Mon Tue Wed Thu Fri Sat Sun

CW

Date: 22 August 2024

Numerical Computing (2)

=> Interpolation & Extrapolation:

x	y
1960	1.2
1970	4.6
1980	9.8
1990	10.1

1972? --> Thru Interpolation.

1940 or 2005? -> Thru Extrapolation.

• Equally Spaced Data:

$$= x_0 \ , \ x_1 = x_0 + h \ , \ x_2 = x_0 + 2h \ , \ x_3 = x_0 + 3h \ , \ \ldots\ldots$$

$$\ldots\ldots\ldots , \ x_i = x_0 + i \cdot h$$

where h is the step size.

=> Operators:

Mon Tue Wed Thu Fri Sat Sun

Date:

• Forward Difference Operator:

$\rightarrow \quad \Delta y_i = y_{i+1} - y_i$

1. $\Delta y_0 = y_1 - y_0$
2. $\Delta y_1 = y_2 - y_1$
3. $\Delta y_2 = y_3 - y_2$

$\vdots$

$\Delta y_{n-1} = y_n - y_{n-1}$

x	y	Δy
x_0	y_0	
x_1	y_1	$y_1 - y_0 = \Delta y_0$ $\boxed{\nabla y_1}$
x_2	y_2	$y_2 - y_1 = \Delta y_1$ $\boxed{\nabla y_2}$
x_3	y_3	$y_3 - y_2 = \Delta y_2$ $\boxed{\nabla y_3}$
x_4	y_4	$y_4 - y_3 = \Delta y_3$ $\boxed{\nabla y_4}$

Continued....

Δy	$\Delta^2 y$	$\Delta^3 y$
Δy_0	$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$ $\boxed{\nabla^2 y_2}$	$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$ $\boxed{\nabla^3 y_3}$
Δy_1	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$ $\boxed{\nabla^2 y_3}$	$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$ $\boxed{\nabla^3 y_4}$
Δy_2	$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$ $\boxed{\nabla^2 y_4}$	
Δy_3		

• Backward Difference Operator:

$\nabla y_i = y_i - y_{i-1}$

$\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$

$\nabla y_3 = y_3 - y_2$

$\vdots$

$\nabla y_n = y_n - y_{n-1}$

$\Delta^3 y$	$\Delta^4 y$
$\Delta^3 y_0$	$\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4 y_0$ $\boxed{\nabla^4 y_4}$
$\Delta^3 y_1$	

∴ Highlited in Blue are the changes that take place in backward difference operator.

Date

- **Central Difference Operator:**

$$y_1 - y_0 = \delta y_{1/2}$$

$$y_2 - y_1 = \delta y_{3/2}$$

$$\vdots$$

$$y_n - y_{n-1} = \delta y_{n-1/2}$$

x	y	Δy	$\Delta^2 y$
x_0	y_0		
x_1	y_1	$y_1 - y_0 = \delta y_{1/2}$	
x_2	y_2	$y_2 - y_1 = \delta y_{3/2}$	$\Delta y_1 - \Delta y_0 = \delta^2 y_{1/2}$
x_3	y_3	$y_3 - y_2 = \delta y_{5/2}$	$\Delta y_2 - \Delta y_1 = \delta^2 y_1$
x_4	y_4	$y_4 - y_3 = \delta y_2$	$\Delta y_3 - \Delta y_2 = \delta^2 y_{3/2}$

- **Gregory - Newton Forward Interpolation Method:**

$\rightarrow x_0$

$\rightarrow x_1 = x_0 + h$

$\rightarrow x_2 = x_0 + 2h$

$\vdots$

$\rightarrow x_i = x_0 + ih$

Consider $y(x)$ to be nth-order polynomial.

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad (1)$$

When $x = x_0$

$y(x_0) = a_0$

$y_0 = a_0$

when $x = x_1$

$y(x_1) = a_0 + a_1(x - x_0)$

$y_1 = a_0 + a_1(h)$

$y_1 = a_0 + a_1 h.$

$y_1 = y_0 + a_1 h$

$y_1 - y_0 = a_1 h$

$\Delta y_0 = a_1 h \Rightarrow a_1 = \dfrac{\Delta y_0}{h}$

when $x = x_2$

$y(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

$y_2 = a_0 + a_1 \cdot 2h + a_2(2h)(h) \Rightarrow a_0 + a_1 \cdot 2h + a_2 \cdot 2h^2$

$y_2 - y_1 = \cancel{a_0} + a_1 \cdot 2h + a_2 \cdot 2h^2 - \cancel{a_0} - a_1 \cdot h$

$\Delta y_1 = a_1 \cdot h + 2a_2 \cdot h^2$

$\Delta y_1 = \dfrac{\Delta y_0}{\cancel{h}} \cdot \cancel{h} + 2a_2 \cdot h^2$

$\Delta y_1 - \Delta y_0 = 2a_2 h^2$

$\Delta^2 y_0 = 2a_2 h^2$

$a_2 = \dfrac{\Delta^2 y_0}{2!\, h^2}$

Similarly

$a_3 = \dfrac{\Delta^3 y_0}{3!\, h^3}$

Mon Tue Wed Thu Fri Sat Sun   Date:

put $a_0, a_1, a_2, a_3 \ldots$ in eq (1), we get

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2! \cdot h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3! \cdot h^3}(x - x_0)(x - x_1)(x - x_2)$$

$$+ \ldots \ldots \quad - (2)$$

Let us Put $x - x_0 = ph$

$x - x_0 = x - x_0 + x_0 - x_1$

$= (x - x_0) + (x_0 - x_1) \Rightarrow ph + (-h)$ → $\because x_1 - x_0 = h$ so $x_0 - x_1 = -h$

$(x - x_1) = h \cdot (p - 1)$

$x - x_2 = (p-2) \cdot h$ → Put these in (2)

$x - x_3 = (p-3) \cdot h$

$$y(x_0 + ph) = y_0 + \frac{\Delta y_0}{\cancel{h}} \cdot p\cancel{h} + \frac{\Delta^2 y_0}{2! \cdot \cancel{h^2}} \cdot p\cancel{h}[\cancel{h}(p-1)] + \frac{\Delta^3 y_0}{3! \cdot \cancel{h^3}} \cdot p\cancel{h}[\cancel{h}(p-1)][(p-2)\cancel{h}]$$

$$+ \ldots \ldots$$

The Final Form.

$$y(x_0 + ph) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots \ldots$$

↓ or y_p

$\ldots \ldots \ldots$

Mon Tue Wed Thu Fri Sat Sun

Date:

- Gregory - Newton Backward Interpolation Formula:

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots\dots$$

Example

$y = \sin\theta$

$\theta = 10, 20, 30, 40, 50$ find y at $\theta = 25°$

θ	$\sin\theta$	Δy
10	0.1736	
20	0.3420	
30	0.5000	
40	0.6428	
50	0.7660	

Mon Tue Wed Thu Fri Sat Sun   Date:

when $x = x_3$

$$y(x_3) = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) + a_3(x_3 - x_0)$$
$$(x_3 - x_1)(x_3 - x_2)$$

$$y_3 = a_0 + a_1 \cdot 3h + a_2 \cdot 3h \cdot 2h + a_3 \cdot 3h \cdot 2h \cdot h$$

$$y_3 = a_0 + 3ha_1 + 6h^2 a_2 + 6h^3 a_3$$

$$y_3 - y_2 = \cancel{a_0} + 3ha_1 + 6h^2 a_2 + 6h^3 a_3 - \cancel{a_0} - 2ha_1 - 2h^2 a_2$$

$$\Delta y_2 = h \cdot a_1 + 4h^2 a_2 + 6h^3 a_3$$

$$\Delta y_2 = \frac{\Delta y_0}{\cancel{h}_1} \cdot \cancel{h} + \cancel{4}^{2}\cancel{h^2} \cdot \frac{\Delta^2 y_0}{\cancel{2!} \cdot \cancel{h^2}} + 6h^3 a_3$$

$$\Delta y_2 = \Delta y_0 + 2\Delta^2 y_0 + 6h^3 \cdot a_3$$

$$\Delta y_2 - \Delta y_0 = 2\Delta^2 y_0 + 6h^3 \cdot a_3$$

$$(\Delta y_2 - \Delta y_1) + (\Delta y_1 - \Delta y_0) = 2\Delta^2 y_0 + 6h^3 \cdot a_3$$

$$\Delta^2 y_1 + \Delta^2 y_0 - 2\Delta^2 y_0 = 6h^3 \cdot a_3$$

$$\Delta^2 y_1 - \Delta^2 y_0 = 6h^3 \cdot a_3$$

$$\Delta^3 y_0 = 6h^3 \cdot a_3 \quad \Rightarrow \quad \frac{\Delta^3 \cdot y_0}{6h^3} = a_3 \text{ or } \frac{\Delta^3 \cdot y_0}{3! \cdot h^3}$$