HW 2: Dynamics and Control

Due Feb 14, 2020

We don't mind if you work with other students on your homework. However, each student must write up and turn in their own assignment (i.e. no copy & paste). If you worked with other students, please **acknowledge who you worked with** at the top of your homework. We recommend using LaTeXto make grading easier. For some of these questions you will need to fill in code in a Matlab file. There is only one autograder that will grade all the Matlab questions and the autograder will generate one .txt file for you to submit to Gradescope. You should submit the other questions to Gradescope separately in a PDF.

1. **Control of a Two Link Manipulator with Series Elastic Actuators**

   In this question, we'll be analyzing the stability and control for the SEA manipulator you derived in Homework 1. For this problem, you will again be using MATLAB; the matrices you're working with in this problem will be a bit too big to manipulate by hand.

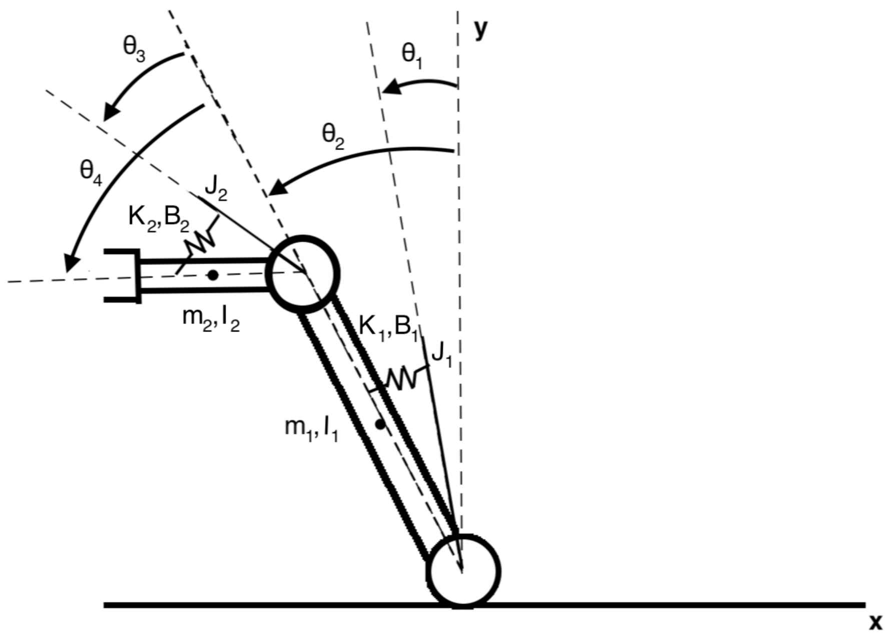

Figure 1: SEA Diagram

We consider the dynamics of this system, which can be derived from the Euler-Lagrange Equations:

$$B\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \text{ External Forces } = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + N(q,\dot{q})$$

Here, B is called the input matrix, and it captures the underactuation in the dynamics. In this case, our dynamics has four dimensions $(\theta_1, \theta_2, \theta_3, \theta_4)$, but we only control two (τ_1 and τ_2, which are the motor torques for each joint). For this problem, we will use Roy Featherstone's notation for the inverse dynamics, where

$$B\tau = D\ddot{q} + H$$

Here $H = C\dot{q} + G + N$. This notation is useful when you don't care about all the individual drift terms in the dynamics (and here we don't), and you'll often see it in papers. Perform the following calculations in the provided MATLAB code, then run the provided autograder and submit the output to gradescope.

(a) Let's define a state variable $x = [q; \dot{q}]$, which includes all four generalized coordinates and their velocities. Compute the forward dynamics of the system

$$\dot{x} = F(x, \tau)$$

(b) We'll now examine the equilibirum point $(x_{eq}, \tau_{eq}) = (0, 0)$, where the arm is pointing straight up and the motors are producing no torque. Verify for yourself that this is an equilibrium point, where $F(x_{eq}, \tau_{eq}) = 0$ (the `subs` command might be useful).

We want to show that this equilibrium point is unstable, and will do so using Lyapunov's Indirect method. Linearize the forward dynamics about this equilibrium point to produce the linear dynamics

$$\dot{x} = A_{lin}x + B_{lin}\tau$$

Remember that B_{lin} is not necessarily the same as B (why?). There should be at least one eigenvalue with a positive real component, which means that the equilibrium point is unstable.

(c) We now want to design a controller to stabilize the system to this equilibrium point, and we'll do so with linear state feedback (the easiest control technique). We'll design a controller

$$\tau = -Kx$$

to stabilize the linearized system. By Lyapunov's Indirect method, a controller that stabilizes the linearization of a system will locally stabilize the nonlinear system. In order to generate our control gain K we'll use an LQR (Linear Quadratic Regulator) controller. Design an LQR controller that minimizes the following cost function:

$$\tau = \text{argmin} \int_0^\infty (x^T Q x + \tau^T R \tau) dt$$

where $Q = I_8$ and $R = I_2$ (identity matrices). Look at the closed loop eigenvalues of the system, and verify that their real parts are all negative.

(d) Consider the closed loop dynamics of your system. What's the best lower bound you can set on the rate of convergence of your system when the state is arbitrarily close to the equilibrium point? By this I mean the largest γ for which this is true.

$$\|x(t)\| \leq \|x(0)\| e^{-\gamma t}$$

(e) *Answer this question in your PDF submission.* This linear controller will only work within some region of attraction close to the equilibrium point, yet intuitively we know that this system must be fully controllable to any equillibrium point. After all, we can control Baxter. Discuss how you might design a controller to globally stabilize this system to any equilibrium point.

2. **Dynamically-Extended Dubins Car**

The Dubins car model (also known as the unicycle model) is one of the most common dynamics models used in mobile robotics. It has three states $q = (x, y, \theta)$ and two inputs $u = (v, \omega)$.

$$\begin{aligned}\dot{x} &= v\cos(\theta)\\ \dot{y} &= v\sin(\theta)\\ \dot{\theta} &= \omega\end{aligned}$$

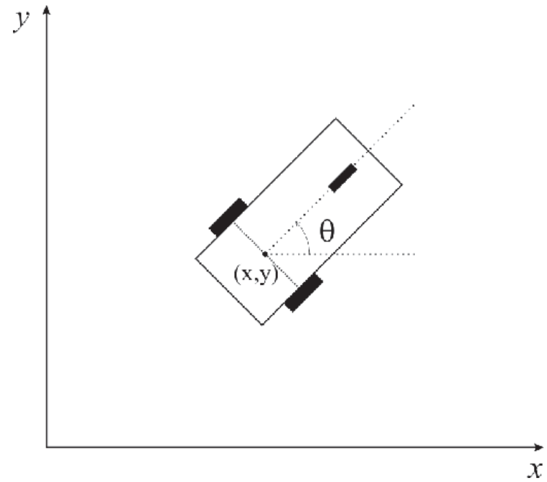

Figure 2: A Dubins car

In particular, you used the Dubins car model to design a controller for a turtlebot in EECS 106A. We've learned about feedback linearization in class, but the Dubins car model is not feedback linearizable. However, we can use a technique called *dynamic extension* to make the model feedback linearizeable. We take the input v and make it a "virtual state". We keep its value stored in memory and instead control its derivative a, the acceleration of the vehicle. The dynamics thus become

$$\begin{aligned}\dot{x} &= v\cos(\theta)\\ \dot{y} &= v\sin(\theta)\\ \dot{v} &= a\\ \dot{\theta} &= \omega\end{aligned}$$

Now we have four states $q = (x, y, v, \theta)$ and two inputs $u = (a, \omega)$.

(a) Derive a feedback linearizing control law for this system using the outputs (x, y), ie a control law $u(q, \tau)$ that will cause the system to behave like the linear system

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\ y\\ \dot{x}\\ \dot{y}\end{bmatrix} + \begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

(we use τ because the variable name v was already taken).

(b) Is this feedback linearizing control law valid at $v = 0$? Why or why not?

3. **Literature Comprehension**

Please first-pass *A Survey of Motion Planning and Control Techniques for Self-driving Urban Vehicles* and answer following questions.

(a) Read pages 5-6 and discuss the advantages and drawbacks of using the kinematic single track model, as opposed to the Dubins car model seen in the previous question?

(b) On page 21, they use an output feedback linearization to control the kinematic single track model (equations V.17 to V.19). However, they only take one derivative before both inputs appear. What does this indicate about the feedback-linearized dynamics?

While the above paper is a great reference, some variants of the kinematic single track model are also used in autonomous driving. One variant is detailed in *Kinematic and Dynamic Vehicle Models for Autonomous Driving Control Design*, and we'll use yet another in project 2.

4. **Short Problems**

   (a) What is the difference between (Jacobian) linearization, and feedback linearization?

   (b) Suppose you have a function $y = h(x)$ and that $\dot{x} = f(x) + g(x)u$. What is the second derivative of y ($\ddot{y}$). Express this using

      i. The chain rule

      ii. The Lie derivatives L_f and L_g

   Don't assume that any terms will go to zero (ie, don't make any assumptions about the relative degree of y).