Discrete Structures

Lecture # 08

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EXERCISE

Define a binary relation E on the set of the integers Z, as follows:

$m, n \in Z$, $m \, E \, n \Leftrightarrow m - n$ is even.

a. Is $0E0$?

   Is $5E2$?

   Does $(6,6) \in E$?

   Does $(-1,7) \in E$?

b. Prove that for any even integer n, $nE0$.

SOLUTION

b. For any even integer, n, we have

$$n - 0 = n, \quad \text{an even integer}$$

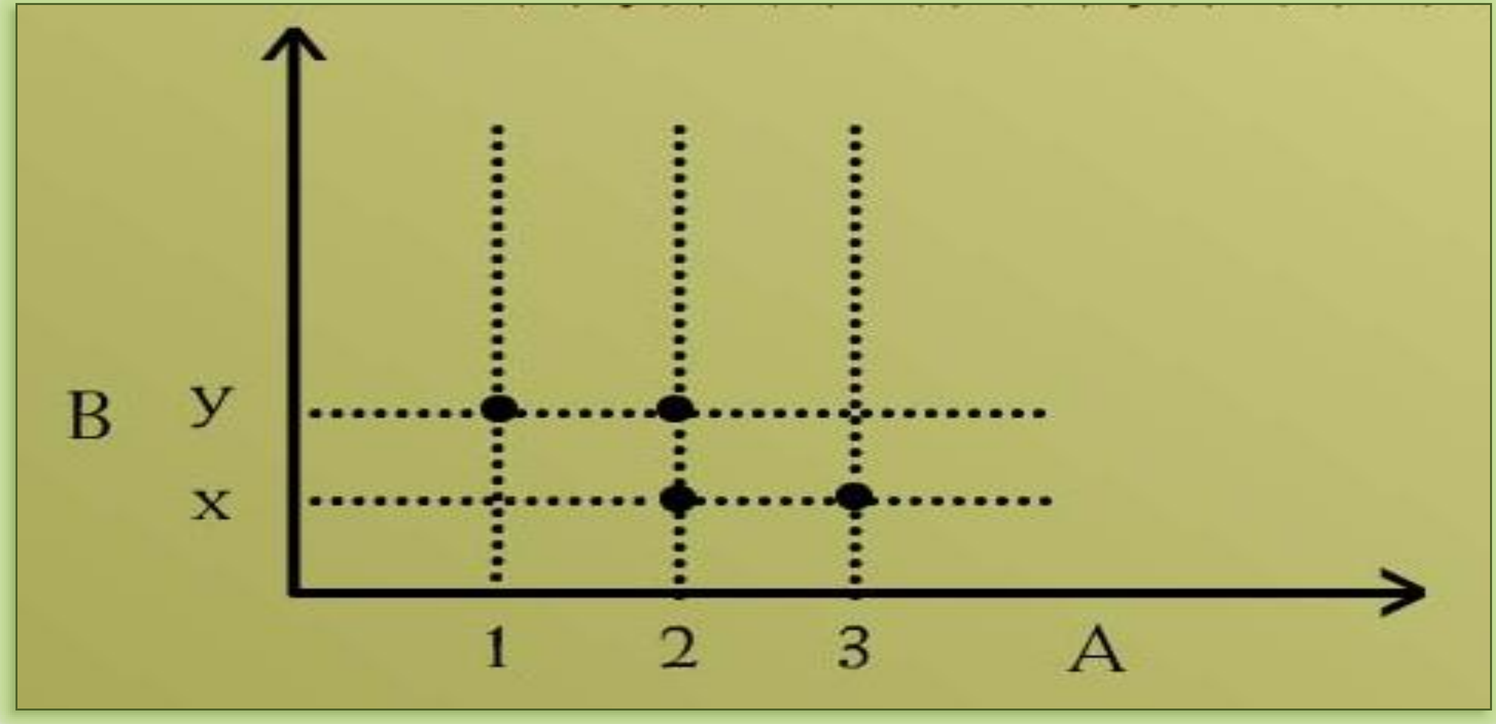
so $(n, 0) \in E$

or equivalently $n \; E \; 0$

CARTESIAN DIAGRAM OF A RELATION

Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$

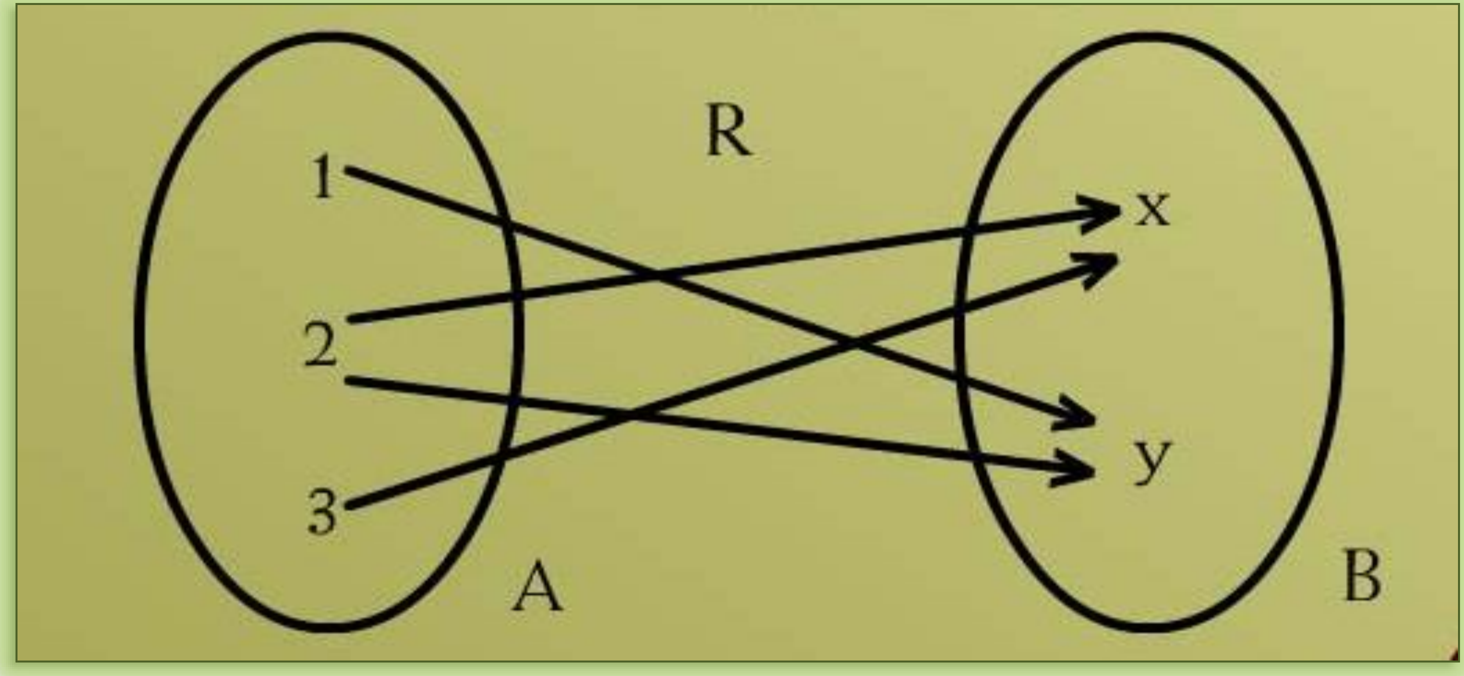
$R = \{ (1, y) , (2, x) , (2, y) , (3, x) \}$



ARROW DIAGRAM OF A RELATION

Let A = {1, 2, 3}   and   B = {x, y}

R = { (1, y) , (2, x) , (2, y) , (3, x) }

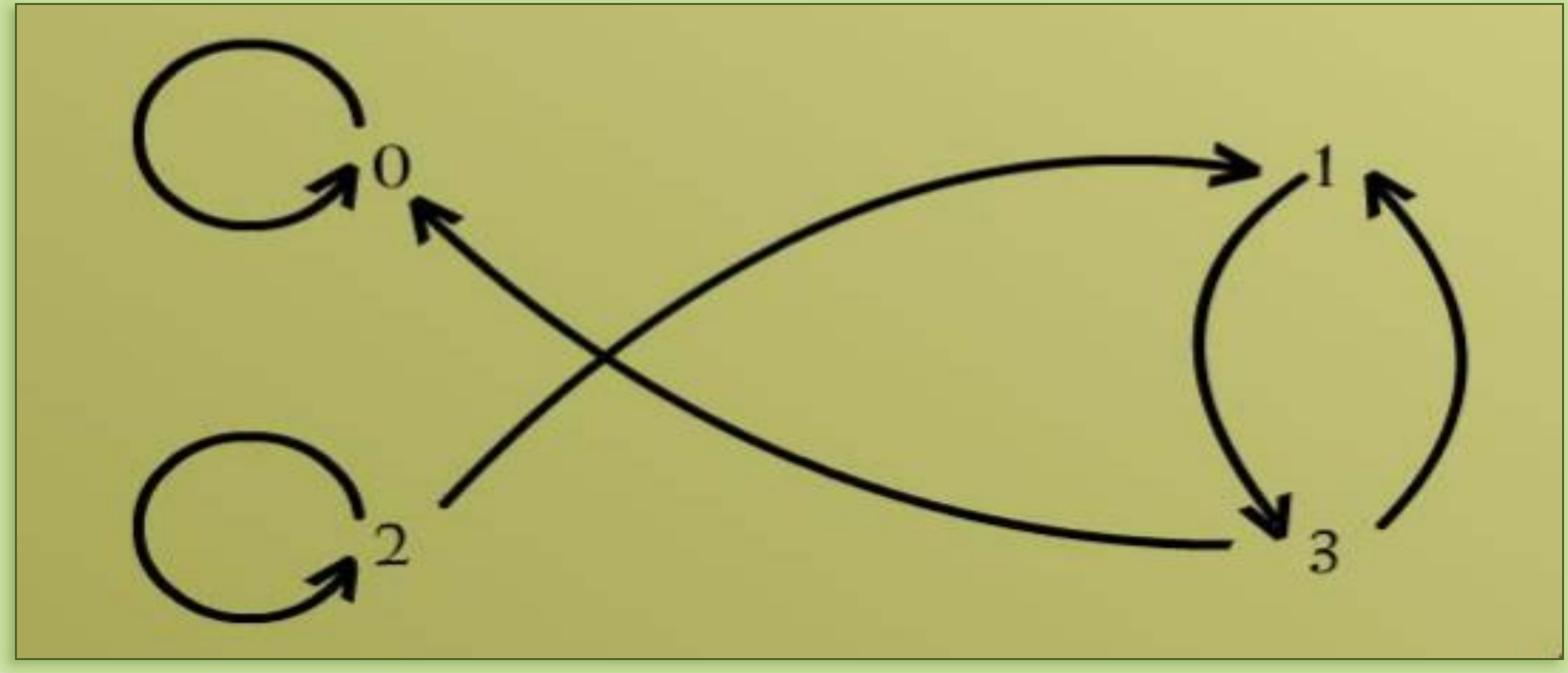


DIRECTED GRAPH OF A RELATION

Let A = {0, 1, 2, 3}

A x A

R = { (0,0) , (1, 3) , (2, 1) , (2, 2) , (3, 0) , (3, 1) }



MATRIX REPRESENTATION OF A RELATION

Let $A = \{a_1, a_2, \ldots, a_n\}$

$B = \{b_1, b_2, \ldots, b_m\}$

Let R be a relation from A to B.

Define the matrix M of order $n \times m$ by

$$M_{(i,j)} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$

EXAMPLE

Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$

$R = \{(1,y), (2,x), (2,y), (3,x)\}$

Order of matrix $= 3 \times 2$

Define A x B first

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\! \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{3 \times 2} \end{array}$$

EXAMPLE

For the relation matrix

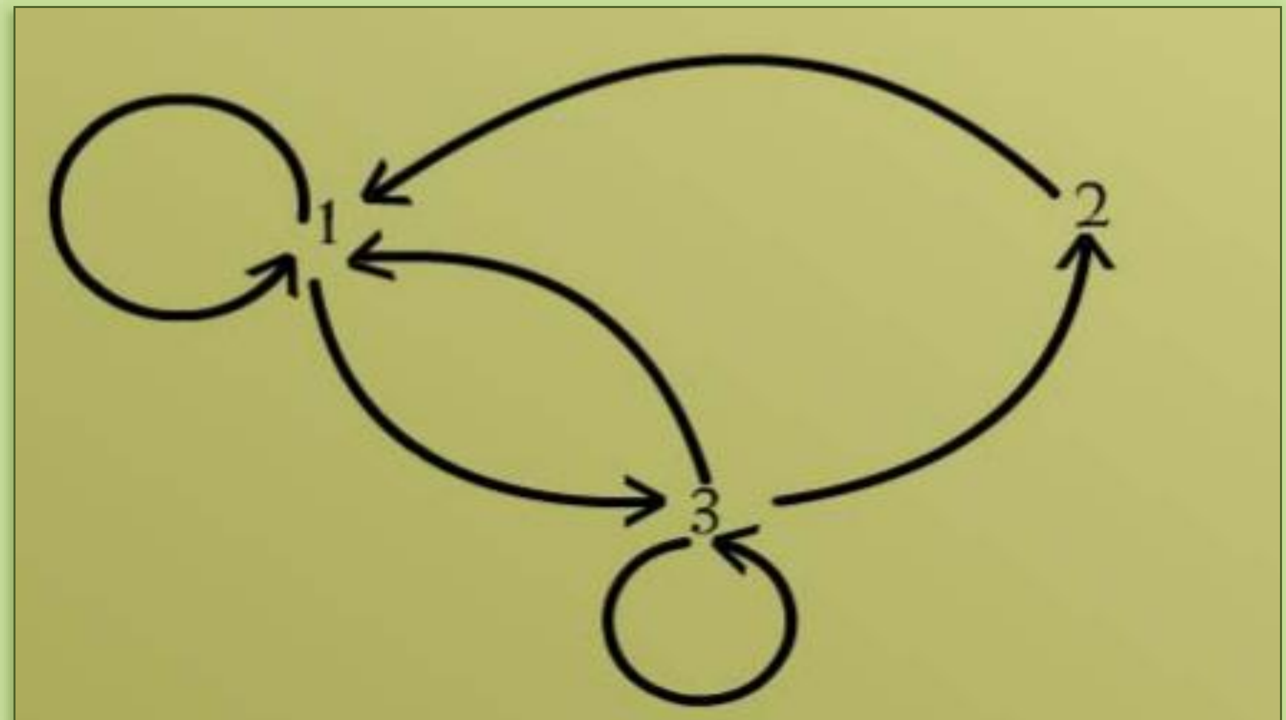
$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

1. List set of ordered pairs represented by M.
2. Draw the directed graph of the relation.

Solution contd…

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

1. R = { (1, 1) , (1, 3) , (2, 1) , (3, 1) , (3, 2) , (3, 3) }



EXERCISE

Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define relations **R** and **S** from **A** to **B** as follows:

$$R = \{(x,y) \in A \times B \mid x\,R\,y \Leftrightarrow x \mid y\}$$

$$S = \{(x,y) \in A \times B \mid x\,S\,y \Leftrightarrow y - 4 = x\}$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$ and $R \cap S$

SOLUTION

$A \times B$

$= \{(2,6), (2,8), (2,10), (4,6), (4,8), (4,10)\}$

$R = \{(2,6), (2,8), (2,10), (4,8)\}$

$S = \{(2,6), (4,8)\}$

$S \subseteq R$

$R \cup S = \{(2,6), (2,8), (2,10), (4,8)\} = R$

$R \cap S = \{(2,6), (4,8)\} = S$

REFLEXIVE RELATION

Let R be a relation on a set A. R is **reflexive if, and only if,** for all $a \in A$, $(a, a) \in R$. Or **equivalently** **aRa**. That is, **each element of A** is related to itself.

REFLEXIVE RELATION

REMARK:

R is not reflexive iff there is an element "a" in A such that $(a, a) \notin R$. That is, some element "a" of A is not related to itself.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

We define relations R_1, R_2, R_3, R_4 on A as follows:

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

EXAMPLE

$$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$$

R_1 is reflexive, since $(a, a) \in R_1$ for all $a \in A$.

$$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$$

R_2 is not reflexive, because $(4, 4) \notin R_2$.

EXAMPLE

$R_3 =$
$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

R_3 is reflexive, since $(a, a) \in R_3$ for all $a \in A$.

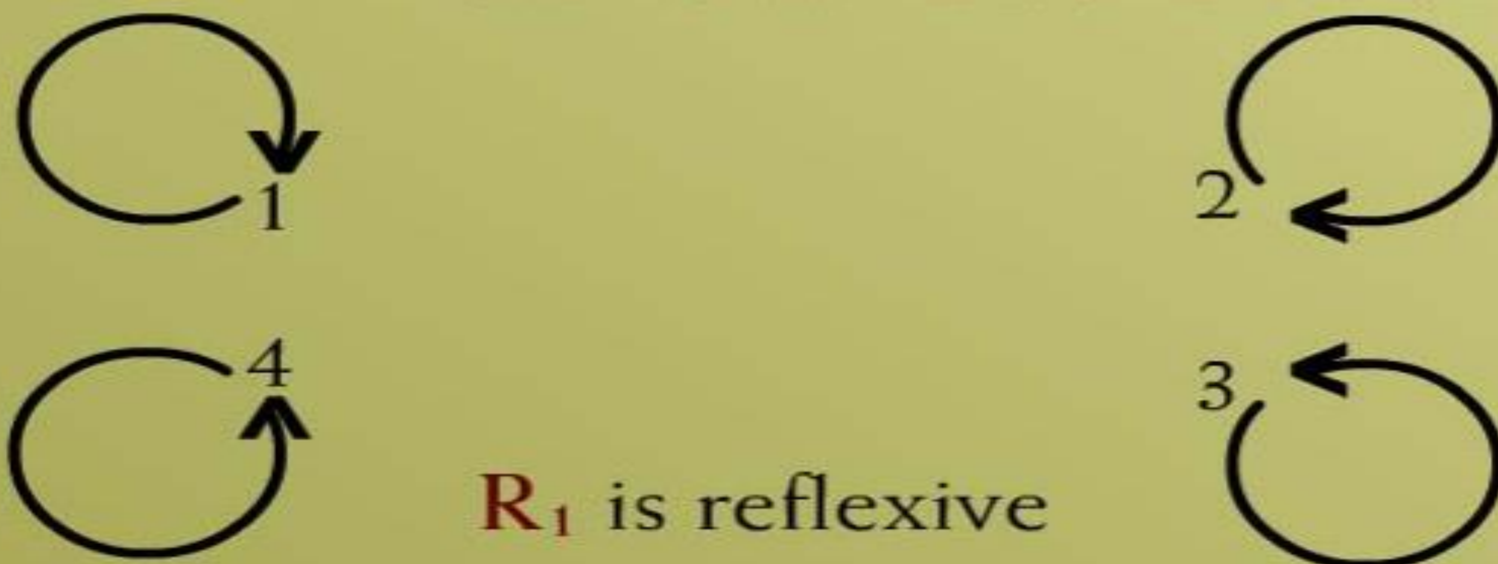
$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

R_4 is not reflexive, because $(1, 1) \notin R_4$,
$(3, 3) \notin R4$

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$



R_1 is reflexive

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

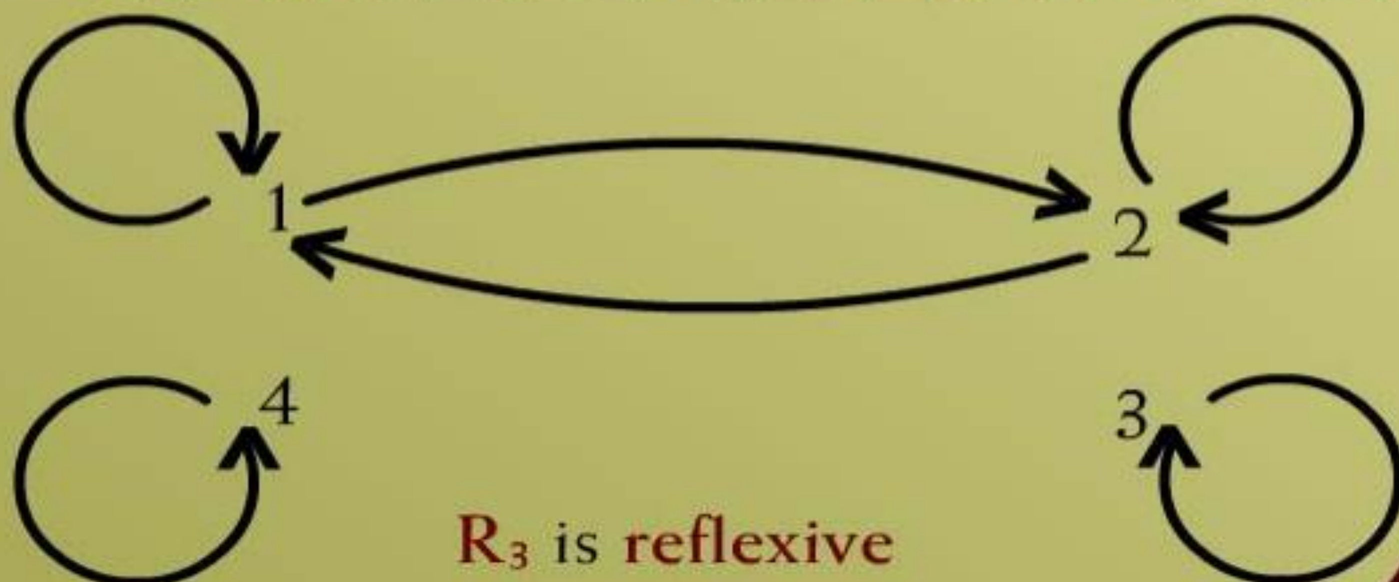
R_2 is not **reflexive**, as there is no loop at 4.

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let

$A = \{1, 2, 3, 4\}$

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$



R_3 is reflexive

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let

$A = \{1, 2, 3, 4\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

R_4 is not reflexive, as there are no loops at 1 and 3.

MATRIX REPRESENTATION OF A REFLEXIVE RELATION

Let $A = \{a_1, a_2, \ldots, a_n\}$. A Relation **R** on **A** is reflexive if and only if $(a_i, a_i) \in R\ \forall\ i = 1, 2, \ldots, n$.

Accordingly, **R** is **reflexive** if all the elements on the **main diagonal** of the matrix **M** representing **R** are equal to 1.

EXAMPLE

Let $A = \{1, 2, 3\}$

$R = \{(1,1), (1,3), (2,2), (3,2), (3,3)\}$

R is reflexive

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

SYMMETRIC RELATION

Let R be a relation on a set A. R is symmetric **if, and only if,** for all a, b $\in$ A, if $(a, b) \in R$ then $(b, a) \in R$. That is, if aRb then bRa.

REMARK:

R is not symmetric iff there are elements a and b in A such that $(a, b) \in R$ but $(b, a) \notin R$.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

R_1, R_2, R_3, R_4

$$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$$

R_1 is symmetric.

$$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

R_2 is symmetric.

EXAMPLE

$R_3 = \{(2, 2), (2, 3), (3, 4)\}$

R_3 is not symmetric, because $(2,3) \in R_3$ but $(3,2) \notin R_3$.

$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$

R_4 is not symmetric, because $(4,3) \in R_4$ but $(3,4) \notin R_3$.

DIRECTED GRAPH OF A SYMMETRIC RELATION

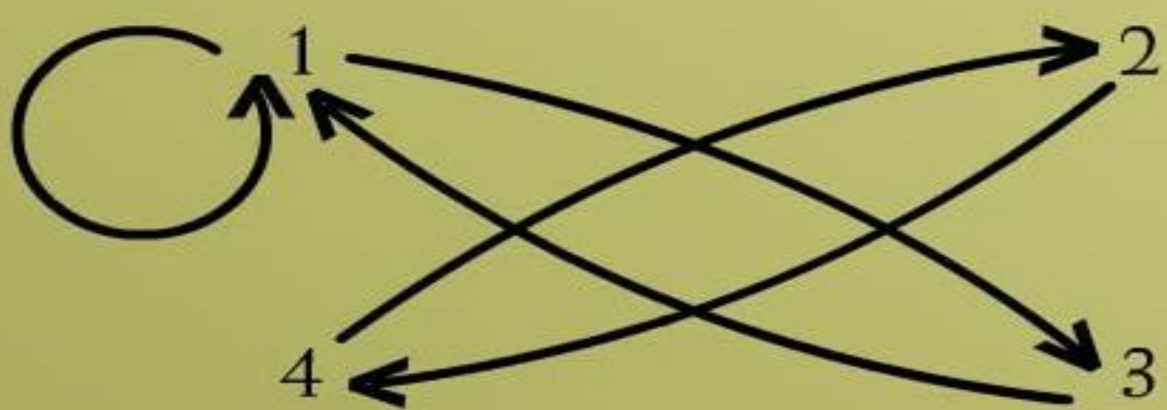
For a symmetric directed graph whenever there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

EXAMPLE

Let

$A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$



R_1 is symmetric

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

R_2 is symmetricis.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_3 = \{(2, 2), (2, 3), (3, 4)\}$

R_3 is not symmetric since there are arrows from 2 to 3 and from 3 to 4 but not conversely.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$

R_4 is not symmetric since there is an arrow from 4 to 3 but no arrow from 3 to 4

MATRIX REPRESENTATION OF A SYMMETRIC RELATION

Let $A = \{a_1, a_2, \ldots, a_n\}$. A relation R on A is symmetric if and only if for all $a_i, a_j \in A$, if $(a_i, a_j) \in R$ then $(a_j, a_i) \in R$.

Accordingly, R is symmetric if the elements in the ith row are the same as the elements in the ith column of the matrix M representing R.

EXAMPLE

Let $A = \{1,2,3\}$

$R = \{(1,3), (2,2), (3,1), (3,3)\}$

$$M = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}$$

M is symmetric i.e $M = M^t$

Transitive Relation

Let R be a relation on a set A.

R is **transitive if and only if**

for all a, b, c $\in$ A, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

That is, if aRb and bRc then aRc.

EXAMPLE

In words, if any one element is related to a second and that second element is related to a third, then the first is related to the third.

REMARKS:

R is not transitive iff there are elements a, b, c in A such that $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$ define relations.

$$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$

$$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$$

EXAMPLE

$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

R_1 is transitive. Because $(1,2) \in R_1$ and $(2,3) \in R_1 \Rightarrow (1,3) \in R_1$

$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

R_2 is not transitive. Because $(1,2) \in R_2$ and $(2,3) \in R_2$ but $(1,3) \notin R_2$

$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

R_3 is transitive. Because $(2,3) \in R_3$ and $(3,4) \in R_3 \Rightarrow (2,4) \in R_3$

DIRECTED GRAPH OF A TRANSITIVE RELATION

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

R_1 is transitive.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

R_2 is not **transitive** since there is **an arrow** from 1 to 2 and from 2 to 3 but **no arrow** from 1 to 3 directly.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

R_3 is transitive

Matrix Representation of Transitive Relation

- Let A be the given matrix.
- Identify all zeros of the A .
- Take the square of the given matrix A .
- The relation is transitive if and only if the squared matrix has no nonzero entry where the original had a zero.

Example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Let A be the given matrix.

Now $$A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

As all zeros in A^2 are intact i.e. zeros in A^2 are at the same position as that of A. Therefore the given relation is transitive.

EXERCISE

Let $A = \{0, 1, 2\}$

$$R = \{(0,2), (1,1), (2,0)\}$$

1. Is R reflexive? Symmetric? Transitive?
2. Which ordered pairs are needed in R to make it a reflexive and transitive relation.

SOLUTION

$$R = \{(0,2), (1,1), (2,0)\}$$

1. R is not reflexive, since $0 \in A$ but $(0, 0) \notin R$.

R is symmetric.

Because $(0,2) \in R \Rightarrow (2,0) \in R$

R is not transitive, since $(0, 2)$ & $(2, 0) \in R$ but $(0, 0) \notin R$.

SOLUTION

$$R = \{(0,2), (1,1), (2,0)\}$$

1. R is not reflexive.
   R to be reflexive it must contains (0, 0) and (2, 2).

2. R is not transitive.
   For R to be transitive it must contain (0, 0) and (2, 2).

EXERCISE

Define a relation L on the set of **real numbers** R be defined as follows:

$$\text{for all } x, y \in R, \qquad x\, L\, y \Leftrightarrow x < y.$$

a. Is L reflexive?
b. Is L symmetric?
c. Is L transitive?

R here is representing Real Numbers and L is representing relation.

SOLUTION

$$x \, L \, y \Leftrightarrow x < y$$

a. L is not reflexive, because $x < x$ for any real number x. is not true.

For example $1 \nless 1$

b. L is not symmetric, because for all $x, y \in R$, if $x < y$ then $y < x$ which can't be true at the same time.

For example $1 < 2$ and $2 \nless 1$

SOLUTION

$$x \; L \; y \Leftrightarrow x < y$$

c. L is transitive, because for all, $x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. (by transitive law of order of real numbers).

Note: These properties are independent of each other.

EXAMPLE

Define a relation **R** on the set of **positive integers Z^+** as follows:

for all a, b $\in Z^+$, **a R b** iff **a $\times$ b** is **odd**.

Determine whether the relation is.

a. **Reflexive**

b. **Symmetric**

c. **Transitive**

SOLUTION

For reflexivity, we have to show that,
aRa iff a x a = odd number.
And it contradicts the definition of reflexivity. Because 2R2 $\leftrightarrow$ 2x2=4 which is not odd.
Hence, R is not reflexive.

SOLUTION

b. **Symmetric**

R is symmetric.

if a R b then $a \times b$ is odd

or equivalently $b \times a$ is odd

$\because (b \times a = a \times b)$

$\Rightarrow$ b R a.

SOLUTION

c. Transitive

R is transitive.

if a R b then a × b is odd

⇒ both "a" and "b" are odd.

bRc then b × c is odd

⇒ both "b "and "c"are odd.

⇒ aRc because a × c is odd.

EXAMPLE

Let "D" be the "divides" relation on Z defined as:

for all m, n $\in$ Z, m D n $\Leftrightarrow$ m | n

Determine whether D is reflexive, symmetric or transitive. Justify your answer.

Note: Here the set of integers do not include 0 i.e. Z-{0}

SOLUTION

Reflexive:

D is clearly **reflexive**.

Let $m \in Z$, since every integer divides itself so $m|m\ \forall\ m \in Z$

therefore $m\ D\ m\ \ \forall\ m \in Z$

For example $2|2$

SOLUTION

Symmetric:

This relation is not symmetric.

For example 2 divides 6.

But 6 does not divides 2.

i.e $2D6$ but $6\not{D}2$.

SOLUTION

Transitive:

If m **divides** n and n **divides** p.

Then m **divides** p.

EQUIVALENCE RELATION

Let A be a non-empty set and R a binary relation on A. R is an equivalence relation if and only if, R is reflexive, symmetric, and transitive.

EXAMPLE

Let

$A = \{1, 2, 3, 4\}$

$R = \{ (1, 1) , (2, 2) , (2, 4) , (3, 3) , (4, 2) , (4, 4) \}$

R is reflexive, symmetric and transitive.

CONGRUENCES RELATION

Let m and n be integers and d be a positive integer.

The notation $m \equiv n \pmod{d}$

means that

$d \mid (m - n)$ {d divides m minus n}

$\Leftrightarrow$ There exists an integer k such that

$(m - n) = d \, . \, k$

EXAMPLE

a. Is $22 \equiv 1 \pmod{3}$? b. Is $-5 \equiv +10 \pmod{3}$?

c. Is $7 \equiv 7 \pmod{4}$? d. Is $14 \equiv 4 \pmod{4}$?

Solution:

a. Since $22-1 = 21$

21 is divisible by 3

Hence $3 \mid (22-1)$, and so $22 \equiv 1 \pmod{3}$

b. Since $-5-10 = -15$

-15 is divisible by 3

Hence $3 \mid ((-5)-10)$, and so $-5 \equiv 10 \pmod{3}$

Solution contd…

c. Since $7 - 7 = 0$

Hence $4 \mid (7-7)$, and so $7 \equiv 7 \pmod 4$

d. Since $14 - 4 = 10$ but 4 does not divide 10

Hence $14 \not\equiv 4 \pmod 4$.

EXERCISE

Define a relation R on the set of all integers Z as follows:

for all integers m and n,

$$m \; R \; n \Leftrightarrow m \equiv n \pmod{3}$$

Prove that R is an equivalence relation.

SOLUTION

R is reflexive.

Every element is related to itself because

$m - m = 0$

Zero is divisible by every number.

Hence $3 \mid (m-m)$, and so $m \equiv m \pmod{3}$

SOLUTION

R is symmetric.

We have to show that

if m R n then n R m.

$mRn \Rightarrow m \equiv n \pmod 3$

$\Rightarrow 3 \mid (m-n)$

$\Rightarrow m-n = 3k$, for some integer k.

$\Rightarrow n - m = 3(-k)$, $-k \in Z$

$\Rightarrow 3 \mid (n-m)$

$\Rightarrow n \equiv m \pmod 3$

$\Rightarrow nRm$

SOLUTION

R is transitive.

mRn and nRp means

$$m \equiv n \pmod 3 \text{ and } n \equiv p \pmod 3$$

$\Rightarrow 3 \mid (m-n)$ and$3 \mid (n-p)$

$\Rightarrow (m-n) = 3r$ and $(n-p) = 3s$

for some $r, s \in Z$

Adding these two equations, we get,

$(m-n) + (n-p) = 3r + 3s$

$\Rightarrow m - p = 3(r + s)$, where $r + s \in Z$

$\Rightarrow 3 \mid (m - p)$

$\Rightarrow m \equiv p \pmod 3 \Leftrightarrow m\,Rp$