MATH 100 Practice Exam 1

Duration: 2.5 hours

Student number: 4 2 6 7 2 8 2 4

First name: Fawaz

Last name: Farookh

Signature: Fawaz

Lecture (large class) section number: **1**

Discussion (small class) section number:

For examiners' use only

Section 1											
Question	1	2	3	4	5	6	7	8	9	10	Total
Points	5	5	5	5	5	5	5	5	5	5	50
Score											

Section 2					
Question	11	12	13	14	Total
Points	14	10	14	12	50
Score					

Overall exam score	

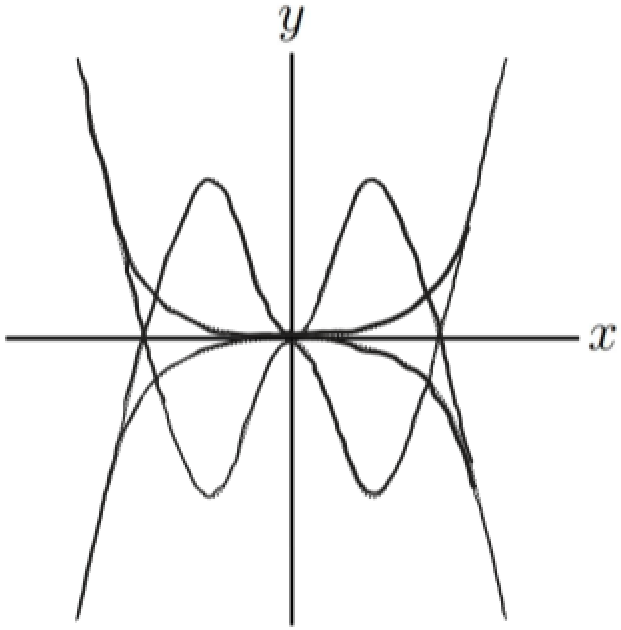
Rules governing UBC examinations:

1. Each candidate must be prepared to produce, upon request, a UBC card for identification.
2. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
3. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
    - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
    - (b) Speaking or communicating with other candidates;
    - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
4. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
5. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Additional rules governing this examination:

1. This is a closed-book exam.
    - (a) Calculators and other calculating devices may not be used.
    - (b) Notes may not be used.
    - (c) Watches must be removed and taken off the table.
2. If an answer box is provided, you must write down your answer (but not its justification) in the box.
3. Answers must be simplified and calculator-ready. For example, write $\log\left(e^{\sqrt{2}}\right) = \sqrt{2}$, but do not write $\sqrt{2} \approx 1.414$.
4. You must justify your answers unless an explicit exception is made.
5. You may use any result proven in class or on assignments.

1. **[5 marks]** Trace the graph of $f(x) = 5x^3 - 5x^4$ on the axes below, using the dotted curves.



Student number: 42672824

2. **[5 marks]** Find all the horizontal and vertical asymptotes of $f(x) = \dfrac{x+2}{\sqrt{4x^2+3x+2}}$.

$$H \cdot A = \frac{n}{\sqrt{4n^2}} \Rightarrow \frac{1}{2}$$

$H \cdot A = \frac{1}{2}$ | $V \cdot A =$ D·N·E

Does not Exist

$$\left(\sqrt{4n^2+3n+2}\right)^2 = 0^2 \qquad V \cdot A = \text{DNE}$$

$$4n^2+3n+2 = 0$$

Answer:

Horizontal = $\frac{1}{2}$, Vertical = DNE

Student number: 4267282

3. **[5 marks]** Find all the values of m such that

$$f(x) = \begin{cases} 6x^3 - 2m & \text{if } x \leq -1 \\ 2x^2 + 5m & \text{if } x > -1 \end{cases}$$

is continuous.

$$f(x) = \begin{cases} 6x^3 - 2m \\ 2x^2 + 5m \end{cases}$$

$$6x^3 - 2m = 2x^2 + 5m$$

$$\Rightarrow \quad 6x^3 - 7m - 2x^2 = 0$$

$$\Rightarrow \quad 6(-1)^3 - 7m - 2(-1)^2 = 0$$

$$\Rightarrow \quad -6 - 7m - 2 = 0$$

$$\Rightarrow \quad -7m = 8$$

$$\boxed{m = \frac{8}{-7}}$$

Answer:

$$m = \frac{8}{-7}$$

Student number: 42672824

4. **[5 marks]** Let $f(x) = \frac{5}{x}$. Use a definition of the derivative to find $f'(x)$. No credit will be given for solutions using differentiation rules, but you can use those to check your answer.

$$f(x) = \frac{5}{x} \qquad f'(x) = \frac{-5}{x^2}$$

$$f'(x) = \frac{(x+h) - f(x)}{h}$$

$$= \frac{\frac{5}{x+h} - \frac{5}{x}}{h} \qquad \frac{-5}{x^2}$$

$$\Rightarrow \frac{\cancel{5x} - \cancel{5x} - 5h}{x(x+h)} \rightarrow \frac{\frac{-5h}{x(x+h)}}{\cancel{h}} \Rightarrow \frac{-5}{x(x+h)}$$

Answer: $\frac{-5}{x^2}$

Student number: 42672824

5. **[5 marks]** Suppose the function $f(x)$ is defined and continuous on all real numbers, and that

$$\lim_{h \to 0} \frac{f(5+h)-3}{h} = -5.$$

Given this information, we can find the equation of the tangent line to the curve $y = f(x)$ at a particular point. What is the point, and what is the slope of the tangent line to the curve $y = f(x)$ at that point?

$$\lim_{h \to 0} \frac{f(5+h)-3}{h} = -5$$

$m = -5$

$f(5) = 3$

$(5,3)$

Answer:

$(5,3)$ $m = -5$

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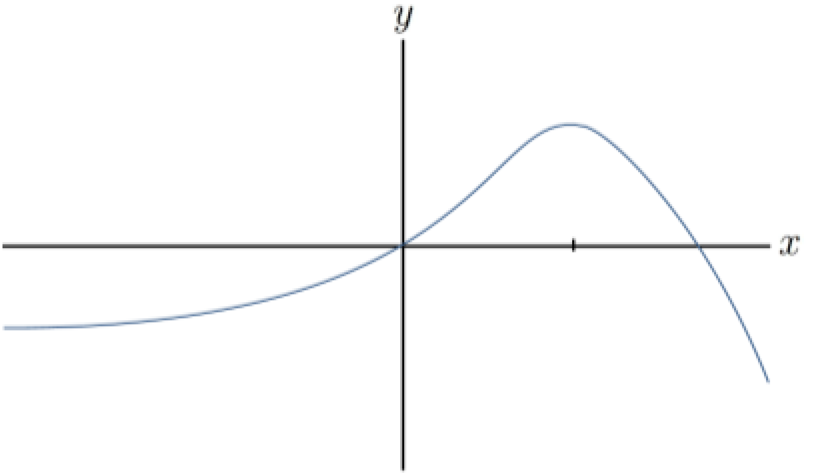
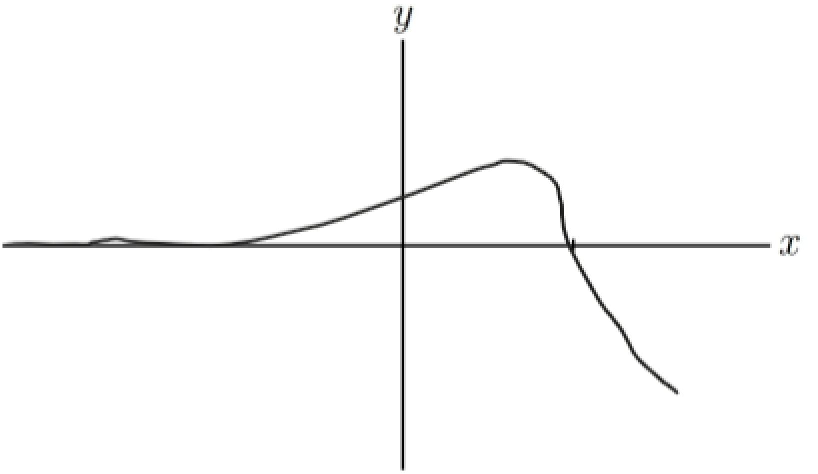
6. **[5 marks]** Pictured below is the graph of a function. On the blank set of axes, sketch the graph of its derivative.





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7. **[5 marks]** Find values for a and b such that such that the line $\underline{y = \overset{m}{2}x - 2}$ is $\underline{\text{tangent}}$ to the curve $\underline{y = ax^2 + b}$ at the point $(1, 0)$.

$y' = 2ax$

$x = 1$

$2a = 2$

$\boxed{a = 1}$

$0 = 1 + b$

$\boxed{b = -1}$

Answer:

$a = 1$ $b = -1$

Student number: 4267284

8. **[5 marks]** Let $f(x) = \frac{4x^3}{5\cos(x)}$. Find $f'(\pi)$.

$$f'(x) = \frac{4x^3 \times 5\cos(x) - 4x^3\, 5\cos(x)y'}{(5\cos(x))^2}$$

$$\rightarrow \boxed{f'(x) = \frac{12x^2\cos(x) + 4x^3\sin x}{5\cos(x)^2}}$$

$$\rightarrow f'(\pi) = \frac{12(\pi)^2\cos(\pi) + 4(\pi)^3\sin(\pi)}{5\cos(\pi)^2}$$

$$\rightarrow -\frac{12\pi^2}{5}$$

Answer: $-\frac{12\pi^2}{5}$

Student number: 42672824

9. [**5 marks**] Let $f(x) = 2xe^x$. Find the equation of the tangent line to the curve $y = f(x)$ at $x = 0$.

$f'(x) = 2e^x + 2xe^x$

$x = 0$

$2e^0 + 2(0)e^0$

$\Rightarrow 2 = m$

$y = 2(0)e^0$

$y = 0$

$0 = 2(0) + c$

$c = 0$

$y = 2x$

Answer:

$y = 2x$

Student number: 42672424

10. **[5 marks]** Let $f(x) = (x^{-4} \overset{u}{+} x^2)(x \overset{v}{-} x^3)$. Find $f'(1)$.

$$f(x) = (x^{-4} + x^2)(x - x^3)$$

$$f'(x) = u'v + v'u$$

$$\Rightarrow \quad x^{-3} - x^{-1} + x^3 - x^5$$

$$\Rightarrow f'(x) = -3x^{-4} - (-1x^{-2}) + 3x^2 - 5x^4$$

$$\Rightarrow f'(x) = \frac{-3 + x^2 + 3x^6 - 5x^8}{x^4}$$

$$\Rightarrow f'(1) = \frac{-3 + (1)^2 + 3(1)^6 - 5(1)^8}{(1)^4}$$

$$\Rightarrow f'(1) = -4$$

Answer:

$f'(x) = -4$

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11. For any positive integer n, the *Hassell model* of exponent n

$$f(x) = \frac{Rx}{(1+\frac{x}{M})^n}, \quad M > 0, R > 1$$

describes the size $f(x)$ of a population given the size $x \geq 0$ of the population in the previous generation. (The special case $n = 1$ gives the Beverton-Holt model from Assignment 1.)

(a) **[3 marks]** Let $n \geq 2$. Solve the equation $f(x) = x$.

$$\frac{Rx}{(1+\frac{x}{m})^n} = x$$

$$\left(\left(1+\frac{x}{m}\right)^n\right)^{\frac{1}{n}} = (R)^{\frac{1}{n}}$$

$$\rightarrow 1+\frac{x}{m} = R^{\frac{1}{n}}$$

$$\rightarrow x = m(R^{\frac{1}{n}} - 1)$$

Answer:

$$x = m\left(R^{\frac{1}{n}} - 1\right)$$

(b) **[2 marks]** Let $n \geq 2$. Find the horizontal asymptote of $f(x)$.

X is close ∞ or $-\infty$

$$f(x) = \frac{Rx}{\left(\frac{x}{m}\right)^n} = \frac{Rx}{\frac{x^n}{m^n}} \Rightarrow \frac{Rxm^n}{x^n}$$

$$\Rightarrow Rx^{1-n}m^n$$

Answer:

$$Rx^{1-n}m^n$$

Student number: 212672824

(c) **[3 marks]** Let n_1 and n_2 be distinct positive integers. Find all points of intersection between the graphs of the Hassell models of exponent n_1 and n_2.

$$\to \frac{\cancel{Rx}}{\left(1+\frac{x}{m}\right)^{n_1}} = \frac{\cancel{Rx}}{\left(1+\frac{x}{m}\right)^{n_2}}$$

$$\to \left(1+\frac{x}{m}\right)^{\frac{1}{n_1-n_2}} = (1)^{\frac{1}{n_1-n_2}}$$

$$1+\frac{x}{m}=1$$

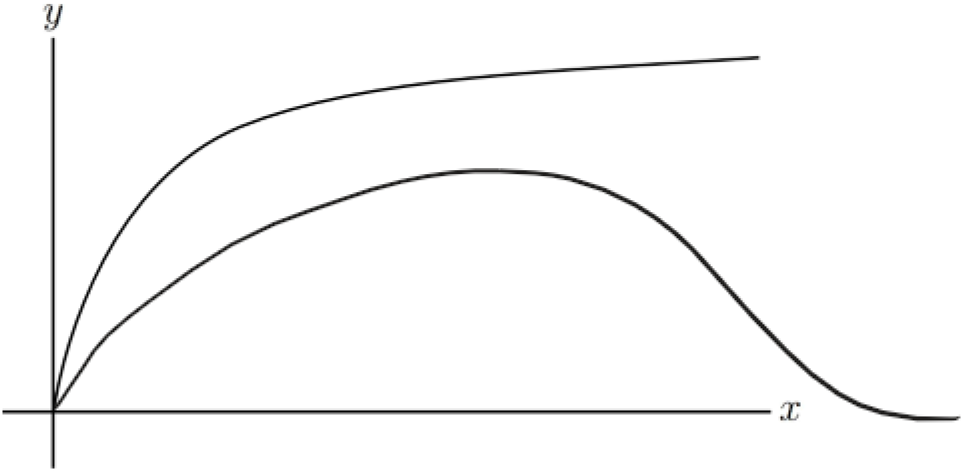
$$\to \frac{x}{m}=0$$

$$(0,0)$$

Answer:

$(0,0)$

(d) **[3 marks]** Pictured below is a graph of the Beverton-Holt model. On the same set of axes, sketch the graph of a Hassell model of exponent $n \geq 2$. (You may assume that the parameters M and R are the same in both models.)



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(e) **[3 marks]** The exponent n is sometimes interpreted to indicate the degree to which scarce resources are distributed equally in a population. Based on your answers to previous parts of this question, do you think an exponent of $n \geq 2$ corresponds to *more equal* or *less equal* distribution of scarce resources, compared to an exponent of $n = 1$ (*i.e.* the Beverton-Holt model)? Justify your answer in a few sentences.

→ I believe a distribution of scarce resources that is equitable would arise from an exponent of

$n \geq 2$. if $n \geq 2$,

a large number of previous generations. which would result in a less equal distribution. if $n=1$, the population of the current generation has a relatively stable relationship with previous generations.

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12. The *cumulative exponential probability distribution* describes the probability $P(t)$ that a particular radioactive atom will decay within t minutes. It is given by

$$P(t) = 1 - e^{-\lambda t},$$

where λ is a positive constant. The domain is restricted to $t \geq 0$.

(a) [**2 marks**] Find the horizontal asymptote of $P(t)$.

$P(t) = 1 - e^{-\lambda t}$ $\Big]$ $e^{-\lambda t} = \frac{1}{e^{\lambda t}} = 0$

H.A = $y = 1$

Answer:

$y = 1$

(b) [**2 marks**] Find the intercepts of $P(t)$.

$y = 0$ | $1 - e^{-\lambda t} = 0$

$-\lambda t = \ln 1$ | $t = -\frac{\ln 1}{t} = 0$

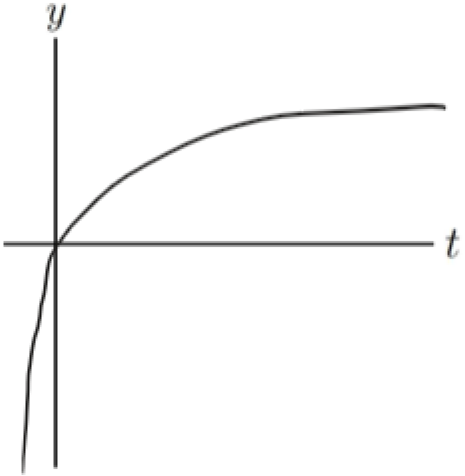
$(0,0)$

Answer:

$(0,0)$

Student number: 42672824

(c) **[3 marks]** Draw a large sketch of the graph of $P(t)$ on the axes below, using the information determined in the previous parts of this question.



(d) **[3 marks]** The *half-life* of a radioactive isotope is the time taken for a sample of the isotope to decay to half of its original mass. It is equal to the time by which the probability of decay of an individual atom is $\frac{1}{2}$. What is the half-life of the isotope whose atoms have the cumulative exponential probability distribution described above?

$$1 - e^{-\lambda t} = \frac{1}{2}$$

$$e^{-\lambda t} = \frac{1}{2}$$

$$\cancel{-}\lambda t = \cancel{-}\ln 2$$

$$t = \frac{\ln 2}{\lambda}$$

Answer:

$$t = \frac{\ln 2}{\lambda}$$

Student number: 42672824

13. Let $f(x) = x^2$.

(a) **[3 marks]** Find the equation of the tangent line to $f(x)$ at the point (x_0, x_0^2). Your answer should be in the form $y = mx + b$, and may include the quantity x_0.

$f'(x) = 2x$

(x_0, x_0^2)

$y = 2(x_0)x - x_0^2$

$m = 2(x_0)$

$x_0^2 = 2(x_0)x + b$

$x_0^2 - 2(x_0)x = b$

$y = mx + b$

$y = 2(x_0)x + b$

Answer: $y = 2(x_0)x - x_0^2$

(b) **[3 marks]** Imagine you were given a numerical value for x_0. Explain in 1-3 sentences how you would determine whether or not the tangent line described in part (a) passes through the point $(2, 3)$.

$3 = 2(x_0)2 - x_0^2$

$\Rightarrow 3 = 4x_0 - x_0^2$

$\Rightarrow 3 - 4x_0 = -x_0^2$

$(x_0 - 1)(x_0 - 3) = 0$

so its 1 or 3

(c) **[1 mark]** Write down the equations of all tangent lines to $f(x)$ that pass through the point $(0, 0)$. Your answer(s) should be in the form $y = mx + b$. Your answer(s) do not have to be justified.

$f'(0) = 0$

Answer: $y = 0x + 0$

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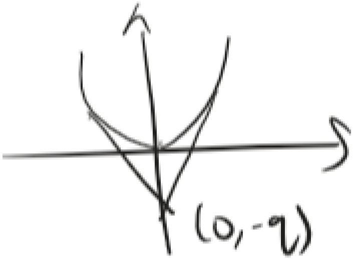
(d) [**3 marks**] Let $q > 0$. How many tangent lines to $f(x)$ pass through the point $(0, q)$? (Remember to justify your answer.)

So the point is inside the graph so its 0

Answer:

0

(e) [**4 marks**] Let $q > 0$. Find the equations of all tangent lines to $f(x)$ that pass through the point $(0, -q)$. Your answer(s) should be in the form $y = mx + b$.



$y_1 = m_1 x - q$

$m_1 = f(x_1) = 2x$

$y_1 = 2x_1, x - q$

$2x_1^2 \cdot q = x_1^2 \quad q = x_1^2 \quad x_1 = q^{\frac{1}{2}}$

Answer:

$y_1 = 2q^{\frac{1}{2}}x - q \quad y_2 = -2q^{\frac{1}{2}}x - q$

$m_1 = 2q^{\frac{1}{2}}x - q$

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14. Hyperbolic functions are analogues to trigonometric functions, defined with respect to the unit hyperbola instead of the unit circle. They are particularly useful in the study of differential equations. They can also be defined in terms of exponential functions. One example is the *hyperbolic cotangent* function

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

defined on the restricted domain $x \neq 0$.

(a) **[3 marks]** Find all horizontal asymptotes of $\coth(x)$.

$x \to \infty$

$\coth(x) = \frac{e^x + e^{\frac{1}{x}}}{e^x - e^{\frac{1}{x}}} \to y = 1$

$\frac{e^x}{e^x} = 1$ | $x \to -\infty \Rightarrow y = -1$

Answer:

$y = 1$, $y = -1$

(b) **[2 marks]** Find all vertical asymptotes of $\coth(x)$.

$e^{2x} - 1 = 0$

$x = 0$

$e^x - e^{-x} = 0$

Answer:

$x = 0$

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(c) [**4 marks**] Determine where coth(x) is increasing and where it is decreasing.

$$f'(x) = \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}$$

$$\rightarrow \quad \frac{-2e^{-x} \cdot 2e^x}{(e^x - e^{-x})^2}$$

$$\Rightarrow \quad \frac{-4e^0}{(e^x - e^{-x})^2} = \frac{-4}{(e^x - e^x)^2}$$

$$\Rightarrow (e^x - e^x)^2 \geq 0$$

so decreasing

Answer:

decreasing

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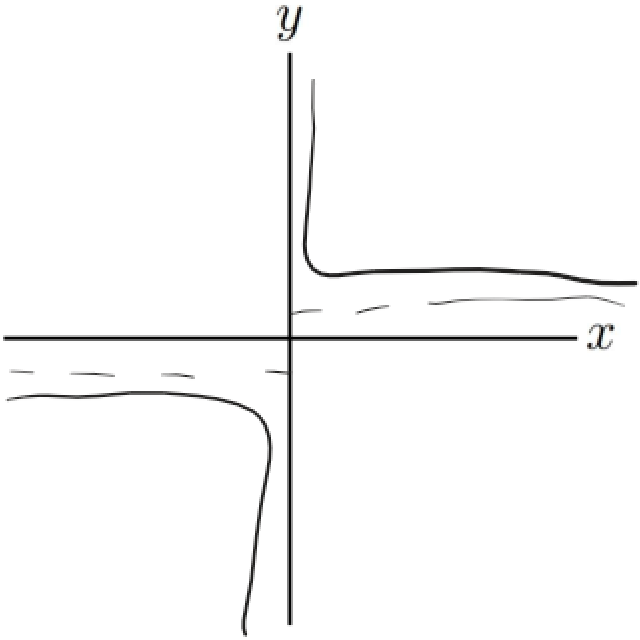
(d) **[3 marks]** Draw a large sketch of the graph of coth(x) on the axes below, using the information determined in the previous parts of this question.



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