# UAVBook Supplement
# Full State Direct and Indirect EKF

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This supplement will explore alternatives to the state estimation scheme presented in the book. In particular, we will derive a full state estimation scheme using both the direct Kalman filter and the indirect Kalman filter. We begin by focusing on the longitudinal states of the aircraft.

# 1 Direct Kalman Filter for Longitudinal States

## 1.1 Aircraft and sensor models

The model for the dynamics of the aircraft are similar to that presented in the book after removing the lateral states. In particular, Equations (5.1)-(5.12) become

$$\dot{p}_n = u \cos \theta + w \sin \theta$$
$$\dot{h} = u \sin \theta - w \cos \theta$$
$$\dot{u} = -qw + \frac{f_x}{\mathsf{m}}$$
$$\dot{w} = qu + \frac{f_z}{\mathsf{m}}$$
$$\dot{\theta} = q.$$

We will assume that the sensors that are available are the $y$-axis gyroscope, the $x$- and $z$-axis accelerometers, the static and differential pressure

sensors, the GPS-north measurement, and the GPS-ground-speed measurement. The models for the sensors, adapted from Equations (7.3), (7.5), (7.9), (7.10), (7.18), and (7.21) are given by

$$y_{\text{accel},x} = \frac{f_x}{\mathsf{m}} + g\sin\theta + \eta_{\text{accel},x}$$

$$y_{\text{accel},z} = \frac{f_z}{\mathsf{m}} - g\cos\theta + \eta_{\text{accel},z}$$

$$y_{\text{gyro},y} = q + b_y + \eta_{\text{gyro},y}$$

$$y_{\text{abs pres}} = \rho g h_{\text{AGL}} + \eta_{\text{abs pres}}$$

$$y_{\text{diff pres}} = \frac{\rho V_a^2}{2} + \eta_{\text{diff pres}}$$

$$y_{\text{GPS},n} = p_n + \nu_n[n]$$

$$y_{\text{GPS},V_g} = V_a + w_n + \eta_V.$$

where $b_y$ is the unknown bias on the $y$-axis gyroscope.

## 1.2 Propagation model for the EKF

The longitudinal states will be defined as

$$x = (p_n, h, u, w, \theta, b_y, w_n)^\top.$$

Given the fact that the accelerometers measure total force minus gravity, we can write the equations of motion for $x$ as

$$\dot{p}_n = u\cos\theta + w\sin\theta$$

$$\dot{h} = u\sin\theta - w\cos\theta$$

$$\dot{u} = -(y_{\text{gyro},y} - b_y)w - g\sin\theta + y_{\text{accel},x} + w\eta_{\text{gyro},y} - \eta_{\text{accel},x}$$

$$\dot{w} = (y_{\text{gyro},y} - b_y)u + g\cos\theta + y_{\text{accel},z} - u\eta_{\text{gyro},y} - \eta_{\text{accel},z}$$

$$\dot{\theta} = y_{\text{gyro},y} - b_y - \eta_{\text{gyro},y}$$

$$\dot{b}_y = 0$$

$$\dot{w}_n = 0.$$

If we define

$$f(x, y) \triangleq \begin{pmatrix} u\cos\theta + w\sin\theta \\ u\sin\theta - w\cos\theta \\ -(y_{\text{gyro},y} - b_y)w - g\sin\theta + y_{\text{accel},x} \\ (y_{\text{gyro},y} - b_y)u + g\cos\theta + y_{\text{accel},z} \\ y_{\text{gyro},y} - b_y \\ 0 \\ 0 \end{pmatrix}$$

and

$$G \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -w & -1 & 0 \\ u & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then we can write

$$\dot{x} = f(x, y) + G(x)\eta_s + \eta_d,$$

where

$$y \triangleq \begin{pmatrix} y_{\text{gyro},y} \\ y_{\text{accel},x} \\ y_{\text{accel},z} \end{pmatrix}$$

and

$$\eta_s \triangleq \begin{pmatrix} \eta_{\text{gyro},y} \\ \eta_{\text{accel},x} \\ \eta_{\text{accel},z} \end{pmatrix}$$

is the noise associated with the gyros and accelerometers, and where $\eta_i$ is a noise term associated with model uncertainty.

The Jacobian of $f$ can be shown to be

$$A(x, y) \triangleq \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 0 & \cos\theta & \sin\theta & [-u\sin\theta + w\cos\theta] & 0 & 0 \\ 0 & 0 & \sin\theta & -\cos\theta & [u\cos\theta + w\sin\theta] & 0 & 0 \\ 0 & 0 & 0 & -(y_{\text{gyro},y} - b_y) & -g\cos\theta & w & 0 \\ 0 & 0 & (y_{\text{gyro},y} - b_y) & 0 & -g\sin\theta & -u & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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The covariance of the noise term $G\eta_s + \eta_d$ is given by

$$E[(G\eta_s + \eta_i)(G\eta_s + \eta_i)^\top] = GQ_sG^\top + Q_d,$$

where

$$Q_s = \text{diag}\left([\sigma^2_{\text{gyro},y}, \sigma^2_{\text{accel}_x}, \sigma^2_{\text{gyro}_z}]\right),$$

$$Q_d = \text{diag}\left([\sigma^2_{p_n}, \sigma^2_h, \sigma^2_u, \sigma^2_v, \sigma^2_\theta, \sigma^2_{b_y}, \sigma^2_{w_n}]\right),$$

and where $Q_d$ are tuning parameters.

## 1.3   Sensor Update Equations

### 1.3.1   Static Pressure Sensor

The model for the static pressure sensor is

$$h_{\text{static}}(x) = \rho g h.$$

Therefore, the Jacobian is given by

$$C_{\text{static}}(x) = \begin{pmatrix} 0 & \rho g & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

### 1.3.2   Differential Pressure Sensor

The model for the differential pressure sensor is

$$h_{\text{diff}}(x) = \frac{1}{2}\rho V_a^2,$$

where $V_a^2 = u_r^2 + w_r^2$, and where $u_r$ and $w_r$ are the wind relative velocities given at the top of page 57 in the uavbook. For the longitudinal case where $w_d = 0$ we can write the relative velocities as

$$u_r = u - w_n \cos\theta$$
$$w_r = w - w_n \sin\theta.$$

Therefore

$$h_{\text{diff}}(x) = \frac{1}{2}\rho(u - w_n \cos\theta)^2 + (w - w_n \sin\theta)^2,$$

with an associated Jacobian of

$$C_{\text{diff}}(x) = \rho\begin{pmatrix} 0 & 0 & u_r & w_r & u_r w_n \sin\theta - w_r w_n \cos\theta & 0 & -u_r \cos\theta - w_r \sin\theta \end{pmatrix}.$$

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### 1.3.3 GPS North sensor

The model for the GPS north sensor is

$$h_{\text{GPS,n}}(x) = p_n$$

with associated Jacobian

$$C_{\text{GPS,n}}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

### 1.3.4 GPS ground-speed sensor

The ground-speed is the projection of the inertial velocity vector $(u, w)^\top$ on to the ground plane. Therefore

$$h_{\text{GPS},V_g}(x) = u \cos\theta + w \sin\theta$$

with associated Jacobian

$$C_{\text{GPS},V_g}(x) = \begin{pmatrix} 0 & 0 & \cos\theta & \sin\theta & (-u\sin\theta + w\cos\theta) & 0 & 0 \end{pmatrix}.$$

## 1.4 Algorithm for Direct Kalman Filter

We assume that the gyro, accelerometers, and pressure sensors are sampled at rate $T_s$, which is the same rate that state estimates are required by the controller. We also assume that GPS is sampled at $T_{\text{GPS}} >> T_s$.

The algorithm for the direct Kalman filter is given as follows.

**0. Initialize Filter.**

Set

$$\hat{x} = (p_n(0), h(0), V_a(0), 0, 0, 0, 0)^\top,$$
$$P = \text{diag}(e_{p_n}^2, e_h^2, e_u^2, e_w^2, e_\theta^2, e_{b_y}^2, e_{w_n}^2),$$

where $e_*$ is the standard deviation of the expected error in $*$.

**1. At the fast sample rate $T_s$**

1.a. Propagate $\hat{x}$ and $P$ according to

$$\dot{\hat{x}} = f(\hat{x}, y)$$
$$\dot{P} = A(\hat{x}, y)P + PA^\top(\hat{x}, y) + G(\hat{x})Q_s Q^\top(\hat{x}) + Q_d$$

1.b. Update $\hat{x}$ and $P$ with the static pressure sensor according to

$$L = P^- C_{\text{static}}^\top(\hat{x}^-)/(\sigma_{\text{abs pres}}^2 + C_{\text{static}}(\hat{x}^-)P^- C_{\text{static}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{static}}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{abs pres}} - h_{\text{static}}(\hat{x}^-)).$$

1.c. Update $\hat{x}$ and $P$ with the differential pressure sensor according to

$$L = P^- C_{\text{diff}}^\top(\hat{x}^-)/(\sigma_{\text{diff pres}}^2 + C_{\text{diff}}(\hat{x}^-)P^- C_{\text{diff}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{diff}}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{diff pres}} - h_{\text{diff}}(\hat{x}^-)).$$

**2. When GPS measurements are received at $T_{\textbf{GPS}}$:**
2.a. Update $\hat{x}$ and $P$ with the GPS north sensor according to

$$L = P^- C_{\text{GPS},n}^\top(\hat{x}^-)/(\sigma_{\text{GPS},n}^2 + C_{\text{GPS},n}(\hat{x}^-)P^- C_{\text{GPS},n}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},n}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{GPS},n} - h_{\text{GPS},n}(\hat{x}^-)).$$

2.b. Update $\hat{x}$ and $P$ with the GPS ground-speed sensor according to

$$L = P^- C_{\text{GPS},V_g}^\top(\hat{x}^-)/(\sigma_{\text{GPS},V_g}^2 + C_{\text{GPS},V_g}(\hat{x}^-)P^- C_{\text{GPS},V_g}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},V_g}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{GPS},V_g} - h_{\text{GPS},V_g}(\hat{x}^-)).$$

Results for the direct Kalman filter are shown in Figure 1. Red is the commanded signal, blue is truth, and green is the estimate. It can be seen from the figure that, with the possible exception of the ground-speed $V_g$, the filter performs well. The simulation files associated with this example are included on the uavbook website.
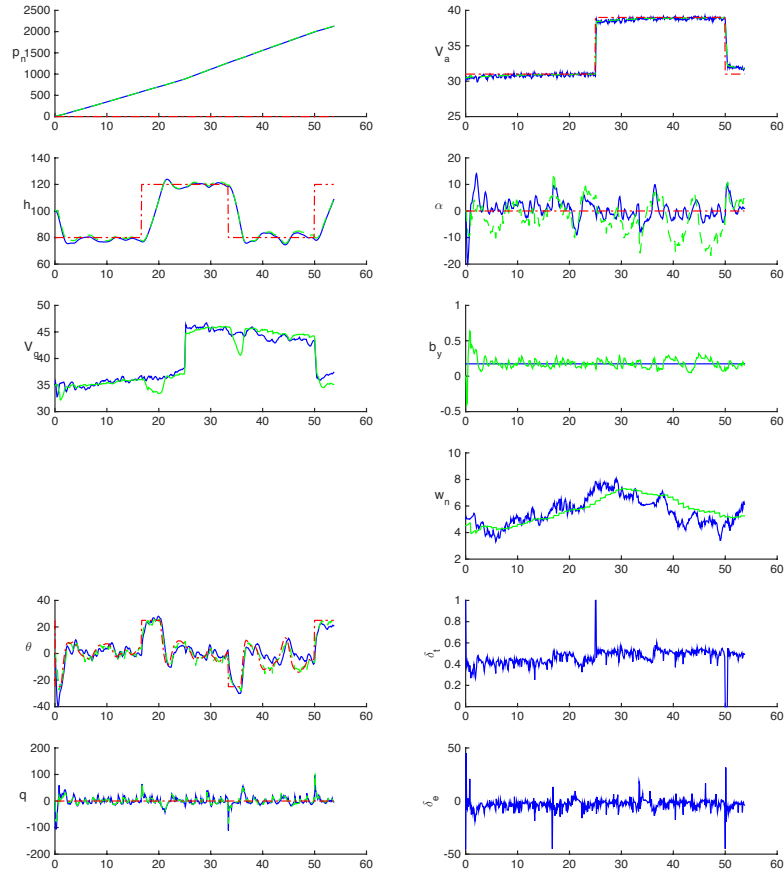
Figure 1: Estimation results for the direct Kalman filter.

# 2 Indirect Kalman Filter for Longitudinal States

For the indirect Kalman filter, the idea is to filter the error states, which should satisfy the linear and Gaussian assumptions better than the direct implementation. Let $x$ be the true state, $\hat{x}$ the estimated state, and let $\tilde{x} = x - \hat{x}$ be the error state. If $x$ satisfies the differential equation

$$\dot{x} = f(x, y) + G(x)\eta_s + \eta_i$$

where all definitions are identical to the previous section, and if $\hat{x}$ satisfies

$$\dot{\hat{x}} = f(\hat{x}, y),$$

then the error state evolves according to

$$\dot{\tilde{x}} = f(x, y) + G(x)\eta_s + \eta_i - f(\hat{x}, y).$$

Using the Taylor series expansion up to the linear term

$$f(x, y) \approx f(\hat{x}, y) + A(\hat{x}, y)\tilde{x}$$
$$G(x)\eta_s \approx G(\hat{x})\eta_s$$

gives

$$\dot{\tilde{x}} = A(\hat{x}, y)\tilde{x} + G(\hat{x})\eta_s + \eta_i.$$

Note that the covariance of the noise term

$$E[(G(\hat{x})\eta_s + \eta_i)(G(\hat{x})\eta_s + \eta_i)^\top] = G(\hat{x})Q_s G(\hat{x})^\top + Q_i,$$

where $Q_i$ is a tuning parameter similar to $Q_d$ for the direct EKF.

The basic idea is to run the Kalman filter on the error state, and then to correct the state estimate at each step by adding in the error. The error state is reset to zero after each correction. The algorithm is outlined below.

## 2.1 Algorithm for the Indirect Kalman Filter

The algorithm for the indirect Kalman filter is given as follows. Terms that differ from the direct Kalman filter are highlighted in blue.

**0. Initialize Filter.**

0.1 Initialize the state estimate according to

$$\hat{x} = (p_n(0), h(0), V_a(0), 0, 0, 0, 0)^\top.$$

0.2 Initialize the error state estimate according to

$$\tilde{x} = (0, 0, 0, 0, 0, 0, 0)^\top,$$
$$P = \text{diag}(e_{p_n}^2, e_h^2, e_u^2, e_w^2, e_\theta^2, e_{b_y}^2, e_{w_n}^2),$$

where $e_*$ is the standard deviation of the expected error in $*$.

**1. At the fast sample rate $T_s$**

1.a. Propagate the estimated state $\hat{x}$ according to

$$\dot{\hat{x}} = f(\hat{x}, y)$$

1.b. Propagate $\tilde{x}$ and $P$ according to

$$\dot{\tilde{x}} = A(\hat{x}, y)\tilde{x}$$
$$\dot{P} = A(\hat{x}, y)P + PA^\top(\hat{x}, y) + G(\hat{x})Q_s Q^\top(\hat{x}) + Q_d$$

1.c. Update $\tilde{x}$ and $P$ with the static pressure sensor according to

$$L = P^- C_{\text{static}}^\top(\hat{x}^-)/(\sigma_{\text{abs pres}}^2 + C_{\text{static}}(\hat{x}^-)P^- C_{\text{static}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{static}}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\text{abs pres}} - h_{\text{static}}(\hat{x}^-) - C_{\text{static}}(\hat{x}^-)\tilde{x}^-).$$

1.d. Update $\tilde{x}$ and $P$ with the differential pressure sensor according to

$$L = P^- C_{\text{diff}}^\top(\hat{x}^-)/(\sigma_{\text{diff pres}}^2 + C_{\text{diff}}(\hat{x}^-)P^- C_{\text{diff}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{diff}}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\text{diff pres}} - h_{\text{diff}}(\hat{x}^-) - C_{\text{diff}}(\hat{x}^-)\tilde{x}^-).$$

**2. When GPS measurements are received at $T_{\text{GPS}}$:**

2.a. Update $\tilde{x}$ and $P$ with the GPS north sensor according to

$$L = P^- C_{\text{GPS},n}^\top(\hat{x}^-)/(\sigma_{\text{GPS},n}^2 + C_{\text{GPS},n}(\hat{x}^-)P^- C_{\text{GPS},n}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},n}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\text{GPS},n} - h_{\text{GPS},n}(\hat{x}^-) - C_{\text{GPS},n}(\hat{x}^-)\tilde{x}^-).$$

2.b. Update $\tilde{x}$ and $P$ with the GPS ground-speed sensor according to

$$L = P^- C_{\text{GPS},V_g}^\top(\hat{x}^-)/(\sigma_{\text{GPS},V_g}^2 + C_{\text{GPS},V_g}(\hat{x}^-)P^- C_{\text{GPS},V_g}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},V_g}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\text{GPS},V_g} - h_{\text{GPS},V_g}(\hat{x}^-) - C_{\text{GPS},V_g}(\hat{x}^-)\tilde{x}^-).$$

2.c. Update $\hat{x}$ according to

$$\hat{x}^+ = \hat{x}^- + \tilde{x}^+.$$

2.d. Set $\tilde{x} = (0, 0, 0, 0, 0, 0, 0)^\top$.

Results for the indirect Kalman filter are shown in Figure 2. Red is the commanded signal, blue is truth, and green is the estimate. It can be seen from the figure that, with the possible exception of the ground-speed $V_g$, the filter performs almost identical to the direct KF. We should note that the results are obtained when the estimator is in the feedback loop. The simulation files associated with this example are included on the uavbook website.
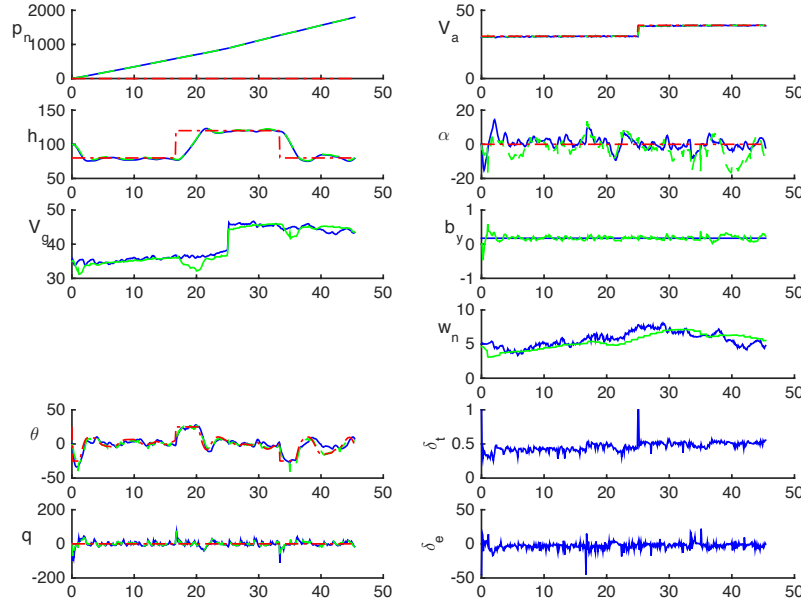


Figure 2: Estimation results for the indirect Kalman filter.

# 3 Direct Kalman Filter for Full States

In this section we will expand upon the previous discussion to include the full aircraft state.

## 3.1 Aircraft and sensor models

The model for the dynamics of the aircraft are similar to that presented in the book. If we define $\mathbf{p} = (p_n, p_e, p_d)^\top$, $\mathbf{v} = (u, v, w)^\top$, $\Theta = (\phi, \theta, \psi)^\top$, $\boldsymbol{\omega} = (p, q, r)^\top$, then we can write the dynamics as

$$\dot{\mathbf{p}} = R(\Theta)\mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{v}^\times \boldsymbol{\omega} + \mathbf{a} + R^\top(\Theta)\mathbf{g}$$
$$\dot{\Theta} = S(\Theta)\boldsymbol{\omega}$$
$$\dot{\mathbf{b}} = 0_{3 \times 1}$$
$$\dot{\mathbf{w}} = 0_{2 \times 1},$$

where we have used the notation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^\times = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

and where

$$R(\Theta) = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$$S(\Theta) = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix},$$

and where we have used the model for the accelerometer in Equation (7.1) to get

$$\frac{1}{\mathsf{m}}\mathbf{f} = \mathbf{a} + R^\top(\Theta)\mathbf{g},$$

where $\mathbf{g} = (0, 0, g)^\top$.

We will assume that the sensors that are available are the gyroscopes, the accelerometers, the static and differential pressure sensors, the GPS north and east measurements, and the GPS ground-speed and course measurement. The models for the sensors are given by

$$\mathbf{y}_{\text{accel}} = \mathbf{a} + \boldsymbol{\eta}_{\text{accel}}$$
$$\mathbf{y}_{\text{gyro}} = \boldsymbol{\omega} + \mathbf{b} + \boldsymbol{\eta}_{\text{gyro}}$$
$$y_{\text{abs pres}} = \rho g h_{\text{AGL}} + \eta_{\text{abs pres}}$$
$$y_{\text{diff pres}} = \frac{\rho V_a^2}{2} + \eta_{\text{diff pres}}$$
$$y_{\text{GPS},n} = p_n + \nu_n[n]$$
$$y_{\text{GPS},e} = p_e + \nu_e[n]$$
$$y_{\text{GPS},V_g} = \sqrt{(V_a \cos\psi + w_n)^2 + (V_a \sin\psi + w_e)^2} + \eta_V$$
$$y_{\text{GPS},\chi} = \text{atan2}(V_a \sin\psi + w_e, V_a \cos\psi + w_n) + \eta_\chi.$$

## 3.2   Propagation model for the EKF

Using the gyro and accelerometer models, the equations of motion can be written as

$$\dot{\mathbf{p}} = R(\Theta)\mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{v}^\times(\mathbf{y}_{\text{gyro}} - \mathbf{b} - \boldsymbol{\eta}_{\text{gyro}}) + (\mathbf{y}_{\text{accel}} - \boldsymbol{\eta}_{\text{accel}}) + R^\top(\Theta)\mathbf{g}$$
$$\dot{\Omega} = S(\Theta)(\mathbf{y}_{\text{gyro}} - \mathbf{b} - \boldsymbol{\eta}_{\text{gyro}})$$
$$\dot{\mathbf{b}} = 0_{3\times 1}$$
$$\dot{\mathbf{w}} = 0_{2\times 1}.$$

Defining

$$\mathbf{x} = (\mathbf{p}^\top, \mathbf{v}^\top, \Theta^\top, \mathbf{b}^\top, \mathbf{w}^\top)^\top$$
$$\mathbf{y} = (\mathbf{y}_{\text{accel}}^\top, \mathbf{y}_{\text{gyro}}^\top)^\top,$$

we get

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{y}) + G_g(\mathbf{x})\boldsymbol{\eta}_{\text{gyro}} + G_a\boldsymbol{\eta}_{\text{accel}} + \boldsymbol{\eta},$$

where

$$f(\mathbf{x},\mathbf{y}) = \begin{pmatrix} R(\Theta)\mathbf{v} \\ \mathbf{v}^\times(\mathbf{y}_{\text{gyro}} - \mathbf{b}) + \mathbf{y}_{\text{accel}} + R^\top(\Theta)\mathbf{g} \\ S(\Theta)(\mathbf{y}_{\text{gyro}} - \mathbf{b}) \\ 0_{3\times 1} \\ 0_{2\times 1,} \end{pmatrix}$$

$$G_g(\mathbf{x}) = \begin{pmatrix} 0_{3\times 3} \\ -\mathbf{v}^\times \\ -S(\Theta) \\ 0_{3\times 3} \\ 0_{2\times 3} \end{pmatrix}$$

$$G_a = \begin{pmatrix} 0_{3\times 3} \\ -I_{3\times 3} \\ 0_{3\times 3} \\ 0_{3\times 3} \\ 0_{2\times 3} \end{pmatrix},$$

and where $\boldsymbol{\eta}$ is the general process noise associated with the model uncertainty and assumed to have covariance $Q$.

The Jacobian of $f$ is

$$A(\mathbf{x},\mathbf{y}) \triangleq \frac{\partial f}{\partial x} = \begin{pmatrix} 0_{3\times 3} & R(\Theta) & \frac{\partial[R(\Theta)\mathbf{v}]}{\partial \Theta} & 0_{3\times 3} & 0_{3\times 2} \\ 0_{3\times 3} & -(\mathbf{y}_{\text{gyro}} - \mathbf{b})^\times & \frac{\partial[R^\top(\Theta)\mathbf{g}]}{\partial \Theta} & -\mathbf{v}^\times & 0_{3\times 2} \\ 0_{3\times 3} & 0_{3\times 3} & \frac{\partial[S(\Theta)(\mathbf{y}_{\text{gyro}}-\mathbf{b})]}{\partial \Theta} & -S(\Theta) & 0_{3\times 2} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 2} \\ 0_{2\times 3} & 0_{2\times 3} & 0_{2\times 3} & 0_{2\times 3} & 0_{2\times 2} \end{pmatrix}.$$

It is straightforward to show that when $\mathbf{z} \in \mathbb{R}^3$ and $A(\mathbf{z})$ is a $3 \times 3$ matrix whose elements are functions of $\mathbf{z}$, then for any $\mathbf{w} \in \mathbb{R}^3$

$$\frac{\partial[A(\mathbf{z})\mathbf{w}]}{\partial \mathbf{z}} = \left( \left(\frac{\partial A(\mathbf{z})}{\partial z_1}\right)\mathbf{w}, \quad \left(\frac{\partial A(\mathbf{z})}{\partial z_2}\right)\mathbf{w}, \quad \left(\frac{\partial A(\mathbf{z})}{\partial z_3}\right)\mathbf{w} \right).$$

Therefore

$$\frac{\partial \left[R(\Theta)\mathbf{v}\right]}{\partial \Theta} = \left( \frac{\partial R(\Theta)}{\partial \phi}\mathbf{v}, \quad \frac{\partial R(\Theta)}{\partial \theta}\mathbf{v}, \quad \frac{\partial R(\Theta)}{\partial \psi}\mathbf{v} \right)$$

$$\frac{\partial \left[R^\top(\Theta)\mathbf{g}\right]}{\partial \Theta} = \left( \frac{\partial R^\top(\Theta)}{\partial \phi}\mathbf{g}, \quad \frac{\partial R^\top(\Theta)}{\partial \theta}\mathbf{g}, \quad \frac{\partial R^\top(\Theta)}{\partial \psi}\mathbf{g} \right) = g \begin{pmatrix} 0 & -\cos\theta & 0 \\ \cos\theta\cos\phi & -\sin\theta\sin\phi & 0 \\ -cos\theta\sin\phi & -\sin\theta\cos\phi & 0 \end{pmatrix}$$

$$\frac{\partial \left[S(\Theta)(\mathbf{y}_{\mathrm{gyro}} - \mathbf{b})\right]}{\partial \Theta} = \left( \frac{\partial S(\Theta)}{\partial \phi}(\mathbf{y}_{\mathrm{gyro}} - \mathbf{b}), \quad \frac{\partial S(\Theta)}{\partial \theta}(\mathbf{y}_{\mathrm{gyro}} - \mathbf{b}), \quad \frac{\partial S(\Theta)}{\partial \psi}(\mathbf{y}_{\mathrm{gyro}} - \mathbf{b}) \right),$$

where $\frac{\partial R(\Theta)}{\partial \phi}$ is the matrix where each element of $R(\Theta)$ has been differentiated by $\phi$.

The covariance of the noise term $G_g\boldsymbol{\eta}_{\mathrm{gyro}} + G_a\boldsymbol{\eta}_{\mathrm{accel}} + \boldsymbol{\eta}$ is

$$E[(G_g\boldsymbol{\eta}_{\mathrm{gyro}}+G_a\boldsymbol{\eta}_{\mathrm{accel}}+\boldsymbol{\eta})(G_g\boldsymbol{\eta}_{\mathrm{gyro}}+G_a\boldsymbol{\eta}_{\mathrm{accel}}+\boldsymbol{\eta})^\top] = G_g Q_{\mathrm{gyro}} G_g^\top + G_a Q_{\mathrm{accel}} G_a^\top + Q,$$

where

$$Q_{\mathrm{gyro}} = \mathrm{diag}([\sigma_{\mathrm{gyro},x}^2, \sigma_{\mathrm{gyro}_y}^2, \sigma_{\mathrm{gyro}_z}^2])$$

$$Q_{\mathrm{accel}} = \mathrm{diag}([\sigma_{\mathrm{accel},x}^2, \sigma_{\mathrm{accel}_y}^2, \sigma_{\mathrm{accel}_z}^2]),$$

and where

$$Q = \mathrm{diag}([\sigma_{p_n}^2, \sigma_{p_e}^2, \sigma_{p_d}^2, \sigma_u^2, \sigma_v^2, \sigma_w^2, \sigma_\phi^2, \sigma_\theta^2, \sigma_\psi^2, \sigma_{b_x}^2, \sigma_{b_y}^2, \sigma_{b_z}^2, \sigma_{w_n}^2, \sigma_{w_e}^2])$$

are tuning parameters.

## 3.3 Sensor Update Equations

### 3.3.1 Static Pressure Sensor

The model for the static pressure sensor is

$$h_{\mathrm{static}}(\mathbf{x}) = \rho g h = -\rho g p_d.$$

Therefore, the Jacobian is given by

$$C_{\mathrm{static}}(\mathbf{x}) = (0, 0, -\rho g, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

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### 3.3.2 Differential Pressure Sensor

The model for the differential pressure sensor is

$$h_{\text{diff}}(\mathbf{x}) = \frac{1}{2}\rho V_a^2,$$

where $V_a^2 = \left(\mathbf{v} - R^\top(\Theta)\mathbf{w}\right)^\top \left(\mathbf{v} - R^\top(\Theta)\mathbf{w}\right)$, and where $\mathbf{w} = (w_n, w_e, 0)^\top$.
Therefore

$$h_{\text{diff}}(\mathbf{x}) = \frac{1}{2}\rho \left(\mathbf{v} - R^\top(\Theta)\mathbf{w}\right)^\top \left(\mathbf{v} - R^\top(\Theta)\mathbf{w}\right).$$

The associated Jacobian is given by

$$C_{\text{diff}}(\mathbf{x}) = \rho \begin{pmatrix} 0_{3\times1} \\ \mathbf{v} - R^\top(\Theta)\mathbf{w} \\ -\frac{\partial\left[R^\top(\Theta)\mathbf{w}\right]}{\partial\Theta}^\top \left(\mathbf{v} - R^\top(\Theta)\mathbf{w}\right) \\ 0_{3\times1} \\ -\left(I_{2\times2}, \ 0_{2\times1}\right) R(\Theta)\left(\mathbf{v} - R^\top(\Theta)\mathbf{w}\right) \end{pmatrix}^\top.$$

### 3.3.3 GPS North sensor

The model for the GPS north sensor is

$$h_{\text{GPS,n}}(\mathbf{x}) = p_n$$

with associated Jacobian

$$C_{\text{GPS,n}}(\mathbf{x}) = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

### 3.3.4 GPS East sensor

The model for the GPS east sensor is

$$h_{\text{GPS,e}}(\mathbf{x}) = p_e$$

with associated Jacobian

$$C_{\text{GPS,e}}(\mathbf{x}) = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

### 3.3.5   GPS ground-speed sensor

The inertial velocity in the world frame, projected onto the north-east plane is given by

$$\mathbf{v}_{g\perp} = \begin{bmatrix} I_{2\times2}, & 0_{2\times1} \end{bmatrix} R(\Theta)\mathbf{v}. \tag{1}$$

The ground-speed is given by

$$V_g(\mathbf{v}, \Theta) = \|\mathbf{v}_{g\perp}\|.$$

The model for the sensor is therefore

$$h_{\text{GPS},V_g}(\mathbf{x}) = \|PR(\Theta)\mathbf{v}\|,$$

where $P = \begin{bmatrix} I_{2\times2}, & 0_{2\times1} \end{bmatrix}$, and the associated Jacobian is given by

$$C_{\text{GPS},V_g}(\mathbf{x}) = \left( 0_{1\times3}, \quad \mathbf{v}^\top R^\top(\Theta) P^\top P R(\Theta), \quad \frac{\partial V_g(\mathbf{v},\Theta)}{\partial\Theta}^\top, \quad 0_{1\times3}, \quad 0_{1\times2} \right),$$

where $\frac{\partial V_g(\mathbf{v},\Theta)}{\partial\Theta}^\top$ is computed numerically as

$$\frac{\partial V_g(\mathbf{v}, \Theta)}{\partial\Theta_i} = \frac{V_g(\mathbf{v}, \Theta + \Delta\mathbf{e}_i) - V_g(\mathbf{v}, \Theta)}{\Delta},$$

where $\mathbf{e}_i$ is the $3 \times 1$ vector with one in the $i^{th}$ location and zeros elsewhere, and $\Delta$ is a small number.

### 3.3.6   GPS course

The course angle is the direction of $\mathbf{v}_{g\perp} = (v_{g\perp n}, v_{g\perp e})^\top$ given in Equation (1). Therefore, the course angle is given by

$$\chi(\mathbf{v}, \Theta) = \text{atan2}(v_{g\perp e}, v_{g\perp n}).$$

The model for the sensor is therefore

$$h_{\text{GPS},\chi}(\mathbf{x}) = \text{atan2}(v_{g\perp e}, v_{g\perp n}),$$

and the associated Jacobian is given by

$$C_{\text{GPS},\chi}(\mathbf{x}) = \left( 0_{1\times3}, \quad \frac{\partial\chi(\mathbf{v},\mathbf{v})}{\partial\mathbf{v}}^\top, \quad \frac{\partial\chi(\mathbf{v},\Theta)}{\partial\Theta}^\top, \quad 0_{1\times3}, \quad 0_{1\times2} \right),$$

where $\frac{\partial\chi(\mathbf{v},\mathbf{v})}{\partial\mathbf{v}}^\top$ and $\frac{\partial\chi(\mathbf{v},\Theta)}{\partial\Theta}^\top$ are computed numerically as

$$\frac{\partial\chi(\mathbf{v}, \Theta)}{\partial\mathbf{v}_i} = \frac{\chi(\mathbf{v} + \Delta\mathbf{e}_i, \Theta) - \chi(\mathbf{v}, \Theta)}{\Delta}$$

$$\frac{\partial\chi(\mathbf{v}, \Theta)}{\partial\Theta_i} = \frac{\chi(\mathbf{v}, \Theta + \Delta\mathbf{e}_i) - \chi(\mathbf{v}, \Theta)}{\Delta}.$$

### 3.3.7   Pseudo Measurement for Side Slip Angle

With the given sensors, the side-slip angle, or side-to-side velocity is not observable and can therefore drift. To help correct this situation, we can add a pseudo measurement on the side-slip angle by assuming that it is zero.

The side slip angle is given by

$$\beta = \frac{v_r}{V_a},$$

therefore an equivalent condition is that $v_r = 0$. The airspeed vector is given by

$$\mathbf{v}_a = \mathbf{v} - R(\Theta)^\top \mathbf{w},$$

which implies that

$$v_r(\mathbf{v}, \Theta, \mathbf{w}) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \left( \mathbf{v} - R(\Theta)^\top \mathbf{w} \right).$$

Therefore, the model for the pseudo-sensor is given by

$$h_\beta(\mathbf{x}) = v_r(\mathbf{v}, \Theta, \mathbf{w}),$$

and the associated Jacobian is given by

$$C_\beta(\mathbf{x}) = \left( 0_{1\times 3}, \quad \frac{\partial v_r(\mathbf{v},\Theta,\mathbf{w})}{\partial \mathbf{v}}^\top, \quad \frac{\partial v_r(\mathbf{v},\Theta,\mathbf{w})}{\partial \Theta}^\top, \quad 0_{1\times 3}, \quad \frac{\partial v_r(\mathbf{v},\Theta,\mathbf{w})}{\partial \mathbf{w}}^\top, \quad \right),$$

where the partial derivatives are computed numerically as

$$\frac{\partial v_r(\mathbf{v}, \Theta, \mathbf{w})}{\partial \mathbf{v}_i} = \frac{v_r(\mathbf{v} + \Delta \mathbf{e}_i, \Theta, \mathbf{w}) - v_r(\mathbf{v}, \Theta, \mathbf{w})}{\Delta}$$

$$\frac{\partial v_r(\mathbf{v}, \Theta, \mathbf{w})}{\partial \Theta_i} = \frac{v_r(\mathbf{v}, \Theta + \Delta \mathbf{e}_i, \mathbf{w}) - v_r(\mathbf{v}, \Theta, \mathbf{w})}{\Delta}$$

$$\frac{\partial v_r(\mathbf{v}, \Theta, \mathbf{w})}{\partial \mathbf{w}_i} = \frac{v_r(\mathbf{v}, \Theta, \mathbf{w} + \Delta \mathbf{e}_i) - v_r(\mathbf{v}, \Theta, \mathbf{w})}{\Delta}.$$

The measurement used in the correction step is $y_\beta = 0$.

## 3.4   Algorithm for Direct Kalman Filter

We assume that the gyro, accelerometers, and pressure sensors are sampled at rate $T_s$, which is the same rate that state estimates are required by the controller. We also assume that GPS is sampled at $T_{\text{GPS}} \gg T_s$.

The algorithm for the direct Kalman filter is given as follows.

**0. Initialize Filter.**

Set

$$\hat{x} = (p_n(0), p_e(0), p_d(0), V_a(0), 0, 0, 0, 0, \psi(0), 0, 0, 0, 0, 0)^\top,$$
$$P = \operatorname{diag}\left([e_{p_n}^2, e_{p_e}^2, e_{p_d}^2, e_u^2, e_v^2, e_w^2, e_\phi^2, e_\theta^2, e_\psi^2, e_{b_x}^2, e_{b_y}^2, e_{b_z}^2, e_{w_n}^2, e_{w_e}^2]\right),$$

where $e_*$ is the standard deviation of the expected error in $*$.

**1. At the fast sample rate $T_s$**

1.a. Propagate $\hat{x}$ and $P$ according to

$$\dot{\hat{x}} = f(\hat{x}, y)$$
$$\dot{P} = A(\hat{x}, y)P + PA^\top(\hat{x}, y) + G_g(\hat{x})Q_{\text{gyros}}G_g^\top(\hat{x}) + G_a Q_{\text{accel}} G_a^\top + Q$$

1.b. Update $\hat{x}$ and $P$ with the static pressure sensor according to

$$L = P^- C_{\text{static}}^\top(\hat{x}^-)/(\sigma_{\text{abs pres}}^2 + C_{\text{static}}(\hat{x}^-)P^- C_{\text{static}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{static}}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{abs pres}} - h_{\text{static}}(\hat{x}^-)).$$

1.c. Update $\hat{x}$ and $P$ with the differential pressure sensor according to

$$L = P^- C_{\text{diff}}^\top(\hat{x}^-)/(\sigma_{\text{diff pres}}^2 + C_{\text{diff}}(\hat{x}^-)P^- C_{\text{diff}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{diff}}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{diff pres}} - h_{\text{diff}}(\hat{x}^-)).$$

1.d. Update $\hat{x}$ and $P$ with the side-slip pseudo measurement according to

$$L = P^- C_\beta^\top(\hat{x}^-)/(\sigma_\beta^2 + C_\beta(\hat{x}^-)P^- C_\beta^\top(\hat{x}^-))$$
$$P^+ = (I - LC_\beta(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(0 - h_\beta(\hat{x}^-)).$$

**2. When GPS measurements are received at $T_{\text{GPS}}$:**

2.a. Update $\hat{x}$ and $P$ with the GPS north measurement according to

$$L = P^- C_{\text{GPS},n}^\top(\hat{x}^-)/(\sigma_{\text{GPS},n}^2 + C_{\text{GPS},n}(\hat{x}^-)P^- C_{\text{GPS},n}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},n}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{GPS},n} - h_{\text{GPS},n}(\hat{x}^-)).$$

2.b. Update $\hat{x}$ and $P$ with the GPS east measurement according to

$$L = P^- C_{\text{GPS},e}^\top(\hat{x}^-)/(\sigma_{\text{GPS},e}^2 + C_{\text{GPS},e}(\hat{x}^-)P^- C_{\text{GPS},e}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},e}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{GPS},e} - h_{\text{GPS},e}(\hat{x}^-)).$$

2.c. Update $\hat{x}$ and $P$ with the GPS groundspeed measurement according to

$$L = P^- C_{\text{GPS},V_g}^\top(\hat{x}^-)/(\sigma_{\text{GPS},V_g}^2 + C_{\text{GPS},V_g}(\hat{x}^-)P^- C_{\text{GPS},V_g}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},V_g}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{GPS},V_g} - h_{\text{GPS},V_g}(\hat{x}^-)).$$

2.d. Update $\hat{x}$ and $P$ with the GPS course measurement according to

$$L = P^- C_{\text{GPS},\chi}^\top(\hat{x}^-)/(\sigma_{\text{GPS},\chi}^2 + C_{\text{GPS},\chi}(\hat{x}^-)P^- C_{\text{GPS},\chi}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{GPS},\chi}(\hat{x}^-))P^-$$
$$\hat{x}^+ = \hat{x}^- + L(y_{\text{GPS},\chi} - h_{\text{GPS},V\chi}(\hat{x}^-)).$$

# 4 Indirect Kalman Filter for Full States

## 4.1 Algorithm for the Indirect Kalman Filter

The algorithm for the indirect Kalman filter is given as follows. Terms that differ from the direct Kalman filter are highlighted in blue.

**0. Initialize Filter.**

0.1 Initialize the state estimate according to

$$\hat{x} = (p_n(0), p_e(0), p_d(0), V_a(0), 0, 0, 0, 0, \psi(0), 0, 0, 0, 0, 0)^\top.$$

0.2 Initialize the error state estimate according to

$$\tilde{x} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^\top,$$

$$P = \text{diag}\left([e_{p_n}^2, e_{p_e}^2, e_{p_d}^2, e_u^2, e_v^2, e_w^2, e_\phi^2, e_\theta^2, e_\psi^2, e_{b_x}^2, e_{b_y}^2, e_{b_z}^2, e_{w_n}^2, e_{w_e}^2]\right),$$

where $e_*$ is the standard deviation of the expected error in $*$.

**1. At the fast sample rate $T_s$**

1.a. Propagate the estimated state $\hat{x}$ according to

$$\dot{\hat{x}} = f(\hat{x}, y)$$

1.b. Propagate $\tilde{x}$ and $P$ according to

$$\dot{\tilde{x}} = A(\hat{x}, y)\tilde{x}$$
$$\dot{P} = A(\hat{x}, y)P + PA^\top(\hat{x}, y) + G_g(\hat{x})Q_{\text{gyros}}G_g^\top(\hat{x}) + G_aQ_{\text{accel}}G_a^\top + Q$$

1.c. Update $\tilde{x}$ and $P$ with the static pressure sensor according to

$$L = P^-C_{\text{static}}^\top(\hat{x}^-)/(\sigma_{\text{abs pres}}^2 + C_{\text{static}}(\hat{x}^-)P^-C_{\text{static}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{static}}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\text{abs pres}} - h_{\text{static}}(\hat{x}^-) - C_{\text{static}}(\hat{x}^-)\tilde{x}^-).$$

1.d. Update $\tilde{x}$ and $P$ with the differential pressure sensor according to

$$L = P^-C_{\text{diff}}^\top(\hat{x}^-)/(\sigma_{\text{diff pres}}^2 + C_{\text{diff}}(\hat{x}^-)P^-C_{\text{diff}}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\text{diff}}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\text{diff pres}} - h_{\text{diff}}(\hat{x}^-) - C_{\text{diff}}(\hat{x}^-)\tilde{x}^-).$$

1.e. Update $\hat{x}$ and $P$ with the side-slip pseudo measurement according to

$$L = P^- C_\beta^\top(\hat{x}^-)/(\sigma_\beta^2 + C_\beta(\hat{x}^-)P^- C_\beta^\top(\hat{x}^-))$$
$$P^+ = (I - LC_\beta(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(0 - h_\beta(\hat{x}^-) - C_\beta(\hat{x}^-)\tilde{x}^-).$$

**2. When GPS measurements are received at $T_{\mathbf{GPS}}$:**

2.a. Update $\hat{x}$ and $P$ with the GPS north measurement according to

$$L = P^- C_{\mathrm{GPS},n}^\top(\hat{x}^-)/(\sigma_{\mathrm{GPS},n}^2 + C_{\mathrm{GPS},n}(\hat{x}^-)P^- C_{\mathrm{GPS},n}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\mathrm{GPS},n}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\mathrm{GPS},n} - h_{\mathrm{GPS},n}(\hat{x}^-) - C_{\mathrm{GPS},n}(\hat{x}^-)\tilde{x}^-).$$

2.b. Update $\hat{x}$ and $P$ with the GPS east measurement according to

$$L = P^- C_{\mathrm{GPS},e}^\top(\hat{x}^-)/(\sigma_{\mathrm{GPS},e}^2 + C_{\mathrm{GPS},e}(\hat{x}^-)P^- C_{\mathrm{GPS},e}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\mathrm{GPS},e}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\mathrm{GPS},e} - h_{\mathrm{GPS},e}(\hat{x}^-) - C_{\mathrm{GPS},e}(\hat{x}^-)\tilde{x}^-).$$

2.c. Update $\hat{x}$ and $P$ with the GPS groundspeed measurement according to

$$L = P^- C_{\mathrm{GPS},V_g}^\top(\hat{x}^-)/(\sigma_{\mathrm{GPS},V_g}^2 + C_{\mathrm{GPS},V_g}(\hat{x}^-)P^- C_{\mathrm{GPS},V_g}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\mathrm{GPS},V_g}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\mathrm{GPS},V_g} - h_{\mathrm{GPS},V_g}(\hat{x}^-) - C_{\mathrm{GPS},V_g}(\hat{x}^-)\tilde{x}^-).$$

2.d. Update $\hat{x}$ and $P$ with the GPS course measurement according to

$$L = P^- C_{\mathrm{GPS},\chi}^\top(\hat{x}^-)/(\sigma_{\mathrm{GPS},\chi}^2 + C_{\mathrm{GPS},\chi}(\hat{x}^-)P^- C_{\mathrm{GPS},\chi}^\top(\hat{x}^-))$$
$$P^+ = (I - LC_{\mathrm{GPS},\chi}(\hat{x}^-))P^-$$
$$\tilde{x}^+ = \tilde{x}^- + L(y_{\mathrm{GPS},\chi} - h_{\mathrm{GPS},V\chi}(\hat{x}^-) - C_{\mathrm{GPS},\chi}(\hat{x}^-)\tilde{x}^-).$$

2.e. Update $\hat{x}$ according to

$$\hat{x}^+ = \hat{x}^- + \tilde{x}^+.$$

2.f. Set $\tilde{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0)^\top.$

# References