UNIVERSIDAD TECNICA
FEDERICO SANTA MARIA

The Shallow Water (Saint-Venant) Equations for Modeling of Tsunamis

Department of Mathematics (MAT-282)
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PRESENTATION OUTLINE

(1) Motivation

(2) Model Definition

(3) The Shallow Water Equations

(4) The Finite Volume Method

(5) First–Order Semi–Discrete Central–Upwind Method

(6) Some Practical Examples

(7) Possible Problems to Address

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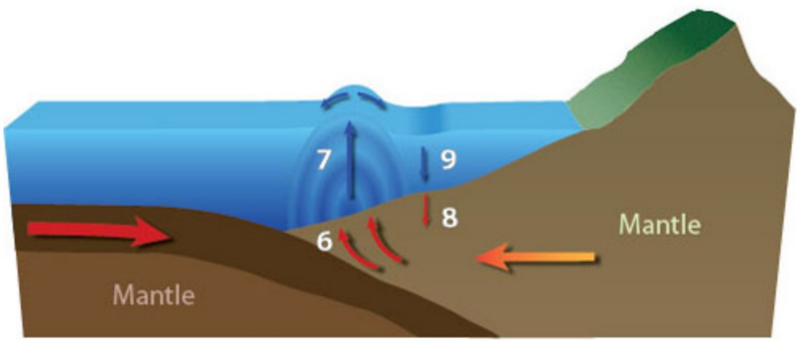
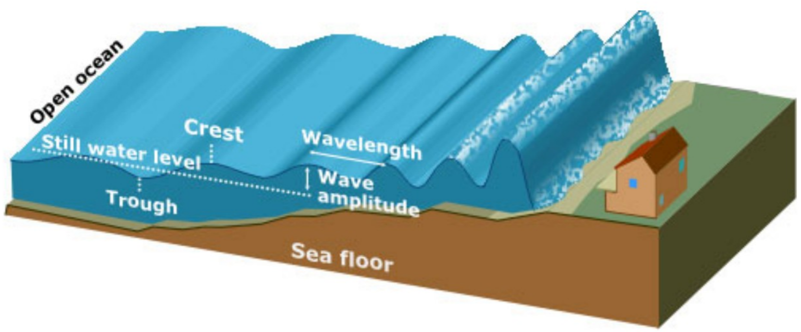
MOTIVATION

Tsunami

A Tsunami is a long wave generated by an earthquake or a sea floor disturbance. In most of the cases, the wavelength of a tsunami is usually large compared to the water depth.



(a) Wave Generation



(b) Wave Propagation

The previous assumption allows the shallow water equations to be adequate for studying tsunami evolutions. In general, explicit numerical finite difference schemes are enough to get a reliable solution.

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MOTIVATION

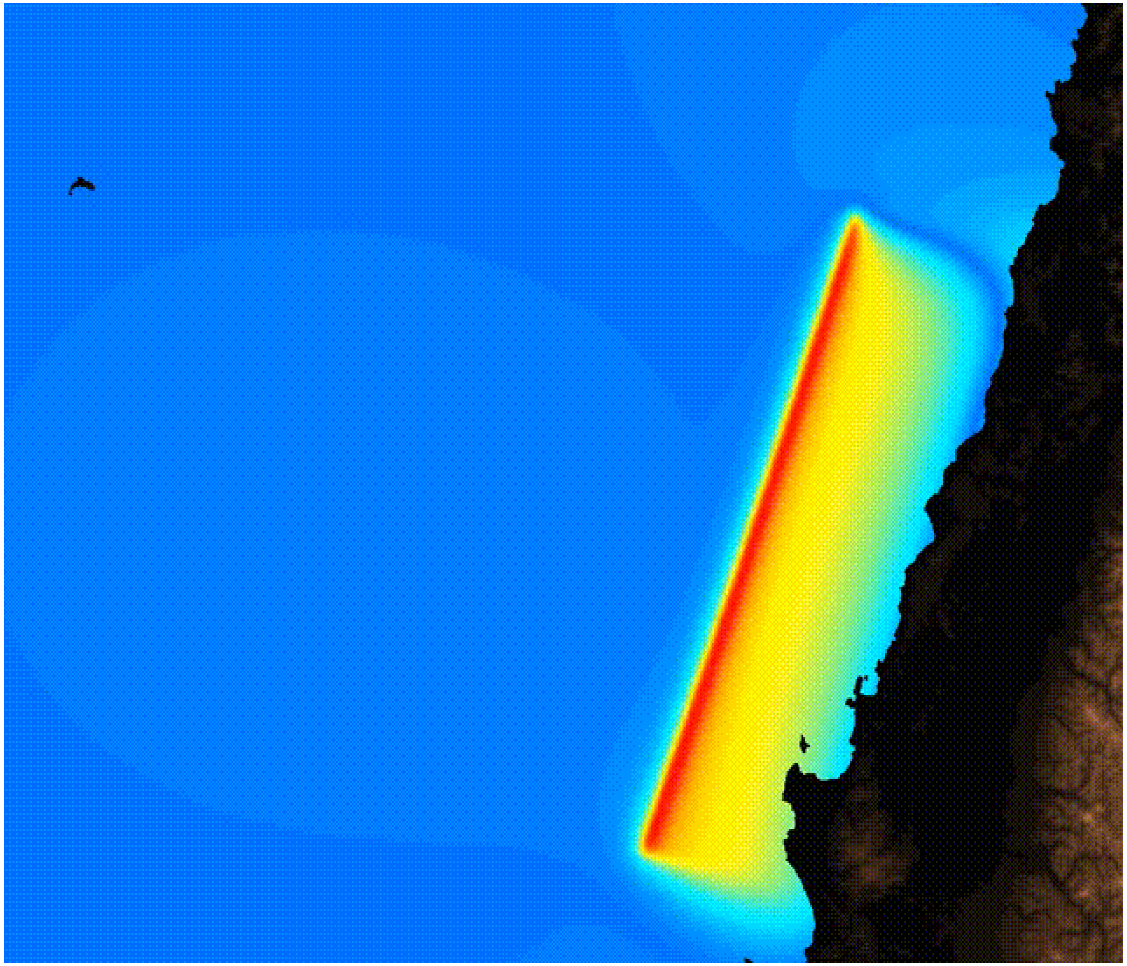
Chile before 2010, Maule

On February 27th , 2010 a major earthquake took place. This phenomenon turned out into three waves which flooded villages and towns in Concepción and Valparaíso.

A poor performance by the Chilean Government was evident. So far, Chile doesn't have a tool to predict the evolution of a tsunami.

The project FONDEF - D11I1119 rises due to the need to avoid the past experiences, and of course to avoid human casualties.

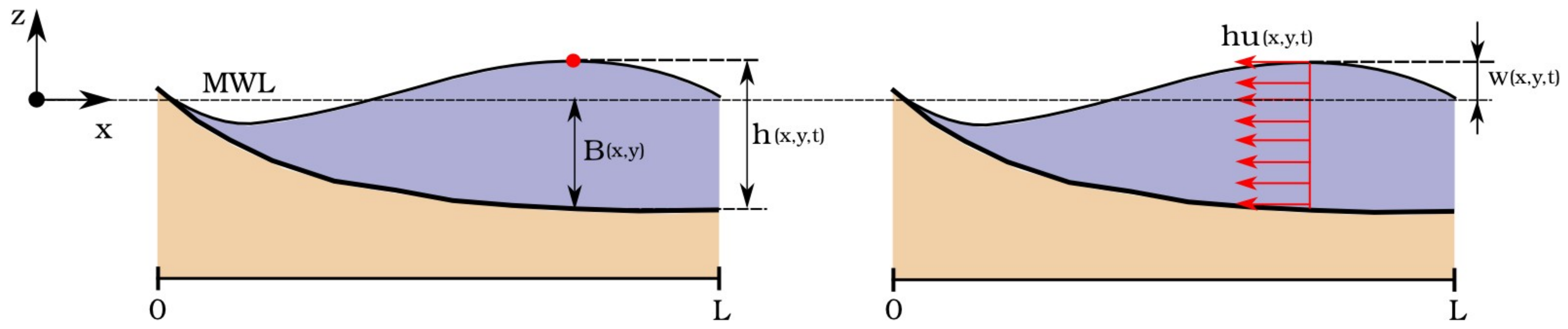
This interdisciplinary project involves several areas such as: High performance computing, Civil engineering, and management.



THE SHALLOW WATER EQUATIONS

Model Definition

- $w(x, y, t)$: Represents the surface elevation. It is measured from mean sea level to the water level.'
- $h(x, y, t)$: Represents the water depth. It is measured from the bed elevation to the water surface.
- $u(x, y, t)$: Represents the velocity field along x-direction.
- $v(x, y, t)$: Represents the velocity field along y-direction.
- $B(x, y)$: Represents the bathymetry. It is measured from the mean sea level to the bottom floor.



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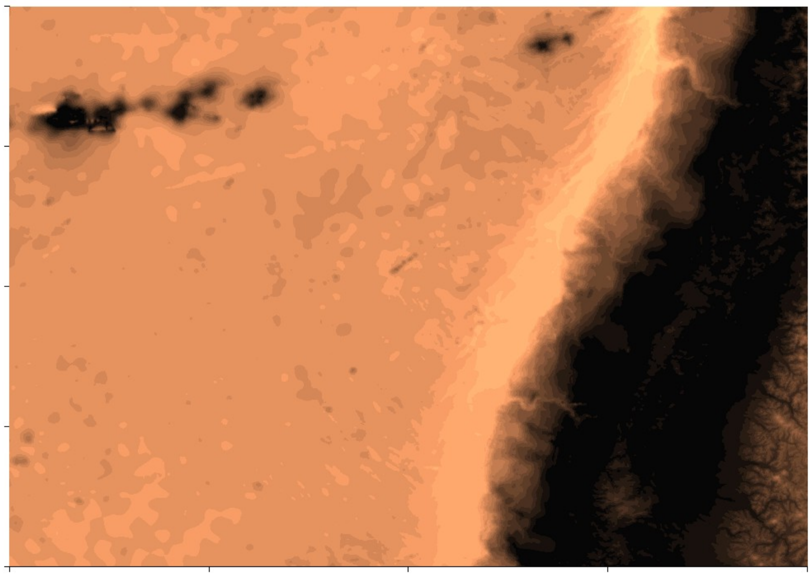
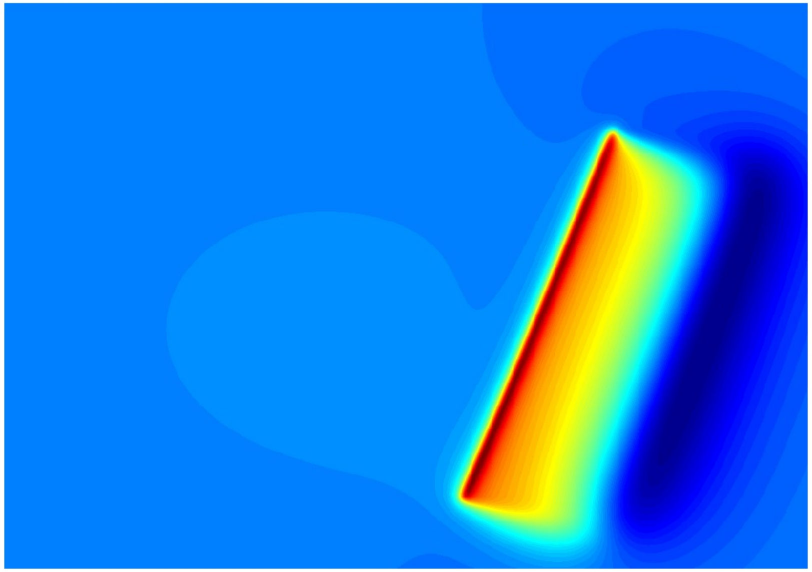
THE SHALLOW WATER EQUATIONS

Model Input Information



(a) Bathymetry



(b) Initial Condition

(C) Boundary Conditions

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THE SHALLOW WATER EQUATIONS

The Saint-Venant Equations

Assuming hydrostatic pressure condition, the SWE are obtained by integrating the Navier-Stokes equations over the water depth. A system of bi-dimensional equations is obtained, where the horizontal velocities are an average of the velocity along the water column. Neglecting the kinematic and turbulent terms, the SWE can be written as:

$$h_t + (hu)_x + (hv)_y = 0 \tag{2}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x + f \cdot hv - gn^2 \frac{(hu)\sqrt{(hu)^2 + (hv)^2}}{h^{7/3}} \tag{3}$$

$$(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghB_y - f \cdot hu - gn^2 \frac{(hv)\sqrt{(hu)^2 + (hv)^2}}{h^{7/3}} \tag{4}$$

The set of equations(2), (3) and (4) can be written down on its conserved vector form:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{q})}{\partial y} = \mathbf{S}(\mathbf{q}) + \mathbf{C}(\mathbf{q}) + \mathbf{R}(\mathbf{q}) \tag{5}$$

Where the previous variables represent:

$$\mathbf{q} = [h, hu, hv]^T \ , \mathbf{f}(\mathbf{q}) = \left[hu, hu + \frac{1}{2}gh^2, huv\right]^T \ , \mathbf{g}(\mathbf{q}) = \left[hv, huv, hv + \frac{1}{2}gh^2\right]^T$$

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The Finite Volume Method

In order to solve the set of equations presented in (2), (3), and (4) the finite volume method will be employed. First, the flux-field will be written in its vector form as: $\overrightarrow{\mathbf{F}}(\mathbf{q}) = (\mathbf{f}, \mathbf{g})$. Then, equation (5) will take the form:

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \overrightarrow{\mathbf{F}}(\mathbf{q}) = \mathbf{S}(\mathbf{q}) + \mathbf{C}(\mathbf{q}) + \mathbf{R}(\mathbf{q}) \tag{12}$$

Integrating by part over a finite triangular control volume Ω, we get:

$$\iint_{\Omega} \frac{\partial \mathbf{q}}{\partial t}\, d\Omega + \iint_{\Omega} \nabla \cdot \overrightarrow{\mathbf{F}}(\mathbf{q})\, d\Omega = \iint_{\Omega} \left(\mathbf{S}(\mathbf{q}) + \mathbf{C}(\mathbf{q}) + \mathbf{R}(\mathbf{q})\right)\, d\Omega \tag{13}$$

Applying the Stokes' theorem to a finite triangular element, we obtain:

$$\frac{\partial \mathbf{q}}{\partial t} \cdot \Omega + \oint_{\Gamma} \mathbf{F}(\mathbf{q}) \cdot \overrightarrow{n}\, d\Gamma = \mathbf{S}(\mathbf{q}) \cdot \Omega + \mathbf{C}(\mathbf{q}) \cdot \Omega + \mathbf{R}(\mathbf{q}) \cdot \Omega \tag{14}$$

Employing the Euler's method to discretize the temporal variable, we get:

$$\mathbf{q}_j^{m+1} = \mathbf{q}_j^{m} - \frac{\Delta t}{\Delta \Omega_j} \sum_{k=1}^{3} \mathbf{F}_{jk}(\mathbf{q}^m) \cdot n_{jk} \cdot l_{jk} + \Delta t \cdot \mathbf{S}_j(\mathbf{q}^m) \tag{15}$$

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First–Order Semi–Discrete Central–Upwind Method

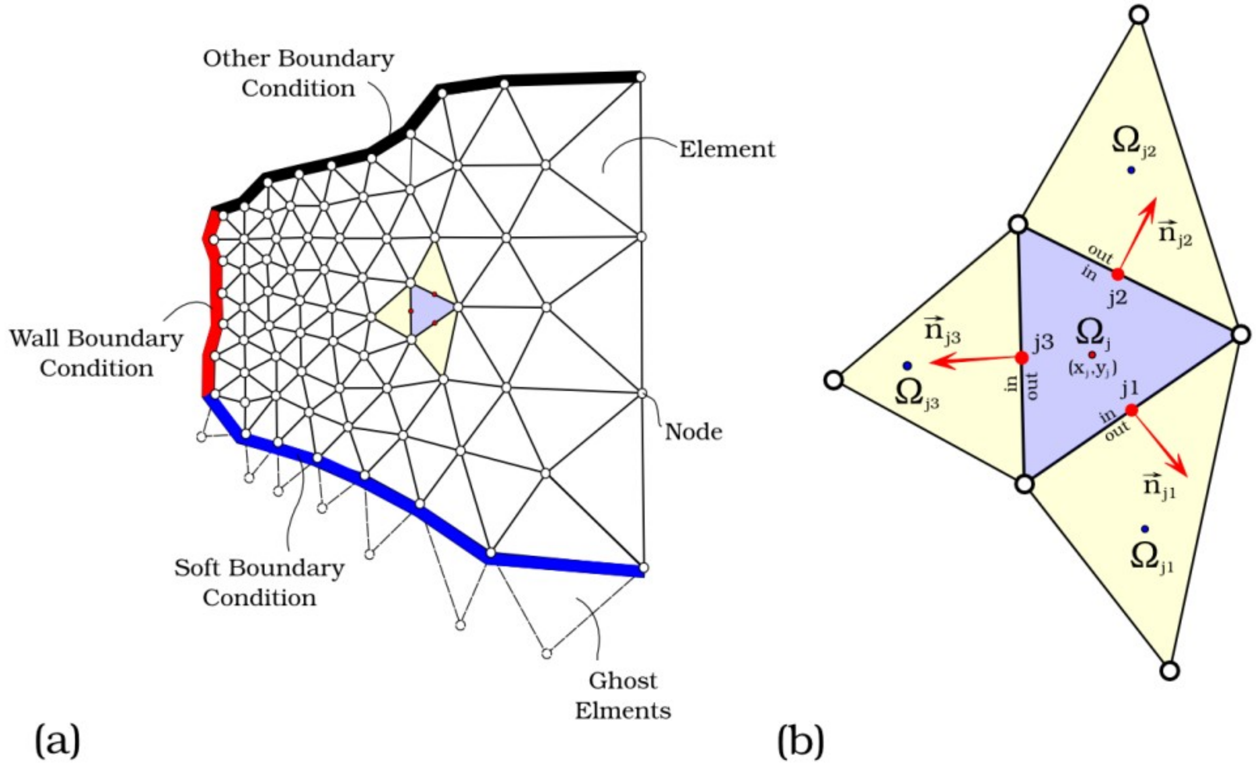

Fig. 2: (a) Finite volume mesh, (b) typical control volume and its associated variables.

THE SHALLOW WATER EQUATIONS

First–Order Semi–Discrete Central–Upwind Method

Wet/dry interface reconstruction

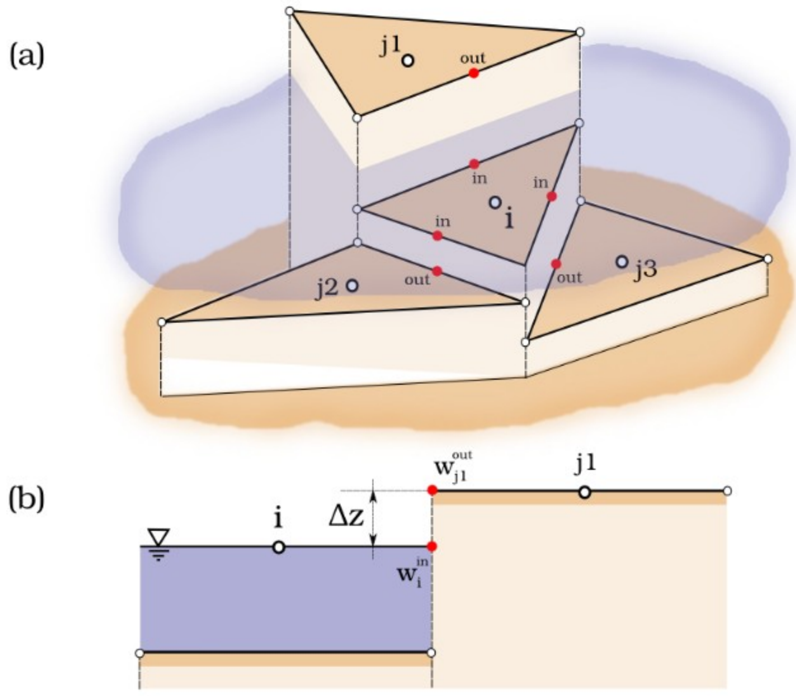

Fig. 3: (a) The i^{th} wet element shares a common edge with the $j1^{th}$ dry element, and the bed elevation of dry element is higher than the water level at the centroid of the i^{th} element. (b) The difference between the actual and fake water level at midpoint $_{jk}$.

THE SHALLOW WATER EQUATIONS

First–Order Semi–Discrete Central–Upwind Method

The complete process of reconstruction/evolution of state variables

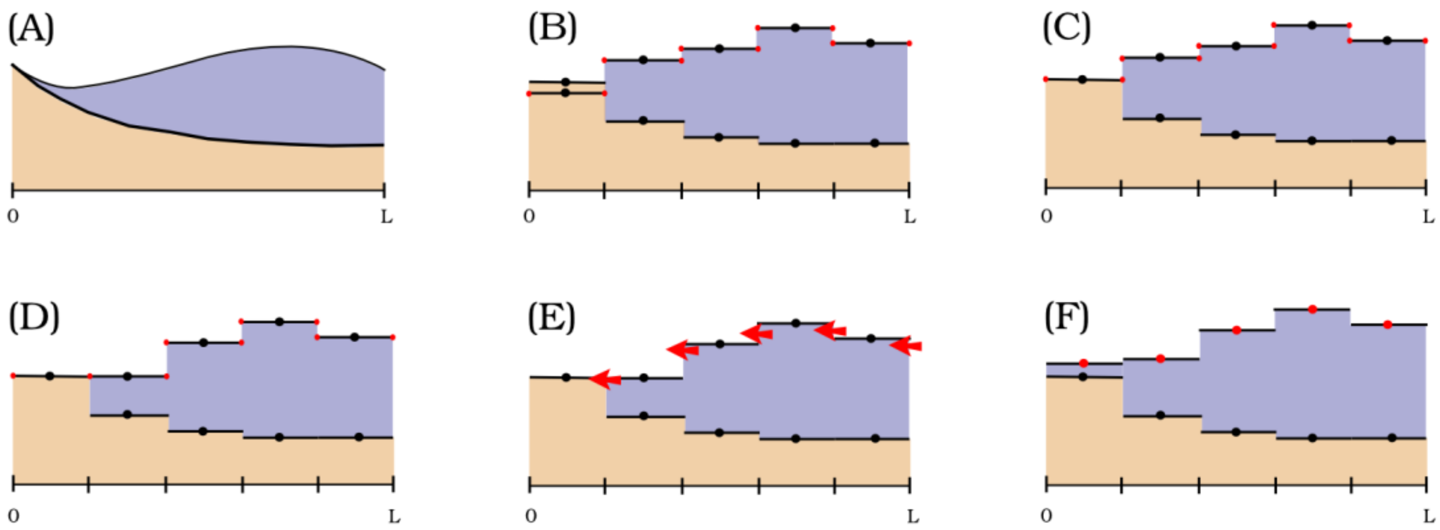

Fig. 4: First–Order Semi–Discrete Central–Upwind Method. (a) Continuous conserved variables, (b) Piece-wise constant reconstruction, (c) Positivity preserving reconstruction, (d) Well-balanced reconstruction, (e) Flux-function computation, (f) Conserved variables update.

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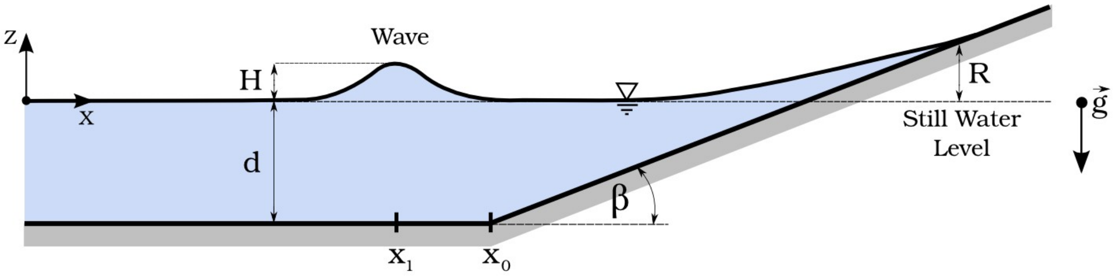
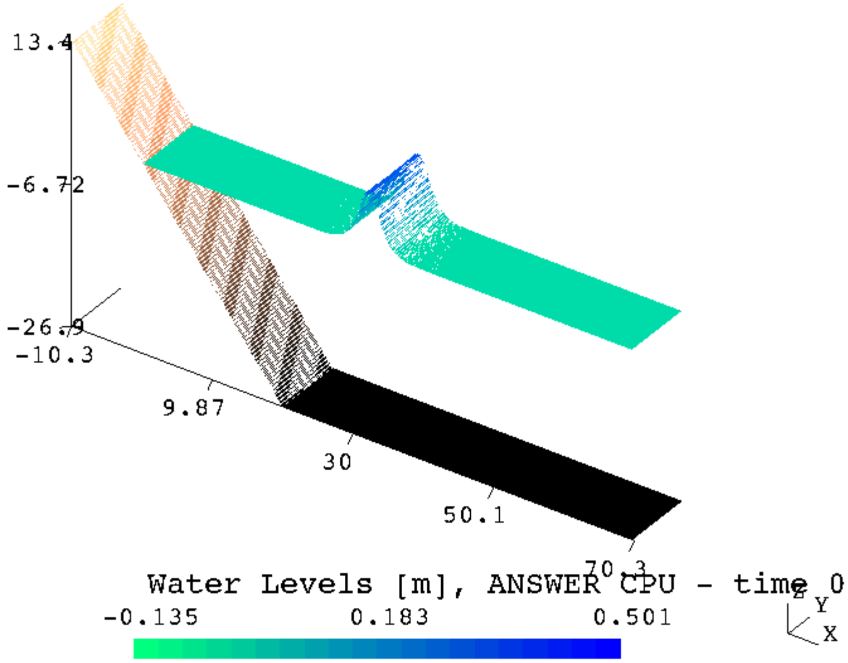
PRACTICAL EXAMPLES

Solitary Wave On A Simple Beach

In our numerical experiment, taken form [16], we consider a 80 $[m]$ long channel with soft conditions at both ends i.e, $x = -10$ $[m]$, and $x = 70$ $[m]$. Periodic condition were applied for the upper and lower boundaries i.e $y = -5$ $[m]$, and $y = 5$ $[m]$. We use the parameters $x_0 = 19.85$ $[m]$, $x_1 = 37.35$ $[m]$, $d = 1.00$ $[m]$, the gravitational constant was taken as $g = 9.80$ $[m/s^2]$. Surface roughness becomes important for runup over harsh slopes and a Manning's coefficient $n = 0.01$ describes the surface condition of the smooth glass beach in the laboratory experiments.





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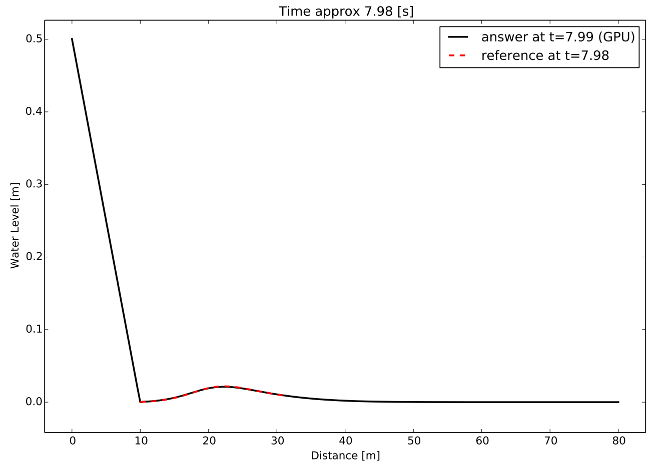
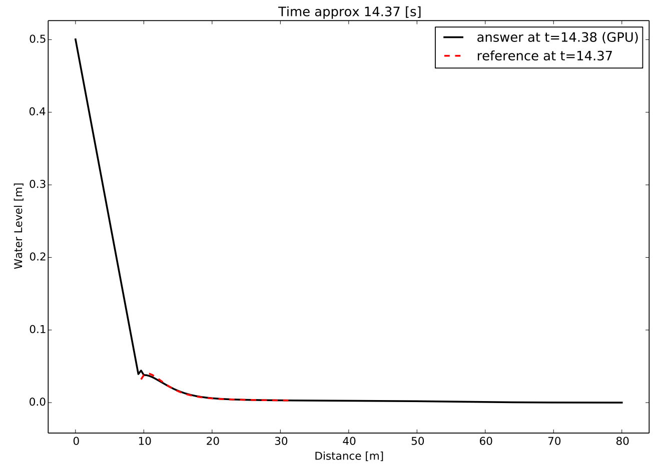
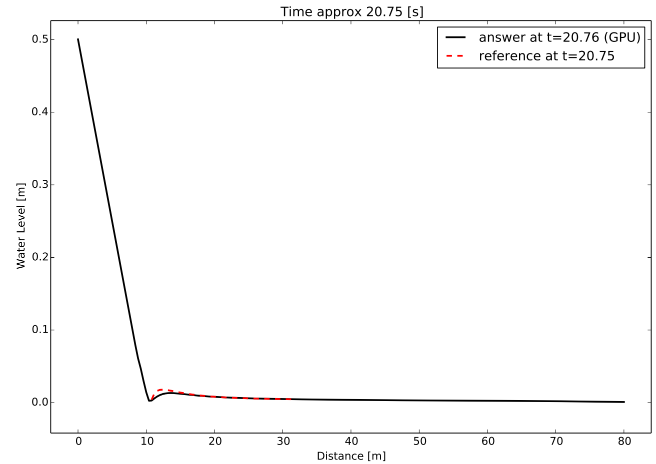
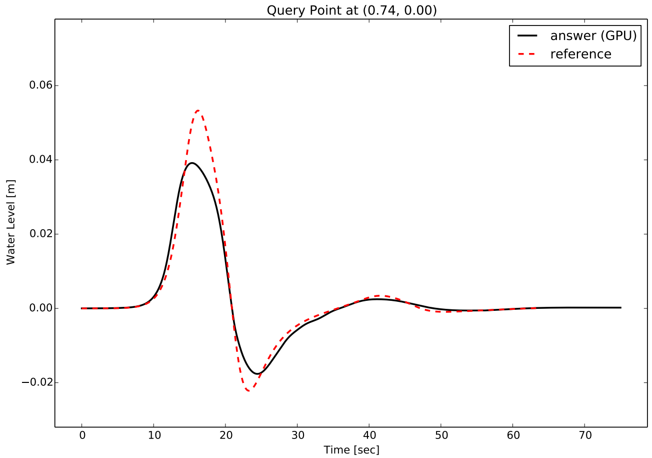
PRACTICAL EXAMPLES

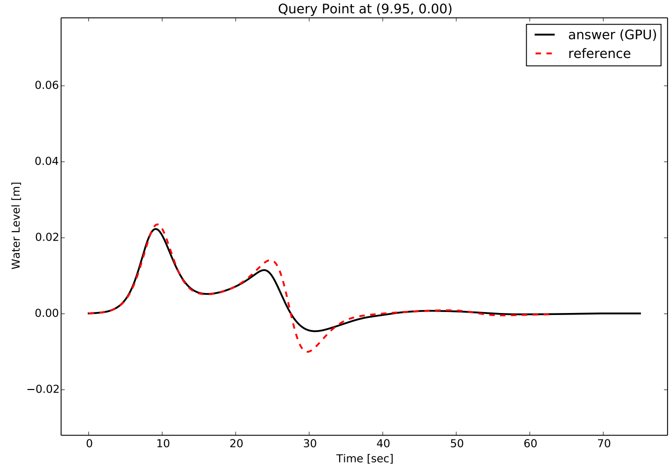
Solitary Wave On A Simple Beach

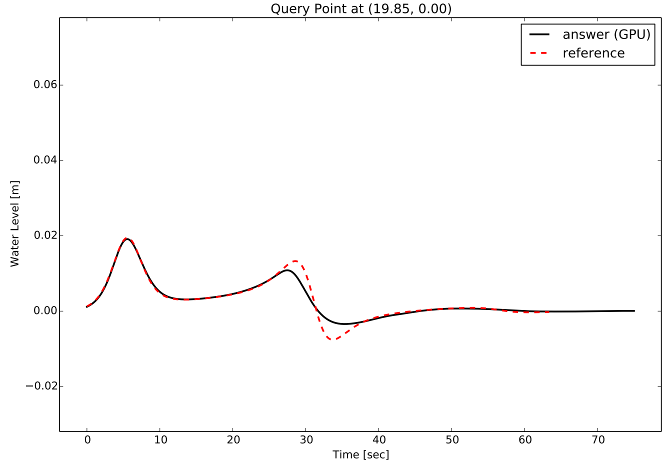






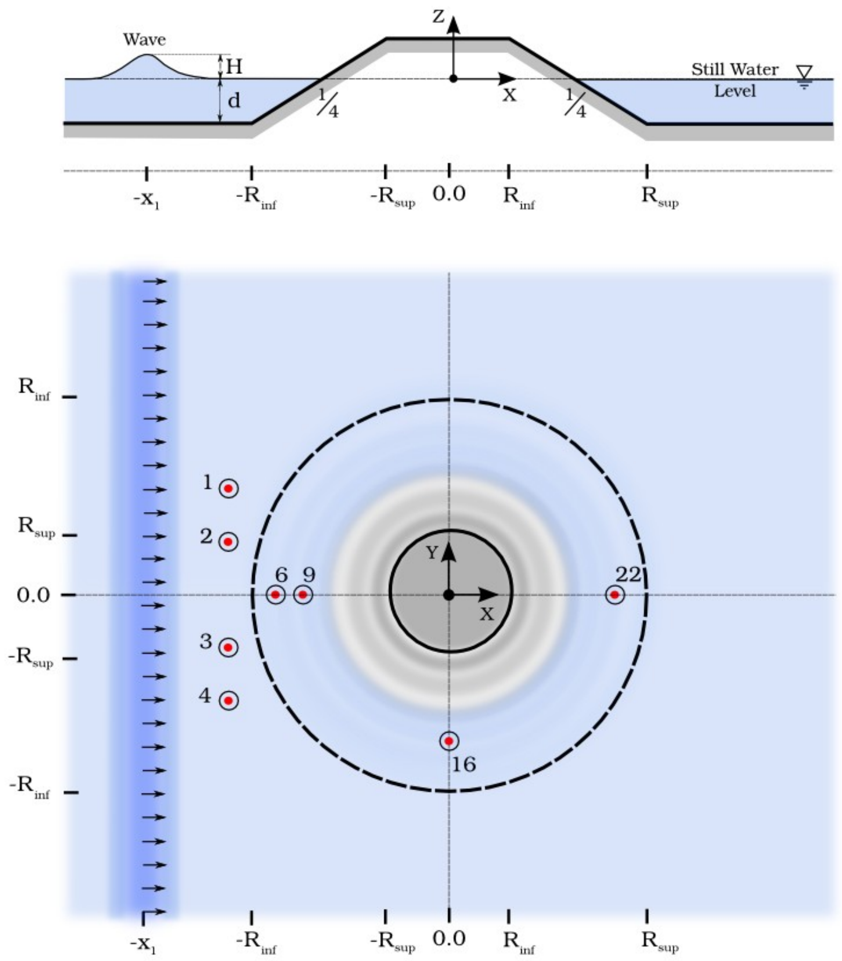


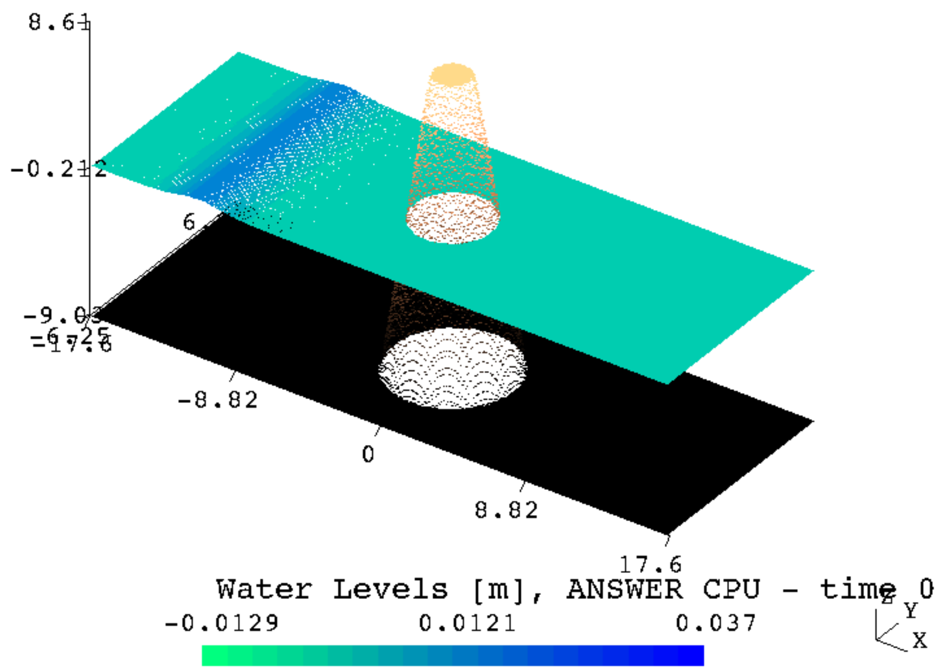




PRACTICAL EXAMPLES

Solitary Wave On A Conical Island





PROBLEMS TO ADDRESS

Apply the methodology to other problems:

- Dam break to study possible inundations (Malpasset case)
- Rain to study the distribution of water in a basin (Santiago/Valparaiso)

Modify/extend the code to improve/implement

- the spatial reconstruction of state variables as described in [1]
- the time evolution algorithm, e.g., Modified Euler, Runge Kutta, etc.
- conservation of water volume for dry/wet interface as explained in [2,3]

Bibliography

[1] Well-balanced positivity preserving central-upwind scheme on triangular grids for the Saint-Venant system.

[2] A well-balanced reconstruction of wet/dry fronts for the shallow water equations.

[3] Well-balanced positivity preserving central-upwind scheme with a novel wet/dry reconstruction on triangular grids for the Saint-Venant system.

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THANKS FOR YOUR ATTENTION

ANY QUESTIONS?

Or feel free to contact me at: danilo.kusanovic@usm.cl for any questions