Q Use Taylor Method of order 3 to estimate $y(1.1)$ and $y(1.2)$ for the following 2nd order ode

$$\frac{d^2y}{dt^2} + y^2\frac{dy}{dt} = t^3 \quad y(1)=1,\ y'(1)=1 \quad -(1)$$

Sol: Let $\frac{dy}{dt} = z$

(1) becomes

$$\frac{dz}{dt} + y^2 z = t^3 \quad y(1)=1 \quad z(1)=1$$

System of odes is

$$\frac{dy}{dt} = z \qquad y(1)=1$$

$$\frac{dz}{dt} = t^3 - y^2 z \qquad z(1)=1$$

Here $f = z$

$g = t^3 - y^2 z$

and $h = 0.1$

$t_0 = 1 \quad y_0 = 1 \quad z_0 = 1$

$i=0 \quad t_1 = 1.1 \quad y_1 = ? \quad z_1 = ?$

$i=1 \quad t_2 = 1.2 \quad y_2 = ? \quad z_2 = ?$

$y' = z$

$y_0' = z_0$

$= 1$

$z' = t^3 - y^2 z$

$z_0' = t_0^3 - y_0^2 z_0$

$= 1^3 - 1^2 * 1$

$= 0$

$y'' = z'$

$y_0'' = z_0'$

$= 0$

$z'' = 3t^2 - 2yy'z - y^2 z'$

$z_0'' = 3t_0^2 - 2y_0 y_0' z_0 - y_0^2 z_0'$

$= 3*1^2 - 2*1*1*1 - 1^2 * 0$

$= 1$

$y''' = z''$

$y_0''' = z_0''$

$= 1$

$z''' = 6t - 2yy'z' - 2(yy'' + y'y')z - y^2 z'' - 2yy'z'$

$= 6t - 4yy'z' - 2yy''z - 2(y')^2 z - y^2 z''$

$z_0''' = 6t_0 - 4y_0 y_0' z_0' - 2y_0 y_0'' z_0 - 2(y_0')^2 z_0 - y_0^2 z_0''$

$= 6 - 0 - 0 - 2 - 1$

$= 3$

i=0

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0'''$$

$$= 1 + 0.1 * 1 + \frac{0.1^2}{2} * 0 + \frac{0.1^3}{6} * 1$$

$$= 1.10017$$

$$z_1 = z_0 + h z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0'''$$

$$= 1 + 0.1 * 0 + \frac{0.1^2}{2} * 1 + \frac{0.1^3}{6} * 3$$

$$= 1.0055$$

④

$y_1' = z_1$

$= 1.0055$

$z_1' = t_1^3 - y_1^2 z_1$

$= 1.1^3 - 1.10017^2 * 1.0055$

$= 0.11397$

$y_1'' = z_1'$

$= 0.11397$

$z_1'' = 3t_1^2 - 2y_1 y_1' z_1 - y_1^2 z_1'$

$= 3 * 1.1^2 - 2 * 1.10017 * 1.0055 * 1.0055$

$- 1.10017^2 * 0.11397$

$= 1.26744$

$y_1''' = z_1''$

$= 1.26744$

$z_1''' = 6t_1 - 4y_1 y_1' z_1' - 2y_1 y_1'' z_1$

$-2(y')^2 z_1 - y_1^2 z_1''$

$= 6 * 1.1 - 4 * 1.10017 * 1.0055 * 0.11397$

$- 2 * 1.10017 * 0.11397 * 1.0055$

$- 2(1.0055)^2 * 1.0055$

$- (1.10017)^2 * 1.26744$

$= 2.27629$

i = 1

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$$

$$= 1.100017 + 0.1 * 1.0055 + \frac{0.1^2}{2} * 0.11397 + \frac{0.1^3}{6} * 1.26744$$

$$= 1.201501$$

$$z_2 = z_1 + h z_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1'''$$

$$= 1.0055 + 0.1 * 0.11397 + \frac{0.1^2}{2} * 1.26744 + \frac{0.1^3}{6} * 2.27629$$

$$= 1.02361$$

Alternate Method:

$y_0 = 1$

$y_0' = 1$

$$\frac{d^2y}{dt^2} + y^2\frac{dy}{dt} = t^3$$

$$\frac{d^2y}{dt^2} = t^3 - y^2\frac{dy}{dt}$$

$$y'' = t^3 - y^2 y' \quad -(1)$$

$$y_0'' = t_0^3 - y_0^2 y_0'$$

$$= 1^3 - 1^2 * 1$$

$$= 0$$

$$y''' = 3t^2 - 2yy'y' - y^2y'' \quad -(2)$$

$$= 3t^2 - 2y(y')^2 - y^2y''$$

$$y_0''' = 3t_0^2 - 2y_0(y_0')^2 - y_0^2y_0''$$

$$= 3*1 - 2*1*1^2 - 1^2*0$$

$$= 1$$

$$y^{(iv)} = 6t - 2y'(y')^2 - 2y(2y'y'') - 2yy'y'' - y^2y'''$$

$$= 6t - 2(y')^3 - 4yy'y'' - 2yy'y'' - y^2y'''$$

$$y^{(iv)} = 6t - 2(y')^3 - 6yy'y'' - y^2y''' \quad -(3)$$

$$y_0^{(iv)} = 6*t_0 - 2(y_0')^3 - 6y_0y_0'y_0'' - y_0^2y_0'''$$

$$= 6*1 - 2*1^3 - 6*1*1*0 - 1^2*1$$

$$= 3$$

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(iv)}$$

$$= 1 + 0.1*1 + \frac{0.1^2}{2}*0 + \frac{0.1^3}{6}*1 + \frac{0.1^4}{24}*3$$

$$= 1.10018$$

$$y_1' = y_0' + h y_0'' + \frac{h^2}{2!} y_0''' + \frac{h^3}{3!} y_0^{(iv)}$$

$$= 1 + 0.1*0 + \frac{0.1^2}{2}*1 + \frac{0.1^3}{6}*3$$

$$= 1.0055$$

① $\rightarrow$ $y_1'' = t_1^3 - y_1^2 y_1'$

$$= 1.1^3 - 1.10018^2 * 1.0055$$

$$= 0.11395$$

② $\rightarrow$ $y_1''' = 3t_1^2 - 2y_1(y_1')^2 - y_1^2 y_1''$

$$= 3*1.1^2 - 2*1.10018*1.0055^2 - 1.10018^2 * 0.11395$$

$$= 1.26744$$

③ $\rightarrow$ $y_1^{(iv)} = 6t_1 - 2(y_1')^3 - 6y_1 y_1' y_1'' - y_1^2 y_1'''$

$$= 6*1.1 - 2*1.0055^3 - 6*1.10018*1.0055*0.11395 - 1.10018^2 * 1.26744$$

$$= 2.27638$$

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{(IV)}$$

$$= 1.10018 + 0.1 * 1.0055 + \frac{0.1^2}{2} * 0.11395$$

$$\frac{0.1^3}{6} * 1.26744 + \frac{0.1^4}{24} * 2.27638$$

$$= 1.20152$$