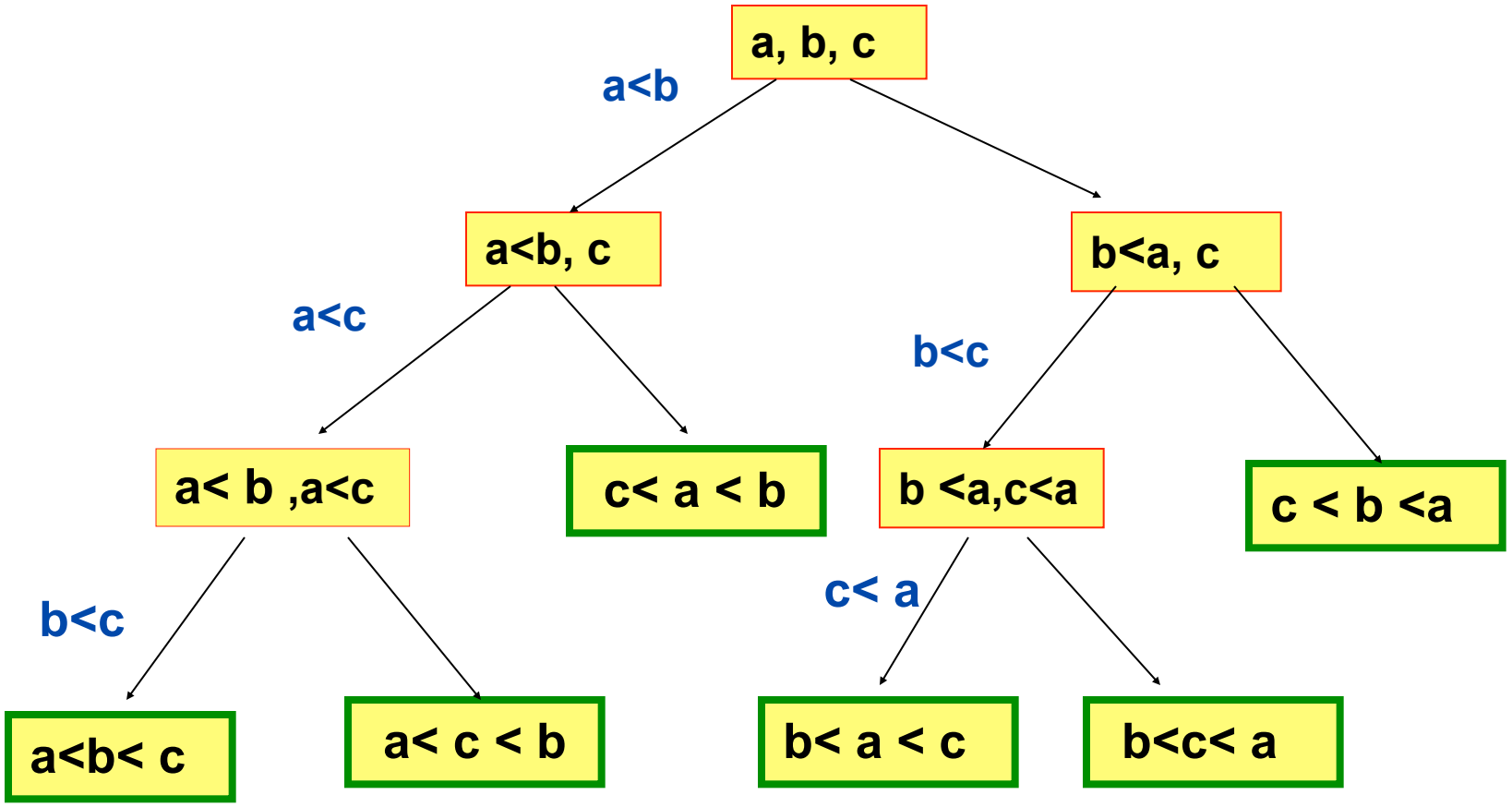
Searching and Sorting

- Searching
  - Linear O(N)
  - bisection O(log(N))
  - dictionary O(log(log(N
- Sorting
  - Insertion, bubble, selection $O(N^2)$ CRLS: 2.1
  - Merge, Quick, Heap O(N log (N)) CRLS: 2.2,
    Proof: $\Omega(N^2)$ near neighbor exchange
- Proof: $\Omega(N \log(N))$ Comparison search
- Median (or k quick selection) Problem CLRS: 9
  - Bin (Count), Radix, Bucket O(N) CLRS: 8
-

Decision Tree

Proof of $\Omega(N\log(N))$



Binary decisions: 3! = 6 possible outcomes. Longest path: log(3!)

Searching: *"Why Sort at All?"*

- int a[0], a[1],a[2],a[3],.... a[m],.... a[2],a[N-1]

Three Algorithms:

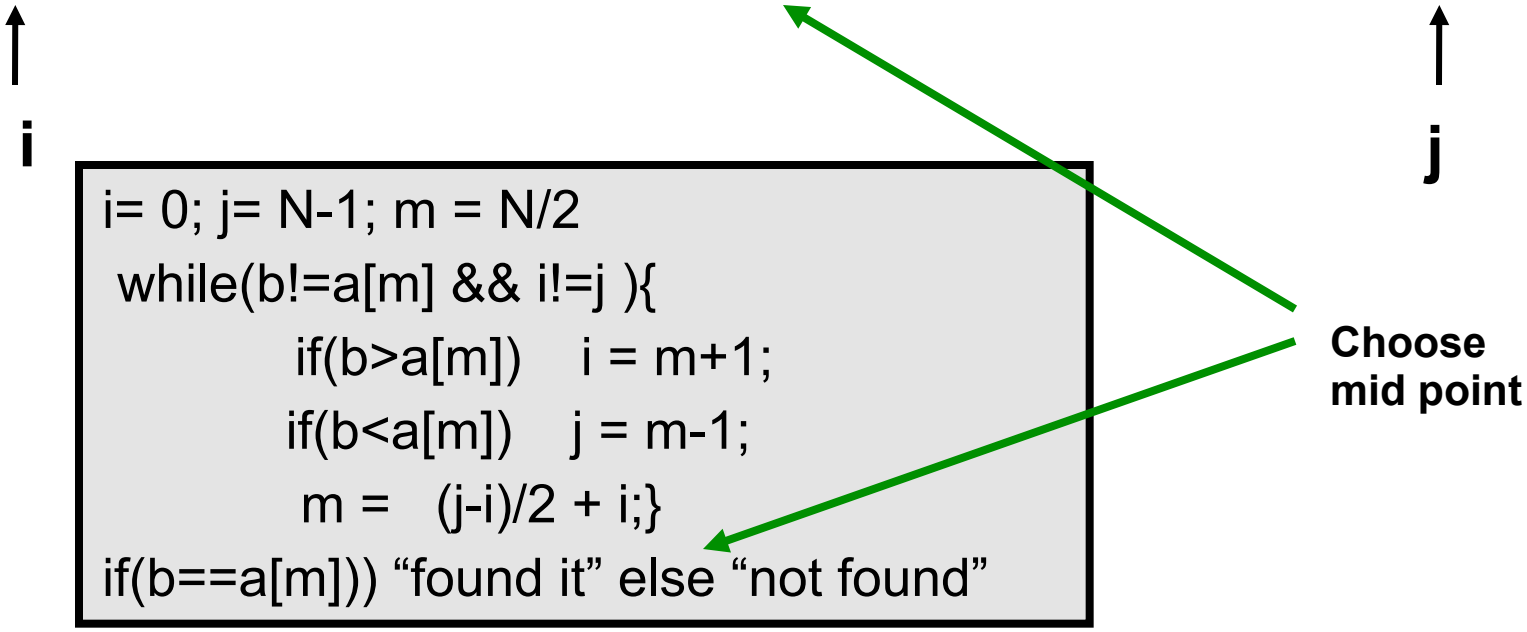
- *Linear Search* ➔ O(N)

  (after Sorting)

- *Bisection Search* ➔ O(log(N)).
- *Dictionary Search* ➔ O(log[log[N]])

Bisection Search of Sorted List

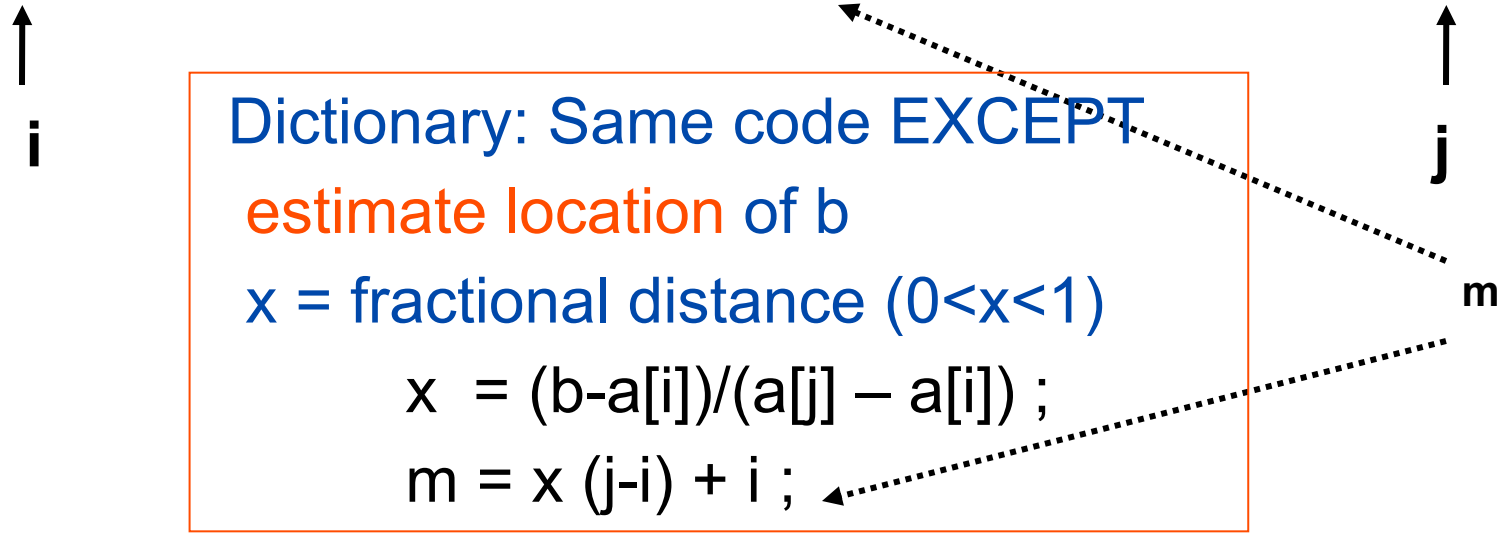
- int a[0], a[1],a[2],a[3],.... a[m],.... a[N-2],a[N-1]



$T(N) = T(N/2) + c_0 \rightarrow T(N) \gg Log(N)$

Dictionary: Sorting a nearly Uniform Sequence

- int a[0], a[1],a[2],a[3],.... a[m],.... a[2],a[N-1]



$$N \to N^{\frac{1}{2}} \to N^{\frac{1}{4}} \to N^{\frac{1}{8}} \cdots \to N^{\frac{1}{2^n}} = 1 \quad \text{or} \quad n = log_2(log_2(N)$$

$$T(N) \simeq T(N^{1/2}) + c_0 \text{ or with } N = 2^n \ , T(n) = T(n/2) + c_0$$
$$\text{Master Equ.} \implies T(n) = O(\log(n)) = O(log(log_2(N)$$

- *Extra Knowledge Helps: % Error » $1/N^{1/2}$*

Insertion Sort --- Deck of Cards

- Insertion Sort(a[0:N-1]):

```
for (i=1;  i < n;  i ++)
        for (j = i;  (j>0) && (a[j]<a[j-1]);  j--)
                swap a[j] and a[j-1] ;
```

Worst case $\Theta(N^2)$ number of "swaps" (i.e. time)

Outer loop trace for Insertion Sort

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	(Swaps)
6 |	5	2	8	3	4	7	1	(1)
5 ←	→ 6							
5	6 |	2	8	3	4	7	1	(2)
	2 ←	→ 6						
2 ←	→ 5							
2	5	6 |	8	3	4	7	1	(0)
2	5	6	8 |	3	4	7	1	(3)
2	3	5	6	8 |	4	7	1	(3)
2	3	4	5	6	8 |	7	1	(1)
2	3	4	5	6	7	8 |	1	(7)

Bubble Sort --- Sweep R to L

- *Bubble Sort(a[0:N-1]):*

```
for i=0 to n-1
        for j = n-1 to i + 1
                if a[j]<a[j-1] then
                        swap a[i] and a[j]
```

Worst case $\Theta(N^2)$ swaps (time)

Outer loop trace for Bubble Sort

a[0] (Swaps)	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	
6	5	2	8	3	4	7	1	(7)
1 |	6	5	2	8	3	4	7	(3)
1	2 |	6	5	3	8	4	7	(3)
1	2	3 |	6	5	4	8	7	(3)
1	2	3	4 |	6	5	7	8	(1)
1	2	3	4	5 |	6	7	8	(0)
1	2	3	4	5	6 |	7	8	(0)
1	2	3	4	5	6	7	8	(17)

- NOTE SAME # OF SWAPS? WHY?

Average Number of N(N-1)/4 swaps

- *Best Case: sorted order ➔ 0 swaps*
- *Worst Case: reverse order ➔ N(N-1)/2 swaps*

  since 1 + 2 + ...+N-1 = N(N-1)/2
- *Average Case: Pair up each of the N! permutations with its reverse order ➔ Every pair must swap in one or the other: Thus average is half of all swaps ➔ (1/2) N(N-1)/2 q.e.d.*

Selection Sort --- (Bubble only the index)

- *Selection Sort(a[0:N-1]):*

```
for i=1 to n-2
    {  min = i
       for j = n-1 to i + 1
              if a[j]<a[min] then
                     min = j;
          swap a[i] and a[min];
       }
```

worst case $\Theta(N)$ swaps + $\Theta(N^2)$ comparisons

Outer loop trace for Selection Sort

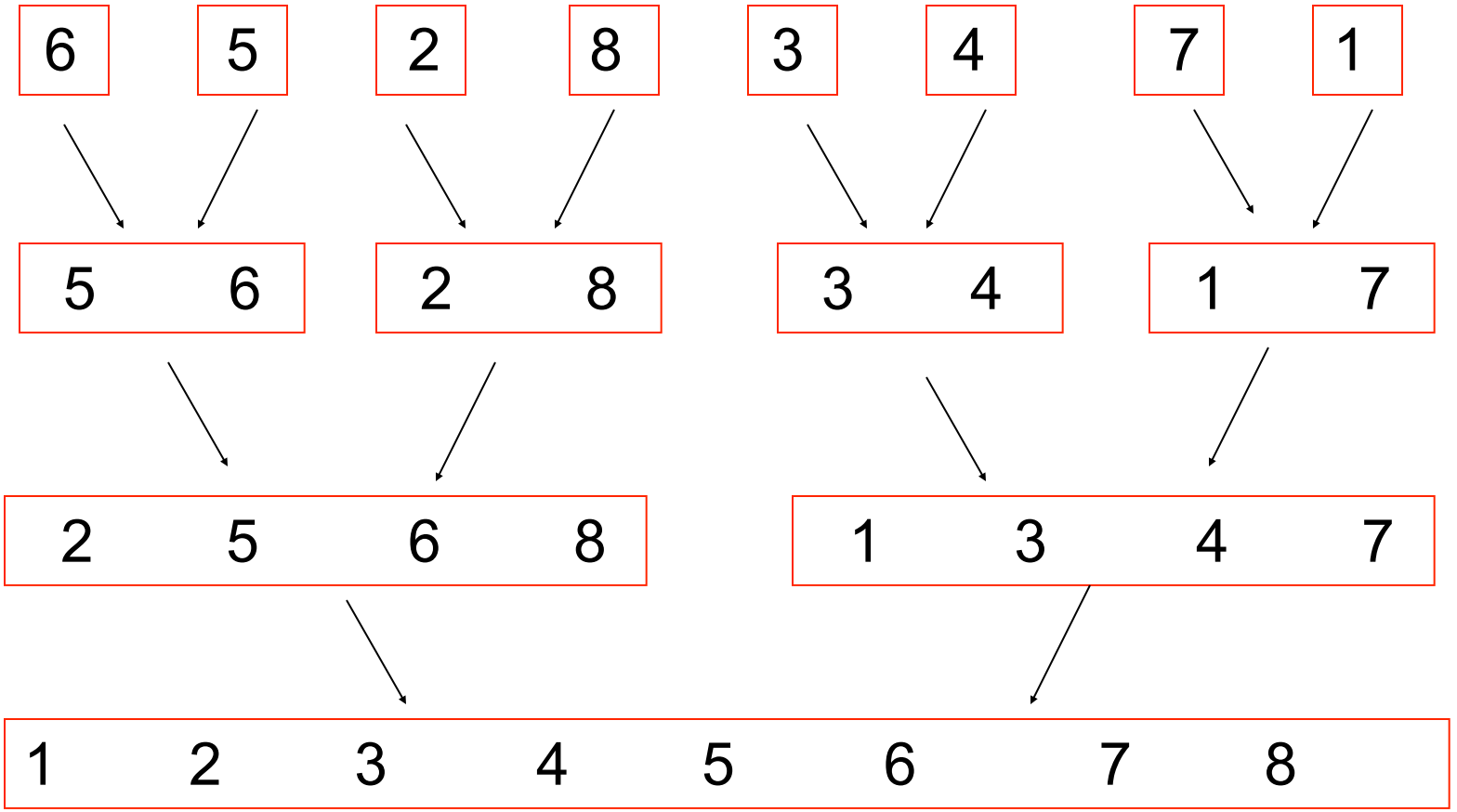
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
(Swaps)							
6	5	2	8	3	4	7	1
(1)							
1 ←						→	6
1	5	2	8	3	4	7	6
(1)							
	2 ←→	5					
1	2 |	5	8	3	4	7	6
(1)							
1	2	3 |	8	5	4	7	6
(1)							
1	2	3	4 |	5	8	7	6
(0)							
1	2	3	4	5 |	6	7	8

Merge Sort: Worst Case Θ(Nlog(N))

```
void mergesort(int a[ ], int l, int r)
    if (r > l)  {
        m = (r+l)/2;
        mergesort(a, l, m);
        mergesort(a, m+1, r);
        for (i = l; i < m+1; l++)  b[i] = a[i];
        for (j = m; j < r; j++)    b[r+m-j] = a[j+1];   // reverse
        for (k = l; k <= r; k++)
                a[k] = (b[i] < b[j]) ? b[i++] : b[j--]; }
```

Outer loop trace for Merge Sort

- a[0] a[1] a[2] a[3] a[4] a[5] a[6] a[7]
- 6 5 2 8 3 4 7 1



Lower Bound Theorem for Camparision Sort

Proof: Compute the maximum depth D of decision tree?

- Need N! leaves to get all possible outcomes of a sorting routine.
- Each level at most doubles: $1 \to 2 \to 4 \to 8 \to \bullet\bullet\bullet \to 2^D$
- Consequently for D levels: $N! \le 2^D \Rightarrow D \ge log_2(N!)$

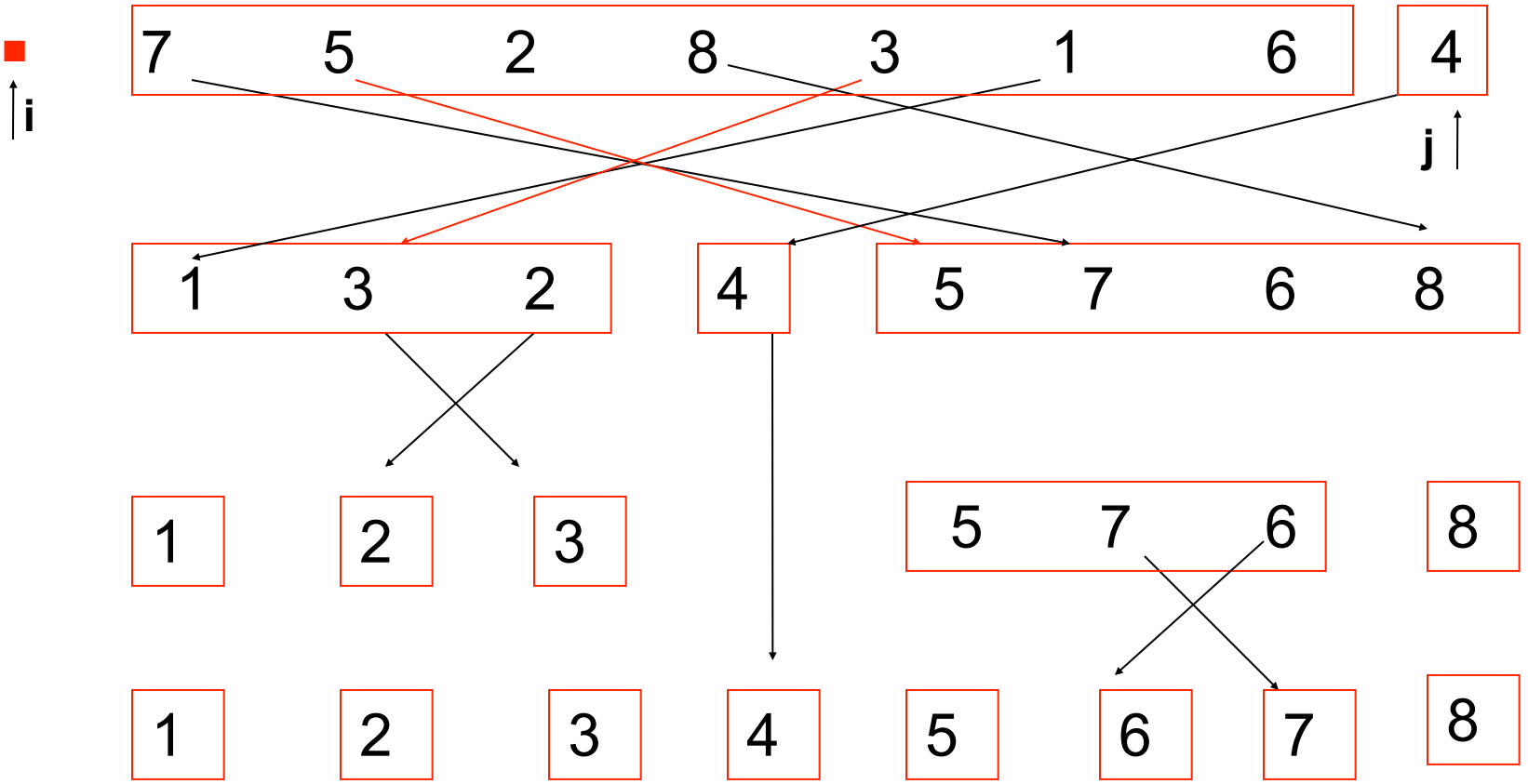
$$\Rightarrow T(N) = \Omega(D) = \Omega(\log_2(N!)) = \Omega(N \log_2(N))$$

$$Information = \log_2(N!) \cong N \log_2(N)$$

Number of bits to encode any (initial) state is information (- Entropy)

Outer loop trace for Quick Sort (i moves before j)

- a[0] a[1] a[2] a[3] a[4] a[5] a[6] a[7]



Quick Sort: original Hoare partition

```
void quicksort(int a[ ], int l, int r)
  if (r > l){
      v = a[r]; i = l-1; j = r;
      for (;;){  while (a[++i] < v);       // move first i to right
                 while (a[--j] > v) ;       // then mover j left
                  if (i >= j) break  ;
                 swap(&a[i],  &a[j]); }
       swap(&a[i], &a[r]);                 // move pivot in to center
       quicksort(a, l, i-1);
       quicksort(a, i+1, r);
    }
```

Quick Sort: My Implementation

```
void quicksort(int a[], int l, int r)
  {  int i, j, vl
    if (r > l)
      {   v = a[r]; i = l-1; j = r;
        for (;;)
          {   while (a[++i] < v && i >= j ) ;  // Move i first from left
              while (a[--j] >= v && i >= j) ;  // Move j second from right
              if (i >= j) break;
              swap(&a[i], &a[j]);   } //  swap root to divide left and right sublist
        swap(&a[i], &a[r]);
        quicksort(a, l, i-1);     quicksort(a, i+1, r);
      }   }
```

Quick Sort: CLRS:7 Implementation

```
void quicksort(int a[ ], int l, int r)
  if (r > l){
      v = a[r]; i = l-1;
      for (int j= l ; j<r ; j++){
          if (a[j] <= v) {
                i +=1 ;                          // move first i to right
               swap(&a[i],  &a[j])  }
        swap(&a[i], &a[r]);                      // move pivot in to center
        quicksort(a, l, i-1);
        quicksort(a, i+1, r);
    }
```

Very cute: See CLRS page 172 Figure 7.1

- Worst Case (choose smallest as pivot!):
  - T(N) = T(N-1) + c N ➔ T(N) =O(N^2)
- Best Case:(choose median as pivot)
  - T(N) = 2 T(N/2) + cN ➔ T(N) = O(N log(N))
- Average Case:
  - T(N) = 2[T(0) + T(1) + …+ T(N-1)]/N + c N
    - ➔ T(N) = O(N log (N))
    - ➔ Using Calculus if you are lazy! (x = N)

$$xT(x) \simeq 2\int_0^x dyT(y) + cx^2 \quad x\frac{dT(x)}{dx} = T(x) + 2cx \Rightarrow T(x) = 2cx\log(x)$$

Can do better-- Worst Case O(N) !

- *Approximate Media Selector for pivot v:*
  - *Partition in 5 rows of N/5*
  - *Sort each column*
  - *Find (Exact) Medium of Middle list!*
- *Result there are (3/5)(1/2)N elements smaller than pivot v*
- *K-th find is O(N) --- Double recursion!*
  - *Sort of N/5 col O(N)*
  - *Find media of T(N/5)*
  - *Find k-th in T(7N/10) at worst*
- *T(N) < C0 * (N/5) + T(N/5) + T(7N/10)*
  - *Try solution: T = C N*
  - *C(N – N/5 – 7N/10) = C N/10 < C0 N/5*
  - *C < 2 C0*

Median Finding: Quick Select

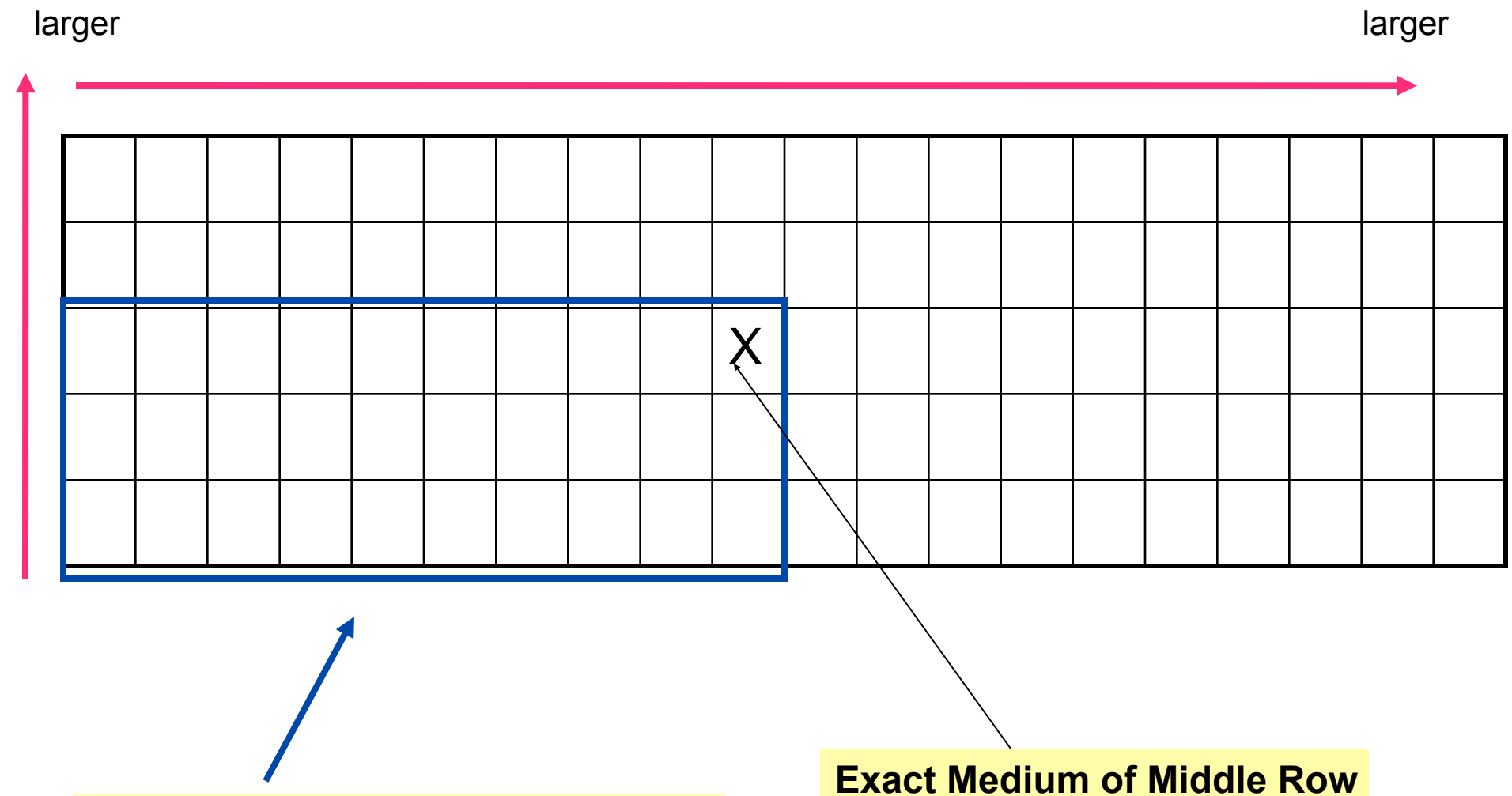
- Median is the element a[m] so that half is less/equal
- Generalize to finding k-th smallest in set S
- Quick(S,k): |S| = size of S
  - If |S| = 1, the k = 1 in S
  - Pick pivot v 2 S & Partition S – {v} into S_L & S_H
    - If k< $|S_L|$ + 1 then k-th 2 S_L: Quick(S_L,k)
    - If k = $|S_L|$ + 1 k-th is v : exit
    - If k > $|S_L|$ + 1 then k-th 2 S_R: Quick(S_R,k- $|S_L|$-1)

Now: T(N) = O(N) is average performance

T(N) = [T(0) + T(1) + … T(N-1)]/N + c N

$$xT(x) \simeq \int_0^x dy T(y) + cx^2 \quad x\frac{dT(x)}{dx} = 2cx \Rightarrow T(x) = 2cx$$

5 row of N/5 Columns



Beating N log N: Radix Sort (IBM Card Sorter!)

- Represent integers in a[i] in base B: $n_0 + n_1 B + n_2 B^2 + ... + n_p B^P$
- Sort into buckets by low digits first: n_0, then n_1, etc.

Queues: B= 10 Example: 64, 8, 216, 512, 27, 729, 0, 1, 343, 125

Bucket:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
#1	0	1	512	343	64	125	216	27	8	729
	8 1 0	216 512	729 27 125		343		64			
#3	64 27 8 1 0	125	216	343		512		729		

O(N P) where $B^P = N$ or $P = \log(N) / \log(B) = O(1)$