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Mathematics of flooding events: Investigating the influence of flood mitigation strategies on return periods and river dynamics

MATH5004M: Assignment in Mathematics

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Contents

1 Introduction

Flood risk in the UK is becoming an increasing concern and according to the Environment Agency "around 5.2 million properties in England, or one in six properties, are at risk of flooding" [11]. With the impact of climate change becoming increasingly severe, this figure is likely to increase in future. Six main types of flooding occur in the UK; these are river, coastal, surface water, groundwater, sewer and reservoir flooding [12]. In this report, we shall focus solely on river flooding, also known as fluvial flooding, which occurs as a result of a river bursting its banks.

There are two key aims of this assignment, with their investigations separated into parts I and II. The sections in part I of this assignment aim to address the concept of return periods - demonstrating what they are and the tools developed to explain these. We also aim to explain strategies of flood mitigation and their influence on return periods. Part II of this assignment aims to use shallow water theory to model the behaviour of fluid over a weir, which will be used to determine the flood storage volume created by a weir.

To begin addressing these aims, we shall introduce some definitions. We define a 'design flood' as a "flood event of a given annual flood probability" [25]. River flooding is one of the most common types of flooding in the UK and there is a 1-in-100 chance of a 'design flood' occurring each year [25]. This period of 1-in-100 years is known as a return period, i.e. the average waiting time between two flooding events of the same magnitude or greater occurring. Return periods for certain areas are calculated based on past data on flooding and using computer modelling to simulate flooding [12]. This means that return periods change and become increasingly accurate as we gather more data on flooding events in a certain location. Return periods can equally be quantified in the form of Annual Event Probability (AEP), where a '1-in-100-year' return period has an AEP of 1%. For the general public, the concept of a return period can be hard to grasp and truly visualise. Understanding the flood risk presented by a return period will be a key element of this report. One tool that has been set up which aims to build understanding in this area is the flood visualisation tool Wetropolis.

Weirs are structures built along the width of rivers to regulate and measure water flow. They regulate flow by creating a pool of water upstream of the weir and this quantity of water stored is known as the flood storage volume. Floodplains are another feature used in flood mitigation that increases flood storage volume by providing an area for excess river water to flow. Flood excess volume describes the quantity of water in a river that exceeds the river's capacity and causes flooding. Therefore, increasing flood storage on a river in turn decreases the flood excess volume that we see during a flooding event. Shallow water theory describes the set of mathematical equations used to examine the behaviour of fluid in a two-dimensional domain between two smoothly varying regions. The equations can be used to derive a river's depth or the flow rate of water within a certain cross-section.

We shall use the same example of region, river and flooding event to apply the concepts and methods learnt throughout this report. This example is chosen as the city of Leeds in West Yorkshire, which the river Aire flows through and where the 2015 Boxing Day Flood took place. The data collected on Leeds for this assignment comes from several sources, including River Levels UK, World Weather Online, the UK Government Website and Professor Onno Bokhove's GitHub page (available here). Python will be used to create most graphs and diagrams seen in this report unless stated otherwise. The code used to create these plots can be found in Appendix A.

The sections in this assignment are grouped into two areas of investigation, parts I and II. The first part, I, focuses on flood risk and management while part II considers river dynamics and shallow water theory. We begin by focusing on the flood risk and management in §2 - §5, starting with an insight into flood risk in the UK in §2. Here the concept of flood risk shall be introduced as well as the common misconceptions faced when measuring flood risk. The next section, §3, presents a comprehensive summary of the flood visualisation tool Wetropolis. Its components shall be defined, as well as the statistical aspect of the model and alterations that can be made to cater for different extreme weather events. In §4 we take a close look at flood hydrographs and the concept of flood excess volume, applying these ideas to produce a hydrograph and estimate the flood excess volume of the 2015 Boxing Day flood in Leeds. The flood risk and management half of the report will conclude in §5 by examining some key methods used in flood management, as seen in the Leeds Flood Alleviation Scheme.

The second area of investigation, §6-§10, focuses on river dynamics and the concept of shallow water theory, which is introduced in §6. Here the assumptions underpinning shallow water theory will be outlined and the two shallow water equations will be introduced and derived. This leads on to §7, where we present the shallow water equations in the context of Wetropolis. Additional aspects of fluid mechanics relating to open channel flow will be introduced in this section, including modelling flow rate and the Bernoulli equation. In §8 we apply the information gathered in the previous two sections to studying the behaviour of fluid over weirs. An equation for the discharge of liquid over the two weir types will be derived, which we apply to the weirs seen in Wetropolis as well as the concept of critical flow. Applying the concepts of shallow water theory further, in §9 we shall outline the method used to model fluids behaviour over a weir using computer programming software. This will be applied to plot the river profile over Kirkstall Valley Weir along the river Aire. Lastly, we shall outline the method required to calculate flood storage for the three separate methods, specific to our example in §10. This will include an investigation into the potential influence of weir height, flow rate and river bed slope on flood storage volume in each case.

I Flood risk and management

2 Flood risk in the UK

The Environment Agency's 2009 report titled 'Flooding in England: A National Assessment of Flood Risk' [11] provides a comprehensive summary of the flood risk citizens face in the United Kingdom, as well as the methods the agency is using to protect the nation against flood damage. It is reported that "the expected annual damages to residential and non-residential properties in England at risk of flooding from rivers and the sea is estimated at more than £1 billion" [11]. Because of this, the agency has pledged an increase in public spending on flood management "from £600 million in 2007-2008 to £800 million in 2010-2011" [11]. Much of this budget has been devoted to Flood Alleviation schemes (FAS), which centralise on regions at high risk of flooding and work to build their resilience to these, see §6 for details on the Leeds FAS. A key element of flood defence seen in these schemes is flood walls and from 2008-2009, approximately two-thirds of the flood management budget was devoted to building flood walls [11]. Other methods of flood defence used in the UK and within flood alleviation schemes include raised embankments, building floodplains, managing current flood walls and river channels, pumps and flood barriers.

We begin by outlining the key misunderstandings or misinterpretations made by the public when examining flood risk in §2.1. A tool proposed by the Government which aims to reduce this will be introduced and explained, as well as a demonstration of this tool centralised on Leeds city in §2.2. This section will also include a map of Leeds' topography and a graph of the average river height over time at a specific location along the river running through central Leeds - the River Aire. Lastly, a plot of the precipitation levels in Leeds will be compared against the river height to demonstrate the influence precipitation has on river levels. Section 2.3 details the influence climate change has on flood and drought risk, particularly in the UK.

2.1 Understanding flood risk

Out of the 5.2 million properties at risk, it is reported that two-thirds of these citizens do not understand their flood risk [12]. A key part of this misunderstanding comes from a lack of clarity around the concept of return periods. To tackle this there must firstly be an understanding of the volume of water required for flooding to occur and secondly, an understanding that the unexpected nature of floods means that they do not occur on a routine basis. Therefore we cannot guarantee that if a '1-in-100-year' flood has just occurred there will not be another flood on this scale for the next 100 years. In reality, a '1-in-100-year' flood has a 63.4% chance of occurring in any 100-year period [12]. To strengthen the public's understanding of the flood risk they experience in their local area, the Environment Agency has assigned risk levels to certain AEP values, as summarised in the table below.

Level of Risk	**Very low**	**Low**	**Moderate**	**High**
AEP bound (%)	< 0.1	$0.1 - 0.99$	$1 - 3.3$	> 3.3
Return period (years)	$< 1:1000$	$1:1000 - 1:100$	$1:100 - 1:33$	$> 1:33$

Table 1: Table to show the level of flood risk in a certain area based on the region's annual event probability and equivalent return period, colour coded according to the flood risk map in figure 1.

The flood risk bands have been applied to an interactive map of the UK, available via the Government website (available here). Areas at high risk of flooding from rivers or the sea have been shaded in the darkest shade of blue and areas of very low risk in the lightest shade of blue, while areas with no shading present virtually no flood risk from rivers or the sea. An image of this map centralised on Leeds is shown in the next section in figure 1. Although this map is beneficial in providing the public with the scale of their flood risk in comparison to other areas of the UK, the bands only show this relative risk and fail to give a true understanding of flood risk. Another tool aimed at giving the public a better understanding of flood risk and return periods is Wetropolis, which shall be introduced in the next section.

2.2 Flood risk in Leeds

We focus the Government's 'risk of flooding from rivers or the sea' map on the region of Leeds in figure 1, where we see that the areas at greatest risk of fluvial flooding are situated upstream of the city centre (red star) in the North West of Leeds, towards Kirkstall. There is also a thin region at high risk of flooding below the city near Knowthorpe that is also along the river's stream. As Leeds is located inland, we can be sure that this flood risk only comes from rivers and not from the sea. These areas of dark blue shading are where the River Aire flows, thus making these zones more susceptible to river flooding as a result of low elevation and high river levels. We see that these regions along the Aire Valley are shown to be areas of low elevation from figure 2, a map of Leeds' topography. Here, ground-level height ranges from 18 to 31 metres above sea level, which is what we would expect to see as the River Aire will naturally be located at the lowest points of elevation inland. Methods of flood prevention have been and are still in the process of being implemented in these shaded regions along the River Aire and around the city, as part of the Leeds FAS (see §6). Also note the light blue shaded region located below the city centre away from the River Aire's stream in figure 1, which indicates that the entirety of this zone is at risk of flooding to some extent, even if it is very low. We see again from figure 2 that this area at risk of flooding has a low elevation, supporting the claim that areas of low elevation have a higher flood risk.

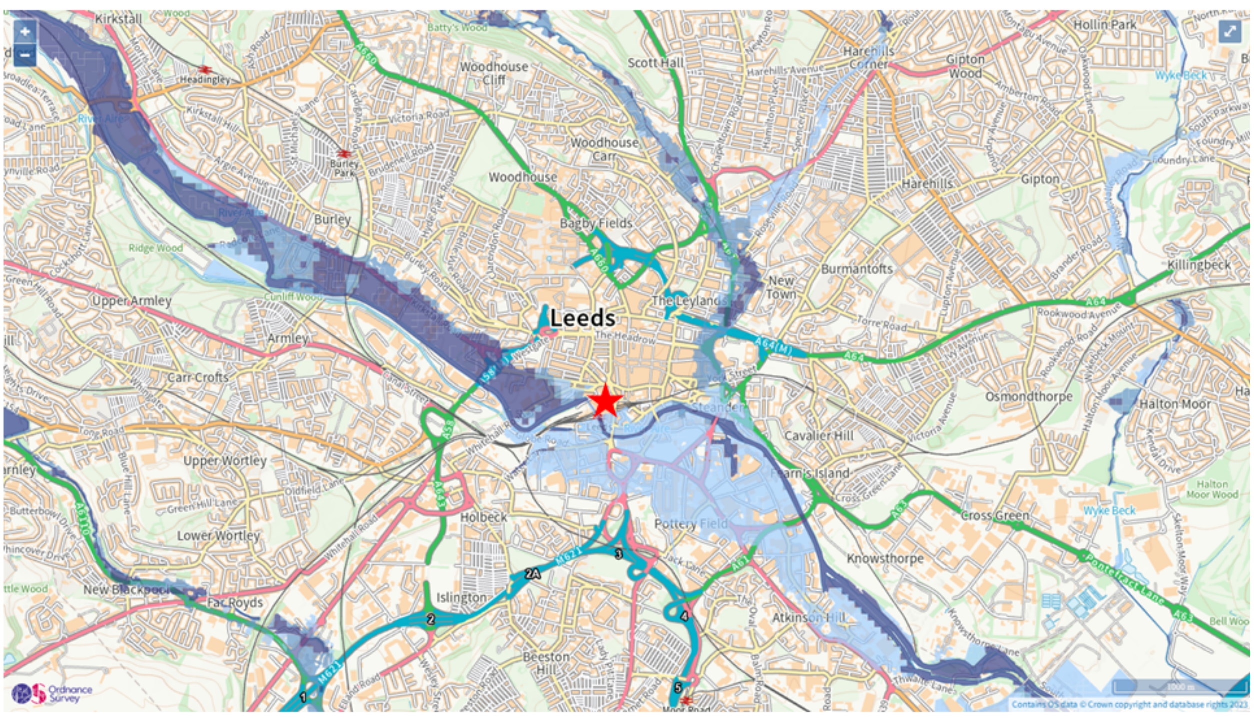


Figure 1: Map of Leeds and areas at risk of river or sea flooding shaded in blue, where the darker the shade of blue is, the higher risk of flooding. The location of Leeds City train station is marked by the red star. This map is a screenshot taken from the Government's flood risk service, available here.

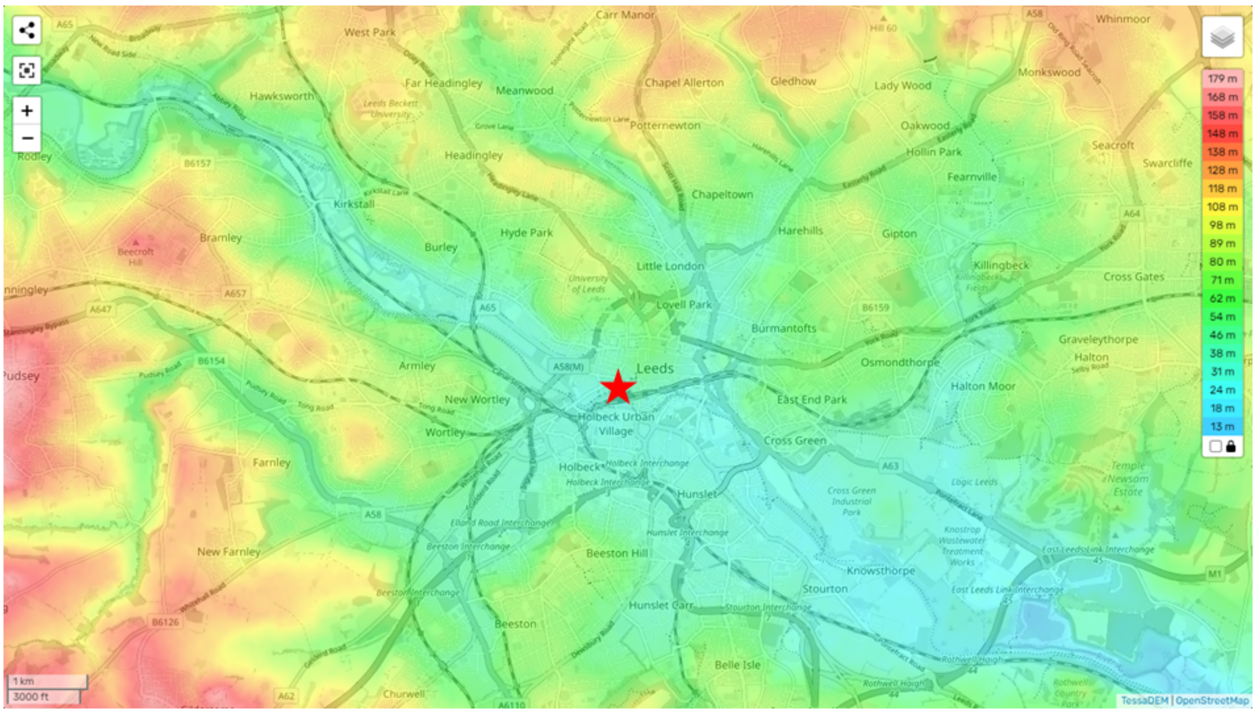


Figure 2: Map of Leeds topography, where the blue shading denotes regions of lower elevation and the red shading for regions of higher elevation, as suggested by the key on the right-hand side. For reference, the location of Leeds City train station is marked by the red star. This map is a screenshot taken from topographic-map.com, available here.

Figure 3 is a plot of the average river height recorded each day over the ten-year period from 1st January 2013 up to 1st January 2023 at the Armley monitoring station on the river Aire. This data has been provided by River Levels UK website, (available here) and the code used to produce this graph can be found in appendix A.1. As shown by the graph, the river height at Armley station greatly varies over the decade, showing an almost yearly cycle of increasing and decreasing river heights as the weather conditions change by season. The red dashed line on the plot denotes the river's height threshold at Armley monitoring station, meaning the maximum level the river can reach before flooding occurs. This value has been suggested by Onno Bokhove [5] to be $h_T = 3.9$ m at this location. There has only been a handful of times where the average river height on a given day exceeds the height threshold. This happened twice during the 2015 Boxing Day flood period (26th and 27th December), but also one further time on the 20th February 2022. The Boxing Day flood shall be examined more closely, with the use of hydrographs and rating curves, in §4.

According to the Government service created to check for flooding in the UK (available here), when the river height exceeds 2.7 metres at Armley monitoring station, flooding may occur on low lying land and one or more flood alerts may be issued. This height is denoted on the plot by the blue dashed line, which we see has been surpassed on multiple occasions over the last decade, 15 occasions to be precise. The 2015 Boxing Day flood on the River Aire was estimated to be a '1-in-200$^+$-year' event [6], or equivalently have an AEP of 0.4%. The river height reaching this threshold on two separate flooding events in only ten years demonstrates the non-routineness of a return period. As although this was a rare flooding event, with a 0.4% chance of occurring each year, a flooding event of similar scale occurred only seven years after this in 2022. Albeit this did not quite reach the magnitude of the 2015 flood, we can still see how rare flooding events of large magnitudes can occur in quick succession despite it being unlikely.

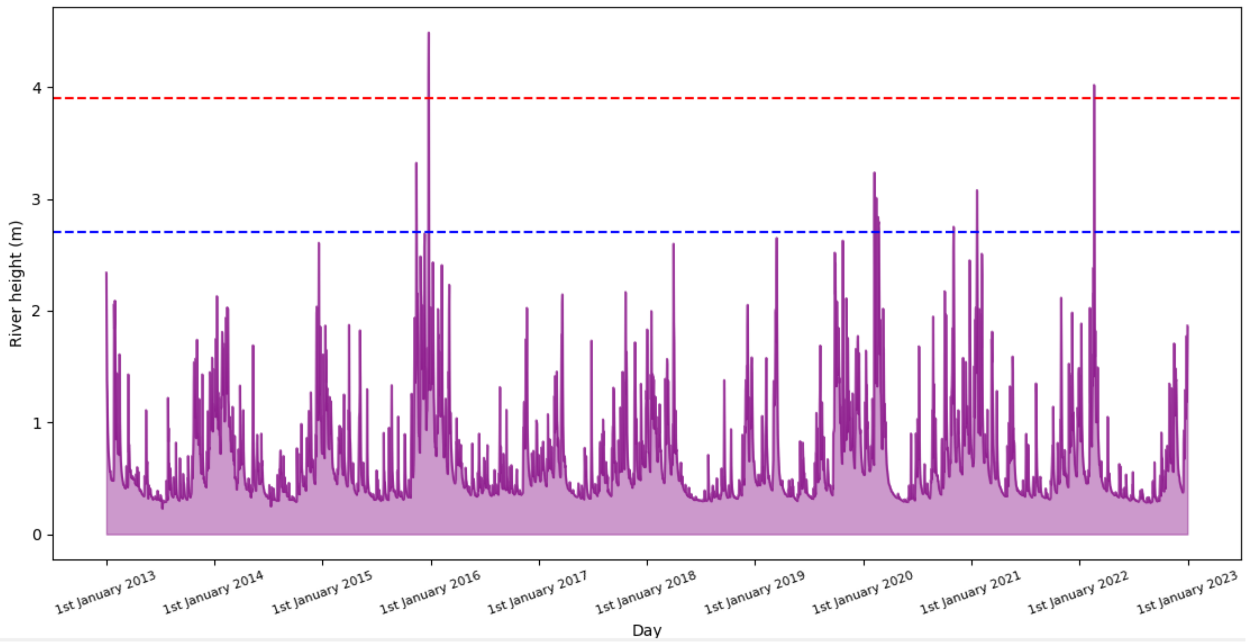


Figure 3: Graph of average river height at Armley Station along the River Aire from 1st January 2013 to 1st January 2023.

Figure 4 compares the average river height for each year against the average rainfall level per month in each year. The data on river height was the same as used in figure 3 and the average rainfall data was provided by World Weather Online (available here). The code for this plot can be found in Appendix A.1. We see by eye that the river height (green squares) and precipitation (orange pentagons) have a positive correlation over the ten-year period. This correlation is found to be 0.76 (2 s.f.), suggesting that there is a positive relationship between rainfall levels and river height over the River Aire. We would expect this to be the case, one key reason being that increased rainfall means a higher volume of water falls directly into the river and thus increases river levels. However, we would also expect this positive relationship due to increased rainfall causing increased soil saturation. As soil can only hold so much moisture, the excess water exhibited in periods of heavy rainfall is directed elsewhere, known as surface runoff, which eventually reaches the lowest point of elevation, i.e. a river, where it is deposited.

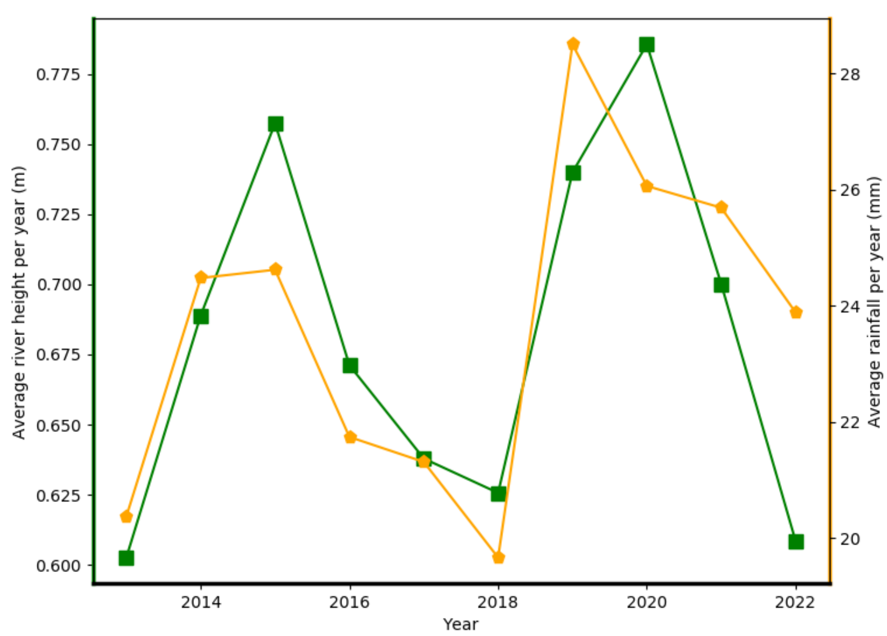


Figure 4: Twin plot of average river height (m) each year (green squares) and average precipitation (mm) per month for each year (orange pentagons), from 2013 to 2022.

2.3 Influence of climate change on flood and drought risk

The Clausius-Clapeyron relationship describes the vaporisation of liquids. A discovery made initially by physicist Rudolf Clausius and later developed by Benoît Clapeyro found that for every one-degree increase in global temperature, the air can hold around 7% more water [21]. Not only does this add to the greenhouse gases in the atmosphere, but an increase in water vapour in the atmosphere leads to an increase in precipitation. It has been predicted by UK scientists that rainfall will increase on the wettest days by $10-20\%$ [21] as a result of increased humidity and other effects of climate change. This will increase the frequency and intensity of flooding events in the UK. We shall investigate how this influence of climate change can be implemented within the flood visualisation tool Wetropolis in §3, to give the public a greater overall understanding of return periods and the influence climate change has on them.

A drought is defined by the World Health Organisation as "a prolonged dry period in the natural climate cycle" [27]. Droughts are becoming more common in occurrence and severity in recent years due to rising temperatures, as a result of climate change. The increase in temperature means that water evaporates more quickly, and combining this with periods of low precipitation leads to prolonged periods of drought. In addition to heavier rainfall, we should also expect to see an increased number of dry days in summer, which could lead to more droughts in the UK. Wetropolis can also be used to demonstrate a drought occurring, which shall be explained in section 3.

3 Wetropolis

The Wetropolis Flood Demonstrator aims to explain the concept of return periods and demonstrate the random nature of flooding events. For instance, a flood with a return period of n years will not occur routinely every n years. Instead, flooding is found to occur once every n years on average, where the time recorded between floods can be much greater or much less than n years. Here a Wetropolis day (WD) is much shorter than the usual 24 hour day, with each day lasting 10 seconds. This means the occurrence of a flooding event is still rare but does not leave the viewer waiting an extended period of time to see a flood.

Wetropolis is made up of two main components, the Galton boards and the tabletop physical model. A photograph of the model is given in figure 5, displaying its four main components; the river, the moor, the reservoir and the city. The river channel runs throughout the board and is approximately four metres in total length, flowing firstly past the reservoir, secondly past the upland moor and lastly into the city, with flow aided by water pumps. Once water passes through the city it flows into the water holding storage, which is a reservoir underneath the board. Water is then pumped back into the model and this cycle repeats, with the option of more water being pumped into the model at the moor or reservoir. We also see a canal running alongside the river, which is made to model the Leeds-Liverpool canal that runs along the River Aire [8]. For each Wetropolis day, a particular duration and location of rainfall will be determined by the Galton boards, with the possible locations being the moor, the reservoir, neither or both. Once decided, rainfall is distributed in the moor or reservoir via water pumps, with equidistant small holes along them to disperse water evenly in the moor or reservoir.

To begin we shall explore statistical modelling in Wetropolis, which relies on the Galton board, as seen in §3.1.1. The probability for each possible outcome, in terms of rainfall location and duration, for each Wetropolis Day shall be presented in this section and derivations of the expectation and standard deviation of a return period in the model will be detailed in §3.1.2. Cases of more extreme flooding shall be examined in §3.1.3, along with their corresponding probabilities. Furthermore, §3.2 will investigate the possible alterations that can be made to Wetropolis and what these aim to show the viewer. In particular, alterations we can make to account for climate change will be outlined in §3.2.1.

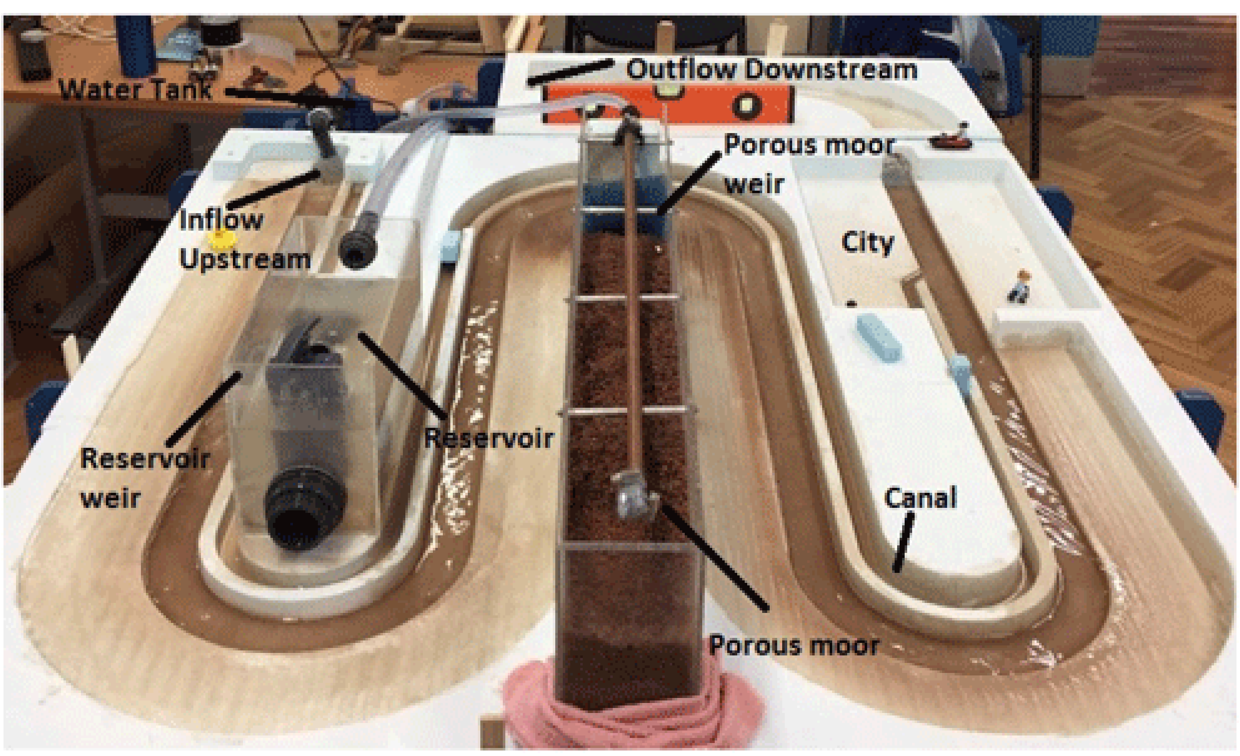


Figure 5: Photograph of the Wetropolis Flood Demonstrator

3.1 Statistical modelling

3.1.1 Galton board

A Galton board is an upright wooden board which contains engraved paths, meaning once a steel ball is dropped into the board from the top, the ball travels through some path of this board before reaching the bottom. The path each ball takes is used to determine the length and duration of each Wetropolis day. In this case, we require two Galton boards and every ten seconds a small steel ball will fall through each Galton board. The board on the left in figure 6 is used to determine the duration of rainfall and the right board decides the location of rainfall. The board is linked to Arduino control units, meaning the duration and location of rainfall decided by the board can be added into the model using the three water pumps, located at the start of the river, at the moor and at the reservoir.

Figure 6: Photograph of Galton Boards

In both boards, the ball has a 50% probability of falling to the right or left. On the left board, if the ball falls into the far left slot, the duration of rain will be one second on that Wetropolis Day. The next slot will give a two-second duration, then four seconds and finally nine seconds for the far right slot. Similarly in the right Galton board, the ball falling in the first slot means rainfall will be located in the reservoir only on that WD. The second slot means rain will be in both the reservoir and moor, the third slot is only in the moor and the final slot means there will be no rainfall regardless of the rainfall duration determined by the left Galton board. The probabilities for the possible durations and locations of rainfall on each WD are summarised in table 2, where duration and location are given in tables (a) and (b) respectively.

Duration of rainfall (seconds)	Probability (p_i)
1	3/16
2	7/16
4	5/16
9	1/16

(a)

Location of rainfall	Probability (q_i)
Reservoir	3/16
Reservoir and Moor	7/16
Moor	5/16
No rain	1/16

(b)

Table 2: Table of probabilities of Galton Board outcomes, duration probabilities are given in sub-table (a) while location probabilities are in sub-table (b).

The total probability for each outcome can also be presented in a four-by-four matrix $P_{ij} = p_i q_j$, where $i, j = 1, \ldots, 4$, p_i is the probability of rainfall duration and q_j the probability of rainfall location. This has been shown in figure 7, with the duration specified by the i-th row and the location by the j-th column. Note that probability is given by $x/256$ for each entry in the matrix x. Also note, when the marble for rainfall location lands on 'No rain', (q_4) the rain duration outcomes are no longer used and therefore the probability of a day with no rain is $\sum_{i=1}^{4} p_i q_4 = q_4 = \frac{1}{16}$.

$$\begin{bmatrix} 9 & 21 & 15 & 3 \\ 21 & 49 & 35 & 7 \\ 15 & 35 & 25 & 5 \\ 53 & 7 & 5 & 1 \end{bmatrix}$$

Figure 7: Matrix to show the probability of each Galton board outcome (x/256 for each matrix entry, x)

When rainfall occurs in one location only, the outflow of water is driven by the water pumps. We can scale the volume of water released per second using r_0. The probability for $0r_0$, i.e. no rainfall, was found above to be q_4. Next, a WD with flow rate $1r_0$ has a probability of $p_1(q_1+q_3)$ to account for the two instances of one second of rainfall occurring in either the moor or the reservoir. The remaining probabilities for r_0, as well as the calculations, are given in table 3, where we see that a flow rate of $4r_0$ has the highest likelihood of occurring on any WD and the least likely is $18r_0$. When the flow rate is $18r_0$, we call this an extreme flooding event. This happens only when the Galton board gives a nine-second duration of rainfall in both locations, the moor and reservoir.

Flow rate	P(Flow rate) Calculation	P(Flow rate)
$0r_0$	q_4	1/16
$1r_0$	$p_1(q_1+q_3)$	24/256
$2r_0$	$p_2(q_1+q_3)+p_1q_2$	77/256
$4r_0$	$p_3(q_1+q_3)+p_2q_2$	89/256
$8r_0$	p_3q_2	35/256
$9r_0$	$p_4(q_1+q_3)$	8/256
$18r_0$	p_4q_2	7/256

Table 3: Probability of rainfall flow rate.

The return period of a one-day flooding event can be found using this probability statistic $p_e = \frac{7}{256} = 2.7\%$ and the time of a WD, T_d = ten seconds. Given the extreme event occurs on day zero, the chance that this will reoccur on day one, two and so on is determined using a geometric distribution with density function $p_n = (1-p_e)^{n-1}p_e$. Using the formulae for the expectation and standard deviation of a geometric distribution, the expectation of time taken over n days $t_n = n\,T_d$ is given by

$$T_r = \mathbb{E}(t_n) = \frac{T_d}{p_e} = 6:06\,\text{mins}, \tag{1}$$

and the standard deviation is as follows;

$$\sigma_r = \mathbb{E}((t_n - \mathbb{E}(t_n))^2) = \sqrt{1-p_e}\,Tr = 6:00\,\text{mins}. \tag{2}$$

3.1.2 Derivation of expectation and standard deviation

Expectation. The above results for expectation and standard deviation of $t_n = n\,T_d$ are determined via manipulation of geometric series. Starting with the fact that $\mathbb{E}(t_n) = \mathbb{E}(N)\,\mathbb{E}(T_d)$ for a discrete random variable N, we use the definition of expectation for geometric series to determine $\mathbb{E}(t_n)$.

$$\begin{aligned}
\mathbb{E}(t_n) &= \sum_{n=1}^{\infty} n p_e (1-p_e)^{n-1}\ \mathbb{E}(T_d) \\
&= p_e \left(-\frac{d}{dp_e} \sum_{n=1}^{\infty} (1-p_e)^n \right)\ \mathbb{E}(T_d) \\
&= p_e \left(-\frac{d}{dp_e} \left(\frac{1-p_e}{p_e} \right) \right)\ \mathbb{E}(T_d) \\
&= p_e \left(\frac{d}{dp_e} \left(1 - \frac{1}{p_e} \right) \right)\ \mathbb{E}(T_d) \\
&= p_e \left(\frac{1}{{p_e}^2} \right)\ \mathbb{E}(T_d) \\
&= \frac{T_d}{p_e},
\end{aligned} \tag{3}$$

where $\mathbb{E}(T_d) = T_d$ as T_d is a constant and the Taylor series for $x = 1 - p_e$ is used in the third line of working.

Standard Deviation. As the standard deviation is given by the square root of the variance, applying the equation $Var(t_n) = \mathbb{E}({t_n}^2) - (\mathbb{E}(t_n))^2$ and finding the square root of this will yield the correct result. To begin we have

$$\begin{aligned}
Var(t_n) &= \mathbb{E}(N^2)\ \mathbb{E}({T_d}^2)\ - \left(\frac{T_d}{p_e} \right)^2 \\
&= \sum_{n=1}^{\infty} n^2 p_e (1-p_e)^{n-1}\ \mathbb{E}({T_d}^2)\ - \left(\frac{T_d}{p_e} \right)^2.
\end{aligned} \tag{4}$$

We shall suppose $q_e = 1 - p_e$ and rewrite $\mathbb{E}(N^2)$ as

$$\begin{aligned}
\mathbb{E}(N^2) &= \sum_{n=1}^{\infty} n^2 p_e q_e^{n-1} \\
&= \sum_{n=1}^{\infty} (n-1+1)^2 p_e q_e^{n-1} \\
&= \sum_{n=1}^{\infty} (n-1)^2 (q_e)^{n-1} p_e + \sum_{n=1}^{\infty} 2(n-1){q_e}^{n-1} p_e + \sum_{n=1}^{\infty} {q_e}^{n-1} p_e
\end{aligned} \tag{5}$$

and given the binomial expansion for a quadratic expression, $(a+b)^2 = a^2 + 2ab + b^2$. Defining $i = n - 1$, we have

$$\begin{aligned}
\mathbb{E}[N^2] &= \sum_{i=0}^{\infty} i^2 q_e^i p_e + 2 \sum_{i=1}^{\infty} i q_e^i p_e + 1 \\
&= q_e\, \mathbb{E}[N^2] + 2q_e\, \mathbb{E}[N] + 1
\end{aligned} \tag{6}$$

as $\mathbb{E}[N] = \sum_{i=1}^{\infty} i q_e^{i-1} p_e$. Further, using the fact that $\mathbb{E}[N] = \frac{1}{p_e}$, the equation becomes

$$\begin{aligned}
p_e\, \mathbb{E}[N^2] &= \frac{2q_e}{p_e} + 1 \\
\implies \mathbb{E}[N^2] &= \frac{2q_e + p_e}{p_e^2} = \frac{q_e + 1}{p_e^2}.
\end{aligned} \tag{7}$$

Substituting back into the variance equation (4) we have the variance of t_n,

$$\begin{aligned}\mathrm{Var}(t_n) &= \frac{q_e+1}{p_e^2}T_d^2 - \left(\frac{T_d}{p_e}\right)^2 \\ &= \frac{T_d^2}{p_e^2}(1-p_e).\end{aligned} \tag{8}$$

As we know $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$ for some constant a and a geometrically distributed random variable X, we can verify that our derivation is correct. Finally, given $T_r = T_d/p_e$, the standard deviation of t_n is given as follows,

$$\begin{aligned}\sigma(t_n) &= \sqrt{\mathrm{Var}(t_n)} \\ &= T_r\sqrt{(1-p_e)}.\end{aligned} \tag{9}$$

3.1.3 Probability of super or mega floods

An even more severe case of flooding occurs in Wetropolis when there are two consecutive days of extreme flooding, known as a superflood. Intuitively, we would expect the return period to be $T_r^{(2)} = T_d/p_e^2 = 3.72$ hours, however, this is not the exact solution and this is calculated instead using a geometric distribution. Suppose we observe k consecutive days of rainfall, for some $k \in \mathbb{Z}$ where $k > 1$, then the expectation is given by

$$\frac{T_r^k}{T_d} = \frac{(1-p_e^k)}{(1-p_e)p_e^k}, \tag{10}$$

[7] and the calculation for standard deviation is as follows

$$\frac{\sigma_r^k}{T_d} = \frac{\sqrt{1-(2k+1)(1-p_e)p_e^k - p_e^{2k+1}}}{(1-p_e)p_e^k} \tag{11}$$

[7]. For small p_e, we have that $T_r^k \approx T_d/p_e^k$ and $\sigma_r^k \approx T_r$, and for $k = 1$ this is exact. Further, using (10) the return period for a superflood is $T_r^{(2)} \approx 3.82$ hours. Taking this even further, we can propose a megaflood, occurring as a result of $k = 3$ days of consecutive flooding. Again using the approximation (10), we find the return period to be $T_r^{(3)} \approx 139.7$ hours.

3.2 Alterations of the model

Adaptations of this model include changing the probability of an extreme event to $p_e = 49/256$, so the occurrence of 9 seconds of rainfall, located in the moor or the reservoir each have a likelihood of 7/16. This decreases the return period to $T_r = \frac{10}{(49/256)} = 52.2$ seconds for one day of flooding and using (10), we find the return period is $T_r^2 \approx 5:25$ minutes for a superflood and $T_r^3 \approx 29:11$ minutes for a megaflood. This alteration helps in allowing the public to witness a flood or superflood, as these will occur much more often.

Another alteration of Wetropolis involves extending the duration of a Wetropolis day from ten to 20 seconds to allow rainfall to vary twice within one day. This is closer in behaviour to the real world, as we often experience a change in weather conditions at a given location throughout the day. We implement this by still using the Galton board to determine rainfall durations every ten seconds, but instead of having the rainfall location decided every ten seconds, it is instead decided every 20 seconds. Therefore, for the first two ten-second periods there will be two separate durations of rainfall carried out in these locations. This adaptation would make superfloods more likely to occur. Given the rainfall location is determined to be both the reservoir and moor, a 20 second Wetropolis day where flooding occurs twice (18 seconds of rainfall) will have probability $p_e = 1/16$ of occurring. Further alterations of the model used to demonstrate to the viewer the influence climate change has on the water cycle are described in the next subsection.

3.2.1 Influence of climate change

We can adapt this model to include the influence of climate change by adding a lake upstream of the moor, in addition to the reservoir. Water flow into this lake will be triggered by the Galton board outcome for the moor, but a weaker pump will be installed in the lake such that the volume of water released into the lake is less than the volume released into the moor. When including this there is found to be a roughly 20% increase in rainfall on these days, which in turn increases the severity of flooding in the city [7]. As outlined in §2.3, we are expecting to see an increase in the intensity of rainfall in the future as a result of climate change. Therefore, adding the lake into Wetropolis accounts for this increase in rainfall intensity and the flooding that occurs as a result.

Furthermore, we saw in section 2.3 that prolonged dry spells and droughts are expected to increase in duration and intensity in the UK due to climate change. Adaptations can be made to Wetropolis to simulate droughts by allowing the no-rain days to fall more often in succession, perhaps having periods up to four days long with no rainfall. The effect this has on the city is made noticeable to the viewer by installing a drinking water pipe in the city. During a drought, the moors water table sinks below the water pipe's inlet, meaning no water can pass downstream leaving the city's residents with a low water supply.

4 Flood Hydrographs

A flood hydrograph shows the change in the flow rate of water over time during a flooding event, this is seen in the top right quadrant of figure 9. Gauge stations located at cross sections along rivers routinely take readings on the height of the river. Plotting these readings over a time frame gives a graph as we see in the bottom left quadrant of figure 9. This data, along with rating curve coefficients, is then used to produce the hydrograph in the top right quadrant and the rating curve in the top left. It is from this data that the flood excess volume (FEV), flood duration, and discharge at a certain point are determined.

We shall begin in §4.1 by introducing rating curves and how they are produced with the use of a rating change report. In §4.2, the concept of flood excess volume will be explained, as well as its role in flood mitigation and how it can be approximated in §4.2.1. As a motivating example, we will reproduce the hydrograph, rating curve and river height readings for the 2015 Boxing Day flooding event on the River Aire in §4.3. Further figures shall be included in this section to analyse the flooding event, these include the river height plotted against precipitation levels and the square lake representation of the flood excess volume for this event. Lastly, in §4.3.3 we shall apply the principles for approximating flood excess volume to the 2015 Boxing Day flood, where we examine multiple estimates of this volume.

4.1 Rating change report

Rating change reports, created and provided to us by the Environment Agency, provide specific coefficients that we use to produce a rating curve for a flooding event. Given the mean river depth $\bar{h}$ over a river's cross-section, the discharge, $Q = Q(\bar{h})$, is calculated by inserting coefficients into the Rating Curve equation

$$Q(\bar{h}) = c_i(\bar{h} - a_i)^{b_i}, \quad i = 1, ..., m, \tag{12}$$

where coefficients a_i, b_i, and c_i fit to the data at pre-determined intervals, known as stages or limbs, such that for each of these stages $h_{i-1} < \bar{h} < h_i$. A Rating Change Report specifies the rating curve coefficients at each interval, as well as the upper and lower limits for h_i. The rating curve specifically is featured in the top left corner of figure 9 and shows the relationship between discharge and river height. It is also from this equation that the hydrograph can be produced. Further, the flood excess volume can be calculated once we provide a river height threshold, h_T, for which flooding is defined to occur when the river depth exceeds this value. Therefore, we determine the duration of a flooding event, T_f, by measuring the time taken from when the river height first exceeds h_T to when it falls back to h_T.

4.2 Flood excess volume

Flood excess volume is defined as 'the volume V_e of water that caused flood damage' [5]. The FEV is the volume that exceeds current mitigation, therefore the aim is to reduce this volume to zero by introducing or improving flood mitigation measures. Once we have determined the FEV for a particular event, it is useful to represent this volume as a cuboid, where the dimensions may be specified to match the river channel being examined. This allows for the partitioning of the shape into different quantities, each corresponding to the volume of water that will be stored or managed using different flood mitigation measures.

4.2.1 Approximating FEV

As the rating curve is deduced indirectly, we are not always able to use this in real-life situations with a high degree of accuracy, and therefore we may look for alternative ways of finding the FEV. Our first estimate of the FEV can be found provided we have recordings of the river depth, $\bar{h}$, taken over some time frame, such that each time t_k has corresponding river depth $\bar{h}_k$ for $k = 1, ..., N$. We also require the rating curve $\bar{Q}(h)$ for this flooding event, which can also include error bars but this is not essential. The estimate is given by the following sum

$$V_e \approx \hat{V}_e = \sum_{k=1}^{N} (Q(\bar{h}_k) - Q_T) \triangle t, \tag{13}$$

where $Q(\bar{h}_k)$ corresponds to the discharge for each river depth $\bar{h}$ and Q_T refers to the flow rate at the chosen height threshold h_T. The accuracy of this approximation increases as the number of readings increases and the time interval between readings decreases, such that taking the limit as $N \to \infty$ and $\triangle t \to 0$ yields the estimate V_e. This can be expressed as

$$\lim_{\substack{N\to\infty \\ \triangle t->0}} \hat{V}_e = V_e = \int_{t_f}^{t_f+T_f} Q(t) - Q_T \, dt, \tag{14}$$

where t_f denotes the initial time that the height threshold is reached, such that $Q(t_f) = Q(h_T) = Q_T$. Therefore, this estimation of the FEV is found by taking the area under the $Q(t)$ curve and above the $Q(h_T)$ line over T_f, as shown by the blue-shaded region in figure 9. In this example, however, we have $\triangle t = 15$ seconds and therefore the estimate of FEV must be calculated using the approximation in (13), which we find to be 9.34 Mm^3, as shown in the bottom right corner.

The second estimation for FEV, V_{e_1}, can be calculated using the same principle of the first estimation, but without the need for a rating curve or readings for river depth over time. Provided we have an estimate for the average discharge during the flooding event, $\bar{Q}(h_T) = Q(h_m) = Q_m$, the discharge at the height threshold, $Q(h_T) = Q_T$, and the flood duration, T_f, we are still able to estimate the FEV using the following equation:

$$V_{e_1} = T_f(\bar{Q}(h_T) - Q_T). \tag{15}$$

This is shown in the top right graph in figure 9 as the area of the boxed-off rectangle. As the discharge and time have units m^3/s and s respectively, multiplying these gives the required units for volume, m^3. Here the idea is that the area of the rectangle is approximately equal to the area covered by the blue-shaded region, so provides an estimate of the FEV which requires us to know less information about the flooding event.

A third approximation of FEV, V_{e_2}, is also found without the need for a rating curve or river depth measurements, albeit estimates for the maximum height h_{max}, height threshold h_T and flood duration T_f are required. It is from h_{max} that we calculate the maximum discharge Q_{max}, using an estimate of maximum mean surface velocity $\bar{u}_{max}$. Assuming for now that we have a rectangular channel with constant river width, w, we find the maximum discharge to be $Q_{max} = h_{max}\, w\, \bar{u}_{max}$. Providing an estimate for the average river height during the flood, h_m, we use linear interpolations to estimate the average discharge, Q_m and discharge at the threshold height, Q_T, which are given as follows,

$$Q_m \approx \frac{h_m}{h_{max}} Q_{max} \quad \& \quad Q_T \approx \frac{h_T}{h_{max}} Q_{max}. \tag{16}$$

Furthermore, the estimate for FEV is given by

$$V_e \approx V_{e_2} = T_f \frac{Q_{max}}{h_{max}} (h_m - h_T). \tag{17}$$

Based on the shape of the hydrograph above Q_T, we can make an educated guess for the average river height h_m. Using the equations of areas for four possible hydrograph shapes - rectangle, triangle, parabola, trapezium - four estimates of h_m can be made. Firstly supposing that the hydrograph has a triangular shape above h_T, we would suggest based on the area of a triangle that the average height lies an equal distance from the height threshold and maximum height, i.e.

$h_m \approx (h_{max} + h_T)/2$. Therefore (17) becomes

$$\begin{aligned} V_e \approx V_{e_2} &= T_f \frac{Q_{max}}{h_{max}} \left(\frac{h_{max} + h_T}{2} - h_T \right) \\ &\implies T_f \frac{Q_{max}}{h_{max}} \left(\frac{h_{max} - h_T}{2} \right). \end{aligned} \tag{18}$$

In the case of a rectangular-shaped hydrograph, we take $h_m \approx h_{max}$ and for a trapezoidal-shaped hydrograph, we suggest $h_m \approx 3h_{max} + h_T/4$. The respective FEV estimates for these are shown below,

$$\begin{aligned} V_{e_2} = T_f \frac{Q_{max}}{h_{max}}(h_{max} - h_T) \quad \& \quad V_{e_2} &= T_f \frac{Q_{max}}{h_{max}} \left(\frac{3h_{max} + h_T}{4} - h_T \right) \\ &\implies T_f \frac{3Q_{max}}{4h_{max}}(h_{max} - h_T). \end{aligned} \tag{19}$$

Lastly, for a parabolic shaped hydrograph, the average height can be taken using the area of a parabola as $h_m \approx 2(h_{max} + h_T)/3$, such that

$$\begin{aligned} V_{e_2} &= T_f \frac{Q_{max}}{h_{max}} \left(\frac{2(h_{max} + h_T)}{3} - h_T \right) \\ &\implies T_f \frac{2Q_{max}}{3h_{max}}(h_{max} - h_T). \end{aligned} \tag{20}$$

This final estimation of FEV becomes particularly useful when examining flooding events that have occurred in developing countries or rural areas, as these areas do not have gauge stations to take routine river depth measurements.

It can be seen that these three estimates for FEV all depend on the chosen threshold height $V_e = V_e(h_T)$. Thus, we find that taking a river height threshold of $h_T = 0$, the FEV equals the total volume of the river and alternately, setting $h_T = h_{max}$, the FEV becomes zero. This occurs when flood defence walls are implemented or raised along a river, as the height threshold is raised dramatically so any excess flood water remains within the channel.

4.3 Hydrograph for Leeds 2015 Boxing Day floods

The Boxing Day flood that took place in 2015 on the River Aire in Leeds was the region's 'biggest flood on record' [5]. Storm Eva lead to extremely high levels of rainfall over the Christmas period, eventually causing the river to burst its banks on Boxing Day. The aftermath of the flooding was of a huge scale and had lasting impacts both socially and economically. Some impacts include major road closure, damaged utilities and it was reported that 3,355 residential and 672 commercial properties were affected [26]. The total cost of infrastructure damage reached a staggering £36.8 million in the inner city region, and an estimated £500 million in the wider city area [26].

The severity of this flood can be visualised in figure 8, where the river levels are almost exceeding the height of the tunnel underneath Apperley Bridge. We shall use the Armley gauge station to extract our data on river height, along with the rating change report for this location. The river height readings from Armley monitoring station over the flooding period have been provided by Professor Onno Bokhove on GitHub (available here). Provided by the Environment Agency to Professor Onno Bokhove (available here), the rating curve coefficients given in the Rating Change Report for the River Aire at Armley monitoring station are summarised in Table 4. This report was produced in August 2016 and is therefore valid for producing a hydrograph based on a flooding event that occurred a few months before in December 2015.

Figure 8: Impact of Boxing Day floods on water height under Apperley Bridge along the river Aire, taken around 09:58 am on 26th December 2015. Photo courtesy of Onno Bokhove [5]

The coefficients for a, b, c and the corresponding upper limits, h_i are given in the rating change report, and shown in table 4. The lower limits for i, as shown in the second column, are taken as the upper limit of $i-1$. The initial lower limit, $h_0 = 0.2$ m, is specified elsewhere in the report, chosen as this reading is just below the lowest check gauging. The river height threshold $h_T = 3.9$m, as chosen by Bokhove [5], is the maximum height the river can reach before flooding. Inserting h_T and the coefficients from table 4 into the hydrograph Python code will produce the following hydrograph. Further details on this code can be found in Appendix A.2.

i	h_{i-1} (m)	h_i(m)	a_i(m)	$b_i(-)$	$c_i(\mathrm{m}^{3-b_i}/\mathrm{s})$
1	0.2	0.685	0.156	1.115	30.69
2	0.685	1.917	0.028	1.462	27.884
3	1.917	4.17	0.153	1.502	30.127

Table 4: Rating Curve Coefficients and lower and upper bounds on height for each stage at Armley monitoring station for the Boxing Day flood, 2015.

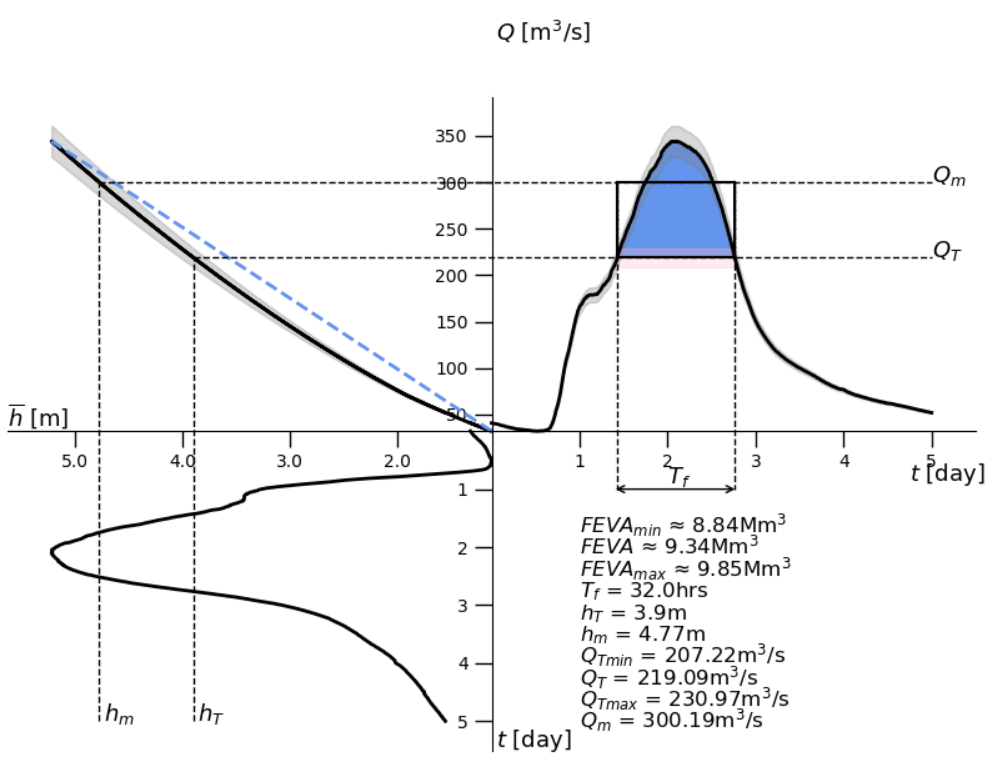


Figure 9: Hydrograph (top right), rating curve (top left) and river levels (bottom left) for the flooding of River Aire on 26th December 2015 at Armley monitoring station with the inclusion of the standard error of 5.42% as provided in the Rating Change Report, shown as the shaded region.

The top right quadrant in figure 9 shows the change in discharge over the time span of the flood event, which shows a steady but fast increase in discharge to begin with, which reaches a peak discharge of 344m^3/s. The bottom left quadrant displays the water level over time, where time increases as we work down the y-axis. We see a similar outcome to the top right graph, where the river height increases rapidly and then falls quickly, reaching a peak in height at around two days. Quadrant two presents the Rating Curve, which combines the discharge and river height axes. There is a smooth, monic increase in discharge, following the increase in water level. Notice also the dotted straight lines within figure 9, where the vertical lines passing through the left quadrants define the river height threshold, $h_T = 3.9$ m, and the average river height during the flood event, $h_m = 4.77$ m. The horizontal lines display the corresponding discharge levels at these heights, which are $Q_T = 219.09$ m^3/s and $Q_m = 300.19$ m^3/s. In addition, the figure also provides the flood duration for this event, $T_f = 32$ hours, which is the length of time that the water level is above the threshold height h_T. The flood excess volume is shown as the boxed-off region of the blue-shaded area, between Q_m and Q_T. The approximation of this volume is given by $FEVA \approx 9.34$ Mm^3, where Mm denotes megametres (a million metres).

The Rating Change Report includes a standard error, specific to each limb limit, $h_i = 0.685$, 1.917, 4.17m, where the standard errors are (5.42, 3.44 and 5.28)% respectively. We shall take the overall standard error as the highest of these values, such that $\sigma = 0.0542$ and the inclusion of this error is shown in figure 9, where the shaded region demonstrates the standard error. This allows for bounds to be made on the estimates for FEV and discharge, where $FEVA \in (8.84, 9.85)$ Mm^3, and at h_T, $Q_T \in (207.22, 230.97)$ m^3/s.

4.3.1 Comparing rainfall and river height during the Boxing Day floods

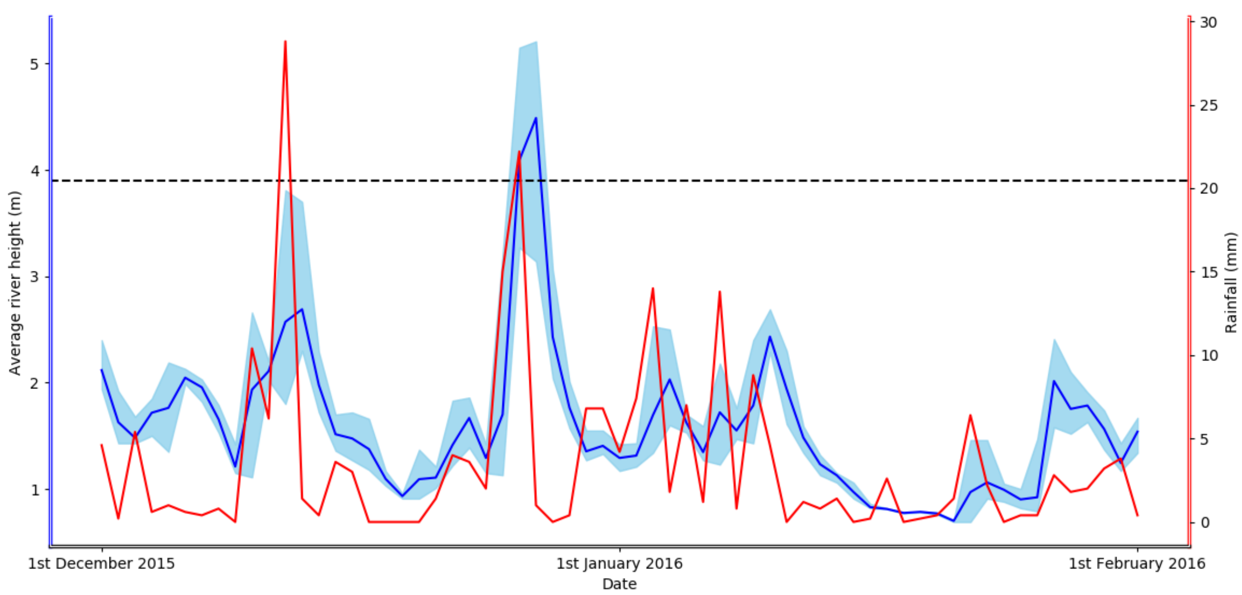


Figure 10: Twin plot of average river height (m) (blue) with maximum and minimum river height (m) (blue shaded region) and rainfall (mm) (red) each day at Armley Station along the river Aire from 1st December 2015 to 1st February 2016, where the horizontal black dashed line represents the river height threshold, $h_T = 3.9$m.

Similarly to figure 4 in §2.2, we can compare the river height and precipitation levels at Armley monitoring station from 1st December 2015 to 1st February 2015. The Python code used to produce this graph can be found in Appendix A.3. The river height data is provided by River Levels UK website, (available here) and the daily precipitation provided by the Government website (available here). Precipitation readings are not available at the exact site of Armley monitoring station, instead we shall use the precipitation readings for a nearby location, Pottery Field, located just south of the city centre. We see in this figure that rainfall peaked twice over these two months, reaching 15mm of precipitation in one day. This occurred around mid-December and again around the Christmas period, when the Boxing Day flood occurred. Similarly, river levels were very high and exceeded the river height threshold at this point, as indicated by the black dashed line. Notice how the river height peaks slightly after these periods of heavy rainfall, similar to what we saw in figure 4. We see this occur again at a smaller magnitude on the 2-3rd January 2015 period. This is to be expected as the increase in rainfall means the volume of the river increases while the rivers capacity remains the same, therefore some of this water overflows, accumulating as flood excess volume.

4.3.2 Graphical representation of FEV

The hydrograph code (Appendix A.2) also includes a plot of the square lake representation of the FEV, as shown in figure 11. This lake is chosen to have a relatively shallow, human-sized depth of $D = 2$ metres, aimed at helping give members of the public a better understanding of the true scale of this FEV. In this case, the lake spans 2.161 km in side length, determined by $L = \sqrt{FEVA/D} = \sqrt{9340000/2} = 2161$ m. To give a greater understanding of the scale of this FEV, assuming an average human walking distance of 12 minutes per kilometre, it would take approximately 25.93 minutes to walk the length of this square lake and 1 hour and 43 minutes to walk the full perimeter.

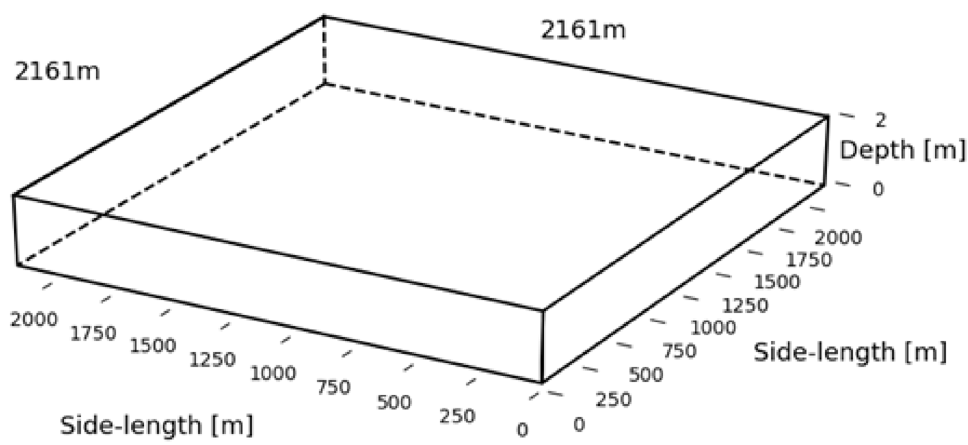


Figure 11: Square lake representation of the FEV for the River Aire flooding on 26th December 2015 at Armley monitoring station.

As the estimate for FEV is a function depending on h_T, we can find the FEV and corresponding square lake length for this volume, which have been plotted for a range of h_T values using the code detailed in Appendix A.4. This is shown in figure 12, where we see that as the threshold height increases, the FEV and square lake length both decrease in size. As the square lake is a function of FEV, we expect a positive correlation between these two variables, as indicated by a 0.973 (3 s.f.) correlation coefficient. We expect that this result holds, as a greater value of h_T means the river can deal with a greater uptake of water over the usual level.

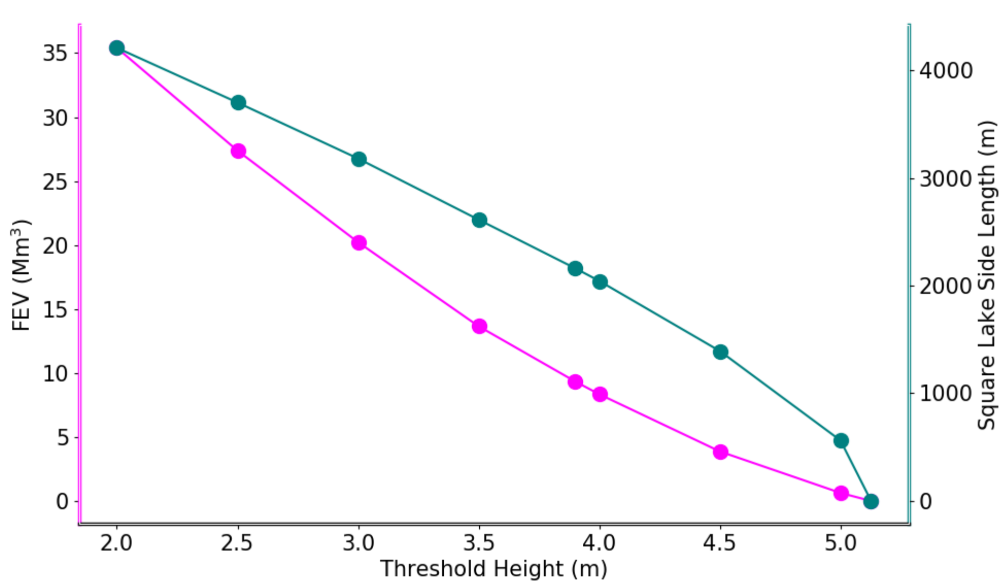


Figure 12: Plot of FEV (pink circles) and square lake length (teal circles) for various height thresholds h_T for the River Aire flooding on 26th December 2015 at Armley monitoring station.

4.3.3 Approximating FEV for the 2015 Boxing Day floods

Given the estimates for Q_m, Q_T and T_f provided in figure 9, the second approximation of the FEV is given in this example as

$$V_{e_1} \approx 32\,\mathrm{x}\,60^2\,(300.19 - 219.09) = 9.34\mathrm{Mm}^3, \tag{21}$$

as shown in the figure as $FEVA$. On the other hand, using the value for h_{max} from the raw data and Q_{max} from the maximum value for the discharge function, found within the Python code, the third approximation, V_{e_2}, is estimated here using (18) as

$$V_{e_2} \approx 32\,\mathrm{x}\,60^2\left(\frac{4.56 - 3.9}{5.217}\right)344.433 = 5.01\mathrm{Mm}^3, \tag{22}$$

where we have taken an estimate for $h_m \approx (h_{max} + h_T)/2 = (5.217 + 3.9)/2 = 4.56$, supposing that the hydrograph has a triangular shape. Alternatively, one could argue this shaped to be more trapezoidal, for which the estimation using (19) becomes

$$V_{e_2} \approx 32\,\mathrm{x}\,60^2\left(\frac{3\,\mathrm{x}\,344.433}{4\,\mathrm{x}\,5.217}\right)(5.217 - 3.9) = 7.51\mathrm{Mm}^3. \tag{23}$$

Or perhaps parabolic, where the FEV is estimated using (20) by

$$V_{e_2} \approx 32\,\mathrm{x}\,60^2\left(\frac{2\,\mathrm{x}\,344.433}{3\,\mathrm{x}\,5.217}\right)(5.217 - 3.9) = 6.68\mathrm{Mm}^3. \tag{24}$$

We find the most appropriate choice of shape is given by the trapezoidal shape, as this gave the closest estimate to V_{e_1}.

5 Flood mitigation - Leeds Flood Alleviation Scheme

We shall use the example of the Leeds FAS to demonstrate the flood mitigation strategies commonly used to protect against fluvial flooding. Leeds FAS has been separated into two phases, phase I shall be covered in §5.2 and phase II in §5.3. In each of these subsections, maps of the proposed regions shall be given and the proposals and methods used to aid flood protection will be explained. These methods can be separated into traditional engineering defences and natural flood management, which shall be defined and explained further below. The effectiveness of both phases shall be evaluated and their influence on return periods in these regions will be described.

In 2013, Leeds had very limited flood defence along the River Aire. This became a cause for concern after the city narrowly missed a serious flooding event in 2000, as well as experiencing dangerously high river levels that were close to flooding on multiple occasions, in 2004, 2007 and 2008 [23]. This encouraged Leeds City Council (LCC) to begin their initiative in flood alleviation strategies. Taking the role of lead authority, LCC partnered with the Environment Agency (EA) to implement the FAS. With 300 acres of developable land at risk [18], the LCC and EA proposed to improve and develop flood protection to make the city better equipped to deal with heavy rainfall and improve the Standard of Protection. Standard of Protection (SoP) is another term we use to refer to the AEP or return period, used to quantify flood risk in any one year. For example, a 1 : 100 year SoP means an area is protected against a one-in-100-year flood occurring, or a flood with 1% AEP. An improved SoP means an area can withstand and protect against floods with a higher return period than could be previously managed.

The FAS scheme involves implementing a combination of traditional engineering strategies and natural flood management schemes to mitigate floods. Traditional engineering strategies used in flood defence usually involve constructing or improving previous flood defence walls, creating flood storage areas (to give room to the river) and making improvements to other existing flood defence structures. On the other hand, natural flood management (NFM) "concerns the use of elements and features drawn from nature to mitigate floods" [6], with the aim to restore or accompany the natural processes that occur along a river, floodplain or catchment area. In fact, the Leeds FAS is the first NFM scheme in the UK to be implemented at catchment scale. [17]

A phased approach was proposed, where the two phases focus on different regions along the river, each with their own targets for Standards of Protection and their own methods for achieving these. Comprised of multiple locations along the River Aire, as shown in figure 13, are the proposed areas for development for each phase, where the blue bracketed region below and within the city concerns phase I and the red bracketed region above and in the city concerns phase II.

Figure 13: Map of proposed FAS area, as provided by the UK Government website, available here.

5.1 Leeds FAS I

As shown in figure 13 the first phase focuses on developing flood protection along the river Aire between Leeds City Centre and Rothwell further downstream. In 2013, £50 million was funded to implement this first phase [18], with donations from multiple sources, including the Environment Agency, DEFRA growth fund, the Regional Growth Fund [14] as well as local businesses. Two key elements that make up this first phase involve the use of traditional engineering methods, these are constructing movable weirs and merging the river and canal. Some methods of NFM are also used in this scheme, like tree planting.

5.1.1 Traditional engineering in FAS I

Movable weirs. These are the first of their kind to be used in the UK, located at Knostrop and Crown Point. At a distance of 2.4 km from each other, the two weirs help to alleviate flood risk with the use of inflatable bladders. These are cylindrical structures made of a rubber-like material that lie across the width of the river underneath a dynamic gate, as featured in figure 14(a). When flooding is expected to occur, these bladders can deflate and lower the gate, allowing more water to flow through the river. These have shown to be successful in reducing water levels, with the potential to decrease flood levels by over a metre [14]. In addition to this, riverside walls were built and given the effectiveness of the movable weirs, the walls could be less than 1.2m high while still providing sufficient flood protection, which is less than 50% of the height initially proposed [3].

Giving room to the river. A 600 metre man-made island between the River Aire and the Leeds-Liverpool canal, known as Knostrop cut located near the Knostrop weir, was removed. This is a form of giving room to the river, an approach aimed at allowing rivers to flow naturally rather than limiting their flow. The removal of the cut increases the river's flowing capacity, which increases flood storage volume and helps slow the flow of water downstream [14]. This walkway was replaced by a 1km stretch on the left bank of the river, as well as a bridge being built over Knostrop weir so walkers and cyclists could still access the region [3]. A before and after photograph of Knostrop cut can be seen in figure 14(b).

(a) Photograph of movable weir mechanism courtesy of New Civil Engineer, available here.

(b) Photograph of Knostrop cut, before and after the island between the canal and river was removed, provided by the Public Sector Executive [23].

Figure 14: Photographs of traditional engineering flood defences in FAS1.

5.1.2 Natural flood management in FAS I

Tree planting. Seven hundred trees were planted among the flood alleviation network locations along the river [3]. This aids flood prevention as the trees absorb the soil's moisture, which in turn increases the amount of water stored in the soil, known as groundwater storage.

5.1.3 Effectiveness of FAS I

Completed in 2017, this scheme was regarded as a great success, demonstrated by the resulting increase in the Standard of Protection. Before the scheme, Leeds could handle a 1-in-75-year flood event occurring up until 2039, while considering the influence of climate change [14]. Now the city's resilience has increased to a 1 : 100 year SoP up until 2069, with the influence of climate change being taken into account [14]. This scheme helped to protect over 3500 homes and 500 businesses from future flood risk in the city centre [3], as well as protecting 300 acres of land for development [14]. The construction of the movable weirs has proven effective, where it has been reported that in the five years after their completion, the weirs have been "readied for deployment over 100 times and operated eight times" [19] to protect the city centre from flooding. In particular, extreme rainfall and high river levels were reported as a result of storm Ciara in 2020 and storms Dudley, Eunice and Franklin in 2022 [19]. Nonetheless, the FAS "served its purpose successfully" [19] in protecting the city from these extreme weather events.

5.2 Leeds FAS II

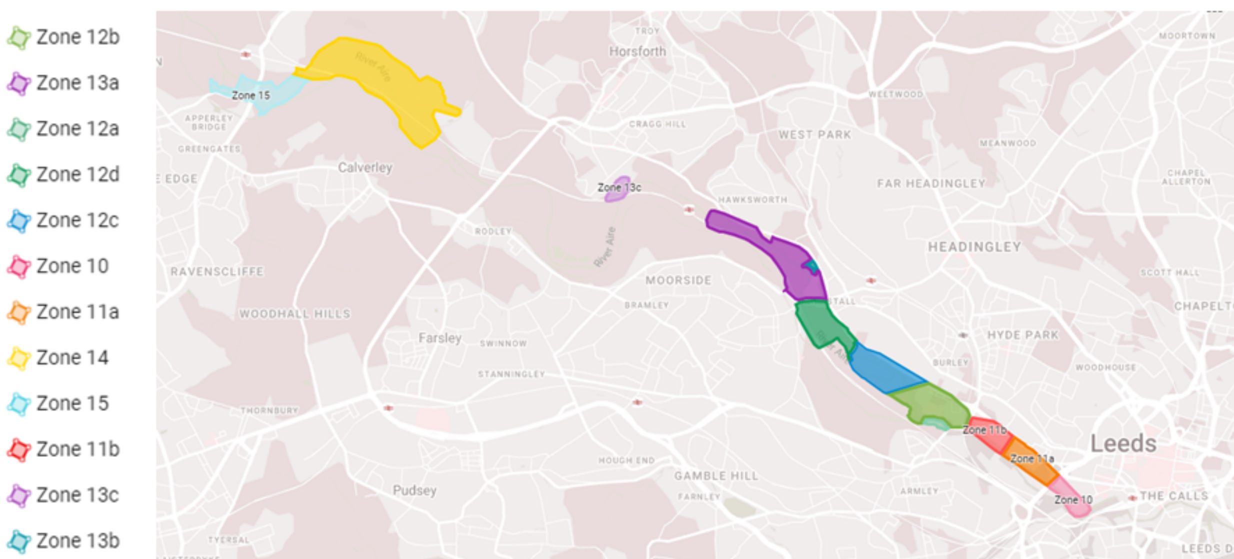


Figure 15: Map of proposed FAS phase 2 area, separated into zones 10-15 and including a key to indicate each zone, as provided by the LCC Flood Resilience website [16]

Again, Leeds City Council partnered with the Environment Agency to develop and implement the second phase of the FAS, with the aim of establishing sufficient protection from a 1-in-200-year flood event, while including an allowance for climate change [16]. This would mean the city is capable of dealing with extreme rainfall, almost to the same scale as the 2015 Boxing Day flood, without causing the same level of flooding and devastation. Successful implementation of this scheme will help to protect a further 1048 homes and 474 businesses from flooding [16]. FAS II centralises its focus on the two regions shown between red brackets in figure 13. These have been defined more clearly in figure 15, where the region is separated into zones 10-15 along the river. Partitioned into two steps, step 1 focuses on developing protection along the 8km stretch of the river Aire between Leeds train station and Newlay conservation centre making up zones 10-13, while step 2 focuses on zones 14 and 15, where Apperley Bridge and Calverley are located. Step 1 was approved first and achieved £87 million in funding, allocated by the UK Government, while step 2 was later approved, securing £21 million in funding in early 2021 [16].

5.2.1 Traditional engineering in FAS II

Flood defence walls. These are featured in most of the labelled zones, where reinforced concrete walls have been covered with masonry to match the surroundings. Notably in zone 13, flood defence walls will be built to assist in blocking underground seepage paths at the Kirkstall Bridge Inn pub that caused a great deal of devastation in the 2015 Boxing Day flood. [15]

Embankments. In zone 10, the region from Wellington Road to the north of Leeds City train station, floating riverbanks were installed along the river. In particular, a four-metre-high earth embankment was built at the Tannery site to replace the collapsed riverbank [15]. Embankments are made up of soil and vegetation to help to soften the riverbank wall, as well as providing new habitats for mammals, birds and insects. They aid flood defence by storing the river's excess water during a flood, which would otherwise overflow into surrounding areas. The riverbanks were completed in October 2022 [15].

Figure 16: River Aire at Armley Mills Industrial Museum before and after one of the proposed flow control structures, courtesy of LCC Flood Resilience [15].

Flood control structures. A significant area in this step of FAS II is zone 12a - where the Armley Mills Industrial Museum is located. In addition to $0.8 - 2.1$ metre high flood walls being built here, two flow control structures are being introduced at either end of the site, due for completion in Spring 2023 [15]. A before and after comparison of the proposed flow control structures is shown in figure 16, using a computer-generated image to show the final result. These structures have sluice gates, which can be opened and closed manually depending on river levels. The gates remain open when experiencing normal river conditions, but once river levels begin to rise the gates can be shut to prevent excess water entering the river at the Museum. A further two flow control structures are being built in zone 12c on opposite sides of Kirkstall's Goit, each with their own penstocks. Penstocks are a form of sluice used to monitor water flow in and out of Kirkstall Goit, which is a channel used to divert water from the river Aire into a smaller inlet, where the public can walk alongside.

Pumping stations. These are another form of technology used to decrease flood risk that will be utilised in zone 12 [15]. Plans for two stations to be built in zones 12c and 12d will decrease flood risk in periods of heavy rainfall, by pumping away large volumes of water to areas less susceptible to flooding. This is particularly necessary in zone 12d as the river lies adjacent to the railway line, therefore flooding here would cause widespread disruption.

Wetlands. The main project in zone 13 is building a wetland located at Kirkstall Meadows. Made up of 2.4 hectares, the area will include kingfisher banks, otter holts and wetland scrapes for fish [26] and is due for completion in 2023. These aid flood defence by absorbing and storing great quantities of water that accumulate as a result of rainfall, therefore preventing a great influx of water downstream.

Flood storage area. Step 2 focuses on the Calverley and Apperley Bridge areas further upstream and is due for completion in late 2023. The main construction here will be the flood storage area located at Calverley, as featured in figure 17 using a computer-generated image. We see from this figure that the area will be able to hold large volumes of water, in fact, 1.8 mm^3 [16], which will be slowly released back into the stream over time. In addition to this, two large floodgates and a 200 metre embankment, with a maximum height of 6.2 metres above ground level [15], will help reduce the volume of water heading downstream.

Figure 17: A computer-generated image of the proposed Calverley flood storage area during normal river levels (top) and during a flooding event (bottom), courtesy of LCC Flood Resilience [15].

5.2.2 Natural flood management in FAS II

Tree planting and soil management. In both steps of FAS II, NFM methods are being delivered by the Environment Agency on behalf of the LCC [26]. This consists of planting trees and implementing soil management practices. The aim is to plant 930 hectares of new woodlands [17] in zones 10-15 to help increase groundwater storage and intercept rainfall, decreasing flood risk as less water flows into the river. Soil management is being implemented in both steps, which consists of aerating the soil by perforating the soil with holes. This helps to increase the supply of air, water and nutrients throughout the soil. Also, the decrease in compactness of the soil allows for greater water storage and improvement in the soil's nutrient density allows for more crops to grow, furthering the groundwater storage. This scheme remains ongoing and is due to be completed in 2025. [20]

Pilot projects. Further NFM practices have been introduced by the Yorkshire Wildlife Trust in areas beyond zones 10-15 and instead at nine alternative sights along the river Aire catchment [28], with many of these located around Skipton in the North West of Leeds. Partnering with landowners in these locations, the trust introduced multiple NFM measures as part of a pilot project scheme. This involved further tree planting, the planting of hedgerows and the building of leaky dams. Leaky dams are made by placing wooded materials along the width of a river or stream which helps to slow the river's flow and expand the wet woodland habitat. The pilot projects have successfully been completed, now delivering 150 metres of hedgerow, 66 leaky dams and 1850 newly planted trees in the wider catchment region. [28]

5.2.3 Effectiveness of FAS II

Despite many elements of this scheme still being in production, the flood control structures and pumping stations built at Redcote Lane in zone 12c have already been completed. This has proven to be successful at increasing the SoP to a 1 : 100 year level of protection [15], which is of great importance here as this site contains Kirkstall's National Grid substation used to power thousands of homes in Leeds.

II River dynamics and shallow water theory

6 Shallow water theory

Also referred to as the St. Venant Equations, named after the 19th-century Mathematician Adhémar Saint-Venant, the shallow water equations are fundamental in hydraulics. Let us begin by stating and explaining the assumptions required to use these equations.

Shallow water. The length of the water is greater than the water's height and width, i.e. the water is of shallow depth.
One-dimensional flow. In any given cross-section velocity remains uniform and the free surface remains horizontal over the channel bed.
Laminar flow. This means that there is little influence of curvature and water moves in a streamlined forward direction, so any vertical acceleration in the fluid is negligible. Resulting from this, the fluid has a hydrostatic pressure distribution.
Small bed slope. The river bed slope must have a shallow gradient, such that we can approximate $\cos\theta \approx 1$ and $\sin\theta \approx \tan\theta$.
Flow varies smoothly and is subcritical. This means that $u < \sqrt{gh}$ for river height h, velocity u and acceleration due to gravity constant g. This shall be explained further in §8.4.
Incompressible flow. The liquid's density remains constant throughout a given cross-section.

The derivations of the continuity (§6.1) and momentum (§6.2) equations have used the principles outlined in Andrew Sleigh's derivation [24], which has been adapted to apply to the case of Wetropolis. Both derivations shall involve the use of diagrams, for which most notation used is defined in table 5. However other notation shall be defined when this is needed. In addition, background information on the Manning and Chézy equations, used within the momentum equation, shall be given in §6.3. This includes derivation of the Chézy equation, which gives an equation to estimate velocity in open channels. Let us begin by introducing the notation in table 5 that will be used to derive these equations.

Table 5: Definitions of notation

Notation	Definition
t	Time, seconds
s	Position of cross-section starting from a point upstream, $s \in [0, L]$ m
$A(s.t)$	Surface area of river cross section
$h(s,t)$	Channel height, m
$b(s,t)$	River bed height, m
L	Length of the river channel
u	Velocity of the fluid, m/s
Q	Discharge/flowrate m^3/s
g	Acceleration due to gravity, m/s^2
C_m	Manning friction coefficient, s/m$^{1/3}$
C_z	Chézy friction coefficient, m$^{1/2}$/s
$R(h)$	Hydraulic radius, wetted area / wetted perimeter
θ	Channel slope
V	Volume , m^3
F	Force , N
z	Elevation from channel bed, m
P_w	Eeir height, m

6.1 Continuity equation

Figure 18 shows the diagram of a river channel, with cross sections 1 and 2 taken at separate points along the river and $\triangle s$ denotes the change in position from cross section 1 to 2. We shall compare discharge at these two cross-sections to derive the continuity equation,

$$\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial s} = 0. \tag{25}$$

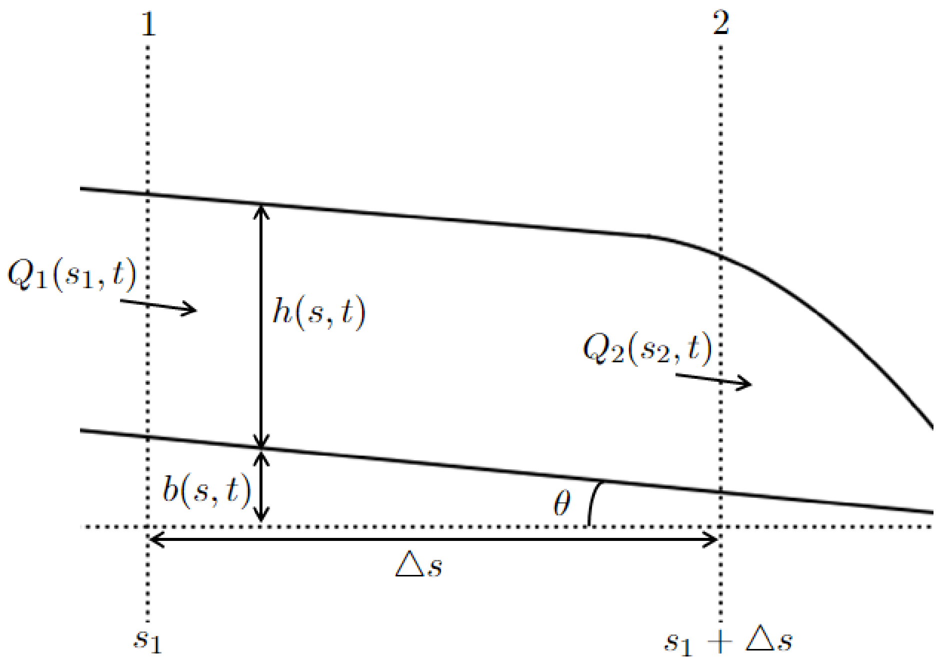


Figure 18: Diagram to show water flow in a river channel with slope θ from cross-section 1 to cross-section 2. This diagram was made in Microsoft Word.

Derivation. We start by noting that the change in discharge from s_1 to $s_1 + \triangle s$ will be equal to the increase in volume over this interval, V_{inc}, but of opposite sign, i.e.

$$Q(s_1, t) - Q(s_1 + \triangle s, t) + V_{inc} = 0. \tag{26}$$

Assuming we have no lateral inflow, the change in discharge is given as follows

$$Q(s_1, t) - Q(s_1 + \triangle s, t) = \frac{\partial Q}{\partial s} \triangle s, \tag{27}$$

where the partial derivative is used as $Q(s, t)$ depends on both s and t. The volume of water between sections 1 and 2 increases at a rate of

$$w \frac{\partial h}{\partial t} \triangle s, \tag{28}$$

where w is given by the cross-sectional area of the channel section, $A = wh$. We can rewrite the area to give the increase in volume as follows,

$$V_{inc} = \frac{\partial A}{\partial t} \triangle s. \tag{29}$$

Substituting (29) and (27) into (26), we have

$$\frac{\partial Q}{\partial s} \triangle s + \frac{\partial A}{\partial t} \triangle s = 0, \tag{30}$$

and as discharge $Q = Au$, where u denotes the velocity $u(s, t)$, dividing by $\triangle s$ leaves the continuity equation

$$u \frac{\partial A}{\partial s} + A \frac{\partial u}{\partial s} + \frac{\partial A}{\partial t} = 0. \tag{31}$$

6.2 Momentum equation

Using figure 19, momentum is calculated by equating Newton's second law equation for force using the forces influencing acceleration in this situation. Below we shall use this method to derive the momentum equation, namely

$$u\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}+g\frac{\partial h}{\partial s}=g\left(\tan\theta-\frac{|u|uC_m^2}{R(h)^{4/3}}\right). \tag{32}$$

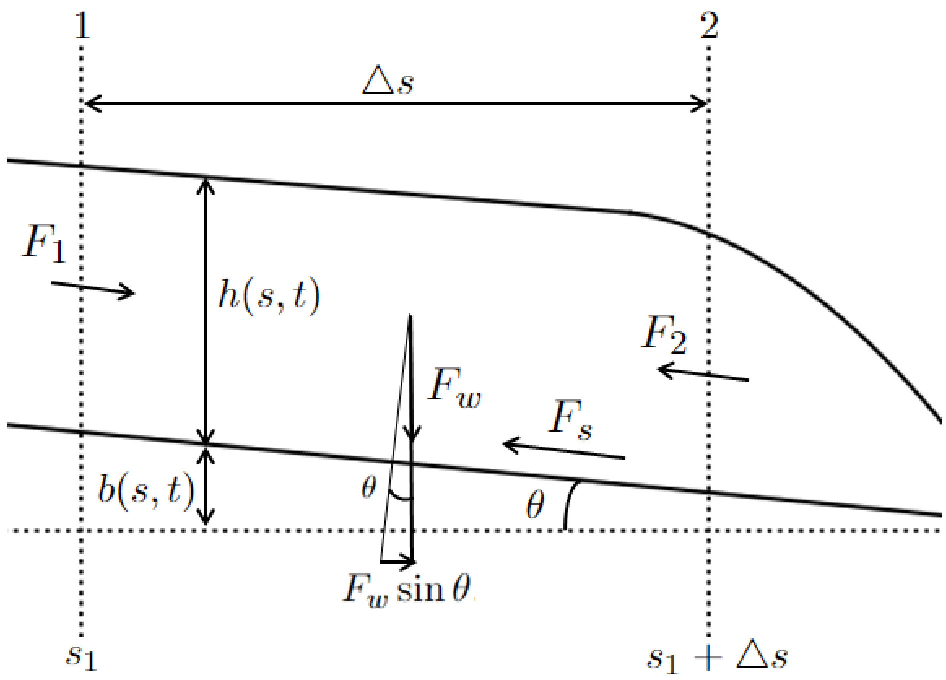


Figure 19: Diagram to show forces acting on water flow in an open channel at cross-sections 1 and 2. This is an extension of figure 18, including the forces acting on the water and the direction in which they act. This diagram was made in Microsoft Word.

Derivation. By Newton's second law, force = mass · acceleration ($F = ma$) and we can rewrite the equation for the sum of external forces acting in this instance as

$$\begin{aligned}\sum_i F_i &= \rho A\triangle s\,\frac{du}{dt}\\ &= \rho A\triangle s\left[u\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right],\end{aligned} \tag{33}$$

as velocity varies both in terms of s and t. There are three external forces, F_i, present here:

1. Change in hydrostatic pressure force - $F_1 - F_2$
2. Shear force on the fluid - F_s
3. Fluid weight - F_w

[22] where F_2 denotes the hydrostatic pressure at $s_1 + \triangle s$ and F_1 the hydrostatic pressure at s_1. The shear force due to the interaction of water and the channel's perimeter occurs up the slope, as shown in figure 19. The fluid weight corresponds to the projection of gravity force acting down the slope, therefore for the bed slope θ we take this force as $F_w \sin\theta$, as demonstrated in the figure. The forces are expressed in this case as

$$F_1-F_2=-\rho gA\frac{\partial h}{\partial s} \quad \& \quad F_s=\rho gAj \quad \& \quad F_w=\rho gA \tag{34}$$

where j denotes the energy loss per unit length of channel per unit weight of fluid [24] and ρ denotes the density of the fluid. The sum of these forces is given by

$$(F_1-F_2\ -F_s+F_w)\triangle s \implies \left(-\rho gA\frac{\partial h}{\partial s}-\rho gAj+\rho gA\sin\theta\right)\triangle s. \tag{35}$$

As we are assuming we have a small bed slope, θ, we take $\sin\theta \approx \tan\theta$. Therefore, returning to equation (33), we have

$$\begin{aligned}&\left(-\rho g A\frac{\partial h}{\partial s}-\rho g A j+\rho g A\tan\theta\right)\triangle s=\rho A\triangle s\left[u\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right]\\ \implies\ &\rho A\triangle s\left(g\frac{\partial h}{\partial s}-gj+g\tan\theta\right)=\rho A\triangle s\left[u\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right].\end{aligned}\tag{36}$$

With further manipulation, this is rewritten as

$$\begin{aligned}&\rho A\triangle s\left(u\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right)=-\rho g A\frac{\partial h}{\partial s}\triangle s-\rho g A j\triangle s+\rho g A\triangle s\,\tan\theta.\\ \implies\ &u\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}+g\frac{\partial h}{\partial s}=g(\tan\theta-j).\end{aligned}\tag{37}$$

We can rewrite j using the Manning coefficient's expression as

$$j=\frac{|u|uC_m^2}{R(h)^{4/3}},\tag{38}$$

where C_m is the Manning friction coefficient and $R(h)$ is the hydraulic radius, as defined in figure 3. Thus we reach the momentum equation, as defined in equation (32). Further explanation of the Manning coefficient shall be detailed in the subsection below, along with the derivation of its predecessor, the Chézy coefficient.

6.3 Manning and Chézy equations

6.3.1 Chézy equation

Chézy's equation, devised in 1768 by the French engineer A. Chézy [9], is used to determine the uniform flow rate in open channels [22]. It gives the following estimate for velocity in open channels

$$u=C_z\sqrt{R\tan\theta}\tag{39}$$

where C_z denotes the Chézy coefficient, with units $\mathrm{m}^{1/2}/\mathrm{s}$. The value of this coefficient typically falls within the range of $30-90\ \mathrm{m}^{1/2}/\mathrm{s}$ [9], where the lower coefficient is used for small rough channels and the higher coefficient for large smooth channels.

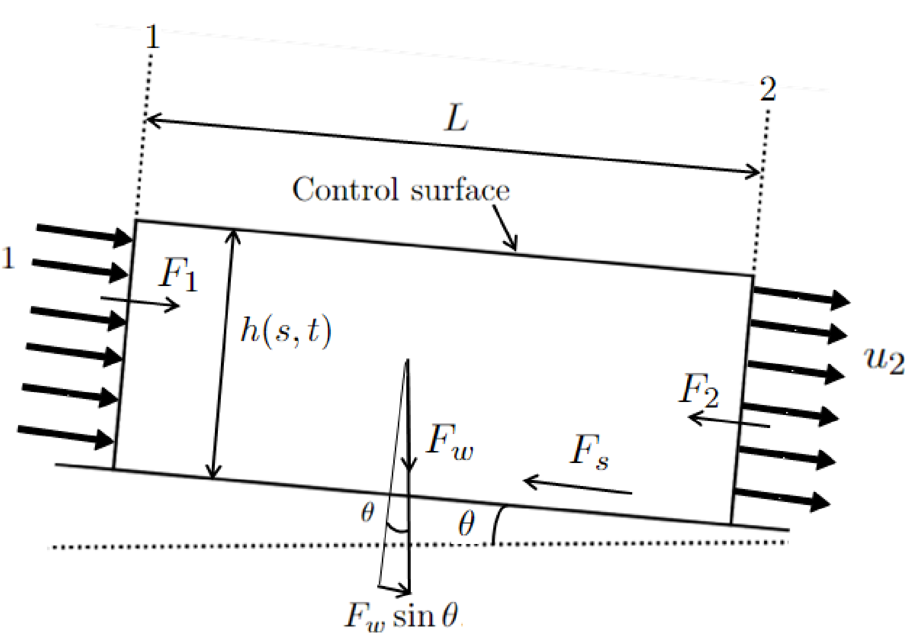


Figure 20: Diagram of control volume for uniform flow in an open channel. This figure builds on the previous, with the addition of L to denote the length of the channel and the inclusion of velocities at cross sections 1 and 2. This diagram was made in Microsoft Word.

Derivation. Figure 20 demonstrates the control volume for uniform flow in an open channel, which we shall be using to aid this derivation. As we are assuming steady uniform flow, we see no variation in velocity over time, meaning $u_1=u_2$ and the horizontal component of the momentum equation becomes

$$\sum_i F_i = \rho Q(u_2 - u_1) = 0. \tag{40}$$

Thus, the forces acting upon the water, as detailed in §6.2 sum to zero, i.e.

$$F_1 - F_2 - F_s + F_w \sin\theta = 0. \tag{41}$$

As flow is at uniform depth, i.e. $h_1(s,t) = h_2(s,t)$, we can take $F_1 = F_2$ and these forces do not contribute to the overall force balance. Therefore it remains that we have

$$-F_s + F_w \sin\theta = 0. \tag{42}$$

Again assuming a small bed slope, we approximate $\sin\theta \approx \tan\theta$. The fluid weight force and shear force are defined here as

$$F_s = \tau_w PL \quad \& \quad F_w = \gamma AL, \tag{43}$$

where τ_w denotes the wall shear stress and γ is the unit weight of water [22]. Substituting these into (42), we have

$$-\tau_w PL + \gamma AL\tan\theta = 0 \implies \tau_w = \frac{\gamma AL\tan\theta}{PL}. \tag{44}$$

Writing the area and perimeter using the hydraulic radius, $R = A/P$, the equation becomes

$$\tau_w = \gamma R\tan\theta. \tag{45}$$

In this case, we shall assume turbulent flow, as opposed to laminar flow, as "most open-channel flows are turbulent" [22]. This is due to the roughness created by the friction slope, which makes the flow irregular. In this case, τ_w is proportional to the dynamic pressure, $q = \rho u^2/2$ [22] and is independent of viscosity. Therefore we can write τ_w as

$$\tau_w = K\rho\frac{u^2}{2}, \tag{46}$$

for some constant $K \in \mathbb{R}$. Equating these two expressions for τ_w and rearranging, we have the Chézy equation,

$$K\rho\frac{u^2}{2} = \gamma R\tan\theta \implies u = C_z\sqrt{R\tan\theta}. \tag{47}$$

6.3.2 Manning equation

After combining the results of many experiments on open channel flow, with the aim of making a single equation that best matched observed velocities of flow in open channels, Manning published a paper in 1889 proposing an improved friction coefficient [22]. This is widely seen as being more accurate for describing the dependence of $R(h)$, as these experiments found that slope dependence of hydraulic radius on the velocity is closer to $u \sim R^{2/3}$ rather than $u \sim R^{1/2}$ [22] as seen in Chézy's equation. We deduce the Manning equation from Chézy's equation by setting

$$C_z = \frac{R^{1/6}}{C_m}, \tag{48}$$

such that velocity can be expressed as

$$\begin{aligned} u =& \frac{R^{1/6}}{C_m}\sqrt{R\tan\theta} \\ =& \frac{R^{2/3}\sqrt{\tan\theta}}{C_m} \end{aligned} \tag{49}$$

and we have the Manning equation, where C_m denotes the Manning friction coefficient which has units $\mathrm{s/m^{1/3}}$. This coefficient is used to account for channel roughness, which varies based on the terrain of the channel's perimeter. We find the rougher the channel is, the larger the Manning coefficient will be. For instance, in a clean and straight natural channel, the Manning coefficient is

0.03 [22], but for a heavy brush floodplain, we'd use a Manning coefficient of 0.075 [22], as shown in the table below. A range of values of the Manning coefficient for different channel surfaces, extracted from [22], have been included here for understanding.

Wetted perimeter		C_m
Natural channels	Clean and straight	0.030
	Sluggish with deep pools	0.040
	Major rivers	0.035
Floodplains	Light brush	0.050
	Heavy brush	0.075
	Trees	0.015
Excavated earth channels	Clean	0.022
	Gravelly	0.025
	Stony, cobbles	0.035
Artificially lined channels	Glass	0.010
	Steel, smooth	0.012
	Concrete, finished	0.012

Table 6: Values of Manning coefficients for different channel types. [22]

7 River dynamics in open channel flow

We see open channel flow being demonstrated in the Wetropolis model, simulated to mirror the behaviour of a river. The specific continuity and momentum equations for the Wetropolis model shall be introduced and explained in §7.1. Furthermore, we will define an alternative expression for discharge that can be used when a river's bed slope and friction are locally balanced, which we see in the Wetropolis model (§3) and will assume when creating a river plot in §10. Lastly in this section, we shall define a key equation in fluid dynamics - the Bernoulli equation. The assumptions needed to use the equation will be stated and explained, along with the notation used in the Bernoulli equation.

7.1 River dynamics in Wetropolis

Along with the river, moor, reservoir and city, we shall introduce a few more components of the model. The river begins at the top right of the tabletop model, this is where water first enters Wetropolis through an aquatic pump. There is a one-sided flood plain along the full length of the river, which allows for the overflow of water seen after heavy rainfall. The model also includes an overflow at the moor and a fixed weir at the reservoir. This is to slow the flow of water and aid flood prevention. The use and influence of weirs in Wetropolis shall be discussed further in §8.3. The notation involved in the model is introduced in table 5 given at the beginning of §6.

The Saint-Venant equations given in [4] are comprised of the continuity and momentum equations, derived from one-dimensional shallow water theory. Note, these two equations have been derived completely in §6 and using these results, along with our introduced notation in table 5, we yield the continuity and momentum equations specific to Wetropolis,

$$\partial_t A + \partial_s(Au) = Q_m\delta(s - s_m) + Q_r\delta(s - s_r) + Q_{L_1}\delta(s - s_{L_1}) \tag{50}$$

$$\partial_t u + u\partial_s u + g\partial_s h = -g\left(\partial_s b + \frac{C_m^2 u|u|}{R(h)^{\frac{4}{3}}}\right). \tag{51}$$

Given that s denotes the position of the cross-section from table five, in the continuity equation s_{L_1} denotes the location along the river, s_m denotes the location of the moor and s_r the location of the reservoir. We also have the discharge, Q, included in (50) for the discharge rates at the river, the moor and the reservoir, given by Q_{L_1}, Q_m, Q_r respectively. The Dirac delta functions, $\delta(s - s_\gamma)$, in the continuity equation are used to model the discharge rates as point sources at each location s_γ along the river. The delta function is defined as

$$\int_{-\infty}^{\infty} \delta(s - s_\gamma) = \begin{cases} 0 & \text{if } s_\gamma < s - \epsilon/2 \\ 1/\epsilon & \text{if } s - \epsilon/2 < s_\gamma < s + \epsilon/2 \\ 0 & \text{if } s_\gamma > s + \epsilon/2, \end{cases} \tag{52}$$

as ϵ tends to zero. For instance, when $s_{x_s} = s_m$ the delta function tends to 0 for all $s \in [0, L]$ except when $s = s_m$, as here the function tends to 1. Therefore the discharge at the moor, Q_m, is included in the momentum function only at point $s = s_m$ and similarly for points s_r and s_{L_1}. In the momentum equation, $-\partial_s b$ denotes the slope of the river channel and the friction Manning coefficient [22], C_m, is chosen from table 6 to be 0.03 as we are dealing with a straight channel. The Hydraulic radius is also included in (51), $R(h)$ in table 5, which is calculated as the wetted area divided by the wetted perimeter. Further investigation of the fluid dynamics in Wetropolis shall be featured in §8.3, where we examine the dynamics of the reservoir and canals in Wetropolis.

7.2 Modelling discharge in open channel flow

In the case of the bed slope and friction being locally balanced, we can create an alternative expression for discharge, where velocity is expressed approximately via the Manning relation. Supposing $u > 0$ and the constant bed slope $-\partial_s b$ is positive, the left side of equation (51) becomes 0 and we have

$$\partial_s b + \frac{C_m u^2}{R(h)^{\frac{4}{3}}} = 0 \quad \Longrightarrow \quad u = \frac{\sqrt{-\partial_s b}\, R(h)^{\frac{2}{3}}}{C_m}. \tag{53}$$

Therefore, discharge can approximately be defined as

$$Q = A(h)u = A(h)\frac{\sqrt{-\partial_s b}\, R(h)^{\frac{2}{3}}}{C_m}. \tag{54}$$

Using this equation for velocity, the continuity equation in Wetropolis (50) becomes

$$\partial_t(A(h)) + \partial_s\left(A(h)\frac{\sqrt{-\partial_s b}\, R(h)^{\frac{2}{3}}}{C_m}\right) = Q_m\delta(s-s_m) + Q_r\delta(s-s_r) + Q_{L_1}\delta(s-s_{L_1}), \tag{55}$$

for some area $A(h(s)) = h\alpha$, where α varies depending on the shape of the cross-section and can be in terms of h. In Wetropolis, a rectangular river channel is used and $\alpha = w$ denotes the channel width. Therefore the hydraulic radius is $R(h) = hw/(2h+w)$ and we have

$$\partial_t(hw) + \partial_s\left((hw)^{\frac{5}{3}}\frac{\sqrt{-\partial_s b}}{C_m(2h+w)^{\frac{2}{3}}}\right) = Q_m\delta(s-s_m) + Q_r\delta(s-s_r) + Q_{L_1}\delta(s-s_{L_1}). \tag{56}$$

7.3 Bernoulli equation

The Bernoulli equation, derived in 1738 by Swiss physicist Daniel Bernoulli [1], is a highly important equation used in aerodynamics and fluid dynamics to this day. We shall introduce and explain the Bernoulli equation in this section, as well as the assumptions we must make when using this equation. The equation shall be later used in §8.1 to determine the flow rate of liquid over three sharp-crested weirs of varying shapes.

The equation can be applied to airflow as well as liquid to determine properties of the air or fluid. Bernoulli's equation can be applied only in specific circumstances and each assumption has been explained below in terms of fluids.

Laminar flow. As seen in shallow water theory (§6), flow is streamlined and we have a hydrostatic pressure distribution.
Steady flow. As seen previously, this means the height, velocity and discharge are independent of time at any given point along the stream.
Inviscid flow. The fluid is frictionless, and as water is regarded as having no viscosity it is acceptable to assume this for river calculations.
Incompressible flow. The liquid remains at a constant density over s, which we can assume for liquids travelling with relatively low velocity.

Given these assumptions, we have the Bernoulli equation;

$$p + \frac{1}{2}\rho u^2 + \gamma z = \text{constant along streamline}. \tag{57}$$

Here p denotes the fluid pressure, ρ is the density of the fluid, z is the elevation above the channel bed and $\gamma = \rho g$ is the specific weight of the fluid [1].

8 Weirs

Weirs are hydraulic structures built along the width of rivers used to monitor flow and provide some flood defence. Weirs are placed within rivers to obstruct the usual flow and hold back a certain volume of water, known as the flood storage volume. Introducing a weir alters the velocity along the river channel, as the water stored slows the velocity of the incoming water from further upstream so much so that the water behind the weir becomes almost stationary. Then once the river height exceeds the weir height, it overflows and water continues to flow downstream.

The Bernoulli equation, introduced in §7.3, will be used to determine the discharge over three types of sharp-crested weirs - a rectangular weir, a v-notch weir and a trapezoidal weir in §8.1. Shallow water theory shall be used to determine the equation for discharge over a broad-crested weir in terms of river height in §8.2. Also using the Shallow Water Theory principles, an exact estimate of river height just before a weir for a given flow rate shall be derived in §8.2.2, which we will use in §9 when plotting river profiles. These principles will be applied to model the dynamics of the moor and canal in Wetropolis, which both involve weirs, in §8.3. Lastly, we define critical, subcritical and supercritical flow in §8.4 and explain when each occurs using the Froude number.

Before this, however, we define the two types of weirs - sharp-crested and broad-crested. The crest of a weir, also known as a nappe, signifies the upstream highest point of the weir, whereas the weir slope refers to the downstream element water flows down after reaching the crest. As suggested by its name, a sharp-crested weir has a sharp-edged flat plate spanning across a river, such that water flows over and immediately drops back into the stream. Whereas a broad crested weir has a smooth or flat horizontal weir plate that water travels over for a brief period before returning to the stream.

8.1 Sharp-crested weirs

In this subsection, the ideas and methods from Munson's 'Fundamentals of Fluid Mechanics' book [22] have been adapted to determine discharge over rectangular and v-notch sharp-crested weirs. Further calculations have been done to find an equation for discharge over a trapezoidal sharp-crested weir. To begin we introduce the diagram of a sharp-crested weir in figure 21 and the possible weir shapes in figure 22. These diagrams were created in Python (Appendix A.5) and annotated later with arrows and notation in Microsoft Word.

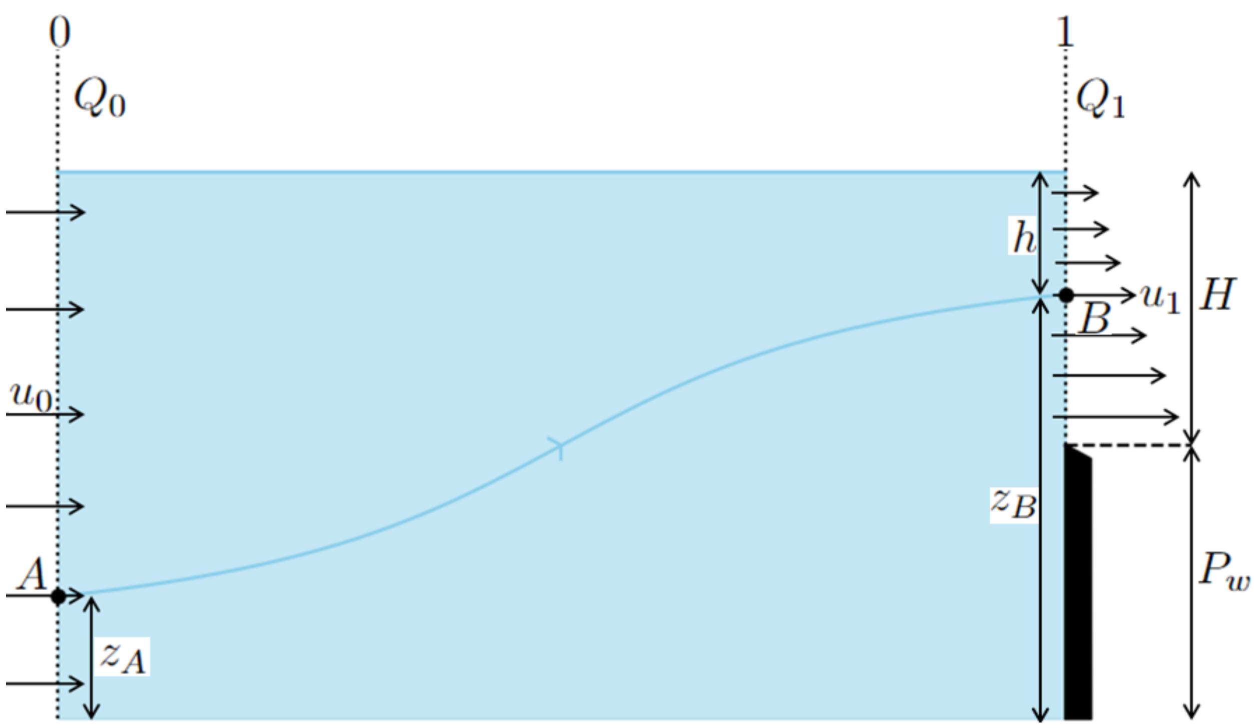


Figure 21: Sharp-crested weir diagram, following the flow of water from point A at cross-section 0, to point B at cross-section 1.

We shall examine the flow of liquid from point A to point B, where cross-section 0 is taken before the weir and cross-section 1 is taken at the weir's crest. Flow in cross-section 0 is unaffected by the introduction of the weir. The distance between point B and the water's surface is denoted by h. The weir height and the height of water above the weir, known as the weir head, are labelled on the right by P_w and H respectively. As we saw from the Bernoulli equation in §7.3, z denotes the height of fluid at a certain point. Therefore z_A and z_B denote the height of water at points A and B respectively. This weir has a sharp edge, small width and its length spans across the entire width of the river. Before we can derive the expression for discharge over the weir, we must state some assumptions that need to be made to use the Bernoulli equation. The discharge at points A and B in cross sections 0 and 1 are denoted by Q_0 and Q_1 respectively and the velocity by u_0 and u_1 respectively. The velocity at various points on each cross-section is shown by the horizontal lines passing through the two cross-sections.

Assumptions. To use the Bernoulli equation, we must suppose the required assumptions in §7.3 hold. This involves assuming flow is steady over this region, meaning h, u and Q no longer depend on time t and assuming atmospheric pressure at the weir, such that $p_B = 0$. Velocity at cross-section 0, u_0, remains constant as we have assumed uniform velocity, as indicated by the equal width velocity arrows. This means velocity remains the same at cross-section 0 no matter where we choose to place A along the cross-section. Whereas, cross-section 1 over the weir has a non-uniform velocity profile, shown by the varying arrow sizes. This means velocity over the weir is dependent on h, such that $u_1 = u_1(h)$.

Equating cross sections 0 and 1 from the diagram and taking $p_B = 0$, the Bernoulli equation from §7.3 (57) becomes

$$\frac{p_A}{\gamma} + \frac{u_1^2}{2g} + z_A = \frac{u_2^2}{2g} + z_B. \tag{58}$$

As h denotes the distance of point B below the surface, the height of point B, z_B, is rewritten in the equation as

$$\frac{p_A}{\gamma} + \frac{u_0^2}{2g} + z_A = \frac{u_1^2}{2g} + H + P_w - h. \tag{59}$$

Rearranging this equation gives an expression for the velocity at the weir,

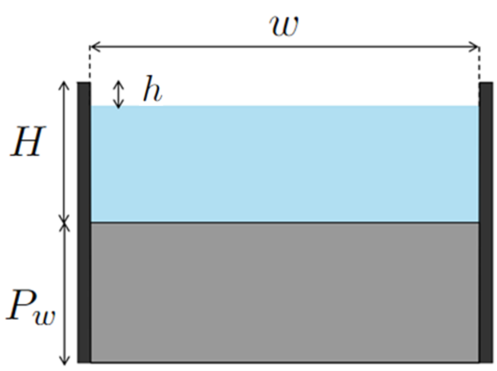
$$\frac{u_0^2}{2g} + H + P_w = \frac{u_1^2}{2g} + H + P_w - h \qquad \Longrightarrow \qquad u_1 = \sqrt{2g\left(h + \frac{u_0^2}{2g}\right)} \tag{60}$$

The discharge at cross-section 2 is found by taking the integral of u_2 with respect to the cross-sectional area A. We take $w(h)$ to be the function of the width of water at a certain cross-section, as indicated by figure 22 for different-shaped weirs. The equation for area can be rewritten in the integral such that at cross-section 1 the integral is given by,

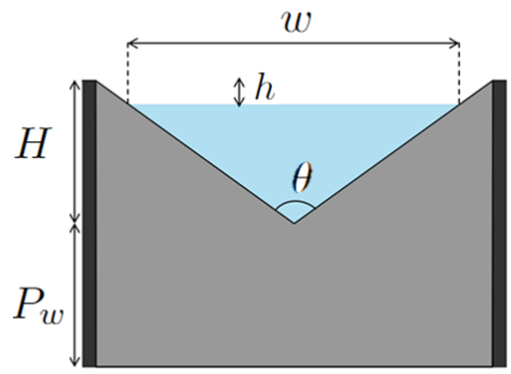
$$Q \quad = \quad \int_{(2)} u_1 \quad dA \quad = \quad \int_{h=0}^{h=H} u_1 w(h) \quad dh. \tag{61}$$

where the width is determined by the specific shape of the weir. A rectangular weir will have a constant width, w, but v-notched and trapezoidal weirs will have a h-dependent width, $w = w(h)$.

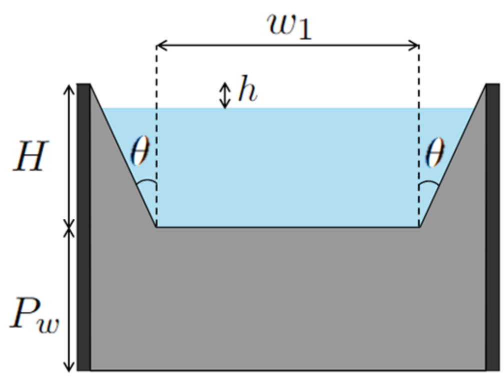
Figure 22 shows the three shapes of weir that we shall study. Here the black lines on each side denote the river walls and the grey region denotes the weir, while water flowing over the weir is shaded light blue. Each subfigure is labelled with the weir height, P_w, weir head, H, and the distance between the river walls height and the water surface height, h. The triangular and trapezoidal shaped weirs also include θ to denote angles within the shapes, which will be used to determine the width of these weirs.



(a) Rectangle



(b) V-notch



(c) Trapezoidal

Figure 22: Sharp-crested weir shapes.

Rectangular sharp-crested weir. For the rectangular weir in figure 22(a), we have $w(h) = w$ and (61) becomes

$$
\begin{aligned}
Q = \int_{h=0}^{h=H} w\sqrt{2g\left(h+\frac{u_0^2}{2g}\right)}\ dh \quad &\Longrightarrow \quad w\sqrt{2g}\int_{h=0}^{h=H}\sqrt{h+\frac{u_0^2}{2g}}\ dh \\
&\Longrightarrow \quad w\sqrt{2g}\left[\frac{2}{3}\left(h+\frac{u_0^2}{2g}\right)^{3/2}\right]_{h=0}^{h=H} \\
&\Longrightarrow \quad \frac{2}{3}w\sqrt{2g}\left[\left(H+\frac{u_0^2}{2g}\right)^{3/2}-\left(\frac{u_0^2}{2g}\right)^{3/2}\right].
\end{aligned} \tag{62}
$$

Since we often see in real-life situations that the weir head is much smaller than the weir height, $H \gg P_w$, we can assume a negligibly small velocity and set this equal to zero. Therefore, the discharge over a rectangular sharp crested weir is given by

$$
Q = \frac{2}{3}w\sqrt{2g}H^{3/2}. \tag{63}
$$

V-notch sharp-crested weir. Supposing we do not know the value of w from figure 22, taking equation (61) and applying it to the diagram for a v-notched sharp-crested weir in figure 22(b) gives us the cross-channel width along the weir. We calculate this trigonometrically to be

$$
w(h) = 2\tan\left(\frac{\theta}{2}\right)(H-h). \tag{64}
$$

Following on from (61) the flow rate is given by

$$
\begin{aligned}
Q =& 2\tan\left(\frac{\theta}{2}\right)\int_{h=0}^{h=H}(H-h)\sqrt{2g\left(h+\frac{u_0^2}{2g}\right)}\ dh \\
\Longrightarrow& 2\tan\left(\frac{\theta}{2}\right)\sqrt{2g}\left(H\int_{h=0}^{h=H}\sqrt{h+\frac{u_0^2}{2g}}\,dh-\int_{h=0}^{h=H}h\sqrt{h+\frac{u_0^2}{2g}}\,dh\right) \\
\Longrightarrow& 2\tan\left(\frac{\theta}{2}\right)\sqrt{2g}\left(\frac{2}{3}H^{\frac{5}{2}}-\frac{2}{5}H^{\frac{5}{2}}\right),
\end{aligned} \tag{65}
$$

where the first integral is determined by multiplying the integral solved in (63) by H and the second integral is found using integration by parts (see Appendix B.1). Therefore discharge over a sharp crested v-notch weir is given by

$$Q = \frac{8}{15} \tan\left(\frac{\theta}{2}\right) \sqrt{2g} H^{5/2}. \tag{66}$$

Trapezoidal sharp-crested weir. Similarly to the v-notch weir, calculating discharge for the trapezoidal weir in figure 22(c) also requires us to define the width in terms of h, which is given trigonometrically by

$$w(h) = w_1 + 2(H - h)\tan\theta, \tag{67}$$

where w_1 denotes the width along the flat region of the weir. The discharge equation for this weir can be found by combining the working of the past two discharge calculations, allowing for the adjustment of some constants. Therefore, the flow rate over a trapezoidal weir is given as follows

$$\begin{aligned} Q = &\int_{h=0}^{h=H} w_1 \sqrt{2g\left(h + \frac{u_0^2}{2g}\right)}\ dh + 2\tan\theta \int_{h=0}^{h=H} (H-h)\sqrt{2g\left(h + \frac{u_0^2}{2g}\right)}\ dh \\ =&\frac{2}{3} w_1 \sqrt{2g} H^{3/2} + \frac{8}{15} \tan\theta \sqrt{2g} H^{\frac{5}{2}} \\ =&\sqrt{2g} H^{\frac{3}{2}} \left(\frac{2}{3} w_1 + \frac{8}{15} \tan\theta H\right). \end{aligned} \tag{68}$$

The full derivation of discharge over a trapezoidal weir can be found in Appendix B.1.

8.2 Broad-crested weirs

In this subsection, we shall examine how flow rate changes over a broad crested weir by looking at a local case, where we focus on a small section of the river channel starting before the weir and up to some point over the weir where flow is critical. Here criticality occurs for some section of water travelling over the broad-crested weir, not necessarily over the entire length of the weir. Using the shallow water equations as defined in §6, an expression for the river height at this critical point shall be determined. From this height, we find an equation for discharge over the weir that can be applied to all broad-crested weirs with constant width. This equation for discharge shall be used in §9, where the discharge will be specified and used to determine the river height. A refined estimate of the discharge equation over a broad-crested weir shall be given, as proposed by Acker [9]. We begin by introducing a diagram used to facilitate this derivation, explaining its notation and stating some assumptions we must make.

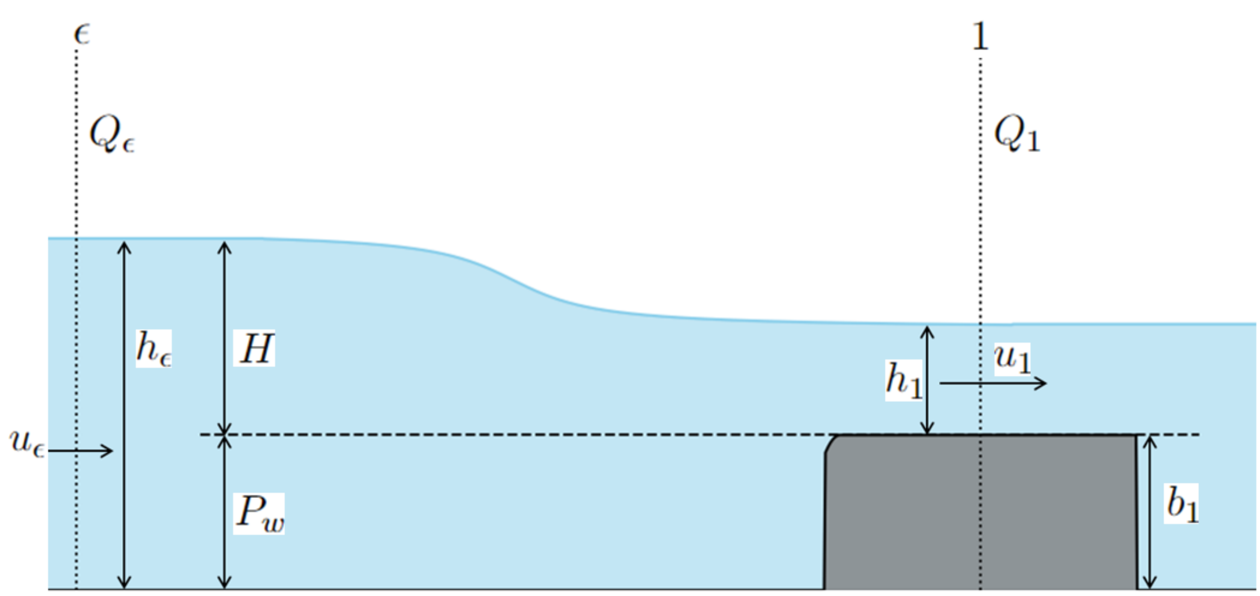


Figure 23: Diagram of a river channel with the introduction of a weir (new with shading, no water just after weir? and change to h_ϵ).

The diagram of the broad crested weir is shown in figure 23, where most notation used is the same as given in §8.1 for the sharp-crested weir. H again denotes the head of water above the top of the weir and P_w the weir height, such that $h_\epsilon = H + P_w$. Note that the bed height at cross-section 1 can also be referred to as b_1, such that $b_1 = P_w$. We have cross-section ϵ, taken before the weir, and cross-section 1, taken at some point over the weir where flow is critical. The respective velocities and flow rates at cross sections ϵ and 1 are denoted by u_ϵ, Q_ϵ, u_1 and Q_1. Before applying the shallow water equations, we must first state some assumptions that will be made.

Assumptions. Similarly to the derivation for sharp-crested weirs, we must assume that h, u and Q no longer depend on time t meaning $\partial_t = 0$ for all three variables. As outlined in the assumptions of shallow water theory, velocity is assumed to be uniform at any point before the weir at cross-section ϵ and non-uniform over the weir at cross-section 1. In this case, we are taking a constant channel slope, such that $\partial_s b(s) = 0$ and fluid flows horizontally over the weir. The width of the river is also assumed to remain constant over s. In addition, we assume the width of the weir, w, is independent of s and given by a constant, such that $A(h) = hw$. Note that we shall calculate the discharge at unit width by referring to the area of the weir as $A(h) = h$. The weir width, w shall be reinstated once we have an equation for discharge per unit width.

Taking these assumptions, as well as the remaining shallow water theory assumptions, the Saint-Venant equations as defined in §6 become

$$\partial_s(hu) = 0 \tag{69}$$

$$g\partial_s h + u\partial_s u = -g\partial_s b. \tag{70}$$

The updated continuity equation (69) tells us that discharge is independent of s, therefore we can write $Q_\epsilon = Q_1 = c \in \mathbb{R}$. Further, we can rewrite the momentum equation as

$$\begin{aligned} &u\partial_s u + g\partial_s(h+b) = 0 \\ \implies &\partial_s\left(\frac{u^2}{2} + g(h+b)\right) = 0. \end{aligned} \tag{71}$$

As the above equation is independent of s, this holds both for $u_\epsilon, h_\epsilon, b_\epsilon$ and for u_1, h_1, b_1. We can equate these such that

$$\begin{aligned} &\frac{{u_\epsilon}^2}{2} + g(h_\epsilon + b_0) = \frac{{u_1}^2}{2} + g(h_1 + b_1) \\ \implies &\frac{{u_\epsilon}^2}{2} + gh_\epsilon = \frac{{u_1}^2}{2} + g(h_1 + b_1), \end{aligned} \tag{72}$$

which follows since $b_\epsilon = 0$. Using the substitution, $u^2 = Q^2/h^2$ in (71), we have

$$\begin{aligned} &\partial_s\left(\frac{Q^2}{2h^2} + g(h+b)\right) = 0 \\ \implies &-\frac{Q^2}{h^3}\partial_s h + g\partial_s h = -g\partial_s b \\ \implies &\left(-\frac{u^2}{h} + g\right)\partial_s h = -g\partial_s b. \end{aligned} \tag{73}$$

where the second line follows from applying the quotient rule to $Q^2/2h^2$. As we are assuming a constant slope, $\partial_s b = 0$, it is required that either $\partial_s h = 0$ or $-\frac{u^2}{h} + g = 0$ and as we have assumed a uniform velocity profile upstream of the weir and $Q_\epsilon = c \in \mathbb{R}$, we know that $A(h)$ must be a constant also, therefore $\partial_s h = 0$. Whereas over the crest of the weir, we have a non-uniform velocity profile with a constant discharge, therefore $\partial_s h \neq 0$ and instead we have the following

$$-\frac{{u_1}^2}{h_1} + g = 0 \implies u_1 = \sqrt{gh_1}, \tag{74}$$

i.e. the flow here is critical (see §8.4). Thus, the discharge at the weir is

$$Q_\epsilon = Q_1 = h_1 u_1 = \sqrt{gh_1}h_1 = \sqrt{g}{h_1}^{3/2}. \tag{75}$$

Although this is perfectly suitable for calculating discharge when the flow is critical (see §8.4), an alternative equation for discharge is required at all other regions, which we shall derive by substituting the equation for u_1 into (72), such that

$$\begin{aligned} \frac{{u_\epsilon}^2}{2} + gh_\epsilon &= \frac{(\sqrt{gh_1})^2}{2} + g(h_1 + b_1) \\ \implies \frac{{u_\epsilon}^2}{2} + gh_\epsilon &= \frac{3}{2}gh_1 + gb_1. \end{aligned} \tag{76}$$

Let us assume we are examining a region of low velocity, u_0 and can ignore this value in the expression. Note that an alternative expression for discharge involving this term shall be shown in §8.2.2. However in this case we rearrange (76) to give the following expression for h_1;

$$\begin{aligned} g(h_\epsilon - b_1) &= \frac{3}{2}gh_1 \\ \implies h_1 &= \frac{2}{3}(h_\epsilon - b_1). \end{aligned} \tag{77}$$

Based on this expression for h_1 and the equation of discharge at cross-section 1 (75), discharge can be expressed as

$$\begin{aligned} Q &= \sqrt{g}\left(\frac{2}{3}(h_\epsilon - b_1)\right)^{3/2} \\ &= \sqrt{g}\left(\frac{2}{3}\right)^{3/2}(h_\epsilon - b_1)^{3/2}. \end{aligned} \tag{78}$$

This equation for Q is what we shall use to find the flow rate along different shaped weirs, which can be found while the velocity u_1 is unknown. For weir head $H = h_0 - b_1$, the discharge at different sections along a river profile per unit width is

$$Q = \sqrt{g}\left(\frac{2}{3}\right)^{3/2} H^{3/2} = \sqrt{g}\,C_f H^{3/2}. \tag{79}$$

As we have derived the equation for discharge per unit width, we must now substitute in the expression of width to give the full equation for discharge over a broad crested weir. Using $A(h) = hw$, the equation for discharge is given by

$$Q = \sqrt{g}\,C_f\,w\,H^{3/2} \tag{80}$$

for the dimensionless coefficient $C_f = (\frac{2}{3})^{\frac{3}{2}}$. As $H = h_\epsilon - b_1$ is to the power of 3/2, we require $h_\epsilon > b_1$. This is the maximum discharge rate, which occurs at the point when flow is critical over the weir. At any other point, flow is subcritical (see §8.4) and it is recommended that we include another constant within the equation to give a more realistic estimate of flow rate. This will depend upon the weir height, P_w, crest length and width, L_w and w, and total weir head, H [9].

When examining subcritical flow, the discharge equation becomes

$$Q = C_D\sqrt{g}\,C_f\,w\,H^{3/2} \tag{81}$$

for the dimensionless discharge coefficient, C_D. Many proposals for calculating this coefficient have been made, for instance through re-analysis of experimental data provided by Acker the coefficient is estimated as $C_D = 0.95$ [9].

8.2.1 Determining the exact value of h_ϵ

Alternatively, we may choose to keep u_ϵ in equation (76) to give the full expression for h_1. It is from this expression that we yield an exact value for h_ϵ, which can be used regardless of the size of u_ϵ. To begin, we rewrite (76) using $u_\epsilon = Q_\epsilon/A(h_\epsilon) = Q_\epsilon/h_\epsilon\alpha$ and rearrange this to put the equation in terms of h_1, i.e.

$$\begin{aligned} \frac{{Q_\epsilon}^2}{2{h_\epsilon}^2w^2} + gh_\epsilon &= \frac{3}{2}gh_1 + gb_1 \\ \implies h_1 &= \frac{{Q_\epsilon}^2}{3g{h_\epsilon}^2w^2} + \frac{2}{3}(h_\epsilon - b_1). \end{aligned} \tag{82}$$

Next, we substitute use this expression for h_1 into the discharge equation per unit width, which we have determined by applying the criticality condition that holds at cross-section 1. Multiplying by width, w, we have the complete expression for Q:

$$\begin{aligned} Q &= \sqrt{g}w{h_1}^{3/2} \\ \implies Q &= \sqrt{g}w\left(\frac{Q^2}{3g{h_\epsilon}^2w^2} + \frac{2}{3}(h_\epsilon - b_1)\right)^{3/2}, \end{aligned} \tag{83}$$

where h_ϵ denotes the height before the weir. As you can see, we have Q on both sides of the equation and given we have values for Q, b_1 and g, we rearrange this equation and solve for h_ϵ. As shown below, (83) becomes

$$\begin{aligned} \left(\frac{Q}{\sqrt{g}w}\right)^{2/3} &= \frac{Q^2}{3g{h_\epsilon}^2w^2} + \frac{2}{3}(h_\epsilon - b_1) \\ \implies \frac{Q^{2/3}}{g^{1/3}w^{2/3}}{h_\epsilon}^2 &= \frac{Q^2}{3gw^2} + \frac{2}{3}({h_\epsilon}^3 - b_1{h_\epsilon}^2) \\ \implies \frac{2}{3}{h_\epsilon}^3 - \left(\frac{Q^{2/3}}{g^{1/3}w^{2/3}} + \frac{2}{3}b_1\right){h_\epsilon}^2 &+ \frac{Q^2}{3gw^2} = 0. \end{aligned} \tag{84}$$

Implementing this equation into the Python worksheet with specific values for Q, g, $b_1 = P_w$ and solving for h_ϵ using the 'fsolve' function in Python, will yield the exact solution of the height before the weir. This shall be done when plotting the river profile over a weir in §9. In comparison, the original equation for Q (80) that did not include velocity can be rearranged to give the following simpler expression for h_ϵ; $\widehat{h_\epsilon}$,

$$\widehat{h_\epsilon} = \frac{3}{2}\left(\frac{Q}{\sqrt{g}\alpha}\right)^{2/3} + P_w. \tag{85}$$

This equation shall also be used for plotting the river profile over a weir in section §9.

8.3 Weirs in Wetropolis

There are a total of four structures in Wetropolis that are modelled as weirs - one located at the reservoir and the remaining three points located along the canal [8]. The reservoir is a rectangular box with dimension h_{res}xw_{res}xL_{res}, denoting the water level, width and length of the reservoir respectively. As seen in figure 5 of the tabletop model, the canal flows alongside the river for the entire length of the river, with berms separating the three sections.

Reservoir. Rainfall enters the reservoir through a pipe with multiple equidistant small holes along its length. Water then leaves the reservoir through an overflow pipe, which is modelled by a straight weir [8]. For simplicity, we do not consider the possibility of the reservoir overflowing in this model. Instead, discharge through the overflow is modelled as flow over a broad-crested weir. This means we can use the original equation for discharge over a broad crested weir (80), such that

$$Q_{res} = C_f\sqrt{g}w_{res}\max(h_{res} - P_w)^{3/2} \tag{86}$$

is the discharge at the reservoir. The final term denotes the maximum height of water over the weir, which is the equivalent of the total water head H_{res}. The water level in the reservoir, $h_{res}(t)$, is dependent upon time. We model the change in volume due to inflow from the rainfall pipe and outflow via the weir as

$$w_{res}L_{res}\frac{dh_{res}}{dt} = w_{res}L_{res}R_{res}(t) - Q_{res}. \tag{87}$$

Canal sections. Each of the three canal sections has its own time-dependent water level, $h_{ic}(t)$, weir height, P_{iw}, and flowrate, Q_{ic}, for $i = 1, 2, 3$. The first canal section is the section located closest to the city, while the second is the next section further upstream and the third is the section at the beginning of the model, starting at $s = 0$. The first canal has inflow from the canal-2, Q_{2c} and outflow of water into the city, Q_{1c}, thus the change in volume is modelled by

$$w_c(L_{1c} - L_{2c})\frac{dh_{1c}}{dt} = Q_{2c} - Q_{1c}. \tag{88}$$

Moving further upstream, water flows into canal-2 from canal-3, Q_{3c}, and leaves out of canal-2 into canal-1, Q_{2c}, such that flow is governed by

$$w_c(L_{2c} - L_{3c})\frac{dh_{2c}}{dt} = Q_{3c} - Q_{2c}. \tag{89}$$

Lastly, water flows into canal-3 via the moor, where rainfall enters through another pipe with multiple holes. Water then leaves canal-3 via a weir in canal-2, Q_{3c} where the change in volume is modelled as

$$w_cL_{3c}\frac{dh_{3c}}{dt} = \gamma Q_{tm} - Q_{3c}. \tag{90}$$

Here Q_{tm} is the flow rate of water flowing out of the moor and γ is the fraction of moor water entering the river [8]. Note that flow at the weir of each canal section is again modelled as flow over a broad-crested weir [8], therefore the flowrate is given by

$$Q_{ic} = C_f\sqrt{g}w_c \max(h_{ic} - P_{iw})^{3/2} \tag{91}$$

for canal $i = [1, 2, 3]$.

8.4 Criticality conditions over weirs

The Froude number is used for scaling open channel flows [9]. It helps us to determine whether the flow at some point within a channel is critical, subcritical or supercritical. Subcritical flow concerns slow and tranquil flow, whereas supercritical flow is fast and rapid. Critical flow takes place between these, occurring at the exact point where flow depth equals critical depth. Critical depth is defined as the flow depth for which the mean specific energy is minimum [9], occurring at the point where velocity and gravitational forces are balanced. The location of this point along a channel depends on multiple factors, including the shape, slope and roughness of the channel.

The Froude number at a certain point on a cross-section is given by

$$Fr = \frac{u}{\sqrt{gh}} \tag{92}$$

for flow depth h, acceleration due to gravity constant g and velocity u height. We find that critical flow conditions are reached when $Fr = 1$, when $Fr < 1$ flow is subcritical and when $Fr > 1$ flow is supercritical.

9 Plotting a river profile using shallow water theory

We shall investigate the influence of a weir on the river profile, focusing on the Kirkstall Valley weir along the River Aire. The open channel flow can be modelled as uniform channel flow, where the height remains constant across the channel, such that $h = h_0$ for all s. However, the introduction of the weir means the flow becomes non-uniform at some points along s and the river height becomes variable depending on s. Let us suppose we have a rectangular broad-crested weir, as shown in figure 23 in §8.2, a gently sloping wall with height P_w at its peak emerging from the river bed. Suppose also that the river bed lies on a shallow uniform downwards slope at an angle of $\alpha > 0$ with gradient $\tan\alpha > 0$. Further, we shall assume for now that the channel is rectangular, meaning its area is given by $A = hw$, and that the river width, w, is constant with no dependence on s. Using these assumptions, we approximate the height of the river $h(s)$ along s by the following equation,

$$h(s) = \begin{cases} h_0 & \text{if } s \leq s_1 \\ h_0 + \tan\alpha(s - s_1) & \text{if } s_1 \leq s \leq s_x, \end{cases} \tag{93}$$

where h_0 is defined as the upstream water depth, taken much further upstream before any influence of the weir. Here s_1 denotes the point at which the upstream uniform channel flow meets the downstream non-uniform channel flow. Further, s_x denotes the location of the weir, meaning the river height above the weir, H, is given by $H = h_0 + \tan\alpha(s_x - s_1) - P_w$.

The location of s_1 will be seen visually in our example in the next section as the point where the two straight lines of different gradients meet. Our aim in this section will be to use the shallow water equations to model river height, such that the height will be a smooth function over s, rather than the piecewise function above. Firstly, however, we shall continue to model the river height approximately using equation (93) and will use shallow water theory to determine values for h_0 and h_e, i.e. the height before the weir.

We shall also be using a narrow yet realistic function for the broad-crested weir than the longer weir used in the diagram in §8.2. The weir will have a narrower width and parabolic shape from the side angle, which will smoothly connect to the sloped river bed using a piecewise function. We require this smooth transition between the river bed and for there not be an extended horizontal section over the weir, to enable the shallow water equations to give a smooth function for river height. Note that choosing a rounded weir will alter our river height function at surrounding regions to the crest of the weir, s_x, with the size of the region affected depending on the decided scale of the weir. Albeit a vertical line of height P_w located at the crest of the weir will be included within the plot for reference. Python will be used to facilitate these calculations and produce our plots, the full script can be found in Appendix A.6.

The general method for approximating river height over a river section taken from further upstream up to the point of critical flow on the weir will be outlined in eight steps in §9.1. The first five steps will lead us to an initial piecewise plot of river height and the final three steps will provide the method needed to produce a plot of river height based on backwards integration. In §9.2, the methods provided in 9.1 will be applied to produce river profile plots for a certain section of the river Aire located at Kirkstall Valley. We will also examine criticality conditions over the weir in §9.2.1, as introduced in §8.3. In §9.2.2, an alternative approximation for the river height at the weir will be calculated using the equation provided in §8.2.2. This height will be used to plot a river profile and compared with the original approximation, which is suitable for small velocities only. Lastly, in §9.3 we will demonstrate how and why the river height approximation found using backwards integration fails for steep river bed slopes.

9.1 Method

9.1.1 Find h_0 and h_ϵ

A plot of this river profile shall be produced using Python, within which the location of s_x along s will be chosen as a constant value, as this means that the river bed remains in the same position when changing the value of discharge, allowing for easier comparison of river profiles.

To plot h_0, the initial river height further upstream at all points previous to s_1, we use Manning's relation (53) in the equation for discharge. As we have assumed a rectangular channel, the expression for discharge in terms of $h(s)$ is given via Manning's relation by

$$Q = wh_0 \left(\frac{\sqrt{-g\partial_s b} \;\; R(h_0)^{2/3}}{C_m} \right), \tag{94}$$

where $-\partial_s b = \tan\alpha$ denotes the gradient of the river profile. Using the definition of hydraulic radius, $R(h)$, for our assumed rectangular channel, (94) can be rearranged to give

$$h_0 w \left(\frac{h_0 w}{2h_0 + w} \right)^{2/3} \frac{\sqrt{-g\partial_s b}}{C_m} - Q = 0. \tag{95}$$

After further manipulation, the equation can be written as an ordinary differential equation, which we shall solve using the 'fsolve' function for defined constants $Q, C_m, w, g, \partial_s b$ in Python to find an estimate for h_0,

$${h_0}^{5/2} - \frac{2h_0}{w^2} \left(\frac{QC_m}{\sqrt{-g\partial_s b}} \right)^{3/2} + \left(\frac{QC_m}{\sqrt{-g\partial_s b}} \right)^{3/2} = 0. \tag{96}$$

Secondly, to determine the height of water before the weir begins, h_ϵ, we shall calculate the height of the river head, as we know $h_\epsilon = H + P_w$. From §8.2, we know the criticality condition holds at some point over the weir, meaning we can replace u with the criticality condition in the equation for discharge. As Q, w and g will be constants given in our example, the equation for discharge on a rectangular weir (80) can be rearranged in this case to give a value for H,

$$H = \left(\frac{Q_0}{\sqrt{g} w C_f} \right)^{2/3}, \tag{97}$$

for $C_f = \left(\frac{2}{3}\right)^{3/2}$. Therefore, depending on the chosen weir height, P_w, the value of h_ϵ is given by

$$h_\epsilon = \left(\frac{Q_0}{\sqrt{g} w C_f} \right)^{2/3} + P_w. \tag{98}$$

It was previously assumed that the velocity was very small and therefore was discounted when calculating h_ϵ in §8.2. However, in §8.2.1 an exact expression for h_ϵ was calculated which took velocity into account. The results of this means that we can find the exact value of h_ϵ in our case by solving equation (84) in Python using 'fsolve' for our chosen constants.

Note that we can also determine h_1, which is the height of water at the weir at the point where flow is critical, given using (80) and replacing H with $\frac{3h_1}{2}$, such that

$$h_1 = \left(\frac{Q_0}{\sqrt{g} w} \right)^{2/3}. \tag{99}$$

9.1.2 Combine the shallow water equations, assuming steady state.

Next we shall begin to derive our alternative river height function by taking the two Saint-Venant equations, which are slightly altered as we are assuming steady state, and combining these to give an expression for the partial derivative of the area over s. The continuity equation and momentum equation given in §6 therefore reduce to the following,

$$\partial_s(Au) = 0 \quad \text{and} \quad u\partial_s u + g\partial_s h = -g \left(\partial_s b + \frac{|u|uC_m^2}{R(h)^{4/3}} \right). \tag{100}$$

As discharge will remain the same regardless of s, the continuity equation reduces to $0 = 0$ and as $Q = Au$, we can express u as a constant in terms of $Q \in \mathbb{R}$ and A. Assuming $u > 0$ and substituting in this expression for u, the momentum equation becomes

$$\begin{aligned} &\left(\frac{-Q^2}{A^3}\partial_s A + g\frac{\partial h}{\partial A}\right)\partial_s A = -g\left(\partial_s b + \frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}}\right) \\ \implies &\frac{1}{A}\left(\frac{-Q^2}{A^2}\partial_s A + gA\frac{\partial h}{\partial A}\right)\partial_s A = -g\left(\partial_s b + \frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}}\right). \end{aligned} \tag{101}$$

We now have an expression for the partial derivative of the area, given by

$$\partial_s A = \frac{-gA\left(\partial_s b + \frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}}\right)}{gA\frac{\partial h}{\partial A} - \frac{Q^2}{A^2}} = \frac{T(A)}{B(A)}. \tag{102}$$

Ultimately, we shall use this expression to determine the cross-sectional area of the channel for a large number of cross-sections along the channel. Then dividing these areas by the channel width, we shall plot the river height as a function of s, which we can compare to our original river height approximation (93). Note that for simplicity, we shall refer to (102) as $T(A)/B(A)$ for the remainder of this section, unless stated otherwise.

9.1.3 Define an equation for the river bed, $b(s)$

We have the original river bed, determined by a given slope $\tan\alpha$, and a parabolic weir, with the shape of a negative x^2 curve centered at s_x. The piecewise function of the new channel bed, $b(s)$, is defined as follows,

$$b(s) = \begin{cases} \tan\alpha(s - s_x) & \text{if } s \le s_a \text{ or } s_b \le s \\ \frac{-(s-s_x)^2}{\sigma} + P_w & \text{if } s_a \le s \le s_b, \end{cases} \tag{103}$$

where the location of the weir s_x decides the y-intercept of the river bed as $\tan\alpha * s_x$. Here $\sigma \in \mathbb{R}$ denotes the scale of the weir, which can vary to give a wider or narrower shape to the weir. Also, s_a and s_b define the two points at which the weir and river slope meet, s_a is the value of s at the first intersection of these two lines and s_b at the second. To plot the river height, we will require a function that adds together the height of the river and the height of the river bed along s. For this we shall exclude the weir function and focus only on the river bed slope, due to the weir's narrowness we shall for now its influence on river height over the weir is negligible. Therefore, adding together (93) and (103) using the equation for river slope only, the river height relative to the x-axis is given by the function

$$h(s) + b(s) = \begin{cases} h_0 + \tan\alpha(s - s_x) & \text{if } s \le s_1, \\ h_0 + \tan\alpha(s_1 - s_x) & \text{if } s_1 \le s \le s_x. \end{cases} \tag{104}$$

Also note that for this function $b(s)$ (103), $\partial_s A = T(A)/B(A)$ is now defined by the following function,

$$\partial_s A = \begin{cases} \dfrac{-gA\left(\tan\alpha + \frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}}\right)}{\frac{gA}{w} - \frac{Q^2}{A^2}} & \text{if } s \le s_a \text{ or } s_b \le s, \\ \dfrac{-gA\left(\frac{-2(s-s_x)}{\sigma} + \frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}}\right)}{\frac{gA}{w} - \frac{Q^2}{A^2}} & \text{if } s_a \le s \le s_b. \end{cases} \tag{105}$$

9.1.4 Set $dA/ds = 0$ and find the area at critical flow, A^*.

Equation (102) is finite everywhere along s except at the critical point $s = s_w$, which occurs when $\partial_s A$ equals zero. We find the cross-sectional area of the river bed at the point $s = s_w$ by setting $B(A) = 0$ and solving for A, which can be seen below

$$gA\frac{\partial h}{\partial A} - \frac{Q^2}{A^2} = 0 \implies \frac{gA}{w} = \frac{Q^2}{A^2} \implies A^* = \sqrt[3]{\frac{Q^2 w}{g}}, \tag{106}$$

where we have expressed h in terms of A, as $h = A/w$. Setting $T(A)$ to zero we find the slope of the channel, $\partial_s b$, at $s = s_w$ is given by

$$-gA\left(\partial_s b + \frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}}\right) = 0 \implies \partial_s b = -\frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}}. \tag{107}$$

Therefore, at $s = s_w$ the slope of the channel can be expressed as a constant, where we substitute in $A = A^*$ as determined by our chosen constants.

9.1.5 Equate $\partial_s b$ with the partial derivative of $b(s)$ and solve numerically to find $s = s_w$.

Given our defined function for $b(s)$ (103), the partial derivative of $b(s)$ is given by

$$\partial_s b(s) = \begin{cases} \tan\alpha & \text{if } s \le s_a \text{ or } s_b \le s, \\ \frac{-2(s-s_x)}{\sigma} & \text{if } s_a \le s \le s_b. \end{cases} \tag{108}$$

Thus, equating the two partial derivatives at the weir, we find

$$\frac{-2(s-s_x)}{\sigma} = -\frac{Q^2 C_m^2}{A^2 R(h(A))^{4/3}} \implies s = \frac{Q^2 C_m^2 \sigma}{2A^2 R(h(A))^{4/3}} + s_x \tag{109}$$

defines the point $s = s_w$ along the channel slope.

9.1.6 Make a mesh from s_w to the beginning of the weir.

Next we take 20 points, s_i, equally spaced along s such that $s_i \in [s_a, s_w]$ and we integrate $\partial_s A$ over this boundary, i.e.

$$\int_{s_i}^{s_w} \frac{-(s-s_x)^2}{\sigma} + P_w \quad ds \tag{110}$$

using the 'spi.quad' function, repeating this calculation for each s_i using a for loop in Python. We can plot this integral over s and extend this further to plot the integral of the whole of $b(s)$ over s, as seen for our specific example in the next section.

9.1.7 Check h^* is where you expect it to be on the river plot.

We first find the estimate for h^* using our equation for A^* given by the criticality condition. Using the usual equation for area we find

$$h^* = \frac{A^*}{w} = \sqrt[3]{\frac{Q^2}{gw^2}}. \tag{111}$$

Note that this is equivalent to h_1, which was derived in §8 when looking at a rectangular channel, meaning the water at cross-section $s = s_w$ is in critical flow. Adding this value of h^* to our river plot at the point $s = s_w$ will help to determine if this is where it should be, which is somewhere between the weir and the head of the river. This is found by adding together h^* and the weir height at $s = s_w$, given by $b(s = s_w)$, i.e.

$$h^* + b(s_w) \quad = \quad h^* + \frac{(s_w - s_x)^2}{\sigma} + P_w. \tag{112}$$

9.1.8 Find the area at different cross sections along the river profile by integrating backwards.

We have found our equation for $\partial_s A$ and next we want to use this to determine the area of the river profile at different cross sections along s. Intuitively, we would integrate $\partial_s A$ to find the values for A, however as $T(A)/B(A)$ is a complicated function of A that cannot be integrated easily, instead we use an iterative formula to find the area at different cross sections starting from after the weir at $s = s_w$. Once we have the area at these cross sections, we can divide each area by the channel width w to give the height along s, which we then add to our plot over river height against s. The iterative equation for the area at cross-section j across the river is given by

$$A_j = A_{j+1} - \triangle s \frac{T(A_{j+1}, s_{j+1})}{B(A_{j+1}, s_{j+1})} \tag{113}$$

where we have taken $N_s + 1$ points from the start of the river profile up to s_w, such that $j = N_s, ..., 0$. Here the s_j are $N_s + 1$ points that are equally spaced over s, from s_w to the start of the river channel s_0 and $\triangle s = (s_w - s_0)/N_s$ denotes the change over s. To begin the iteration, we take the initial value for A as the area at s_w, which from the previous steps is $A_{N_s} = A^*$. For this first estimate, we cannot use the proposed iterative formula (113) as $T(A)/B(A)$ blows up at this point for $A = A^*$, as we originally determined A* by setting $B(A) = 0$. Instead we use an approximation of $T(A)/B(A)$ via L'Hopital's rule and evaluate this at $A = A^*$, which means the first iteration is given as follows,

$$A_{N_s-1} = A^* + \triangle s \frac{T'(A^*, s_w)}{B'(A^*, s_w)}. \tag{114}$$

For the remaining points along s, $A_{N_s-2}, ..., A_0$ shall be determined using (113). Given we have found the cross-sectional area of the river profile at $N_s + 1$ points along s found by integrating backwards, we may take these values and multiply by the change in s, $\triangle s$, to give an approximation of the area under the curve i.e. the volume of water in the river. This is the method we shall use to determine the flood storage volume created by the introduction of a weir, which we shall explore further in the §10 with our specific example.

Alternatively, we may choose to approximate the area at different cross sections using forward integration, taking an initial estimate of $A_0 = wh_0$ from the start of the river profile. We find subsequent areas of cross sections along s up to N_s using the equation

$$A_{j+1} = A_j + \triangle s \frac{T(A_j, s_j)}{B(A_j, s_j)}. \tag{115}$$

However, as we shall see this in the next section this causes problems further down the line.

9.2 Plotting a river profile - River Aire at Kirkstall Valley

The River Aire is 148km in length [2] and flows from the Yorkshire Dales near Malham [2] in North Yorkshire until it forms a single channel with the River Ouse at Airmyn near Goole [2] in the East Riding of Yorkshire. Here we shall demonstrate our workings in a 1.097 km stretch of the river, located between Kirkstall Bridge and the Kirkstall Valley Nature Reserve Weir in Leeds, with photographs of the two locations in figure 24 and these points starred on a map in figure 25.

(a) Kirkstall Bridge

(b) Kirkstall Valley Weir

Figure 24: Photographs of the two locations between which we will examine the river profile, taken by myself.

To create a river profile that is true to life and similar to what we see along a river, choosing appropriate values for the constants involved is of great importance. Therefore using the Leeds topological map (available here), which gives the elevation at any given point on the map of Leeds, we shall calculate the river slope, $-\tan\alpha$, with a great deal of accuracy. We do this by taking

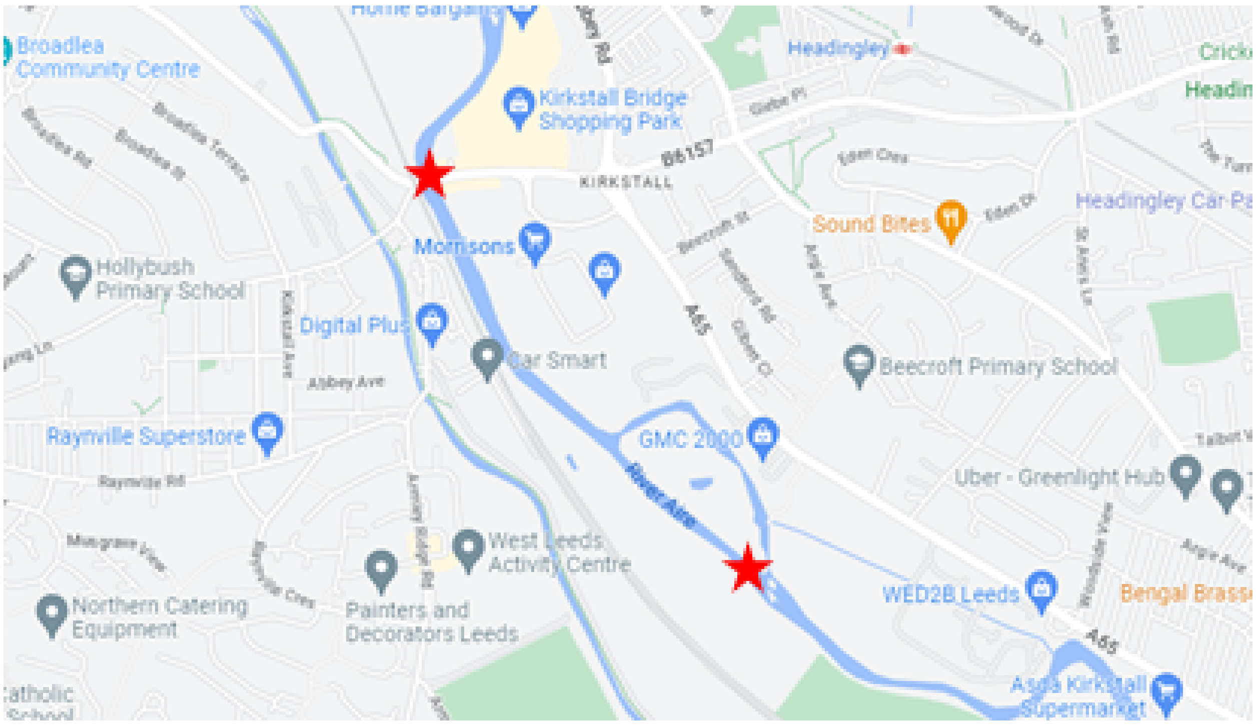


Figure 25: Map of Kirkstall, Leeds and surrounding areas, with the location of Kirkstall Bridge (left) and Kirkstall Valley Weir (right) starred.

the river elevation at these two locations and measuring the distance between them, and since the elevation is 37 metres at Kirkstall Bridge and 34 metres at Kirkstall Valley Weir, and the river length between these two locations is roughly 1097 metres, the river slope is $-\tan\alpha = \partial_s b = -93/34000 \approx -0.0027$ for $s < s_a$. Here s_a denotes the first point where the weir and river bed meet, and s_b, the second point where the weir and river bed meet, shall be defined numerically in step 3.

Our first assumption we shall make with this specific river profile is that the slope remains uniform over s. The width of the river is also found from the map to be roughly $w = 25.8$m, on average and as it can be seen on the map, the width stays rather uniform between the two stars, therefore we shall also assume that the width remains uniform along s. Next, the value of the Manning coefficient is $C_m = 0.03$, chosen from table 6 as the terrain at this area of the River Aire matches the description of a 'clean and straight natural channel', which we see in figure 24(b) that the profile matches this description. Further, a reasonable constant for the discharge must be selected, which we can take an informed estimate for using the hydrograph in figure 9. Since this hydrograph is taken from Armley monitoring station, a location not far from the area we are focusing on and along the same river, we may take a low value of discharge recorded here, before the effects of the storm were felt. In this case, $Q = 100\text{m}^3/\text{s}$ seems a reasonable estimate and for the weir height, we shall choose $P_w = 2$ m. This value for P_w is chosen because, as we shall see, this is a small height above the original height of the stream, h_0, without being a much greater height than the stream. If we chose a P_w that is a lot higher than h_0 it would cause the water to 'back up' for a very long distance along s, therefore for simplicity we shall use $P_w = 2$ metres. The constants and some previously defined constants that we will use have been summarised in table 7.

Table 7: Values of constants

Variable	Value
Q_0	$100\ \text{m}^3/\text{s}$
w	$\frac{800}{31} \approx 25.8$ m
$\tan\alpha$	$\frac{-93}{34000} \approx -0.0027$
σ	200
P_w	2 m
C_m	$0.03\ \text{s}/\text{m}^{1/3}$
g	9.81 m/s
C_f	$\left(\frac{2}{3}\right)^{3/2}$

Step 1. Following the first step outlined in §9.1, we use our constants and the equations in this section to find the estimates of h_0, H, h_ϵ, h_1. We find that $h_0 = 1.70$ metres, $h_1 = 1.15$ m $h_\epsilon = 3.73$ m and as $P_w = 2$ m, we have $H = 1.73$ m. Note that in this section the calculated values will be accurate up to three significant figures, unless stated otherwise.

Step 2. We assume that the channel shape here is rectangular, such that $A = wh$ and our equation for T(A)/B(A) (102) becomes

$$\partial_s A = \frac{-gA\left(\partial_s b + \frac{Q^2 {C_m}^2 \left(\frac{2A}{w}+w\right)^{4/3}}{A^{10/3}}\right)}{\frac{gA}{w} - \frac{Q^2}{A^2}} = \frac{T(A)}{B(A)}. \tag{116}$$

Step 3. To plot the river bed function, $b(s)$, we first must determine the location of the weir along the river bed by selecting a constant for the crest of the weir s_x. Let us take $s_x = 1100$, as this is roughly the length of the channel we are studying. Despite this, the value we choose here is not of great importance, as we only use this value as a point of reference that can be altered. Instead, we are focusing on the distance between the crest of the weir, s_x, and the point where upstream and downstream flow meet, s_1. This distance, denoted by L_s, can be found geometrically using the equation

$$L_s = \frac{h_\epsilon - h_0}{-\tan\alpha} = \frac{3.73... - 1.70...}{93/34000} = 743\,\text{metres}, \tag{117}$$

and as $s_x = 1100$, we have $s_1 = s_x - L_s = 357$ m. Taking this value for s_x, as well as the slope and the height of the weir, the equation for b(s) becomes

$$b(s) = \begin{cases} -\frac{93}{34000}(s - 1100) & \text{if } s \leq s_a \text{ or } s_b \leq s, \\ \frac{-(s-1100)^2}{200} + P_w & \text{if } s_a \leq s \leq s_b, \end{cases} \tag{118}$$

where we define the scale parameter $\sigma = 200$ to produce a relatively narrow weir in comparison to the full channel s, which is still of realistic size. In addition, we have defined the function such that the river bed reaches a height of P_w at the crest of the weir, s_x. This makes it easier to determine the height of water above the weir crest, as this is relative to $P_w = 2$ only, however where we choose is not of great significance. Albeit, choosing to define the function this way means the y-intercept of the river bed is given by $93 * 1100/34000$. Next, we set the two sides of the piecewise equation (118) equal to determine the values of s_a and s_b and solving this in Python yields the values $s_a = 1080$ m (4 s.f.) and $s_b = 1120$ m (4 s.f.).

The function for the water height relative to the x-axis (104) becomes

$$h(s) + b(s) = \begin{cases} h_0 - \frac{93}{34000}(s - 1100) & \text{if } s \leq s_1, \\ h_0 - \frac{93}{34000}(s_1 - 1100) & \text{if } s_1 \leq s \leq s_x, \end{cases} \tag{119}$$

for our values of h_0, s_1 and s_x. As you can see, the second part of the function produces a constant, which is what we expect as the water starts to 'back up' along the river bed just before the weir. This constant is equal to h_ϵ, i.e. the height of the river just before the weir begins.

Step 4. Substituting the values of Q, g and w from table 7 into (106), we find the area at $s = s_w$ is $A^* = 29.7$ m^2. From setting T(A) equal to zero, the slope of the channel at $s = s_w$ is given by (107) and as we have a rectangular channel, this is equivalently

$$\partial_s b = \frac{Q^2 {C_m}^2 \left(\frac{2A^*}{w} + w\right)^{4/3}}{{A^*}^{10/3}}. \tag{120}$$

Therefore we have $\partial_s b = -0.00944$. Furthermore, as we are studying a rectangular channel, we find the height of the weir at $s = s_w$ is given by $h^* = A^*/w = 29.7.../25.8... = 1.15$ m. Notice that this equals the value of h_1, as we would expect to see as h^* is equal to the height of water at the point of critical flow, h_1.

Step 5. Taking the value of $\partial_s b$ found in step four and setting this equal to the partial derivative of $b(s)$ (118) concerning s, we use the 'fsolve' function in Python to find $s = s_w$. This is found to be $s_w = 1100.94$ m (6 s.f.), given with greater accuracy to show it does not lie at the same point as s_x. As we can see, this is very close to s_x but slightly further down past the crest of the weir. The corresponding height of the weir at this location is given by simply inserting this value of s into $b(s)$, such that $b(s_w) = 1.996$ m (4 s.f.). We now have sufficient information to produce a plot of the approximated river height, which includes a marker on the weir and river height at $s = s_w$, a line to denote the initial weir and a dotted line to signify the height of the river previous to the introduction of a weir. This plot along with a zoomed-in version at the weir section are shown in figures 26 and 27 respectively.

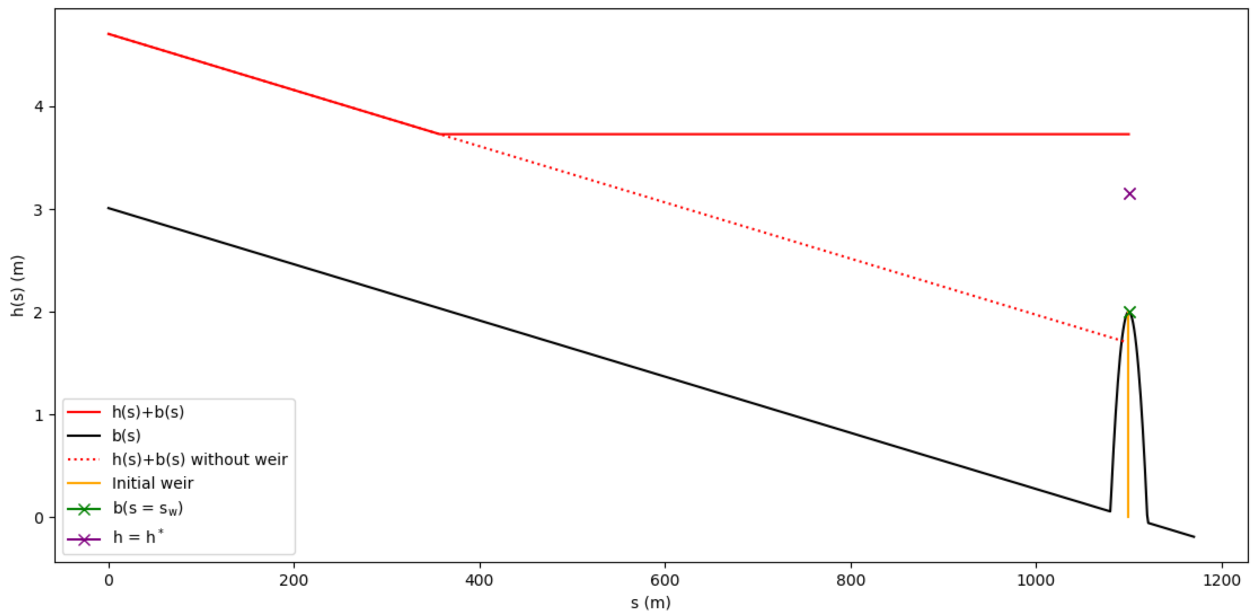


Figure 26: Diagram of a river channel with a weir, where the initial weir is a line at the position of the crest of the weir, $s = s_x$ from $h = 0$ to the height at the crest of the weir, $h(s_x)$.

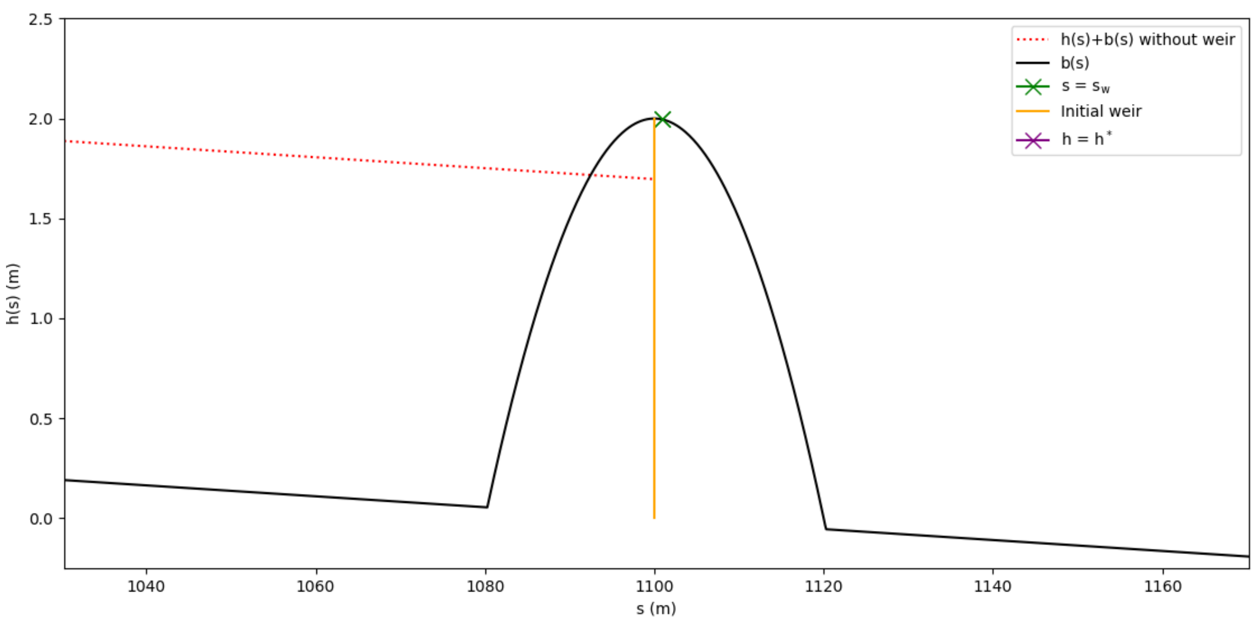


Figure 27: Zoom in on the diagram of the river channel with the introduction of a weir.

We see from the black line in figure 26 that the river bed has a constant slope from $s = 0$ up until around $s = 1075$, where the weir begins. The weir quickly increases in height and levels off at $s = 1100 = s_x$, and then falls rapidly to meet the river bed again at around $s = 1125$. The height of the river (red line) begins as a constant downward slope along s and taking the difference between the y-intercepts of the river height, roughly 4.7m and river bed, roughly 3m, gives the initial height of the river, $h_0 \approx 4.7 - 3 = 1.7$m, as expected. As seen by the red dotted line, the river height would continue at this height for the remainder of s if the weir had not been introduced. We also see the point at which the river height levels off over the river bed, which is what happens in real life when a weir is introduced, as the water being 'held back' behind the weir accumulates. This triangle-shaped section of water, seen between the red horizontal line and red dotted line from s_1 to s_w, is an area we shall use to approximate the flood storage volume created by the weir. Multiplying this area by the river width, w, will give the approximated flood storage volume and calculating this shall be covered in §10. Note that we have included the orange line underneath the weir in both plots as a reference to an oversimplified initial weir, which is an infinitesimally thin 'wall' of height P_w. The purple cross on the plot in figure 26 denotes the height of the river at the point $s = s_w$, which shall be discussed further in step 7. Lastly, the green cross denotes the weir height at $s = s_w$, shown with greater accuracy on the zoomed-in figure, where it can be seen that the point $s = s_w$ is slightly after the crest of the weir $s = s_x$.

Step 6. The method detailed in §9.1.6 has been carried out in Python using our specified constants to give the integral at 20 points along the weir, working backwards from $s = s_w$. The integral to solve in this case is given as follows,

$$\int_{s_i}^{s_w} \quad \frac{-(s-1100)^2}{200} + 2 \quad ds. \tag{121}$$

The results of this for each $s_i \in [s_a, s_w]$ are shown by the magenta line in figure 28 along with a black dotted line to denote the point $s = s_x$.

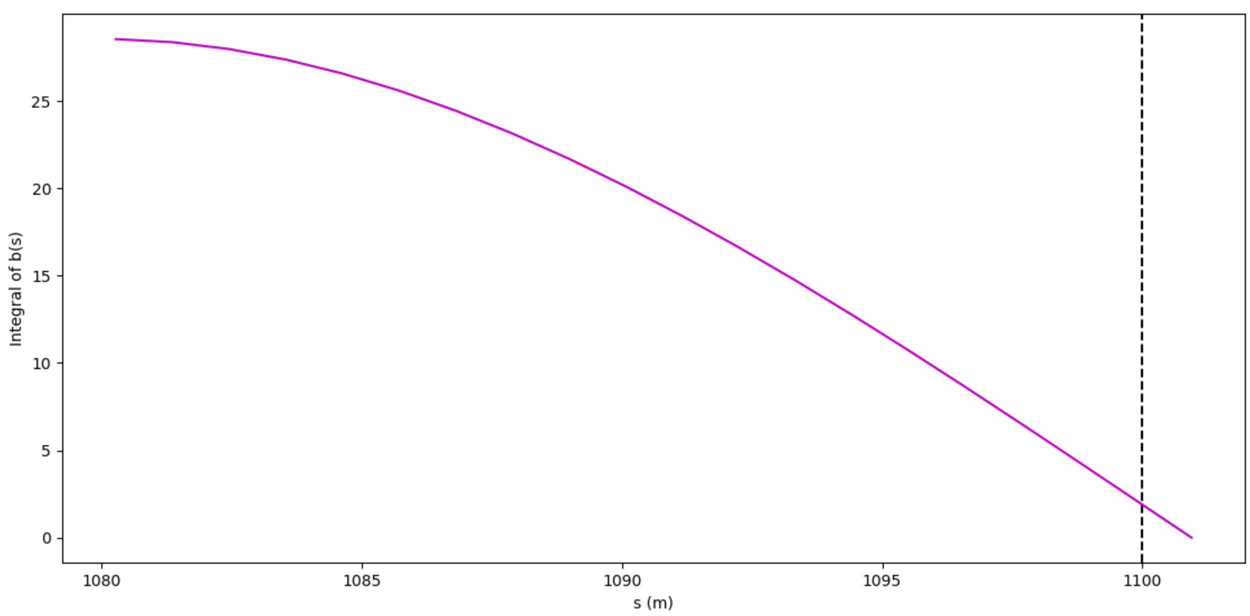


Figure 28: Integral of b(s) along the weir (magenta) with a vertical line to denote the crest of the weir at $s = s_x$

We see that the area under the curved weir steadily increases as the region of the weir we are integrating over increases and levels off as s_i approaches s_a, which is to be expected as the height of the weir decreases as we move closer to s_a. Therefore, we can estimate the area under the entire weir from where the magenta line begins, which gives an approximate area of 28 m^2. We can take this one step further and create a mesh over the whole of $b(s)$, where for $s_i \in [s_0, s_a]$ the area under the river bed and weir is given by the following integral,

$$\int_{s_i}^{s_a} \quad \frac{-93}{34000}(s-1100) \quad ds \quad + \quad \int_{s_a}^{s_w} \quad \frac{-(s-1100)^2}{200} + 2 \quad ds. \tag{122}$$

Moreover, the total area under $b(s)$ is found by taking $s_i = s_0$, i.e.

$$\int_{s_0}^{s_a} \quad \frac{-93}{34000}(s-1100) \quad ds \quad + \quad \int_{s_a}^{s_w} \quad \frac{-(s-1100)^2}{200} + 2 \quad ds, \tag{123}$$

gives the total area under the river bed and weir. The plot of the integral over the whole $b(s)$

function is shown in figure 29, where the green line denotes the integral of $b(s)$ for values of s that lie on the river bed. Similarly to the figure above, we see a steady increase in area as s_i moves increasingly backwards. At $s_i = 0$, we have the approximation for the area under the whole of $b(s)$, which we can read from the graph to be approximately 1700 m^2. Here we can compare this estimate to the actual area underneath the river bed section of the plot (figure 26), before the weir. We see in this figure that the area underneath $b(s)$ is given by the triangle between the x-axis and the river bed slope and by eye we can estimate this area to be $3 * (1075)/2 = 1612.5$ m^2. Whereas taking our result of the entire integral (123) and subtracting the area of $b(s)$ under the weir, we have that the area under the river bed is approximately $1700 - 28 = 1672$ m^2. A relative error of 3.56% between these values suggests our calculations of area are correct.

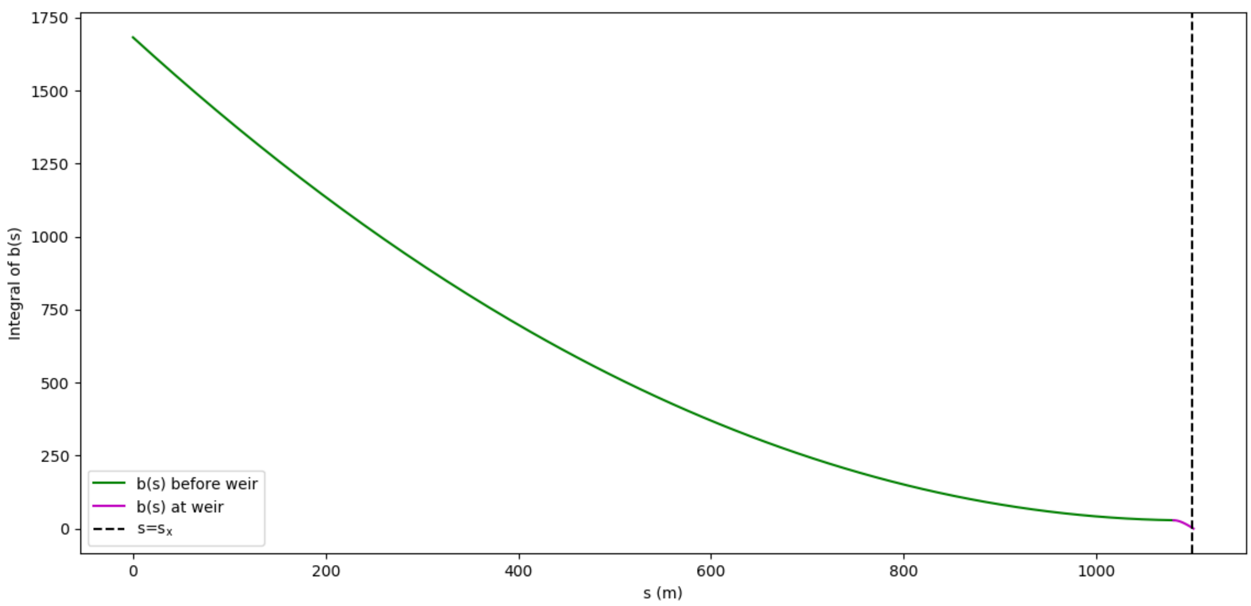


Figure 29: Integral of b(s) at different points along the river profile.

Step 7. We divide our value of A^* by the fixed river width, w, to find the river height at $s = s_w$, i.e. $h^* = A^*/w = 29.7.../25.8... = 1.15$ metres and as we expect, this is equal to our h_1 value. Putting this height onto the river plot requires us to find the height of the weir at this point also, which we find using the equation (112)

$$\begin{aligned} & h^* + \frac{(s_w - 1100)^2}{200} + 2 \\ = \ & 1.15... + \frac{(1100.47... - 1100)^2}{200} + 2 \ = \ 3.15\text{m}. \end{aligned} \tag{124}$$

This value of $h^* + b(s_w)$ has already been added to the river plot (purple cross in figure 26). This shows that h^* indeed lies somewhere between the weir and the head of the weir, therefore we have evidence to suggest our calculations are correct thus far.

Step 8. The larger the value of N_s, the more cross-sectional areas we will have of the channel and therefore the greater accuracy we will achieve when calculating the area under the curve, which we shall do in §10. Therefore, let us choose $N_s = 50000$ and we shall use backward integration to find A_i for $i = 0, ..., 50000$, which will give us the area at 50001 cross sections along s. Starting with our initial estimate of area, $A^* = 29.7$m, at $s = s_w$, we find the successive area term using equation (114) is given by

$$A_{49999} = A^* + \frac{(s_w - s_0)}{50000} \frac{\partial_A T(A, s)}{\partial_A B(A, s)}\bigg|_{A=A^*,\, s=s_w}. \tag{125}$$

To see the point at which the original approximation and backward integration align in the river plot, we shall see below that we must move the start of the plot to $s_0 = -400$. Therefore we have $\triangle s = (s_w - s_0)/N_s = 1100.94... + 400/50000 = 0.030$. For the remainder of the cross-sectional areas, we use the usual iterative formula given in equation (113). Putting these formulas into a for loop within Python, as shown in Appendix A.6, we produce the values for A_i at their corresponding locations s_i, as well as the values for $b(s_i)$ which we use to check that our iterative formula has been done correctly. The calculated values of A_i have been divided by w to give the value of each h_i, which has been plotted against s_i, laid over the original river plot (figure in step 5?) as shown by the blue dashed line in figure 30.

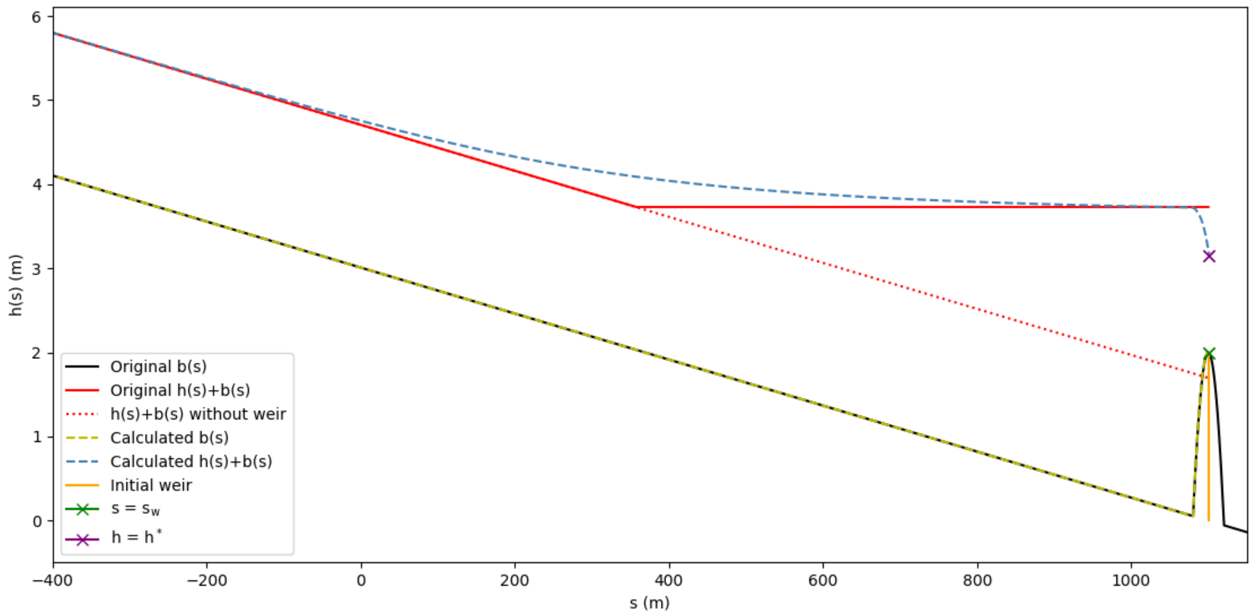


Figure 30: Diagram of the cross-sectional area at different points along s, as found using backwards integration from the stationary point on the weir.

The yellow dashed line shows the function for the river bed, $b(s_i)$, given in the for loop and we can see that this has been done correctly as it aligns with the original river bed function, as shown by the black line. Note that extending the start of the river plot to begin from $s_0 = -400$ in this figure does not change the location of the river bed or the river height functions, but simply extends our view of the original river channel and slope before the influence of the weir. The blue dashed line starts at h^*, as shown by the purple cross, and increases rapidly over the weir as we work backwards along s until it reaches the point where it aligns with the horizontal line, h_ϵ. Once the weir ends, the cross-sectional height steadily increases as s_i decreases, all the way up to $s_0 = -400$ where it aligns with the red solid line. We see from the graph that the integration gives a smooth function for the river height, which is a better depiction of what we would see in real life compared to the height approximation we first used to model the river height. Further, the blue line sits slightly higher than or in alignment with the red line over the whole of $s \in [s_0, s_a]$ and only at the location of the weir does this line fall below the original approximation. Therefore, we can estimate that the area between the blue dashed line and the river bed will be slightly larger than the area given by the solid red line. A thorough evaluation of this claim will be covered in §10, where we shall estimate the volume of water found from both methods.

In addition to this, we attempt to use the forward integration method to plot the river height, which we do using the iterative formula (115). Again we select $N_s = 50000$ and here the initial estimate is given by $A_0 = h_0 w = 1.70... * 25.8... = 43.8$ m^2. Using a for loop for this iterative formula and plotting the results for h_i against s, we produce the river plot in figure 31.

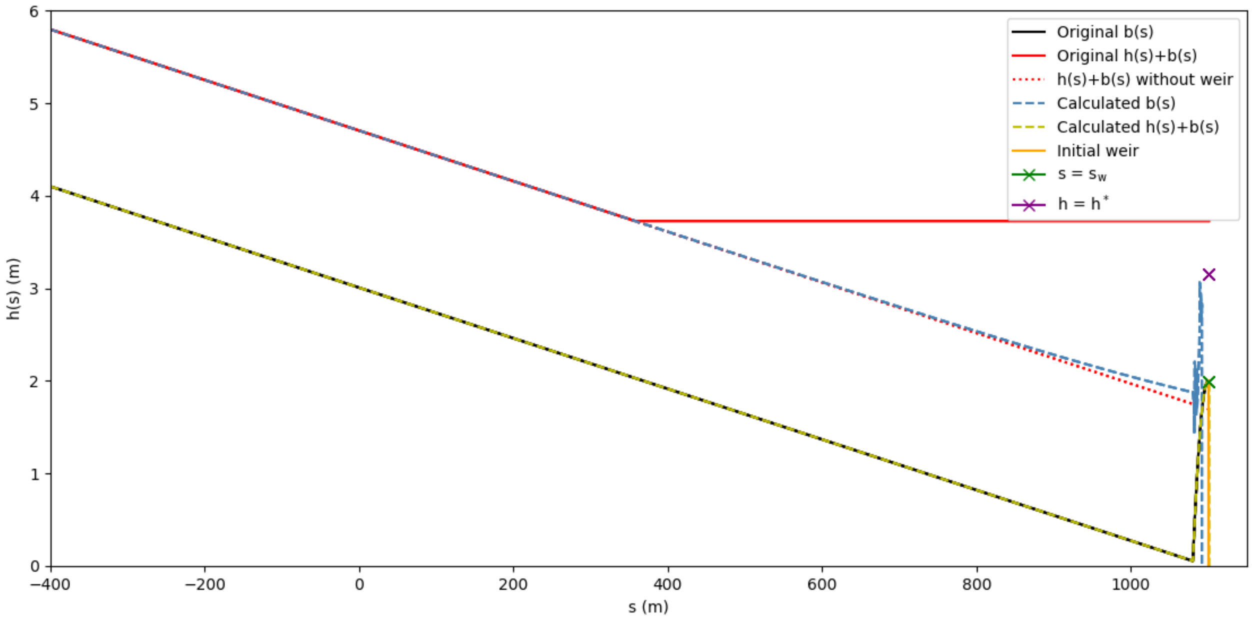


Figure 31: Diagram of the cross-sectional area at different points along s, as found using forwards integration from the s_0.

It can be seen that the yellow dashed line, again showing the $b(s)$ function derived using our iterative formula, aligns with the original river bed plot. However, the forward integration method fails to correctly calculate the cross-sectional area and therefore river height along s, as shown by the blue dashed line. The river height here begins at $s_0 = -400$ in line with the original river slope at a height of h_0, which we would expect initially, but then continues to stay at this height and aligns with the red dotted line. This changes once we reach the beginning of the weir at $s = s_a$, where the river height rapidly fluctuates over the weir. As this does not meet $h = h^*$ at $s = s_w$, it is clear that this method provides an incorrect estimate of the cross-sectional area over s and would be inappropriate to use to find the volume of water in the channel.

9.2.1 Determining subcritical or supercritical flow

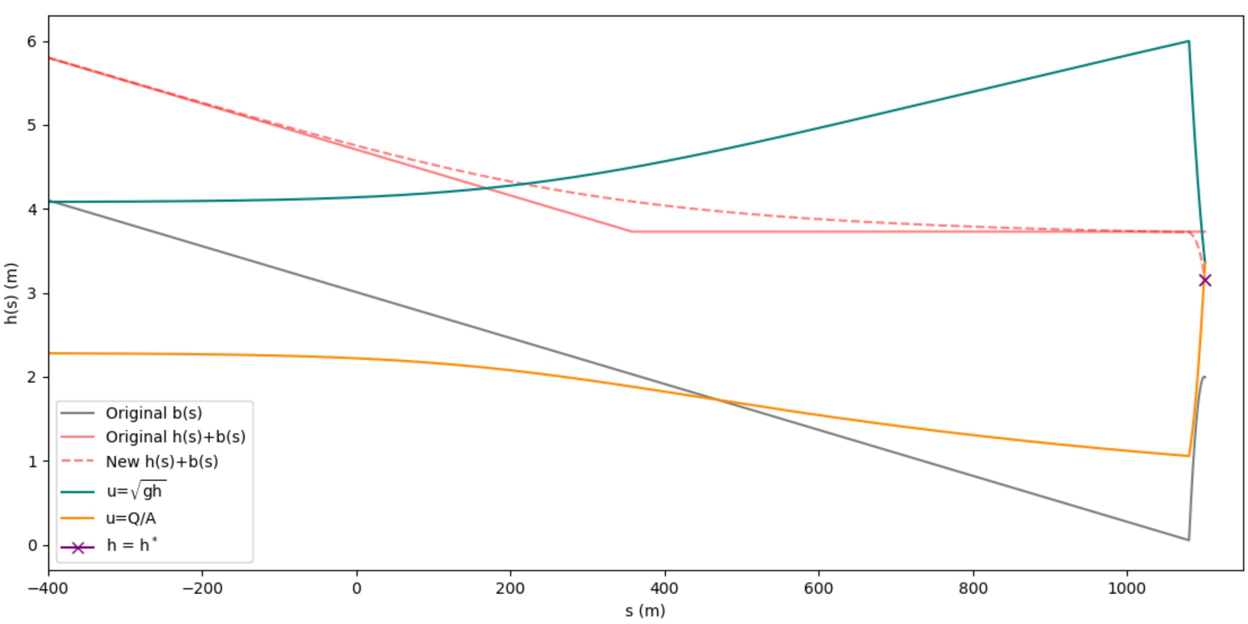
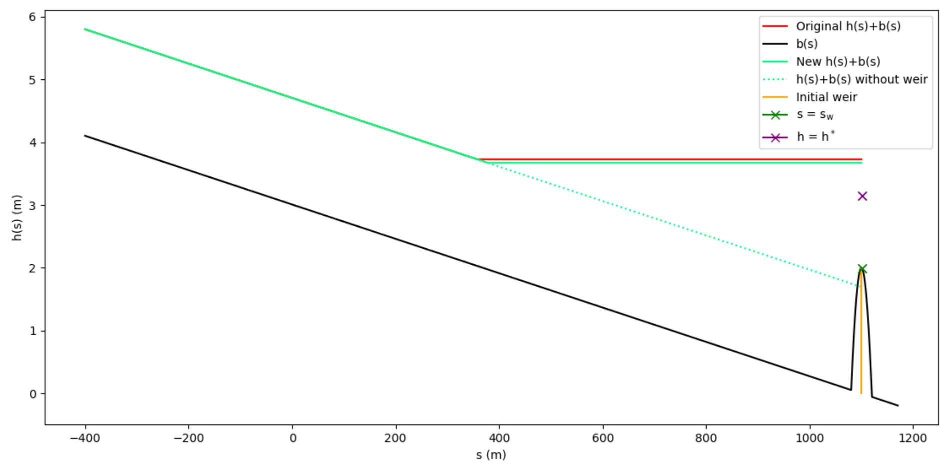


Figure 32: Diagram of calculations of velocity over s, $u = Q/A$ (orange) compared to the calculation of $\sqrt{gh}$ (teal). These lines are overlaid on the original river plot, as shown by the shaded black and red lines.

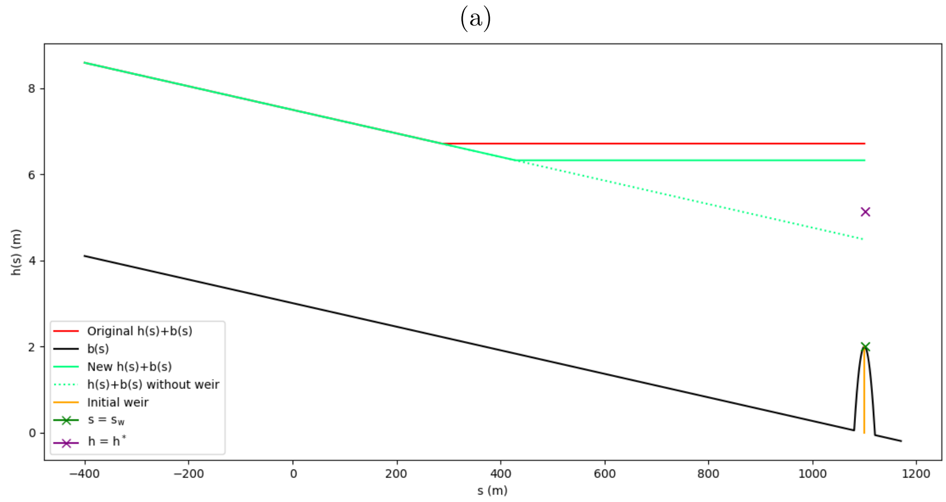
The velocity of the water over s is given at each cross-section s_i by dividing the discharge constant by the cross-sectional area values, A_i, as determined in step 8. The velocity values u_i have been plotted for s in figure 32, as indicated by the orange line. To determine whether the flow over s is critical, supercritical or subcritical, we calculate $\sqrt{gh}$ for $h(s)$, as indicated by the teal line. From section 8.3, we know the flow is considered supercritical when $u < \sqrt{gh}$, subcritical when $u > \sqrt{gh}$

and critical when they are equal. In figure 32, we see that $Q/A(h) = u < \sqrt{gh}$ over $s \in [s_0, s_w)$, thus flow is subcritical in this region. This is what we would expect to see, as outlined in the assumptions of the shallow water equations (§6), we require subcritical flow. The two lines meet at $s = s_w$, and we have $Q/A(h) = u = \sqrt{gh}$, i.e. critical flow. This is the only point of critical flow along s.

9.2.2 Alternative approximation of river height, h_ϵ



(a)



(b)

Figure 33: River profiles comparing the calculated height just before the weir, the initial estimate $h_\epsilon = \widehat{h_{\epsilon 2}}$ (red) and the exact value $h_{\epsilon 2}$ (green). Subfigure (a) shows the river profile when $Q = 100$ m$^3/s$ and (b) is when $Q = 450$ m$^3/s$.

We can provide an exact expression for h_0 without having to remove the u_0 term, meaning we get a more accurate estimate for the river height, h_0, in real-life scenarios when velocity is high. Using the cubic equation as defined in (84) and implementing this code into Python with our defined values of Q, g and P_w, we find that h_ϵ is given by $h_\epsilon = 2.617$ m (4 s.f.). Further, the exact values are given by $h_{\epsilon 2} = 3.67$ m and $H_2 = 1.67$, meaning we have a relative error between $h_\epsilon = \widehat{h_{\epsilon 2}}$ m and $h_{\epsilon 2}$ of 1.52% and an error between H and H_2 of 3.28%.

We found in step 1 of section (prev to this) that the height of water immediately before the weir is estimated as $h_\epsilon = 3.729$ m (4 s.f.). This is very similar to the exact value for $h_\epsilon = 3.67$ m and we have a relative error between h_ϵ and $\widehat{h_\epsilon}$ of 1.52%. Therefore suggesting we are correct in assuming a small velocity u in this case and it is suitable to use h_ϵ instead of $h_{\epsilon 2}$ for the remainder of our analysis. Both of these have been plotted in figure 33, where we see that the new approximation using $h_{\epsilon 2}$ becomes increasingly further from the original approximation using h_ϵ as the velocity increases.

9.3 River profile for steep bed slopes

For too steep of a slope the backwards integration fails, as this violates the small bed slope assumption of the shallow water equations given in section 6. When keeping the normal discharge and weir height from table 7 but changing the slope to $\tan\alpha = -0.01$, the resulting river plot is shown in figure 34. We see that the river height found using this method fails to align with the original red line approximation from $s \in [0, 800]$. This leads to a smaller flood storage volume estimate given from this method compared to the normal approximation, which will be seen in figure 38 for steeper slopes.

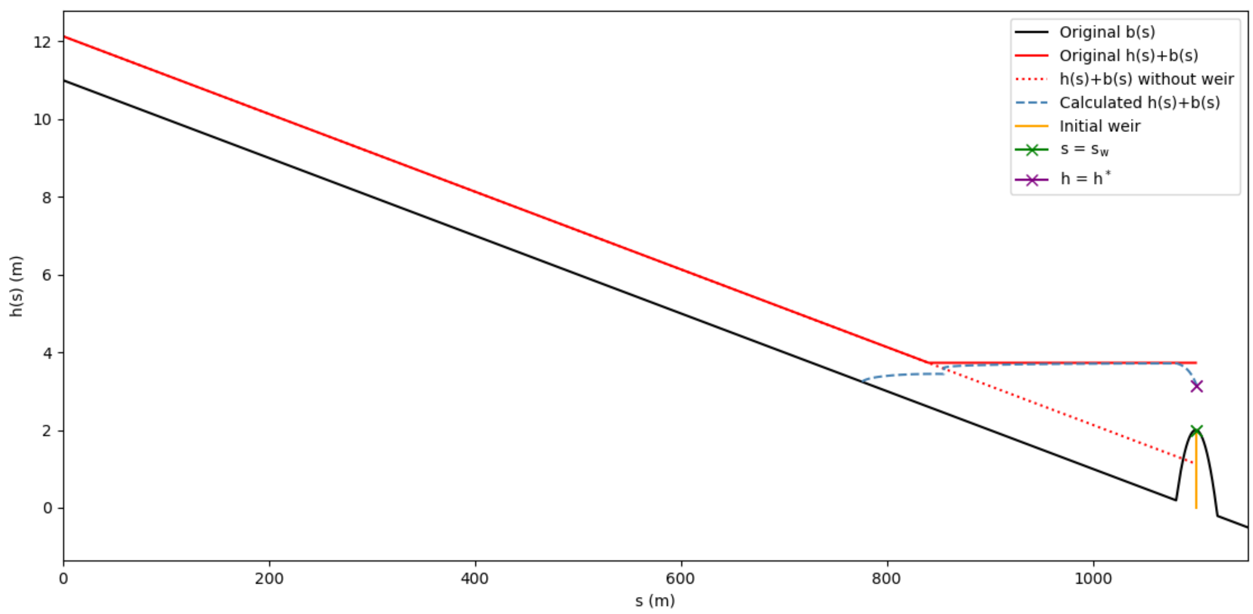


Figure 34: River plot for bed slope $\tan\alpha = -0.01$.

10 Calculating flood storage volume using shallow water theory

In §10.1, we shall find two formulae for the flood storage volume along the river profile, one using the original river height approximation and the other using the backwards integration method, as applied to the river Aire profile in §9.2. These estimates will be compared against one another and we will see how changing the values for discharge, weir height and river bed slope affect this relationship in §10.1.1. Further, the flood storage volume of the refined river height approximation, as detailed in §9.2.2, will be calculated and again compared against the flood storage given by the previous two methods.

10.1 Calculating flood storage volume - River Aire at Kirkstall Valley

As introduced in step 5 in the previous section, the triangle-shaped section of water between the red horizontal line and red dotted line in figure 26 is the flood storage area and we multiply this area by the river's width to find the flood storage volume, FSV_1. After this, we shall look at varying the weir height, P_w, flow rate, Q, and river bed slope, $\tan\alpha$, to see the influence this has on the flood storage volume. The general formula for the FSV using this approximation, as applied to the P_w, Q and $\tan\alpha$ values given in table 7 is shown below,

$$FSV_1 = \frac{w(h_e - h_0)}{2}(s_x - s_1), \tag{126}$$

using the value for s_1 as determined in step 3, we find $FSV_1 = 19478$ m^3 (5 s.f.). Next, we must calculate the flood storage volume given by the backward integration method covered in step 8, denoted FSV_2. We do this by finding the area between the blue dashed line and river bed in figure 30 over the region $s \in [s_0, s_x]$, which we shall denote Vol_1 and taking away the area between the river bed and red dotted line over the same region, denoted Vol_2. As our integration produced the cross-sectional areas along s, we find the volume by adding the cross-sectional areas A_i together and multiplying by the change in s, $\triangle s$. However, as we are only focusing on the region $s \in [s_0, s_x] = [-400, 1100.94]$, we must first determine the volume of the region encompassing $s \in [s_x, s_w]$, Vol_γ which we will then minus from the full volume over the whole of s, Vol_β. Vol_β is given as follows,

$$Vol_\beta = \sum_{i=0}^{N_s} A_i \,\frac{(s_w - s_0)}{N_s}, \tag{127}$$

which for $N_s = 50000$ becomes $Vol_\beta = 70625$ m^3 (5 s.f.). To find Vol_β, we apply the backwards iterative formula to $Ns + 1 = 50001$ points for $s \in [s_x, s_w]$ only, which produces the cross sectional area $\tilde{A}_i$ of 50001 points along $[s_x, s_w]$. Summing these together over the specified region, we find Vol_γ is given by

$$Vol_\gamma = \sum_{i=0}^{N_s} \tilde{A}_i \,\frac{(s_w - s_x)}{N_s}. \tag{128}$$

Applying this formula to Python gives the result $Vol_\gamma = 28.7$ m^3. Therefore $Vol_1 = Vol_\beta - Vol_\gamma =$ 70596 m^3 (5 s.f.). The volume of the region between the red dotted line and the river bed, i.e. the original stream, has a parallelogram shape, meaning its area is found by simply multiplying the height by the length of the section, which we then multiply by the river's width to find the volume,

$$Vol_2 = wh_0(s_x - s_0). \tag{129}$$

We find $Vol_2 = 48205$ m^3 (5 s.f.). Therefore, the calculation of flood storage volume via the method of backwards integration is $FSV_2 = Vol_1 - Vol_2 = 70596... - 48205... = 22391$ m^3 (5 s.f.). As expected, the flood storage volume calculated via backwards integration is greater than the flood storage volume given by the river height approximation. Albeit, these volumes are similar in size and if we take FSV_2 as the actual volume over s, we have a relative error of 13%.

10.1.1 Flood storage volume for varying parameters

Presented in §10.2 are three figures of the flood storage volume on the y-axes, compared with a range of values for either the discharge, the weir height or the river bed slope on the x-axis. Estimations of the flood storage volume for each of these parameters are given for FSV_1 and FSV_2 by the red and blue lines respectively.

Varying weir height. Let us suppose the weir height is a variable that can range from one to five metres, increasing in increments of 0.5 metres. We shall remain with the current discharge rate and slope as seen previously, $Q = 100$ m^3/s and $\tan\alpha = -93/34000$. The two values of flood storage volume, FSV_1 and FSV_2 can be plotted against different values for the weir height, P_w, as shown in figure 36. The blue line represents FSV_2 and the red line represents FSV_1 for a range of weir heights, note that we will not discuss the green FSV line for now. We see that the flood storage volume produced by the approximation is slightly smaller than the volume calculated via the backwards integration method for weirs of height $P_w = [0.5, 5.5]$.

Note that for the higher values of P_w, we must move the start of the plot, s_0, back to include the point at which the river height approximation and integration meet. For instance when $P_w = 5$ an appropriate point to start the plot from is $s_0 = -1300$, as the red solid line and dashed blue lines align at this point instead of $s_0 = -400$. If we chose $s_0 = -400$, the region between the two river height readings along $s_0 \in [-1300, -400]$ would not be included, thus giving an inaccurate estimate for FSV_2 that is smaller than its true value.

Varying discharge. Staying with the original weir height of 2 metres, we can instead vary the flow rate, from $Q = 50$ m^3/s up to $Q = 450$ m^3/s, increasing in increments of 50 m^3/s and plot the corresponding FSV against Q. This is shown for both methods of finding flood storage in figure 37, again by the blue line for the integration method, FSV_2, and the red line for FSV_1. We see that both volumes increase steadily as the discharge rate increases, slowing in increase once we reach $Q = 450$ m^3/s. The flood storage volume found using the integration method is consistently larger than the flood storage given from the original approximation.

Varying river bed slope. For discharge $Q = 100$ m^3/s and weir height $P_w = 2$ m, we alter the river bed slope and find the corresponding flood storage volume for each gradient. Taking $\tan\alpha$ in the range [-0.009,-0.001], we find the corresponding values for FSV_1 (red line) and FSV_2 (blue line) at each slope in figure 38. There is a steady increase in flood storage volume FSV_1 as the river bed slope becomes shallower, similarly to FSV_2 for slopes $\tan\alpha \in [-0.009, -0.005]$. Despite this, we see a much larger increase in FSV_2 as the gradient becomes shallower, for $\tan\alpha \in [-0.005, -0.001]$. This was shown to us in the original river plots, where at a small section around s_1 the river height was higher from the backwards integration method compared to the original approximation. Therefore, we found that a larger area of the plot lay between the river bed and integration line than the river bed and approximation line, which leads us to a larger flood storage volume for the integration method. The disparity between FSV_1 and FSV_2 becomes increasingly larger for these values of $\tan\alpha$, which we expect as the flood storage volume itself increases for gentler slopes.

In comparison, the steeper slopes for $\tan\alpha$ on the left side of the plot show that $FSV_2 < FSV_1$ for $\tan\alpha \in [-0.009, -0.006]$. In comparison to the original river plot we used, where the gradient is smaller than this at $\tan\alpha \approx 0.0027$ (2 s.f.), the river plot at a gradient in this range is shown in figure 35. For $\tan\alpha = -0.006$, we see that the river height given by the integration method (blue dashed line) is very close to the original river height approximation (red line), only diverting from this at a small region around s_1 and over the weir at $s \in [s_a, s_w]$. This explains why FSV_2 is smaller than FSV_1, as the river height readings are almost the same, except near the weir when the blue dashed line dips below the horizontal red line. This small difference in area between the two lines leads us to get a smaller estimate of flood storage volume for the integration method, compared to the original approximation.

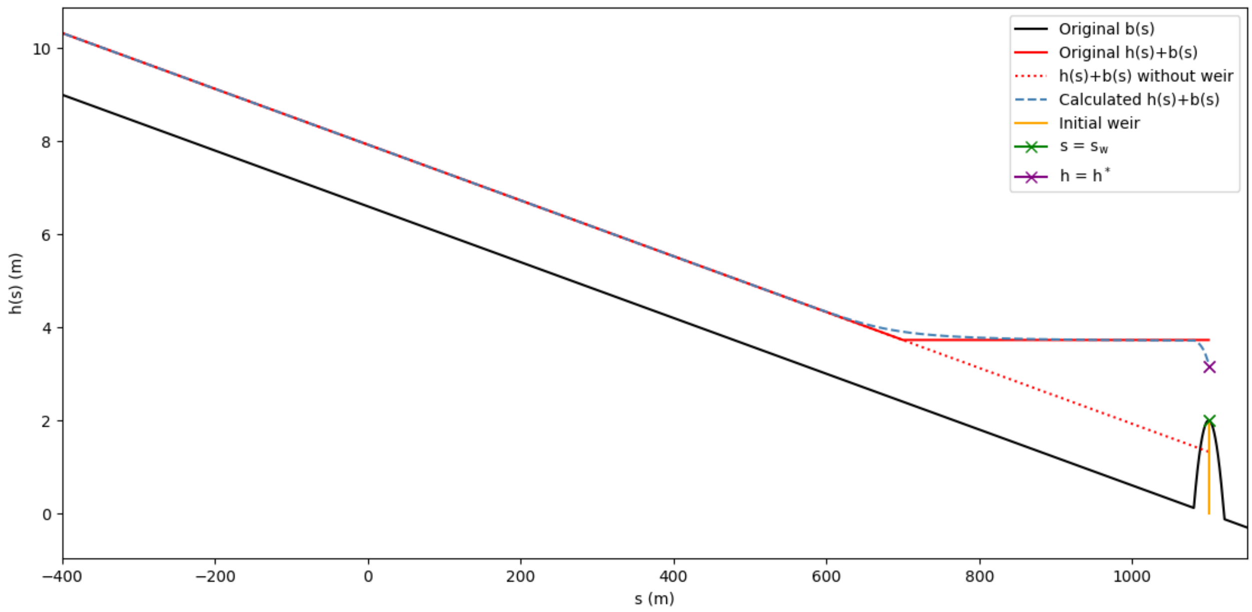


Figure 35: River plot for bed slope $\tan\alpha = -0.006$

10.2 Flood storage volume using exact h_ϵ

We can also find the estimated flood storage volume of the river using the exact value for h_ϵ, as presented in §9.2.2. Plots of FSV_1 for exact h_ϵ are featured in figures 36, 37 and 38 for varying discharge, weir heights and bed slopes, as shown by the green squares. The Python code used to create these graphs can be found in Appendix A.6.

Varying weir height. The flood storage volume estimated using the exact h_ϵ value is shown to lie very close to the original FSV_1 approximation for the whole range of weir heights, $P_w = [0.5, 5.5]$. The disparity between the two however does very slightly narrow as P_w increases, transitioning from an approximate 760 m^3 difference in FSV at $P_w = 0.5$ m to a 766 m^3 difference at $P_w = 5.5$ m. This suggests that an increased weir height increases the total flood storage volume found using the h_ϵ estimator slightly more than the FSV found with the exact value for h_ϵ.

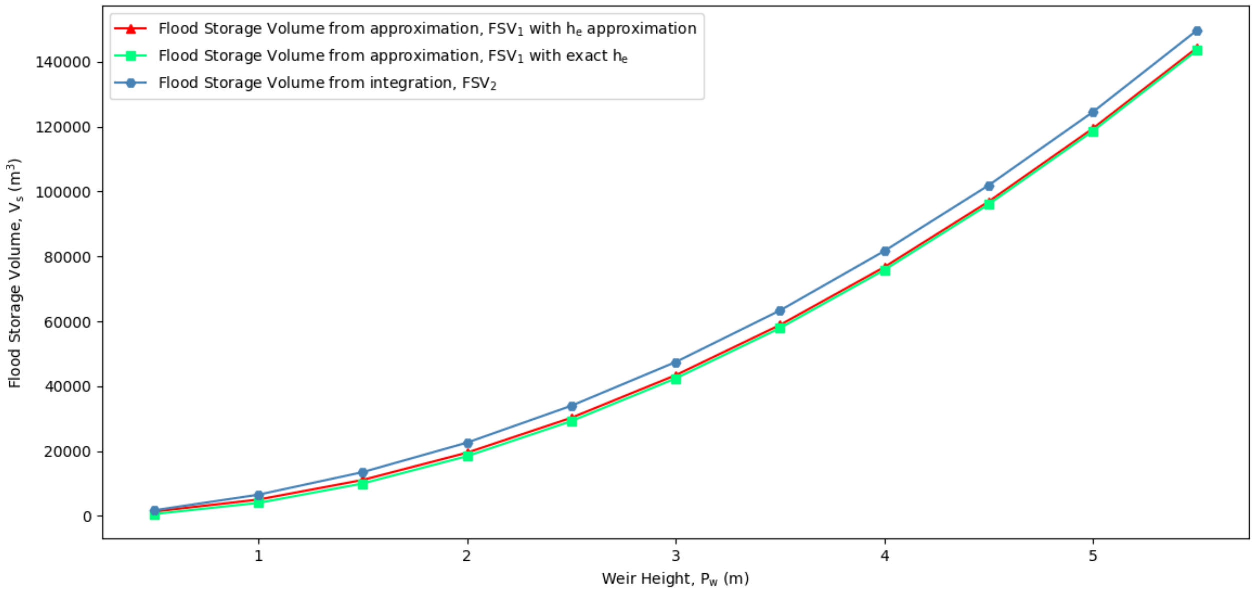


Figure 36: Plot to compare the flood storage volume from the river height approximation with approximate h_ϵ for small velocities (red triangles), river height approximation with exact h_ϵ (green squares) and backwards integration method (blue squares) for different weir heights, P_w.

Varying discharge. We see that the exact value for h_ϵ yields a much smaller flood storage volume compared to the original FSV_1 and FSV_2. Although for a low discharge of $Q = 50$, the value for FSV_1 calculated using both h_ϵ values is approximately equal, showing that the estimate of h_ϵ denoted by the red line is accurate for small velocities. Figure 33 also aligns with these findings, as we saw that for a higher discharge value, the triangle of flood storage shown by the green line using the exact h_ϵ value is smaller than the red line using the h_ϵ estimate.

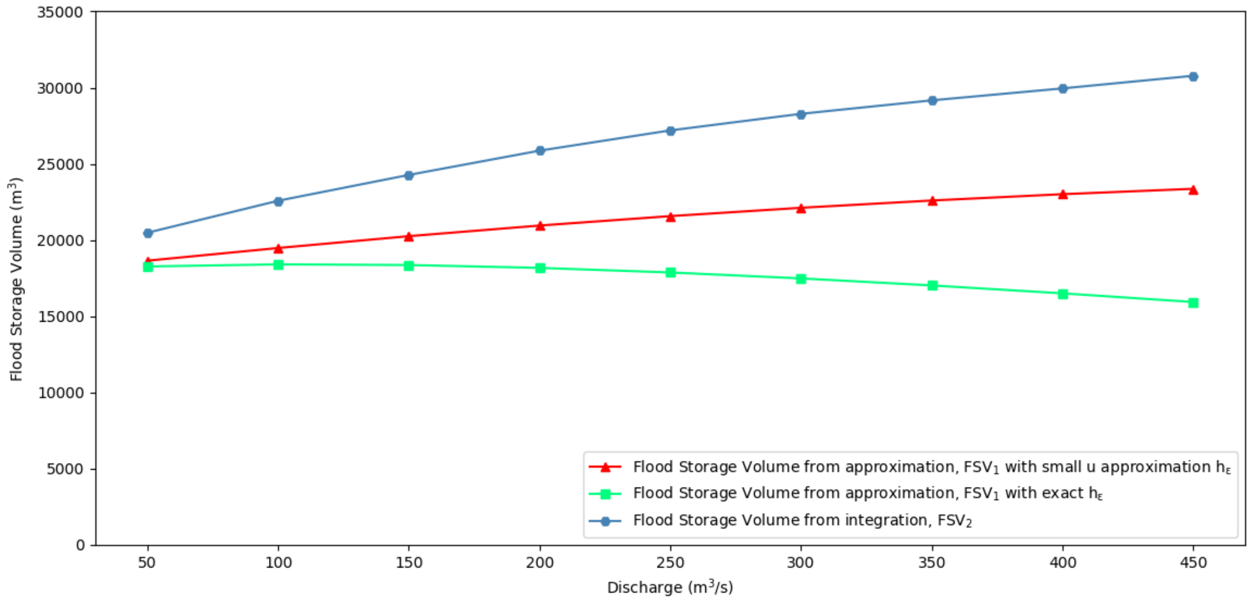


Figure 37: Plot to compare the flood storage volume from the river height approximation with approximate h_ϵ for small velocities (red triangles), river height approximation with exact h_ϵ (green squares) and backwards integration method (blue hexagons) for different discharge rates, Q.

Varying river bed slope. Similarly to figure 36, the new value for FSV_1 shown by the green squares is just less than the original FSV_1 estimate for all bed slopes. Therefore suggesting the river bed slope has little influence on the difference in flood storage volume found using the h_ϵ estimate compared to the true value.

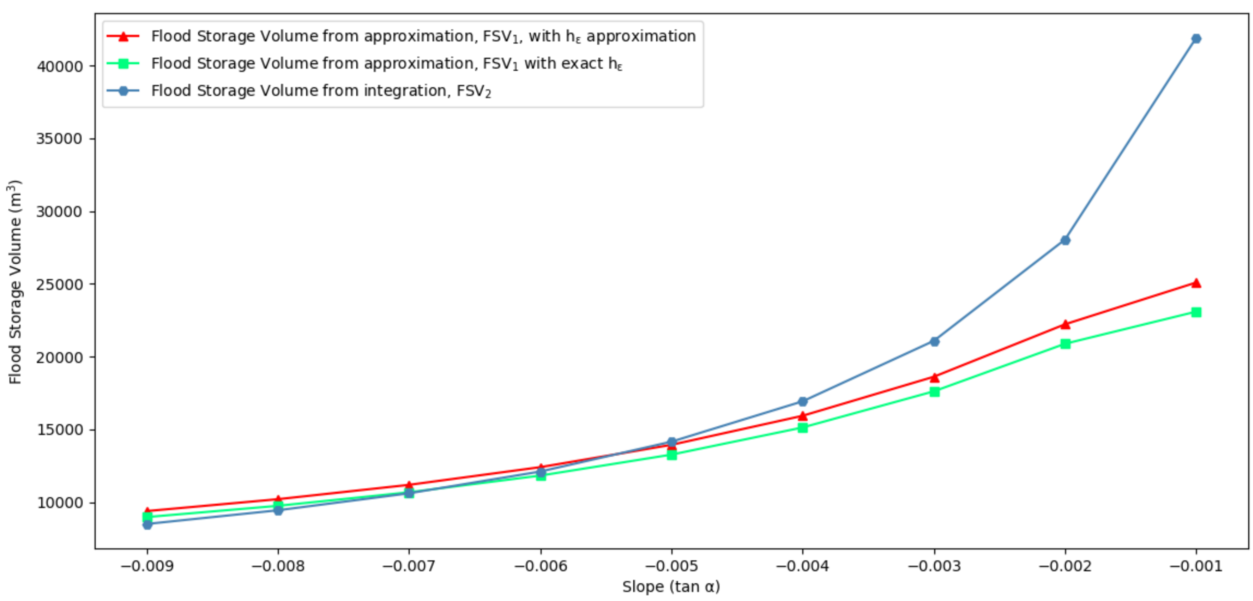


Figure 38: Plot to compare the flood storage volume from the river height approximation with approximate h_ϵ for small velocities (red triangles), river height approximation with exact h_ϵ (green squares) and backwards integration method (blue hexagons) for different river bed slopes, $\tan\alpha$.

11 Conclusion

In this report, we have found that flooding in the UK is becoming an increasing concern, socially and economically, only to be accelerated by the influence of climate change in future years. In particular, the flood risk currently facing the city of Leeds in West Yorkshire is high and the impacts of past flooding here have been examined, focusing on the 2015 Boxing Day flood. The increase in the river's height during this flood was demonstrated through the use of a flood hydrograph, where we estimated the flood excess volume for this flood to be 9.34 Mm^3. Furthermore, we explored the flood mitigation methods put in place here to reduce fluvial flooding, as part of the flood alleviation scheme. These strategies were found to be beneficial in the Leeds FAS, where the standard of protection for multiple locations in Leeds improved as a result of the scheme. A particular focus was seen on one method of mitigation - weirs, where we used shallow water theory to find how flow rate changes over a weir in an open channel. These concepts were applied to a cross-section of the river Aire, for which we plotted a model of the river profile and calculated the flood storage volume created by the Kirkstall Valley weir.

The first half of this assignment aimed to explain the concept of return periods. This has been done successfully, we have introduced the tool aimed at helping the general public understand this risk, Wetropolis and explained the workings behind this in terms of statistical and dynamical theory. In addition, we have successfully investigated the flood mitigation strategies used in the UK and the impact they have on return periods. These strategies include traditional engineering practices like building flood defence walls, weirs and floodplains, but also make use of NFM strategies such as tree planting and leaky dams. We have seen that most methods of flood mitigation work with the common aim of increasing flood storage, for example, weirs, for which we have used shallow water theory and the Bernoulli equation to model the discharge over weirs. Using shallow water theory, a method of calculating flood storage over a weir has been developed, that we can apply to any straight channel with a smooth rectangular weir, given the channel has a constant width, slope and rectangular shape. This addresses the second aim of this assignment, for which we wanted to use shallow water theory to model fluid passing over a weir. In addition, we were able to use this model to examine the flood storage volume created by weirs and investigated the influence changing river bed slope, flow rate and weir height has on this volume.

As part of further study, I would apply the method of plotting the river profile over a broad-crested weir to broad-crested weirs which v-notched, parabolic or trapezoidal in shape. From here I would investigate the influence this has on flow rate and the resulting flood storage provided by this weir. Alternatively, I can model the flow of water over river sections where the width or bed slope depend on s, or have a different channel shape to the rectangle shape we have assumed. The resulting flood storage can be calculated for these river sections and the influence of weir height, channel slope and flow rate, can be investigated to see if these parameters help to increase or decrease storage. In addition to this, another future direction of work could involve the inclusion of a floodplain in the model to accompany the weir in increasing flood storage volume. This can be included as a gentle slope located on one side of the river once the river exceeds a certain height. The influence of this floodplain on river height and subsequently flood storage volume can be studied.

These findings have given us a greater understanding of the factors that increase a river's flood storage. In particular, we found that as a weir's height increases and flow rate increases, the flood storage increases. Furthermore, as a river's slope becomes shallower we saw an increase in flood storage compared to a slope with steeper gradient. Albeit this model only examines flow over already gentle slopes, as only gentle slopes fit within the constraints of shallow water theory. Therefore, as part of further study, I would investigate how we can plot a river profile for rivers with steeper slopes and high flow rates, where supercritical flow may be experienced. Another aspect we could not include in this modelling is the behaviour of fluid over sharp-crested weirs. Therefore, further study would involve modelling fluid over sharp-crested weirs using the derived discharge equations.

12 References

1. O. Agamenon. *History of the Bernoulli Principle.* 2019. [Accessed 28 Feb. 2023] Available from: https://www.researchgate.net

2. Aire Rivers Trust. *Interesting facts and figures.* [Accessed 21 Mar. 2023] Available from: https://aireriverstrust.org.uk

3. ARUP. *Leeds Flood Alleviation Scheme.* [Accessed 8 Apr. 2023] Available from: https://www.arup.com/projects/leeds-flood-alleviation-scheme

4. P. D. Bates, M. S. Horritt, T. J. Fewtrell. *A simple inertial formulation of the shallow water equations for efficient two-dimensional flood inundation modelling.* J. Hydrol., 387, 33–45, 2010. [Accessed 10 Nov. 2022] Available from: https://www.sciencedirect.com/science/article/pii/S0022169410001538

5. O. Bokhove, M. A. Kelmanson, T. Kent, G. Piton, J. Tacnet. *A Cost-Effectiveness Protocol for Flood-Mitigation Plans Based on Leeds' Boxing Day 2015 Floods.* 2020. [Accessed 11 Dec. 2023] Available from: https://www.mdpi.com/2073-4441/12/3/652

6. O. Bokhove, M.A. Kelmanson, T. Kent. *Using flood-excess volume in flood mitigation to show that upscaling beaver dams for protection against extreme floods proves unrealistic.* 2018. [Accessed 9 Dec. 2022] Available from: https://eartharxiv.org/repository/view/1281/

7. O. Bokhove, M.A. Kelmanson, G. Piton, J.-M. Tacne. *Visualising Flood Frequency, Flood Volume and Mitigation of Extreme Events.* 2022. [Accessed 17 Oct. 2022]

8. O. Bokhove, T. Hicks, W. Zweers, T. Kent. *Wetropolis extreme rainfall and flood demonstrator: from mathematical design to outreach.* 2020. [Accessed 17 Oct. 2022] Available from: https://hess.copernicus.org/articles/24/2483/2020/hess-24-2483-2020.html

9. H. Chanson. *The Hydraulics of open channel flow: an introduction* 2004. [Accessed 1 Feb. 2023] Available from: https://www.sciencedirect.com/book/9780750659789/hydraulics -of-open-channel-flow

10. Environment Agency. *Aire at Armley Rating Change Report.* 2016. [Accessed 14 Nov. 2022] Available from: https://github.com/obokhove/floodproject5872math

11. Environment Agency. *Flooding in England: A National Assessment of Flood Risk.* 2009. [Accessed 12 Dec. 2022] Available from: https://assets.publishing.service.gov.uk/government

12. The Flood Hub. *Understanding Flood Risk.* [Accessed 20th Nov. 2023] Available from: https://thefloodhub.co.uk/blog-understanding-flood-risk/

13. S. Kennet. *How to do FEV Analysis.* 2019. [Accessed 16 Oct. 2022] Available from: https://github.com/Rivers-Project-2018/How-to-do-FEV-Analysis

14. Leeds City Council. *Leeds Flood Alleviation Scheme (Phase 1) in numbers.* [Accessed 11 Apr. 2023] Available from: https://www.youtube.com/watch?v=TyxGKsyVGJU

15. Leeds City Council Flood Resilience. *Leeds FAS2 Engineering Works Updates.* [Accessed 12 Apr. 2023] Available from: https://leedscitycouncilfloodresilience.commonplace.is/en-GB/proposals/Leeds-FAS2-engineering-works-updates/step1

16. Leeds City Council Flood Resilience. *Leeds FAS2 General Information.* 2017. [Accessed 10 Apr. 2023] Available from: https://leedscitycouncilfloodresilience.commonplace.is/proposals/leeds-FAS2-general-information

17. Leeds City Council Flood Resilience. *Leeds FAS2 Natural Flood Management.* [Accessed 12 Apr. 2023] Available from: https://leedscitycouncilfloodresilience.commonplace.is/proposals/leeds-fas2-natural-flood-management/step2

18. Leeds Government UK. *£50 million flood defence scheme opens in Leeds* 2017. Available from: https://news.leeds.gov.uk/news/50million-flood-defence-scheme-opens-in-leeds

19. Leeds Government UK. *Celebrating five years of the Leeds Flood Alleviation Scheme.* 2022. [Accessed 13 Apr. 2023] https://news.leeds.gov.uk/news/celebrating-five-years-of-the-leeds-flood-alleviation-scheme

20. Leeds Government UK. *Leeds Flood Alleviation Scheme Phase 2.* [Accessed 12 Apr. 2023] https://www.leeds.gov.uk/flooding/leeds-flood-alleviation-scheme

21. M. McGrath. *Climate change: Warming signal links global floods and fires.* BBC News. 2019. [Accessed 12 Apr. 2023] Available from: https://www.bbc.co.uk/news/science-environment-50407508

22. B. R. Munson, D. F. Young, T. H. Okiishi's. *Fundamentals of fluid mechanics.* 8e edition. Hoboken, NJ: Wiley, 2019. [Accessed 17 Oct. 2022] Available from: http://course.sdu.edu.cn

23. Public Sector Executive. *Leeds pressing ahead with cutting-edge flood defence scheme.* 2014. [Accessed 12 Apr. 2023] Available from: https://www.publicsectorexecutive.com/Public-Sector-News/leeds-pressing-ahead-with-cutting-edge-flood-defence-scheme

24. A. Sleigh, I. Goodwill. *The St Venant Equations.* 2000. [Accessed 17 Oct. 2022] Available from: https://www.researchgate.net/publication/242606536

25. UK Government. *Flood Risk and Coastal Change.* 2022. [Accessed 12 Dec. 2022] Available from: https://www.gov.uk/guidance/flood-risk-and-coastal-change

26. UK Government. *Green light for next stage of Leeds Flood Alleviation Scheme.* 2019. [Accessed 12 Apr. 2023] Available from: https://www.gov.uk/government/news/green-light-for-next-stage-of-leeds-flood-alleviation-scheme

27. World Health Organisation. *Drought.* 2021. [Accessed 13 Dec. 2022] Available from: https://www.who.int/health-topics/droughttab=tab

28. Yorkshire Wildlife Trust. *Leeds Flood Alleviation Scheme Phase 2 Pilot projects.* [Accessed 19 Apr. 2023] Available from: https://upperaire.org.uk/wp-content/uploads/2021/06/210518-FAS2-Pilots.pdf

29. Z. Zhang. *Analyzing 2007 and 2019 River Don flood events in Rotherham and Sheffield; and discussing the cost-efficient schemes of flooding mitigation for Rotherham.* 2020. [Accessed 16 Oct. 2022] Available from: https://github.com/obokhove/floodproject5872math/blob/Data-and-code/ZhemingZhang

A Python codes

A.1 Code for river height plots at Armley monitoring station

The following code has taken the data on river height levels at a certain location station each day and plotted them over time. This has been applied to the Armley monitoring station located along the river Aire and has been applied over the period 01/01/2013 – 01/01/2023. The code also produces a twin plot of the average river height compared to average precipitation levels per month each year, which has again been applied in the same location over the same time frame.

```
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np

###########################################################################################
#Plot of river height from 2013-2023
###########################################################################################

Data=pd.read_csv('file:///C:/Users/fayew/Documents/_Year 4 Semester 2/5004M Project in
Mathematics/Making graphs/river-aire-leeds-armley-2013-2022.csv')
date=Data['date']
height=Data['avg_level']
heightmin=Data['min_level']
heightmax=Data['max_level']

plt.figure(1)
plt.plot(date, height, color="purple", alpha=0.65)
plt.axhline(y=3.9, color='red', linestyle='--') #River height threshold
plt.axhline(y=2.7, color='blue', linestyle='--') #Low level flooding threshold
years = ['1st January 2013', '1st January 2014', '1st January 2015', '1st January 2016',
'1st January 2017', '1st January 2018', '1st January 2019', '1st January 2020',
'1st January 2021', '1st January 2022',  '1st January 2023']
plt.xticks(np.linspace(0,3649,11), years, rotation = 20, size= 8)
plt.fill_between(date, height, color="purple", alpha=0.4)
plt.xlabel("Day")
plt.ylabel("River height (m)")

###########################################################################################
#Plot of average river height compared to average monthly precipitation for each year
###########################################################################################

fig, ax = plt.subplots()
ax2 = ax.twinx()
Year = np.arange(2013, 2023, 1)
average_h = [0.602658, 0.688907, 0.757394, 0.671161, 0.637934, 0.62569, 0.739896,
0.78547, 0.700131, 0.6086]
average_r = [20.37083, 24.47917, 24.62167, 21.74167, 21.31167, 19.66417, 28.5025,
26.06417, 25.68917, 23.88833]

ax.plot(Year, average_h, 'black', color='green', marker = 's', markersize=8)
ax.set_ylabel("Average river height per year (m)",fontsize=10)
ax.set_xlabel("Year",fontsize=10)
ax2.plot(Year, average_r, 'black', color='orange', marker = 'p', markersize=8)
ax2.set_ylabel("Average rainfall per year (mm)",fontsize=10)
```

```
ax.tick_params(labelsize=10)
ax2.tick_params(labelsize=10)
ax.spines['bottom'].set_color('black')
ax.spines['bottom'].set_linewidth(2.5)
ax.spines['left'].set_color('green')
ax.spines['left'].set_linewidth(2.5)
ax.spines['right'].set_color('orange')
ax.spines['right'].set_linewidth(2.5)
plt.show()
```

A.2 Code for hydrograph

The original hydrograph code used to produce this figure was written by Sophie Kennet [13] and provided to me by my supervisor via GitHub. I used Zheming Zhang's [29] code to include the error bounds on the hydrograph. The code produces a hydrograph, with the inclusion of error bounds and a corresponding square lake representation of the FEV. I have applied this to create the hydrograph and square lake representation for the River Aire 2015 Boxing Day Flood, monitored at Armley.

```
# -*- coding: utf-8 -*-

#Your chosen threshold height.
ht=3.9

#Your rating curve coefficients, listed starting from those corresponding to
#the lowest range of heights up until the highest range.
a=[0.156,0.028,0.153]
b=[1.115,1.462,1.502]
c=[30.69,27.884,30.127]

#Upper and lower limits of the ranges of the river heights given for your
#rating curve
lower_limits=[0.2, 0.685, 1.917]
upper_limits=[0.685, 1.917, 4.17]

#You do not have to change the following.
import matplotlib.pyplot as plt
import pandas as pd
fig, ax = plt.subplots()

#Import your data, your river height data must be saved into a csv file
#(we are using the file 'Aire Data.csv' but you must change what your csv file
#is saved as) within the same folder as this code is saved. The first column must have the
#heading 'Time', with time values converted into days (with the digits beyond the
#decimal point representing what the hours and seconds elapsed are in terms of a
#fraction of a day, more information on how to do this can be found at
#https://github.com/Rivers-Project-2018/How-to-do-FEV-Analysis/blob/master/README.md)
#and the second column must have the heading 'Height'.
Data=pd.read_csv('Armley2015.csv')
time=Data['Time']
height=Data['Height']
#Imput your own data here:##

#####You do not have to change any of the rest of the code.#####
#But if you wish to change elements such as the colour or the x and y axis
```

```
#there are instructions on how to do so.
import bisect
import numpy as np

plt.rcParams["figure.figsize"] = [11,8]
plt.rcParams['axes.edgecolor']='white'
ax.spines['left'].set_position(('zero'))
ax.spines['bottom'].set_position(('zero'))
ax.spines['left'].set_color('black')
ax.spines['bottom'].set_color('black')

time_increment=(time[1]-time[0])*24*3600

number_of_days=int((len(time)*(time[1]-time[0])))

def scale(x):
    return ((x-min(x))/(max(x)-min(x)))

error = 0.0542

error_height_up = [i * (1+error) for i in height]
error_height_down = [i * (1-error) for i in height]
scaledtime=scale(time)
scaledheight=scale(height)

w=[]
for i in range(len(a)):
    w.append(i)

def Q(x):
    z=0
    while z<w[-1]:
        if x>lower_limits[z] and x<=upper_limits[z]:
            y = (c[z]*((x-a[z])**b[z]))
            break
        elif x>upper_limits[z]:
            z = z+1
    else:
        y = (c[w[-1]]*((x-a[w[-1]])**b[w[-1]]))
    return(y)

qt = Q(ht)
qtmin = qt * (1-error)
qtmax = qt * (1+error)

Flow = []
for i in height:
    Flow.append(Q(i))

#Flow_lower = []
#for i in error_height_down:
#    Flow_lower.append(Q(i))
```

```
Flow_lower = [i * (1-error) for i in Flow]

#Flow_upper = []
#for i in error_height_up:
#    Flow_upper.append(Q(i))

Flow_upper = [i * (1+error) for i in Flow]

scaledFlow = []
for i in Flow:
    scaledFlow.append((i-min(Flow))/(max(Flow)-min(Flow)))

scaledFlow_up = [i*(1+error) for i in scaledFlow]
scaledFlow_down = [i*(1-error) for i in scaledFlow]
negheight=-scaledheight
negday=-(scaledtime)

#To change the colour, change 'conrflowerblue' to another colour such as 'pink'.
ax.plot(negheight,scaledFlow,'black',linewidth=2)
ax.plot([0,-1],[0,1],'cornflowerblue',linestyle='--',marker='',linewidth=2)
ax.plot(scaledtime, scaledFlow,'black',linewidth=2)
ax.plot(negheight, negday,'black',linewidth=2)

scaledht = (ht-min(height))/(max(height)-min(height))
scaledqt = (qt-min(Flow))/(max(Flow)-min(Flow))

QT=[]
for i in scaledFlow:
    i = scaledqt
    QT.append(i)

SF=np.array(scaledFlow)
e=np.array(QT)

ax.fill_between(scaledtime,SF,e,where=SF>=e,facecolor='cornflowerblue')

idx = np.argwhere(np.diff(np.sign(SF - e))).flatten()

f=scaledtime[idx[0]]
g=scaledtime[idx[-1]]

def unscaletime(x):
    return (((max(time)-min(time))*x)+min(time))

C=unscaletime(f)
d=unscaletime(g)

Tf=(d-C)*24

time_increment=(time[1]-time[0])*24*3600
```

```
flow = []
for i in Flow:
    if i>=qt:
        flow.append((i-qt)*(time_increment))

flow_lower = []
for i in Flow_lower:
    if i>=qtmin:
        flow_lower.append((i-qtmin)*(time_increment))

flow_upper = []
for i in Flow_upper:
    if i>=qtmax:
        flow_upper.append((i-qtmax)*(time_increment))

FEV=sum(flow)
FEV_min = sum(flow_lower)
FEV_max = sum(flow_upper)

Tfs=Tf*(60**2)

qm=(FEV/Tfs)+qt
scaledqm = (qm-min(Flow))/(max(Flow)-min(Flow))

hm=((qm/c[-1])**(1/b[-1]))+a[-1]
scaledhm = (hm-min(height))/(max(height)-min(height))

ax.plot([-scaledht,-scaledht],[-1,scaledqt],'black',linestyle='--',linewidth=1)
ax.plot([-scaledhm,-scaledhm],[-1,scaledqm],'black',linestyle='--',linewidth=1)
ax.plot([-scaledht,1],[scaledqt,scaledqt],'black',linestyle='--',linewidth=1)
ax.plot([-scaledhm,1],[scaledqm,scaledqm],'black',linestyle='--',linewidth=1)

ax.plot([f,f,f],[scaledqt,scaledqm,-1/5], 'black', linestyle='--', linewidth=1)
ax.plot([g,g,g],[scaledqt,scaledqm,-1/5], 'black', linestyle='--', linewidth=1)
ax.plot([f,f],[scaledqm,scaledqt], 'black',linewidth=1.5)
ax.plot([f,g],[scaledqm,scaledqm], 'black',linewidth=1.5)
ax.plot([f,g],[scaledqt,scaledqt], 'black',linewidth=1.5)
ax.plot([g,g],[scaledqm,scaledqt], 'black',linewidth=1.5)
plt.annotate(s='', xy=(f-1/100,-1/5), xytext=(g+1/100,-1/5), arrowprops=dict(arrowstyle='<->'))

h=[]
for i in np.arange(1,number_of_days+1):
    h.append(i/number_of_days)

#If you wish to set the flow to be shown on the axis by a certain increment, change all
#appearances of 50 in lines 153 and 157 to the desired increment, e.g 25 or 100.
#Otherwise leave as is.
l=np.arange(0,max(Flow)+50,50)
m=bisect.bisect(l,min(Flow))

n=[]
for i in np.arange(l[m],max(Flow)+50,50):
```

```
    n.append(int(i))

#If you wish to set the height to be shown on the axis by a certain increment, change all
#appearances of 1 in lines 163 and 167 to the desired increment, e.g 0.25 or 0.5.
#Otherwise leave as is.
o=np.arange(0,max(height)+1,1)
p=bisect.bisect(o,min(height))

q=[]
for i in np.arange(o[p],max(height)+1,1):
    q.append(i)

k=[]
for i in q:
    k.append(-(i-min(height))/(max(height)-min(height)))

j=[]
for i in n:
    j.append((i-min(Flow))/(max(Flow)-min(Flow)))

ticks_x=k+h

r=[]
for i in h:
    r.append(-i)

ticks_y=r+j

#Change s as needed
s=[]
for i in np.arange(1,number_of_days+1):
    s.append(i)

Ticks_x=q+s
Ticks_y=s+n

ax.set_xticks(ticks_x)
ax.set_yticks(ticks_y)
ax.set_xticklabels(Ticks_x)
ax.set_yticklabels(Ticks_y)


lists1 = sorted(zip(*[negheight, scaledFlow_down]))
negheight1, scaledFlow_down1 = list(zip(*lists1))
lists2 = sorted(zip(*[negheight, scaledFlow_up]))
negheight1, scaledFlow_up1 = list(zip(*lists2))
ax.fill_between(negheight1,scaledFlow_down1,scaledFlow_up1,color="grey", alpha = 0.3)
ax.fill_between(scaledtime,scaledFlow_up,scaledFlow_down,color="grey", alpha = 0.3)
QtU = scaledqt*(1+error)
QtD = scaledqt*(1-error)
ax.fill_between([scaledtime[idx[0]], scaledtime[idx[-1]]], QtU, QtD, color = "pink", alpha = 0
```

```
ax.tick_params(axis='x',colors='black',direction='out',length=9,width=1)
ax.tick_params(axis='y',colors='black',direction='out',length=10,width=1)

plt.text(-scaledht+1/100, -1,'$h_T$', size=13)
plt.text(-scaledhm+1/100, -1,'$h_{m}$', size=13)
plt.text(1, scaledqm,'$Q_{m}$', size=13)
plt.text(1, scaledqt,'$Q_T$', size=13)
plt.text(((f+g)/2)-1/50,-0.18,'$T_{f}$',size=13)

plt.text(0.01, 1.35,'$Q$ [m$^3$/s]', size=13)
plt.text(0.95, -0.17,'$t$ [day]', size=13)
plt.text(0.01, -1.09,'$t$ [day]', size=13)
plt.text(-1.1, 0.02,'$\overline {h}$ [m]', size=13)

ax.scatter(0,0,color='white')

A=round(FEV/(10**6),2)
B=round(Tf,2)
C=round(ht,2)
D=round(hm,2)
E=round(qt,2)
F=round(qm,2)
Amax=round(FEV_max/(10**6),2)
Amin=round(FEV_min/(10**6),2)
Emin=round(qtmin,2)
Emax=round(qtmax,2)


#Change labels as needed for indivual peaks of a double peak event
plt.text(0.2,-0.35,'$FEVA_{min}$  '+ str(Amin) +'Mm$^3$', size=12)
plt.text(0.2,-0.425,'$FEVA$  '+ str(A) +'Mm$^3$', size=12)
plt.text(0.2,-0.5,'$FEVA_{max}$  '+ str(Amax) +'Mm$^3$', size=12)
plt.text(0.2,-0.575,'$T_{f}$ = '+ str(B) +'hrs', size=12)
plt.text(0.2,-0.65,'$h_T$ = '+ str(C) +'m', size=12)
plt.text(0.2,-0.725,'$h_{m}$ = '+ str(D) +'m', size=12)
plt.text(0.2,-0.8,'$Q_{Tmin}$ = '+ str(Emin) +'m$^3$/s', size=12)
plt.text(0.2,-0.875,'$Q_T$ = '+ str(E) +'m$^3$/s', size=12)
plt.text(0.2,-0.95,'$Q_{Tmax}$ = '+ str(Emax) +'m$^3$/s', size=12)
plt.text(0.2,-1.025,'$Q_{m}$ = '+ str(F) +'m$^3$/s', size=12)

from mpl_toolkits.mplot3d import Axes3D

fig = plt.figure(figsize=plt.figaspect(1)*0.7)
ax = Axes3D(fig)
plt.rcParams['axes.edgecolor']='white'
plt.rcParams["figure.figsize"] = [10,8]

ax.grid(False)
ax.xaxis.pane.fill = False
ax.yaxis.pane.fill = False
ax.zaxis.pane.fill = False
```

```
ax.xaxis.pane.set_edgecolor('w')
ax.yaxis.pane.set_edgecolor('w')
ax.zaxis.pane.set_edgecolor('w')

sl = (FEV/2)**0.5

a = [sl, sl]
b = [sl, sl]
c = [2, 0]

d = [sl, 0]
e = [sl, sl]
f = [0, 0]

g = [sl, sl]
h = [sl, 0]
i = [0, 0]

ax.plot(a, b, c, '--', color = 'k')
ax.plot(d, e, f, '--', color = 'k')
ax.plot(g, h, i, '--', color = 'k')

x = [sl, sl, sl, 0, 0, 0, sl, sl, 0, 0, 0, 0]
y = [sl, 0, 0, 0, 0, sl, sl, 0, 0, 0, sl, sl]
z = [2, 2, 0, 0, 2, 2, 2, 2, 2, 0, 0, 2]

ax.plot(x, y, z, color = 'k')

ax.text(5*(sl/9), -5*(sl/9), 0, 'Side-length [m]', size=13)
ax.text(-sl/4, sl/4, 0, 'Side-length [m]', size=13)
ax.text(-0.02*sl, 1.01*sl, 0.8, 'Depth [m]',size=13)

ax.text(7*(sl/10), 5*(sl/4), 1, ''+str(int(round(sl)))+'m', size=13)
ax.text(14*(sl/10), 6*(sl/10), 1, ''+str(int(round(sl)))+'m', size=13)

ax.set_zticks([0, 2])

ax.set_xlim(sl,0)
ax.set_ylim(0,sl)
ax.set_zlim(0,10)
```

A.3 Code for twin plot of river height and precipitation at Armley monitoring station

This code was created to plot the river height and precipitation levels at some location on a river over a specified period. Taking the period to be 01/12/2015-01/02/2016, I have plotted the average river height each day against the recorded rainfall each day.

```
fig, ax = plt.subplots()
ax2 = ax.twinx()
Data3=pd.read_csv('file:///C:/Users/fayew/Documents/_Year 4 Semester 2/5004M Project in Mathema
date3=Data3['date']
height3=Data3['avg_level']
heightmin3=Data3['min_level']
```

```
heightmax3=Data3['max_level']
Data4=pd.read_csv('file:///C:/Users/fayew/Documents/_Year 4 Semester 2/5004M Project in Mathema
average_r=Data4['Qry_PotteryField_Daily.RAIN_mm_TOT']

ax.plot(date3, height3, 'black', color='blue', markersize=8)
ax.axhline(y=3.9, color='black', linestyle='--') #River height threshold
ax.fill_between(date3, heightmax3, heightmin3, color="skyblue", alpha=0.75)
ax.set_ylabel("Average river height (m)",fontsize=10)
ax.set_xlabel("Date",fontsize=10)
ax2.plot(date3, average_r, 'black', color='red', markersize=8)
ax2.set_ylabel("Rainfall (mm)",fontsize=10)
#ax.tick_params(labelsize=10)
months = ['1st December 2015', '1st January 2016', '1st February 2016']
ax.set_xticks(range(0,63,31))
ax.set_xticklabels(months)
#ax.bar(months, height3)
#ax.set_xticklabels(np.linspace(0,63,3), months, rotation=45)
ax.spines['bottom'].set_color('black')
ax.spines['bottom'].set_linewidth(2.5)
ax.spines['left'].set_color('blue')
ax.spines['left'].set_linewidth(2.5)
ax.spines['right'].set_color('red')
ax.spines['right'].set_linewidth(2.5)
plt.show()
```

A.4 Code for FEV and square lake length for varying h_T

Provided by Rebecca Horrocks on the Flood Projects GitHub page (available from: https://github.com), this code produces a twin plot comparing FEV and square lake length with varying river threshold heights. I have adapted the values for h_T, square lake length and FEV to fit my data, for which the FEV values for different h_T were determined by the Hydrograph code seen in Appendix A.2.

```
import matplotlib.pyplot as plt
import pandas as pd
fig, ax = plt.subplots()
ax2 = ax.twinx()

height=[2, 2.5, 3, 3.5, 3.9, 4, 4.5, 5, 5.127]
squarelake=[4208, 3700, 3180, 2612, 2161, 2043, 1392, 562, 0]
fev=[35.41, 27.38, 20.22, 13.65, 9.34, 8.34, 3.87, 0.63, 0]

ax.plot(height, fev, 'black', color='red', marker = 'o', markersize=10) #New
ax.set_xlabel("Threshold Height (m)",fontsize=15)
ax.set_ylabel("FEV (Mm$^3$)",fontsize=15)
ax2.plot(height, squarelake, 'black', color='blue', marker = 'x', markersize=10)
ax2.set_ylabel("Square Lake Side Length (m)",fontsize=15)
ax.tick_params(labelsize=15)
ax2.tick_params(labelsize=15)
ax.spines['bottom'].set_color('black')
ax.spines['bottom'].set_linewidth(2.5)
ax.spines['left'].set_color('red')
ax.spines['left'].set_linewidth(2.5)
ax.spines['right'].set_color('blue')
ax.spines['right'].set_linewidth(2.5)
```

```
plt.show()
```

A.5 Code for weir diagrams

I have created this Python code that first plots the river profile before a sharp-crested weir (figure 21), along with diagrams for a rectangular, v-notch and trapezoidal weir (figure 22). The code also provides a diagram of the river profile over a broad crested weir, shown in (figure 23). Note that the labels and arrows in the figures were added using Microsoft Word.

```
import matplotlib.pyplot as plt
import numpy as np
import math

##########################################################################################
#River diagram over a sharp-crested weir
##########################################################################################

plt.figure(1)
def water(e):
    return 4
e = np.linspace(0,4,1000)
plt.plot(e, list(map(water, e)), color="skyblue")
plt.fill_between(e, list(map(water,e)), color="skyblue", alpha=0.5)
plt.plot([0,0], [0,5], 'k--')
plt.plot([4,4], [2,5], 'k--')
def weir(p):
    if p<4.1:
        return -p+6
    else:
        return 0
p = np.linspace(4,4.1,100)
plt.plot(p, list(map(weir, p)), 'k-')
def curve(e):
    return math.atan(e-2)+2
e2 = np.linspace(0.01, 4, 1000)
plt.plot(e2, list(map(curve, e2)), color="skyblue")
plt.fill_between(p, list(map(weir,p)), color="black")
plt.plot([4,4], [0,2], 'k-')
plt.plot([0,0], [curve(0.01), curve(0.01)], 'ko')
plt.plot([4,4], [curve(4), curve(4)], 'ko')

##########################################################################################
#Rectangular sharp-crested weir
##########################################################################################

plt.figure(2)
def rec(t):
        return 3
t = np.linspace(0, 3, 1000)
plt.fill_between(t, list(map(rec, t)), color="grey", alpha=0.8)
plt.plot(t, list(map(rec,t)), 'k-')
def wall(r):
    if r<0:
        return 6
```

```
    if r>3:
        return 6
    else:
        return 0
r = np.linspace(-0.1,3.1,1000)
plt.fill_between(r, list(map(wall,r)), color="black", alpha=0.8)
plt.plot(r, list(map(wall,r)), 'k-')
def water(r):
    return 5.5
def weir(r):
    return 3
r = np.linspace(0,3,1000)
plt.fill_between(r, list(map(water, r)), list(map(weir, r)), color="skyblue", alpha=0.65)

#####################################################################################
#V-notch sharp-crested weir
#####################################################################################

plt.figure(3)
def tri(t):
    if t<3/2:
        return -2*t +6
    else:
        return 2*t
t = np.linspace(0,3,1000)
plt.fill_between(t, list(map(tri, t)), color="grey", alpha=0.8)
plt.plot(t, list(map(tri,t)), 'k-')
def wall(r):
    if r<0:
        return 6
    if r>3:
        return 6
    else:
        return 0
r = np.linspace(-0.1,3.1,1000)
plt.fill_between(r, list(map(wall,r)), color="black", alpha=0.8)
plt.plot(r, list(map(wall,r)), 'k-')
def water(t):
        return 5.5
t2=np.linspace(.25,2.75,1000)
plt.fill_between(t2, list(map(water, t2)), list(map(tri, t2)), color="skyblue", alpha=0.65)

######################################################################################
#Trapezoidal sharp-crested weir
######################################################################################

plt.figure(4)
def trap(t):
    if t<1/2:
        return -6*t + 6
    if t>5/2:
        return 6*t - 12
```

```
    else:
        return 3
t = np.linspace(0,3,1000)
plt.fill_between(t, list(map(trap, t)), color="grey", alpha=0.8)
plt.plot(t, list(map(trap,t)), 'k-')
def wall(r):
    if r<0:
        return 6
    if r>3:
        return 6
    else:
        return 0
r = np.linspace(-0.1,3.1,1000)
plt.fill_between(r, list(map(wall,r)), color="black", alpha=0.8)
plt.plot(r, list(map(wall,r)), 'k-')
def water(t):
    return 5.5
t2=np.linspace(1/12,35/12,1000)
plt.fill_between(t2, list(map(water, t2)), list(map(trap, t2)), color="skyblue", alpha=0.65)

######################################################################################
#Broad crested weir
######################################################################################

plt.figure(5)
def water(x):
    if x<-2:
        return 3
    if -2<x<0:
        return -(x+2)**2/10+3
    if 0<x<5:
        return 2
    else:
        return -(x-5)**2/10+2
x = np.linspace(-7.5,12,1000)
#plt.plot(x, list(map(water, x)), color="skyblue")
#plt.fill_between(x, list(map(water, x)), color="skyblue", alpha=0.5)
def bed(x):
    if 5<x<5.25:
        return -2*(x-5.25)**2+1
    if 5.25<x<10:
        return 1
    else:
        return 0
def curve(x):
    if 8>x>-4:
        return -math.atan(x)/5+2
    if x>8:
        return 1.715
    else:
        return 2.267
plt.plot(x, list(map(curve, x)), color="skyblue")
```

```
plt.fill_between(x, list(map(curve, x)), color="skyblue", alpha=0.5)
plt.plot(x, list(map(bed, x)), 'k')
plt.fill_between(x, list(map(bed, x)), color="grey", alpha=0.8)
plt.ylim(0,4)
```

A.6 Code for plotting a river profile

The method outlined in §9 uses this code to produce its results. I have created a code that plots the height of a river along a stream when a weir is introduced. This code calculates and models the river height using an approximation method at first, then an alternative profile is found using a more precise approximation and lastly, a third profile is found using a backwards integration method. Plots of flood storage volume, as covered in §10, using the three river profiles have also been coded. These vary depending on the chosen discharge, weir height and river bed slope. I have applied this code to a certain river section along the River Aire at Kirkstall Valley, but this can be altered for any gently sloping river section with a weir.

```
# -*- coding: utf-8 -*-
"""
Created on Fri Feb 10 17:50:18 2023

@author: fayew
"""

import matplotlib.pyplot as plt
import numpy as np
from scipy.optimize import fsolve
from sympy import *
import scipy.integrate as spi

#Define values
w=800/31 #river width
slope=-93/34000 #river bed slope
g=9.81 #acceleration due to gravity constant
Q_0=100 #discharge
C_m=0.03 #Manning coefficient
P_w=2 #weir height
C_f = (2/3)**(3/2) #discharge coefficient
s_0=0 #where weir plot begins
sigma = 200 # scale of weir
s_x = 1100 # location of weir's crest

#Find h_0 by solving Q_0 = ... replacing u with constants via Manning relation
f = lambda h_0: h_0*w * (h_0*w/(2*h_0 + w))**(2/3) * ((-slope)**(1/2) / C_m) - Q_0
h_0_ans = fsolve(f, [0, 100])
h_0 = h_0_ans[0]

h_1 = (Q_0/(g**(1/2)*w))**(2/3)
#Original estimate for height just before weir, excluding u term, h_e
hepsilon = lambda h: h - (3*Q_0**(2/3))/(2*g**(1/3)*w**(2/3)) - P_w
h_e_ans = fsolve(hepsilon, [-10000,10000])
h_e = h_e_ans[0]
#O.E.
h_e_0 = (Q_0/(g**(1/2)*C_f*w))**(2/3) + P_w
```

```
#Head of water over the weir
H = h_e - P_w

#Updated estimate for height just before weir including u term, h_e2
hepsilon2 = lambda h: h - (3*Q_0**(2/3))/(2*g**(1/3)*w**(2/3)) + (Q_0**2/(2*g*h**2*w**2)) - P_w
h_e2_ans = fsolve(hepsilon2, [-10000,10000])
h_e2 = h_e2_ans[0]

#Updated value for head of water over weir
H_2 = h_e2 - P_w

#Error of h_e estimate relative to h_e2 estimate
error = abs((h_e2-h_e)/h_e)
error2 = abs((H_2-H)/H)

#Length from s_1 (where upstream and downstream flow meet) to weir
length_s = abs((h_e-h_0)/slope)
#Location of s_1
s_1 = s_x - length_s
#y-intercept of the river bed, decided by taking s_x as the point at which the river slope =0
coef = -slope*(s_x)

#Height function relative to x-axis
def hb(s):
    if s<s_1:
        return h_0 + -(-slope*s)+coef
    else:
        return P_w + H
s = np.linspace(s_0,s_x,10000)

#Points s_a and s_b are the points at which the river bed and weir meet,
#found by setting the weir equation equal to the river bed equation
s_meet = lambda s: -(s-s_x)**2/sigma + P_w - (coef + slope*s)
s_meet_ans = fsolve(s_meet, [0,100000])
s_a = s_meet_ans[0]
s_b = s_meet_ans[1]
#Length of weir over s
s_dist = s_b-s_a

#Full eq for river bed b(s), using the points at which the weir and bed meet, s_a and s_b
def b(x):
 if(x < s_a):
     return coef+slope*x
 if(x > s_b):
     return coef+slope*x
 else:
     return (-(x-s_x)**2/sigma)+P_w
x=np.linspace(s_0, s_b+50, 10000)

#continuation of line of intial slope for reference
h_0_cont = h_0 + -(-slope*s)+coef
```

```
#Estimate for A*, h*, hydraulic radius and dsb at the point A=A*, s=s_w using shallow water th
A_star = (Q_0**2*w/g)**(1/3)
h_star = A_star/w
R = A_star/((2*A_star/w) + w)
dsb = -((C_m**2)*(Q_0**2))/((A_star**2)*(R**(4/3)))

#Formula for s_w, point at which criticality holds, using estimate for dsb given at s=s_w via
s_w1 = lambda s: -2*(s-s_x)/sigma - dsb
s_w_ans = fsolve(s_w1, [0,10000])
s_w = s_w_ans[0]

Ns=50000
#Q_1 = lambda Q: (Q**2*w/g)**(1/3) - ((-C_m**2*w**(10/9)*g**(10/9)**(2*((Q**2*w/g)**(1/3))/w +
#Q_1_ans = fsolve(Q_1, [199.5,200.5])
#Q_o = Q_1_ans[0]

#Height of the weir at s=s_w
height_weir = -((s_w-s_x)**2/sigma)+P_w

##############################################################################
#Plot of river profile
##########################################################################

plt.figure(1)
plt.plot(s, list(map(hb, s)), 'r-')
plt.plot(x, list(map(b, x)), 'k-')
plt.plot(s, h_0_cont, 'r:')
plt.plot([s_x,s_x], [0,P_w],  color='orange')
plt.plot([s_w,s_w], [height_weir,height_weir],  color='green', marker = 'x', markersize=7)
plt.plot([s_w,s_w], [h_star+P_w,h_star+P_w],  color='purple', marker = 'x', markersize=7)
plt.legend(['h(s)+b(s)','b(s)', 'h(s)+b(s) without weir','Initial weir',  'b(s = $\mathregular
plt.xlabel("s (m)")
plt.ylabel("h(s) (m)")
plt.show()

length_s_2 = abs((H_2+P_w-h_0)/slope)
s_1_2 = s_x - length_s_2 #Point at which water flowing from river and water held back by weir
coef_2 = -slope*(s_x)

def h2(s):
    if s<s_1_2:
        return h_0 + -(-slope*s)+coef_2
    else:
        return P_w + H_2

h_0_2_cont = h_0 + -(-slope*s)+coef_2

#####################################################################################
#Plot of river profile for exact estimate of height immediately before the weir, h_e2
#####################################################################################

plt.figure(2)
```

```
plt.plot(s, list(map(hb, s)), 'r-')
plt.plot(x, list(map(b, x)), 'k-')
plt.plot(s, list(map(h2, s)), color="springgreen")
plt.plot(s, h_0_2_cont, linestyle='dotted',  color="springgreen")
plt.plot([s_x,s_x], [0,P_w],  color='orange')
plt.plot([s_w,s_w], [height_weir,height_weir],  color='green', marker = 'x', markersize=7)
plt.plot([s_w,s_w], [h_star+P_w,h_star+P_w], color='purple', marker = 'x', markersize=7)
plt.legend(['Original h(s)+b(s)','b(s)','New h(s)+b(s)', 'h(s)+b(s) without weir', 'Initial wei
plt.xlabel("s (m)")
plt.ylabel("h(s) (m)")
plt.show()

###############################################################################
#Plot of zoomed in weir section of river profile
###########################################################################

plt.figure(3)
plt.plot(s, h_0_cont, 'r:')
plt.plot(x, list(map(b, x)), 'k-')
plt.plot([s_w,s_w], [height_weir,height_weir], color='green', marker = 'x', markersize=10)
plt.plot([s_x,s_x], [0,P_w],  color='orange')
plt.plot([s_w,s_w], [h_star+P_w,h_star+P_w], color='purple', marker = 'x', markersize=10)
plt.legend(['h(s)+b(s) without weir','b(s)', 's = $\mathregular{s_w}$', 'Initial weir', 'h = $\
plt.xlabel("s (m)")
plt.ylabel("h(s) (m)")
plt.xlim(s_a-50, s_b+50)
plt.ylim(-0.25, 2.5)
plt.show()


###############################################################################
#Plot of river profile via backwards integration
##############################################################################

#Approximation of dA/ds via L'Hopital's rule
A = Symbol('A')
y_1 = -g*A*(dsb + ((C_m**2*Q_0**2*(2*A/w + w)**(4/3))/A**(10/3))) #T(A)
df_A_t = y_1.diff(A) #T'(A)
y_2 = (g*A)/w - Q_0**2/A**2 #B(A)
df_A_b = y_2.diff(A) #B'(A)

#Evaluating this at A=A*, we have
df_A_start = df_A_t.subs(A, A_star) #T'(A*)
df_A_starb = df_A_b.subs(A, A_star) #B'(A*)
df_split = df_A_start/df_A_starb #T'(A*)/B'(A*)

#Number of points along s we shall find the cross sectional area of
Ns = 50000
#Section of s each estimate of A will cover
ds = (s_w-s_0)/Ns

#Have initial estimate A* and using L'Hopital's rule the second estimate, A_49999 is given by
```

```
A_49999 = A_star + ds * abs(df_split)


plt.figure(4)
#T(A)/B(A) for river bed before s_a and weir after s_a
def f(A, s):
    if s < s_a:
        return (-g*A*(slope+ C_m**2*Q_0**2*(2*A/w + w)**(4/3)/A**(10/3))) / (g*A/w - Q_0**2/A*
    else:
        return (-g*A*(-(2*(s-s_x)/sigma)+ (C_m**2*Q_0**2*(2*A/w + w)**(4/3)/A**(10/3)))) / (g*

#Points along s that we will find cross sectional area of
s_values = np.linspace(s_0, s_w, Ns)
A = 0.0*s_values
bb = 0.0*s_values
A[Ns-1] = A_star
A[Ns-2] = A_49999
bb[Ns-1]= b(s_w)
bb[Ns-2]= b(s_w - ds)
#Loop to carry out iterative formula to find A and to find b(s) along s (to check)
for i in range(Ns-3, 0, -1):
    A[i] = A[i+1] - (ds * f(A[i+1], s_values[i+1]))
    bb[i] = b(s_values[i+1])

plt.plot(x, list(map(b, x)), 'k')#to check they align
plt.plot(s, list(map(hb, s)), 'r-')
plt.plot(s, h_0_cont, 'r:')
plt.plot(s_values, (A/w)+bb, color='steelblue', ls='--')
plt.plot([s_x,s_x], [0,P_w], color='orange')
plt.plot([s_w,s_w], [height_weir,height_weir], color='green', marker = 'x', markersize=7)
plt.plot([s_w,s_w], [h_star+P_w,h_star+P_w], color='purple', marker = 'x', markersize=7)
plt.xlabel('s (m)')
plt.ylabel('h(s) (m)')
plt.legend(['Original b(s)', 'Original h(s)+b(s)', 'h(s)+b(s) without weir', 'Calculated h(s)+
plt.xlim(s_0,s_w+50)
plt.show()


###############################################################################
#Plot of river profile via forwards integration
##############################################################################

plt.figure(5)

#Using same Ns, ds and s_values as seen in the backwards integration method
A2 = 0.0*s_values
bb2 = 0.0*s_values
#Initial estimate for A and b(s) at s=Y
A2[0] = h_0*w
bb2[0]= b(s_0)

#Loop to find cross sectional area and b(s) values along s
for ii in range(1, Ns-1, 1):
```

```
    A2[ii] = A2[ii-1] + ds*f(A2[ii-1], s_values[ii-1])
    bb2[ii] = b(s_values[ii-1] + ds)
plt.plot(s_values, list(map(b, s_values)), 'k-') #to check they align
plt.plot(s_values, list(map(hb, s_values)), 'r-')
plt.plot(s, h_0_cont, 'r:')
plt.plot(s_values, (A2/w)+bb2, color='steelblue', ls='--')
plt.plot(s_values, bb2, 'y--')
plt.plot([s_x,s_x], [0,P_w], color='orange')
plt.plot([s_w,s_w], [height_weir,height_weir], color='green', marker = 'x', markersize=7)
plt.plot([s_w,s_w], [h_star+P_w,h_star+P_w], color='purple', marker = 'x', markersize=7)
plt.xlabel('s (m)')
plt.ylabel('h(s) (m)')
plt.legend(['h(s)+b(s)','h(s)', 'b(s)'])
plt.legend(['Original b(s)', 'Original h(s)+b(s)', 'h(s)+b(s) without weir', 'Calculated b(s)'
plt.xlim(s_0,s_w+50)
plt.ylim(0,6)
plt.show()

##############################################################################################
#Plot of velocity and sqrt(gh) along s, using the value of h along s determined by backwards ir
##############################################################################################

plt.figure(6)
plt.plot(s_values, list(map(b, s_values)), color="black", alpha=0.5)
plt.plot(s_values, list(map(hb, s_values)), color="red", alpha=0.5)
plt.plot(s_values, (A/w)+bb, '--', color="red", alpha=0.5)
plt.plot(s_values, ((A/w)*g)**(1/2), color='teal')
plt.plot(s_values, Q_0/(A), color='darkorange')
plt.plot([s_w,s_w], [h_star+P_w,h_star+P_w], color='purple', marker = 'x', markersize=7)
plt.legend(['Original b(s)', 'Original h(s)+b(s)', 'New h(s)+b(s)', 'u=$\mathregular{\sqrt{gh}]
plt.xlim(s_0,s_w+50)
plt.xlabel('s (m)')
plt.ylabel('h(s) (m)')

##############################################################################################
#Plot of mesh over weir up to s_w
##############################################################################################

plt.figure(7)
X3 = np.linspace(s_a, s_w, 20)
result = [spi.quad(lambda x: b(x), c, s_w)[0] for c in X3]
plt.axvline(x=s_x, color='k', ls='--')
plt.plot(X3, result, 'm')
plt.xlabel("s (m)")
plt.ylabel("Integral of b(s)")

##############################################################################################
#Plot of mesh over the whole of s, up to s_w
##############################################################################################

plt.figure(8)
X4 = np.linspace(s_0, s_a, 200)
```

```
result2 = [spi.quad(lambda x: b(x), c, s_w)[0] for c in X4]
plt.plot(X4, result2, 'g')
plt.plot(X3, result, 'm')
plt.axvline(x=s_x, color='k', ls='--')
plt.xlabel("s (m)")
plt.ylabel("Integral of b(s)")
plt.legend(['b(s) before weir', 'b(s) at weir', 's=$\mathregular{s_x}$'])

#################################################################################################
#Flood storage volume calculations from start of river profile up tp crest of the weir, s_x
#################################################################################################

FSV_1 = w*(h_e-h_0)*(s_x-s_1)/2
#FSV calculated from the refined h_e2 estimation
FSV_1_e = w*(h_e2-h_0)*(s_x-s_1)/2
#Volume underneath the backwards integration line
Vol_beta = sum(A)*ds

#Backwards integration method to find the volume underneath the section s_x to s_w,
#as this will not be included in the FSV calculation
Ns2=5000 #Smaller value as we are looking at a smaller section of s
ds2 = (s_w-s_x)/Ns2
A_4999 = A_star + ds * abs(df_split)

s_values2 = np.linspace(s_x, s_w, Ns2)

A3 = 0.0*s_values2
bb3 = 0.0*s_values2
A3[Ns2-1] = A_star
A3[Ns2-2] = A_4999
bb3[Ns2-1]= b(s_w)
bb3[Ns2-2]= b(s_w - ds2)
for i in range(Ns2-2, 0, -1):
    A3[i] = A3[i+1] - (ds2 * f(A3[i+1], s_values2[i+1]))
    bb3[i] = b(s_values2[i+1])
Vol_gamma = sum(A3)*ds2

Vol_1 = Vol_beta - Vol_gamma
Vol_2 = h_0*(s_w-s_0)*w
FSV_2 = Vol_1 - Vol_2
FSV_error = abs(FSV_1-FSV_2)/FSV_2

#############################################################################
#Plotting river heights h_0, h_e, h_e2 for different discharge rates
############################################################################

from scipy.optimize import root
#Define values for Q we want to find FSV at
Q_values = np.arange(50,500,50)
#Keep the same weir height as previously seen
P_w=2
```

```
#Finding h_0 values for corresponding discharge value
def func(h_0, Q):
    return h_0*w * (h_0*w/(2*h_0 + w))**(2/3) * ((-slope)**(1/2) / C_m) - Q
h_0_values = []
for Q in Q_values:
    sol = root(func, 1, args=(Q,))
    h_0_values.append(list({sol.x[0]}))

#Finding h_e values for corresponding discharge value
def func(h_e, Q):
    return h_e - ((3*Q**(2/3))/(2*g**(1/3)*w**(2/3)) + P_w)
h_e_values = []
for Q in Q_values:
    soly = root(func, 1, args=(Q,))
    h_e_values.append(list({soly.x[0]}))

#Finding h_e2 values for corresponding discharge value
def func(h_e2, Q):
    return h_e2 - (3*Q**(2/3))/(2*g**(1/3)*w**(2/3)) + (Q**2/(2*g*h_e2**2*w**2)) - P_w
h_e2_values = []
for Q in Q_values:
    solx = root(func, 3, args=(Q,))
    h_e2_values.append(list({solx.x[0]}))

##############################################################################
#Plotting flood storage volume for different discharge rates
#############################################################################

#To find FSV_1 = w*(h_e-h_0)**2/(2*-slope) we require h_0 and h_e values
#at each discharge rate, found in section above
#Meanwhile, slope and w remain the same

#To find h_e - h_0 = h_diff
h_0_values = np.array(h_0_values)
h_diff_values = []
h_0_neg = list(-h_0_values)
for x, y in zip(h_e_values, h_0_neg):
    h_diff_values.append(x+y)
FSV_Q_values = ((np.array(h_diff_values)**2)/(2*-slope))*w

#For h_e2
h_0_values = np.array(h_0_values)
h_2_diff_values = []
for x, y in zip(h_e2_values, h_0_neg):
    h_2_diff_values.append(x+y)
FSV_Q_he2_values = ((np.array(h_2_diff_values)**2)/(2*-slope))*w

plt.figure(9)
FSV_2_Q = [20479.06438, 22577.97906, 24273.14242, 25865.54181, 27194.82248, 28282.71072, 29166
plt.plot(Q_values, FSV_Q_values, color = 'r', marker='^')
plt.plot(Q_values, FSV_Q_he2_values, color = 'springgreen', marker='s')
plt.plot(Q_values, FSV_2_Q, color = 'steelblue', marker='H')
```

```
plt.xlabel("Discharge ($\mathregular{m^3/s}$)")
plt.ylabel("Flood Storage Volume ($\mathregular{m^3}$)")
plt.legend(["Flood Storage Volume from approximation, $\mathregular{FSV_1}$ with small u appro
plt.ylim(0, 35000)

###############################################################################
#Plotting flood storage volume for different weir heights up to s_x
###########################################################################

#Want FSV_1 = w*(h_e-h_0)**2/(2*-slope)
#slope, w and h_0 do not depend on P_w, so we must find h_e and h_e2 only

plt.figure(10)
#Define constant discharge
Q_0=100
#Values of P_w we shall use
P_w_values = np.arange(0.5,6,.5)

#Values of h_e2 for different weir heights, take equal to h_0 at P_w=0 i.e. normal stream
def funcx(h_e2, P_w):
    return h_e2 - (3*Q_0**(2/3))/(2*g**(1/3)*w**(2/3)) + (Q_0**2/(2*g*h_e2**2*w**2)) - P_w
h_e2_Pw_values = []
for P_w in P_w_values:
    solx = root(funcx, 3, args=(P_w,))
    h_e2_Pw_values.append(list({solx.x[0]}))

#Find values of h_e for different P_w values
def funcx(h_e_Pw, P_w):
    return h_e_Pw - (3*Q_0**(2/3))/(2*g**(1/3)*w**(2/3)) - P_w
h_e_Pw_values = []
for P_w in P_w_values:
    solx = root(funcx, 3, args=(P_w,))
    h_e_Pw_values.append(list({solx.x[0]}))

#Equation for FSV of approximation using h_e
FSV_Pw_values = (h_e_Pw_values - h_0)**2*w/(2*-slope)
#Equation for FSV of approximation using h_e2
FSV_Pw_he2_values = (h_e2_Pw_values - h_0)**2*w/(2*-slope)
#Values for FSV of backwards integration for different P_w values
FSV_2_Pw = [1706.293784, 6523.688909, 13484.32036, 22577.97906, 33941.84605, 47422.63388, 6332
plt.plot(P_w_values, FSV_Pw_values, 'r', marker='^')
plt.plot(P_w_values, FSV_Pw_he2_values, color='springgreen', marker='s')
plt.plot(P_w_values, FSV_2_Pw, color='steelblue', marker='H')
plt.legend(['Flood Storage Volume from approximation, $\mathregular{FSV_1}$ with $\mathregular
plt.xlabel("Weir Height, $\mathregular{P_w}$ (m)")
plt.ylabel("Flood Storage Volume, $\mathregular{V_s}$ ($\mathregular{m^3}$)")

###############################################################################
#Plotting flood storage volume for different slopes up to weir crest s_x
###########################################################################

#Want FSV_1 = w*(h_e-h_0)**2/(2*-slope)
```

```
#w, h_e and h_e2 do not depend on slope, but h_0 does

#Define constants and slope values
Q_0=100
P_w=2
slope_values = np.linspace(-0.001,-.009, 9)

#Finding h_0 for each slope value
def func(h_0, slope_s):
    return h_0*w * (h_0*w/(2*h_0 + w))**(2/3) * ((-slope_s)**(1/2) / C_m) - Q_0
h_0_values_slope = []
for slope_s in slope_values:
        sol = root(func, 1, args=(slope_s,))
        h_0_values_slope.append(list({sol.x[0]}))

#Find the difference between h_e and h_0 squared for each value of h_0
height_diff_slope = (h_e-h_0_values_slope)**2
#Put into FSV_1 equation
slope_values_denom = list(-2*np.array(slope_values))
FSV_values_slope = []
for x, y in zip(height_diff_slope, slope_values_denom ):
    FSV_values_slope.append((x/y)*w)

#Find the difference between h_e and h_0 squared for each value of h_0
height_diff_he2_slope = (h_e2-h_0_values_slope)**2
#Put into FSV_1 equation)
FSV_values_he2_slope = []
for x, y in zip(height_diff_he2_slope, slope_values_denom ):
    FSV_values_he2_slope.append((x/y)*w)

plt.figure(11)
FSV_2_slope = [41906.38517, 28047.84924, 21085.51063, 16931.3361, 14157.6442, 12106.3664, 10613
plt.plot(slope_values, FSV_values_slope, color = 'r', marker='^')
plt.plot(slope_values, FSV_values_he2_slope, color = 'springgreen', marker='s')
plt.plot(slope_values, FSV_2_slope, color = 'steelblue', marker='H')
plt.xlabel("Slope (tan \u03B1)")
plt.ylabel('Flood Storage Volume ($\mathregular{m^3}$)')
plt.legend(["Flood Storage Volume from approximation, $\mathregular{FSV_1}$, with $\mathregula
```

B Further calculations

B.1 Integrals for calculating discharge over a sharp-crested weir

Integral calculation for discharge over a v-notch sharp-crested weir.

$$\begin{aligned}\int_{h=0}^{h=H} h\sqrt{h+\frac{u_0^2}{2g}}\quad dh =&\left[h\quad\frac{2}{3}\left(h+\frac{u_0^2}{2g}\right)^{3/2}\right]_{h=0}^{h=H}-\frac{2}{3}\int_{h=0}^{h=H}\left(h+\frac{u_0^2}{2g}\right)^{3/2}dh\\ =&\frac{2}{3}H\left(H+\frac{u_0^2}{2g}\right)^{3/2}-\frac{2}{3}\left[\frac{2}{5}\left(h+\frac{u_0^2}{2g}\right)^{5/2}\right]_{h=0}^{h=H}\\ =&\frac{2}{3}H\left(H+\frac{u_0^2}{2g}\right)^{3/2}-\frac{4}{15}\left[\left(H+\frac{u_0^2}{2g}\right)^{5/2}-\left(\frac{u_0^2}{2g}\right)^{5/2}\right]\\ =&\frac{2}{3}H^{5/2}-\frac{4}{15}H^{5/2}\\ =&\frac{2}{5}H^{5/2}.\end{aligned} \tag{130}$$

Integral calculation for discharge over a trapezoidal sharp-crested weir. Splitting the integral into two, we find

$$\begin{aligned}I_1 =&\quad\int_{h=0}^{h=H} w_1\sqrt{2g\left(h+\frac{u_0^2}{2g}\right)}\quad dh\\ \Longrightarrow&\ w_1\sqrt{2g}\int_{h=0}^{h=H}\sqrt{h+\frac{u_0^2}{2g}}\quad dh\\ \Longrightarrow&\ w_1\sqrt{2g}\left[\frac{2}{3}\left(h+\frac{u_0^2}{2g}\right)^{3/2}\right]_{h=0}^{h=H}\\ \Longrightarrow&\ \frac{2}{3}w_1\sqrt{2g}\left[\left(H+\frac{u_0^2}{2g}\right)^{3/2}-\left(\frac{u_0^2}{2g}\right)^{3/2}\right]\\ \Longrightarrow&\ \frac{2}{3}w_1\sqrt{2g}H^{3/2}\end{aligned} \tag{131}$$

and

$$\begin{aligned}I_2 =&2\tan\theta\int_{h=0}^{h=H}(H-h)\sqrt{2g\left(h+\frac{u_0^2}{2g}\right)}\quad dh\\ \Longrightarrow&\ 2\tan\theta\sqrt{2g}\left(H\int_{h=0}^{h=H}\sqrt{h+\frac{u_0^2}{2g}}\,dh-\int_{h=0}^{h=H}h\sqrt{h+\frac{u_0^2}{2g}}\,dh\right)\\ \Longrightarrow&\ 2\tan\theta\sqrt{2g}\left(\frac{2}{3}H^{\frac{5}{2}}-\frac{2}{5}H^{\frac{5}{2}}\right)\\ \Longrightarrow&\ \frac{8}{15}\tan\theta\sqrt{2g}H^{\frac{5}{2}}.\end{aligned} \tag{132}$$

Therefore,

$$\begin{aligned}Q &= I_1+I_2\\ &=\frac{2}{3}w_1\sqrt{2g}H^{3/2}+\frac{8}{15}\tan\theta\sqrt{2g}H^{\frac{5}{2}},\end{aligned} \tag{133}$$

as required.

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