Homework 1 for Algebra II

Winter Quarter, 2024

(Due date: Jan. 10.)

1. Let R be a noetherian ring. Show that the power series ring $R[[x]]$ is noetherian. Conclude that for a field k, the ring of power series in n variables, $k[[x_1, \dots, x_n]]$, is noetherian.

2. Let X be a compact Hausdorff space, and let $C(X, \mathbb{C})$ be the ring of continuous complex-valued functions on X. Show that the maximal ideals of $C(X, \mathbb{C})$ are all of the form $\{f \in C(X, \mathbb{C}) | f(x) = 0\}$ for some $x \in X$; in particular, the maximal ideals are in bijection with X. (Hint: Let $\mathfrak{m} \subset C(X, \mathbb{C})$ be a maximal ideal which is not contained in one of the above. Use partitions of unity to derive a contradiction.)

3. Consider the collection $\mathrm{Gr}(2, 4)$ of 2-dimensional subspaces W of a 4-dimensional vector space V (over an algebraically closed field k). We construct a map

$$\mathrm{Gr}(2,4) \to \mathbb{P}^5_k \tag{1}$$

called the *Plücker embedding* as follows: send a subspace $W \subset V$ to the line $\bigwedge^2 W \subset \bigwedge^2 V$ (where $\bigwedge^2 V$ is six-dimensional). Show that:

   (a) The map (1) is injective.

   (b) The image of (1) is a quadric hypersurface. Specifically, the image consists of those $\omega \in \bigwedge^2 V \setminus \{0\}$ (up to scalars) such that $\omega \wedge \omega = 0$ in $\bigwedge^4 V$.

4. Let k be an algebraically closed field of characteristic $\neq 2$. In the following, the bijections should be produced using natural "algebraic" operations.

   (a) Let C be the conic in $\mathbb{P}^2_k$ defined as the vanishing locus of the equation $X^2 + Y^2 + Z^2 = 0$. Produce a bijection $\mathbb{P}^1_k \simeq C$.

   (b) Let Q be the quadric in $\mathbb{P}^3_k$ defined by the homogeneous equation $XY = ZW$. Produce a bijection $Q \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$.