Discrete Structures

Lecture # 07

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EXERCISE

A number of computer users are surveyed to find out if they have a printer, modem or scanner. Draw separate Venn diagrams and shade the areas, which represent the following configurations. **Keep in mind that all three are not mutually exclusive/disjoint.**

(i) modem and printer but no scanner

(ii) scanner but no printer and no modem

(iii) scanner or printer but no modem.

(iv) no modem and no printer.

SOLUTION

We have the sets

Computer users having a printer.

Computer users having a modem.

Computer users having a scanner.

SOLUTION

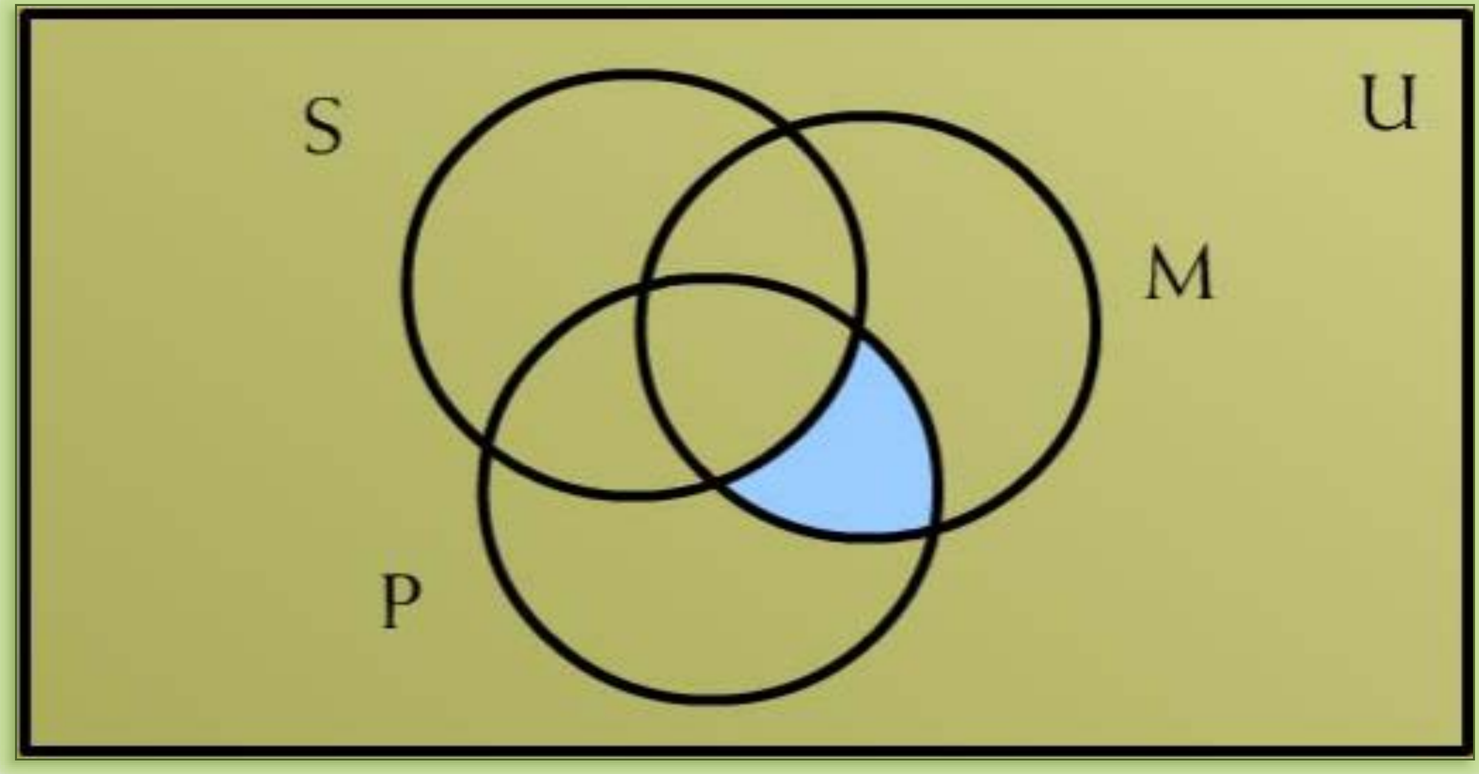
P represent the set of computer users having printer.

M represent the set of computer users having modem.

S represent the set of computer users having scanner.

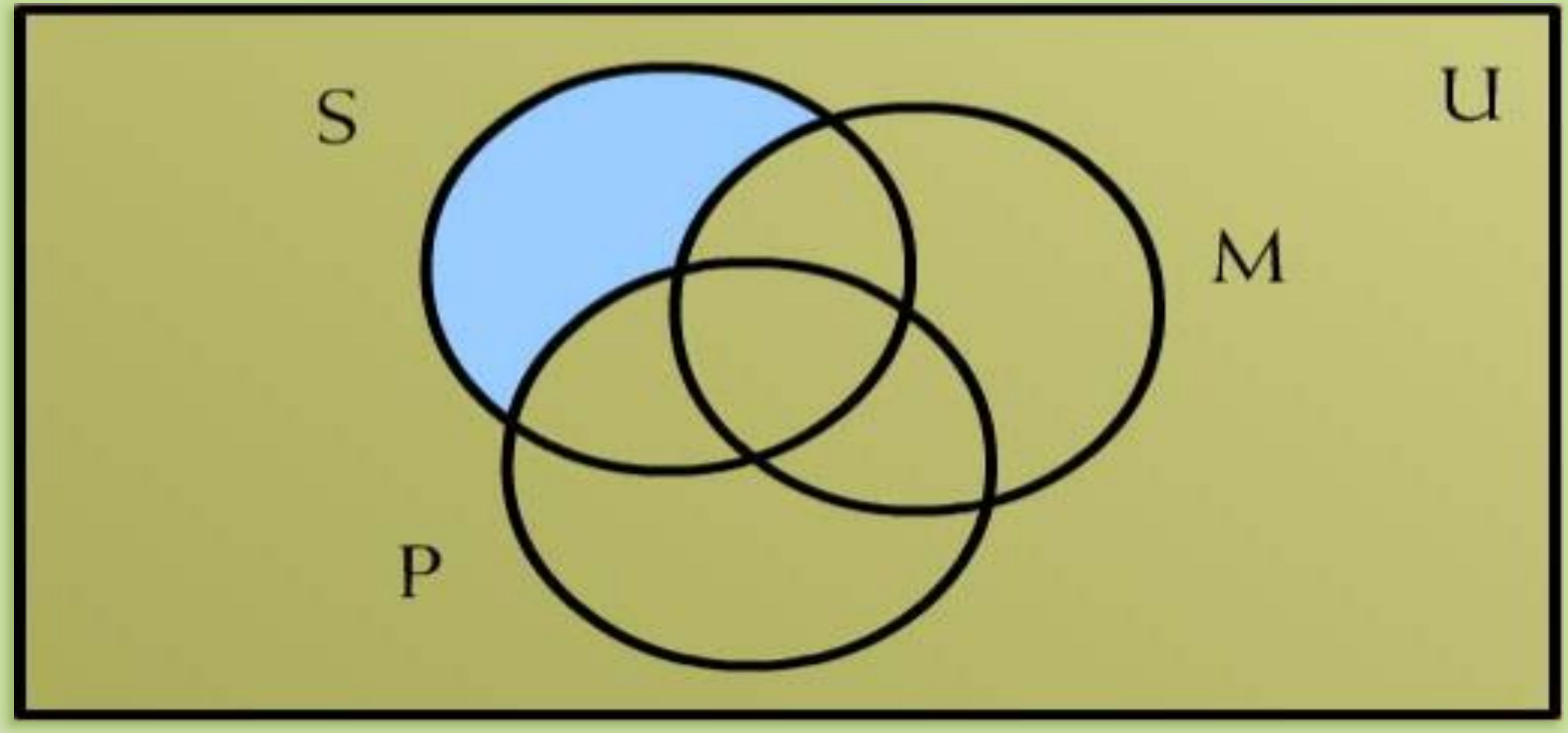
SOLUTION

(i) modem and printer but no scanner.



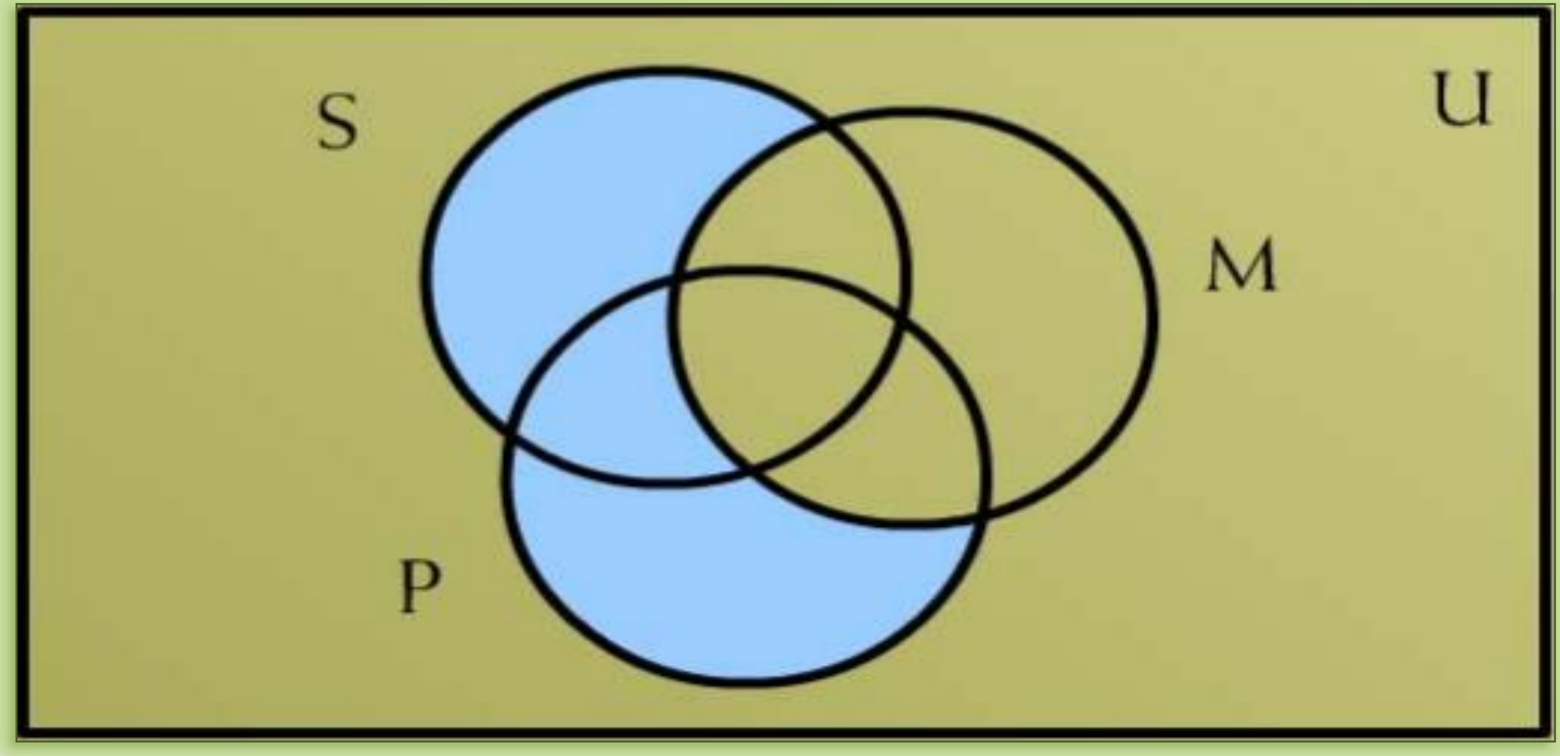
SOLUTION

(ii) scanner but no printer and no modem.



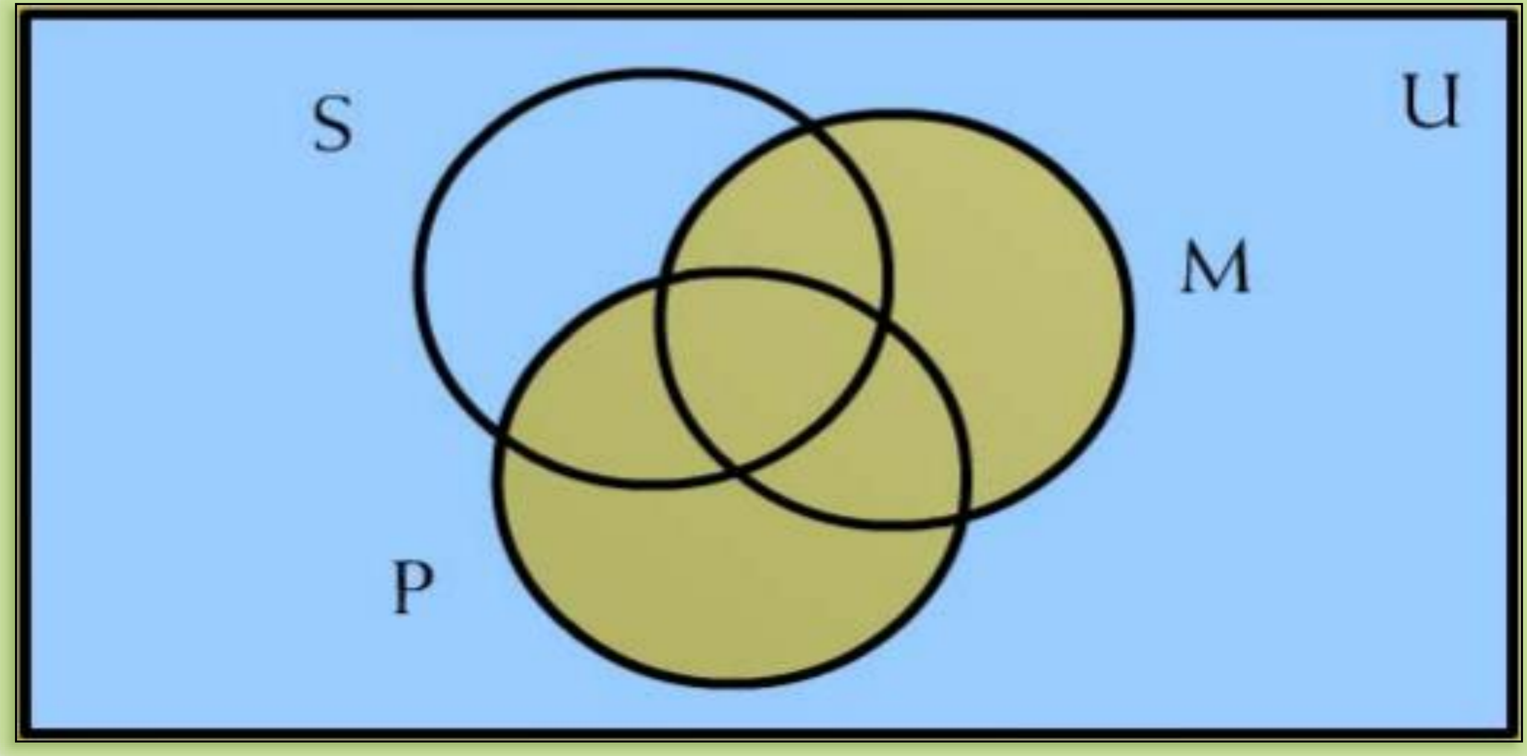
SOLUTION

(iii) scanner or printer but no modem.



SOLUTION

(iii) no modem no printer.



EXERCISE

21 typists are in an office, 5 use all manual typewriters (M), electronic typewriters (E) and word processors (W); 9 use E and W; 7 use M and W; 6 use M and E; but no one uses M only.

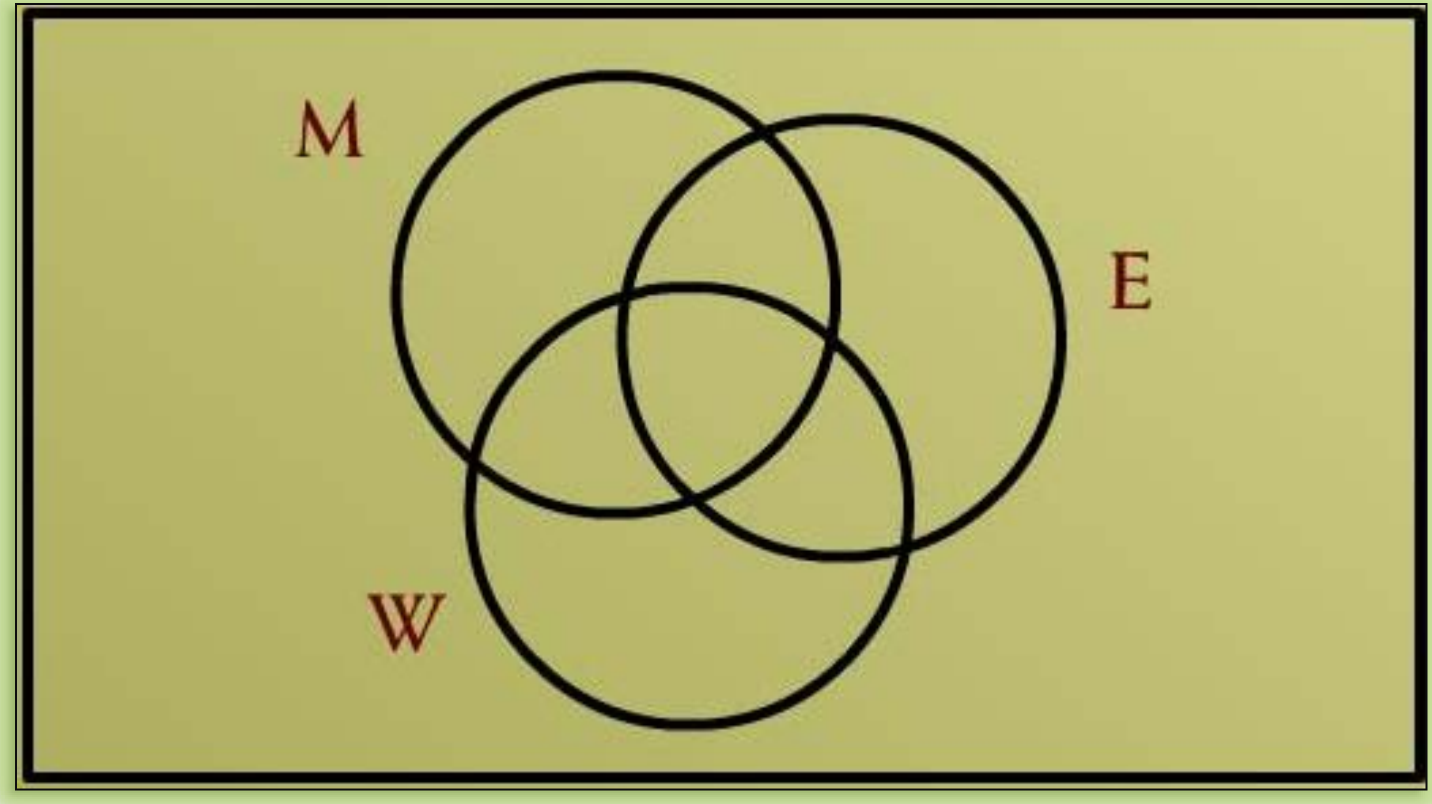
(i) Represent this information in a Venn Diagram.

(ii) If the same number of typists use electronic as use word processors, then

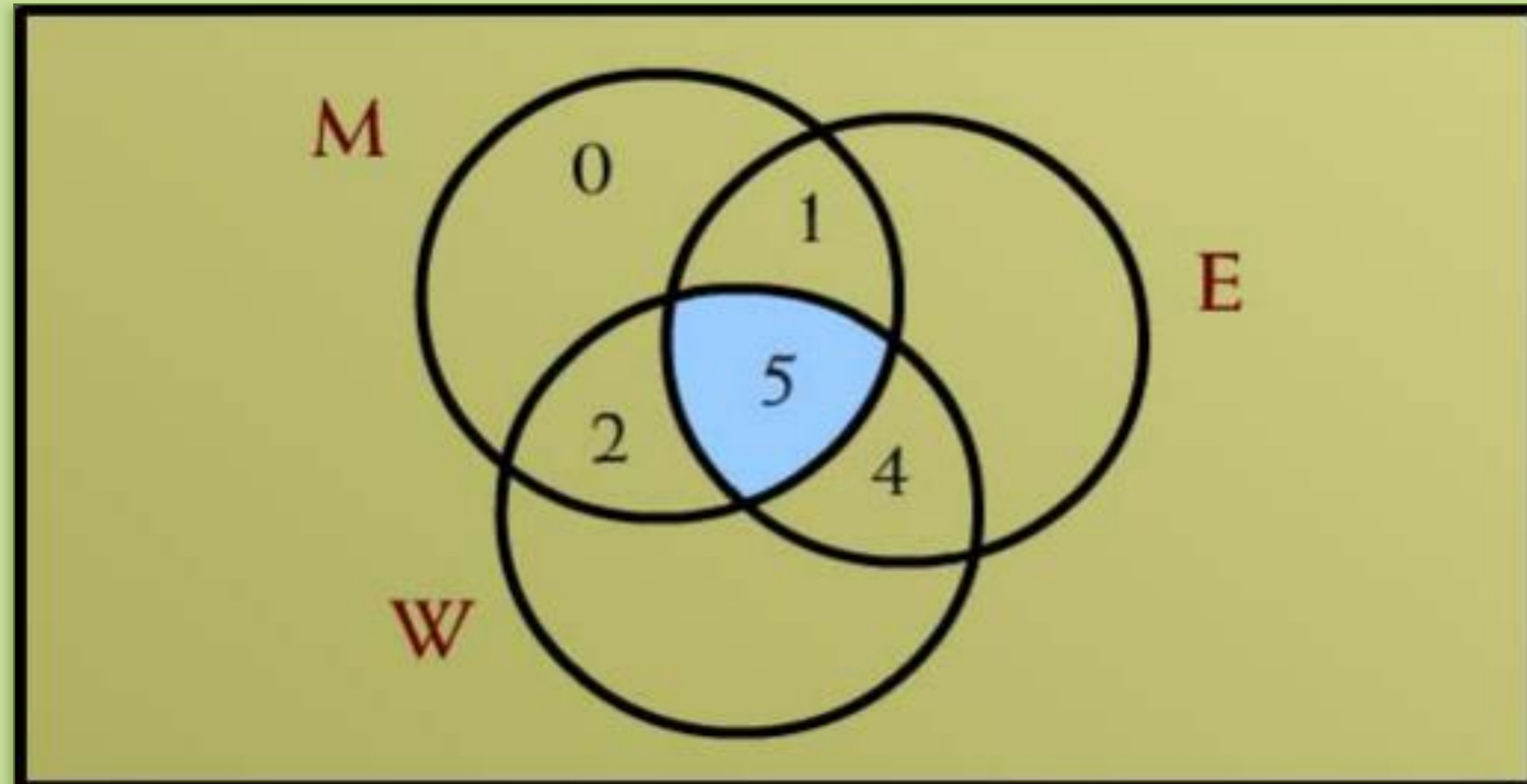
(a) how many use word processors only,

(b) how many use electronic typewriters?

SOLUTION

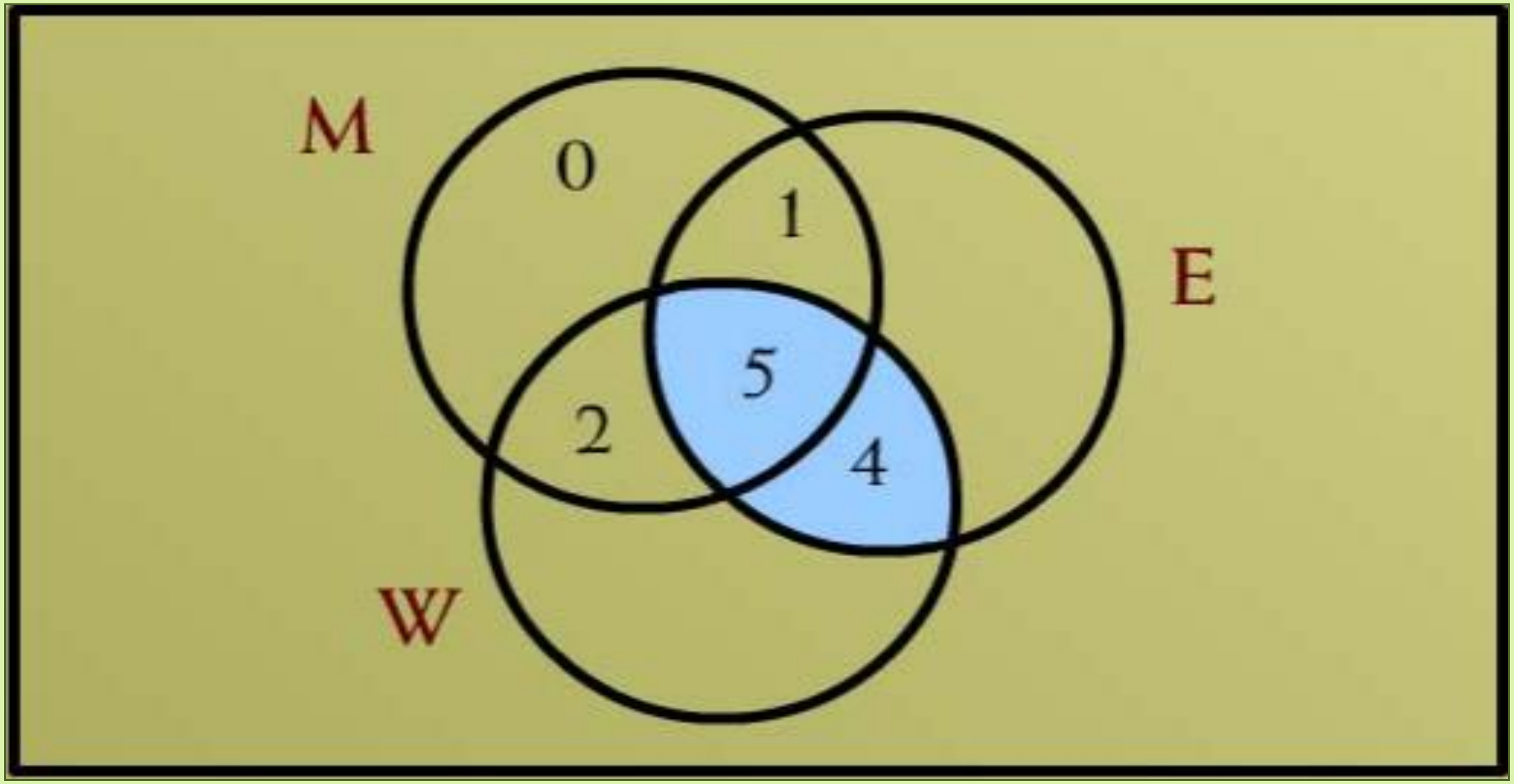


SOLUTION



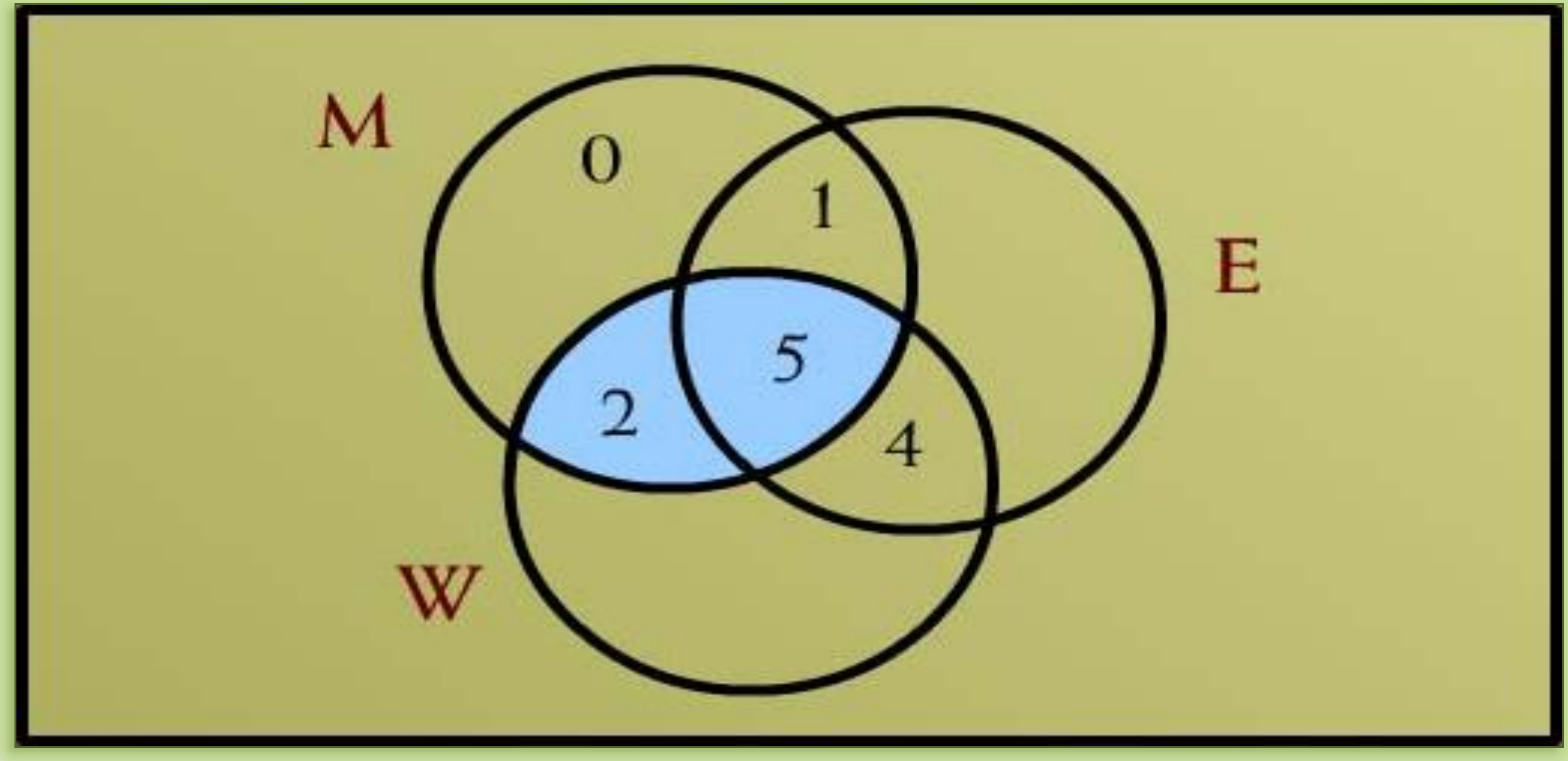
5 use all M, E and W

SOLUTION



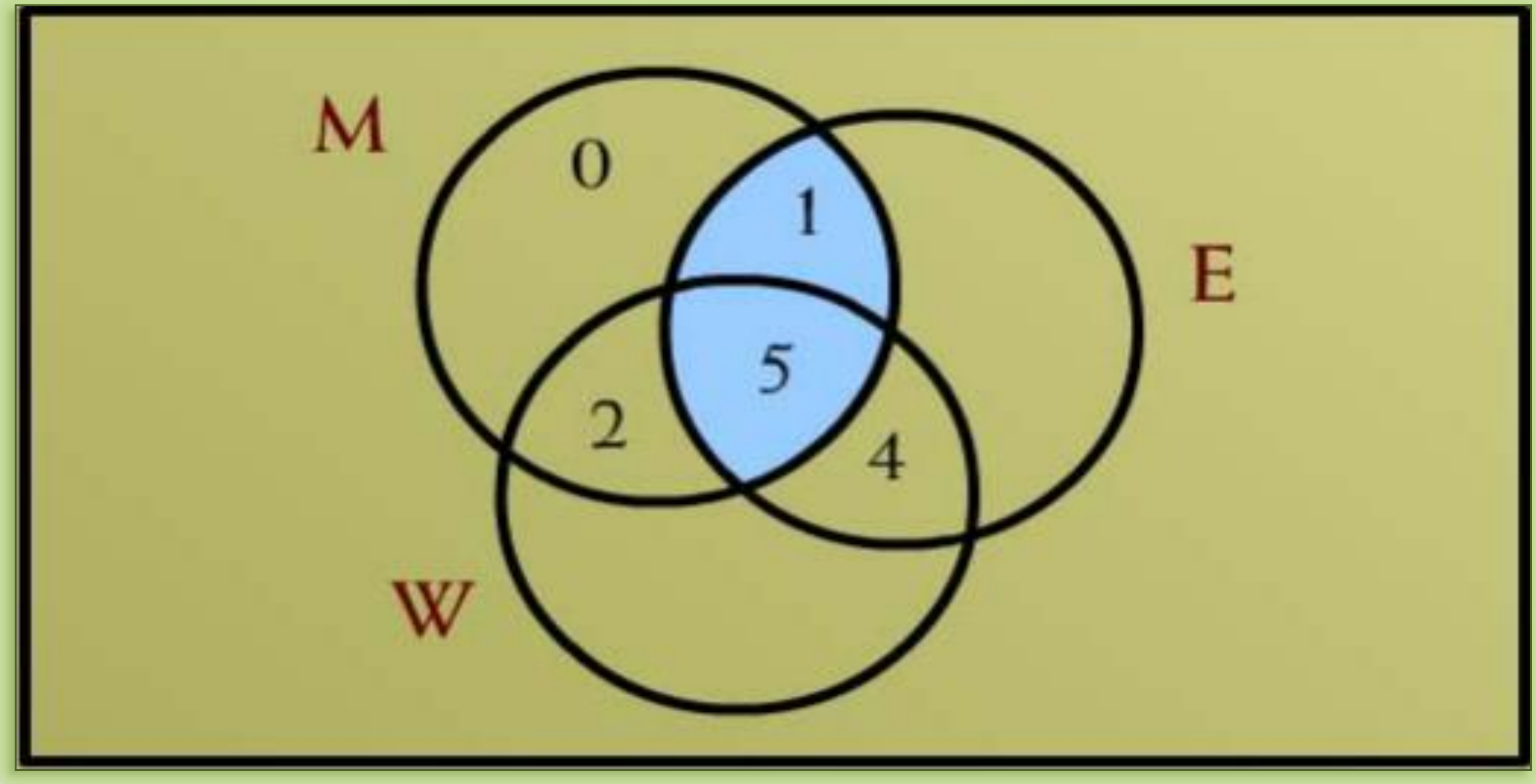
9 use E and W

SOLUTION



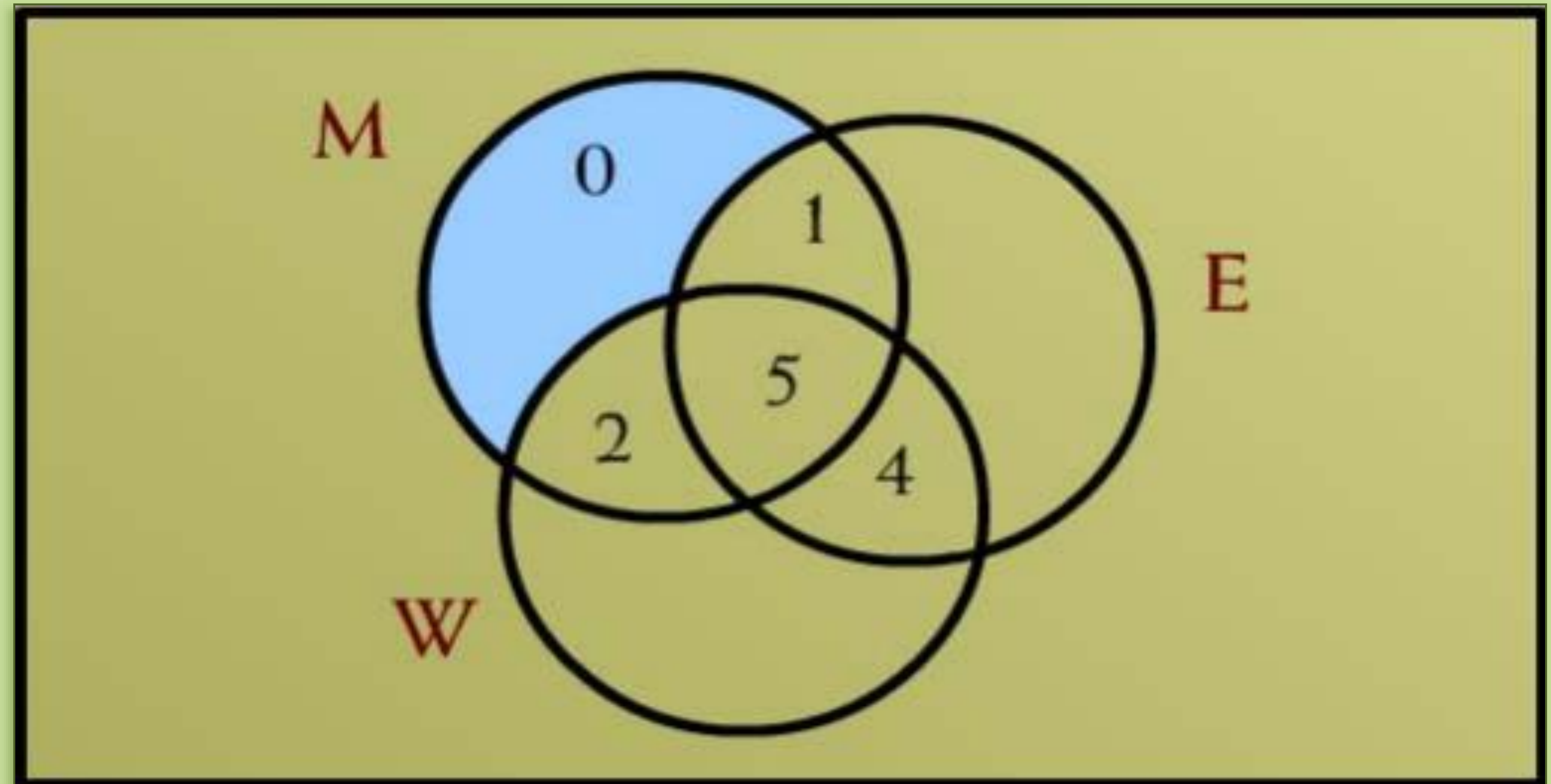
7 use M and W

SOLUTION



6 use M and E

SOLUTION



no one uses M only

SOLUTION

(ii)-(a) If same number of typists use electronic typewriters as word processors, how many use word processors only?

SOLUTION:

Let

x = number of typists using electronic typewriters (E)

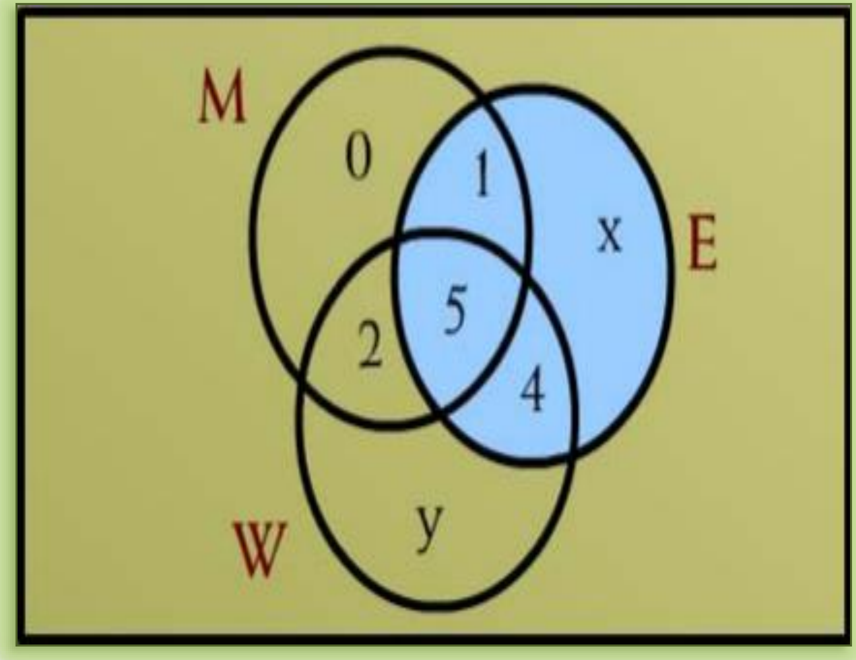
y = number of typists using word processors (W)

Solution contd…

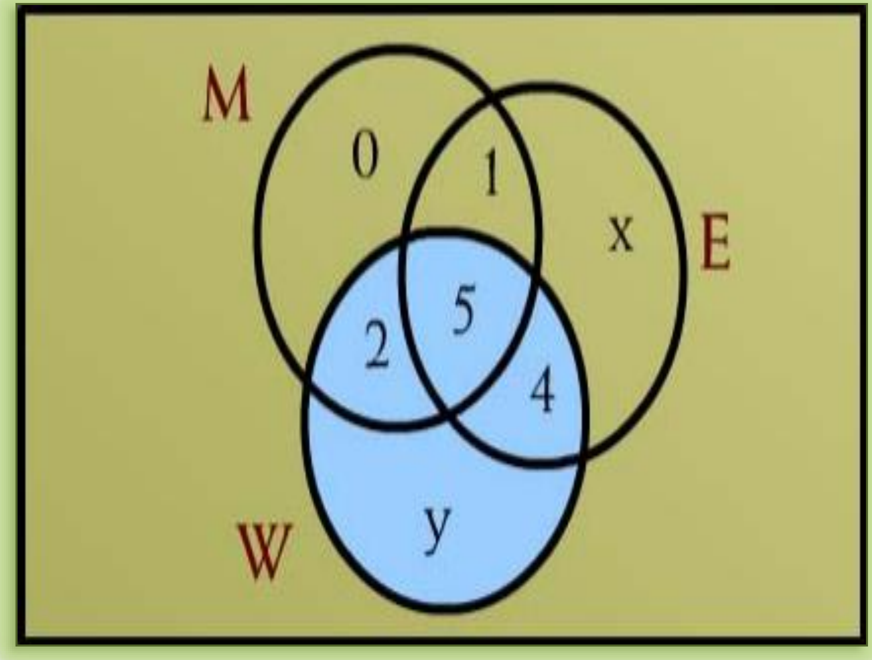
Total number of typists using E =

Total number of typists using W

$$1 + 5 + 4 + x = 2 + 5 + 4 + y$$



Electronic users only (In diagram)



Word processor users only (In diagram)

Solution contd…

or $\qquad x - y = 1 \ldots\ldots\ldots\ldots\ldots (1)$

total number of typists $= 21$

$$\Rightarrow 0 + x + y + 1 + 2 + 4 + 5 = 21$$

or $\qquad x + y = 9 \ldots\ldots\ldots\ldots\ldots (2)$

Solving (1) & (2), we get

$$x = 5,\ y = 4$$

$\therefore$ Number of typists using word processor only is $y = 4$

SOLUTION

(ii)-(b) How many typists use electronic typewriters?

SOLUTION:

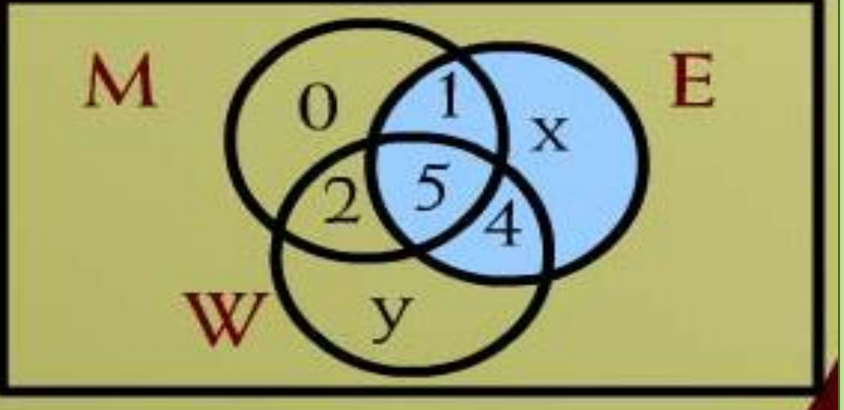
Typists using electronic typewriters

= No. of elements in E

= 1 + 5 + 4 + x

= 1 + 5 + 4 + 5

= 15



EXERCISE

In a school, 100 students have access to three software packages, A , B and C

28 did not use any software

8 used only packages A

26 used only packages B

7 used only packages C

10 used all three packages

13 used both A and B

EXERCISE

(i) Draw a **Venn diagram** with all sets enumerated as for as possible. Label the two subsets which cannot be enumerated as x and y, in any order.

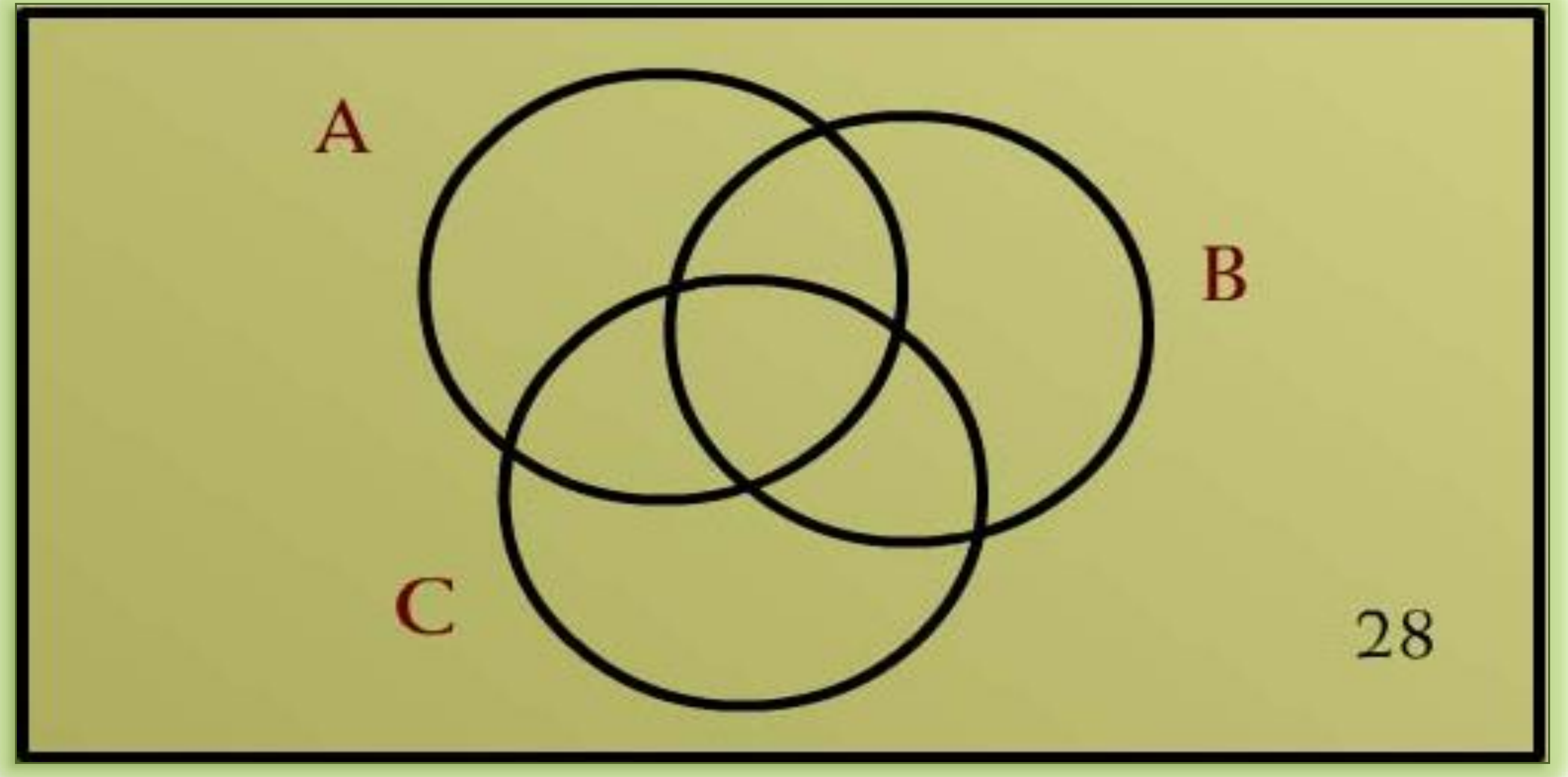
(ii) If twice as many students used **package B** as **package A**, write down a pair of simultaneous equations in x and y.

(iii) Solve these equations to find x and y.

(iv) How many students used **package C**?

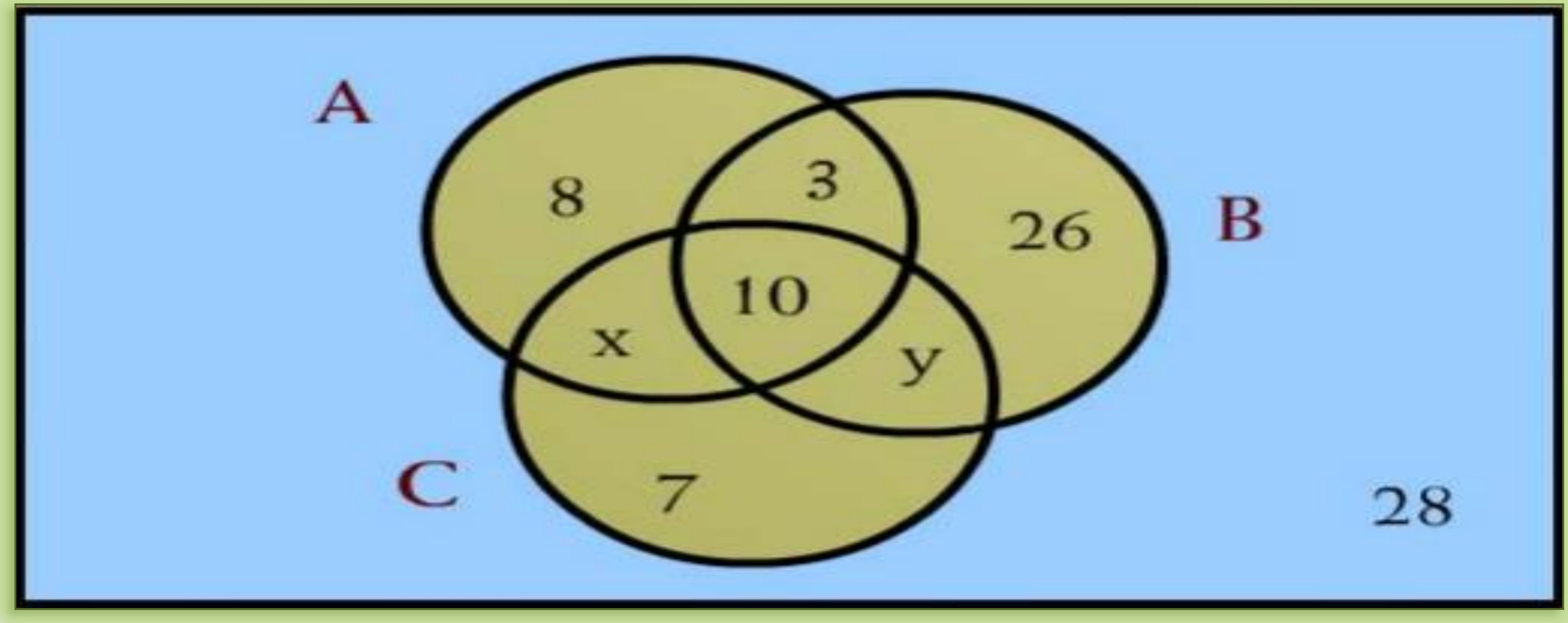
SOLUTION

(i) Venn Diagram with all sets enumerated.



SOLUTION

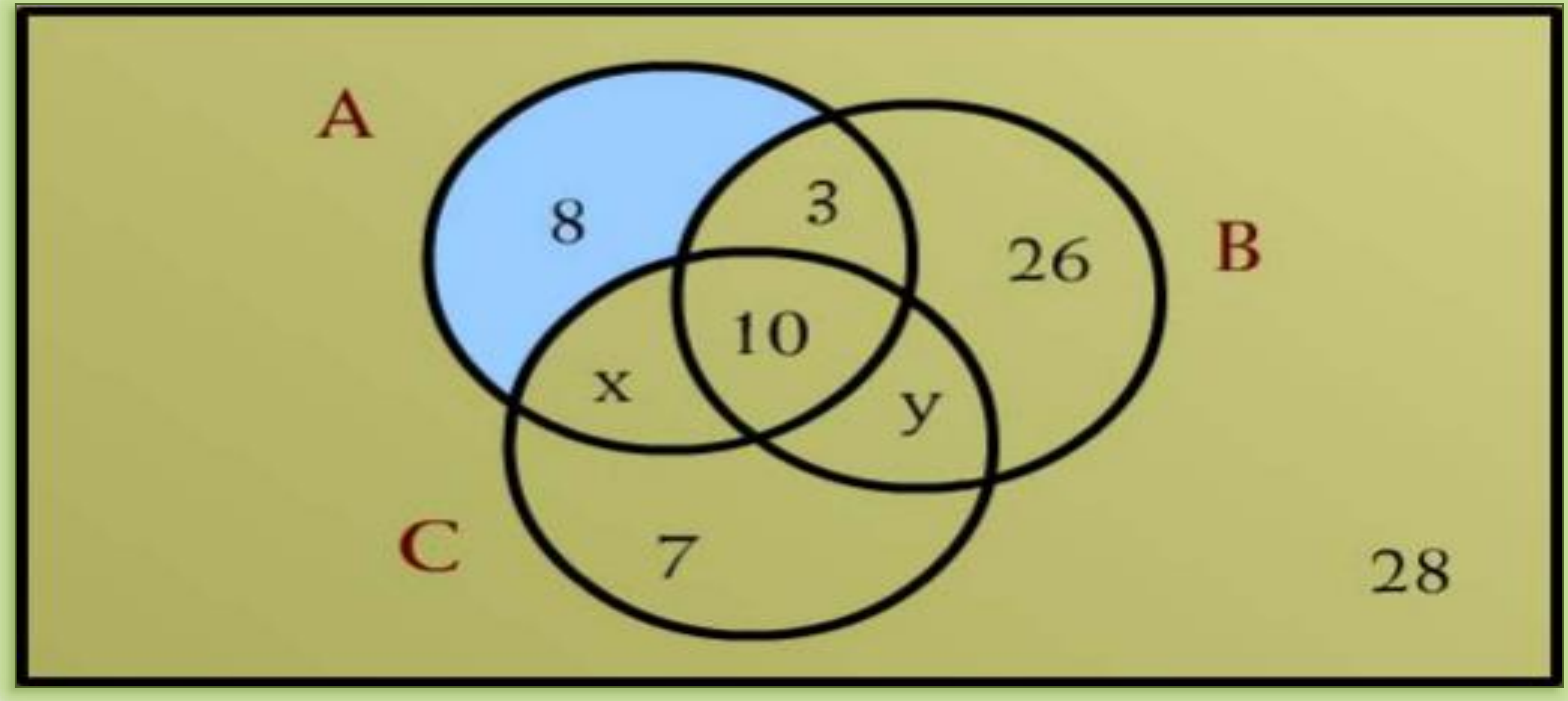
(i) Venn Diagram with all sets enumerated.



28 did not use any software

SOLUTION

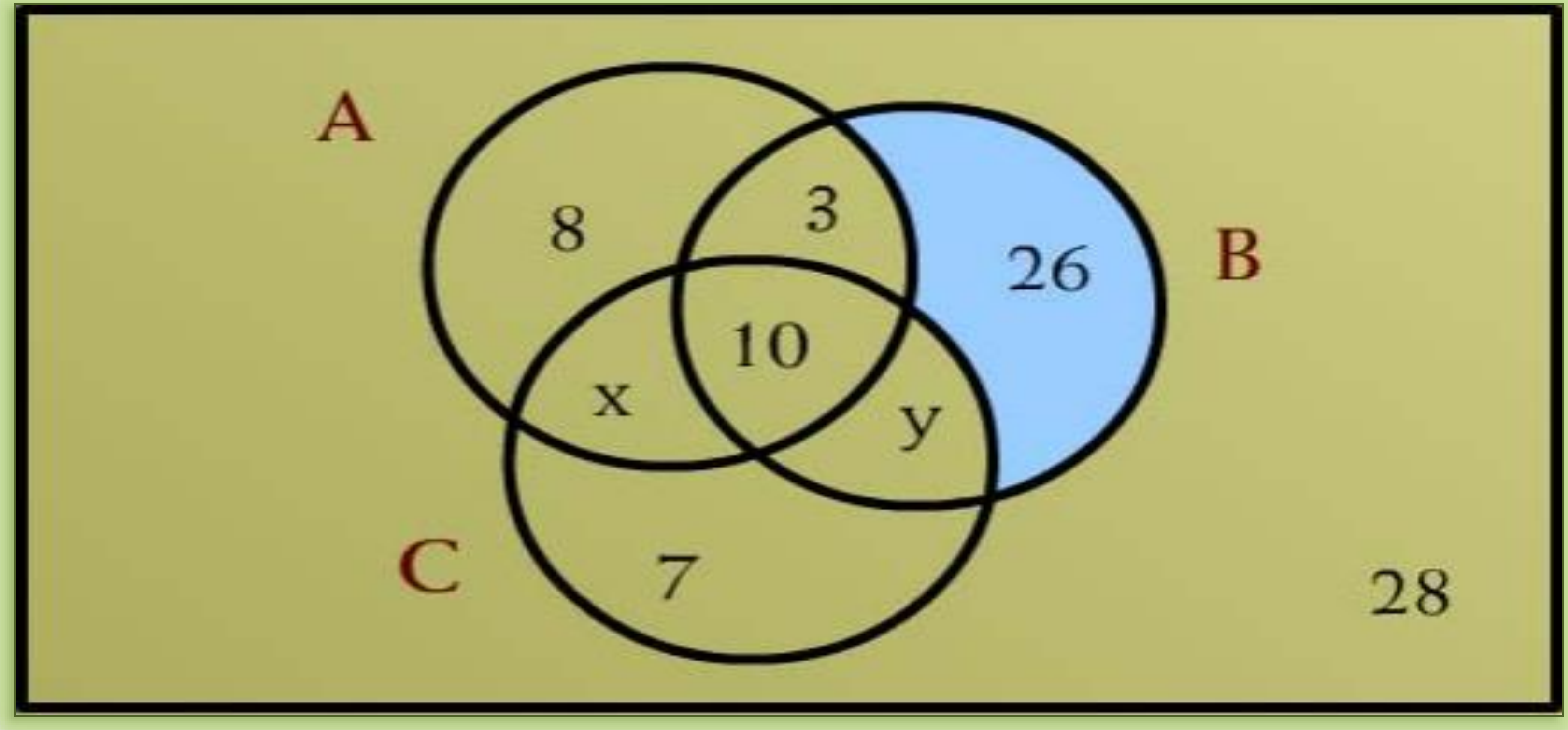
(i) Venn Diagram with all sets enumerated.



8 used only package A

SOLUTION

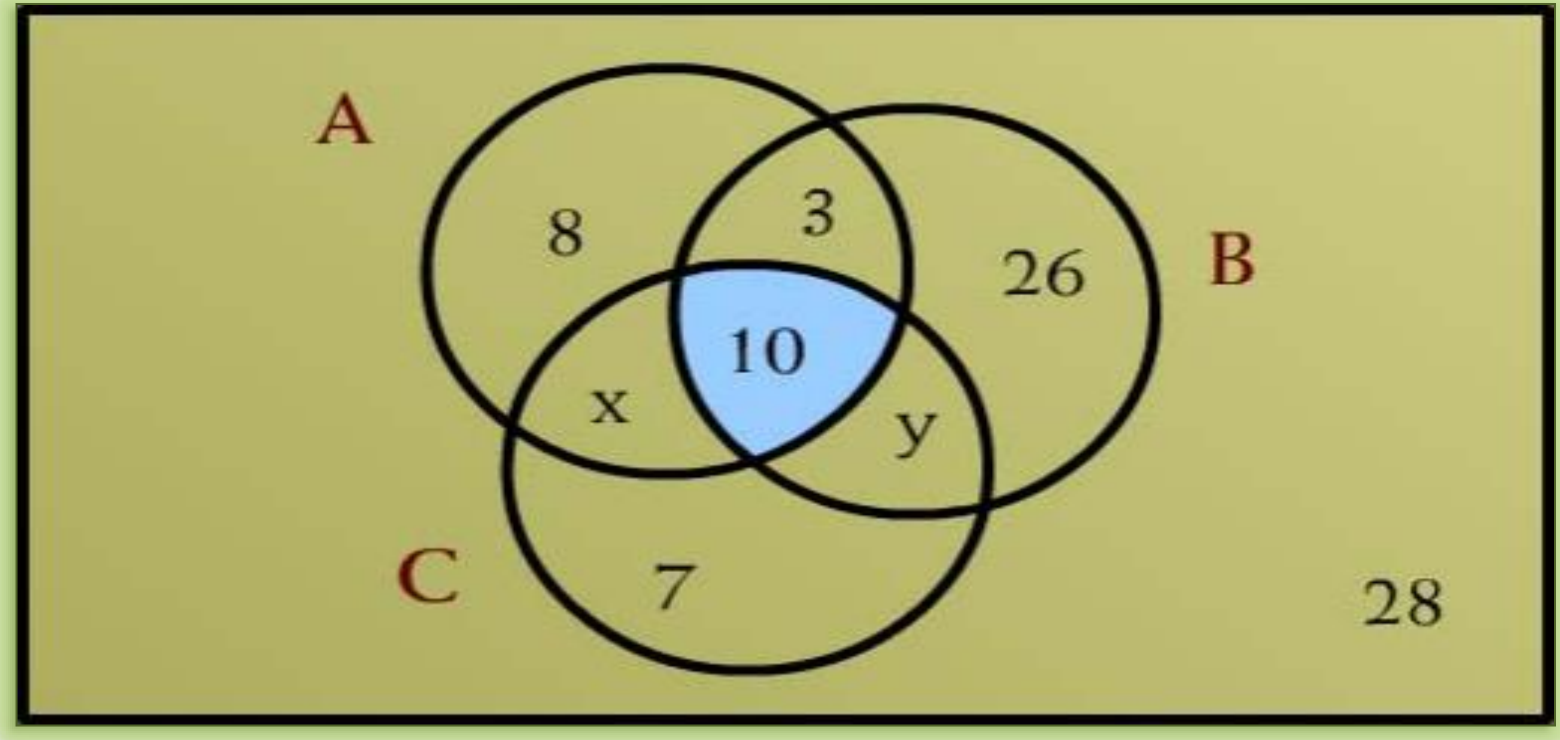
(i) Venn Diagram with all sets enumerated.



26 used only package B

SOLUTION

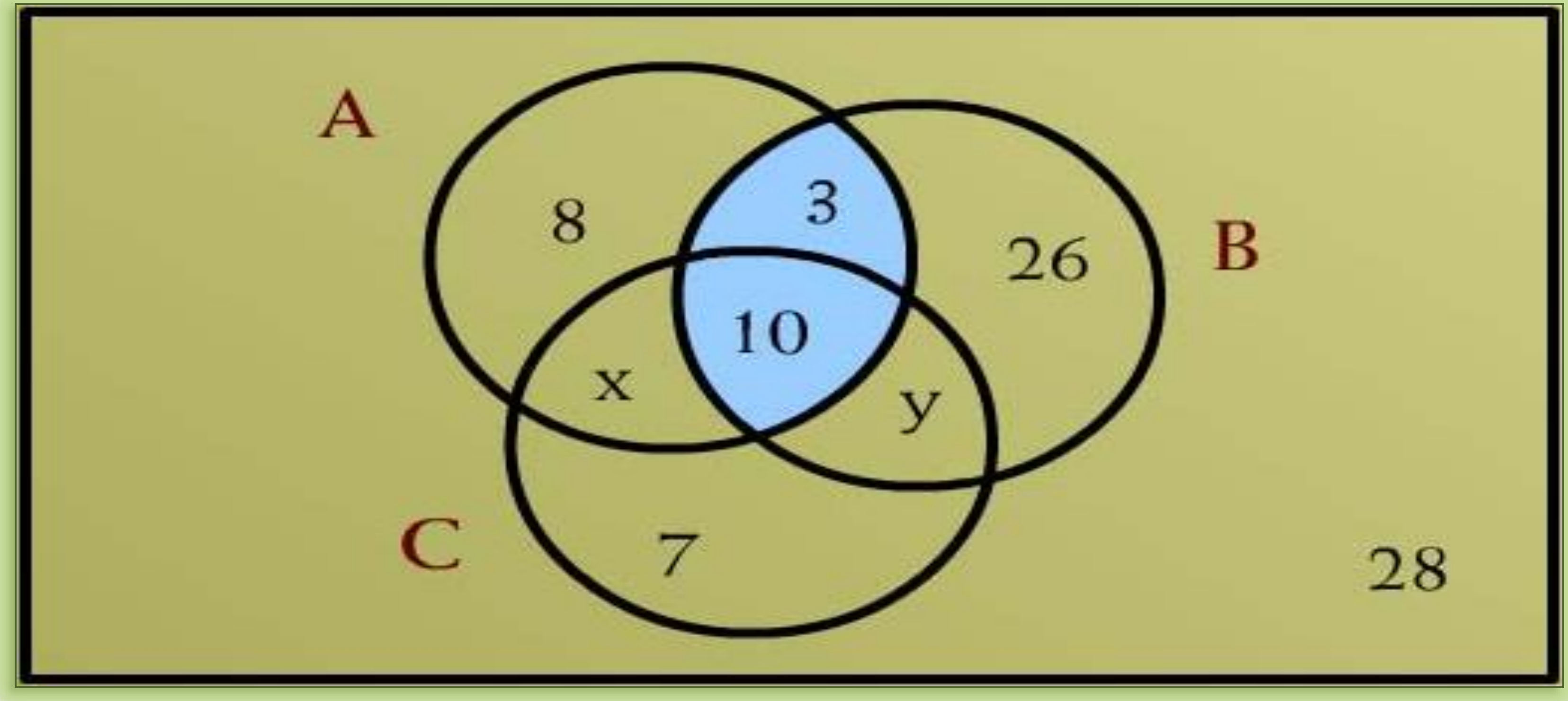
(i) Venn Diagram with all sets enumerated.



10 used all three packages

SOLUTION

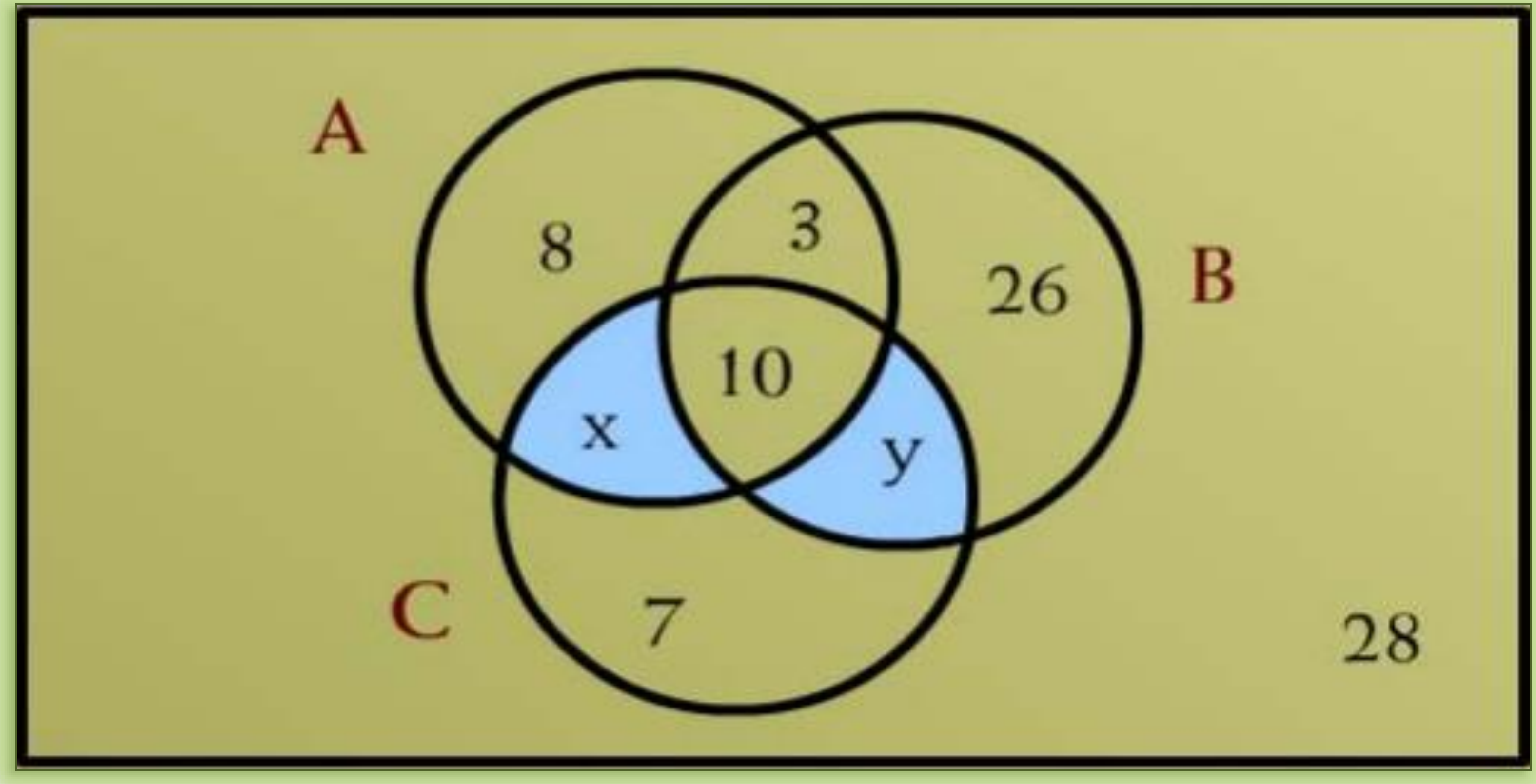
(i) Venn Diagram with all sets enumerated.



13 used both A and B

SOLUTION

(i) Venn Diagram with all sets enumerated.



Solution contd…

(ii) If twice as many students used package B as package A, write down a pair of simultaneous equations in x and y.

SOLUTION:

students using package B

=2(# students using package A)

$$\Rightarrow \quad 3 + 10 + 26 + y = 2\,(8 + 3 + 10 + x)$$

Solution contd…

$\Rightarrow \quad 39 + y = 42 + 2x$

or $\quad y = 2x + 3 \ldots\ldots\ldots\ldots (1)$

Also, total number of students = 100.

Hence,

$$8 + 3 + 26 + 10 + 7 + 28 + x + y = 100$$

or $\quad 82 + x + y = 100$

or $\quad x + y = 18 \ldots\ldots\ldots\ldots (2)$

Solution contd…

(iii) Solving simultaneous equations for x and y.

SOLUTION:

$$y = 2x + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x + y = 18 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Using (1) in (2), we get,

$$x + (2x + 3) = 18$$

or $3x + 3 = 18$

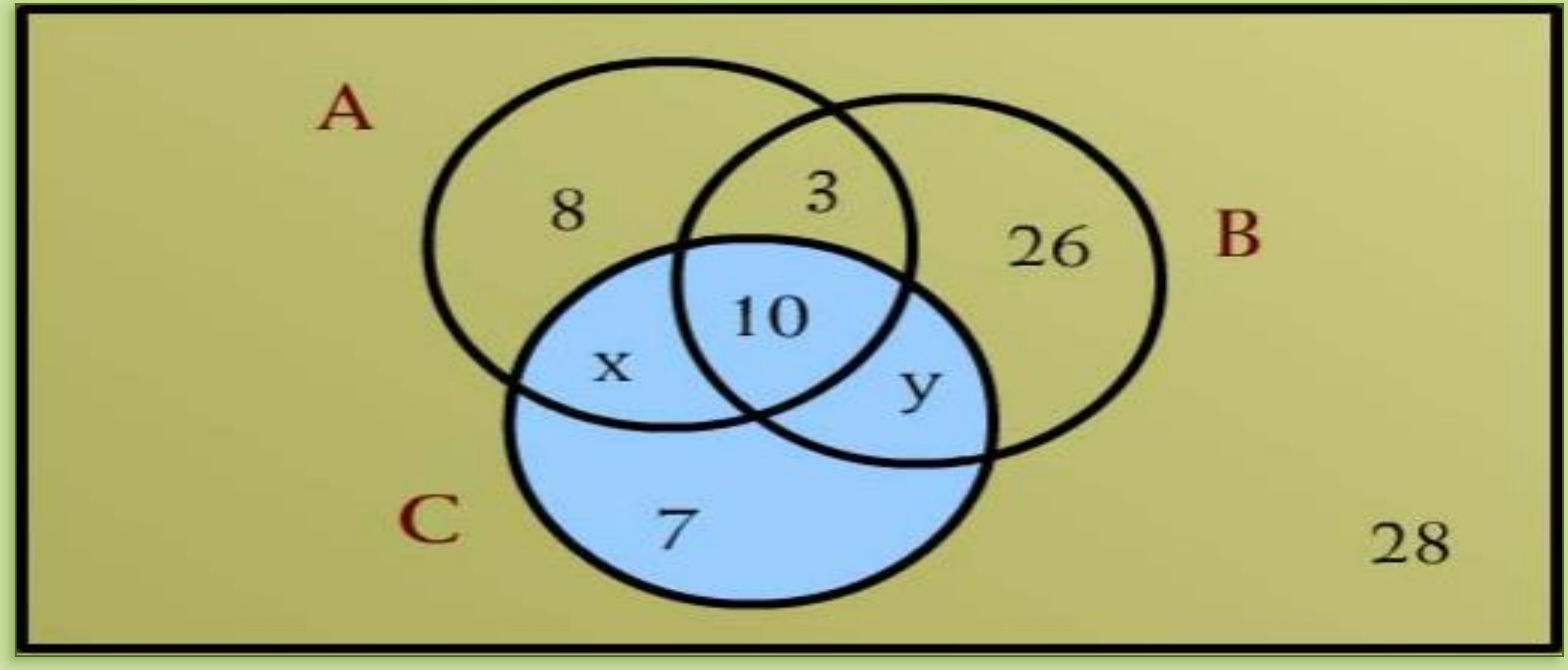
or $3x = 15$

$\Rightarrow$ $x = 5$

Consequently $y = 13$

SOLUTION

(iv) How many students used package C?



$$x + 10 + y + 7 = 35$$

PARTITION OF A SET

A set may be divided up into its **disjoint subsets**. Such division is called a **partition**.

More precisely,

> A partition of a set A is a collection of non-empty subsets $\{A_1, A_2, \ldots A_n\}$ of A, such that

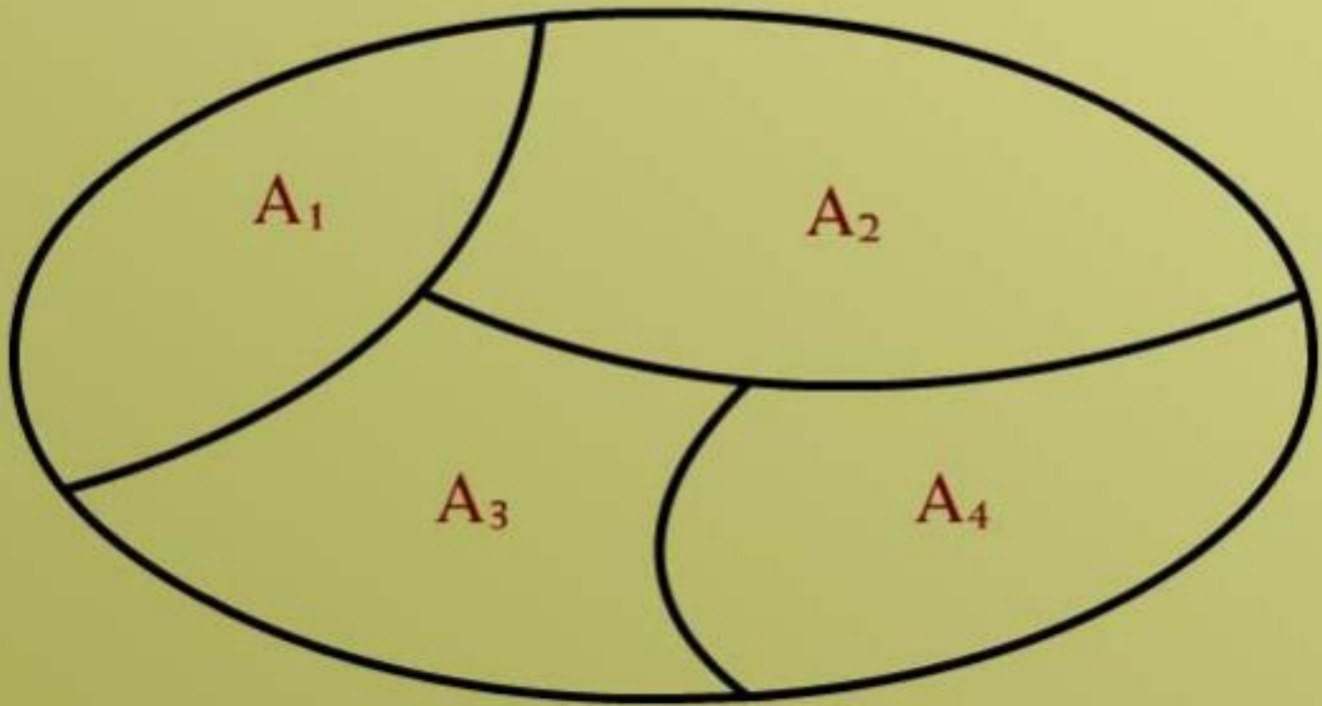
PARTITION OF A SET

1. $A = A_1 \cup, A_2 \cup \dots \cup A_n$

2. $A_1, A_2, \dots, A_n$ are **mutually disjoint** (or pair wise disjoint),
   i.e., $\forall$ i, j = 1, 2, …, n

   $$A_i \cap A_j = \varnothing \text{ whenever } i \neq j$$

PARTITION OF A SET



Partition of a set A

EXAMPLE

$$\text{Let } A = \{1, 2, 3, 4, 5, 6\}$$

$$A_1 = \{1, 2\}$$

$$A_2 = \{3, 4, 5\}$$

$$A_3 = \{6\}$$

Then

$$A_1 \cup A_2 \cup A_3 = \{1, 2\} \cup \{3, 4, 5\} \cup \{6\}$$
$$= A$$

EXAMPLE

$$A_1 \cap A_2 = \varnothing$$

$$A_1 \cap A_2 = \varnothing$$

and

$$A_2 \cap A_3 = \varnothing$$

i.e. A_1, A_2, A_3 are **mutually disjoint**.

$\{A_1, A_2, A_3\}$ is a **partition** of A.

EXAMPLE

Let E be the set of all even integers and O be the set of all odd integers. Is {E, O} a partition of Z, the set of all integers? Justify your answer.

SOLUTION

$$Z = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$$

$$E = \{0, \pm 2, \pm 4, \pm 8, \ldots\}$$

$$O = \{\pm 1, \pm 3, \pm 5, \pm 7, \ldots\}$$

$$E \cup O = Z$$

$$E \cap O = \varnothing$$

Then $\{E, O\}$ is a partition of Z.

POWER SET

The power set of a set A is the set of all subsets of A, denoted by P(A).

EXAMPLE:

Let $A = \{1, 2\}$, then

$$P(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$$

REMARK:

If A has **n elements** then P(A) has 2^n **elements**.

ORDERED PAIR

An ordered pair (a, b) consists of two elements “a” and “b” in which “a” is the first element and “b” is the second element.

ORDERED n-TUPLE

The **ordered n-tuple**, $(a_1, a_2, \ldots, a_n)$ consists of elements a_1, a_2, ..a_n together with the ordering: first a_1, second a_2, and so forth up to a_n.

In particular, an ordered **2-tuple** is called an **ordered pair**, and an ordered **3-tuple** is called an **ordered triple**.

EQUALITY OF ORDERED n-TUPLE

Two **ordered n-tuples**, $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if the corresponding elements are equal, that is

$$a_1 = b_1$$
$$a_2 = b_2$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$a_n = b_n$$

CARTESIAN PRODUCT OF TWO SETS

Let A and B be sets. The Cartesian product of A and B, denoted A x B (read "A cross B") is the set of all ordered pairs (a, b), where a is in A and b is in B.

Symbolically:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

EXAMPLE

$$A = \{1, 2\} \qquad B = \{a, b, c\}$$

$$A \times B = \{(x,y) \in A \times B \mid x \in A \text{ and } y \in B\}$$

$$= \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$

$$B \times A = \{(x,y) \in A \times B \mid x \in B \text{ and } y \in A\}$$

$$= \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

EXAMPLE

$$A = \{1, 2\} \qquad B = \{a, b, c\}$$

$$A \times A = \{(x,y) \in A \times A \mid x \in A \text{ and } y \in A\}$$

$$= \{(1, 1), (1,2), (2, 1), (2, 2)\}$$

$$B \times B = \{(x,y) \in B \times B \mid x \in B \text{ and } y \in B\}$$

$$= \{(a,a), (a,b), (a,c), (b,a), (b, b), (b, c), (c,a), (c,b), (c,c)\}$$

REMARKS

1. $A \times B \neq B \times A$

for non-empty and unequal sets A and B.

2. $A \times \varnothing = \varnothing \times A = \varnothing$

3. $|A \times B| = |A| \times |B|$

CARTESIAN PRODUCT OF MORE THAN TWO SETS

The **Cartesian product** of sets $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered **n-tuples** $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{ for } i = 1, 2, \ldots, n\}$$

RELATION

Let A and B be sets. A (binary) **relation R** from A to B is a subset of $A \times B$.

When $(a, b) \in R$, we say a is related to b by R, written a R b.

Otherwise if $(a, b) \notin R$, we write $a \not R b$. $a \not R b$ means that a is not related to b by R.

The total no. of relations that can be defined from a set A to a set B is the no. of possible subsets of A x B. If n(A)=p and n(B)=q then n(A x B)=pq and the total no. of relations is 2^(pq).

EXAMPLE

$$A = \{1, 2\} \qquad B = \{1, 2, 3\}$$

$$A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$$

$$R_1 = \{(1,1), (1, 3), (2, 2)\}$$

$$R_2 = \{(1, 2), (2, 1), (2, 2), (2, 3)\}$$

$$R_3 = \{(1, 1)\}$$

$$R_4 = A \times B$$

$$R_5 = \varnothing$$

DOMAIN OF A RELATION

The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R denoted Dom(R)

Symbolically:

$$\mathrm{Dom}(R) = \{a \in A \mid (a,b) \in R\}$$

RANGE OF RELATION

The range of A relation R from A to B is the set of all second elements of the ordered pairs which belong to R denoted Ran(R).

Symbolically:

$$\mathrm{Ran}(R) = \{b \in B \mid (a,b) \in R\}$$

EXERCISE

$A = \{1, 2\}$ $\qquad$ $B = \{1, 2, 3\}$

$$R = \{(a, b) \in A \times B \mid a < b\}$$

Then

a. Find the ordered pairs in R.

b. Find the Domain and Range of R.

c. Is 1R3, 2R2?

SOLUTION

$A = \{1, 2\}$ $\quad$ $B = \{1, 2, 3\}$

$A \times B =$
$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

a. $R = \{(a,b) \in A \times B \mid a < b\}$

$R = \{(1,2), (1,3), (2,3)\}$

Solution contd…

b. $\mathrm{Dom}(R) = \{1,2\}$

$\mathrm{Dom}(R) = A$

and

$\mathrm{Ran}(R) = \{2, 3\}$

$\mathrm{Ran}(R) \subseteq B$

c. Since $(1,3) \in R$ so $1R3$

But $(2, 2) \notin R$ so $2R3$

EXAMPLE

Let $A = \{\text{eggs, milk, corn}\}$

and

$B = \{\text{cows, goats, hens}\}$

Define a relation R from A to B by $(a, b) \in R$ iff a is produced by b.

EXAMPLE

Let $A = \{eggs, milk, corn\}$

and

$B = \{cows, goats, hens\}$

Define a relation R from A to B by $(a, b) \in R$ iff a is produced by b.

Then **Define A x B first**

$R = \{(eggs,hens),(milk,cows),(milk,goats)\}$

eggs R hens,

milk R cows, milk R goats etc.

EXERCISE

$A = \{0,1\}$ $B = \{1\}$

Find all binary relations from A to B

SOLUTION:

$$A \times B = \{(0,1),\ (1,1)\}$$

All binary relations from A to B are in fact all subsets of $A \times B$, which are:

$$R_1 = \varnothing$$

$$R_2 = \{(0,1)\}$$

$$R_3 = \{(1,1)\}$$

$$R_4 = \{(0,1),\ (1,1)\} = A \times B$$

RELATION ON A SET

A **relation** on the set A is a **relation** from A to A.
In other words, a **relation** on a set A is a **subset** of $A \times A$.

EXAMPLE:

Let $A = \{1, 2, 3, 4\}$

$(a,b) \in R$ iff **a divides b** {symbolically written as $a \mid b$}

Define A x A first

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

REMARK

For any set A

1. $A \times A$ is known as the **universal relation.**

2. $\varnothing$ is known as the **empty relation.**