments (assuming zero wind)

$$\text{Moment of inertia about the } y \text{ axis is } I_{yy} \quad (33)$$

$$\text{Aerodynamic forces in stability frame: } \begin{bmatrix} -D \\ -L \end{bmatrix} \quad (34)$$

$$\text{Aerodynamic forces in body frame: } \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \times \begin{bmatrix} -D \\ -L \end{bmatrix} \quad (35)$$

$$\text{Force in } x_B \text{ direction } F_X = L \sin \alpha - D \cos \alpha + T - mg \sin \theta \quad (36)$$

$$\text{Force in } z_B \text{ direction } F_Z = -L \cos \alpha - D \sin \alpha + mg \cos \theta \quad (37)$$

We're going to simplify by ignoring the forces and moments from the fuselage and fuselage-wing interference, and compute the forces and moments as:

$$\text{Lift: } L = C_L \bar{q} S \quad (38)$$

$$C_L = C_{L_0} + C_{L_\alpha} \alpha + C_{L_{\delta E}} \delta E \quad (39)$$

$$\text{Drag: } D = C_D \bar{q} S \quad (40)$$

$$C_D = C_{D_0} + \epsilon C_L^2 \quad (41)$$

$$= (C_{D_0} + \epsilon C_{L_0}^2) + C_{D_\alpha} + C_{L_\alpha^2} \alpha^2 \quad (42)$$

$$\text{with induced drag factor } \epsilon \quad (43)$$

$$C_{D_\alpha} = 2\epsilon C_{L_0} C_{L_\alpha} \quad (44)$$

$$C_{L_\alpha^2} = \epsilon C_{L_\alpha}^2 \quad (45)$$

$$\text{Pitch: } M = C_M \bar{q} S \bar{c} \quad (46)$$

$$C_M = C_{M_0} + C_{M_\alpha} \alpha + C_{M_{\delta E}} \delta E \quad (47)$$

$$\text{where the elevator deflection is } \delta E \quad (48)$$

Quantities needed to implement this model:

$$\text{Initial conditions: } \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}_I, \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix}_I, \Theta_0, \omega_0$$

$$\text{Vehicle properties: mass } m, \text{ wing area } S, \text{ mean aerodynamic chord } \bar{c}, \text{ inertial matrix } I$$

$$\text{Coefficients: } C_{L_0}, C_{L_\alpha}, C_{D_0}, \epsilon, C_{L_\alpha^2},$$

$$\text{Air density: } \rho(x)$$

$$\text{Gravity: } g$$

$$\text{Controls: thrust } T, \text{ elevator deflection } \delta E$$

### 3 Linearized Longitudinal Model:

$$\begin{aligned}
\dot{x}_I &= u \cos \theta + w \sin \theta \\
\dot{z}_I &= -u \sin \theta + w \cos \theta \\
\dot{\theta} &= q \\
\dot{u} &= -g \sin \theta + \frac{\rho V^2 S}{2m} \left[ C_{X_0} + C_{X_\alpha} \alpha + C_{X_q} \frac{\bar{c}q}{2V} + C_{X_{\delta_e}} \delta_e \right] + (T + \delta T)/m - qw \\
\dot{w} &= -g \cos \theta + \frac{\rho V^2 S}{2m} \left[ C_{Z_0} + C_{Z_\alpha} \alpha + C_{Z_q} \frac{\bar{c}q}{2V} + C_{Z_{\delta_e}} \delta_e \right] + qu \\
\dot{q} &= \frac{\rho V^2 \bar{c} S}{2I_{yy}} \left[ C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \frac{\bar{c}q}{2V} + C_{m_{\delta_e}} \delta_e \right]
\end{aligned}$$

$$w = V \sin \alpha$$

$$\bar{w} = V^* \cos \alpha^* \bar{\alpha}$$

$$\begin{bmatrix} \dot{\bar{x}}_I \\ \dot{\bar{z}}_I \\ \dot{\bar{\theta}} \\ \dot{\bar{u}} \\ \dot{\bar{\alpha}} \\ \dot{\bar{q}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -u^* \sin \theta^* + w^* \cos \theta^* & \cos \theta^* & V^* \sin \theta^* \cos \alpha^* & 0 \\ 0 & 0 & -u^* \cos \theta^* - w^* \sin \theta^* & -\sin \theta^* & V^* \cos \theta^* \cos \alpha^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g \cos \theta^* & X_u & X_w V^* \cos \alpha & X_q \\ 0 & 0 & \frac{-g \sin \theta^*}{V^* \cos \alpha^*} & \frac{Z_u}{V^* \cos \alpha^*} & Z_w & \frac{Z_q}{V^* \cos \alpha^*} \\ 0 & 0 & 0 & M_u & M_w V^* \cos \alpha^* & M_q \end{bmatrix} \cdot \begin{bmatrix} \bar{x}_I \\ \bar{z}_I \\ \bar{\theta} \\ \bar{u} \\ \bar{\alpha} \\ \bar{q} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ X_{\delta_e} & X_{\delta T} \\ \frac{Z_{\delta_e}}{V \cos \alpha} & 0 \\ M_{\delta_e} & 0 \end{bmatrix} \cdot \begin{bmatrix} \bar{\delta e} \\ \bar{\delta T} \end{bmatrix}$$

Coefficient	Formula
$X_u$	$\frac{u^* \rho S}{m} [C_{X_0} + C_{X_\alpha} \alpha^* + C_{X_{\delta_e}} \delta e^*] - \frac{\rho S w^* C_{X_\alpha}}{2m} + \frac{\rho S \bar{c} C_{X_q} u^* q^*}{4m V^*}$
$X_w$	$-q^* + \frac{w^* \rho S}{m} [C_{X_0} + C_{X_\alpha} \alpha^* + C_{X_{\delta_e}} \delta e^*] + \frac{\rho S u^* C_{X_\alpha}}{2m} + \frac{\rho S \bar{c} C_{X_q} w^* q^*}{4m V^*}$
$X_q$	$-w^* + \frac{\rho V^* S C_{X_q} \bar{c}}{4m}$
$X_{\delta_e}$	$\frac{\rho V^{*2} S C_{X_{\delta_e}}}{2m}$
$X_{\delta T}$	$\frac{1}{m}$
$Z_u$	$q^* + \frac{u^* \rho S}{m} [C_{Z_0} + C_{Z_\alpha} \alpha^* + C_{Z_{\delta_e}} \delta e^*] - \frac{\rho S w^* C_{Z_\alpha}}{2m} + \frac{\rho S \bar{c} C_{Z_q} u^* q^*}{4m V^*}$
$Z_w$	$\frac{w^* \rho S}{m} [C_{Z_0} + C_{Z_\alpha} \alpha^* + C_{Z_{\delta_e}} \delta e^*] + \frac{\rho S u^* C_{Z_\alpha}}{2m} + \frac{\rho S \bar{c} C_{Z_q} w^* q^*}{4m V^*}$
$Z_q$	$u^* + \frac{\rho V^* S C_{Z_q} \bar{c}}{4m}$
$Z_{\delta_e}$	$\frac{\rho V^{*2} S C_{Z_{\delta_e}}}{2m}$
$M_u$	$\frac{u^* \rho S \bar{c}}{I_{yy}} [C_{m_0} + C_{m_\alpha} \alpha^* + C_{m_{\delta_e}} \delta e^*] - \frac{\rho S w^* C_{m_\alpha}}{2I_{yy}} + \frac{\rho S \bar{c}^2 C_{m_q} u^* q^*}{4I_{yy} V^*}$
$M_w$	$\frac{w^* \rho S \bar{c}}{I_{yy}} [C_{m_0} + C_{m_\alpha} \alpha^* + C_{m_{\delta_e}} \delta e^*] - \frac{\rho S u^* C_{m_\alpha}}{2I_{yy}} + \frac{\rho S \bar{c}^2 C_{m_q} w^* q^*}{4I_{yy} V^*}$
$M_q$	$\frac{\rho V^* S \bar{c}^2 C_{m_{\delta_e}}}{4I_{yy}}$
$M_{\delta_e}$	$\frac{\rho V^{*2} S \bar{c} C_{m_{\delta_e}}}{4I_{yy}}$

## 4 Lateral-Directional motion:

Lateral-directional equations of motion are

$$\dot{x}_I = u(\cos \theta \cos \psi) + v(\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi) + w(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) \quad (49)$$

$$\dot{y}_I = u(\cos \theta \sin \psi) + v(\sin \phi \sin \theta \cos \psi + \cos \phi \cos \psi) + w(\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) \quad (50)$$

$$\dot{z}_I = -u \sin \theta + v \sin \phi \cos \theta + w \cos \phi \cos \theta \quad (51)$$

$$\dot{\phi} = p + r \cos \phi \tan \theta \quad (52)$$

$$\dot{\theta} = -r \sin \phi \quad (53)$$

$$\dot{\psi} = r \cos \phi \sec \theta \quad (54)$$

$$\dot{u} = -g \sin \theta + \frac{\rho V^2 S}{2m} [C_{X_0} + C_{X_\alpha} \alpha] + T/m + rv \quad (55)$$

$$\dot{v} = g \cos \theta \sin \phi + \frac{\rho V^2 S}{2m} \left[ C_{Y_0} + C_{Y_\beta} \beta + C_{Y_p} \frac{bp}{2V} + C_{Y_r} \frac{br}{2V} + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \right] + pw - ru \quad (56)$$

$$\dot{w} = g \cos \theta \cos \phi + \frac{\rho V^2 S}{2m} [C_{Z_0} + C_{Z_\alpha} \alpha] - pv \quad (57)$$

$$\dot{p} = (I_{zz}L + I_{xz}N) / (I_{xx}I_{zz} - I_{xz}^2) \quad (58)$$

remembering that  $L$  and  $N$  are the rolling and yawing moments.

$$\dot{q} = \frac{\rho V^2 \bar{c} S}{2I_{yy}} \left[ C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \frac{\bar{c} q}{2V} + C_{m_{\delta_e}} \delta_e \right] \quad (59)$$

$$\dot{r} = (I_{xz}L + I_{xx}N) / (I_{xx}I_{zz} - I_{xz}^2) \quad (60)$$

And we can write the rolling (not lift) and yawing moments as

$$L = \frac{1}{2} \rho V^2 S b \left[ C_{l_0} + C_{l_\beta} \beta + C_{l_p} \frac{bp}{2V} + C_{l_r} \frac{br}{2V} + C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r \right] \quad (61)$$

$$N = \frac{1}{2} \rho V^2 S b \left[ C_{r_0} + C_{r_\beta} \beta + C_{r_p} \frac{bp}{2V} + C_{r_r} \frac{br}{2V} + C_{r_{\delta_a}} \delta_a + C_{r_{\delta_r}} \delta_r \right] \quad (62)$$

$$(63)$$

If we are in level flight, with a small sideslip angle  $\beta$ , then our forces and moments are:

$$Y = C_{Y_\beta} \bar{q} S \beta + C_{Y_{\delta_R}} \delta R \quad (64)$$

It's weird that the sideforce should be given with respect to the reference area  $S$  and dynamic pressure  $\bar{q}$ . Unpacking the co-efficient, it gets renormalised for the tail, as

$$C_{Y_\beta} = \left( \frac{\bar{q}_{vt}}{\bar{q}} \right) \left( 1 + \frac{\partial \sigma}{\partial \beta} \right) \eta_{vt} \left( \frac{S_{vt}}{S} \right) (C_{Y_{\beta_{vt}}}) . \quad (65)$$

where  $\eta_{vt}$  is a tail efficiency parameter,  $\sigma$  is the sidewash angle which we can ignore, and the parameters  $(\cdot)_{vt}$  are the relevant parameters of the vertical tail. Similarly, the rolling and yawing moments get translated to the tail, for example as

$$C_{N_{\beta_{vt}}} = -C_{Y_{\beta_{vt}}} \eta_{vt} \frac{S_{vt} l_{vt}}{S b} \quad (66)$$

where  $l_{vt}$  is the vertical tail length, i.e., the distance from centre of mass to tail centre of pressure. But if we are banked, then the longitudinal forces will have lateral-directional effect too.

(67)

Additional quantities needed to implement this model:

Coefficients:  $C_{Y_0}, C_{Y_\beta}, C_{Y_p}, C_{Y_r}, C_{Y_{\delta a}}, C_{Y_{\delta r}}, C_{l_0}, C_{l_\beta}, C_{l_p}, C_{l_r}, C_{l_{\delta a}}, C_{l_{\delta r}}, C_{r_0}, C_{r_\beta}, C_{r_p}, C_{r_r}, C_{r_{\delta a}}, C_{r_{\delta r}}$   
Wing parameters:  $b, S$   
Tail parameters:  $S_{vt}, \bar{q}_{vt}, \eta_{vt}, l_{vt}$ .

## 5 Linearized Lateral-Directional Model:

Because we have the trim state and the longitudinal model, for the lateral dynamics case we can basically forget about the  $x, z$  and pitch variables in the state. Given a trim state that contains all the relevant variables, we can build and analyze a linear model that only contains the lateral velocity  $v$ , the roll and yaw  $\phi$  and  $\psi$ , and the corresponding rates  $p$  and  $r$ .

$$\begin{aligned}\dot{v} &= g \cos \theta \sin \phi + \frac{\rho V^2 S}{2m} \left[ C_{Y_0} + C_{Y_\beta} \beta + C_{Y_p} \frac{bp}{2V} + C_{Y_r} \frac{br}{2V} + C_{Y_{\delta a}} \delta_a + C_{Y_{\delta r}} \delta_r \right] + pw - ru \\ \dot{p} &= (I_{zz}L + I_{xz}N) / (I_{xx}I_{zz} - I_{xz}^2) \\ \dot{r} &= (I_{xz}L + I_{xx}N) / (I_{xx}I_{zz} - I_{xz}^2) \\ \dot{\phi} &= p + r \cos \phi \tan \theta \\ \dot{\psi} &= r \cos \phi \sec \theta\end{aligned}$$

But

$$v = V \sin \beta$$

Linearizing around  $\beta = \beta^*$ :

$$\begin{aligned}\bar{v} &= V^* \cos \beta^* \bar{\beta} \\ \dot{\bar{\beta}} &= \frac{1}{V^* \cos \beta^*} \dot{\bar{v}}\end{aligned}$$

$$\begin{bmatrix} \dot{\bar{\beta}} \\ \dot{\bar{p}} \\ \dot{\bar{r}} \\ \dot{\bar{\phi}} \\ \dot{\bar{\psi}} \end{bmatrix} = \begin{bmatrix} Y_v & \frac{Y_p}{V^* \cos \beta^*} & \frac{Y_r}{V^* \cos \beta^*} & \frac{g \cos \theta^* \cos \phi^*}{V^* \cos \beta^*} & 0 \\ L_v V^* \cos \beta^* & L_p & L_r & 0 & 0 \\ N_v V^* \cos \beta^* & N_p & N_r & 0 & 0 \\ 0 & 1 & \cos \phi^* \tan \theta^* & q^* \cos \phi^* \tan \theta^* & -r^* \sin \phi^* \tan \theta^* \\ 0 & 0 & \cos \phi^* \sec \theta^* & p^* \cos \phi^* \sec \theta^* & -r^* \sin \phi^* \sec \theta^* \end{bmatrix} \cdot \begin{bmatrix} \bar{\beta} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \\ \bar{\psi} \end{bmatrix} + \begin{bmatrix} \frac{Y_{\delta a}}{V^* \cos \beta^*} & \frac{Y_{\delta r}}{V^* \cos \beta^*} \\ L_{\delta a} & L_{\delta r} \\ N_{\delta a} & N_{\delta r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \bar{\delta a} \\ \bar{\delta r} \end{bmatrix}$$

Aircraft are often symmetric about the plane spanned by  $x_b$  and  $z_b$ , which means that

$$I = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}$$

We can further define the following terms

$$\begin{aligned}
\Gamma &= I_{xx}I_{zz} - I_{xz}^2 \\
\Gamma_1 &= \frac{I_{xz}(I_{xx} - I_{yy} + I_{zz})}{\Gamma} \\
\Gamma_2 &= \frac{I_{zz}(I_{zz} - I_{yy}) + I_{xz}^2}{\Gamma} \\
\Gamma_3 &= \frac{I_{zz}}{\Gamma} \\
\Gamma_4 &= \frac{I_{xz}}{\Gamma} \\
\Gamma_5 &= \frac{I_{zz} - I_{xx}}{I_{yy}} \\
\Gamma_6 &= \frac{I_{xz}}{I_{yy}} \\
\Gamma_7 &= \frac{(I_{xx} - I_{yy})I_{xx} + I_{xz}^2}{\Gamma} \\
\Gamma_8 &= \frac{I_{xx}}{\Gamma}
\end{aligned}$$

We can therefore rewrite the angular rate derivatives as follows:

$$\begin{aligned}
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr \\
\dot{q} &= \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr
\end{aligned}$$

Coefficient	Formula
$Y_v$	$\frac{v^* \rho S b}{4m V^*} [C_{Y_p} p^* + C_{Y_r} r^*] + \frac{v^* \rho S}{m} [C_{Y_0} + C_{Y_\beta} \beta^* + C_{Y_{\delta a}} \delta a^* + C_{Y_{\delta r}} \delta r^*] + \frac{\rho S C_{Y_\beta}}{2m} \sqrt{u^{*2} + w^{*2}}$
$Y_p$	$w^* + \frac{\rho V^* S b}{4m} C_{Y_p}$
$Y_r$	$-u^* + \frac{\rho V^* S b}{4m} C_{Y_r}$
$Y_{\delta a}$	$\frac{\rho V^{*2} S}{2m^2} C_{Y_{\delta a}}$
$Y_{\delta r}$	$\frac{\rho V^{*2} S}{2m} C_{Y_{\delta r}}$
$L_v$	$\frac{v^* \rho S b^2}{4V^*} [C_{p_p} p^* + C_{p_r} r^*] + v^* \rho S b [C_{p_0} + C_{p_\beta} \beta^* + C_{p_{\delta a}} \delta a^* + C_{p_{\delta r}} \delta r^*] + \frac{\rho S b C_{p_\beta}}{2} \sqrt{u^{*2} + w^{*2}}$
$L_p$	$\Gamma_1 q^* + \frac{\rho V^* S b^2}{4} C_{p_p}$
$L_r$	$-\Gamma_2 q^* + \frac{\rho V^* S b^2}{4} C_{p_r}$
$L_{\delta a}$	$\frac{\rho V^{*2} S b}{2} C_{p_{\delta a}}$
$L_{\delta r}$	$\frac{\rho V^{*2} S b}{2} C_{p_{\delta r}}$
$N_v$	$\frac{v^* \rho S b^2}{4V^*} [C_{r_p} p^* + C_{r_r} r^*] + v^* \rho S b [C_{r_0} + C_{r_\beta} \beta^* + C_{r_{\delta a}} \delta a^* + C_{r_{\delta r}} \delta r^*] + \frac{\rho S b C_{r_\beta}}{2} \sqrt{u^{*2} + w^{*2}}$
$N_p$	$\Gamma_7 q^* + \frac{\rho V^* S b^2}{4} C_{r_p}$
$N_r$	$-\Gamma_1 q^* + \frac{\rho V^* S b^2}{4} C_{r_r}$
$N_{\delta a}$	$\frac{\rho V^{*2} S b}{2} C_{r_{\delta a}}$
$N_{\delta r}$	$\frac{\rho V^{*2} S b}{2} C_{r_{\delta r}}$

## 6 Unified Model without Forces:

$$\begin{aligned}
\dot{x}_I &= (\cos \theta \cos \psi)u + (\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi)v + (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)w \\
\dot{y}_I &= (\cos \theta \sin \psi)u + (\sin \phi \sin \theta \sin \psi - \cos \phi \cos \psi)v + (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi)w \\
\dot{z}_I &= u \sin \theta - v \sin \phi \cos \theta - w \cos \phi \cos \theta \\
\dot{u} &= rv - qw \\
\dot{v} &= pw - ru \\
\dot{w} &= qu - pv \\
\dot{\phi} &= p + (q \sin \phi + r \cos \phi) \tan \theta \\
\dot{\theta} &= q \cos \phi - r \sin \phi \\
\dot{\psi} &= (q \sin \phi + r \cos \phi) \sec \theta \\
\dot{p} &= (-[I_{xz}(I_{yy} - I_{xx} - I_{zz})p + [I_{xz}^2 + I_{zz}(I_{zz} - I_{yy})]r]q) / (I_{xx}I_{zz} - I_{xz}^2) \\
\dot{q} &= \frac{-(I_{xx} - I_{zz})pr - I_{xz}(p^2 - r^2)}{I_{22}} \\
\dot{r} &= (-[I_{xz}(I_{yy} - I_{xx} - I_{zz})r + [I_{xz}^2 + I_{xx}(I_{xx} - I_{yy})]p]q) / (I_{xx}I_{zz} - I_{xz}^2)
\end{aligned}$$

## 7 Unified Model:

$$\begin{aligned}
\dot{x}_I &= (\cos \theta \cos \psi)u + (\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi)v + (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)w \\
\dot{y}_I &= (\cos \theta \sin \psi)u + (\sin \phi \sin \theta \sin \psi - \cos \phi \cos \psi)v + (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi)w \\
\dot{z}_I &= u \sin \theta - v \sin \phi \cos \theta - w \cos \phi \cos \theta \\
\dot{u} &= rv - qw - g \sin \theta + \frac{\bar{q}}{m} \left[ C_X(\alpha) + C_{X_q}(\alpha) \frac{\bar{c}q}{2V_a} + C_{X_{\delta_e}}(\alpha) \delta_e \right] + T \\
\dot{v} &= pw - ru + g \cos \theta \sin \phi + \frac{\bar{q}}{m} \left[ C_{Y_0} + C_{Y_\beta}(\beta) + C_{Y_p} \frac{bp}{2V_a} + C_{Y_r} \frac{br}{2V_a} + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \right] \\
\dot{w} &= qu - pv - g \cos \theta \cos \phi + \frac{\bar{q}}{m} \left[ C_Z(\alpha) + C_{Z_q}(\alpha) \frac{\bar{c}q}{2V_a} + C_{Z_{\delta_e}}(\alpha) \delta_e \right] \\
\dot{\phi} &= p + (q \sin \phi + r \cos \phi) \tan \theta \\
\dot{\theta} &= q \cos \phi - r \sin \phi \\
\dot{\psi} &= (q \sin \phi + r \cos \phi) \sec \theta \\
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr + \bar{q}b \left[ C_{p_0} + C_{p_\beta} \beta + C_{p_p} \frac{bp}{2V_a} + C_{p_r} \frac{br}{2V_a} + C_{p_{\delta_a}} \delta_a + C_{p_{\delta_r}} \delta_r \right] \\
\dot{q} &= \Gamma_5 pr - \Gamma_6 (p^2 - r^2) + \bar{q} \frac{\bar{c}}{I_{yy}} \left[ C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \frac{\bar{c}q}{2V_a} + C_{m_{\delta_e}} \delta_e \right] \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr + \bar{q}b \left[ C_{r_0} + C_{r_\beta} \beta + C_{r_p} \frac{bp}{2V_a} + C_{r_r} \frac{br}{2V_a} + C_{r_{\delta_a}} \delta_a + C_{r_{\delta_r}} \delta_r \right]
\end{aligned}$$

where

$$\begin{aligned}
C_{p_0} &= \Gamma_3 C_{l_0} + \Gamma_4 C_{n_0} \\
C_{p_\beta} &= \Gamma_3 C_{l_\beta} + \Gamma_4 C_{n_\beta} \\
C_{p_p} &= \Gamma_3 C_{l_p} + \Gamma_4 C_{n_p} \\
C_{p_r} &= \Gamma_3 C_{l_r} + \Gamma_4 C_{n_r} \\
C_{p_{\delta_a}} &= \Gamma_3 C_{l_{\delta_a}} + \Gamma_4 C_{n_{\delta_a}} \\
C_{p_{\delta_r}} &= \Gamma_3 C_{l_{\delta_r}} + \Gamma_4 C_{n_{\delta_r}} \\
C_{r_0} &= \Gamma_4 C_{l_0} + \Gamma_8 C_{n_0} \\
C_{r_\beta} &= \Gamma_4 C_{l_\beta} + \Gamma_8 C_{n_\beta} \\
C_{r_p} &= \Gamma_4 C_{l_p} + \Gamma_8 C_{n_p} \\
C_{r_r} &= \Gamma_4 C_{l_r} + \Gamma_8 C_{n_r} \\
C_{r_{\delta_a}} &= \Gamma_4 C_{l_{\delta_a}} + \Gamma_8 C_{n_{\delta_a}} \\
C_{r_{\delta_r}} &= \Gamma_4 C_{l_{\delta_r}} + \Gamma_8 C_{n_{\delta_r}}
\end{aligned}$$

and we push the dependence on the angle of attack into the lift and drag co-efficients, so that

$$\begin{aligned}
C_X(\alpha) &= -C_D(\alpha) \cos \alpha + C_L(\alpha) \sin \alpha \\
C_{X_q}(\alpha) &= -C_{D_q}(\alpha) \cos \alpha + C_{L_q}(\alpha) \sin \alpha \\
C_{X_{\delta_e}}(\alpha) &= -C_{D_{\delta_e}}(\alpha) \cos \alpha + C_{L_{\delta_e}}(\alpha) \sin \alpha \\
C_Z(\alpha) &= -C_D(\alpha) \sin \alpha - C_L(\alpha) \cos \alpha \\
C_{Z_q}(\alpha) &= -C_{D_q}(\alpha) \sin \alpha - C_{L_q}(\alpha) \cos \alpha \\
C_{Z_{\delta_e}}(\alpha) &= -C_{D_{\delta_e}}(\alpha) \sin \alpha - C_{L_{\delta_e}}(\alpha) \cos \alpha
\end{aligned}$$

## 8 Cascaded Control Loops

- Roll attitude hold

$$\begin{aligned}
E_{\phi_t} &= \phi_t^c - \phi_t \\
D_t &= \frac{E_{\phi_t} - E_{\phi_{t-1}}}{\Delta t} \\
\delta a_t &= k_{p_\phi} E_t - k_{d_\phi} D_t
\end{aligned}$$

- Course hold

$$\begin{aligned}
E_{\chi_t} &= \chi_t^c - \chi_t \\
I_{\chi_t} &= I_{\chi_{t-1}} + E_{\chi_t} \Delta t \\
\phi^c &= k_{p_\chi} E_{\chi_t} + k_{i_\chi} I_{\chi_t}
\end{aligned}$$

- Sideslip Hold

$$\begin{aligned}
E_{\beta_t} &= \beta_t^c - \beta_t \\
I_{\beta_t} &= I_{\beta_{t-1}} + E_{\beta_t} \Delta t \\
\delta r &= -k_{p_\beta} E_{\beta_t} - k_{i_\beta} I_{\beta_t}
\end{aligned}$$



- Pitch attitude hold

$$\begin{aligned}
E_{\theta_t} &= \theta_t^c - \theta_t \\
D_t &= \frac{E_{\theta_t} - E_{\theta_{t-1}}}{\Delta t} \\
\delta e &= k_{p_\theta} E_{\theta_t} + k_{d_\theta} D_{\theta_t}
\end{aligned}$$

- Altitude hold

$$\begin{aligned}
E_{z_t} &= z_t^c - z_t \\
I_{z_t} &= I_{z_{t-1}} + E_{z_t} \Delta t \\
\theta^c &= k_{p_z} E_{z_t} + k_{i_z} I_{z_t}
\end{aligned}$$

- Airspeed hold using commanded pitch

$$\begin{aligned}
E_{V_t} &= V_t^c - V_t \\
I_{V_t} &= I_{V_{t-1}} + E_{V_t} \Delta t \\
\theta^c &= k_{p_{V_2}} E_{V_t} + k_{i_{V_2}} I_{V_t}
\end{aligned}$$

- Airspeed hold using commanded throttle

$$\begin{aligned}
E_{V_t} &= V_t^c - V_t \\
I_{V_t} &= I_{V_{t-1}} + E_{V_t} \Delta t \\
\delta t &= \delta^* t + k_{p_V} E_{V_t} + k_{i_V} I_{V_t}
\end{aligned}$$