Discrete Structures

Lecture # 15

Dr. Muhammad Ahmad

Department of Computer Science

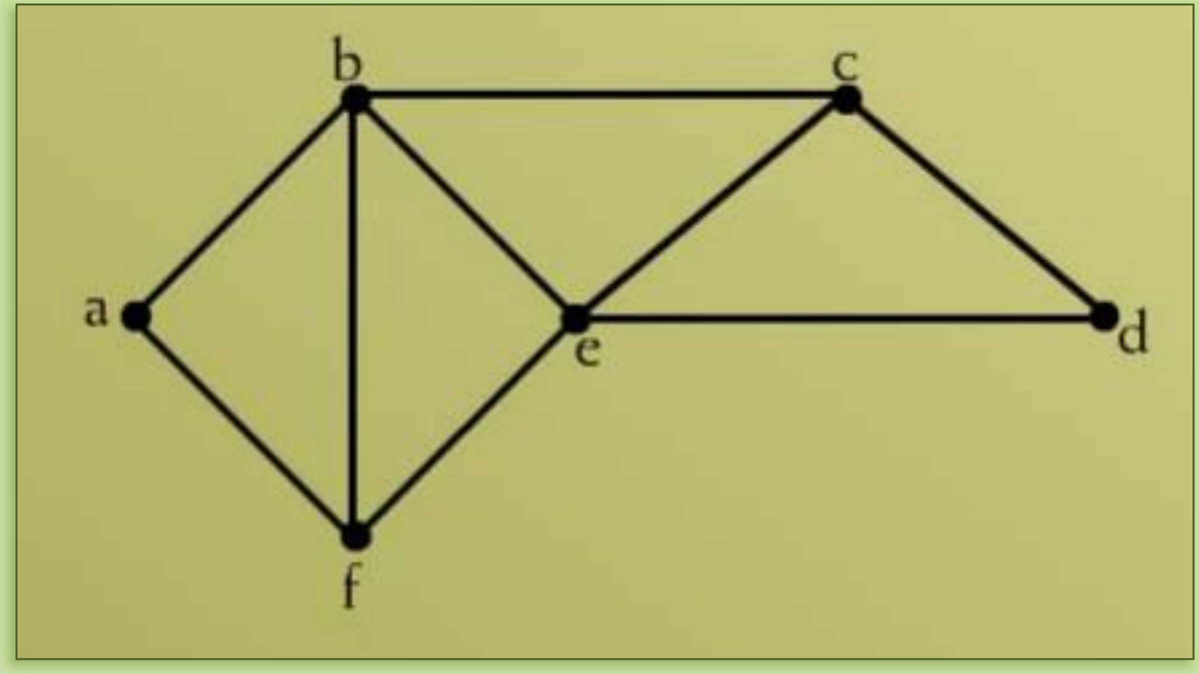
FAST -- National University of Computer and Emerging Sciences. CFD Campus

SPANNING TREES

Suppose it is required to develop a system of roads between six major cities.

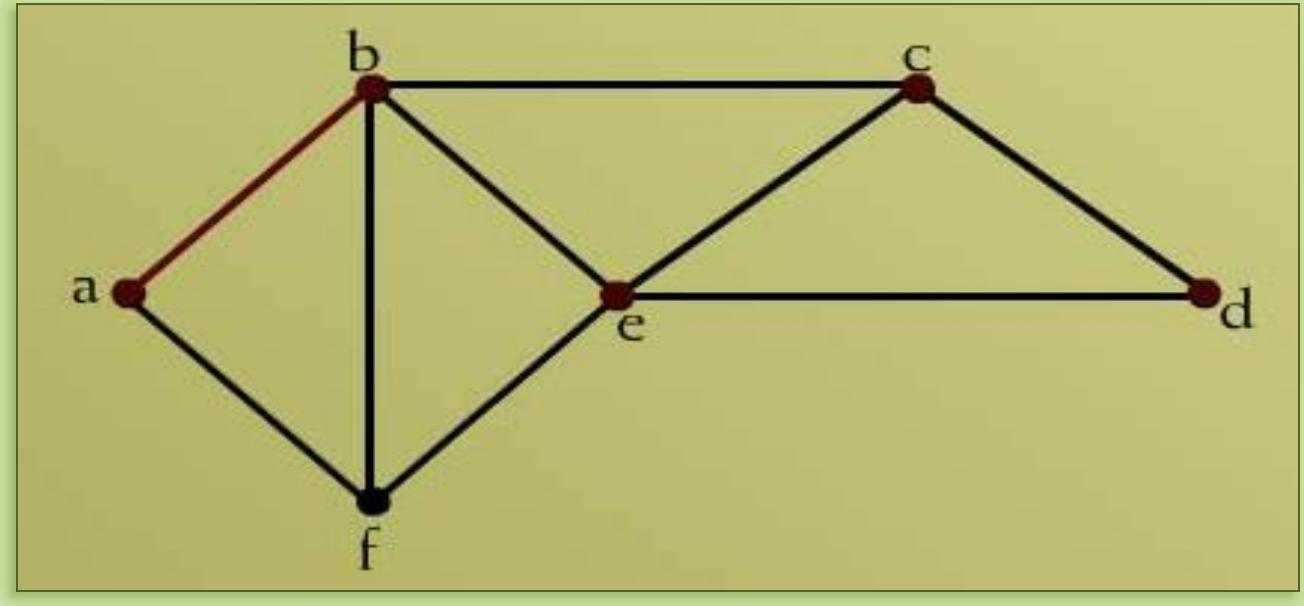
A survey of the area revealed that only the roads shown in the graph could be constructed.
For economic reason, it is desired to construct the least possible number of roads to connect the six cities. One such set of roads is

SPANNING TREES



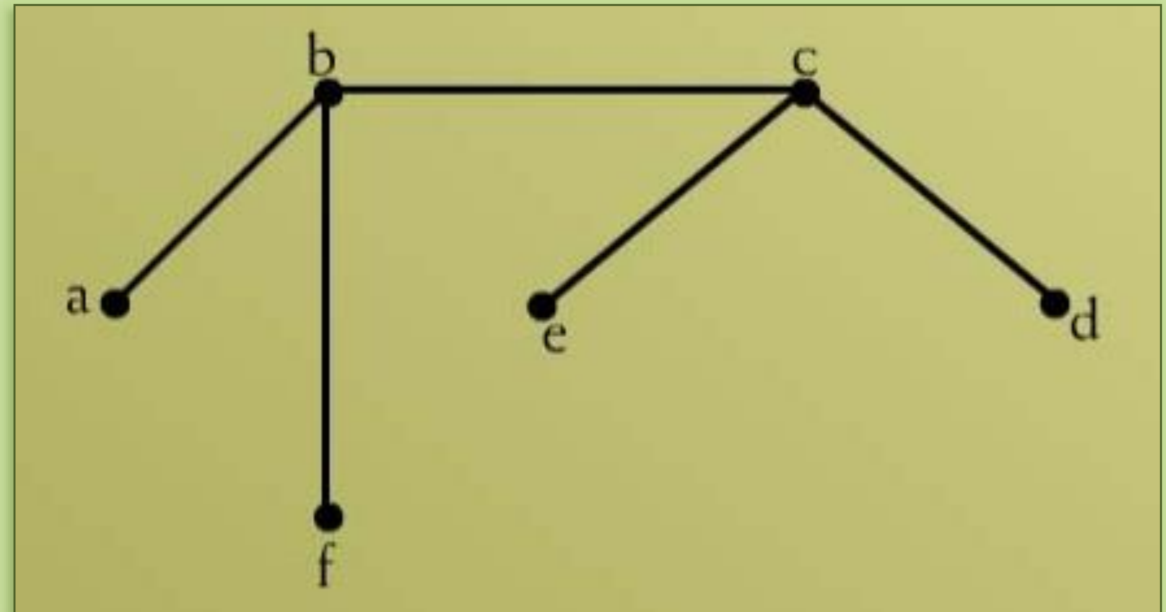
We have six vertices a, b, c, d, e, f.

SPANNING TREES



we can construct road between "a" and "b" but we can not construct road from "a" to "e" or "a" to "d" or "a" to "c".

SPANNING TREES



This is a spanning tree with number of vertices 6 and edges 5=6-1.

FORMAL DEFINITION OF SPANNING TREES

A spanning tree for a simple graph G is a sub-graph of G that contains every vertex of G and is a tree.

REMARKS

1- Every connected graph has a spanning tree.

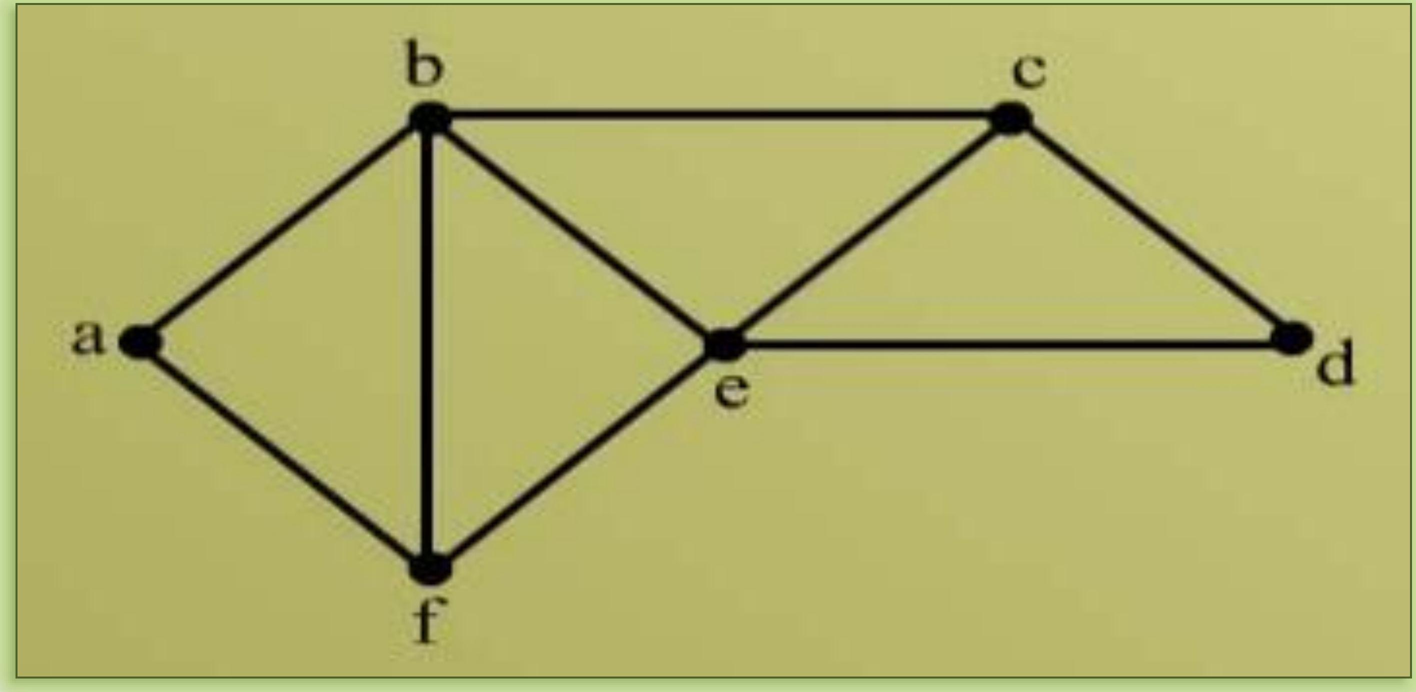
2- A graph may have more than one spanning trees.

3- Any two spanning trees for a graph have the same number of edges.

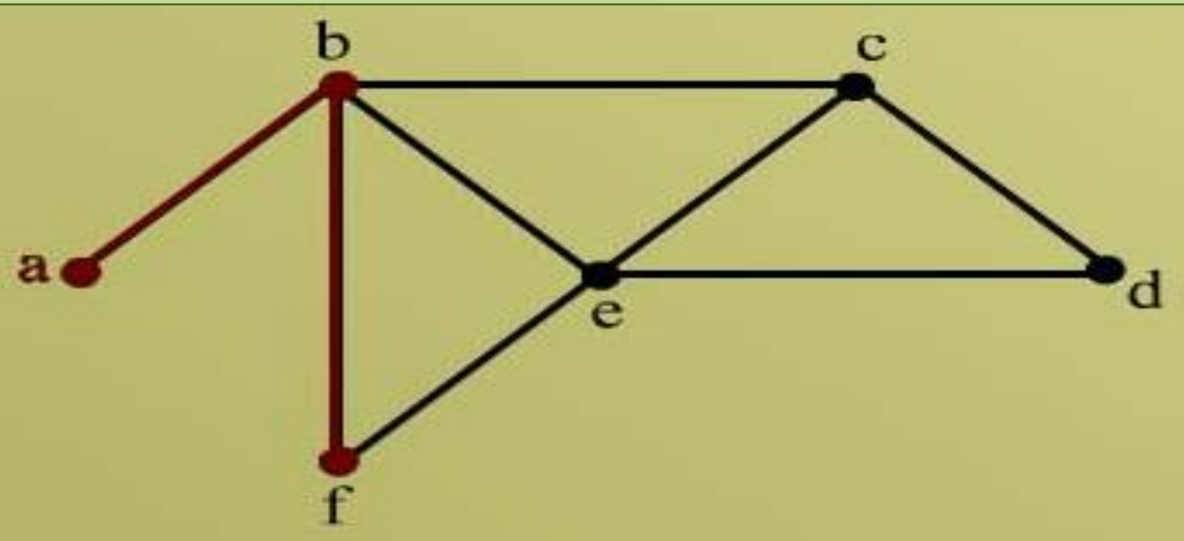
4- If a graph is a tree, then its only spanning tree is itself.

EXAMPLE

Find a spanning tree for the graph below:

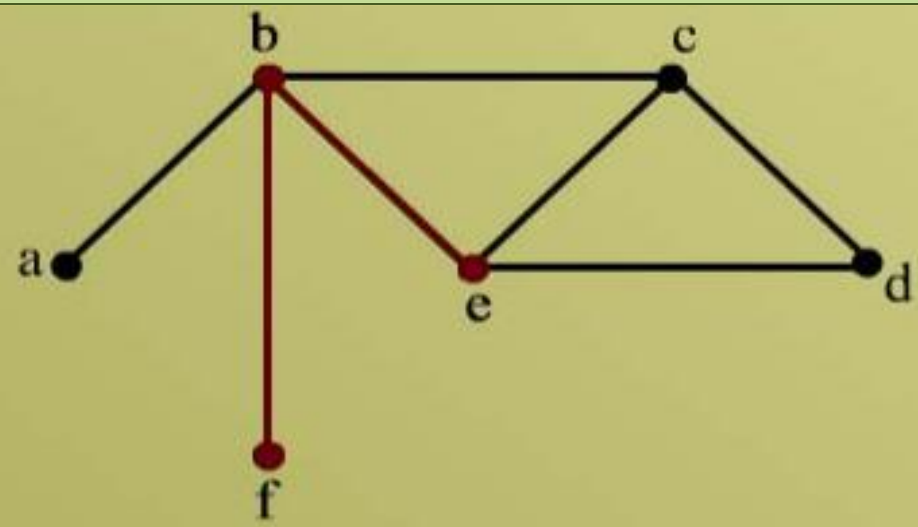


EXAMPLE



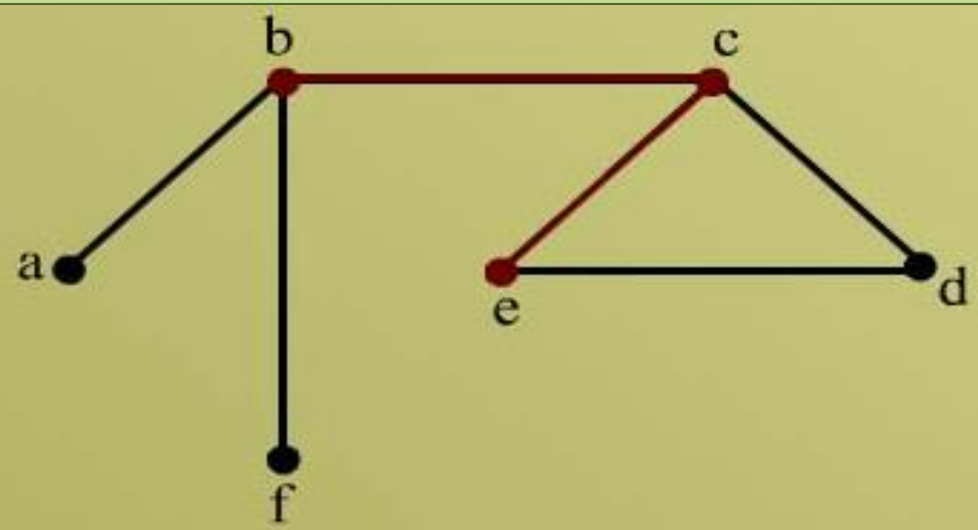
abf is a triangle so we remove af .

EXAMPLE



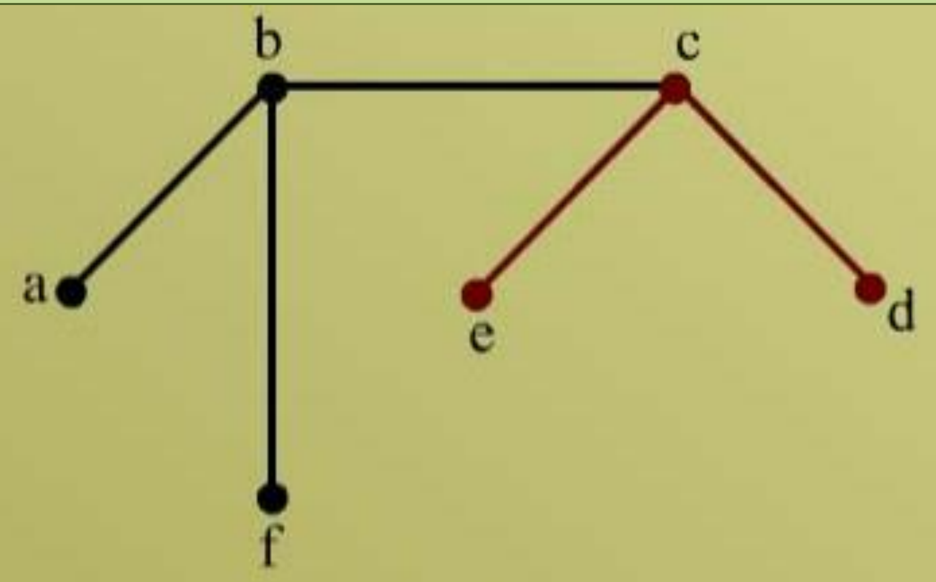
bef is another circuit so we remove fe .

EXAMPLE



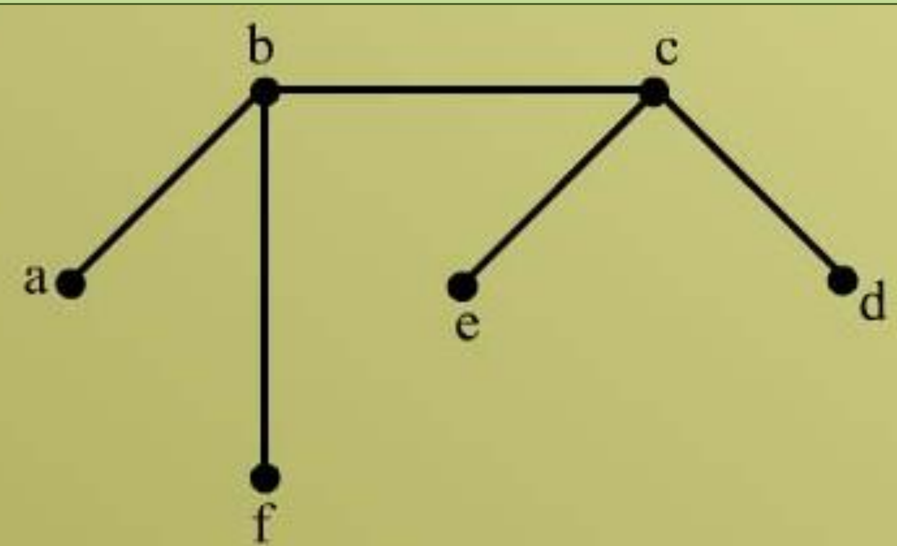
"bce" is another circuit so we remove "be".

EXAMPLE



"cde" is another circuit so we remove "ed" .

EXAMPLE

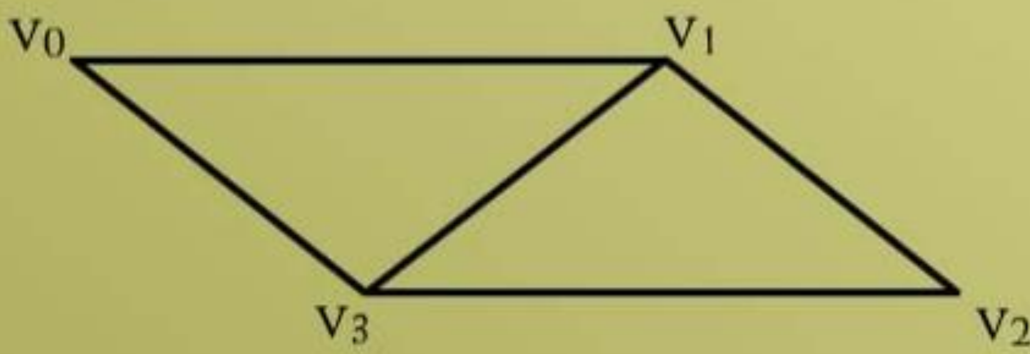


Note that we have removed

$9 - 6 + 1 = 4$ edges.

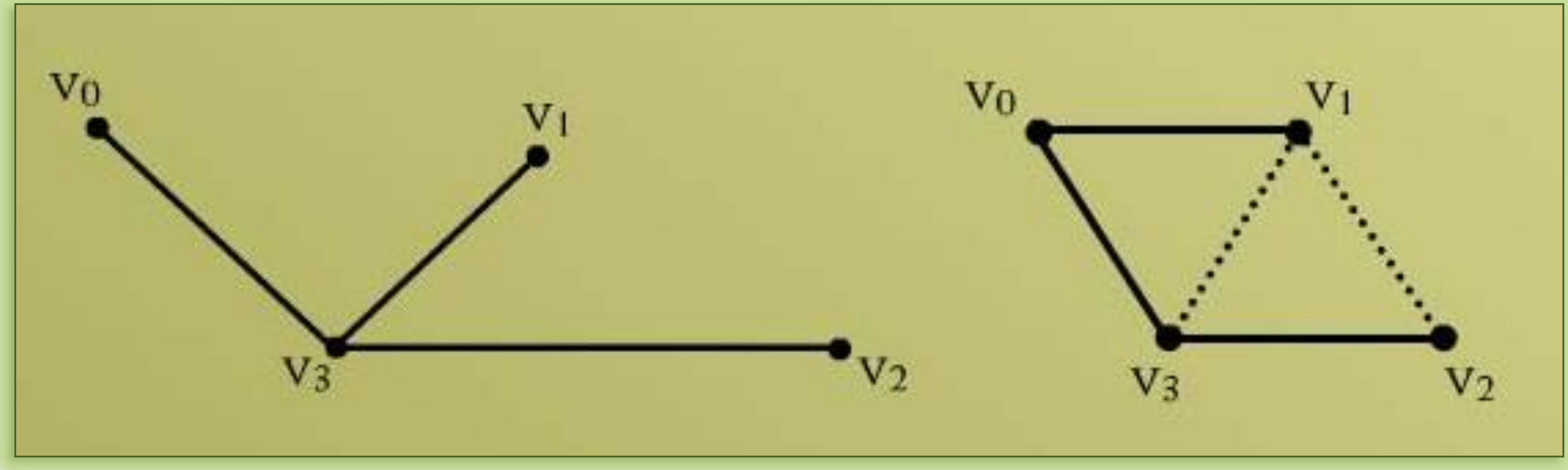
EXERCISE

Find all the spanning trees of the graph given below.



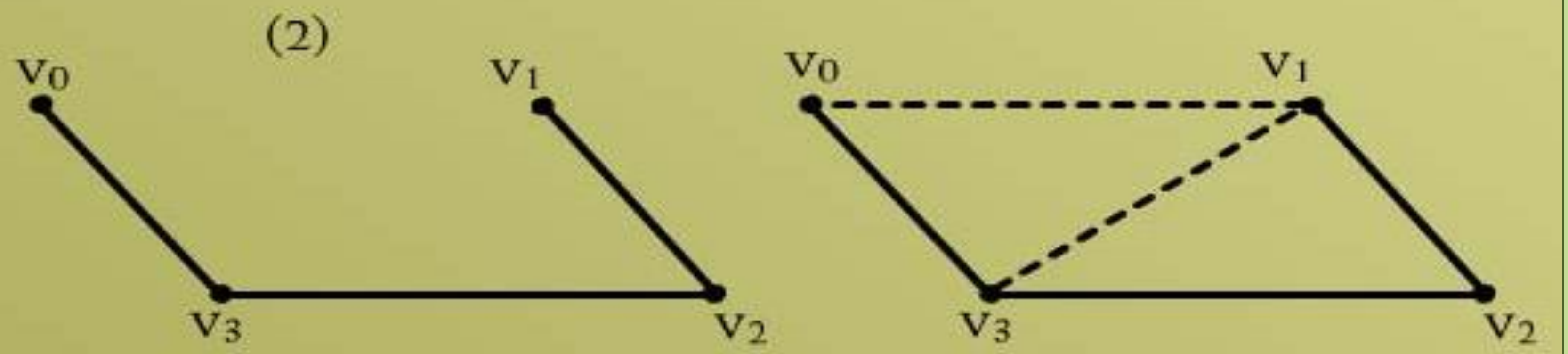
We have to remove $e - v + 1$ edge that is $5 - 4 + 1 = 2$ to obtain spanning tree.

SOLUTION



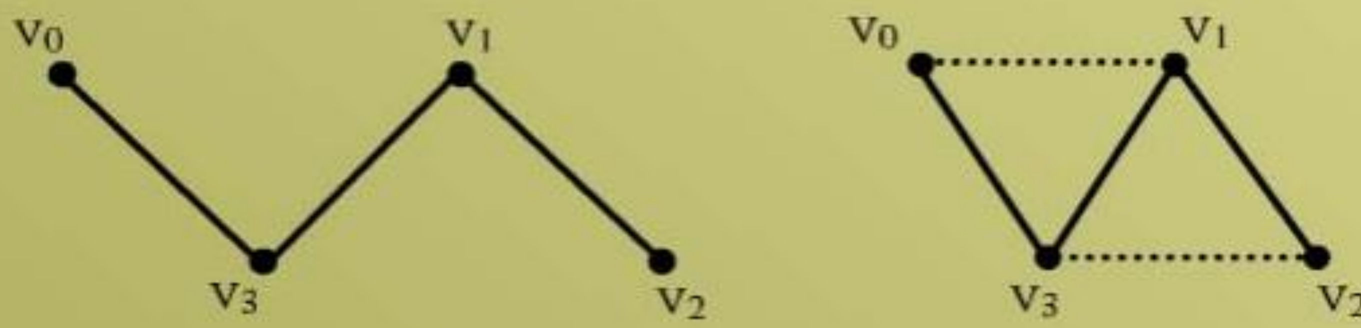
We have removed edges v_0v_1 and v_1v_2.

SOLUTION



We have removed edges v_0v_1 and v_1v_3.

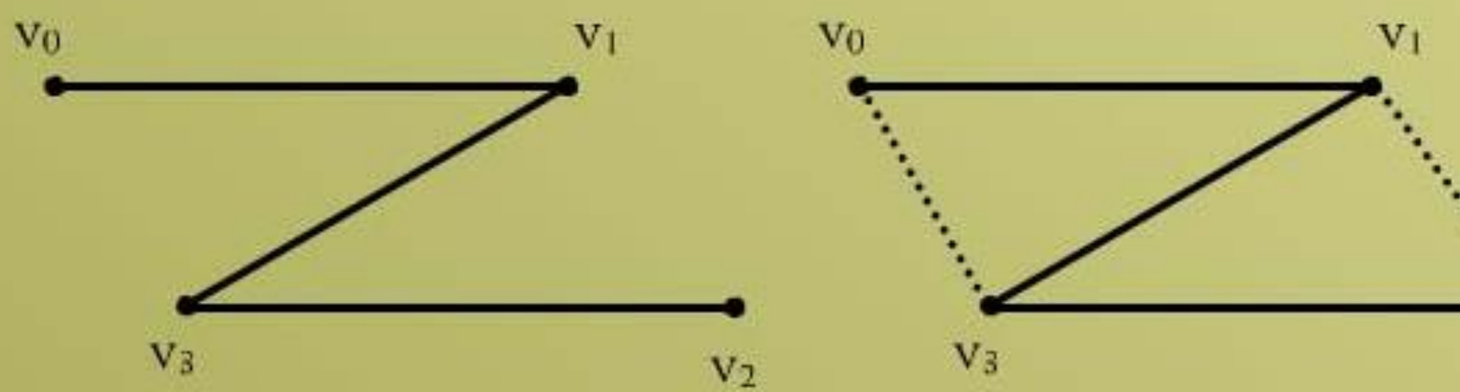
SOLUTION



(3)

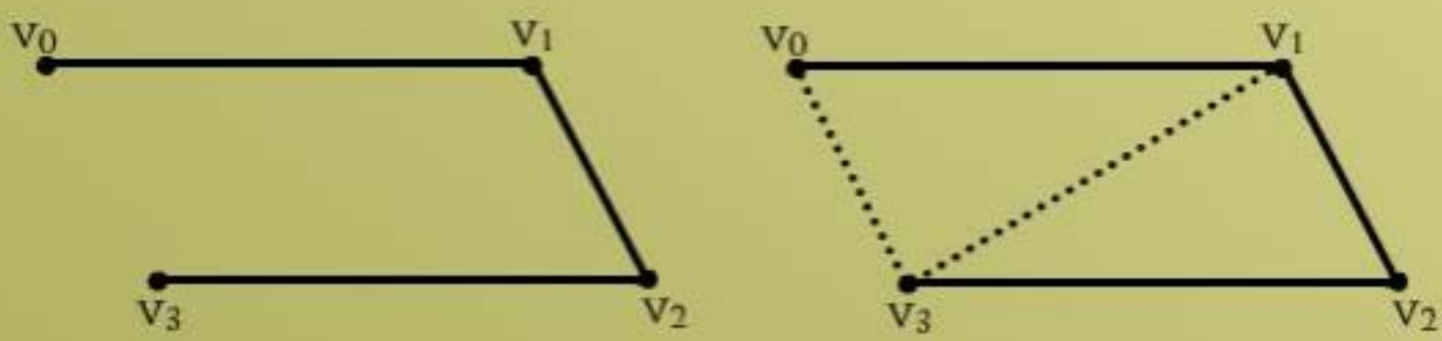
We have removed v_0v_1 and v_2v_3.

SOLUTION



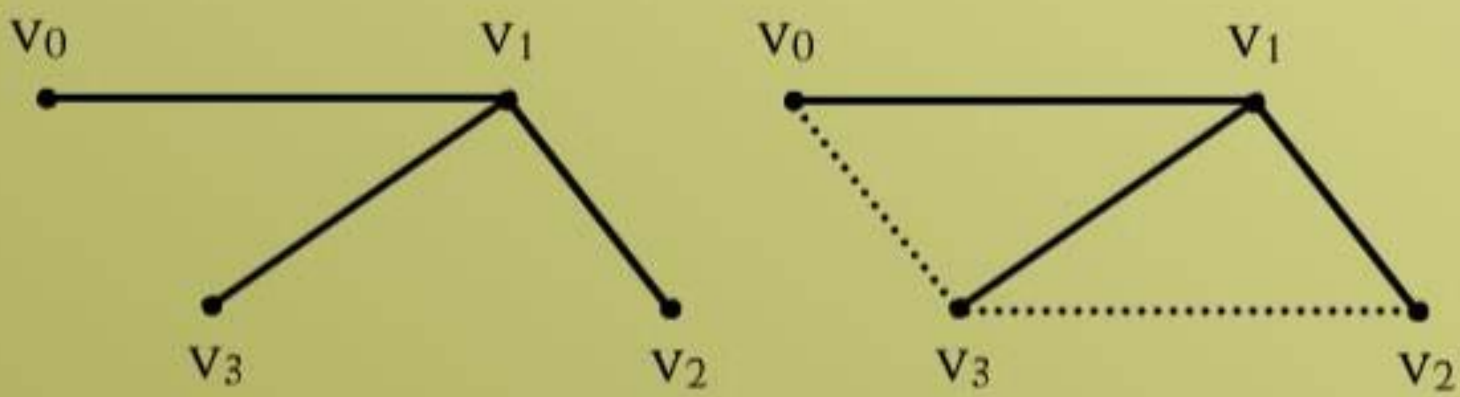
We have removed v_0v_3 and v_1v_2.

SOLUTION



We have removed edges v_0v_3 and v_1v_3.

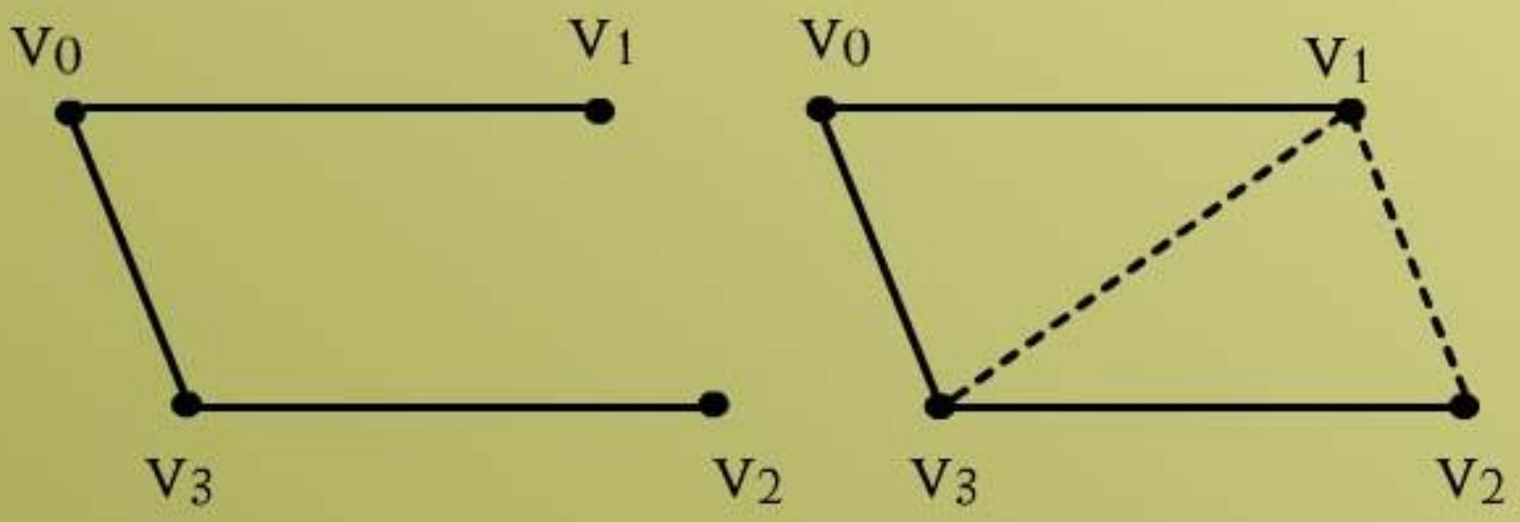
SOLUTION



We have removed edges v_0v_3 and v_2v_3.

SOLUTION

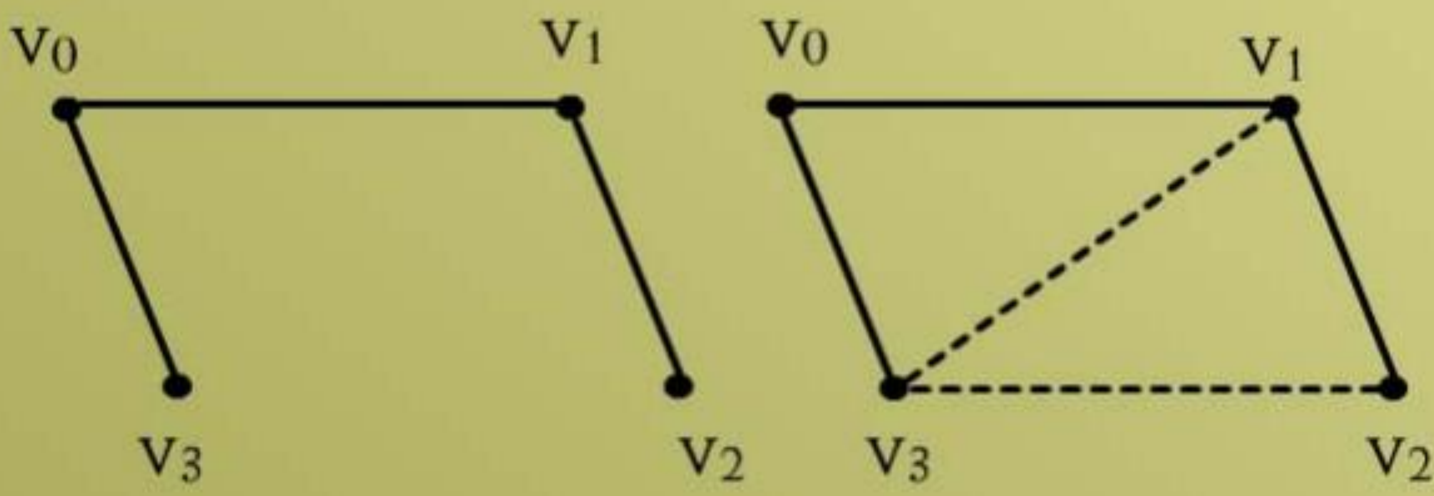
(7)



We delete v_1v_3 and v_1v_2 .

SOLUTION

(8)



We delete v_1v_3 and v_2v_3.

EXAMPLE

Find a spanning tree for each of the following graphs.

(a) $k_{1,5}$ (b) k_4

$k_{1,5}$ is bipartite graph having one vertex in one set and five vertices in other set.

SOLUTION

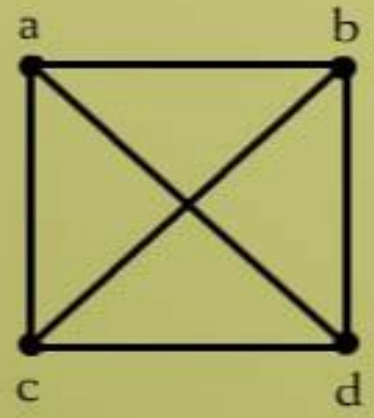
$k_{1,5}$ represents a complete bipartite graph on 6 vertices, drawn below:

$k_{1,5}$ has 6 vertices and 5 edges.

Hence already a spanning tree.

SOLUTION

k_4 represents a complete graph on four vertices.



we have to remove $e\text{-}v+1=6\text{-}4+1=3$ edegs to get spanning tree.

SOLUTION

we have removed ab,bd and cd.

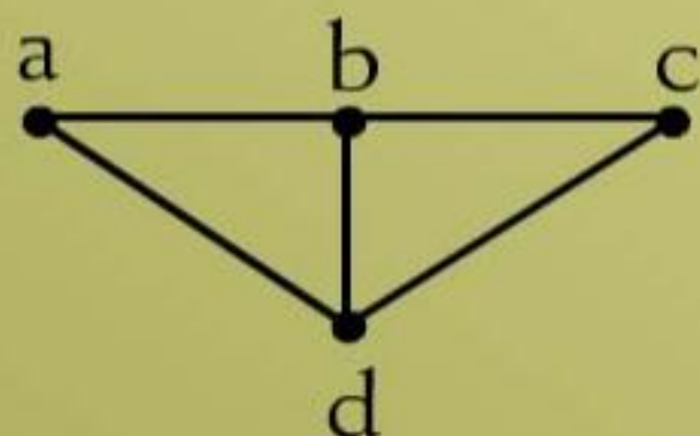
KIRCHHOFF'S THEOREM OR MATRIX-TREE THEOREM

Let M be the matrix obtained from the adjacency matrix of a connected graph G by changing all 1's to -1's and replacing each diagonal 0 by the degree of the corresponding vertex.

Then the number of spanning trees of G is equal to the value of any cofactor of M.

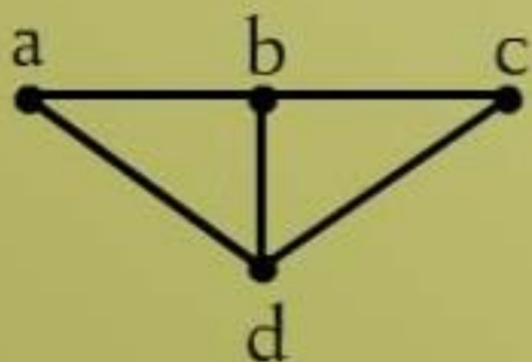
EXAMPLE

Find the number of spanning trees of the graph G.



SOLUTION

The adjacency matrix of G is



$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4 \times 4} \end{array}$$

SOLUTION

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The matrix specified in Kirchhoff's theorem is

$$M = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

SOLUTION

$$M = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

The first 0 of diagonal correspond to "a" and degree of "a" is two so we replace 0 by 2 in M.

$$M = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

SOLUTION

$$M = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

The second 0 of diagonal correspond to "b" and degree of "b" is three so we replace 0 by 3 in M.

$$M = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

SOLUTION

$$M = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

The third 0 of diagonal correspond to "c" and degree of "c" is two so we replace 0 by 2 in M.

$$M = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

SOLUTION

$$M = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

The fourth 0 of diagonal correspond to "d" and degree of "d" is three so we replace 0 by 3 in M.

$$M = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

SOLUTION

Now cofactor of the element at $(1,1)$ in M is

$$\begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

SOLUTION

Expanding by first row, we get

$$= 3\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= 3(6-1) + (-3-1) + (-1)(1+2)$$

$$= 15 - 4 - 3$$

$$= 8$$

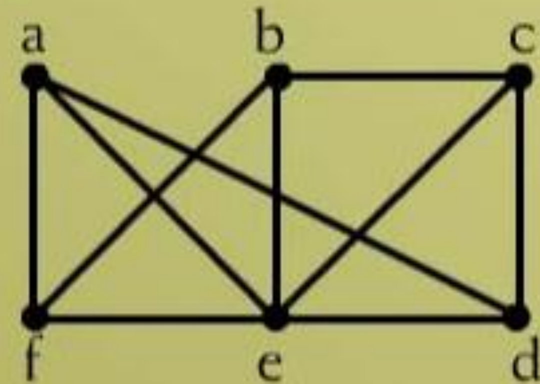
EXAMPLE

Suppose an oil company wants to build a series of pipelines between six storage facilities in order to be able to move oil from one storage facility to any of the other five.

For environmental reasons it is not possible to build a pipeline between some pairs of storage facilities.

EXAMPLE

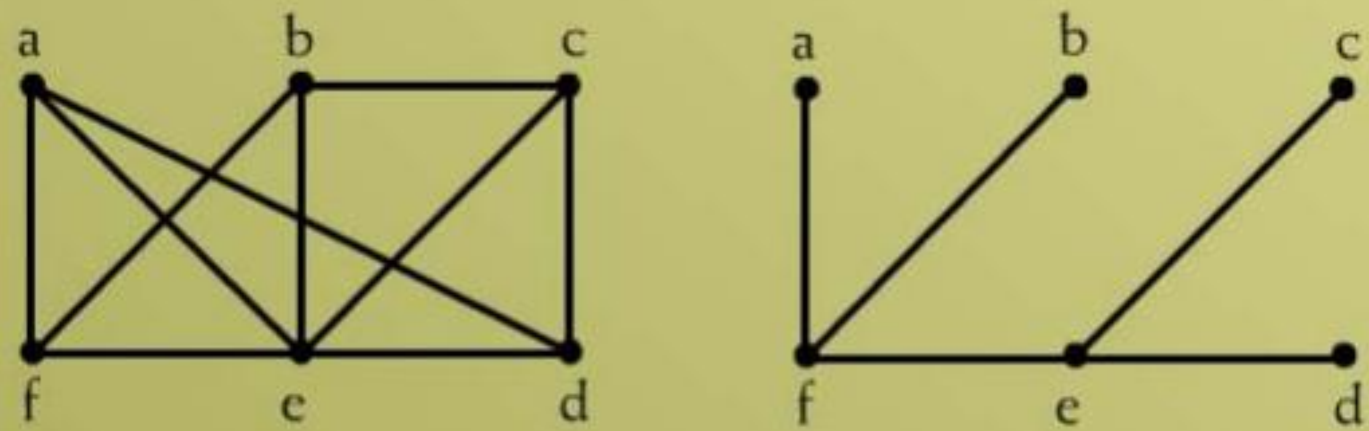
The possible pipelines that can be build are.



Because the construction of a pipeline is very expensive, construct as few pipelines as possible.

The company does not mind if oil has to be routed through one or more intermediate facilities.

SOLUTION



We have removed five edges and graph is spanning tree now.