Stochastic Processes 2023: Assignment
Deadline: 12 noon on 10 March 2023

Attempt all questions.

The assignment carries 90 marks and contributes 20% of your final assessment.

1. Let $X_i, i = 1, 2, ..., n$ denote random variables which are independent and Bernoulli distributed with $P(X_i = 1) = p$ and $P(X_i = 0) = q = 1 - p$. Suppose that p is a random variable with a uniform distribution on [0,1].

   (a) What is the distribution of S_n?

   [10 Marks].

2. Let N be a Poisson random variable with mean λ, with characteristic function $\exp(\lambda(e^{it} - 1))$. Let $X_1, X_2, ...$ be i.i.d. with characteristic function ϕ and independent of N.

   (a) Compute the characteristic function of the random sum

   $$S = \sum_{i=1}^{N} X_i.$$

   [10 Marks].

3. Recall a problem we did in the second problem solving session. Let $(\Omega, \mathcal{F}, \mathbf{P}; \{\mathcal{F}_n\}, \{X_n\})$ be our standard infinite product space with $\mathbf{P} = \mu^\infty$ where μ is some probability law on $\mathbf{R}$. On the same standard space, let τ be the stopping time $\inf\{k \geq 2 : X_k > X_{k-1}\}$. Which of the following events are in $\mathcal{F}_\tau$? Please justify your answer.

   (a) $\{S_{\tau-1} > 10\}$ (as usual, S_n are the partial sums)
   (b) $\inf\{k \geq 2 : X_k \leq X_{k-1}\}$

   [15 Marks].

4. Let $X(t), t \geq 0$ be a standard Poisson process and let τ_n denote the first t for which $X(t) \geq n$.

   (a) What is the mean and variance of τ_n?

   [20 Marks].

5. You start with a fortune of one million dollars. At every time $n \geq 1$ you decide on a risk factor $0 \leq q_n \leq 1$; then you flip a fair coin and your fortune is multiplied by 3^{q_n} if you win and by $(1/9)^{q_n}$ if you lose. Your goal is to get to nine million dollars.

   We proceed to construct a rigorous probability model for this scenario, as follows. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the infinite product $(\Omega_0, \mathcal{F}_0, \mathbf{P}_0)^\infty$ where Ω_0 is the set $\{0, 1\}$, $\mathcal{F}_0 = 2^{\Omega_0}$ and each $\mathbf{P}_0$ is the uniform measure on Ω_0. Let $\mathcal{F}_n = \sigma(\omega_1, \ldots, \omega_n)$ be the standard filtration. Let $\{q_n\}$ be any predictable process. Let us define Z_n, your fortune at time n, in units of millions. Let us also define a stopping time τ that occurs if and when your goal is achieved.

   (a) Give an explicit expression for both Z_n and τ.

   (b) Under what conditions on $\{q_n\}$ is Z_n a submartingale?

   (c) Under what conditions on $\{q_n\}$ is Z_n a supermartingale?

[20 Marks].

6. We return to the homework in Week 2. Let $X(t), t \geq 0$ to be a standard Poisson process with rate 1. Let $X(0) = 0$, and let $Y(t) = x + X(t) - t, t \geq 0$, where $0 < x < a$. Let τ_a denote the first time $t > 0$ for which $Y(t) \geq a$, and let τ_0 denote the first time t that $Y(t) = 0$.

   (a) If $a = 2$, find the probability that τ_0 is less than τ_a, starting at $x \in (0, a)$.

   HINT: Let $Q(x) = P_x(\tau_0 < \tau_2)$. Note that, for $0 \leq x \leq 1$, one may write:

$$Q(x) = e^{-x} + \int_0^x e^{-y} Q(x - y + 1) dy.$$

[15 Marks].