8.1 INTRODUCTION

We assume that a function $f(x)$ is given in a tabular form at a set of $n + 1$ distinct points $x_0, x_1, \ldots, x_n$. From the given tabular data, we require approximations to the derivatives $f^{(r)}(x')$, $r \geq 1$, where x' may be a tabular or a non-tabular point. We consider the cases $r = 1, 2$.

INSIDE THIS CHAPTER

- Introduction
- Numerical Differentiation
- Derivative Using forward Difference Formula
- Derivatives using Backward Difference Formula
- Derivatives Using Central Difference Formula

In many applications of science and engineering, we require to compute the value of the definite integral $\int_a^b f(x)\,dx$, where $f(x)$, may be given explicity or as a tabulated data. Even when $f(x)$ is given explicity, it may be a complicated function such that integration is not easily carried out.

In this chapter, we shall derive numerical methods to compute the derivatives or evaluate an integral numerically.

8.2 NUMERICAL DIFFERENTIATION

There are two approaches by which approximation to the derivatives can be obtained numerically:

(*i*) Methods based on finite differences for equispaced data.

(*ii*) Methods based on divided differences or Lagrange interpolation for non-uniform data.

Formula for Derivatives

Consider the function $y = f(x)$ which is tabulated for the values x_i $(= x_0 + ih)$, $i = 0, 1, 2, \ldots, n$

8.3 DERIVATIVE USING FORWARD DIFFERENCE FORMULA

Newton's forward interpolation formula is

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. u, we have

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{2!}\Delta^3 y_0 + \ldots$$

Since $u = \dfrac{x-x_0}{h}$, therefore $\dfrac{du}{dx} = \dfrac{1}{h}$

Now, $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \ldots\right] \quad \ldots(1)$$

At $x = x_0$, $u = 0$. Hence putting $u = 0$

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \quad \ldots(2)$$

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6u-6}{3!}\Delta^3 y_0 + \frac{12u^2-36u+22}{4!}\Delta^4 y_0 + \ldots\right]_h$$

Putting $u = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right] \quad \ldots(3)$$

Similarly

$$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right] \quad \ldots(4)$$

Otherwise: we know that

$$1 + \Delta = E = e^{hD}$$

$$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots$$

or $$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]$$

and $$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots\right]$$

and $$D^3 = \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \ldots\right]$$

Now applying the above identities to y_0, we get

$$Dy_0 = \left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \ldots\right]$$

and $$\left(\frac{d^3 y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

which are the same as (2), (3) and (4) respectively.

8.4 DERIVATIVES USING BACKWARD DIFFERENCE FORMULA

Newton's backward interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)}{3!}(u+2)\nabla^3 y_n + \ldots$$

Differentiating both sides w.r.t. u, we get

$$\frac{dy}{du} = \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \ldots$$

Since $u = \dfrac{x - x_n}{h}$, therefore $\dfrac{du}{dx} = \dfrac{1}{h}$

Now $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \ldots\right] \quad \ldots(5)$$

At $x = x_n$, $u = 0$. Hence putting $u = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \ldots\right] \quad \ldots(6)$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2 y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\nabla^2 y_n + \frac{6u+6}{3!}\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \ldots\right]$$

Putting $u = 0$, we obtain

$$\left(\frac{d^2 y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad \ldots(7)$$

Similarly, $$\left(\frac{d^3 y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right] \quad \ldots(8)$$

Otherwise:

Since $$1 - \nabla = E^{-1} = e^{-hD}$$

$\therefore$ $$-hD = \log(1-\nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$\therefore$ $$D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\right]^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\right]$$

Similarly, $$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \ldots\right]$$

Applying these identities to y_n, we get

$$Dy_n \text{ i.e., } \left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right]$$

8.5 DERIVATIVES USING CENTRAL DIFFERENCE FORMULA

Stirling's formula is

$$y_u = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Differentiating both sides w.r.t u, we get

$$\frac{dy}{du} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2u}{2!}\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4u^3-2u}{4!}\Delta^4 y_{-2} + \ldots$$

Since $$u = \frac{x-x_0}{h} \quad \therefore \quad \frac{du}{dx} = \frac{1}{h}$$

Now $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{3u^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2u^3-u}{12}\Delta^4 y_{-2} + \ldots\right]$$

At $x = x_0$, $u = 0$. Hence putting $u = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right] \quad \ldots(9)$$

Similarly,

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\right]$$

Example 1. *Find* $\frac{dy}{dx}$ *at x = 1 from the following table of values*

x	*1*	2	3	4
y	*1*	8	27	64

Solution. We have the following forward difference table:

x	**y**	**Δ y**	**Δ² y**	**Δ³ y**
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		
4	64			

we have $h = 1, x_0 = 1$ since $u = 0$

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right]$$

$$= 7 - \frac{1}{2}(12) + \frac{1}{3}(6) = 3$$

Remark 1. Numerical differentiation must be done with care. When a data is given, we do not know whether it represents a continuous function or a piecewise continuous function. It is possible that the function may not be differentiable at some points in its domain.

Remark 2. We use the forward difference formulas for derivatives, when we need the values of the derivatives at points near the top of the table of the values.

Example 2. *Find f′ (3) and f″ (3) for the following data:*

x	3.0	3.2	3.4	3.6	3.8	4.0
f (x)	– 14	– 10.032	– 5.296	– 0.256	6.672	14

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3.0	– 14					
		3.968				
3.2	– 10.032		0.768			
		4.736		– 0.464		
3.4	– 5.296		0.304		2.048	
		5.040		1.584		–5.120
3.6	– 0.256		1.888		– 3.072	
		6.928		– 1.488		
3.8	6.672		0.400			
		7.328				
4.0	14					

Since $u = 0$ we have the following results:

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \dots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6} y_0 + \frac{137}{180}\Delta^6 y_0 - \dots\right]$$

$$\left(\frac{dy}{dx}\right)_{3.0} = \frac{1}{0.2}\left[3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(-0.464) - \frac{1}{4}(2.048) + \frac{1}{5}(-5.120)\right]$$

$$= 9.4667$$

$$\left(\frac{d^2 y}{dx^2}\right)_{3.0} = \frac{1}{0.04}\left[0.768 + 0.464 + \frac{11}{12}(2.048) - \frac{5}{6}(-5.12)\right] = 184.4$$

Remark 3. Numerical differentiation has two main drawbacks: (*i*) on the right hand side of the approximation to $f'(x)$, we have multiplying factor $\frac{1}{h}$, and on the right hand side of the approximation to $f''(x)$ we have the multiplying factor $\frac{1}{h^2}$. Since h is small, this implies that we may be multiplying by a large number. For example if $h = 0.01$, the multiplying factor on the right hand side of the approximation to $f'(x)$ is 100, while the multiplying factor on the right hand side of the approximation to $f''(x)$ is 10000. Therefore, the round off errors in the values of $f(x)$ and hence in the forward differences, when multiplied by these multiplying factors may seriously effect the solution and the numerical process may become unstable.

Example 3. *The following data gives the velocity of a particle for 8 seconds at an interval of 2 seconds. Find the initial acceleration using the entire data.*

Time (sec.)	*0*	*2*	*4*	*6*	*8*
Velocity (m/sec)	*0*	*172*	*1304*	*4356*	*10288*

Solution. If v is the velocity, then initial acceleration is given by $\left(\frac{dv}{dt}\right)_{t=0}$

Using forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		172			
2	172		960		
		1132		960	
4	1304		1920		0
		3052		960	
6	4356		2880		
		5932			
8	10288				

at $x = x_0$ we have the following result:

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \ldots\right]$$

$$\left(\frac{dy}{dx}\right)_0 = \frac{1}{2}\left[172 - \frac{1}{2}(960) + \frac{1}{3}(960)\right] = 6$$

Example 4. *Find $f'(3)$ using the Newton's backward difference formula, for the data*

x	1.0	1.5	2.0	2.5	3.0
$f(x)$	– 1.5	– 2.875	– 3.5	– 2.625	0.5

Solution. $h = 0.5$

for $x = 3.0$ $\quad x = x_n + uh$

$$3.0 = 3.0 + u\,(0.5)$$

$\Rightarrow$ $\quad u = 0$

We have the following backward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	– 1.5				
		– 1.375			
1.5	– 2.875		0.75		
		– 0.625		0.75	
2.0	– 3.5		1.5		0.0
		0.875		0.75	
2.5	– 2.625		2.25		
		3.125			
3.0	0.5				

From the formula at $x = x_0$ we have

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + ...\right]$$

$$\left(\frac{dy}{dx}\right)_{(3)} = \frac{1}{0.5}\left[3.125 + \frac{1}{2}(2.25) + \frac{1}{3}(0.75)\right] = 9$$

Example 5. *Given that*

x	*1.0*	*1.1*	*1.2*	*1.3*	*1.4*	*1.5*	*1.6*
y	*7.989*	*8.403*	*8.781*	*9.129*	*9.451*	*9.750*	*10.031*

Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at*

(a) $x = 1.1$ *(b)* $x = 1.6$

Solution. The difference table is:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		−0.036				
		0.378		0.006			
1.2	8.781		−0.030		−0.002		
		0.348		0.004		0.001	
1.3	9.129		−0.026		−0.001		−0.002
		0.322		0.003		0.003	
1.4	9.451		−0.023		0.002		
		0.299		0.005			
1.5	9.750		−0.018				
		0.281					
1.6	10.031						

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + ...\right] \quad ...(1)$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - ...\right] \quad ...(2)$$

Here, $h = 0.1$, $x_0 = 1.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$ etc.

Substituting these values in (1) and (2), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right]$$

$$= 3.95$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03-(0.004)+\frac{11}{12}(-0.001)-\frac{5}{6}(+0.003)\right]$$

$$= -3.7416$$

(*b*) at $x = 1.6$ we apply the backward difference formula

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \quad \ldots(1)$$

and
$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad \ldots(2)$$

Here, $h = 0.1, x_n = 1.6, \nabla y_n = 0.281, \nabla^2 y_n = -0.018$ etc.

Putting these values in (3) and (4), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281+\frac{1}{2}(-0.018)+\frac{1}{3}(0.005)+\frac{1}{4}(0.002)+\frac{1}{5}(0.002)+\frac{1}{6}(0.002)\right]$$

$$= 2.727$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018+0.005+\frac{11}{12}(0.002)+\frac{5}{6}(0.003)+\frac{137}{180}(0.002)\right]$$

$$= -0.7144$$

Example 6. *A slider in a machine moves along a fixed straight rod. Its distance x c.m. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 second.*

t =	*0*	*0.1*	*0.2*	*0.3*	*0.4*	*0.5*	*0.6*
x =	*30.13*	*31.62*	*32.87*	*33.64*	*33.95*	*33.81*	*33.24*

Solution. The difference table is:

x	u	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	–3	30.13						
			1.49					
0.1	–2	31.62		– 0.24				
			1.25		– 0.24			
0.2	–1	32.87		– 0.48		0.26		
			0.77		0.02		– 0.27	
0.3	0	33.64		– 0.46		– 0.01		0.29
			0.31		0.01		0.02	
0.4	1	33.95		– 0.45		0.01		
			– 0.14		0.02			
0.5	2	33.81		– 0.43				
			– 0.57					
0.6	3	33.24						

Using Stirling's formulae:

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots \quad \ldots(1)$$

$$\left(\frac{d^2x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} - \ldots\right] \quad \ldots(2)$$

Here, $h = 0.1, \quad x_0 = 0.3, \Delta x_0 = 0.31, \Delta x_{-1} = 0.77, \Delta^2 x_{-1} = -0.46$

Putting these values in (1) and (2), we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\left(\frac{0.31+0.77}{2}\right) - \frac{1}{6}\left(\frac{0.01+0.2}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right) - \ldots\right]$$

$$= 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \ldots\right] = -45.6$$

Hence, the required velocity is 5.33 c.m/sec and acceleration is – 45.6 c.m/sec^2

Example 7. *Using Bessel's formula, find $f'(7.5)$ from the following table:*

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$f(x)$	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Solution. $x_0 = 7.50, h = 0.01$

$$u = \frac{x - x_0}{h} = \frac{x - 7.50}{0.01} \quad \text{at } x = 7.5, u = 0$$

The difference table is

x	u	y_u	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.47	– 3	0.193						
			0.002					
7.48	– 2	0.195		0.001				
			0.003		– 0.001			
7.49	– 1	0.198		0.000		0.000		
			0.003		– 0.001		0.003	
7.50	0	0.201		– 0.001		0.003		– 0.01
			0.002		0.002		– 0.007	
7.51	1	0.203		0.001		– 0.004		
			0.003		– 0.002			
7.52	2	0.206		– 0.001				
			0.002					
7.53	3	0.208						

$$y_p = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(u+1)\,u\,(u-1)(u-2)}{4!}\,\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{\left(u-\frac{1}{2}\right)(u+1)\,u\,(u-1)(u-2)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(u+2)(u+1)\,u\,(u-1)(u-2)(u-3)}{6!}\cdot\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \ldots$$

Since $u = \dfrac{x - x_0}{h}$ $\quad\therefore\quad \dfrac{dy}{dx} = \dfrac{1}{h}$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{1}{h}\dfrac{dy}{du}$

Differentiating (*i*) w.r.t u and putting $u = 0$, we get

$$\frac{dy}{dx} = \frac{1}{h}\left(\frac{dy}{du}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{0.01}\left[0.002 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002) + \frac{1}{24}(-0.004 + 0.003) - \frac{1}{120}(-0.007) - \frac{1}{240}(-0.010 + 0)\right]$$

$\therefore\quad (\Delta^6 y_{-2} = 0)$

$$= 0.2 + 0 + 0.01666 - 0.0416 + 0.00583 + 0.00416 = 0.223$$

Example 8. *Find f′(10) from the following data:*

x	3	5	11	27	34
$f(x)$	– 13	23	899	17315	35606

Solution. As the values of x are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3	– 13				
		18			
5	23		16		
		146		1	
11	899		40		0
		1026		1	
27	17315		69		
		2613			
34	35606				

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)$$
$$(x - x_2)(x) f[x_0, x_1, x_2, x_3, x_4]$$

Differentiating w.r.t 'x' we get

$$f'(x) = f[x_0, x_1] + (2x - x_0 - x_1) f[x_0, x_1, x_2] + [3x^2 - 2x(x_0 + x_1 + x_2)$$
$$+ x_0x_1 + x_1x_2 + x_2x_0)] \times f[x_0, x_1, x_2, x_3] + [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3)$$
$$+ 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2) - (x_0x_1x_2 + x_1x_2x_3$$
$$+ x_2x_3x_0 + x_0x_1x_3)] f[x_0, x_1, x_2, x_3, x_4]$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27$ and $x = 10$, we obtain

$$f'(10) = 18 + 12 \times 16 + 23 \times 1 - 426 \times 0 = 233$$

Example 9. *Find the first and second derivatives at $x = 1.6$, for the function represented by the following tabular data:*

x	1.0	1.5	2.0	3.0
$f(x)$	0.0	0.40547	0.69315	1.09861

Solution. Since the data is not equispaced. We apply the Newton's divided difference formula to find the derivatives we have the following difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1.0	0.0			
		0.81094		
1.5	0.40547		– 0.235580	
		0.57536		0.061157
2.0	0.69315		– 0.113267	
		0.40546		
3.0	1.09861			

Substituting $x = 1.6$ in the formula

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + [(x - x_0)(x - x_1)] f[x_0, x_1, x_2]$$
$$+ [(x - x_0)(x - x_1)(x - x_2)] f[x_0, x_1, x_2, x_3]$$

$$f(x) = 0 + (x - 1) \times 0.81094 + (x - 1)(x - 1.5) \times (-0.235580)$$
$$+ (x - 1)(x - 1.5)(x - 2) \times 0.061157$$

$$f'(1.6) = 0.81094 + [(1.6 - 1.0) + (1.6 - 1.5)](-0.23558) + (1.6 - 1.5)(1.6 - 2.0)$$
$$+ (1.6 - 1.0)(1.6 - 1.5) + (1.6 - 1.0)(1.6 - 2.0)] (0.061157)$$
$$= 0.81094 + 0.7(-0.23558) - 0.22(0.061157) = 0.63258$$

$$f''(1.6) = 2(-0.23558) + 2[(1.6 - 1.0) + (1.6 - 1.5) + (1.6 - 2.0)](0.061157)$$
$$= -0.47116 + 0.03669 = -0.43447$$

EXERCISE

1. Find the first and second derivatives of the function tabulated below at the point $x = 19$.

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	0.128	0.544	1.296	2.432	4.00

2. The following data gives corresponding values of pressure and specific volume of superheated steam.

V	2	4	6	8	10
P	105	42.07	25.3	16.7	13

(*i*) Find the rate of change of pressure with respect to volume when V = 2.

(*ii*) Find the rate of change of volume with respect to pressure when P = 105.

3. Find $y'(0)$ and $y''(0)$ from the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

4. From the values in the table given below, find the value of sec 31° using numerical differentiation.

$\theta°$	31	32	33	34
$\tan\theta$	0.6008	0.6249	0.6494	0.6745

5. By using the following table evaluate $\frac{d}{dx} P_n(x)$ at $x = 0.1$

x	0.0	0.1	0.2	0.3	0.4
$P_n(x)$	1.0	0.9975	.9900	0.9776	0.9604

6. Find the angular velocity $\frac{d\theta}{dt}$ at $t = 0.06$, if θ and t are given by the following

θ	.052	0.105	0.168	0.292	0.327	0.408	0.489
t	0	0.02	0.04	0.06	0.08	0.10	0.12

7. Find $\frac{dy}{dx}$, when $x = 6$ from the following table

x	4.5	5.0	5.5	6.0	6.5	7.0	7.5
y	9.69	12.90	16.71	21.18	26.37	32.34	39.15

8. From the following table of values of x and y, find $y'(1.25)$ and $y''(1.25)$.

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

9. Obtain the value of $f'(0.04)$ using Bessel's formula given the table below:

x	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

10. For the following pairs of values of x and y, find numerically the first derivative at $x = 4$.

x	1	2	4	8	10
y	0	1	5	21	27

11. Find the value of $f'(7.60)$ from the following table using Gauss's formula.

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$f(x)$	0.193	0.195	0.198	0.201	0.203	0e.206	0.208

Answers

1. 5.44
2. 1.6101
3. 0.8687
4. 24256.53
5. – 0.05
6. 4.054
7. 9.66
8. 0.44733, 0.158332
9. 0.25625
10. 2.8326
11. 0.223