Fixed Point Theorem:

Let $g \epsilon C[a,b]$ be such that $g(x) \epsilon [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g'exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad for\ all\ x \ \epsilon\ (a,b)$$

Then for any number $p_0 \ \epsilon\ [a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$.

Corollary:

If g satisfies the hypotheses of Fixed Point Theorem, then bounds for the error in using p_n to estimate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \quad for\ all \quad n \geq 1.$$

QUESTION: Use Fixed Point Theorem to show that $g(x) = \pi + 0.5\ sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

SOLUTION:

Existence of p:

x	$g(x) = \pi + 0.5\,sin(0.5x)$
0	3.1415926536
$\pi/2$	3.495146044
π	3.6415926536
$3\pi/2$	3.4951460442
2π	3.1415926532

It follows form the above data that $g(x) \ \epsilon\ [0, 2\pi]$, for all x in $[0, 2\pi]$.
Hence there is at least one fixed-point p in the interval $[0, 2\pi]$.

Uniqueness of p:

$$g'(x) = 0 + 0.5\cos(0.5x) * 0.5$$
$$g'(x) = 0.25\cos(0.5x)$$

x	$\lvert g'(x)\rvert = \lvert 0.25\cos(0.5x)\rvert$
0	0.25
$\pi/2$	0.1767766953
π	0
$3\pi/2$	0.1767766952
2π	0.25

It follows form the above data that $|g'(x)| \le 0.25, \quad for\ all\ x \in (0,2\pi)$.
Hence there is a unique fixed- point p of g in $[0, 2\pi]$.

Here upper bound of $g'(x)$ is $k \approx 0.25$.

Estimation of Fixed-point p:

To approximate the fixed point p of function g, we choose an initial approximation $p_0 = 0$, and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting

$$p_n = g(p_{n-1}), \quad n \ge 1$$

Python Code:

```
import numpy as np

import sys

def g(x): return np.pi + 0.5 * np.sin(0.5 * x)


p0 = float(input("Enter p0: "))
n = 1
N = 10
TOL = 1e-2

print('n          pn')
while n <= N:
    p = g(p0)
    print(f'{n} \
        {p:.15f} ')
    if abs(p - p0) <= TOL:
        print('Fixed-Point has converged')
        print('final p =', p)
        sys.exit(0)
    else:
        p0 = p
    n += 1

print('zero not found to the desired tolerance')
```

Output:

n=1:

$p = g(p0)$

$p = g(0)$

$p = \pi + 0.5\, sin(0.5 * 0)$

$= \pi = 3.1415926536$

n	pn
1	3.141592653589793

$|p - p_0| = |3.1415926536 - 0| = 3.1415926536 > TOL$

$p0 = 3.1415926536$

n=2:

$p = g(p0)$

$p = g(\pi)$

$p = \pi + 0.5\, sin(0.5 * \pi)$

$= \pi + 0.5 = 3.6415926536$

n	pn
1	3.141592653589793
2	3.641592653589793

$|p - p_0| = |3.6415926536 - 3.1415926536| = 0.5 > TOL$

$p0 = 3.6415926536$

n=3:

$p = g(p0)$

$p = g(3.6415926536)$

$p = \pi + 0.5\, sin(0.5 * 3.6415926536)$

$= 3.6260488644$

n	pn
1	3.141592653589793
2	3.641592653589793
3	3.626048864445115

$|p - p_0| = |3.6260488644 - 3.6415926536| = 0.015543789145 > TOL$

$p0 = 3.6260488644$

n=4:
$p = g(p0)$
$p = g(3.6260488644)$
$p = \pi + 0.5\, sin(0.5 * 3.6260488644$
$= 3.6269956224$

```
n      pn
1      3.141592653589793
2      3.641592653589793
3      3.626048864445115
4      3.626995622438735
```

$|p - p_0| = |3.6269956224 - 3.6260488644| = 0.0009467579936 < TOL$

```
Fixed-Point has converged
final p = 3.626995622438735
final n = 4

Process finished with exit code 0
```

Error bound for fixed- point method is:

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|, \quad for\ all \quad n \ge 1.$$

$$\frac{k^n}{1-k}|p_1 - p_0| = 10^{-2}$$

$$\frac{0.25^n}{1-0.25}|\pi - 0| = 10^{-2}$$

$$\frac{0.25^n}{0.75} * \pi = 0.01$$

$$0.25^n = 0.0023873241$$

$$\ln 0.25^n = \ln 0.0023873241$$

$$n * \ln 0.25 = \ln 0.0023873241$$

$$n = \frac{\ln 0.0023873241}{\ln 0.25}$$

$$n = 4.3551949093$$

For the bound to be less than 1e-2, we need $n \ge 4$, and $p_4 = 3.6269956224$ is accurate to within 10^{-2}.