Discrete Structures

Lecture # 14

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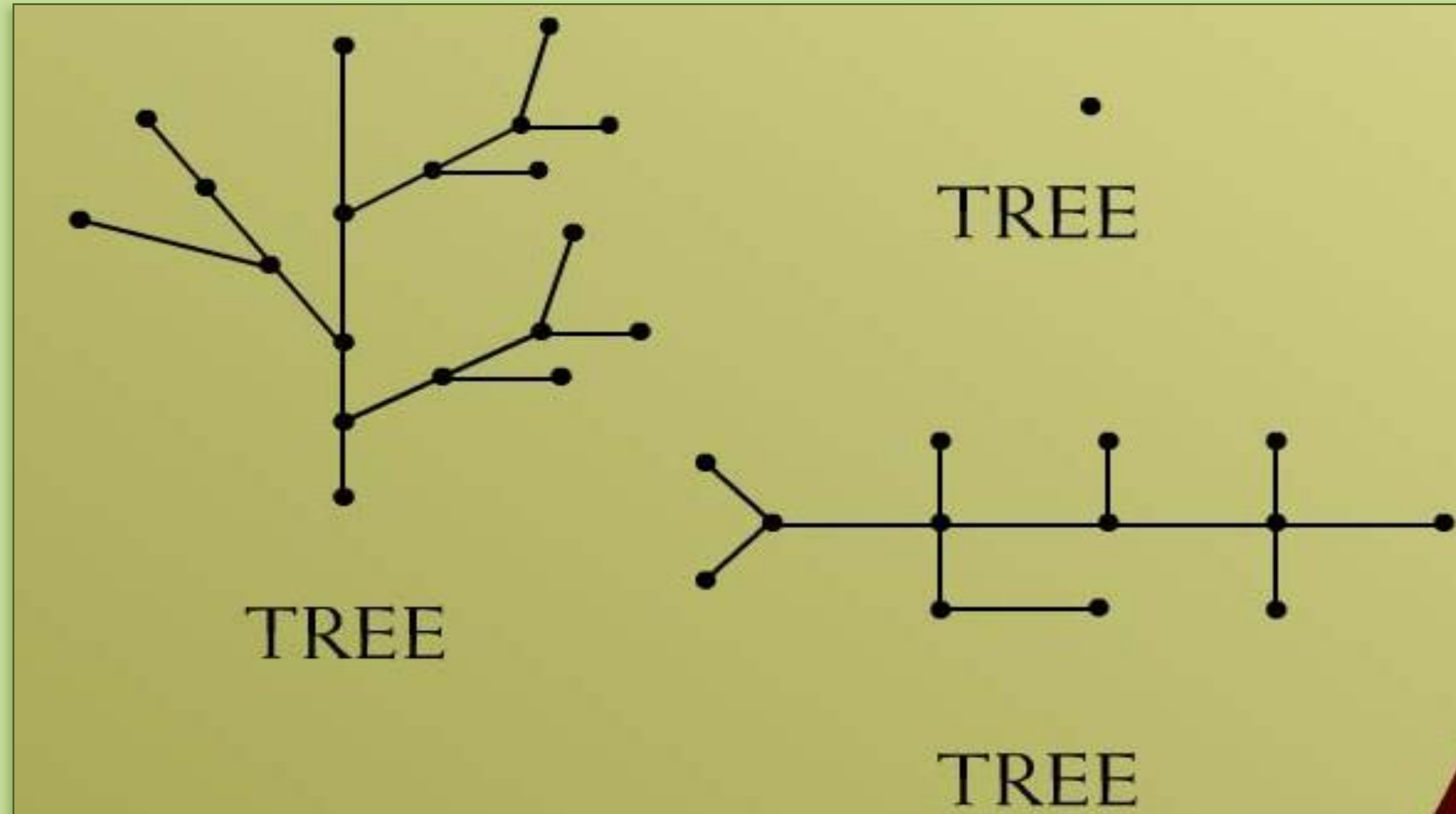
TREE

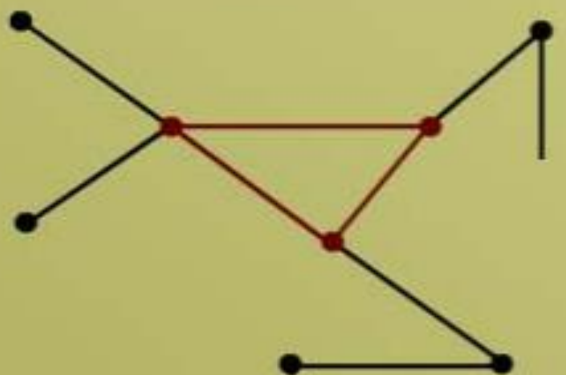
A tree is a connected graph that does not contain any nontrivial circuit. (it is circuit-free).

A trivial circuit is one that consists of a single vertex.

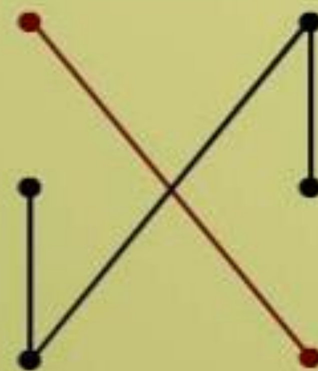
EXAMPLE



EXAMPLES OF NON TREES



(a) Graph with a circuit.



(b) Disconnected graph.

(c) Graph with a circuit.

SOME SPECIAL TREES

1. TRIVIAL TREE

   A **graph** that consists of a **single vertex** is called a **trivial tree** or **degenerate tree**.

2. EMPTY TREE

   A **tree** that does not have any **vertices or edges** is called **an empty tree**.

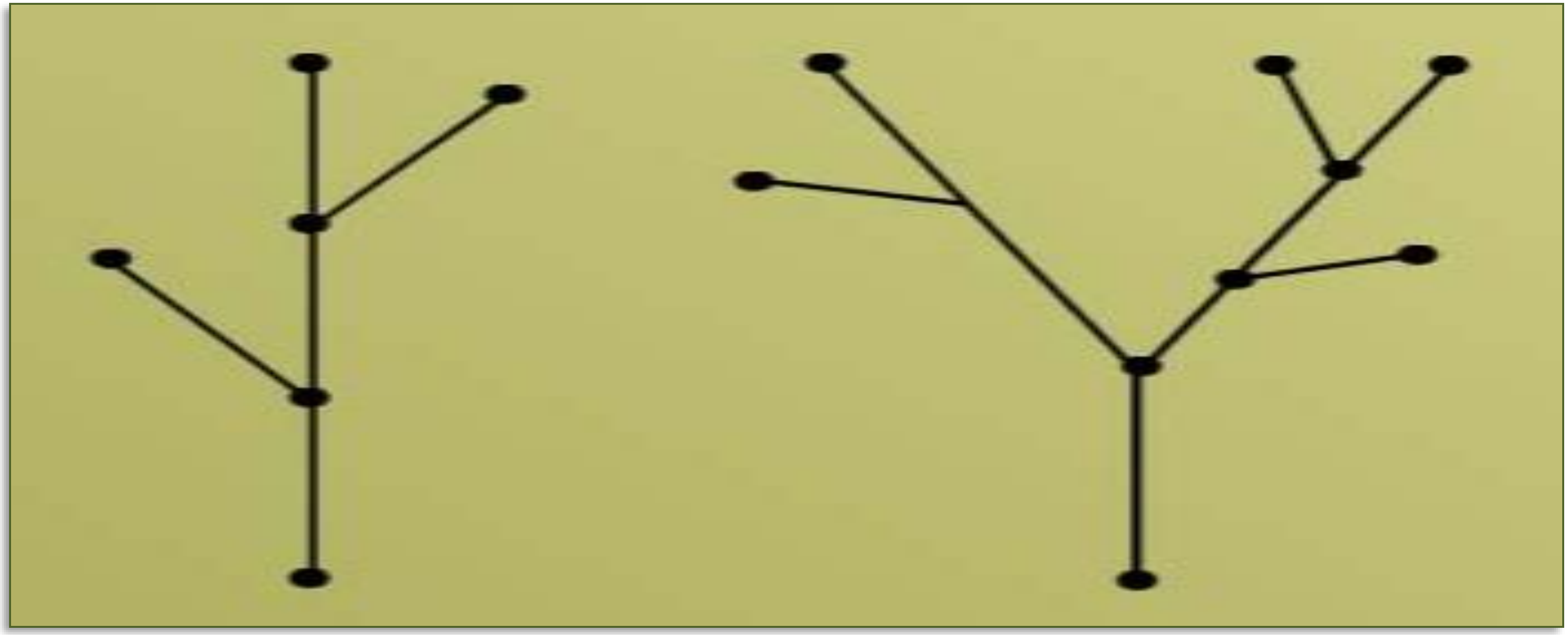
SOME SPECIAL TREES

FOREST:

A graph is called a forest if, and only if, it is circuit-free.

Hence, the connected components of a forest are trees.

FOREST



Forest has only two trees in it.

PROPERTIES OF TREES

1. A tree with n vertices has n − 1 edges.

2. Any connected graph with n vertices and n − 1 edges is a tree.

3. A tree has no nontrivial circuit; but if one new edge is added to it, then the resulting graph has exactly one nontrivial circuit.

PROPERTIES OF TREES

4. A tree is connected, but if any edge is deleted from it, then the resulting graph is not connected.

5. Any tree that has more than one vertex has at least two vertices of degree 1.

6. A graph is a tree iff there is a unique path between any two of its vertices.

EXERCISE

Explain why graphs (Tree) with the given specification do not exist.

1. Tree with twelve vertices and fifteen edges.
2. Trees with five vertices and total degree 10.

SOLUTION:

1. Any tree with 12 vertices will have $12 - 1 = 11$ edges.
   Hence solution of first is not possible.

SOLUTION

Tree with five vertices and total degree 10.

Any tree with five vertices will have 5 - 1 = 4 edges.

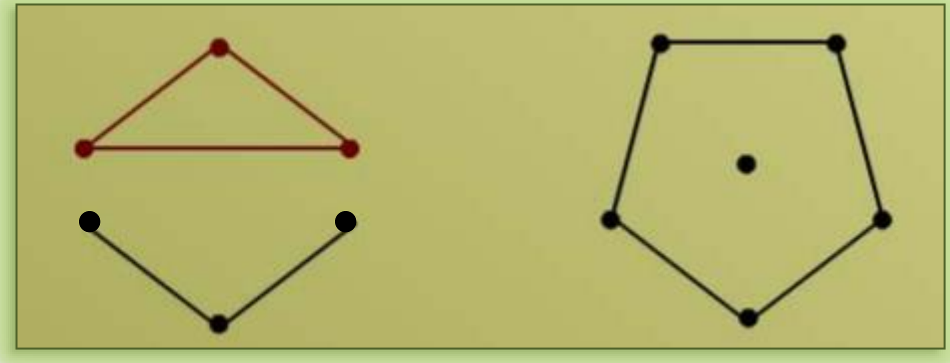
We are given total degree of graph is 10. So it must have edges 10/2 = 5.

The two conditions contradict each other.

SOLUTION

Draw a graph with six vertices, five edges that is not a tree.

SOLUTION:

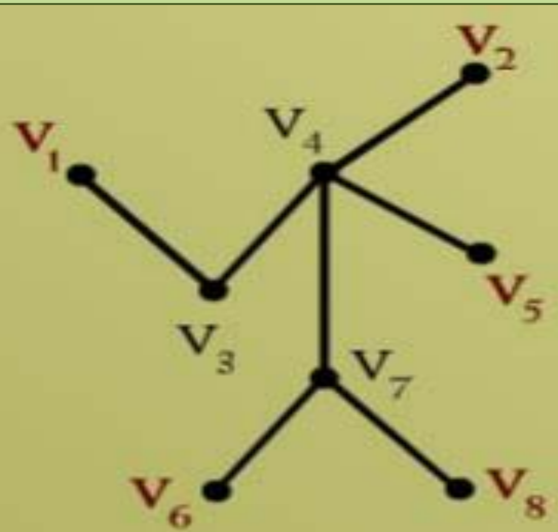


First is not tree because it is not connected and also has a circuit similarly for second.

TERMINAL AND INTERNAL VERTEX

A vertex of degree ‘1’ in a tree is called terminal vertex or a leaf (leaf has no children) and a vertex of degree greater than ‘1’ in a tree is called an internal vertex or a branch vertex.

EXAMPLE



v_1, v_2, v_5, v_6 and v_8 are terminal vertices.

v_3, v_4, v_7 are internal vertices.

ROOTED TREE

A rooted tree is a tree in which one vertex is distinguished from the others and is called the root.

The level of a vertex is the number of edges along the unique path between it and the root.

The height of a rooted tree is the maximum level to any vertex of the tree.

ROOTED TREE

The children of any internal vertex v are all those vertices that are adjacent to v and are one level farther away from the root than v. If w is a child of v, then v is called the parent of w.

Two vertices that are both children of the same parent are called siblings.

Given vertices v and w, if v lies on the unique path between w and the root, then v is an ancestor of w and w is a descendant of v.

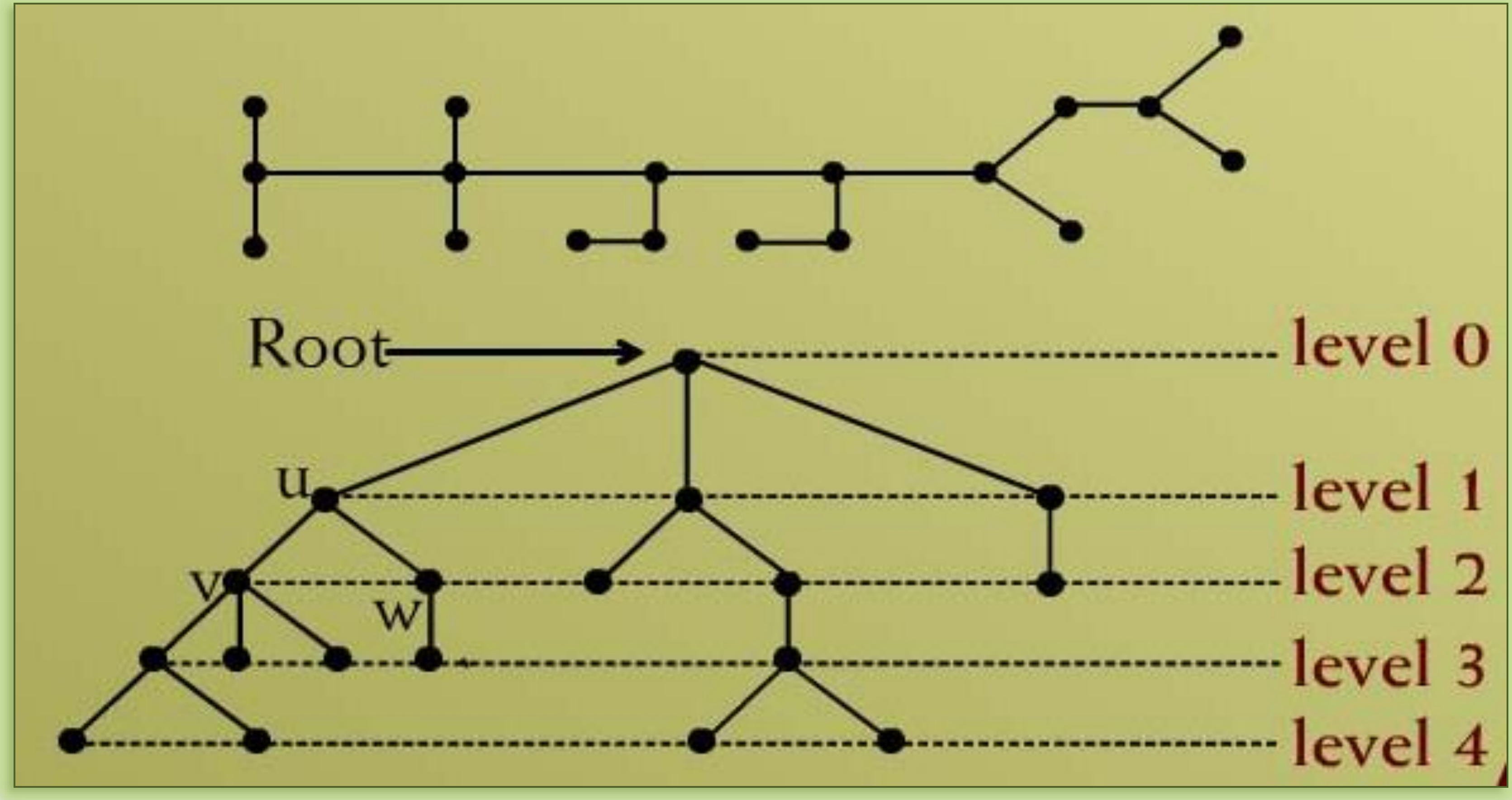
ROOTED TREE

The root is an internal vertex unless it is the only vertex in the graph, and in that case it will be a leaf.

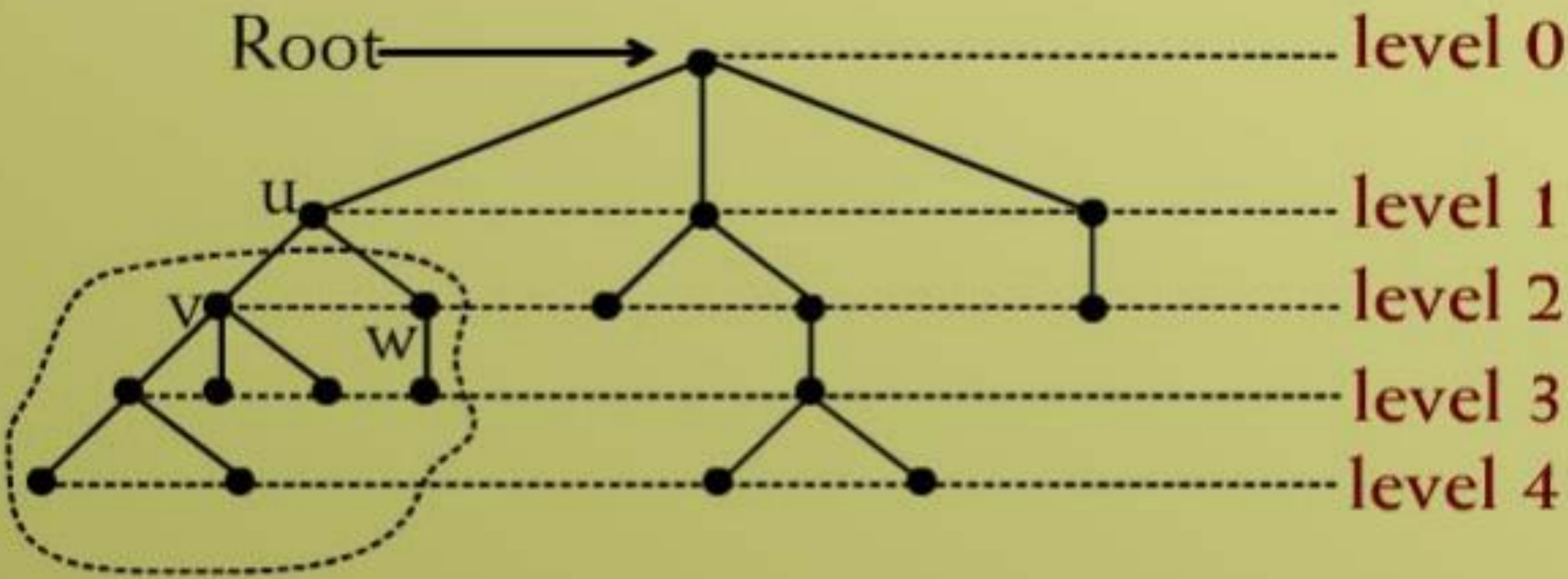
Subtree

If *a* is a vertex in a tree, the subtree with *a* as its root is the subgraph of the tree consisting of *a* and its descendants and all edges incident to these descendants.

EXAMPLE

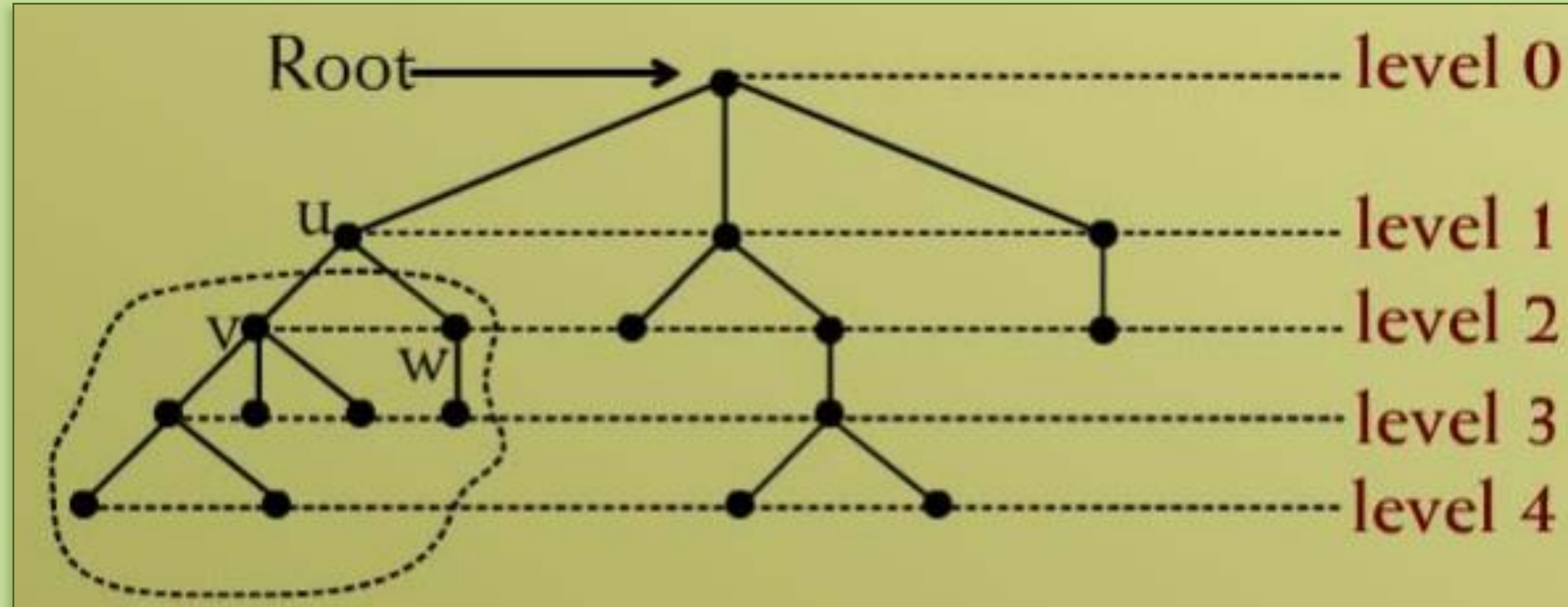


EXAMPLE



Vertices in enclosed region are descendants of u, which is an ancestor of each.

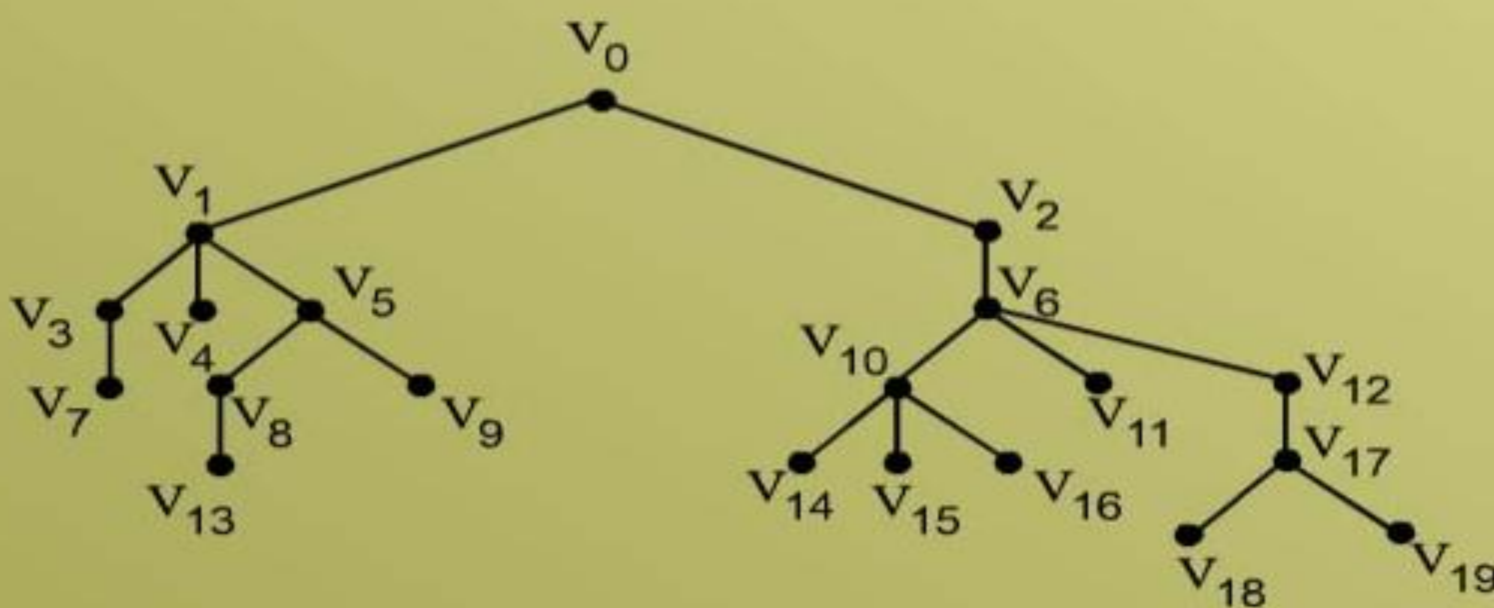
EXAMPLE



v is a child of u,u is the parent of v,v and w are sublings,height = 4.

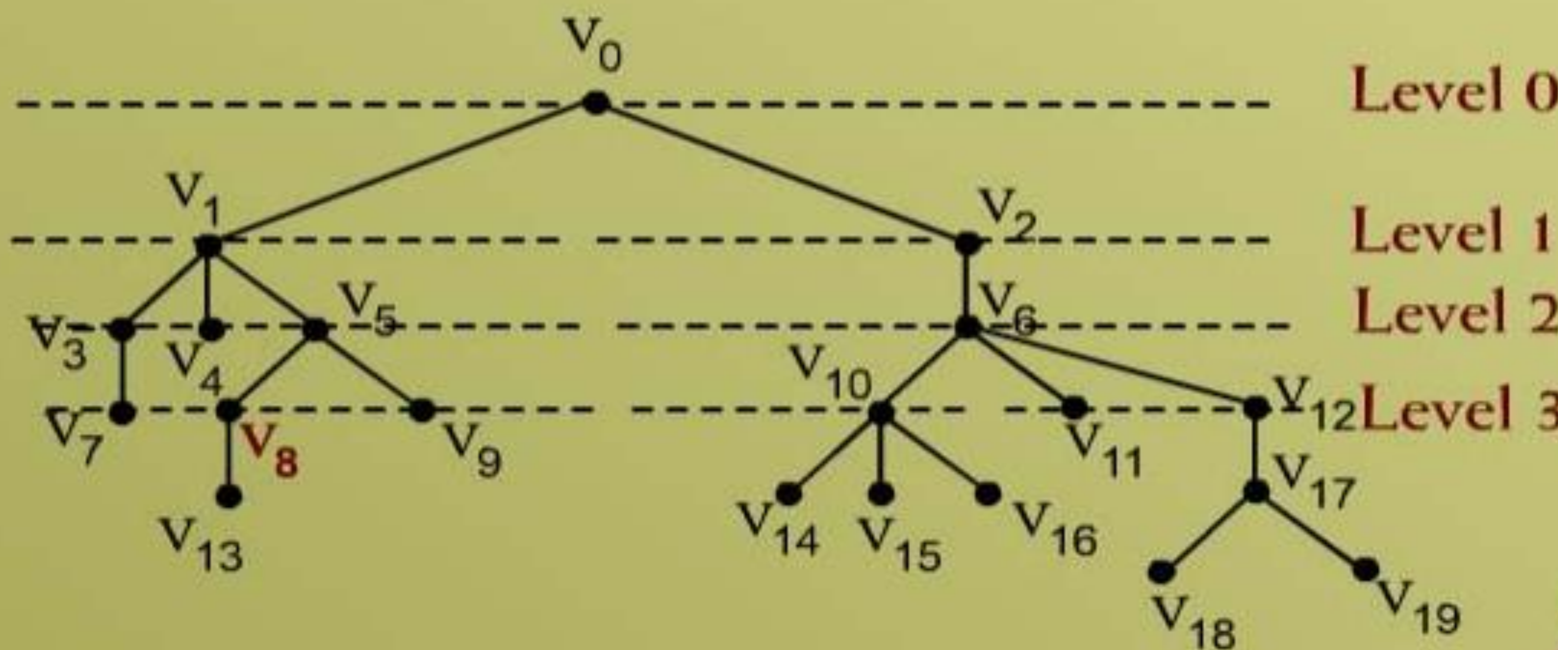
EXERCISE

Consider the **rooted tree** shown below with root v_0.



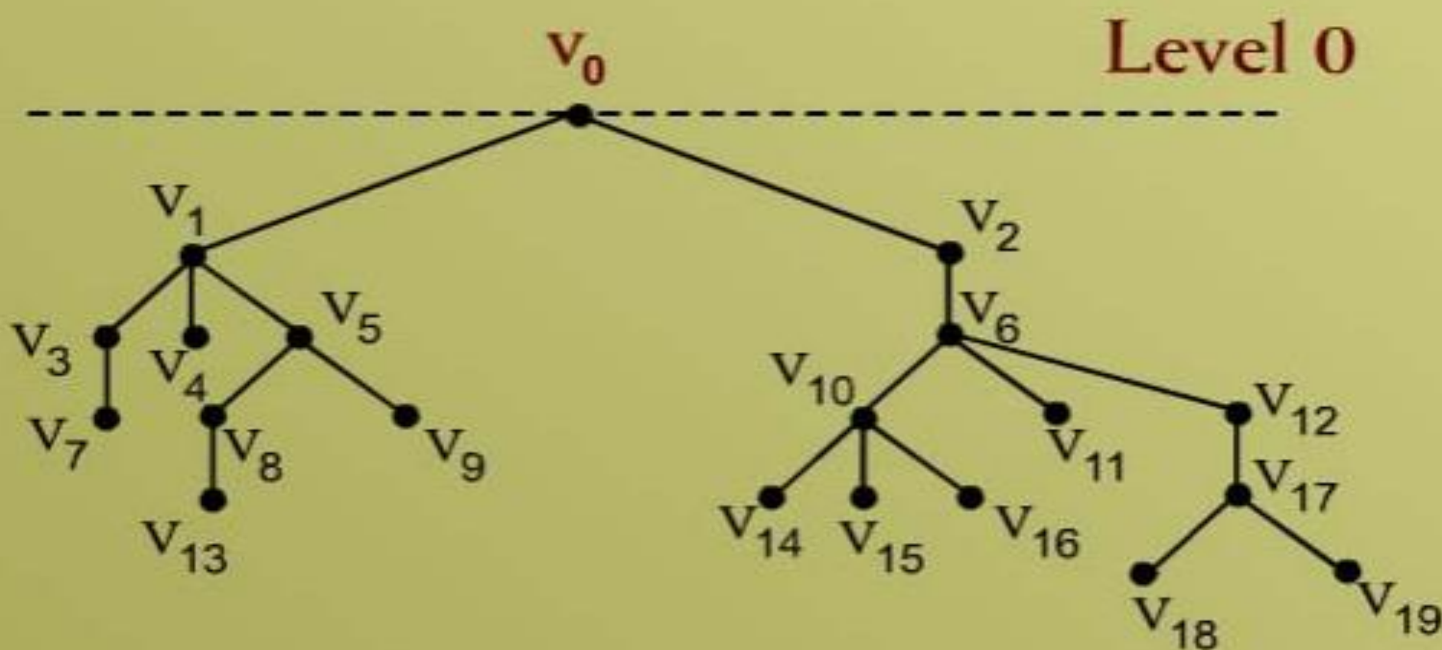
EXERCISE

a. What is the level of v_8?



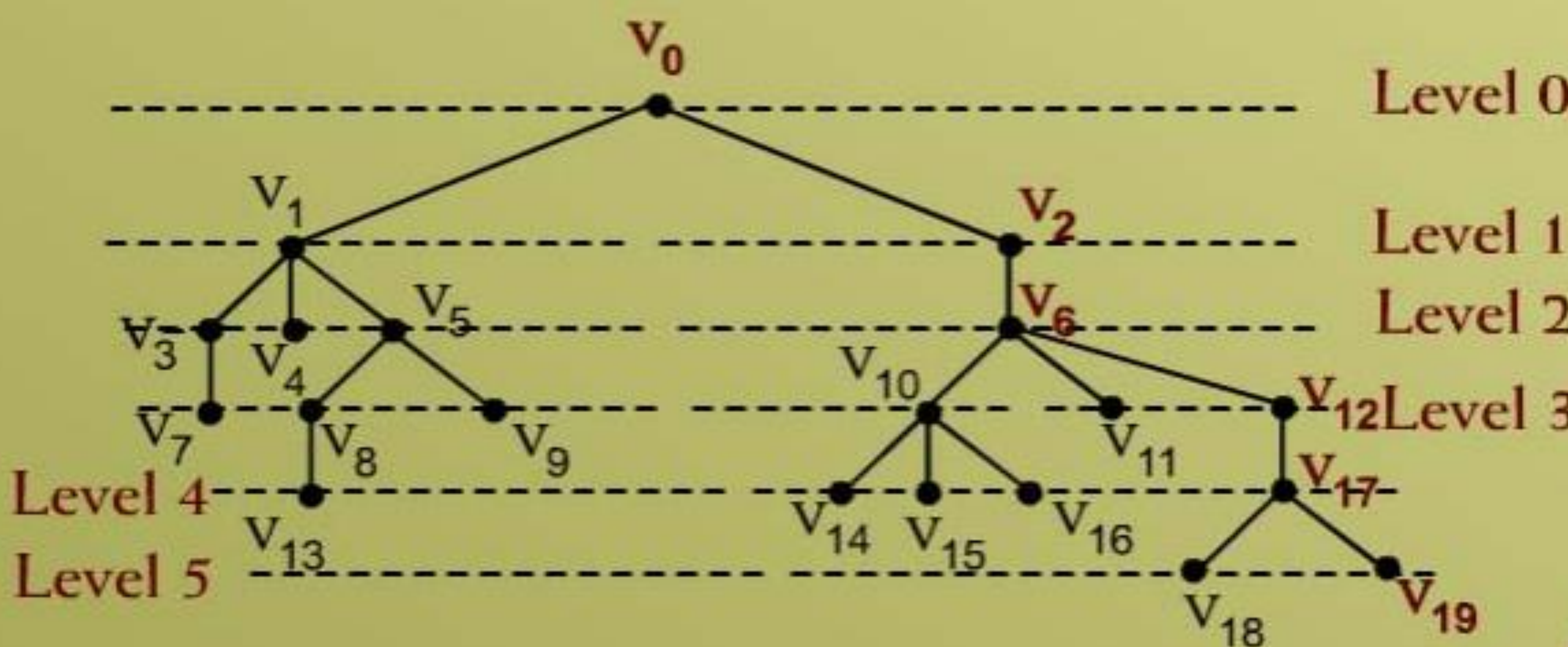
EXERCISE

b. What is the level of v_0?



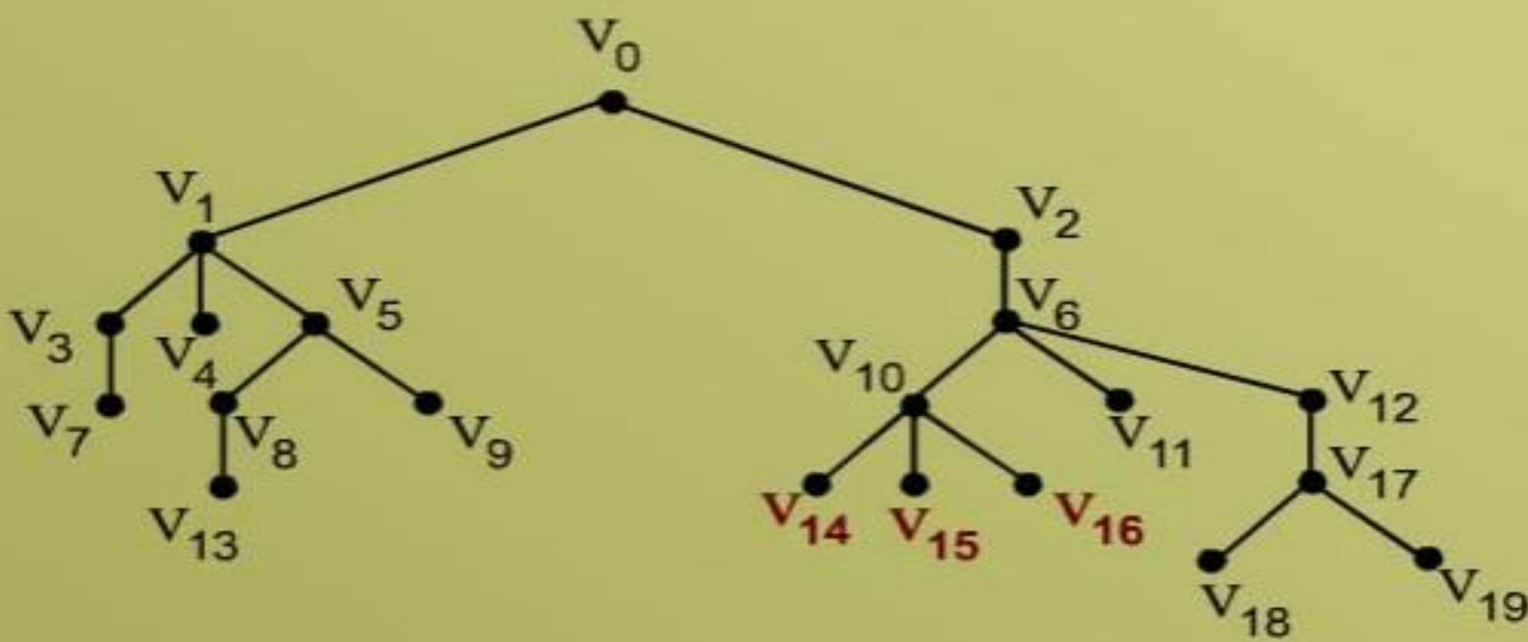
EXERCISE

c. What is the height of this tree?



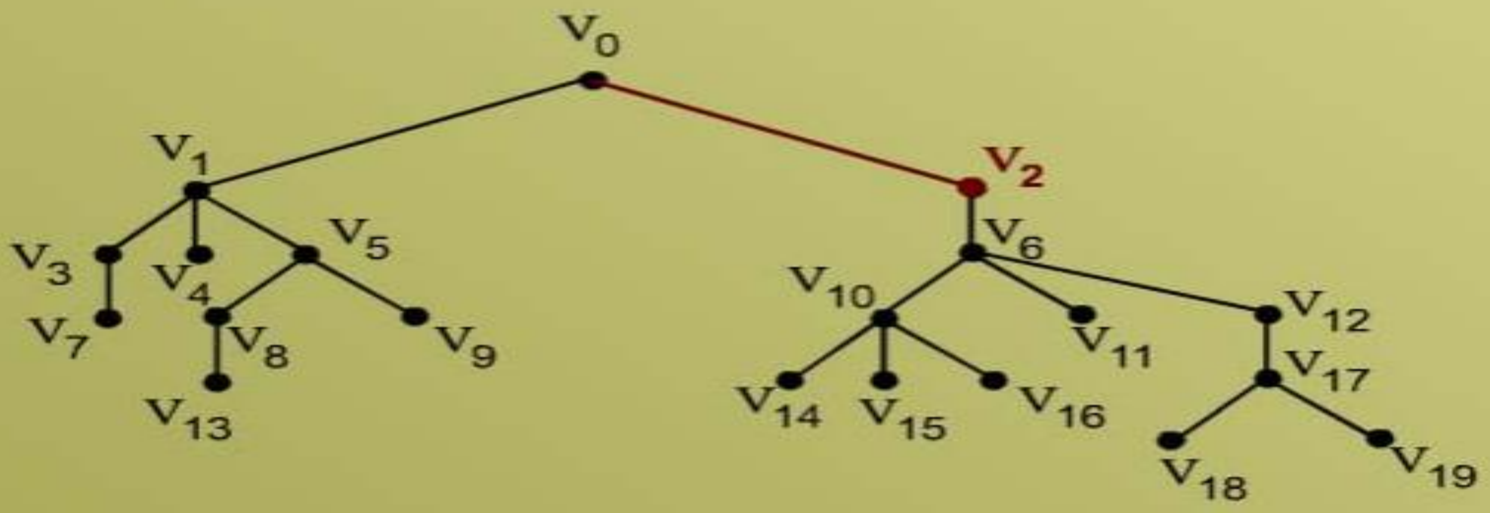
EXERCISE

d. What are the children of v_{10}?



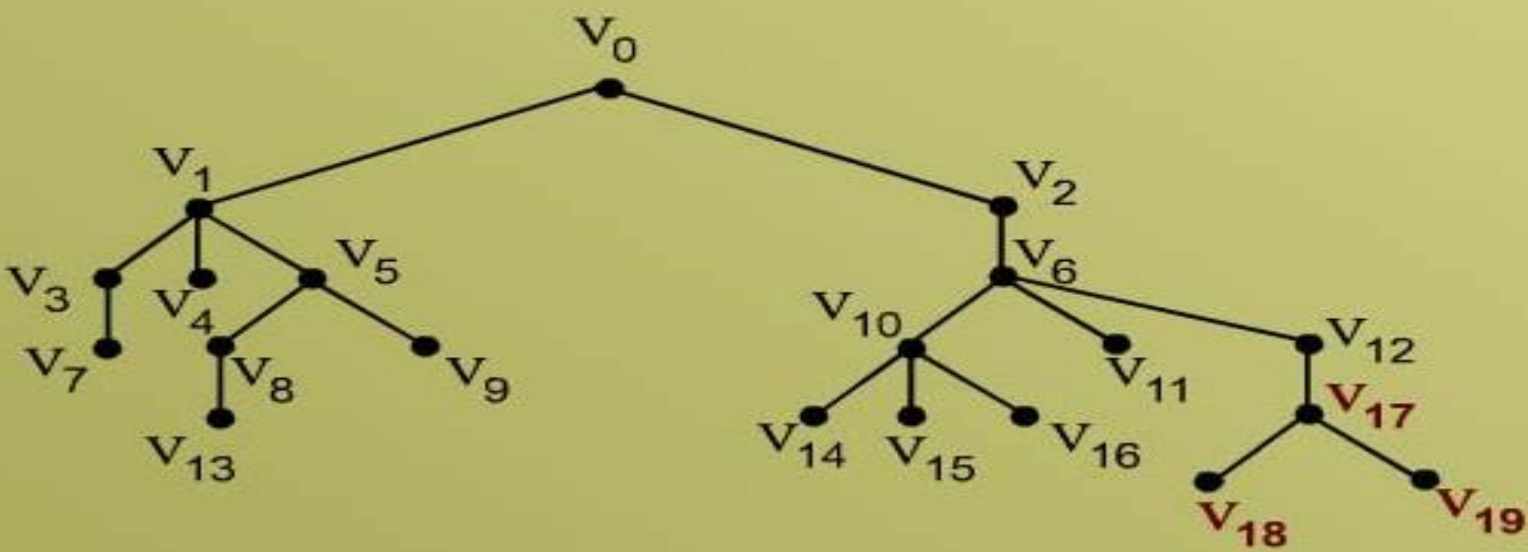
EXERCISE

e. What are the siblings of v_1?



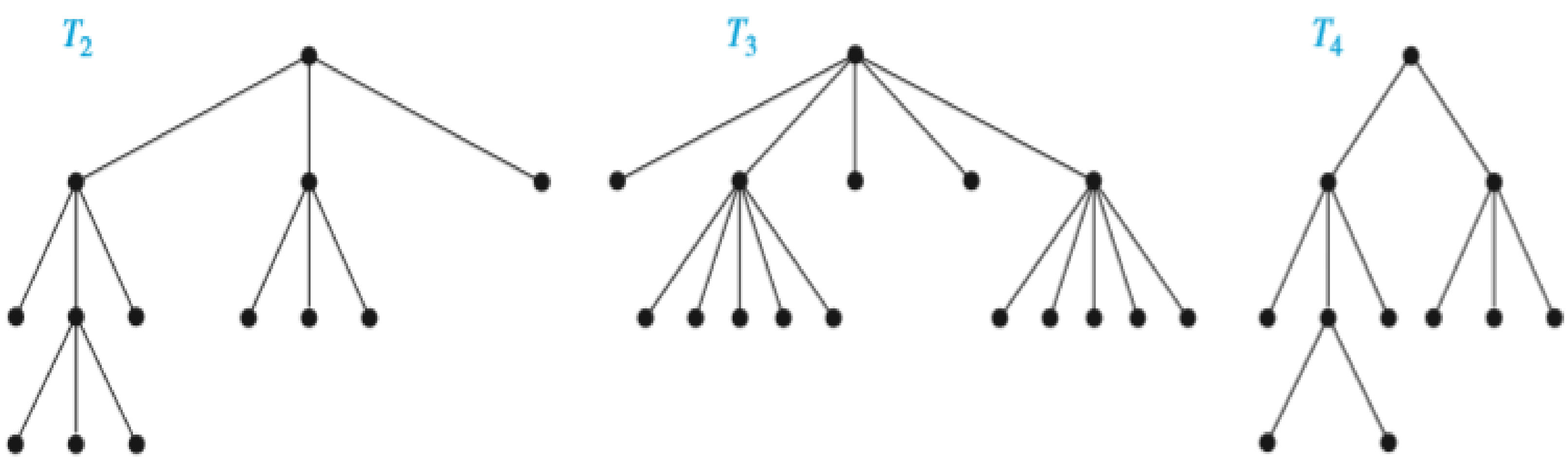
EXERCISE

f. What are the **descendants** of v_{12}?



m-ary TREE

- A rooted tree is called *m-ary* tree if every internal vertex has no more than *m* children.
- The tree is called *full m-ary tree* if every internal vertex has exactly *m* children.
- An *m-ary* tree with *m=2* is called a binary tree.



BINARY TREE

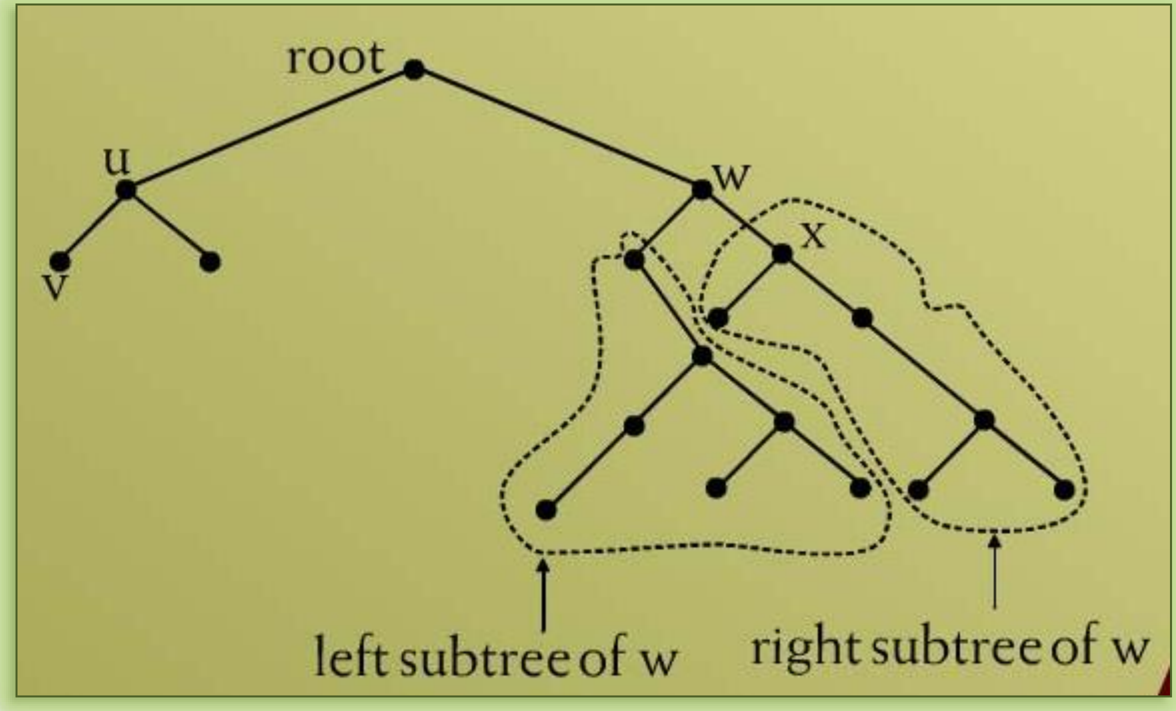
A binary tree is a rooted tree in which every internal vertex has at most two children.

Every child in a binary tree is designated either a left child or a right child.

FULL BINARY TREE

A **full binary tree** is a **binary tree** in which each **internal vertex** has **exactly two children**.

EXAMPLE



THEOREM

A *full m-ary* tree with *k* internal vertices contains $n=mk+1$ vertices.

1. If k is a positive integer and T is a full binary tree with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices.

THEOREM

A *full m-ary* tree with

1. n vertices has $k = (n-1)/m$ internal vertices and $l = [(m-1)n + 1]/m$ leaves.
2. k internal vertices has $n = mk+1$ vertices and $l = (m-1)k + 1$ leaves.
3. l leaves has $n = (ml-1)/(m-1)$ vertices and $k = (l-1)/(m-1)$ internal vertices.

THEOREM

There are at most m^h leaves (terminal vertices) in an *m-ary* tree of height h.

THEOREM

2. If T is a binary tree that has t terminal vertices and height h, then

$$t \leq 2^h$$

Equivalently,

$$\log_2 t \leq h$$

EXERCISE

Explain why graphs with the given specification do not exist.

1. Full binary tree with nine vertices and five internal vertices.

2. Binary tree with height 4 and eighteen terminal vertices.

SOLUTION

1. Any full binary tree with five internal vertices has six terminal vertices, for a total of eleven, not nine vertices in all.

   Thus there is no full binary tree with the given properties.

2. Any binary tree of height 4 has at most $2^4 =$ 16 terminal vertices.

   Hence, there is no binary tree that has height 4 and eighteen terminal vertices.

EXERCISE

Draw a full binary tree with seven vertices.

SOLUTION:

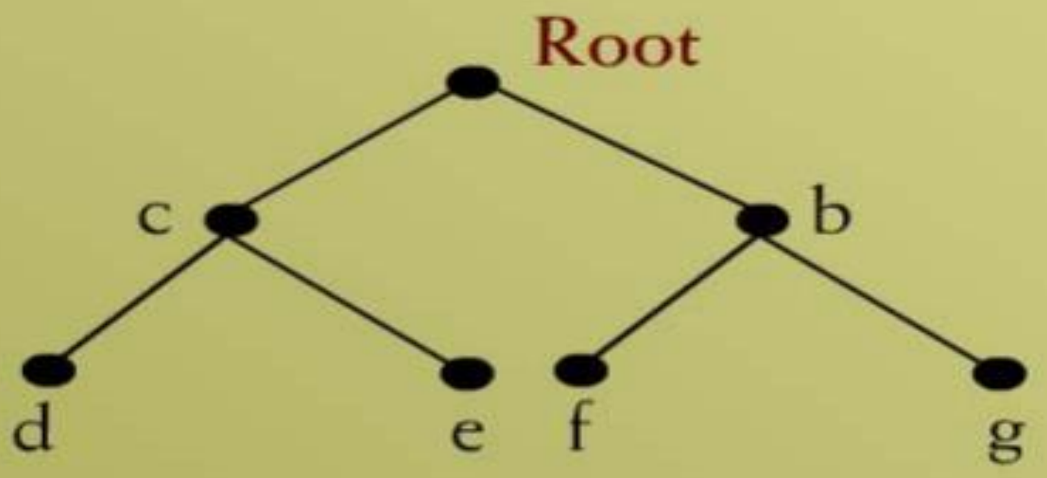
Total vertices $= 2k + 1 = 7$

$\Rightarrow \quad k = 3$

Hence, number of internal vertices $= k = 3$

Number of terminal vertices $= k + 1 = 4$

EXERCISE

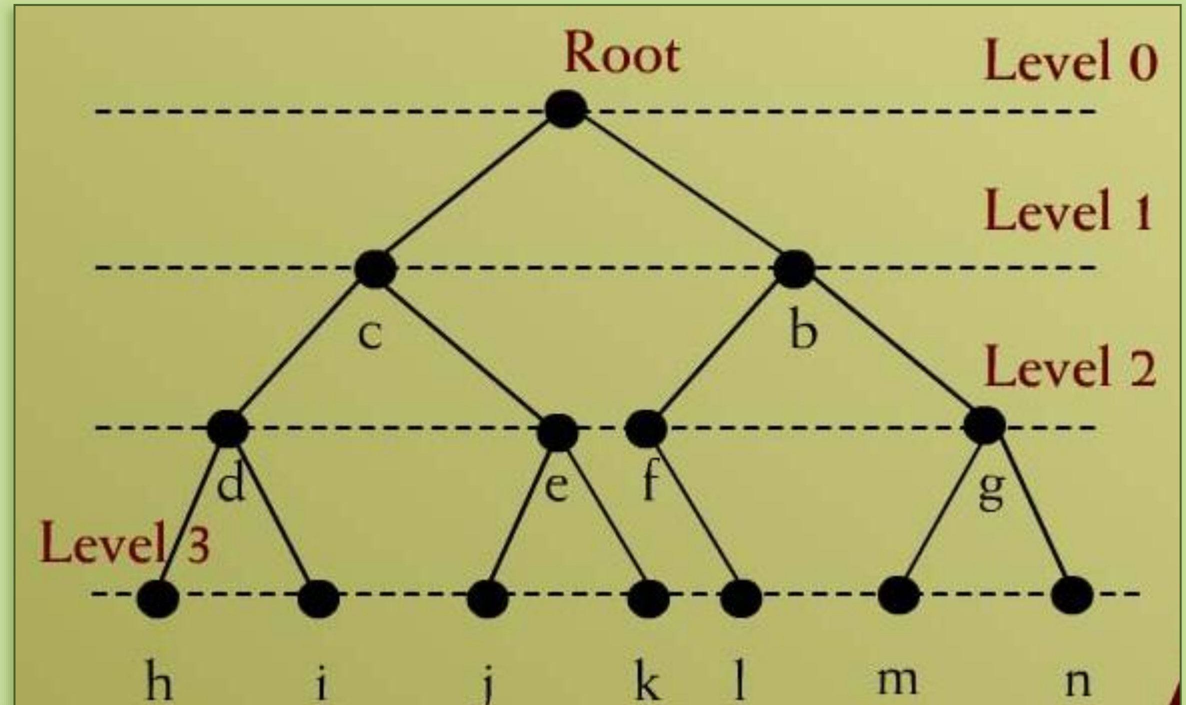


Which is required full binary tree with seven vertices.

EXAMPLE

Draw a binary tree with height 4 (level 3) and having seven terminal vertices.

SOLUTION



Balanced Rooted Tree

A rooted *m-ary* tree of height *h* is balanced if all leaves are at levels *h* or *h-1*.