MT1006 Differential Equations

Saturday, September 23, 2023

Serial No:

1st Sessional Exam
Total Time:1 Hour
Total Marks: 45

Course Instructors

Dr. Arfan Shahzad
Dr. Saima Aslam
Mr. M. Munawar
Mr. M. Shoaib
Mr. Sameer Munir

Solution
R Aamir

Signature of Invigilator

Roll No ________ Section ________ Signature ________

DO NOT OPEN THE QUESTION BOOK OR START UNTIL INSTRUCTED.

Instructions:

1. Verify at the start of the exam that you have a total of Five (5) questions printed on Nine (09) pages including this title page.
2. Attempt all questions on the question-book and in the given order.
3. The exam is closed books, closed notes. Please see that the area in your threshold is free of any material classified as 'useful in the paper' or else there may a charge of cheating.
4. Read the questions carefully for clarity of context and understanding of meaning and make assumptions wherever required, for neither the invigilator will address your queries, nor the teacher/examiner will come to the examination hall for any assistance.
5. Fit in all your answers in the provided space. You may use extra space on the last page if required. If you do so, clearly mark question/part number on that page to avoid confusion.
6. Use only your own stationery and calculator. If you do not have your own calculator, use manual calculations.
7. Use only permanent ink-pens. Only the questions attempted with permanent ink-pens will be considered. Any part of paper done in lead pencil cannot be claimed for checking/rechecking.

	Q-1	Q-2	Q-3	Q-4	Q-5	Total
Total Marks	10	05	10	10	10	**45**
Marks Obtained						

Vetted By: ____________________ **Vetter Signature:** ____________________

University Answer Sheet Required: **No** ☐ **Yes** ☐

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Question No. 1	10 Points

a) Determine the convergence or divergence of the sequence $a_n = \frac{(-1)^{n+1}+5^{n+1}}{(-1)^n+5^n}$.

$$a_n = \frac{(-1)^{n+1} + (5)^{n+1}}{(-1)^n + 5^n}$$

$\div 5^n$

$$a_n = \frac{-\left(-\frac{1}{5}\right)^n + 5}{\left(-\frac{1}{5}\right)^n + 1}$$

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} -\left(-1/5\right)^n + \lim_{n\to\infty} 5}{\lim_{n\to\infty} \left(-\frac{1}{5}\right)^n + \lim_{n\to\infty} 1}$$

$\because \lim_{n\to\infty} x^n = 0$ if $|x|<1$.

$$= \frac{0+5}{1+1} = 5$$

Seq is Convergent.

b) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}+5^{n+1}}{(-1)^n+5^n}$. Justify your answer in detail.

$$a_n = \frac{(-1)^{n+1} + 5^{n+1}}{(-1)^n + 5^n}$$

By using Divergent test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1} + 5^{n+1}}{(-1)^n + 5^n}$$

$$= \lim_{n\to\infty} \frac{-\left(\frac{-1}{5}\right)^n + 5}{\left(-\frac{1}{5}\right)^n + 1}$$

$$= 5 \neq 0$$

So the given series is divergent,

$\because$ $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fail to exist

or is differend from zero

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Question No. 2	5 ~~10~~ Points

Identify the type of series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n^2+n}}$ and find the sum(if possible).

$$\sum \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \sum_{n=1}^{\infty}\left(\frac{\sqrt{n+1}}{\sqrt{n}\sqrt{n+1}} + \frac{\sqrt{n}}{\sqrt{n}\sqrt{n+1}}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}\right)$$

$$= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{1+n}}$$

$$1 + \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{3}} + \cdots + \frac{2}{\sqrt{n}} + \frac{1}{\sqrt{1+n}}$$

$$1 + 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}\right) + \frac{1}{\sqrt{1+n}}$$

$$1 + 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}\right) + \frac{1}{\sqrt{1+n}}$$

$$1 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}\right) + \frac{1}{\sqrt{1+}}$$

$$1 + 2\sum_{n=2}^{\infty} \frac{1}{n^{1/2}} + \frac{1}{\sqrt{1+n}}$$

Where $\sum \frac{1}{n^{1/2}}$

is a divergent P-Series. Hence the given series is divergent

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Question No. 3	10 Points

Apply integral test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$:

Let $a_n = \frac{\ln(n)}{\sqrt{n}}$

i) a_n is +ve $\forall$ $n \geq 1$.

ii) a_n is continuous $\forall$ $n \geq 1$.

iii) a_n is decreasing $\forall$ $n \geq 2$.

for Decreasing

$$f(n) = \frac{\ln n}{\sqrt{n}}$$

$$f'(n) = \frac{\frac{1}{n}\sqrt{n} - \frac{1}{2\sqrt{n}} \cdot \ln n}{n}$$

$$= \frac{\frac{1}{\sqrt{n}} - \frac{\ln n}{2\sqrt{n}}}{n}$$

$$= \frac{2 - \ln n}{2n\sqrt{n}} < 0$$

$2 < \ln n$

$e^2 < e^{\ln n}$ $\boxed{e^2 < n}$

$7.3 < x$

$$\int_N^\infty f(n)\,dn = \int_8^\infty \frac{\ln(n)}{\sqrt{n}}\,dn$$

Let $u = \ln n$

$du = \frac{1}{n}\,dn$

$du = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}}\,dn$

$\sqrt{n}\,du = \frac{1}{\sqrt{n}}\,dn$

when $n = 8$

$u = \ln(8)$

when $n \to \infty$

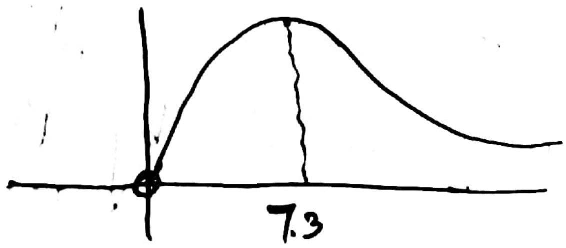
$u \to \infty$

$u = \ln(n)$

$e^u = n$

$e^{u/2} = \sqrt{n}$

$e^{u/2}\,du = \frac{1}{\sqrt{n}}\,dn$



$$\int_2^\infty \frac{\ln(n)}{\sqrt{n}}\,dn = \int_{\ln(2)}^{\infty} e^{u/2} \cdot u \, du$$

$$= \lim_{b \to \infty} \int_{\ln(2)}^{b} u \cdot e^{u/2} \, du$$

Integration by Parts

$$= u \frac{e^{u/2}}{1/2} - \int \frac{e^{u/2}}{1/2} \cdot 1 \, du$$

$$= 2\left[u e^{u/2} - \int e^{u/2} \, du\right]$$

$$= \lim_{b \to \infty} 2\left[u e^{u/2} - 2 e^{u/2}\right]\Big|_{\ln(2)}^{b}$$

$$= 2 \lim_{b \to \infty} e^{u/2}(u-2)\Big|_{\ln(2)}^{b}$$

$$= 2 \lim_{b \to \infty} \left[e^{b/2}(b-2) - 2 e^{\frac{\ln(2)}{2}}(\ln(2) - 2)\right]$$

$$= \infty$$

So, the Series is divergent.

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Question No. 4	10 Points

Apply limit comparison test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$.

Let $a_n = \frac{2^n+3^n}{3^n+4^n}$ and $b_n = \frac{3^n}{4^n}$ or $\left(\frac{3}{4}\right)^n$

and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

is a Convergent geometric Series.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{2^n+3^n}{3^n+4^n} \times \frac{4^n}{3^n} \right)$$

$$= \lim_{n\to\infty} \frac{8^n + 12^n}{9^n + 12^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{12^n}\left[(8/12)^n + 1\right]}{\cancel{12^n}\left[(9/12)^n + 1\right]}$$

$$= \lim_{n\to\infty} \frac{(2/3)^n + 1}{(3/4)^n + 1}$$

$$= \frac{0+1}{0+1} = 1 > 0,$$

$\because \lim_{n\to\infty} x^n = 0$; $|x| < 1$

$\Rightarrow \lim_{n\to\infty} (2/3)^n = 0$

And

$\lim_{n\to\infty} (3/4)^n = 0$

As $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)$ is a Convergent Series,

Hence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ also Converges.

Question No. 5	10 Points

Apply root test and ratio test to determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4^n}{(3n)^n}$ converges absolutely or not? justify your answer.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{(3n)^n}$$

By Root Test

let $a_n = (-1)^{n+1} \frac{4^n}{(3n)^n}$

$$|a_n| = \left| (-1)^{n+1} \frac{4^n}{(3n)^n} \right| = \frac{4^n}{(3n)^n}$$

So

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{4^{n\times\frac{1}{n}}}{(3n)^{n\times\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{4}{3n}$$

$$= \frac{4}{3} \lim_{n\to\infty} \frac{1}{n}$$

$$= \frac{4}{3} \cdot 0 = 0 < 1$$

Hence Series Converges absalutely by Root Test.

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By Ratio Test

Let $a_{n+1} = \frac{(-1)^{n+2} 4^{n+1}}{(3n+3)^{n+1}}$ & $a_n = \frac{(-1)^{n+1} 4^n}{(3n)^n}$

So,

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2} 4^{n+1}}{(3n+3)^{n+1}} \times \frac{(3n)^n}{(-1)^{n+1} 4^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} (-1) \cancel{4^n} \cdot 4^1}{(3n+3)^n \cdot (3n+3)^1} \cdot \frac{(3n)^n}{\cancel{(-1)^{n+1}} \cdot \cancel{4^n}} \right|$$

$$= 4 \lim_{n\to\infty} \left| \frac{(3n)^n}{(3n+3)^n} \cdot \frac{1}{3n+3} \right|$$

$$= 4 \left| \lim_{n\to\infty} \frac{1}{\left(\frac{3n+3}{3n}\right)^n} \cdot \lim_{n\to\infty} \frac{1}{3n+3} \right|$$

$$= 4 \left| \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \lim_{n\to\infty} \frac{1}{3n+3} \right|$$

$$= 4 \left| \frac{1}{e} \cdot 0 \right| = 0 < 1$$

Hence Series Converges absolutely by Ratio Test.

$\because \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$