MAT 116: Precalculus

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What are Polynomial Functions?

A polynomial is an expression consisting of variables and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents of variables.

What are Polynomial Functions?

Polynomials appear in a wide variety of areas of mathematics and science. They appear broadly in basic chemistry and physics to economics and soci al science; they are used in calculus and numerical analysis to approximate other functions.

Identify Polynomial Functions and Their Degree

Define with examples

A polynomial function is a function whose rule is given by a polynomial in one variable. The **degree** of a polynomial function is the largest power of x that appears. The zero polynomial function $f(x) = 0 + 0x + 0x^2 + \cdots + 0x^n$ is not assigned a degree.

Polynomial functions are among the simplest expressions in algebra. They are easy to evaluate: only addition and repeated multiplication are required. Because of this, they are often used to approximate other, more complicated functions. In this section, we investigate properties of this important class of functions.

Identifying Polynomial Functions

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

(a) $f(x) = 2 - 3x^4$ (b) $g(x) = \sqrt{x}$ (c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

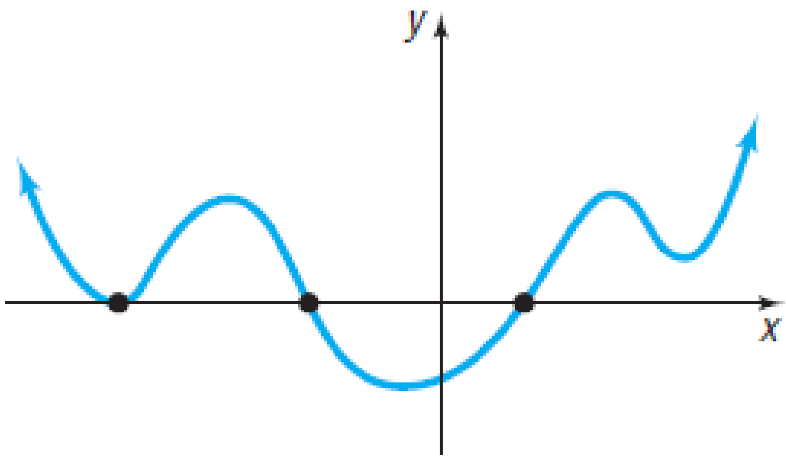
(d) $F(x) = 0$ (e) $G(x) = 8$ (f) $H(x) = -2x^3(x - 1)^2$

Polynomial Functions

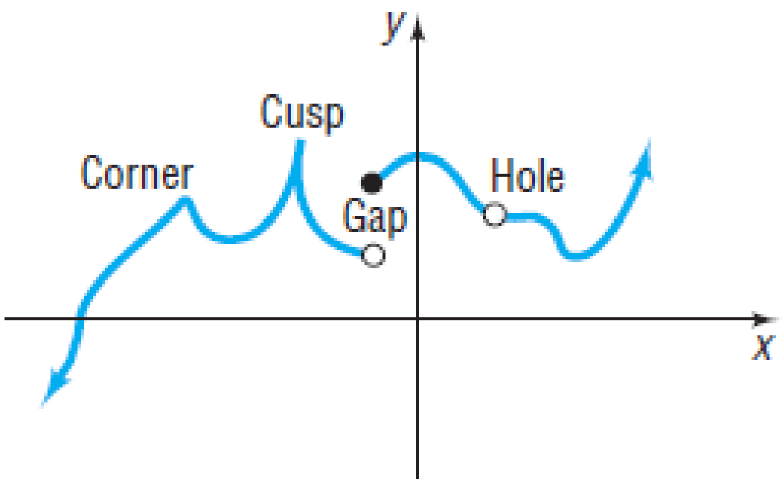
The following table gives a summary of the properties of the graphs of these polynomial functions.

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1 x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2 x^2 + a_1 x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$

Polynomial Functions



(a) Graph of a polynomial function: smooth, continuous



(b) Cannot be the graph of a polynomial function

Power Functions

A **power function of degree n** is a monomial function of the form

$$f(x) = ax^n \tag{2}$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

Power Functions

Examples of power functions are

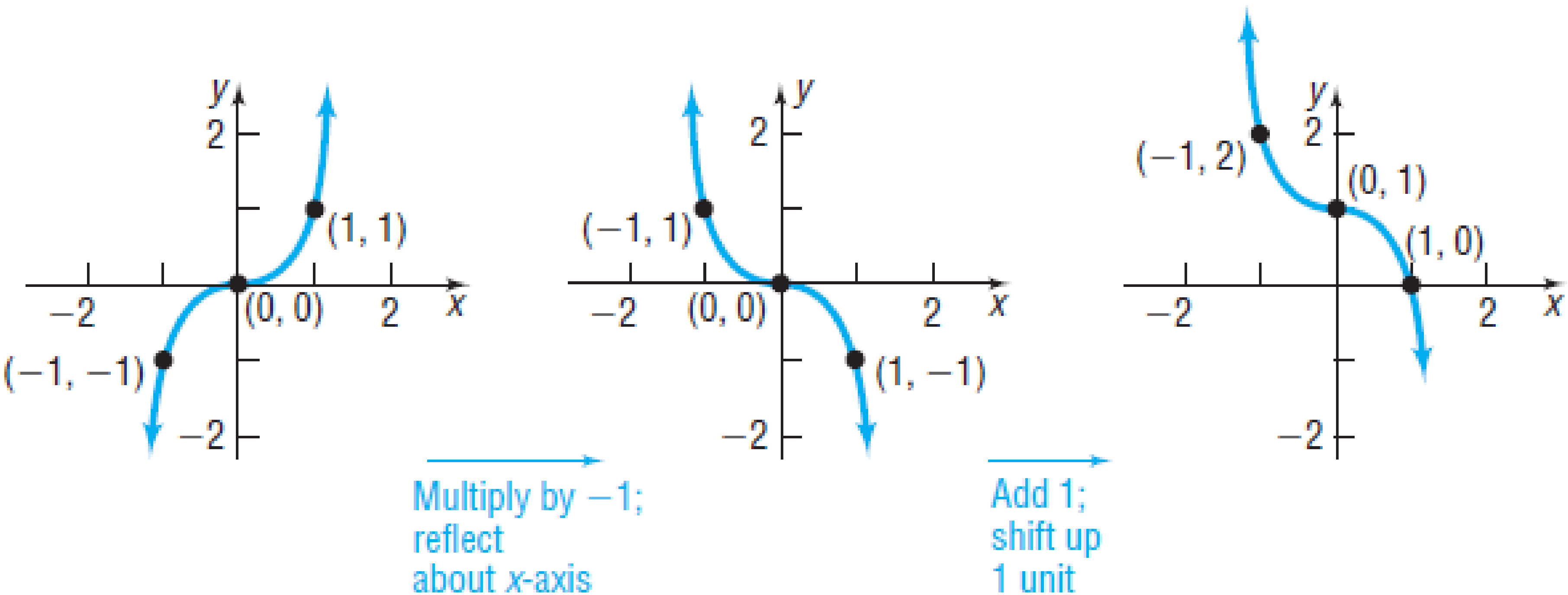
$$f(x) = 3x \qquad f(x) = -5x^2 \qquad f(x) = 8x^3 \qquad f(x) = -5x^4$$

degree 1 degree 2 degree 3 degree 4

The graph of a power function of degree 1, $f(x) = ax$, is a straight line, with slope a, that passes through the origin. The graph of a power function of degree 2, $f(x) = ax^2$, is a parabola, with vertex at the origin, that opens up if $a > 0$ and down if $a < 0$.

Graphing a Polynomial Function Using Transformations

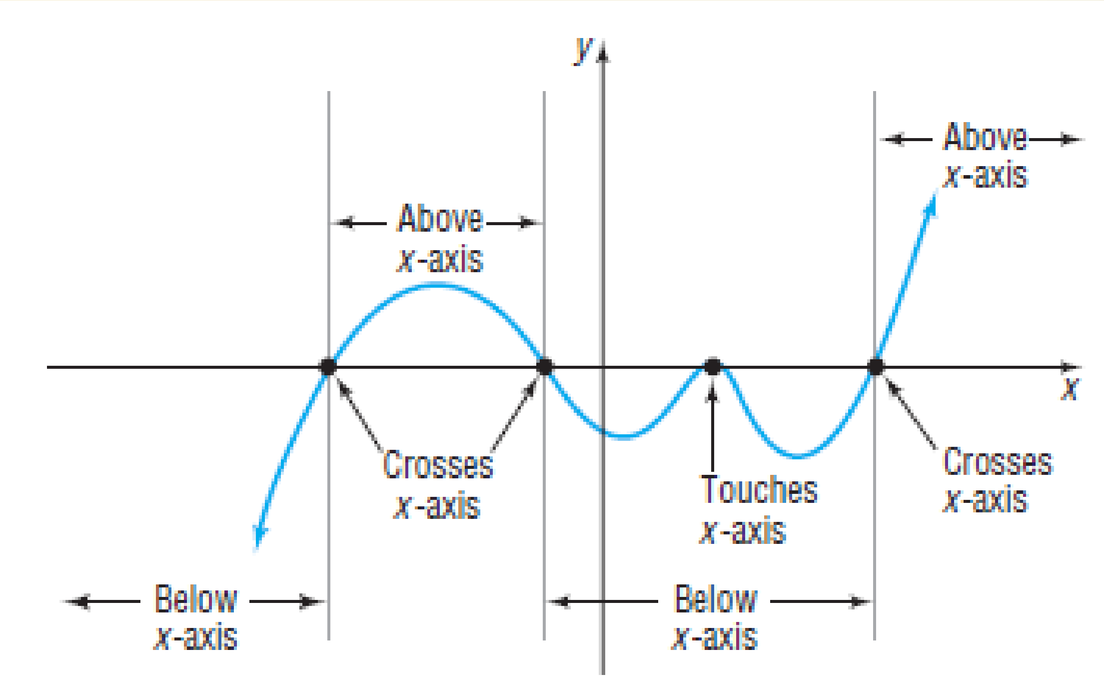
Graph: $f(x) = 1 - x^5$



(a) $y = x^5$ (b) $y = -x^5$ (c) $y = -x^5 + 1 = 1 - x^5$

Identify the Real Zeros of a Polynomial Function and Their Multiplicity

Following Figure shows the graph of a polynomial function with four *x*-intercepts. Notice that at the *x*-intercepts the graph must either cross the *x*-axis or touch the *x*-axis. Consequently, between consecutive *x*-intercepts the graph is either above the *x*-axis or below the *x*-axis.



Identify the Real Zeros of a Polynomial Function and Their Multiplicity

If a polynomial function f is factored completely, it is easy to locate the x-intercepts of the graph by solving the equation $f(x) = 0$ using the Zero-Product Property. For example, if $f(x) = (x-1)^2(x+3)$, then the solutions of the equation

$$f(x) = (x-1)^2(x+3) = 0$$

are identified as 1 and -3. That is, $f(1) = 0$ and $f(-3) = 0$.

Identify the Real Zeros of a Polynomial Function and Their Multiplicity

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f .

As a consequence of this definition, the following statements are equivalent.

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x - r$ is a factor of f.
4. r is a solution to the equation $f(x) = 0$.

Finding a Polynomial Function from Its Zeros

Problem: Find a polynomial of degree 3 whose zeros are -3, 2, and 5.

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Identifying Zeros and Their Multiplicities

For the polynomial function

$$f(x) = 5(x-2)(x+3)^2\left(x-\frac{1}{2}\right)^4$$

Finding a Polynomial Function from Its Zeros

If the same factor $x - r$ occurs more than once, r is called a **repeated,** or **multiple, zero of** f. More precisely, we have the following definition.

If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity** m **of** f.*

Turning Points

If f is a polynomial function of degree n, then f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, the degree of f is at least n.

Based on the first part of the theorem, a polynomial function of degree 5 will have at most $5 - 1 = 4$ turning points. Based on the second part of the theorem, if a polynomial function has 3 turning points, then its degree must be at least 4.

Identifying the Graph of a Polynomial Function

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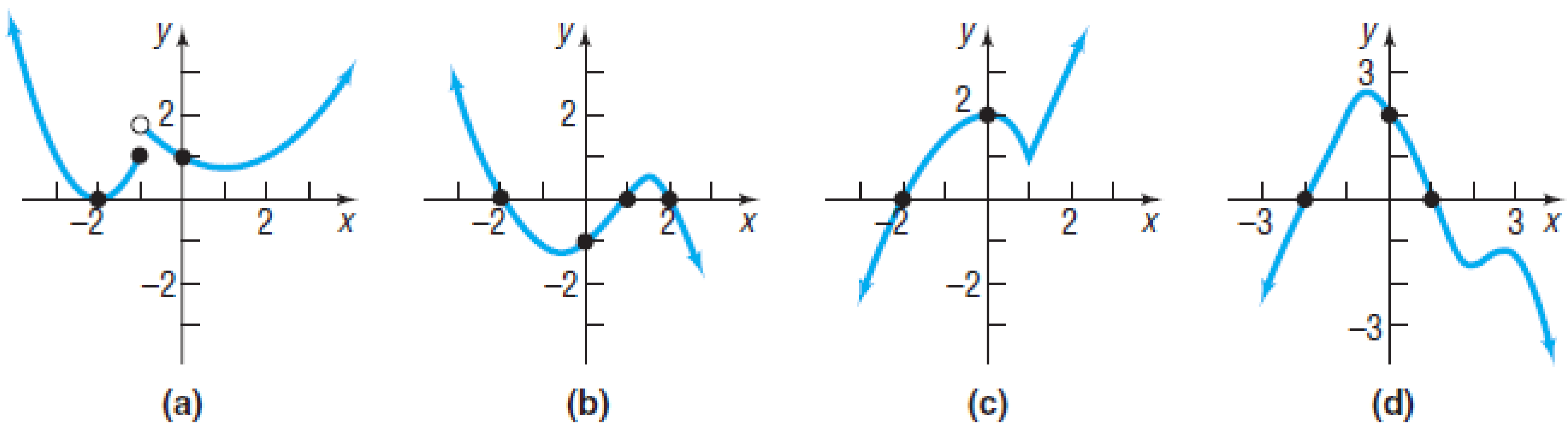
Which of the graphs in Figure 10 could be the graph of a polynomial function? For those that could, list the real zeros and state the least degree the polynomial function can have. For those that could not, say why not.

Figure 10



End Behaviour

Class work

For large values of x, either positive or negative, the graph of the polynomial function

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$$

resembles the graph of the power function

$$\boxed{y = a_n x^n}$$

End Behaviour

For example, if $f(x) = -2x^3 + 5x^2 + x - 4$, then the graph of f will behave like the graph of $y = -2x^3$ for very large values of x, either positive or negative. We can see that the graphs of f and $y = -2x^3$ "behave" the same by considering Table

x	$f(x)$	$y = -2x^3$
10	−1494	−2000
100	−1,949,904	−2,000,000
500	−248,749,504	−250,000,000
1000	−1,994,999,004	−2,000,000,000

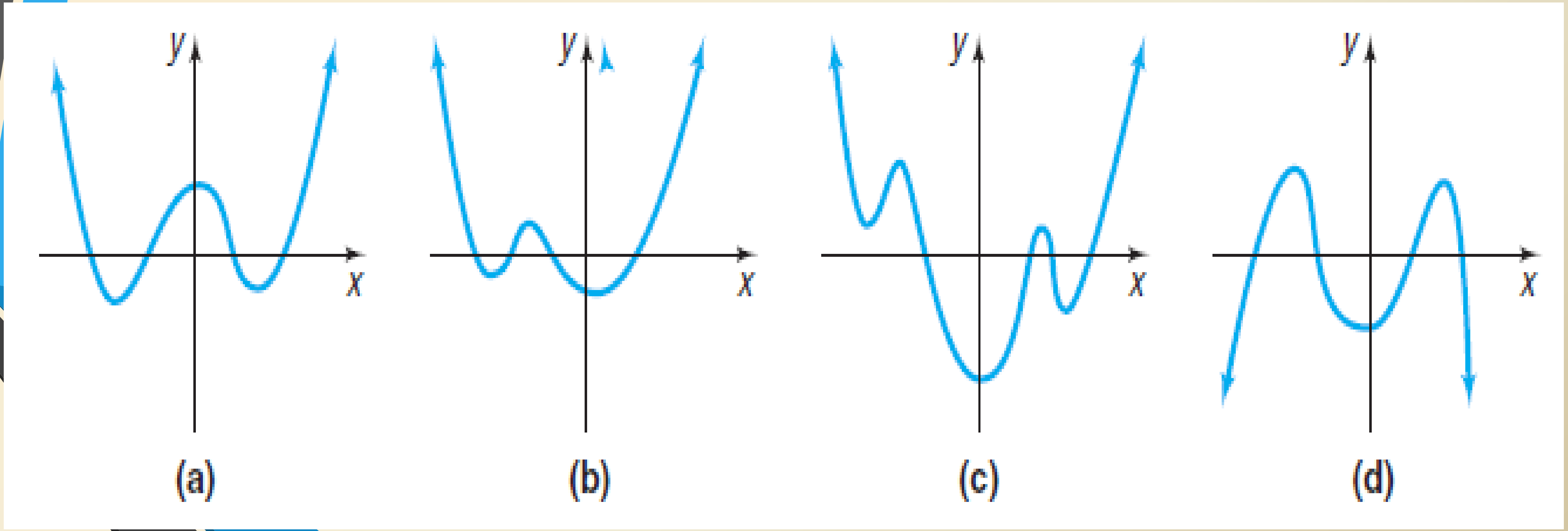
Identifying the Graph of a Polynomial Function

Problem: Class work

Which of the graphs in Figure 13 could be the graph of

$$f(x) = x^4 + ax^3 + bx^2 - 5x - 6$$

where $a > 0$, $b > 0$?



How to analyse the Graph of Polynomial Function

Class work

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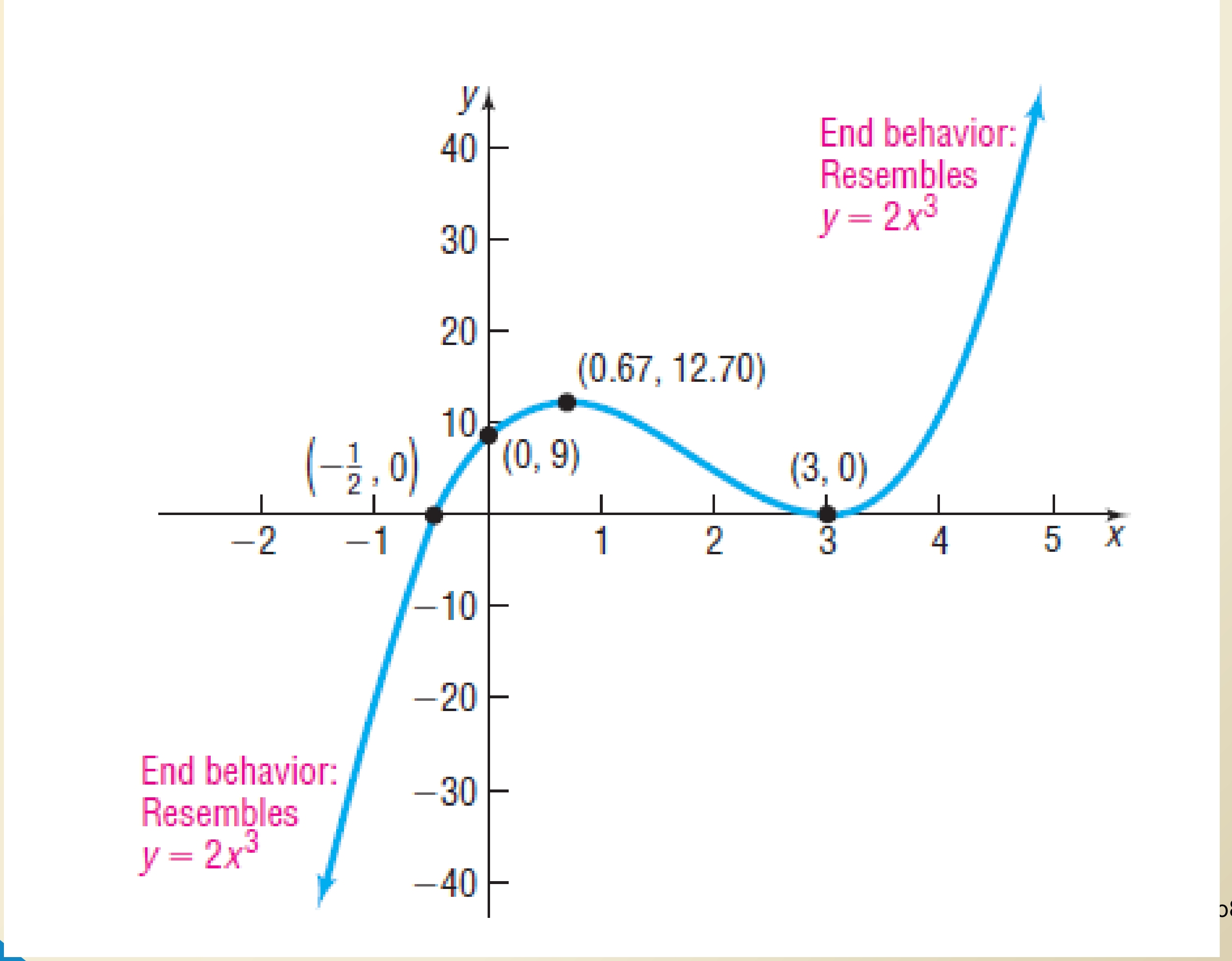
How to analyse the Graph of Polynomial Function



Remainder and Factor Theorems

When we divide one polynomial (the dividend) by another (the divisor), we obtain a quotient polynomial and a remainder, the remainder being either the zero polynomial or a polynomial whose degree is less than the degree of the divisor. To check our work, we verify that

$$(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}$$

Remainder Theorem

Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Problem (Class work)

Using the Remainder Theorem

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x - 3$ (b) $x + 2$

Factor Theorem

Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

The Factor Theorem actually consists of two separate statements:

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

Problem (Class Work)

Using the Factor Theorem

Use the Factor Theorem to determine whether the function

$$f(x) = 2x^3 - x^2 + 2x - 3$$

has the factor

(a) $x - 1$ (b) $x + 2$

*****THEOREM*****

Number of Real Zeros

A polynomial function cannot have more real zeros than its degree.

*****Prove the Theorem (H/W)*****

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*****Home Work*****

Rational Zeros of a Polynomial Function

Rational Zeros Theorem

Find the Real Zeros of a Polynomial Function

Find the Real Zeros of a Polynomial Function

Problem (Class work)

Find the real zeros of the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$. Write f in factored form.

Find the Real Zeros of a Polynomial Function

Step 1 Use the degree of the polynomial function to determine the maximum number of zeros.

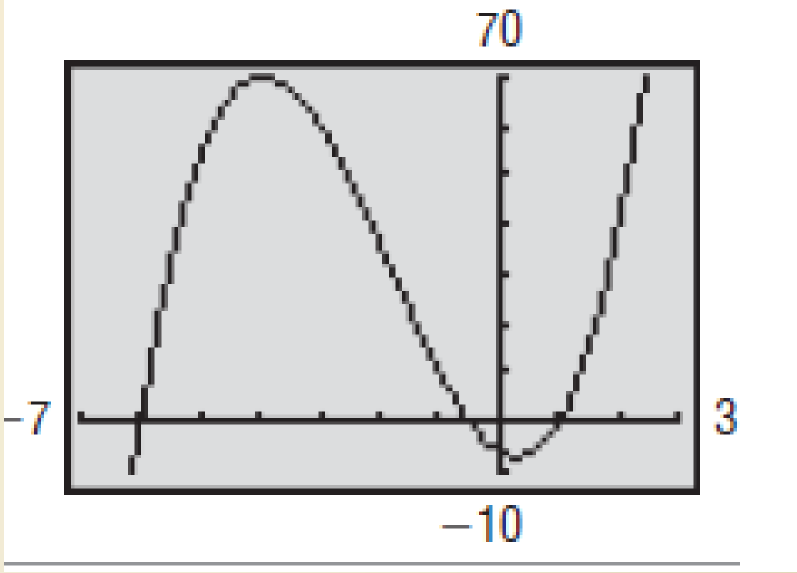
Step 2 If the polynomial function has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

Find the Real Zeros of a Polynomial Function

Step 3: Graph the polynomial function

Figure 24 shows the graph of f. We see that f has three zeros: one near –6, one between –1 and 0, and one near 1.

Step 3

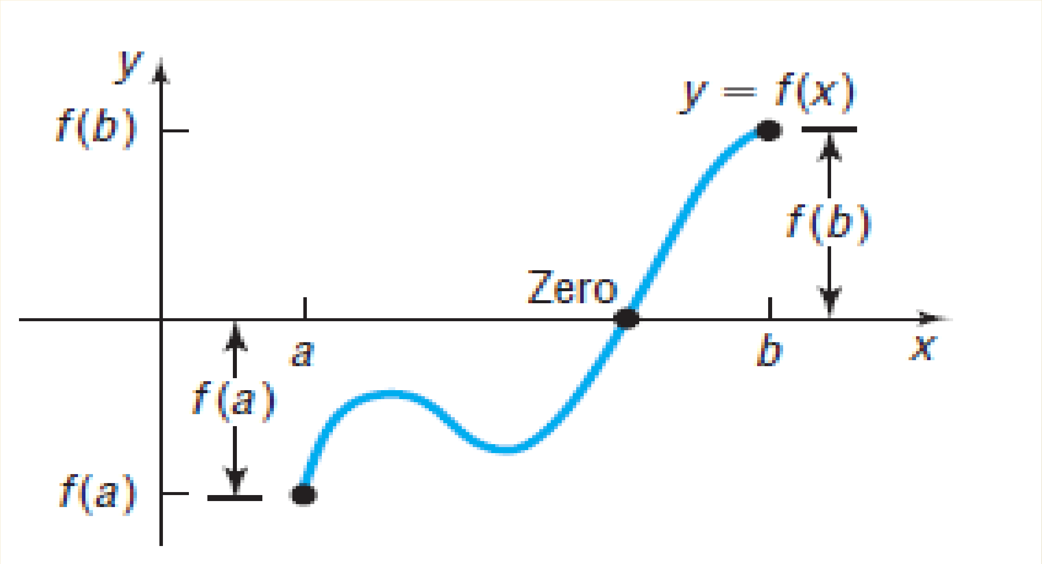


Find the Real Zeros of a Polynomial Function

Step 4 Use the Factor Theorem to determine if the potential rational zero is a zero. If it is, use synthetic division or long division to factor the polynomial function. Repeat Step 4 until all the zeros of the polynomial function have been identified and the polynomial function is completely factored.

Intermediate Value Theorem (Class Work)

Let f denote a continuous function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, then f has at least one zero between a and b.



If $f(a) < 0$ and $f(b) > 0$ and if f is continuous, there is at least one zero between a and b.

Complex Polynomial Function

(Class Work)

Fundamental Theorem of Algebra (Class work)

Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n) \qquad \textbf{(2)}$$

where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Conjugate Pairs Theorem

Theory (Class work)

Rule 1 The conjugate of a sum equals the sum of the conjugates

Rule 2 The conjugate of a product equals the product of the conjugates.

Rule 3 The conjugate of a real number equals the real number.

Finding a Polynomial Function Whose Zeros Are Given

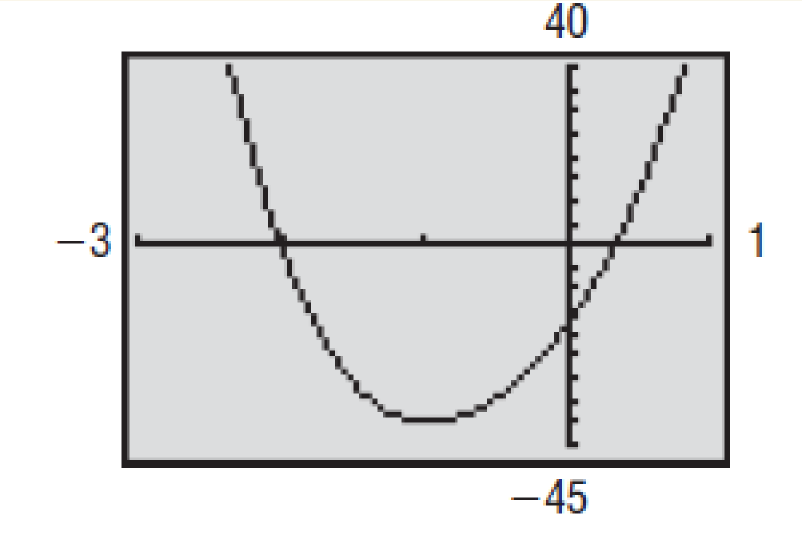
Problem: Find a polynomial function *f* of degree 4 whose coefficients are real numbers and that has the zeros 1, 1, and -4 + *i*. (Class Work)

Finding the Complex Zeros of a Polynomial Function

Problem: Class Work

Find the complex zeros of the polynomial function

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$



Rational Functions

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Domain of a Rational Function

The domain of $R(x) = \dfrac{2x^2 - 4}{x + 5}$ is the set of all real numbers x except -5; that is, the domain is $\{x | x \neq -5\}$.

The domain of $R(x) = \dfrac{1}{x^2 - 4}$ is the set of all real numbers x except -2 and 2; that is, the domain is $\{x | x \neq -2, x \neq 2\}$.

The domain of $R(x) = \dfrac{x^3}{x^2 + 1}$ is the set of all real numbers.

The domain of $R(x) = \dfrac{x^2 - 1}{x - 1}$ is the set of all real numbers x except 1; that is, the domain is $\{x \mid x \neq 1\}$. ■

Thank You for Listening

Any Questions?