

EC504 Algorithms and Data Structure

Rich Brower Tu-Tr 11:00- 12:45

EC504 Course Organization

- Why Algorithm \Rightarrow Data Structures
- CRSL text with Slide Summaries
- Scaling, Math and Empirical Analysis on Simple Cases.
- Use [GitHub](#) (EC405), [Slack](#) and [CCS and Unix Tools](#)
- HW's pencil and paper handed at CCS
- Software delivered CCS — Must run from Makefile.
- Basic Unix environment — useful for computer engineers to know!

Course Organization

- Text:

- Cormen, Leiserson, Rivest & Stein (CLRS), Fundamental text!
“Introduction to Algorithms” 3rd Edition MIT Pres

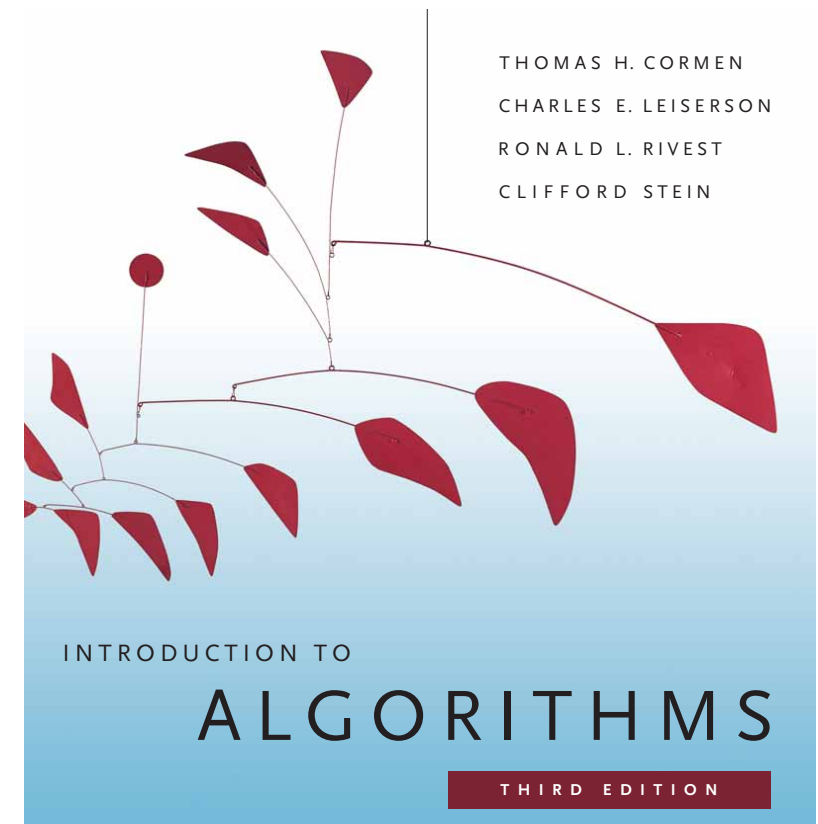
- Keynote Slides guide to CLRS

- Reference:

- Wikipedia!
- Mark Allen Weiss “Data Structure and Algorithms in C”.
- UNIX, Makefiles, very basic C/C++ and gnuplot:

- Grading:

- HW with Coding Programs: 30%
- Class participation 10%++
- Project 25%
- Midterm (up to trees) 15%
- Final (comprehensive) 20%



C vs C++ Advice — C++ compiler is C

1. To understand what is under the hood and how algorithms work go to C (The C++ compiler generates C!)
3. **Bottom up:** Simpler the better to see how the computer executes an algorithm and to optimized performance.
5. (KISS) Avoid **C++ sugar**. Adopt C++ism only when that add value.
7. Of course you can always use **C++ libraries!**
9. Standards are always better **Avoid re-inventing the wheel!**

Course Outline

- Algorithms Analysis CRLS 2-4 (5) (HW1?)
 - Definition of Problem Class of Size N
 - Math for large N Asymptotics:
- I. 1-D Data Structures CRLS 6,7,8,9 (HW2?)
 - Arrays, Lists, Stacks, Queues CRLS 10
 - Searching, Sorting, String Matching, Scheduling
- II. 1.5 D Trees CRLS 12 -`14 (HW3?)
 - BST, AVL,
 - Coding, Union/Join CRLS 18-21, midterm (HW4?)
- III. 2D Graphs CRLS 22,23,24,25, (HW 5?)
 - Traversal, Min Spanning Tree, Shortest Path, Capacity, Min Flow CRLS 26, (HW6?)
- IV Selected Advanced Topics & Projects
 - Spatial Data Structures, FFT's, Complexity, Approx. Solutions, Quantum Computing etc

INTRODUCTORY READING IN CRLS

- CRLS 1.2
- CRLS 1.3
- CRLS 1.4
- CRLS 1.5 Just a bit of averaging!

What is an algorithm? An unambiguous list of steps (program) to transform some input into some output.



- Pick a Problem (set)
- Find method to solve
 1. Correct for all cases (elements of set)
 2. Each step is finite ($\Delta t_{\text{step}} < \text{max time}$)
 - Next step is unambiguous
 - Terminate in finite number of steps
- ◆ You know many examples:
GCD, Multiply 2 N bit integers, ...

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi
Bagdad (Iraq) 780-850

Searching Sorted List :

- int a[0], a[1], a[2], a[3], a[m], a[2], a[N-1]

Three Algorithms:

- *Linear Search* → O(N)
- *(after Sorting)*
- *Bisection Search* → O(log(N)).
- *Dictionary Search* → O(log[log[N]])

Euclid's Algorithm GCD

(325-265 BC in Egypt)

The Greatest Common Divisor (gcd) of positive integers p and q is the largest integer which divides p and q evenly.

Can assume $p > q$

If $p = nq + r$, then
 $\gcd(p, q) = \gcd(q, r)$

```
int gcd(int p, int q)
{ int r;
  while(q!=0){
    r = p%q;
    p=q; q=r;}
  Return p; }
```

$$\gcd(22, 8): 22 = 2 * 8 + 6$$

$$\gcd(8, 6): 8 = 1 * 6 + 2$$

$$\gcd(6, 2): 6 = 3 * 2 + 0$$

Answer = 2

Complexity: $q = N$: $T(N)$ = calls to gcd

Worst case: $T(N) < 1.44 \log_2(N)$

Average: $T(N) \gg (12 \ln 2 / \pi^2) \ln N$

Proof of Euclid's Algorithms

- With $r = p \% q$, $p = n q + r$ and $k = \gcd(p, q)$ then
- Therefore $p = k p'$ and $q = k q'$
 -) $k p' = n k q' + r$
 -) $r = k$ & q has k as a factor so
 -) $\gcd(q, r) = K \geq k$
- BUT K can't be bigger than k since
 -) $p = n q + r = n (K q' + K r)$
 -) $k = n K \geq K$
 -) $\gcd(q, r) = \gcd(p, q)$

Also note $p \bmod q < p/2$ so $T(N) < 2 \log_2(N)$

Halting Problem: is this an algorithm?

Examples:

$x=1 \rightarrow 1$

$x=2 \rightarrow 2, 1$

$x=3 \rightarrow 3, 10, 5, 16, 8, 4, 2, 1$

...

$x=27 \rightarrow 27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484,$
242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233,
700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 618,
309, 928, 464, 232, 116, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10,
5, 16, 8, 4, 2, 1

The $3x + 1$ Problem by L. Collatz (1937)

ENDS?(x):

while $x > 1$:

print x

if x is even

then x gets $x/2$

else x gets $3x+1$

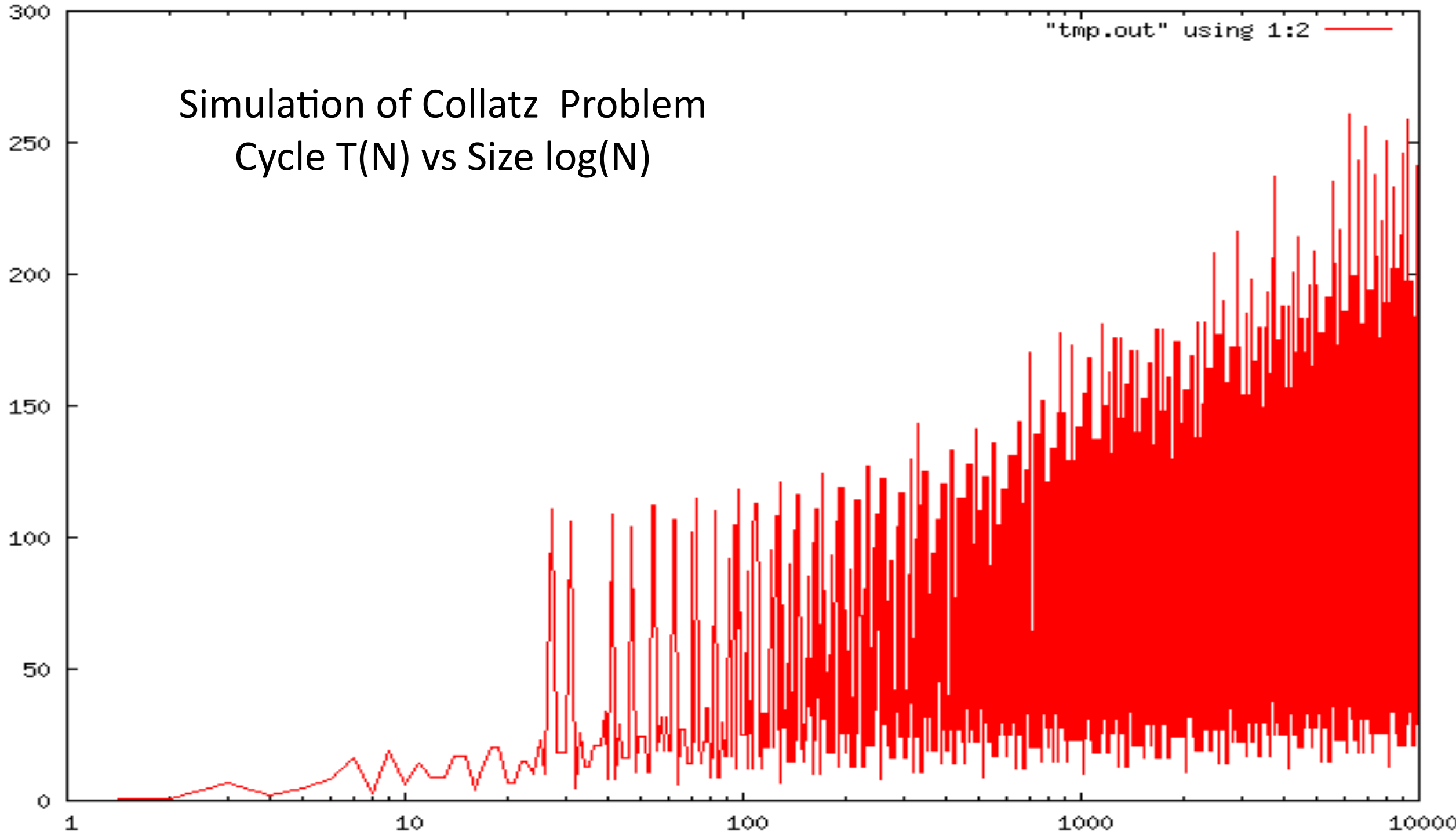
print x

halt!

Simulation of Collatz Problem

Cycle $T(N)$ vs Size $\log(N)$

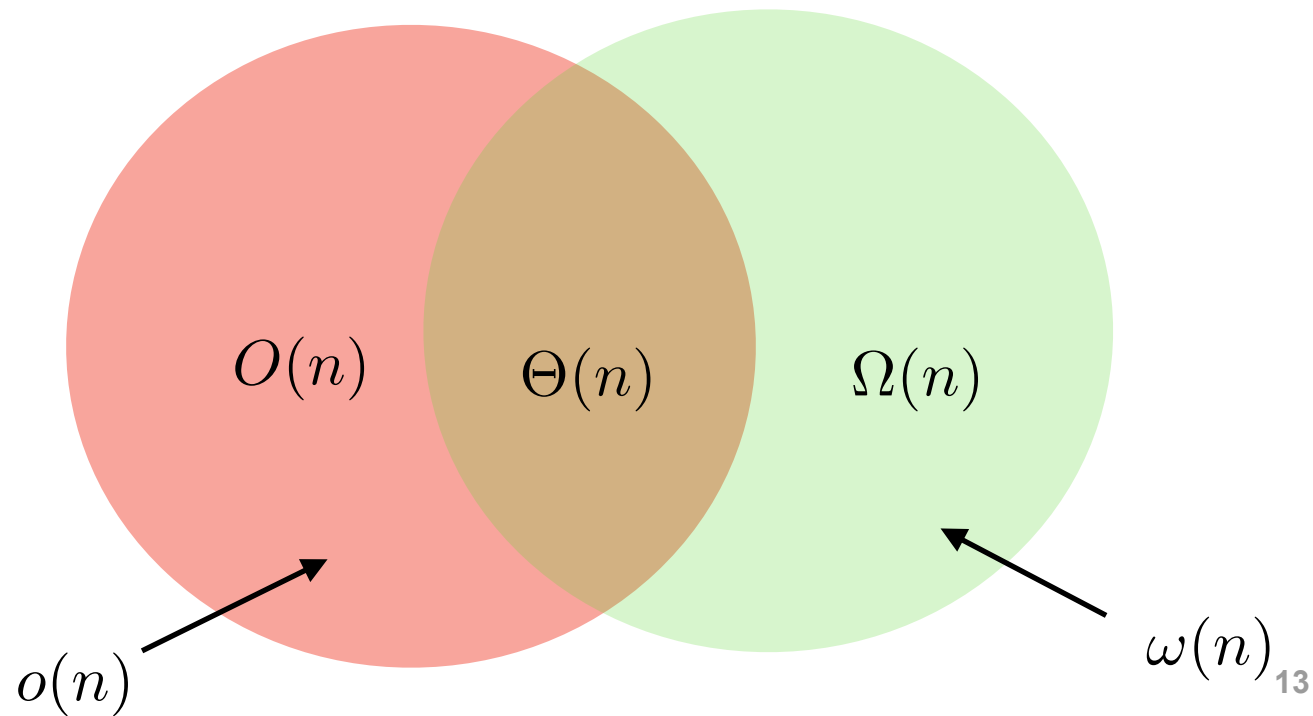
"tmp.out" using 1:2



Growth of Algorithm with Size n

$$T(n) = O(g(n)) \quad \text{or} \quad T(n) \in O(g(n))$$

- $T(n)$ in set $O(g(n))$
 - like $T(n) \leq g(n)$ for large
 - e.g. n^a $\log(n)$ $\exp[n]$ etc.



Why is big-O important?

time

input size

(processor doing ~1,000,000 steps per second)

N	10	20	30	40	50	60
log n	3.3μsec	4.4μsec	5μsec	5.3μsec	5.6μsec	5.9μsec
n	10μsec	20μsec	30μsec	40μsec	50μsec	60μsec
n ²	100μsec	400μsec	900μsec	1.5msec	2.5msec	3.6msec
n ⁵	0.1sec	3.2sec	24.3sec	1.7min	5.2min	13min
3 ⁿ	59msec	48min	6.5yrs	385,500yrs	2x10 ⁸ centuries...	
n!	3sec 7.8x10 ⁸ millennia					

Non polynomial algorithms are terrible!
Logs are great!

All Logarithms are the “same”

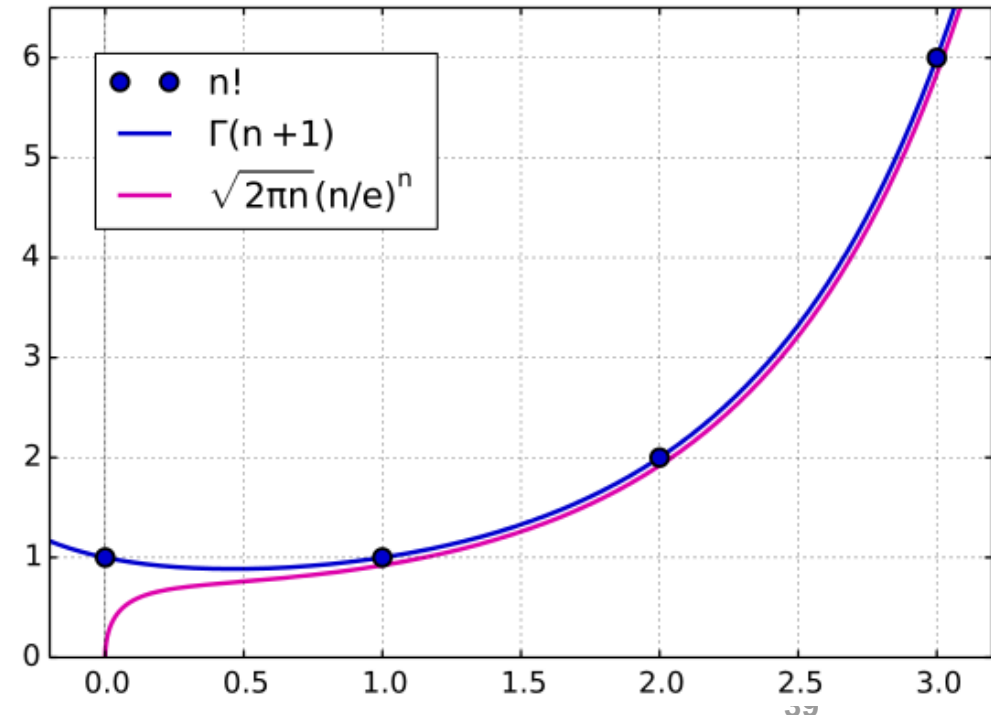
$$N = b^{\log_b(N)}$$

Therefore $\log_a(N) = \log_a(b^{\log_b(N)}) = \log_a(b) \log_b(N)$

Factorial: Worse than Exponential

Decisions $2^D > N!$ implies

$$D > \log_2(N!) \simeq N \log N \dots$$





Rules of thumb

- For polynomials, only the largest term matters.

$$a_0 + a_1N + a_2N^2 + \dots + a_kN^k \in O(N^k)$$

- $\log N$ is in $o(N)$

Proof: As $N \rightarrow \infty$ the ratio $\log(N)/N \rightarrow 0$

- Some common functions in increasing order:

1 $\log N$ \sqrt{N} N $N \log N$ N^2 N^3 N^{100} 2^N 3^N $N!$ N^N

Insertion Sort --- Deck of Cards

- Insertion Sort(a[0:N-1]):
for (i=1; i < n; i++)
 for (j = i; (j>0) && (a[j]<a[j-1]); j--)
 swap a[j] and a[j-1] ;

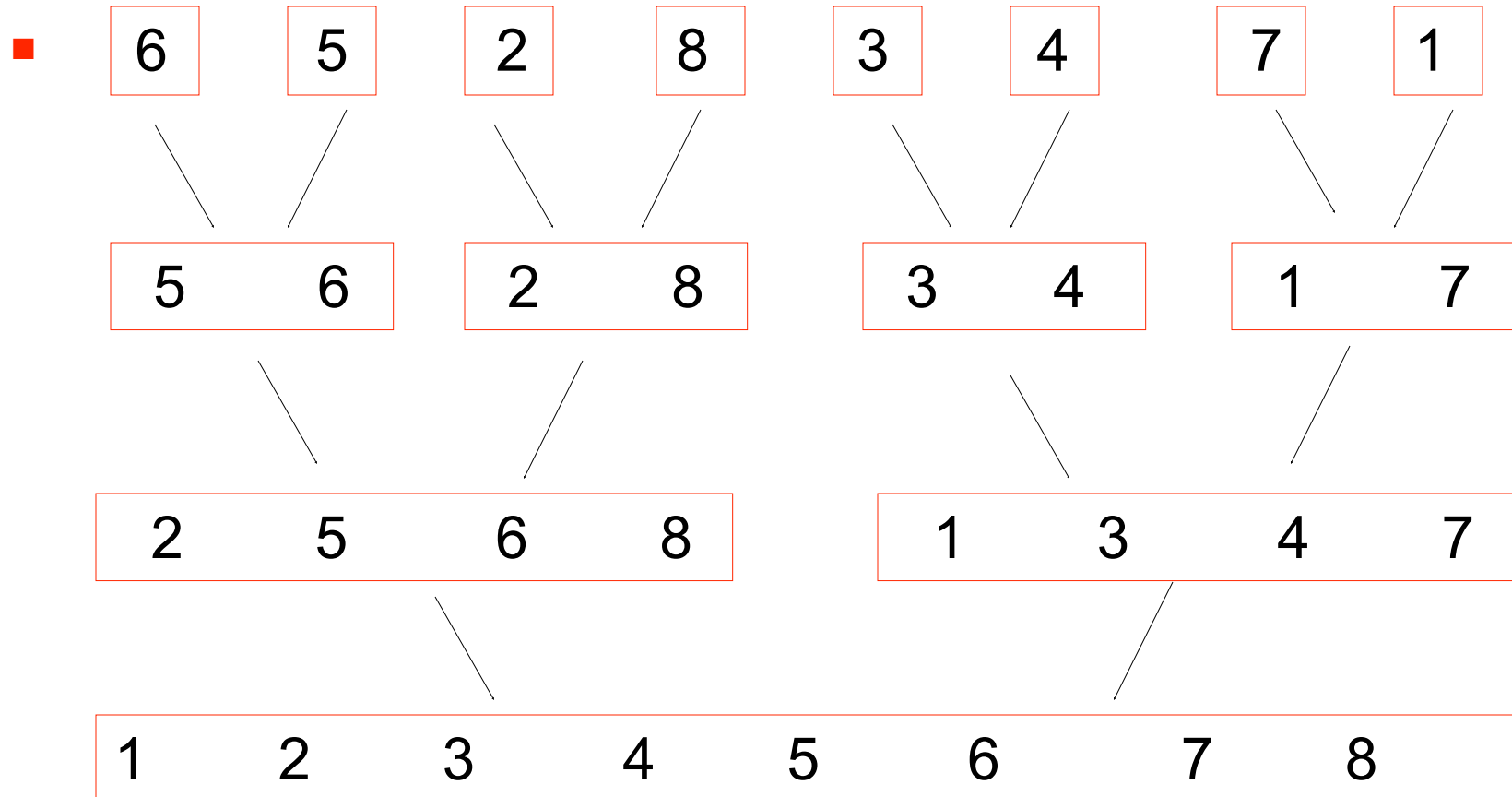
Worst case $\Theta(N^2)$ number of “swaps” (i.e. time)

Outer loop trace for Insertion Sort: $O(n^2)$

■	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	
	(Swaps)								
■	6	5	2	8	3	4	7	1	(1)
	5	← → 6							
■	5	6	2	8	3	4	7	1	
			2	← → 6					
		2	← → 5						
■	2	5	6	8	3	4	7	1	(0)
■	2	5	6	8	3	4	7	1	(3)
■	2	3	5	6	8	4	7	1	(3)
■	2	3	4	5	6	8	7	1	(1)
■	2	3	4	5	6	7	8	1	(7)

Merge Sort - Recursive $O(n \log(n))$

■ $a[0]$ $a[1]$ $a[2]$ $a[3]$ $a[4]$ $a[5]$ $a[6]$ $a[7]$



How do we find $T(n)$? What is big Oh ?

- Count the number of steps:
 - What is a step? RAM serial model.

- Iterative loops: Sum series like

$$\sum_{i=0}^N i^k = 1 + 2^k + 3^k + \dots + N^k \sim O(N^{k+1})$$

but $k = -1 \rightarrow O(\log(n))$

- Solve Recursive Relations:

$$T(n) = a T(n/b) + O(f(n))$$

Sums

- Cases:
$$\sum_{i=1}^N 1 = N \approx \frac{1}{1} N$$
$$\sum_{i=1}^N i = \frac{1}{2} N(N+1) \approx \frac{1}{2} N^2$$
$$\sum_{i=1}^N i^2 = \frac{1}{6} N(N+1)(2N+1) \approx \frac{1}{3} N^3$$
$$\sum_{i=1}^N i^k \approx \frac{1}{k+1} N^{k+1}$$

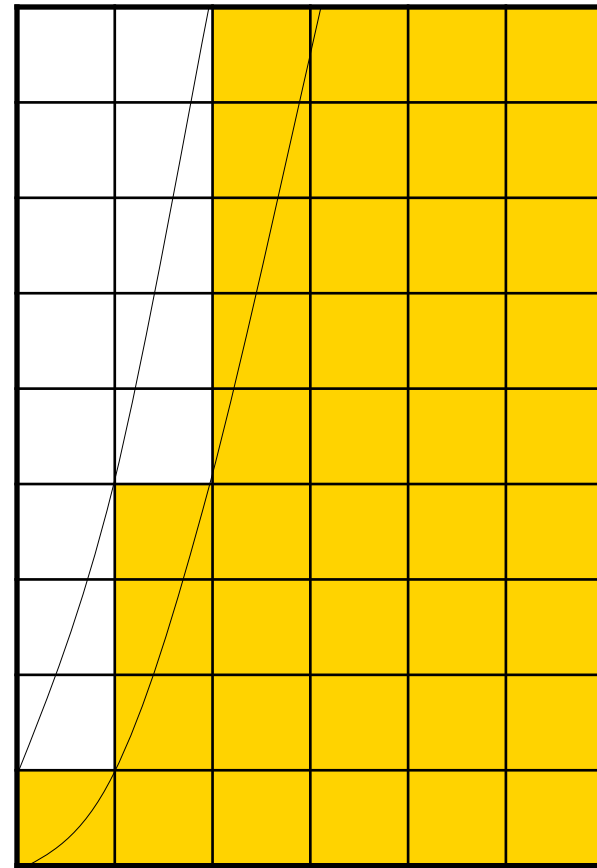
Prove this by Integration:

Estimating Sums

- Integral Bounds:

$$S_k = \sum_{i=1}^N i^k$$

Estimate by integrating $S_k(x) = x^k$



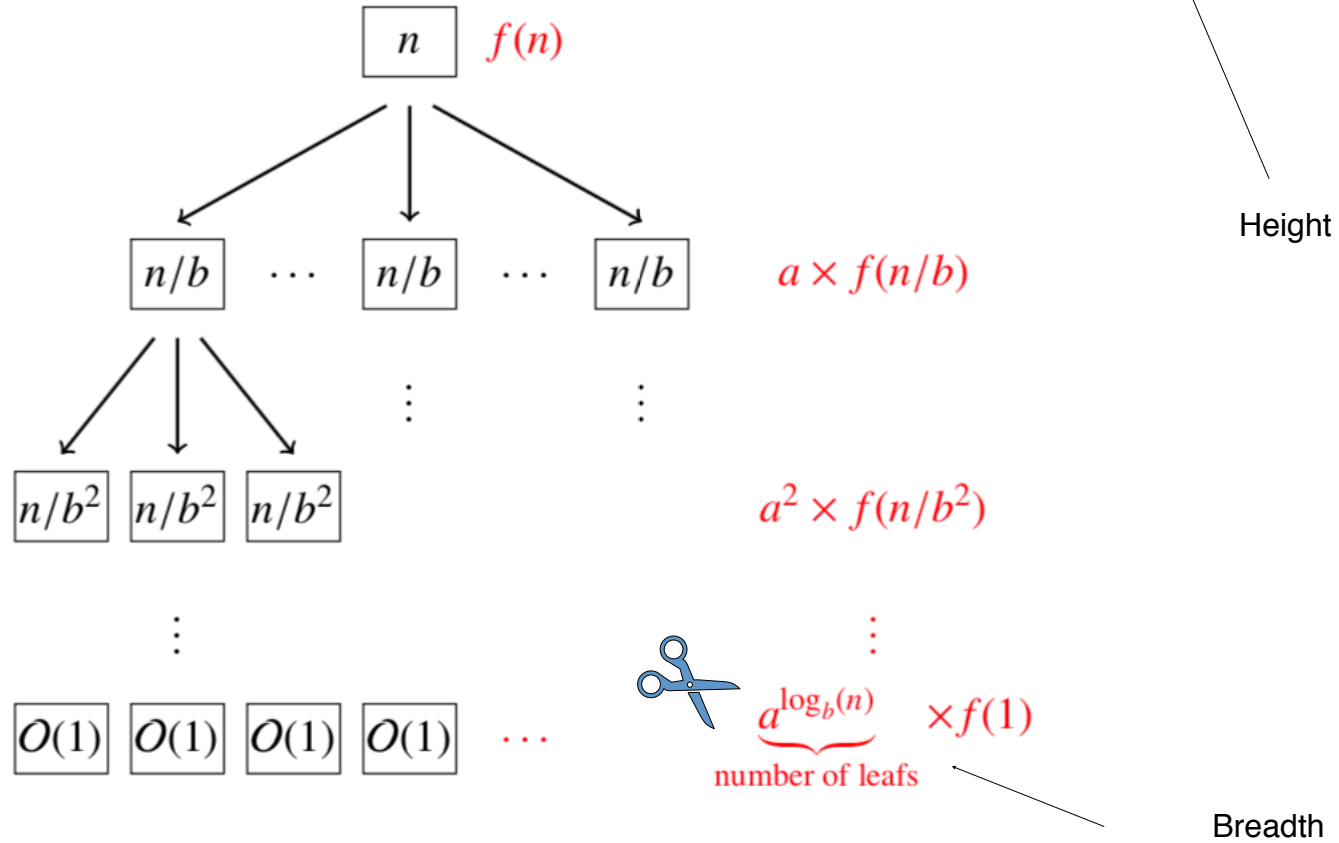
$$\int_0^N x^k dx \leq S_k = \sum_{i=1}^N i^k \leq \int_0^N (x+1)^k dx$$

$$\frac{1}{k+1} N^{k+1} \leq S_k \leq \frac{1}{k+1} ((N+1)^{k+1} - 1)$$

Build Tree to Solve

$$T(n) = aT(n/b) + f(n)$$

$$n/b^h = 1 \implies h = \log_b(n)$$



$$T(n) = f(n) + af(n/b) + \dots + a^{\log_b(n)-1}f(b^2) + a^hT(1)$$

Master Equation (brute force): $T(n) = aT(n/b) + f(n)$

$$T(n) = aT(n/b) + f(n)$$

$$aT(n/b) = a^2T(n/b^2) + af(n/b)$$


$$a^2T(n/b^2) = a^3T(n/b^3) + a^2f(n/b^2)$$

...

$$a^{h-2}T(b^2) = a^{h-1}T(b) + a^{h-2}f(b^2)$$

$$a^{h-1}T(b) = a^hT(1) + a^{h-1}f(b)$$

$$T(n) = a^hT(1) + f(n) + af(n/b) + a^2f(n/b^2) + \dots + a^{h-1}f(b)$$


$$a^h = n^\gamma$$

using: $n/b^h = 1 \implies h = \log_b(n)$

Let's be very careful for $f(n) = cn^k$

$$T(n) = aT(n/b) + cn^k$$

$$aT(n/b) = a^2T(n/b^2) + c an^k / b^k$$

$$a^2T(n/b^2) = a^3T(n/b^3) + c a^2 n^k / b^{2k}$$

...

$$a^{h-2}T(b^2) = a^{h-1}T(b) + c a^{h-2} n^k / b^{(h-2)k}$$

$$a^{h-1}T(b) = a^h T(1) + c a^{h-1} n^k / b^{(h-1)k}$$

Therefore

$$T(n) = a^h T(1) + c n^k \frac{(a/b^k)^h - 1}{a/b^k - 1}$$

$$a^h = n^\gamma \longrightarrow = n^\gamma T(1) + c \frac{n^\gamma - n^k}{a/b^k - 1}$$

$$\text{since } 1 + a/b^k + (a/b^k)^2 + (a/b^k)^3 + \dots + (a/b^k)^{h-1} = \frac{(a/b^k)^h - 1}{a/b^k - 1}$$

Master Equation:

$$T(N) = a T(N/b) + \Theta(g(N))$$

Theorem: The asymptotic Solution is:

- $T(N) \in \Theta(N^\gamma)$ if $g(N) \in O(N^{\gamma-\epsilon}) \forall \epsilon > 0$
- $T(N) \in \Theta(g(N))$ if $g(N) \in \Omega(N^{\gamma+\epsilon}) \forall \epsilon > 0$
- $T(N) \in \Theta(N^\gamma \log(N))$ if $g(N) \in \Theta(N^\gamma)$

where $a = b^\gamma$ or $\gamma = \log(a)/\log(b)$

L'Hospital's Rule

Limit for ratio is same as for ratio of derivatives!

$$\lim_{N \rightarrow \infty} \frac{f(N)}{g(N)} = \lim_{N \rightarrow \infty} \frac{\frac{df(N)}{dN}}{\frac{dg(N)}{dN}}$$

e.g. $\lim_{N \rightarrow \infty} \frac{\log^2(N)}{N} =$
 $\lim_{N \rightarrow \infty} \frac{2 \log(N)/N}{1} = \lim_{N \rightarrow \infty} \frac{2/N}{1} = 0$

$$\gamma - k \rightarrow 0, \quad \text{where} \quad a = b^\gamma$$

$$T(N) = N^\gamma T(1) + c_0(N^\gamma - N^k)/(a/b^k - 1)$$

$$T(N) = N^\gamma T(1) + c_0 N^k \frac{N^{\gamma-k} - 1}{b^{\gamma-k} - 1}$$

Take derivative with respect to $x = \gamma - k$



$$T(N) = N^\gamma T(1) + c_0 N^k \log(N)/\log(b)$$

More useful stuff

- Logarithmic sum (Harmonic Series):

$$H_N = \sum_{n=1}^N \frac{1}{n} = \ln(N) + \gamma_{Euler} + \Theta(1/N)$$

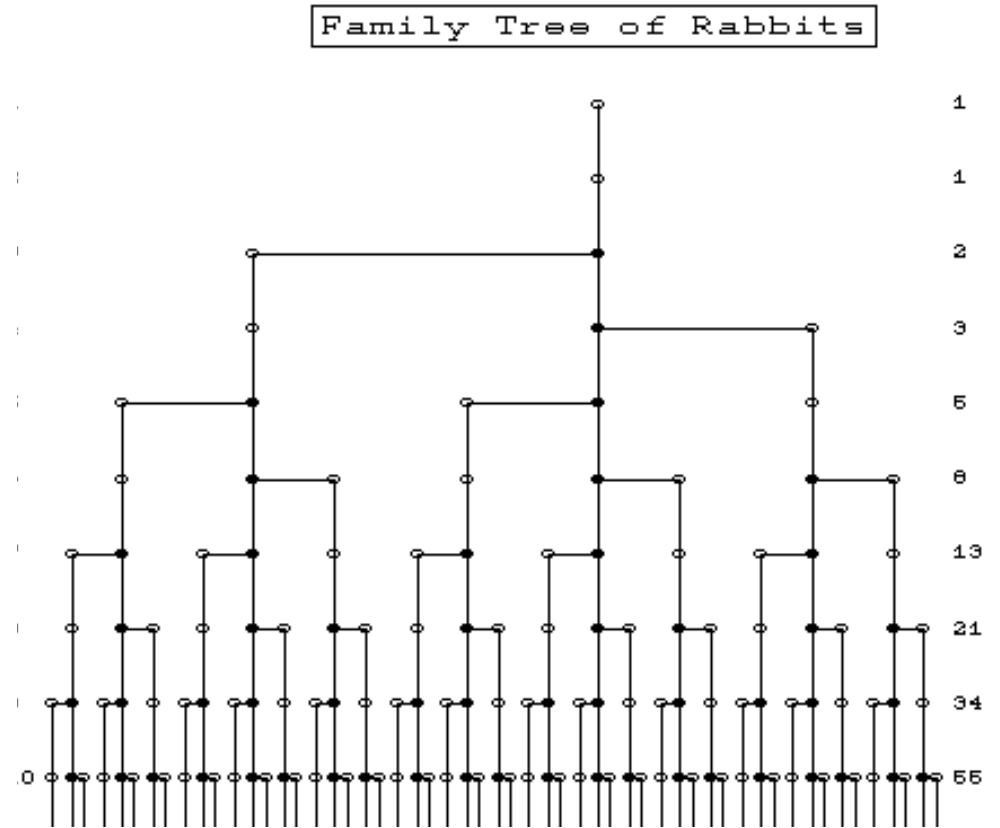
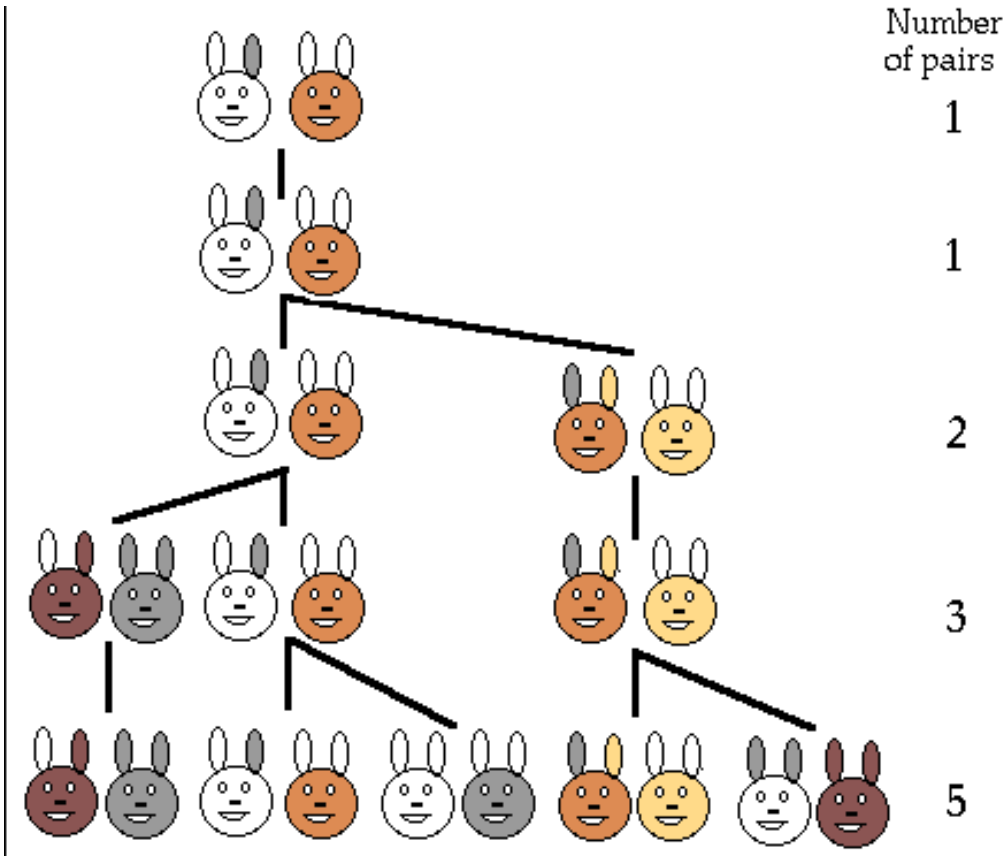
$$\gamma_{Euler} = 0.577215664901532860606512090082$$

■ Stirling's Approx: $N! \simeq \sqrt{2\pi N} N^N e^{-N} (1 + O(1/N))$

$$\log(N!) = N \log(N) - N \log(e) + \frac{\log(2\pi N)}{2} + \Omega(1/N)$$

Rabbits

Fibonacci investigated (in the year 1202) was about how fast rabbits could breed in ideal circumstances.
 Females take one month to mature: Pairs mate and produce a male and female in a month



Fibonacci: $F_k = F_{k-1} + F_{k-2} \Rightarrow 0, 1, 1, 2, 3, 5, 8, \dots$

Characteristic equation, try:

$$F_k = \phi^k \implies \phi^k = \phi^{k-1} + \phi^{k-2}$$

$$\phi^2 - \phi + \phi = 0 \quad \phi = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$$

$$F_k = c_1 \left(\frac{1 + \sqrt{5}}{2} \right)^k + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^k$$

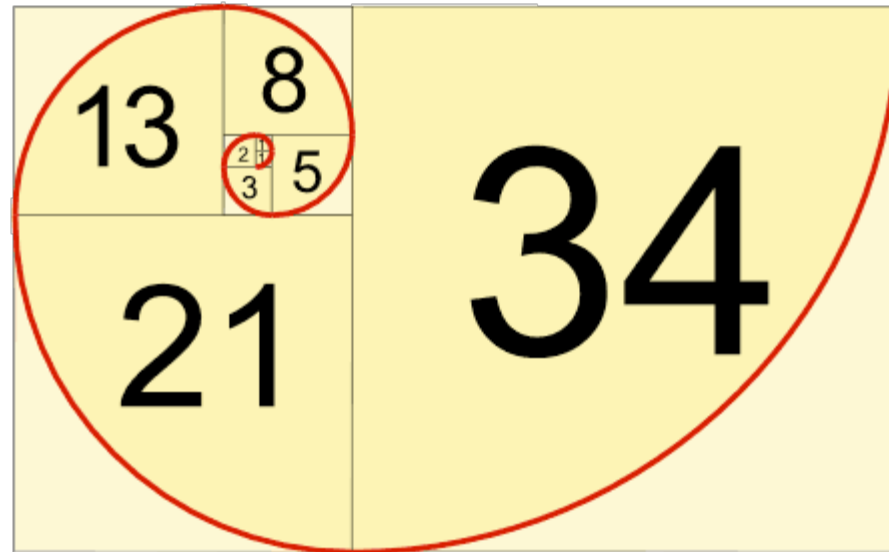
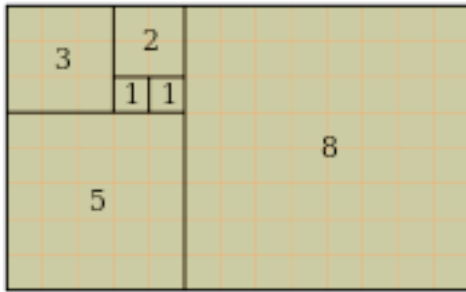
$$F_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right]$$

$$ax^2 + bx + c = 0 \implies x = -b/2a \pm \sqrt{(b/2a)^2 - c/a} \quad 42$$

Fibonacci: $F(N) = F(N-1) + F(N-2)$ →

0,1,1,2,3,5,8 , for $N = 0,1,2,3,....$

- Many examples in nature!



Rabbits, Bees and Double Window Panes

Maximum Subsequence Sum: CLRS 4.1

- Given $a[0], a[1], \dots, a[N-1]$ find max

$$\sum_{k=i}^j a[k]$$

– Dumbest

$$O(N^3)$$

– Dumb

$$O(N^2)$$

– Smart

$$O(N \log(N))$$

– Smartest

$$O(N)$$

```

for ( i = 0; i < N; i++)
    for(j = i; j < N; j++)
    { Sum = 0;
      for(k=i; k<j+1; k++)
          Sum += a[k];
      if(Sum > MaxSum)
          MaxSum = Sum;
    }

```

$$O(N^3)$$

$$\sum_{k=i}^j 1 = j - i + 1$$

$$\sum_{j=i}^{N-1} (j - i + 1) = \frac{1}{2}(N - i)(N - i + 1)$$

$$\frac{1}{2} \sum_{i=0}^{N-1} i(i + 1) = \frac{1}{6}(N^3 + 3N + 2N)$$

```

for ( i = 0; i < N; i++)
{ Sum = 0;
  for(j=i; j<N; j++)
      Sum += a[j];
  if(Sum > MaxSum)
      MaxSum = Sum;
}

```

$$O(N^2)$$

$$\sum_{j=i}^{N-1} 1 = N - 1 - i + 1 = N - i$$

$$\sum_{i=0}^{N-1} (N - i) = N^2 - \left(\frac{N(N-1)}{2}\right)$$

$$= \frac{1}{2}(N^2 + N)$$

Recursion versus Single Pass

- $T(N) = 2 T(N/2) + c N$
 - Large left/right + sum to left and right for split screen.

$$O(N \log(N))$$

$$O(N)$$

- On line:
 - Quit when you are in debt and start over.

```
Sum = 0;
for(j=0; j<N; j++){
    Sum += a[j];
    if(Sum > MaxSum)
        MaxSum = Sum
    else if (Sum < 0)
        Sum = 0;
}
```