Discrete Structures

Week#2 (Lec3 & Lec4)

FAST -- National University of Computer

and Emerging Sciences. CFD Campus

Material Partially Collected by Prof. Dr. A.M. Khan, Slides Prepared by Dr. M. Ahmad

Today’s Topics

Conditional Statement (Implication)

Examples with Exercise

Implication equivalence

Converse, Inverse, and Contrapositive for Implication with examples

Bi-Conditional statements

Laws of Logics

Conditional Statements

- "If you earn an A in Math, then I'll buy a computer."

p: "You earn an A in Math,"

and

q: "I will buy you a computer."

Conditional Statements

- The original statement is then saying:

if p is true, then q is true

Or

If p, then q

We can also phrase this as p implies q, and we write $p \Longrightarrow q$.

Conditional Statement or Implication

- If p and q are statement variables, the conditional of q by p is "If p then q" or "p implies q" and is denoted $p \Rightarrow q$.

Conditional Statement or Implication

- The arrow “$\Rightarrow$ “ is the conditional operator

  p is called the hypothesis (or antecedent)

  q is called the conclusion (or consequent)

Truth Table for Implication

$$p \to q$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Examples

STATEMENTS	TRUTH VALUES
1. "If $1 = 1$, then $3 = 3$."	TRUE
2. "If $1 = 1$, then $2 = 3$."	FALSE
3. "If $1 = 0$, then $3 = 3$."	TRUE
4. "If $1 = 2$, then $2 = 3$."	TRUE
5. "If $1 = 1$, then $1 = 2$ and $2 = 3$."	FALSE
6. "If $1 = 3$ or $1 = 2$ then $3 = 3$."	TRUE

Alternative Ways of Expressing Implications

- “If p then q”
- “p implies q”
- “if p, q”
- “p only if q”
- “p is sufficient for q”
- “not p unless q”
- “q follows from p”
- “q if p”
- “q whenever p”
- “q is necessary for p”

Exercise

a) Your guarantee is good only if you bought your CD less than 90 days ago.

   if your guarantee is good, then you must have bought your CD player less than 90 days ago.

b) To get tenure as a professor, it is sufficient to be world-famous.

   If you are world-famous, then you will get tenure as a professor.

Exercise

c) That you get the job implies that you have the best credentials.

If you get the job, then you have the best credentials.

d) It is necessary to walk 8 miles to get to the top of the peak.

If you get to the top of the peak, then you must have walked 8 miles.

Translating English Sentences To Symbols

- Let p and q be propositions:

  p = "you get an A on the final exams."

  q = "you do every exercise in this book."

  r = "you get an A in this class."

- To get an A in this class it is necessary for you to get an A on the final.

Solution: $r \rightarrow p$

Translating English Sentences To Symbols

You do every exercise in this book and You get A on the final, implies, you get an A in the class.

Solution:

$$p \wedge q \rightarrow r$$

Translating English Sentences To Symbols

Getting an A on the final and doing every exercise in this book is sufficient for getting an A in the class.

Solution:

$$p \wedge q \rightarrow r$$

Translating English Sentences To Symbols

Let p, q, and r be the propositions:

p = “you have the flu”
q = “you miss the final exam”
r = “you pass the course”

$p \to q$ If you have flu, then you will miss the final exam.

$\sim q \to r$ If you don’t miss the final exam, then you will pass the course.

$\sim p \wedge \sim q \to r$ If you neither have flu nor miss the final exam, then you will pass the course.

Hierarchy of Operations for Logical Connectives

1) $\sim$ (negation)

2) $\wedge$ (conjunction), $\vee$ (disjunction)

3) $\rightarrow$ (conditional)

Truth Table -- Example 1

$$p \vee \sim q \rightarrow \sim p$$

$p \vee \sim q \rightarrow \sim p$ means $(p \vee (\sim q)) \rightarrow (\sim p)$

p	q	$\sim q$	$\sim p$	$p \vee \sim q$	$p \vee \sim q \rightarrow \sim p$
T	T	F	F		
T	F	T	F		
F	T	F	T		
F	F	T	T		

Truth Table -- Example 1

$$p \vee \sim q \rightarrow \sim p$$

$p \vee \sim q \rightarrow \sim p$ means $(p \vee (\sim q)) \rightarrow (\sim p)$

p	q	$\sim q$	$\sim p$	$p \vee \sim q$	$p \vee \sim q \rightarrow \sim p$
T		F		T	
T		T		T	
F		F		F	
F		T		T	

Truth Table -- Example 1

$$p \vee \sim q \rightarrow \sim p$$

$p \vee \sim q \rightarrow \sim p$ means $(p \vee (\sim q)) \rightarrow (\sim p)$

p	q	$\sim q$	$\sim p$	$p \vee \sim q$	$p \vee \sim q \rightarrow \sim p$
			F	T	F
			F	T	F
			T	F	T
			T	T	T

Truth Table -- Example 2

$$(p \rightarrow q) \wedge (\sim p \rightarrow r)$$

p	q	r	$p \rightarrow q$	$\sim p$	$\sim p \rightarrow r$	$(p \rightarrow q) \wedge (\sim p \rightarrow r)$
T	T	T	T			
T	T	F	T			
T	F	T	F			
T	F	F	F			
F	T	T	T			
F	T	F	T			
F	F	T	T			
F	F	F	T			

Truth Table -- Example 2

$$(p \to q) \wedge (\sim p \to r)$$

p	q	r	$p \to q$	$\sim p$	$\sim p \to r$	$(p \to q) \wedge (\sim p \to r)$
		T		F	T	
		F		F	T	
		T		F	T	
		F		F	T	
		T		T	T	
		F		T	F	
		T		T	T	
		F		T	F	

Truth Table -- Example 2

$$(p \to q) \wedge (\sim p \to r)$$

p	q	r	$p \to q$	$\sim p$	$\sim p \to r$	$(p \to q) \wedge (\sim p \to r)$
T	T	T	T	F	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

Truth Table -- Example 3

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

p	q	~q	~p	p → q	~q → ~p
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

Implication Law -- Truth Table -- Example 4

$$p \rightarrow q \equiv \sim p \vee q$$

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Negation of a Conditional Statement

Since $p \to q \equiv \sim p \vee q$ therefore

$$\sim (p \to q) \equiv \sim (\sim p \vee q)$$

$$\equiv \sim (\sim p) \wedge (\sim q) \quad \text{De Morgan's law}$$

$$\equiv p \wedge \sim q \quad \text{Double Negative law}$$

Example 1

1) If Ali lives in Pakistan then he lives in Lahore.
2) If my car is in the repair shop, then I cannot get to class.
3) If x is prime then x is odd or x is 2.
4) If n is divisible by 6, then n is divisible by 2 and n is divisible by 3.

Solutions:

1) Ali lives in Pakistan and he does not live in Lahore.
2) My car is in the repair shop and I can get to class.
3) x is prime but x is not odd and x is not 2.
4) n is divisible by 6 but n is not divisible by 2 or by 3.

Inverse of a Conditional Statement

The inverse of the conditional statement $p \rightarrow q$ is $\sim p \rightarrow \sim q$

For instance; for an implication $p \rightarrow q$,

Its **inverse** is: $\neg\, p \rightarrow \neg\, q$

Inverse of a Conditional Statement

$p \rightarrow q$ is not equivalent to $\sim p \rightarrow \sim q$

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

Writing Inverse of a Conditional Statement

1. If Today is Friday, Then $2+3 = 5$.
   - If today is not Friday, Then $2 + 3 \neq 5$
2. If it Snows today, I will ski tomorrow
   - If it does not snow today, I will not ski tomorrow.
3. If P is a square, then P is a rectangle.
   - If P is not a square, then P is not a rectangle.
4. If my car is in the repair shop, then I can get to class.
   - If my car is not in the repair shop, then I shall not get to the class.

Converse of a Conditional Statement

The converse of the conditional statement $p \rightarrow q$ is $q \rightarrow p$

For instance; for an implication $\boldsymbol{p} \rightarrow \boldsymbol{q}$,

Its **converse** is: $\boldsymbol{q} \rightarrow \boldsymbol{p}$

Converse of a Conditional Statement

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Writing Converse of a Conditional Statement

1. If Today is Friday, Then 2+3 = 5.
   - If 2 + 3 = 5, Then today is Friday.

2. If it Snows today, I will ski tomorrow
   - I will ski tomorrow If it does snow today.

3. If P is a square, Then P is a rectangle.
   - If P is a rectangle, Then P is a square.

4. If my car is in the repair shop, Then I can get to class.
   - If I can go to class, Then my car is in the repair shop.

Contrapositive of a Conditional Statement

The contrapositive of the conditional statement
$p \to q$ is

$$\sim q \to \sim p$$

A conditional and its contrapositive are equivalent.
Symbolically,

$$p \to q \equiv \sim q \to \sim p$$

For instance; for an implication $p \to q$,

Its **contrapositive** is: $\neg\, q \to \neg\, p$

Writing Contrapositive of a Conditional Statement

1. If Today is Friday, Then $2+3 = 5$.
   - If $2 + 3 \neq 5$, Then today is not Friday.

2. If it Snows today, I will ski tomorrow
   - I will not ski tomorrow only If it does not snow today.

3. If P is a square, then P is a rectangle.
   - If P is not a rectangle, then P is not a square.

4. If my car is in the repair shop, then I can get to class.
   - If I don't get to the class, then my car is not in the repair shop.

Converse, Inverse, Contrapositive

Some terminology, for an implication $p \to q$

- Its **converse** is: $q \to p$
- Its **inverse** is: $\neg p \to \neg q$
- Its **contrapositive** is: $\neg q \to \neg p$

p	q	$\neg$p	$\neg$q	$p \to q$	$q \to p$ (converse)	$\neg p \to \neg q$ (inverse)	$\neg q \to \neg p$ (contrapositive)
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

SAME

Example of Converse, Inverse, Contrapositive

Write the **converse, inverse and contrapositive** of the statement

"if $x \neq 0$, then John is a programmer"

Note: The negation operation ($\neg$) is different from the inverse operation.

Example of Converse, Inverse, Contrapositive

“if $x \neq 0$, then John is a programmer”

- **Converse**: “if John is a programmer, then $x \neq 0$”
- **Inverse**: “if $x = 0$, then John is not a programmer”
- **Contrapositive**: “if John is not a programmer, then $x = 0$”

Takeaway

1. An implication is logically equivalent to it’s contrapositive.

2. The inverse of an implication are logically equivalent.

3. An implication is not equivalent to it’s converse.

Bi-Conditional Statement

If **p and q** are statement variables, the **bi-condition** of p and q is "**p if and only if q**" and is denoted as "**$p \leftrightarrow q$**"

The word "**if and only if**" are sometimes abbreviated "iff".

The double headed arrow "$\leftrightarrow$" is bi-conditional operator.

Truth Table for Bi-Conditional Statement $p \leftrightarrow q$

p	**q**	**$p \leftrightarrow q$**
F	F	T
F	T	F
T	F	F
T	T	T

Bi-Conditional Statement $p \leftrightarrow q$ -- Example

1. $1 + 1 = 3$ if and only if earth is flat	True
2. Sky is blue if and only if $1 = 0$	False
3. Milk is white if and only if birds lay eggs	True
4. 33 is divisible by 4 iff horse has four legs	False
5. $x > 5$ iff $x^2 > 25$	False

Bi-Conditional Statement – Logical Equivalence

$$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \land (q \rightarrow p)$
T	T	T			
T	F	F			
F	T	F			
F	F	T			

Bi-Conditional Statement – Logical Equivalence

$$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \land (q \rightarrow p)$
T	T	T	T	T	
T	F	F	F	T	
F	T	F	T	F	
F	F	T	T	T	

Bi-Conditional Statement – Logical Equivalence

$$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \land (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Rephrasing Bi-Conditional

$p \leftrightarrow q$ is also expressed as:

"p is necessary and sufficient for q"

"If p then q, and conversely"

"p is equivalent to q"

Bi-Conditional – Examples

1. **If** it is hot outside you buy an ice cream cone, **and if** you buy an ice cream cone it is hot outside.
   - **Sol:** You buy an ice cream cone **if and only if** it is hot outside.

2. For you to win the contest it is **necessary and sufficient** that you have the only winning ticket.
   - **Sol:** You win the contest **if and only if** you hold the only winning ticket.

3. If you read the news paper every day, you will be informed and conversely.
   - **Sol:** You will be informed **if and only if** you read the news paper every day.

Truth Table for $(p \to q) \leftrightarrow (\sim q \to \sim p) \equiv t$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T					
T	F					
F	T					
F	F					

Truth Table for $(p \to q) \leftrightarrow (\sim q \to \sim p) \equiv t$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \to \sim q$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T		
T	F	F	T	F		
F	T	T	F	T		
F	F	T	T	T		

Truth Table for $(p \to q) \leftrightarrow (\sim q \to \sim p) \equiv t$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \to \sim q$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	
T	F	F	T	F	F	
F	T	T	F	T	T	
F	F	T	T	T	T	

Truth Table for $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p) \equiv t$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim p \rightarrow \sim q$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Truth Table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Truth Table for (p ↔ q) ↔ (r ↔ q)

p	q	r	p ↔ q	r ↔ q	(p ↔ q) ↔ (r ↔ q)
T	T	T	T		
T	T	F	T		
T	F	T	F		
T	F	F	F		
F	T	T	F		
F	T	F	F		
F	F	T	T		
F	F	F	T		

Truth Table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
T	T	T	TDF	T	
T	T	F	T	F	
T	F	T	F	F	
T	F	F	F	T	
F	T	T	F	T	
F	T	F	F	F	
F	F	T	T	F	
F	F	F	T	T	

Truth Table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	T	T

Truth Table for $p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Truth Table for $p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T	F			
T	T	F	T			
T	F	T	F			
T	F	F	T			
F	T	T	F			
F	T	F	T			
F	F	T	F			
F	F	F	T			

Truth Table for $p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T	F	F		
T	T	F	T	T		
T	F	T	F	F		
T	F	F	T	T		
F	T	T	F	F		
F	T	F	T	F		
F	F	T	F	F		
F	F	F	T	F		

Truth Table for $p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T	F	F	T	
T	T	F	T	T	T	
T	F	T	F	F	T	
T	F	F	T	T	F	
F	T	T	F	F	T	
F	T	F	T	F	T	
F	F	T	F	F	T	
F	F	F	T	F	F	

Truth Table for $p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	~r	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

Truth Table for $\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
T	T	F	F		
T	F	F	T		
F	T	T	F		
F	F	T	T		

Truth Table for $\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
T	T	F	F	F	
T	F	F	T	T	
F	T	T	F	T	
F	F	T	T	F	

Truth Table for $\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
T	T			
T	F			
F	T			
F	F			

Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
T	T	F		
T	F	T		
F	T	T		
F	F	F		

Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
T	T	F	T	
T	F	T	F	
F	T	T	F	
F	F	F	T	

Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Laws of Logics

Laws of Logic

Commutative Law: $P \wedge Q \equiv Q \wedge P$ and $P \vee Q \equiv Q \vee P$

Associative Law: $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ and $(P \vee Q)$

Distributive Law: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (Q \wedge R)$ and $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (Q \vee R)$

Identity Law: $P \wedge t \equiv P$ and $P \vee c \equiv P$

Negation Law: $P \vee \sim P \equiv t$ and $P \wedge \sim P \equiv c$

Double Negation Law: $\sim(\sim P) \equiv P$

Idempotent Law: $P \wedge P \equiv P$ and $P \vee P \equiv P$

DeMorgan's Law: $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ and $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$

Universal Bound Law: $P \vee t \equiv t$ and $P \wedge c \equiv c$

Absorption Law: $P \vee (P \wedge Q) \equiv P$ and $P \wedge (P \vee Q) \equiv P$

Negations of "t" and "c": $\sim t \equiv c$ and $\sim c \equiv t$

Laws of Logic

1. ***Commutative Law***: $p \leftrightarrow q \equiv q \leftrightarrow p$

2. ***Implication Law***: $p \rightarrow q \equiv \sim p \vee q$

2. ***Negation of Implication Law***: $\sim(p \rightarrow q) \equiv \sim(\sim p \wedge q)$

4. ***Exportation Law***: $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

5. ***Equivalence Law***: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

6. ***Reduction and Absurdum***: $p \rightarrow q \equiv (p \wedge \sim q) \rightarrow c$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Proving Equivalence using Logic Laws

Example: Show that $\neg (P \vee (\neg P \wedge Q))$ and $(\neg P \wedge \neg Q)$ are logically equivalent.

$$\neg \ (P \vee (\neg P \wedge Q))$$

$$\equiv \neg P \wedge \neg (\neg P \wedge Q) \quad \text{De Morgan}$$

$$\equiv \neg P \wedge (\neg(\neg P) \vee \neg Q) \quad \text{De Morgan}$$

$$\equiv \neg P \wedge (P \vee \neg Q) \quad \text{Double negation}$$

$$\equiv (\neg P \wedge P) \vee (\neg P \wedge \neg Q) \quad \text{Distributive}$$

$$\equiv \mathbf{F} \vee (\neg P \wedge \neg Q) \quad \text{Negation}$$

$$\equiv (\neg P \wedge \neg Q) \quad \text{Identity}$$

Proving Equivalence using Logic Laws

Example: Show that $\neg(\neg(P \to Q) \to \neg Q)$ is a contradiction.

$$\neg(\neg(P \to Q) \to \neg Q)$$

$$\equiv \neg(\neg(\neg P \vee Q) \to \neg Q) \text{ Equivalence}$$

$$\equiv \neg((P \wedge \neg Q) \to \neg Q) \text{ De Morgan}$$

$$\equiv \neg(\neg(P \wedge \neg Q) \vee \neg Q) \text{ Equivalence}$$

$$\equiv \neg(\neg P \vee Q \vee \neg Q) \text{ De Morgan}$$

$$\equiv \neg(\neg P \vee \mathbf{T}) \text{ Trivial Tautology}$$

$$\equiv \neg(\mathbf{T}) \text{ Domination}$$

$$\equiv \mathbf{F} \quad \text{Contradiction}$$

Application

Simplify: $p \vee [\sim(\sim p \wedge q)]$

Solution:

$p \vee [\sim(\sim p \wedge q)]$	
$\equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan'sLaw
$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law
$\equiv [p \vee p] \vee (\sim q)$	Associative Law for $\vee$
$\equiv p \vee (\sim q)$	Indempotent Law

Which is the simplified statement form.

Application

<u>Verify:</u> $\sim(\sim p \land q) \land (p \lor q) \equiv p$

$\sim(\sim p \land q) \land (p \lor q)$	
$\equiv (\sim(\sim p) \lor \sim q) \land (p \lor q)$	DeMorgan's Law
$\equiv (p \lor \sim q) \land (p \lor q)$	Double Negative Law
$\equiv p \lor (\sim q \land q)$	Distributive Law in Reverse
$\equiv p \lor c$	Negation Law
$\equiv p$	Identity Law

Applications

Prove that $\boldsymbol{p \wedge \sim q \to r \equiv \sim(p \wedge \sim q) \vee r}$

Solution.

$p \wedge \sim q \to r \equiv (p \wedge \sim q) \to r$ *Order of Operation*

$\equiv \sim(p \wedge \sim q) \vee r$ *Implication Law*

Applications

Prove that $(p \to r) \leftrightarrow (q \to r) \equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$

Solution.

$(p \to r) \leftrightarrow (q \to r) \equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$ *Implication*

$\equiv [(\sim p \vee r) \to (\sim q \vee r)] \wedge [(\sim q \vee r) \to (\sim p \vee r)]$ *Equivalence of bi-conditional*

$\equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$ *Implication Law*

Applications

Prove that $\boldsymbol{\sim(p \to q) \to p \equiv t}$

Solution.

$\sim(p \to r) \to p \equiv \sim[\sim(p \wedge \sim q)] \to p$	*Implication Law*
$\equiv (p \wedge \sim q) \to p$	*Double Negation*
$\equiv \sim(p \wedge \sim q) \vee p$	*Implication Law*
$\equiv (\sim p \vee q) \vee p$	*De Morgan's Law*
$\equiv (q \vee \sim p) \vee p$	*Commutative Law of* $\vee$
$\equiv q \vee (\sim p \vee p)$	*Associative law of* $\vee$
$\equiv q \vee t$	*Negation Law*
$\equiv t$	*Universal Bound Law*