(9)

BESSEL'S FORMULA:

$$y_p = \sum_{i=0}^{n}\left[\binom{P+i-1}{2i}\delta^{2i}\mu y_{1/2} + \left(\frac{1}{2i+1}\right)\left(P-\frac{1}{2}\right)\binom{P+i-1}{2i}\delta^{2i+1} y_{1/2}\right]$$

Collocation is at $P = -n, \ldots, n+1$

suppose

$$\begin{array}{cccccccc} P = -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ x_{-3} & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$

$\underline{\underline{n=3}}$

$$y_p = \mu y_{1/2} + \left(P-\frac{1}{2}\right)\delta y_{1/2}$$

$$+\binom{P}{2}\delta^2\mu y_{1/2} + \frac{1}{3}\left(P-\frac{1}{2}\right)\binom{P}{2}\delta^3 y_{1/2}$$

$$+\binom{P+1}{4}\delta^4\mu y_{1/2} + \frac{1}{5}\left(P-\frac{1}{2}\right)\binom{P+1}{4}\delta^5 y_{1/2}$$

$$+\binom{P+2}{6}\delta^6\mu y_{1/2} + \frac{1}{7}\left(P-\frac{1}{2}\right)\binom{P+2}{6}\delta^7 y_{1/2}$$

Uj.

(b)

$$\binom{P}{2} = \frac{P!}{2!\,(P-2)!} = \frac{P(P-1)}{2!}$$

$$\binom{P+1}{4} = \frac{(P+1)!}{4!\,(P-3)!} = \frac{(P+1)P(P-1)(P-2)}{4!}$$

$$\binom{P+2}{6} = \frac{(P+2)!}{6!\,(P-4)!} = \frac{(P+2)(P+1)P(P-1)(P-2)(P-3)}{6!}$$

$$\begin{aligned}
y_P = \mu y_{1/2} &+ \left(P-\tfrac{1}{2}\right)\delta y_{1/2} \\
&+ \frac{P(P-1)}{2!}\delta^2 \mu y_{1/2} \\
&+ \frac{1}{3}\left(P-\tfrac{1}{2}\right)\frac{P(P-1)}{2!}\delta^3 y_{1/2} \\
&+ \frac{(P+1)P(P-1)(P-2)}{4!}\delta^4 \mu y_{1/2} \\
&+ \frac{1}{5}\left(P-\tfrac{1}{2}\right)\frac{(P+1)P(P-1)(P-2)}{4!}\delta^5 y_{1/2} \\
&+ \frac{(P+2)(P+1)P(P-1)(P-2)(P-3)}{6!}\delta^6 \mu y_{1/2} \\
&+ \frac{1}{7}\left(P-\tfrac{1}{2}\right)\frac{(P+2)(P+1)P(P-1)(P-2)(P-3)}{6!}\delta^7 y_{1/2}
\end{aligned}$$

(c) $y_0 \quad y_{1/2} \quad y_1$

$$\mu y_{1/2} = \frac{y_0 + y_1}{2}$$

$$\Rightarrow \delta^{2n} \mu y_{1/2} = \frac{\delta^{2n} y_0 + \delta^{2n} y_1}{2}$$

$$y_p = \left(\frac{y_0 + y_1}{2}\right) + \left(p - \frac{1}{2}\right) \delta y_{1/2}$$

$$+ \frac{p(p-1)}{2!} \left(\frac{\delta^2 y_0 + \delta^2 y_1}{2}\right)$$

$$+ \left(p - \frac{1}{2}\right) \frac{p(p-1)}{3!} \delta^3 y_{1/2}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!} \left(\frac{\delta^4 y_0 + \delta^4 y_1}{2}\right)$$

$$+ \left(p - \frac{1}{2}\right) \frac{(p+1)p(p-1)(p-2)}{5!} \delta^5 y_{1/2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)(p-3)}{6!} \left(\frac{\delta^6 y_0 + \delta^6 y_1}{2}\right)$$

$$+ \left(p - \frac{1}{2}\right) \frac{(p+2)(p+1)p(p-1)(p-2)(p-3)}{7!} \delta^7 y_{1/2}$$

$+$

$\vdots$

$+$

(d) +

|

|

$$+ \frac{P(P-1)(P+1)(P-2)(P+2)\cdots(P-n)(P+n-1)}{(2n)!}\left(\frac{\delta^{2n}y_0+\delta^{2n}y_1}{2}\right)$$

$$+ \frac{(P-0.5)P(P-1)(P+1)(P-2)(P+2)\cdots(P-n)(P+n-1)}{(2n+1)!}\delta^{2n+1}y_{1/2}$$

Bessel formula is used for interpolation in the middle of the table for the values of p close to 0.5. In practical applications it is used for $0.25 \leq p \leq 0.75$. The formula has the simplest form for $p=0.5$, Since all the terms, containing the odd differences disappear

Here $p = \frac{x - x_0}{h}$

Uj.

(e)

Q) Use Bessel's formula to estimate sinh 1.45224, from the following data by constructing a polynomial of degree 5.

1	1.17520
1.1	1.33565
1.2	1.50946
1.3	1.69838
1.4	1.90430
1.5	2.12928
1.6	2.37557
1.7	2.64563
1.8	2.94217

Sol $X = 1.45224 \quad x_0 = 1.4 \quad h = 0.1$

$$P = \frac{X - x_0}{h} = \frac{1.45224 - 1.4}{0.1} = 0.5224$$

For degree 5, n will be 2.

i.e collocation is at $-2, \ldots, 3$

Uj.

-4	1	1.17520								
			0.16045							
-3	1.1	1.33565		0.01336						
			0.17381		0.00175					
-2	1.2	1.50946		0.01511		0.00014				
			0.18892		0.00189		0.00003			
-1	1.3	1.69838		0.017		0.00017		−0.00001		
			0.20592		0.00206		0.00002		0.00001	
0	1.4	1.90430 y_0		0.01906 $\delta^2 y_0$		0.00019 $\delta^4 y_0$		0		0.00001
			0.22498 $\delta y_{1/2}$		0.00225 $\delta^3 y_{1/2}$		0.00002 $\delta^5 y_{1/2}$		0.00002	
1	1.5	2.12928 y_1		0.02131 $\delta^2 y_1$		0.00021 $\delta^4 y_1$		0.00002		
			0.24629		0.00246		0.00004			
2	1.6	2.37557		0.02377		0.00025				
			0.27006		0.00271					
3	1.7	2.64563		0.02648						
			0.29654							
4	1.8	2.94217								

⑨

$$y_p = \frac{1.90430 + 2.12928}{2}$$

$$+ (0.5224 - 0.5) * 0.22498$$

$$+ \frac{0.5224(0.5224-1)}{2}\left[\frac{0.01906 + 0.02131}{2}\right]$$

$$+ \frac{(0.5224-0.5)\,0.5224(0.5224-1)}{6} * 0.00225$$

$$+ \frac{(0.5224+1)\,0.5224(0.5224-1)(0.5224-2)}{24}\left[\frac{0.00019 + 0.00021}{2}\right]$$

$$+ \frac{(0.5224-0.5)(0.5224+1)(0.5224)(0.5224-1)(0.5224-2)}{120} * 0.00002$$

$$y_p = 2.01679 + 0.005039552 + 0.0025180609 87$$

$$- 0.0000020957 9 + 0.000004677049$$

$$+ 0.0000000002095$$

$$= 2.019314074$$

$$\sinh 1.45224 = 2.019314074$$