CS-2001

Data Structures

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Tree

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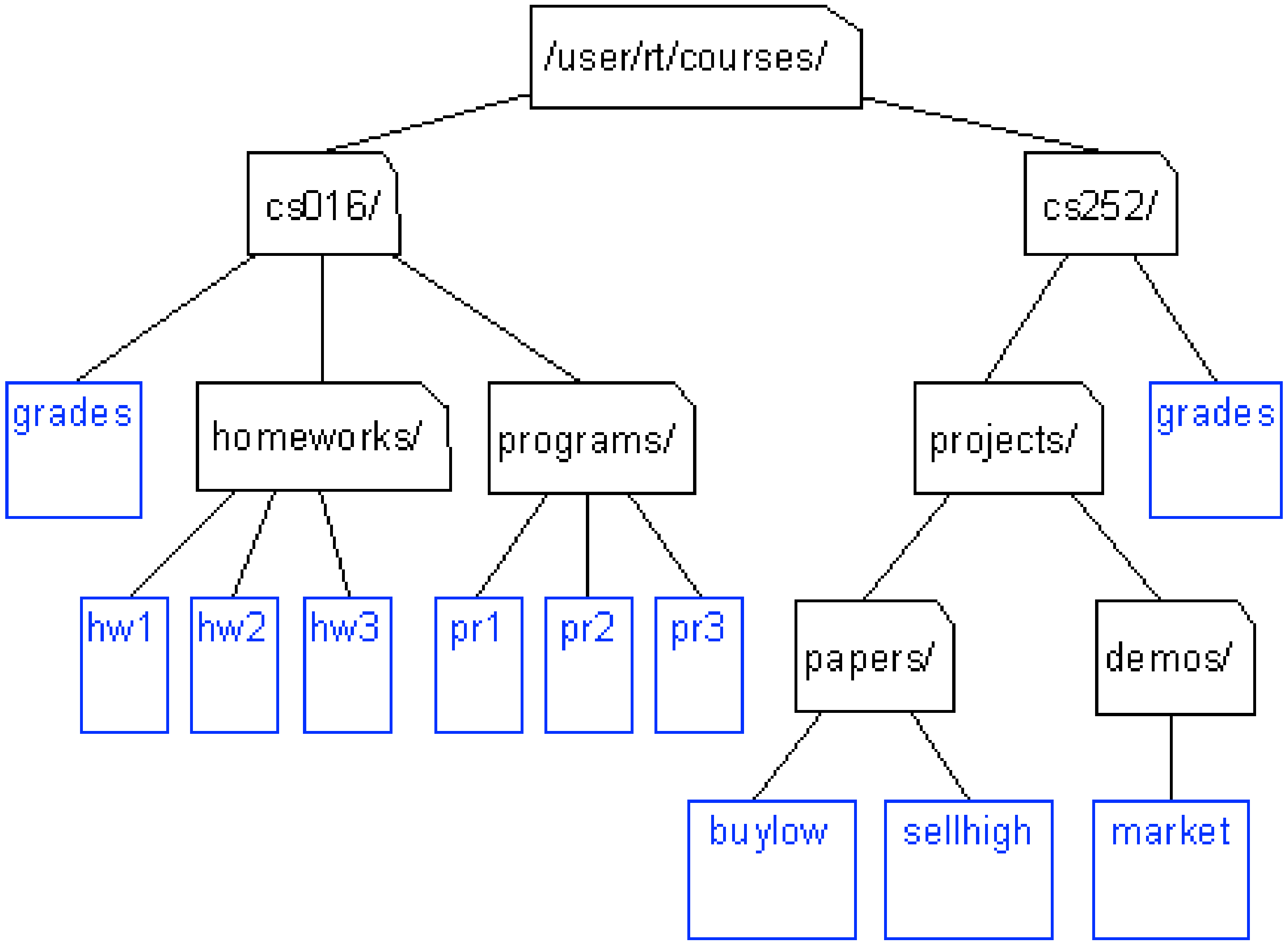
Trees

- Hierarchical data structure

- Examples:
  - Indexes in a book have a shallow tree structure
  - A family tree
  - Tree structure of University
- Others?

Unix / Windows file structure

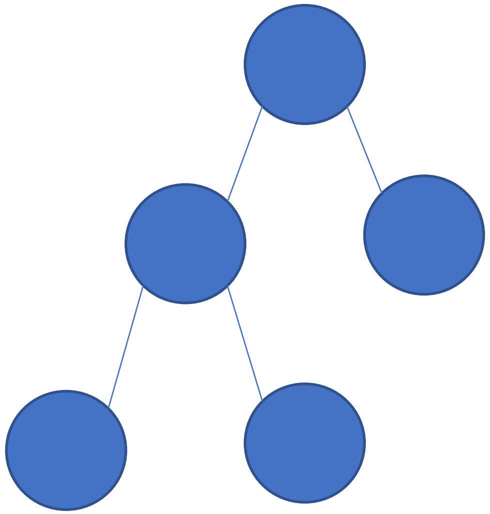


Trees: Basic terminology

- Hierarchical data structure
- Each position in the tree is called a **node**
- The “top” of the tree is called the **root**
- The nodes immediately below a node are called its **children**; nodes with no children are called **leaves** (or terminal nodes), and the node above a given node is its **parent** (or father)
- A node x is **ancestor** of node y if x is father of y or father of some ancestor of y and y is called descendent of x
    - **Note:** Ancestor of a node is its parent, grand parent or grand-grand parent or so on….

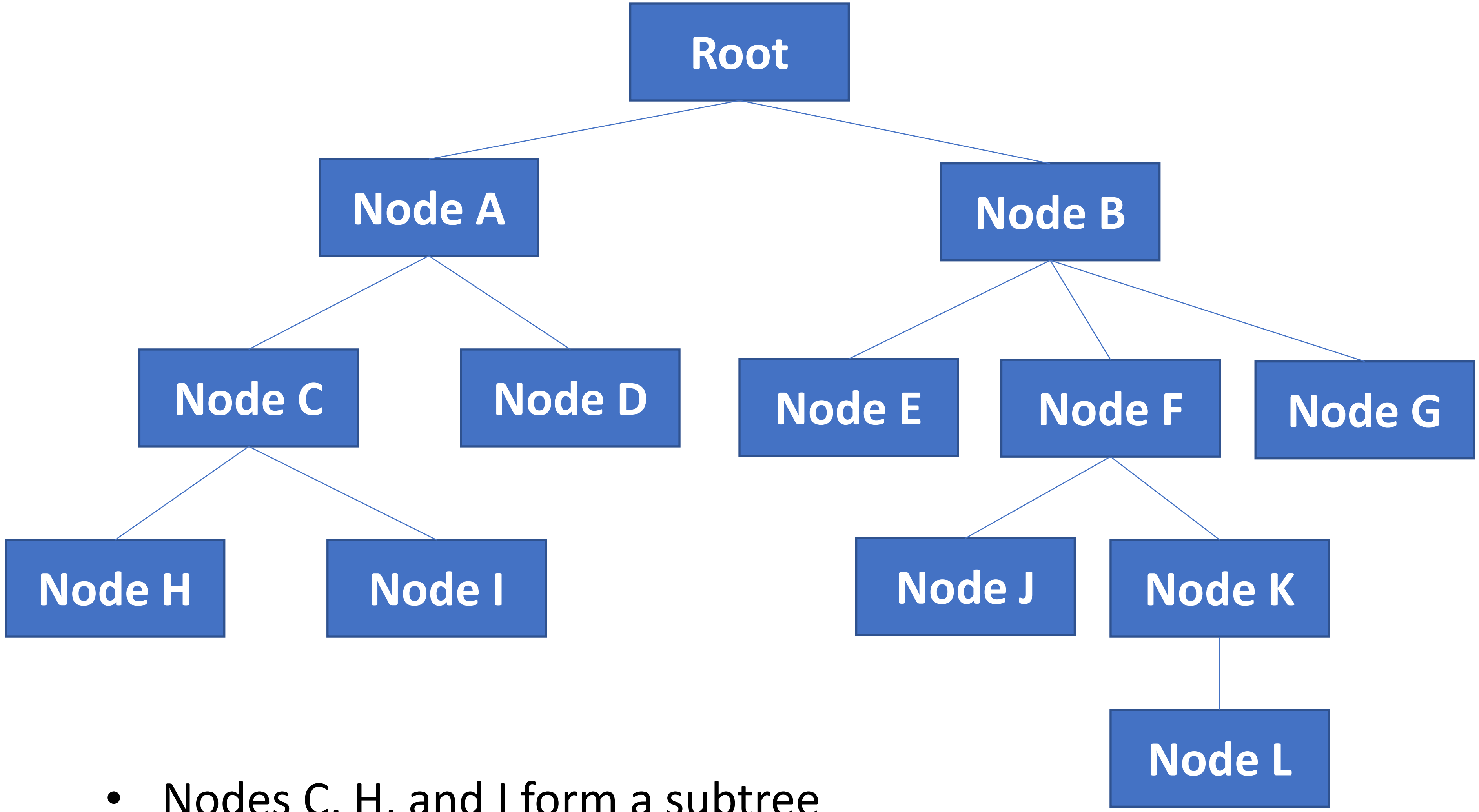
Trees: Basic terminology

- Nodes with the same parent are siblings
- A node and collection of nodes beneath it is called a subtree
- The number of nodes in the longest path from the root to a leaf is the depth (or height) of the tree
- is the depth 2 or 3?
- depends on the author



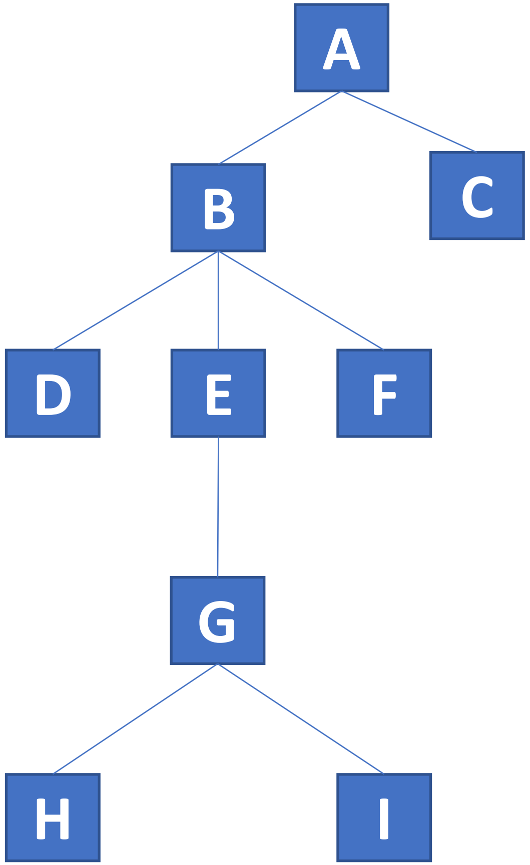
Text book definition of **depth** of a tree

- Depth of a tree is maximum level of any leaf in the tree
- Root of tree is at level 0, and level of any other node in the tree is one more than the level of its father



- Nodes C, H, and I form a subtree
- Nodes H and I are siblings
- C is H’s parent, H and I are children of C
- What is the depth of this tree?

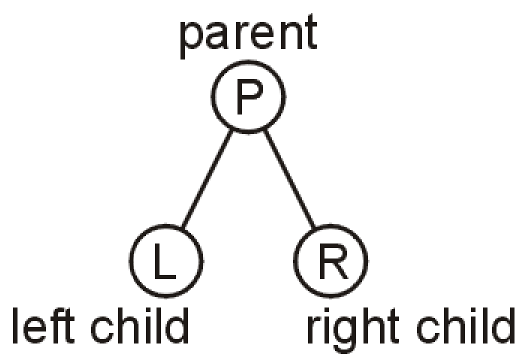
Tree Properties



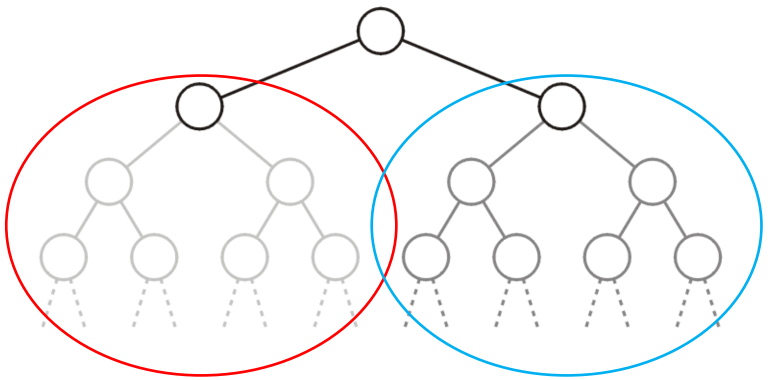
Property	Value
• Number of nodes	9
• Height	4
• Root Node	A
• Leaves	C, D, F, H, I
• Max level number	4
• Ancestors of H	G, E, B, A
• Descendants of B	D, E, F, G, H, I
• Siblings of E	D, F
• Right subtree	C

Binary Tree

- A commonly used type of tree is a binary tree
  - In a binary tree each node has at most two children
  - Allows to label the children as left and right

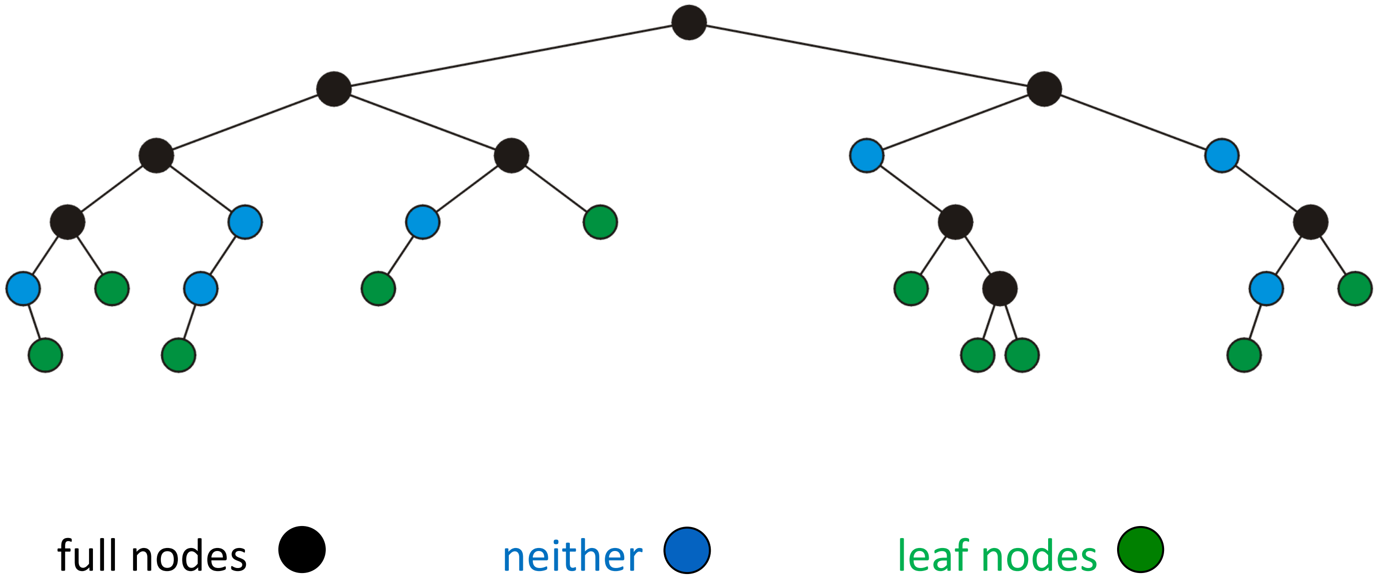


- Likewise, the two sub-trees are referred as
  - Left sub-tree
  - Right sub-tree



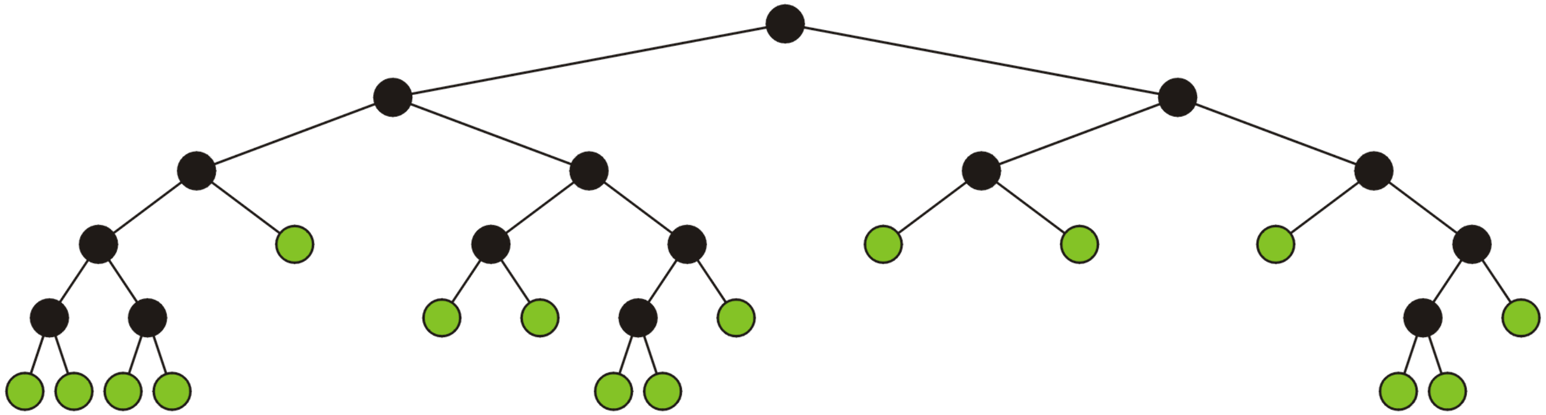
Binary Tree: Full Node

- A full node is a node where both the left and right sub-trees are non-empty trees
- (OR) if it has exactly two child nodes



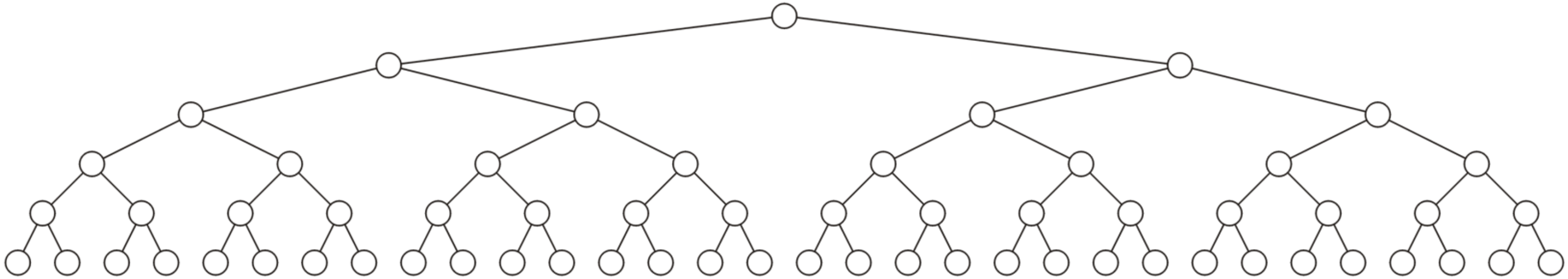
Binary Trees

- A tree is called **Strictly Binary Tree** if every non leaf node has exactly two children
  - A strictly binary tree having **n** leaves always contain $\mathbf{2^n-1}$ nodes
  - It is also known as **Full Binary Tree, Proper Binary Tree or 2-Tree**
    - Every node is a **full node** or a **leaf node**



Perfect/Complete Binary Tree

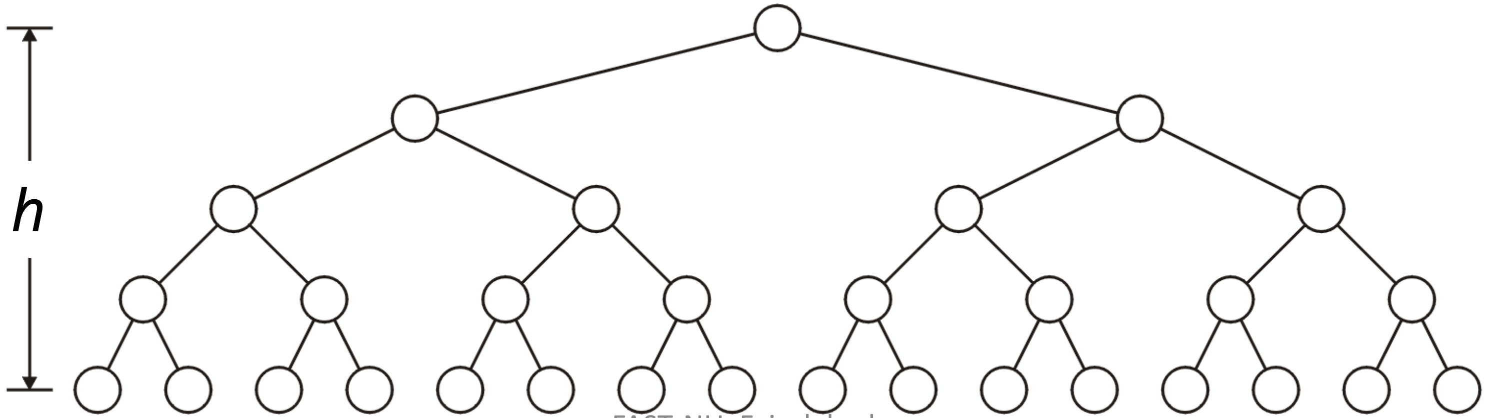
- A **perfect/Complete** binary tree of height h is a binary tree where
  - All leaf nodes have the same depth or level *L*
  - All other nodes are full-nodes



Binary Tree: Properties

- A perfect/complete binary tree with height h has 2^h leaf nodes

- A perfect/complete binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$



Binary Tree: Properties (4)

- A perfect/complete binary tree with height h has 2^h leaf nodes
- A perfect/complete binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$
- A perfect/complete binary tree with n nodes has **height** $\log_2(n+1)-1$

$$
\begin{aligned}
&n = 2^{h+1} - 1 \\
&2^{h+1} = n + 1 \\
&h + 1 = \log_2(n + 1) \\
&\Rightarrow h = \log_2(n + 1) - 1
\end{aligned}
$$

Binary Tree: Properties (4)

- A perfect binary tree with height h has 2^h leaf nodes

- A perfect binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$

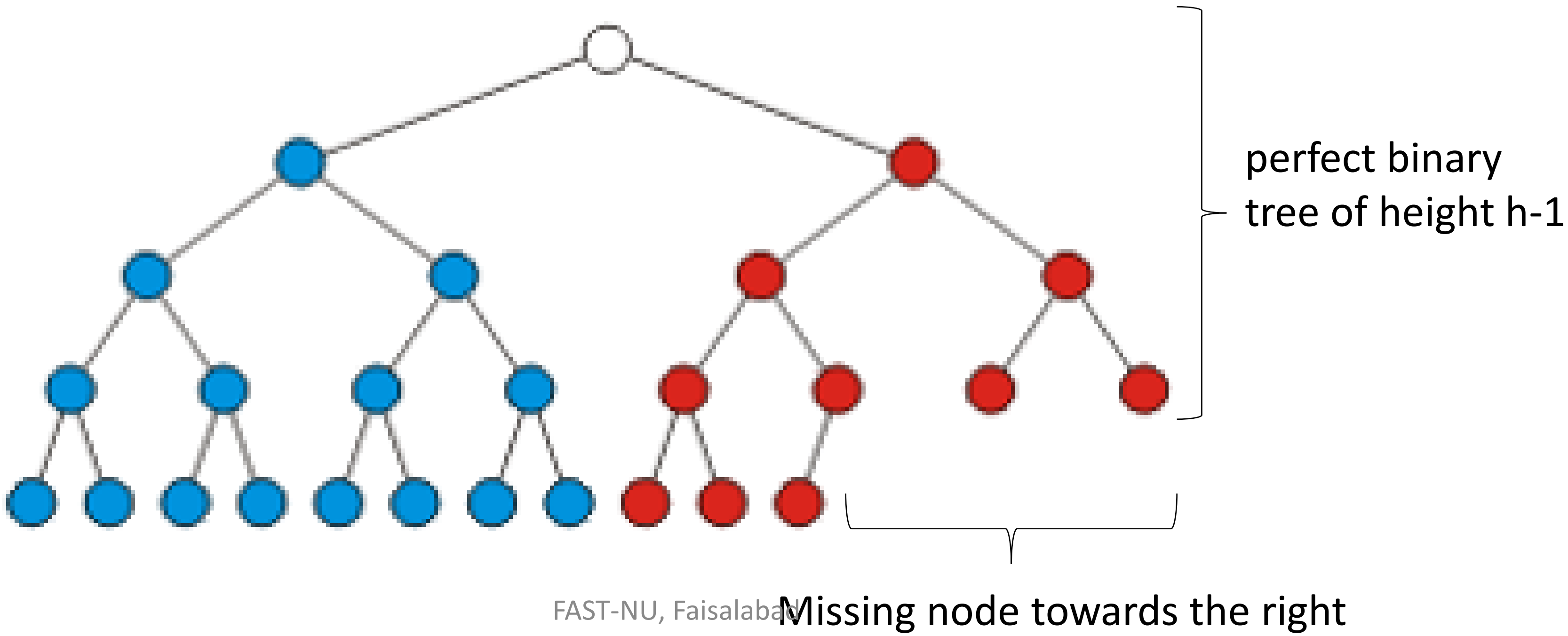
- A perfect binary tree with n nodes has height $\log_2(n + 1) - 1$

- **Number n of nodes in a binary tree of height h is at least h+1 and at most $2^{h+1} - 1$**

[Almost] Complete Binary tree

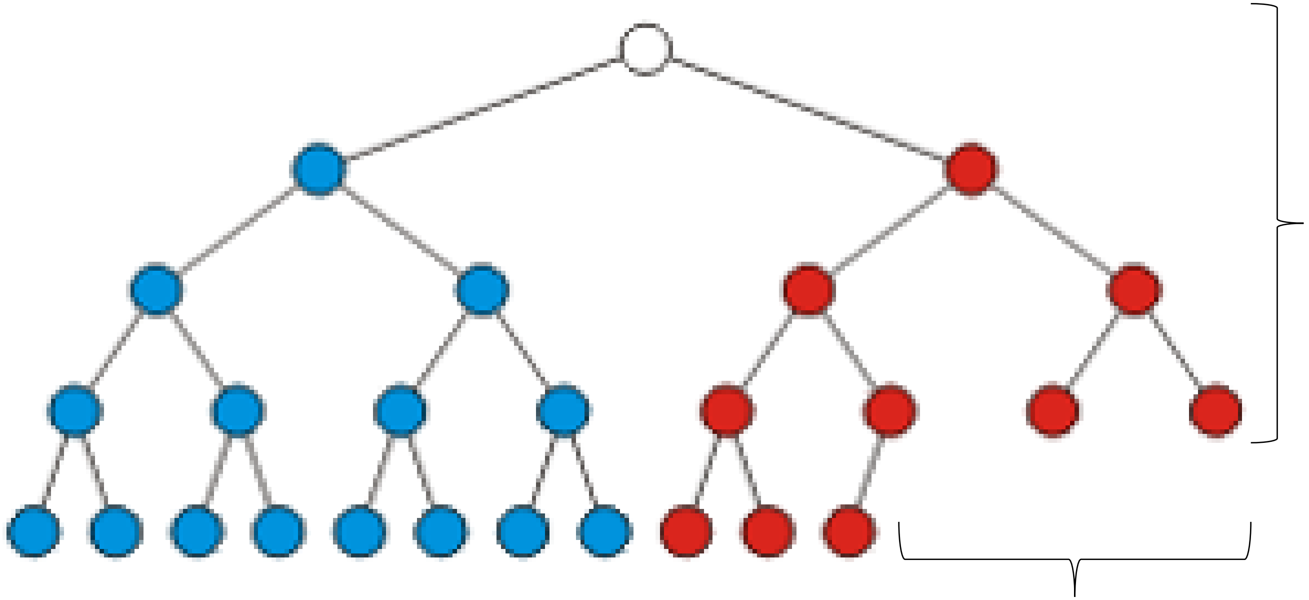
Almost (or Nearly) Complete Binary Tree

- Almost complete binary tree of height h is a binary tree in which
  1. There are 2^d nodes at depth d for d = 1,2,...,h−1
     - Each leaf in the tree is either at level h or at level h − 1
  2. The nodes at depth h are as far left as possible



Complete Binary Tree

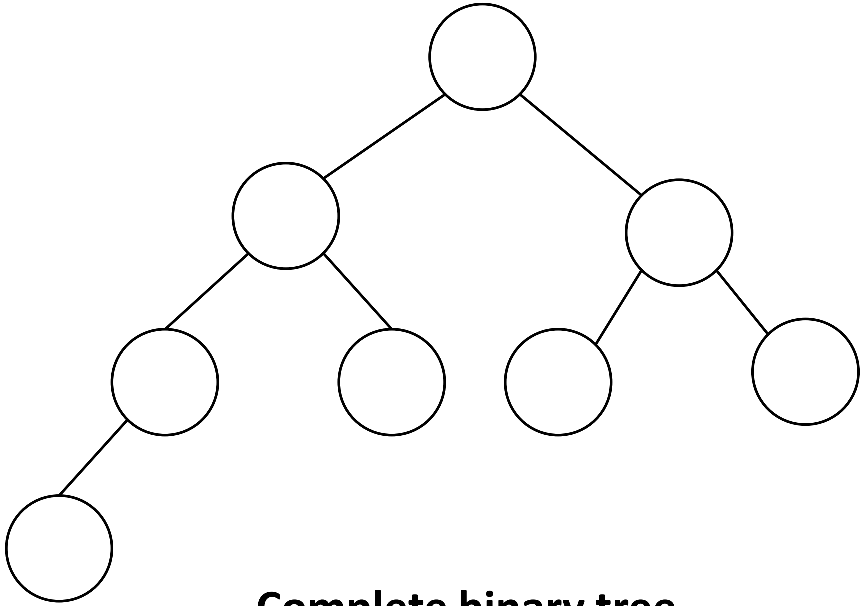
- Complete binary tree of height h is a binary tree in which
  1. There are 2^d nodes at depth d for d $= 1,2,...,h-1$
     - Each leaf in the tree is either at level h or at level $h - 1$
  2. The nodes at depth h are as far left as possible



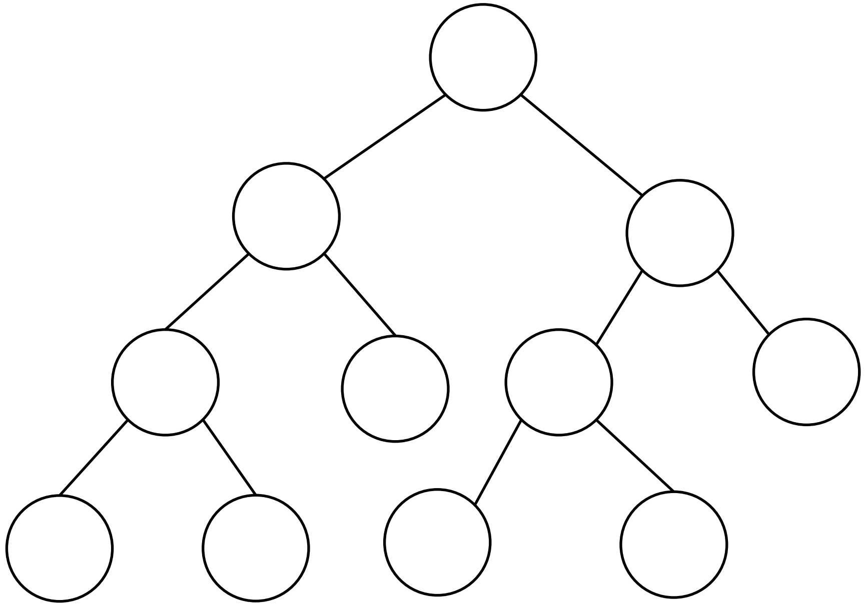
Complete Binary Tree

Condition 2: The nodes at depth h are as far left as possible

- If a node p at depth h−1 has a left child
  - Every node at depth h−1 to the left of *p* has 2 children
- If a node at depth h−1 has a right child
  - It also has a left child

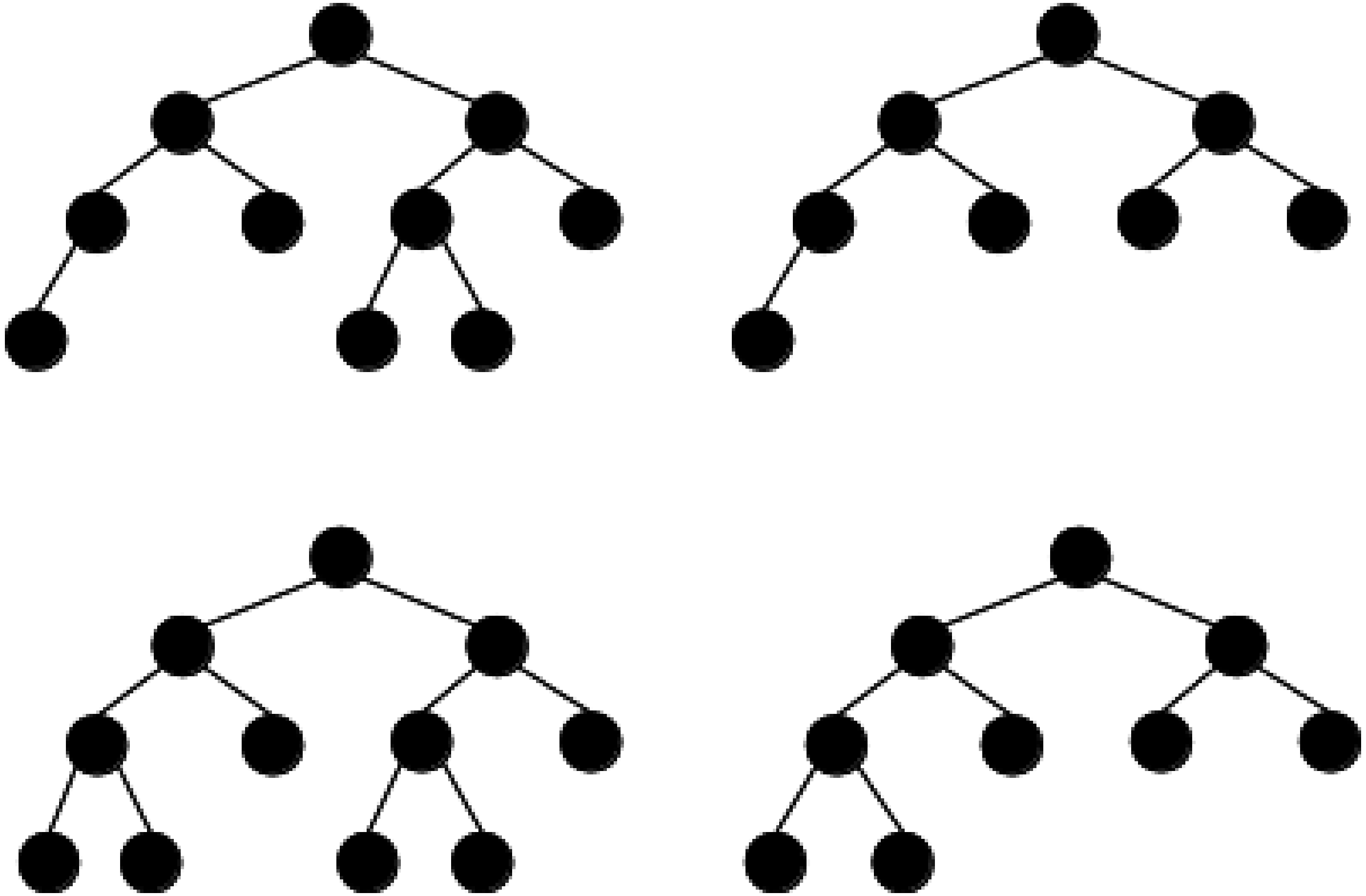


Complete binary tree

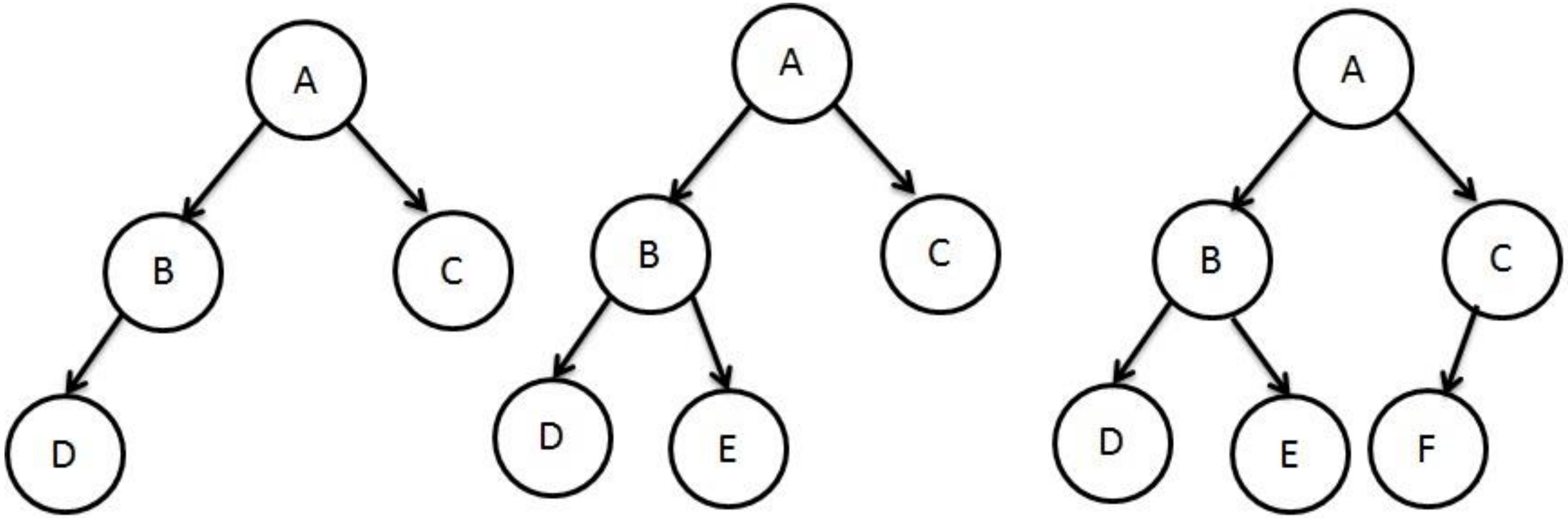


Not a Complete binary tree (condition 2 violated)

Full vs. Complete Binary Tree



Complete Binary Trees…



What is the height and number of nodes for each tree?

Complete Binary Tree: Properties

- Total number of nodes n are between
  - **At least:** perfect binary tree of height h-1 + 1 (i.e., 1 node in the next level) → $2^h - 1 + 1 = 2^h$ nodes
  - **At most:** perfect binary tree of height h, i.e., $2^{h+1} - 1$ nodes

- Height h is equal to $\lfloor \boldsymbol{Log_2(n)} \rfloor$

Balanced Binary Tree

- **Balanced binary tree**
  - For each node, the difference in height of the right and left sub-trees is no more than one
  - Both Perfect binary trees and complete binary trees are balanced as well

- **Completely balance binary tree**
  - Left and right sub-trees of every node have the same height
  - A perfect binary tree is completely balanced

Tree ADT

Tree ADT

- **Data Type:** Any type of objects can be stored in a tree

- Accessor methods
  - **`root()`** – returns the root of the tree
  - **`parent(p)`** – returns the parent of a node
  - **`children(p)`** – returns the children of a node

- Query methods
  - **`size()`** – returns the number of nodes in the tree
  - **`isEmpty()`** – returns true if the tree is empty
  - **`elements()`** – returns all elements
  - **`isRoot(p)`** – returns true if node p is the root

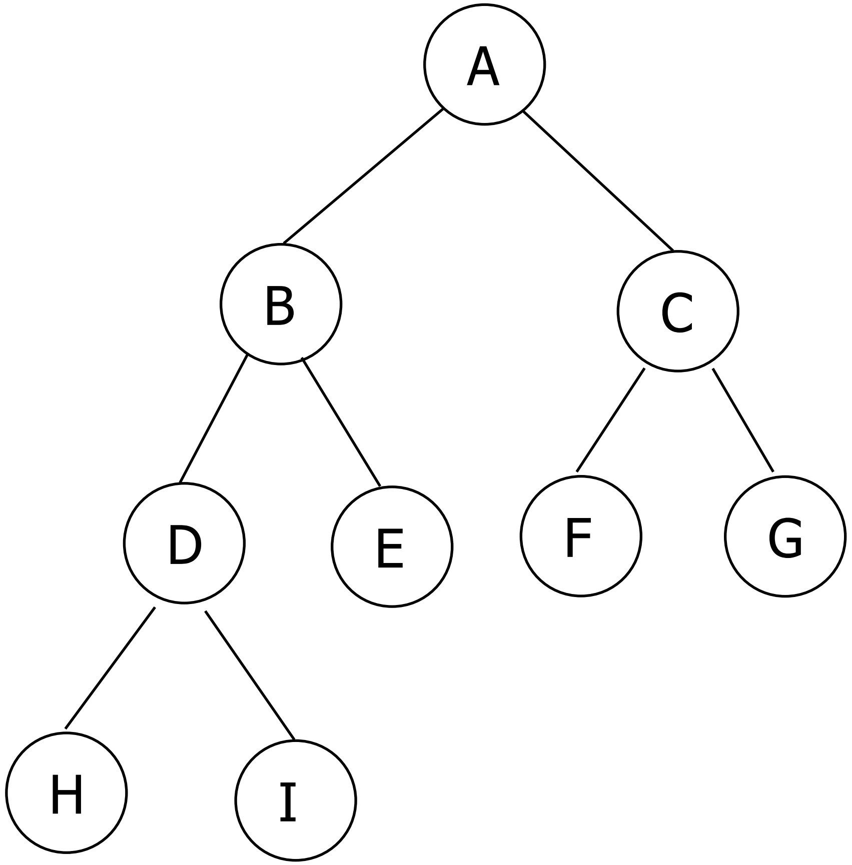
- Other methods
  - Tree traversal, Node addition/deletion, create/destroy

Binary Tree Storage

- Contiguous storage
- Linked-list based storage

Contiguous Storage

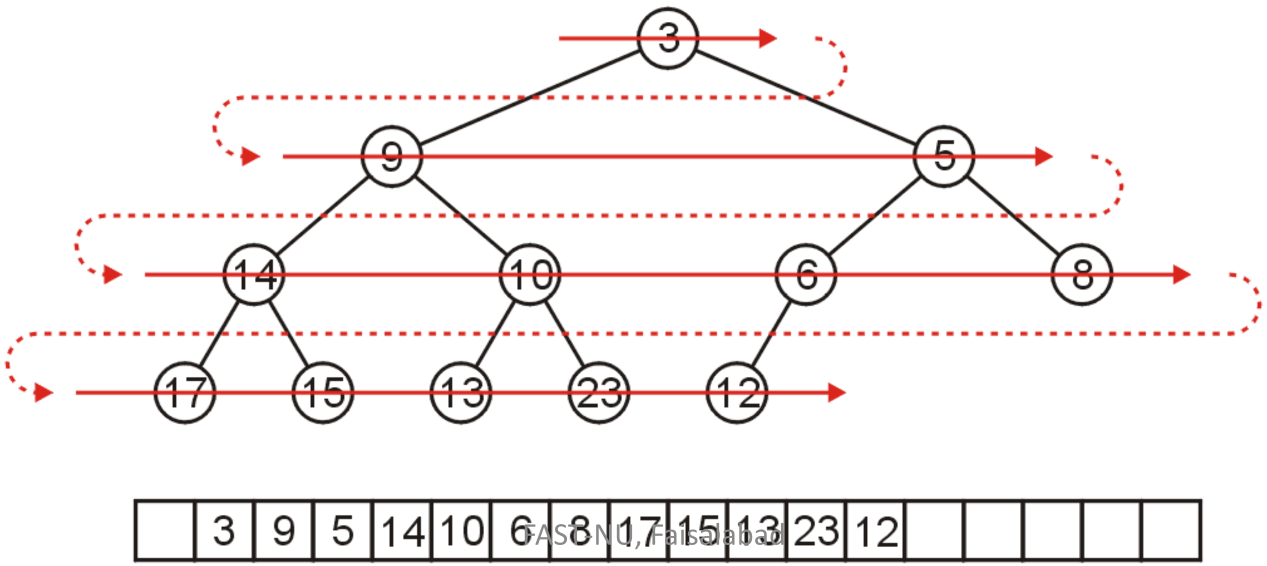
Array Storage Example



[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

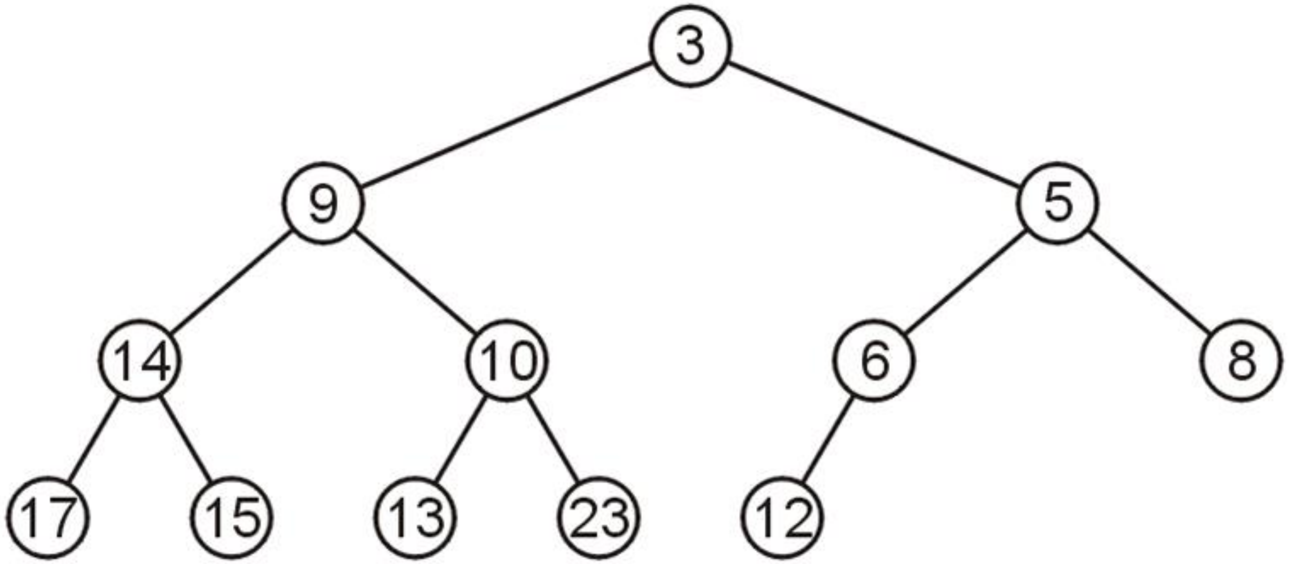
Array Storage

- We can store a binary tree as an array
- Traverse tree in breadth-first order, placing the entries into array
  - Storage of elements (i.e., objects/data) starts from root node
  - Nodes at each level of the tree are stored left to right



Array Storage

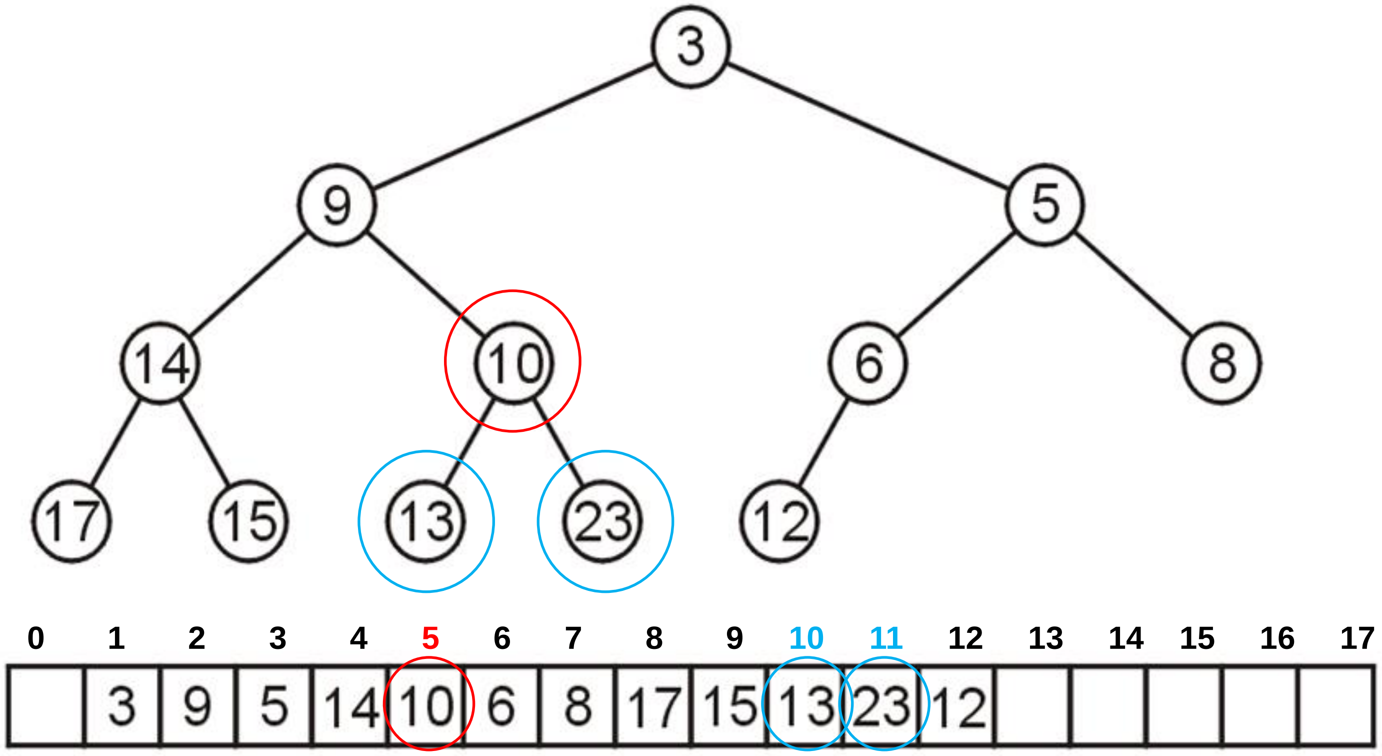
- The children of the node with index k are in 2k and 2k + 1
- The parent of node with index k is in k/2



	3	9	5	14	10	6	8	17	15	13	23	12					

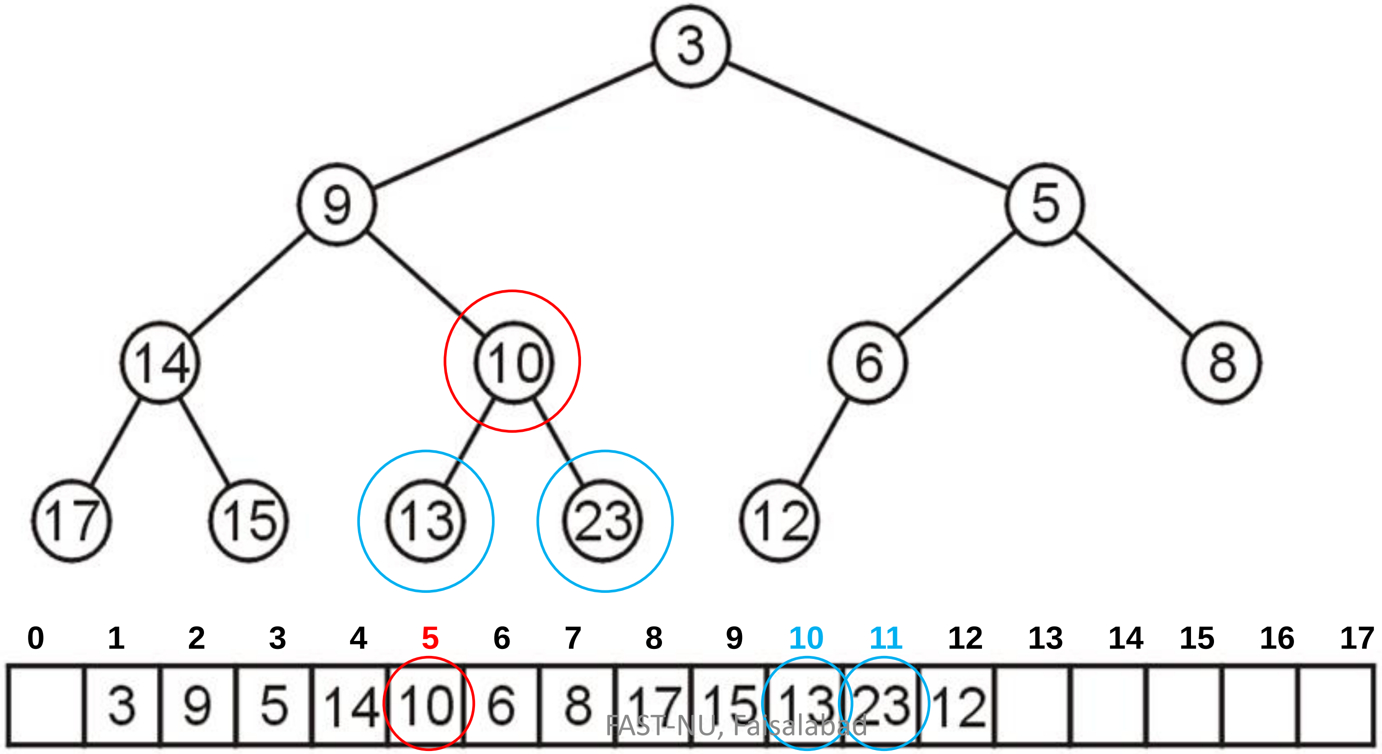
Array Storage Example

- Node 10 has index 5
  - Its children 13 and 23 have indices 10 and 11, respectively



Array Storage Example

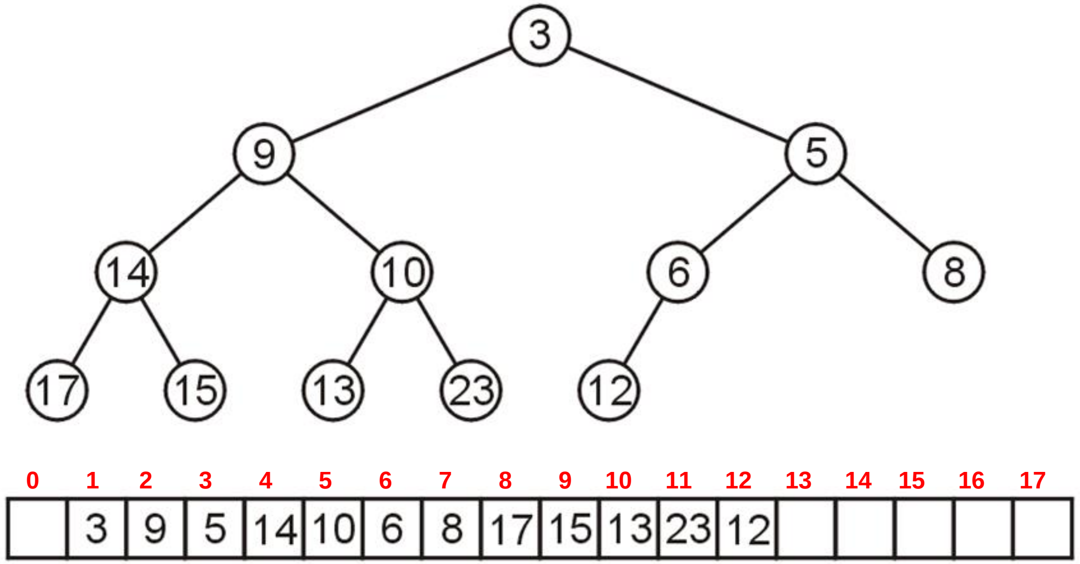
- Node 10 has index 5
  - Its children 13 and 23 have indices 10 and 11, respectively
  - Its parent is node 9 with index 5/2 = 2



Array Storage

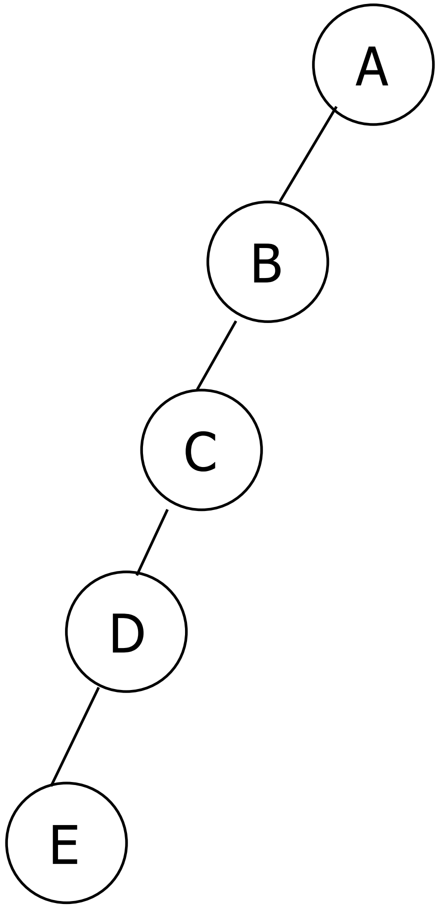
- Why array index is not started from 0
  - In C++, this simplifies the calculations

```
parent = k >> 1;
left_child = k << 1;
right_child = left_child | 1;
```



Array Storage Example

- Unused nodes in tree represented by a predefined bit pattern

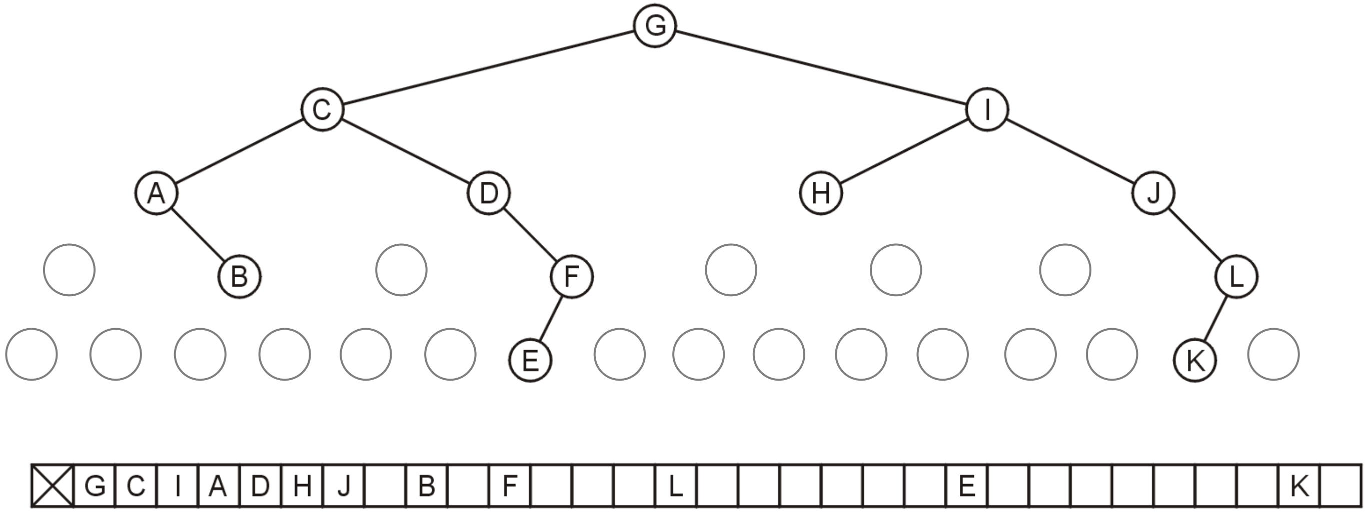


[1]	A
[2]	B
[3]	-
[4]	C
[5]	-
[6]	-
[7]	-
[8]	D
[9]	-
…	…
[16]	E

Array Storage: Disadvantage

- Why not store any tree as an **array** using breadth-first traversals?
  - Because, there is a significant potential for a lot of wasted memory

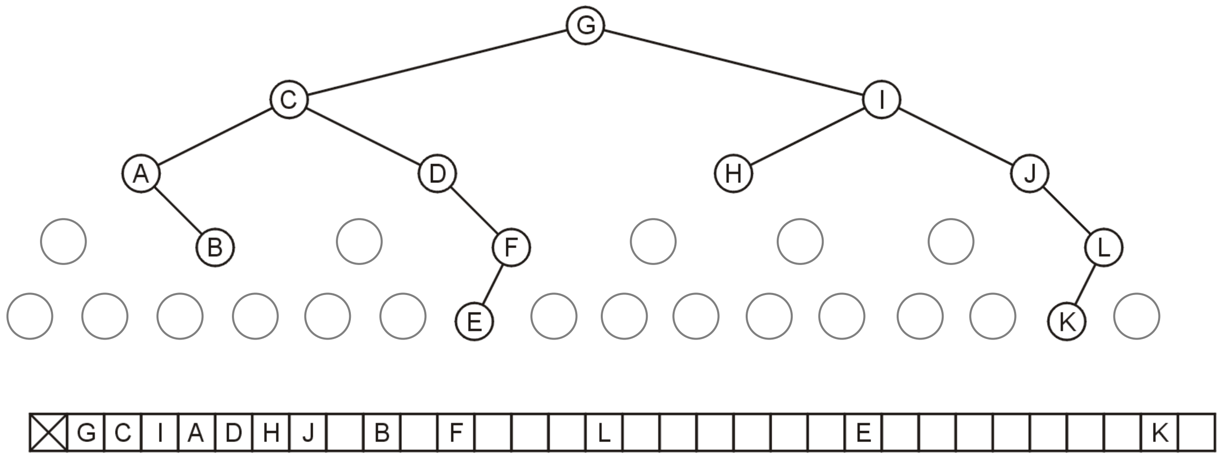
- Consider the following tree with 12 nodes
  - What is the required size of array?



Array Storage: Disadvantage

- Why not store any tree as an array using breadth-first traversals?
  - There is a significant potential for a lot of wasted memory

- Consider the following tree with 12 nodes
  - What is the required size of array? **32**
  - What will be the array size if a child is added to node K? **(Double it)**



Linked List Storage

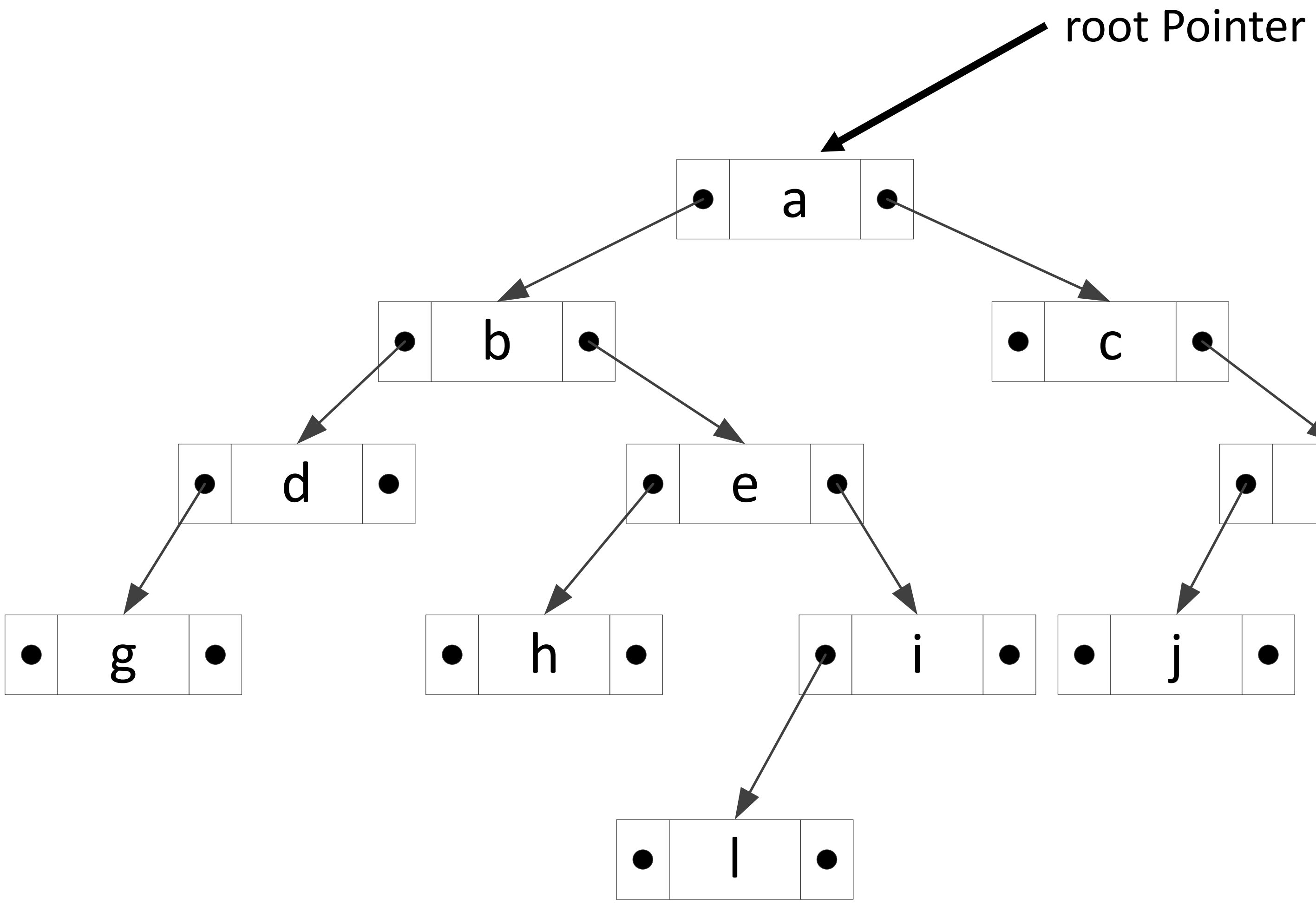
As Linked List Structure

- We can implement a binary tree by using a struct which stores:
  - An element
  - A left child pointer (pointer to first child)
  - A right child pointer (pointer to second child)

```
struct Node {
     Type value;
     Node *LeftChild, *RightChild;
}*root;
```

- The root pointer points to the root node
  - Follow pointers to find every other element in the tree
- Leaf nodes have `LeftChild` and `RightChild` pointers set to NULL

As Linked List Structure: Example



Tree Traversal

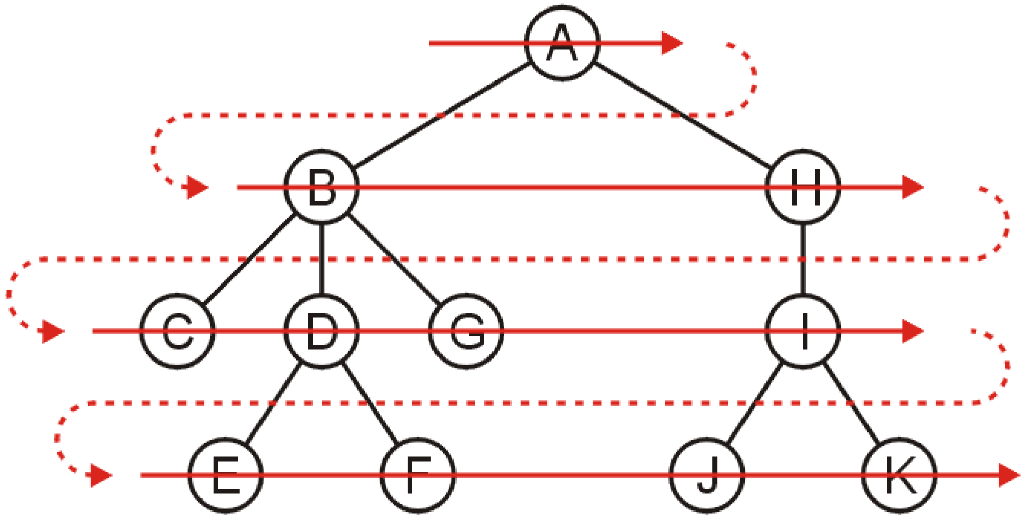
Tree Traversal

- To traverse (or walk) the tree is to visit (printing or manipulating) each node in the tree exactly once
  - Traversal must start at the root node
    - There is a pointer to the root node of the binary tree

- Two types of traversals
  - Breadth-First Traversal
  - Depth-First Traversal

Breadth-First Traversal (For Arbitrary Trees)

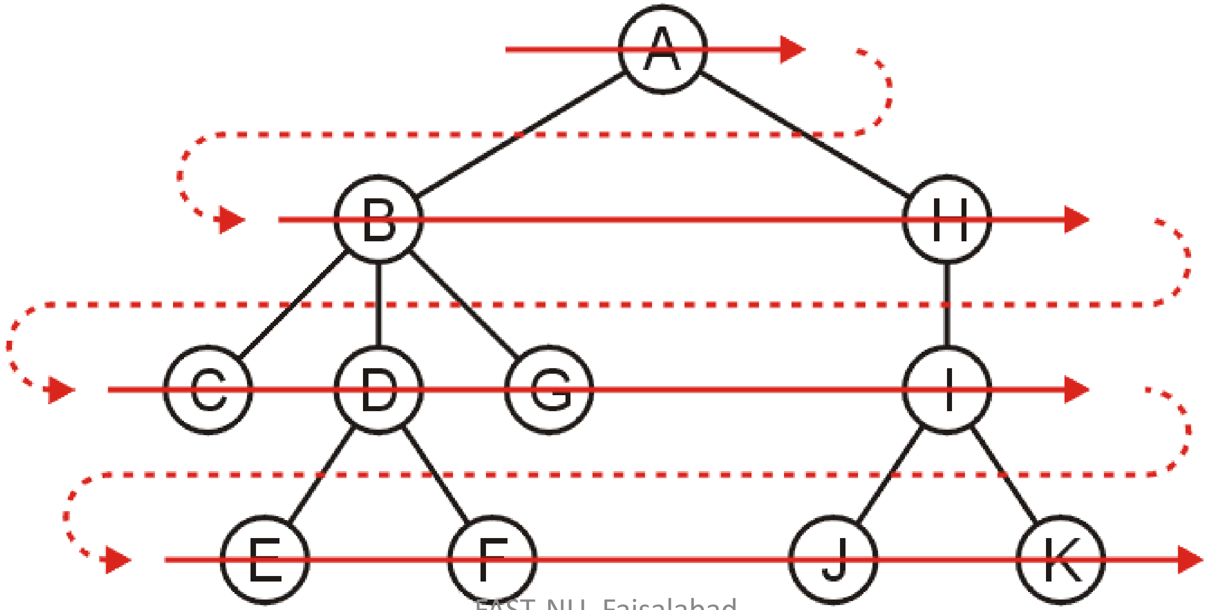
- All nodes at a given depth d are traversed before nodes at d+1
- Can be implemented using a queue



- Order: A B H C D G I E F J K

Breadth-First Traversal – Implementation

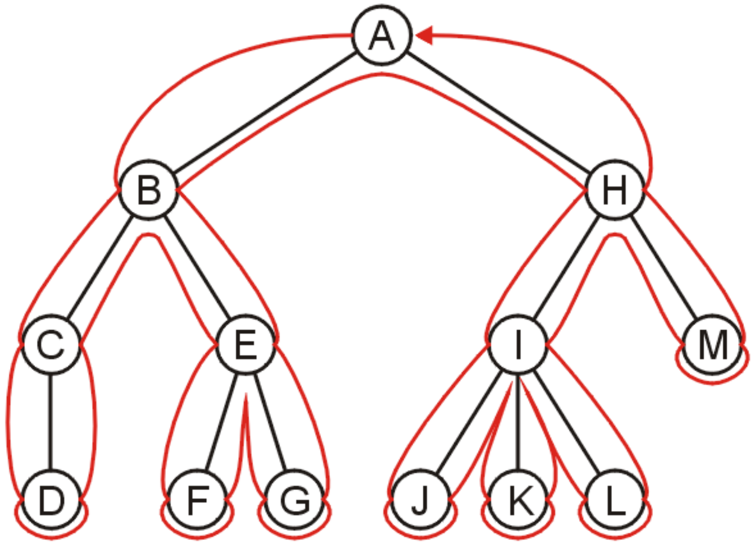
- Create a queue and push the root node onto the queue
- While the queue is not empty:
  - Enqueue all of its children of the front node onto the queue
  - Dequeue the front node



Depth-First Traversal (For Arbitrary Trees)

- Traverse as much as possible along the branch of each child before going to the next sibling
  - Nodes along one branch of the tree are traversed before backtracking

- Each node could be approached multiple times in such a scheme
  - The first time the node is approached (before any children)
  - The last time it is approached (after all children)



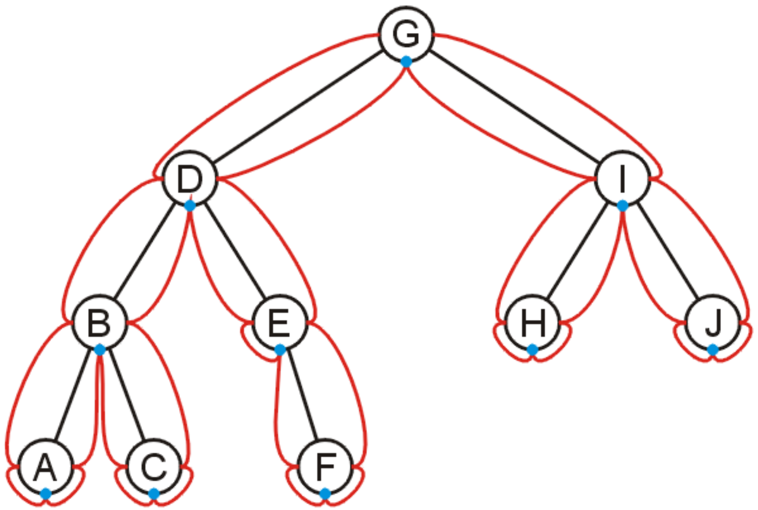
Depth-First Tree Traversal (Binary Trees)

- For each node in a binary tree, there are three choices
  - Visit the node first
  - Visit the node after left subtree
  - Visit the node after both the subtrees

- These choices lead to three commonly used traversals
  - Preorder traversal: visit Root (Left subtree) (Right subtree)
  - In-order traversal: (Left subtree) visit Root (Right subtree)
  - Post-order traversal: (Left subtree) (Right subtree) visit Root

Inorder Traversal

- Algorithm
  1. Traverse the left subtree in inorder
  2. Visit the root
  3. Traverse the right subtree in inorder



A, B, C, D, E, F, G, H, I, J

Inorder Traversal

- Algorithm
  1. Traverse the left subtree in inorder
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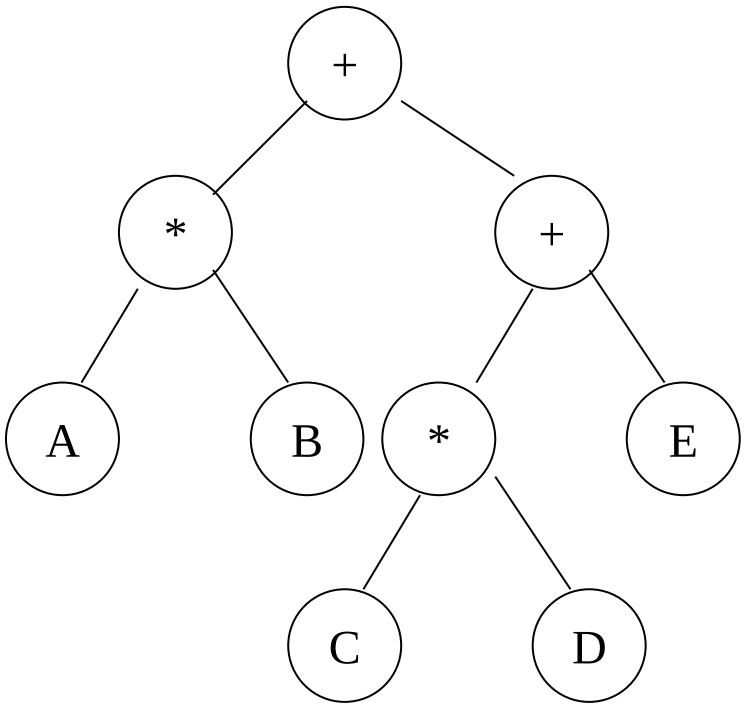
  3. Traverse the right subtree in inorder



- Example
  - Left + Right
  - [Left * Right] + [Left + Right]
  - (A * B) + [(Left * Right) + E)
  - (A * B) + [(C * D) + E]

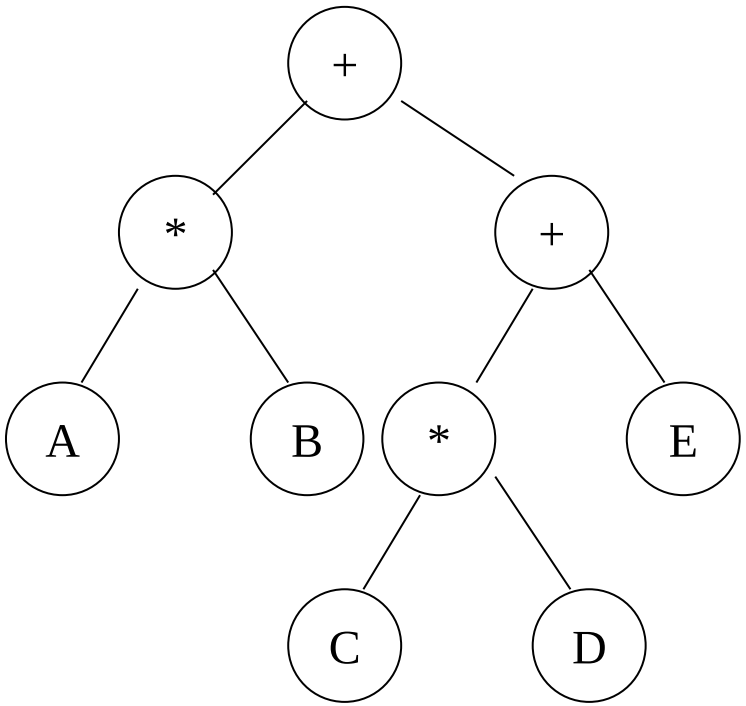
Inorder Traversal – Implementation

```
void inorder(Node *p) const
{
   if (p != NULL)
   {
      inorder(p->leftChild);
      cout << p->info << " ";
      inorder(p->rightChild);
   }
}

void main () {
   . . .
   inorder (root);
}
```

Preorder Traversal

- Algorithm
  1. Visit the node
  2. Traverse the left subtree
  3. Traverse the right subtree



- Example
  - + Left Right
  - + [* Left Right] [+ Left Right]
  - + (* AB) [+ * Left Right E]
  - +*AB + *C D E

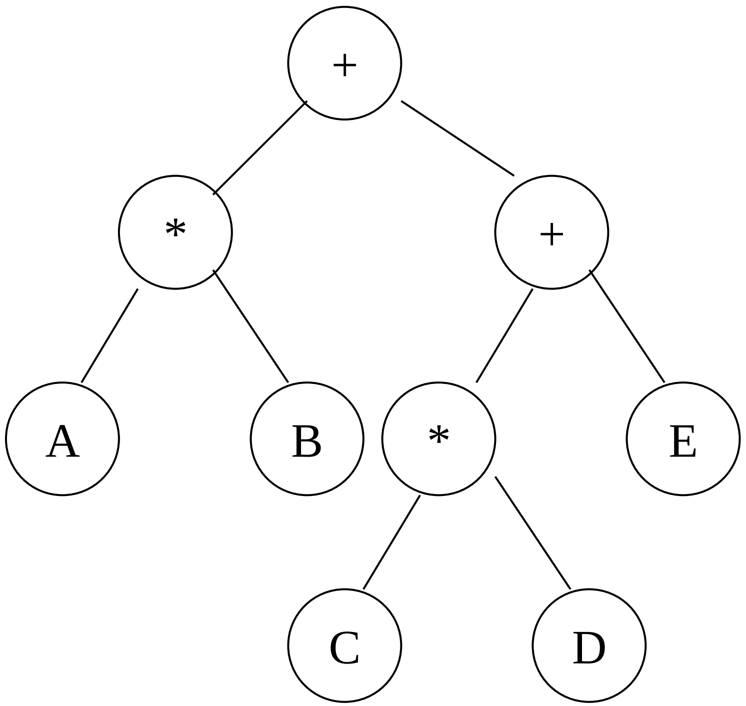
Preorder Traversal – Implementation

```
void preorder(Node *p) const
{
   if (p != NULL)
   {
      cout << p->info << " ";
      preorder(p->leftChild);
      preorder(p->rightChild);
   }
}

void main () {
   . . .
   preorder (root);
}
```

Postorder Traversal

- Algorithm
  1. Traverse the left subtree
  2. Traverse the right subtree
  3. Visit the node



- Example
  - Left Right +
  - [Left Right *] [Left Right+] +
  - (AB*) [Left Right * E +]+
  - (AB*) [C D * E +]+
  - AB* C D * E + +

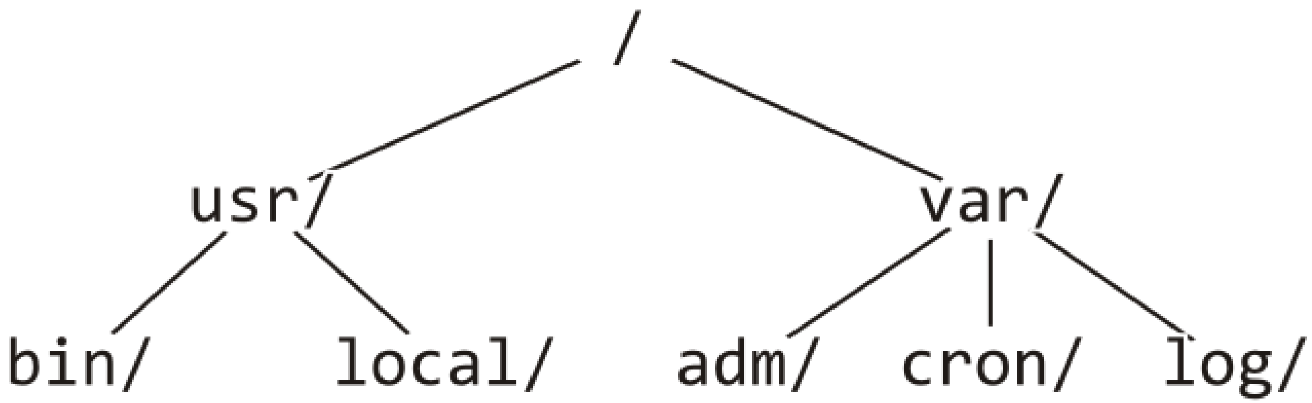
Postorder Traversal – Implementation

```
void postorder(Node *p) const
{
   if (p != NULL)
   {
      postorder(p->leftChild);
      postorder(p->rightChild);
      cout << p->info << " ";
   }
}

void main () {
   . . .
   postorder (root);
}
```

Example: Printing a Directory Hierarchy

- Consider the directory structure presented on the left
  - Which traversal should be used?




```
/
   usr/
      bin/
      local/
   var/
      adm/
      cron/
      log/
```