

Definition of Schemes

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1 Category-theoretical stuff

Let \mathcal{C} be a sub-category of **Set** that has small limits and colimits, and let X be a topological space.

Definition 1.1. A *sheaf* on X is a functor

$$F : \text{Top}(X)^{\text{op}} \rightarrow \mathcal{C}$$

that satisfies a local-to-global condition, i.e. for $(U_i)_i$ open and $s_i \in F(U_i)$ such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique $s \in F(\bigcup_i U_i)$ such that

$$s|_{U_i} = s_i$$

Here we write $s|_V$ for the image of $s \in F(U)$ under the map $F(U) \rightarrow F(V)$ (which we get from the inclusion morphism $V \subseteq U$ in $\text{Top}(X)$).

Definition 1.2. Let F be a sheaf on X . Then the *stalk* of F at some $x \in X$ is the colimit

$$F_x := \text{colim}_{U \ni x \text{ open}} F(U)$$

Definition 1.3. Let B be a basis of X . A B -sheaf is a functor

$$F : \underbrace{\text{Top}(X)|_B}_{\text{The subcategory of Top}(X) \text{ containing only the objects from } B}^{\text{op}} \rightarrow \mathcal{C}$$

The subcategory of $\text{Top}(X)$ containing only the objects from B

that satisfies a local-to-global condition, i.e. for $(U_i)_i$ in B and $s_i \in F(U_i)$ such that

$$\forall x \in U_i \cap U_j \quad \underbrace{\exists x \in V \subseteq U_i \cap U_j : s_i|_V = s_j|_V}_{\text{Since } B \text{ is a basis, there is always at least one such } V} \quad \text{and} \quad U := \bigcup_i U_i \in B$$

there exists a unique $s \in F(U)$ such that

$$s|_{U_i} = s_i$$

Theorem 1.4. Let F be a B -sheaf for some basis B of X . Then there exists a unique (up to unique isomorphism) sheaf \tilde{F} on X that extends F .

Proof. First, we show Existence. For an open U in X , define

$$\tilde{F}(U) := \lim_{V \subseteq U, V \in B} F(V)$$

For an inclusion $U_1 \subseteq U_2$ and $s \in \tilde{F}(U_2)$, define then $\tilde{F}(U_1 \subseteq U_2)$ as the unique map such that

$$\tilde{F}(U_2) \xrightarrow{\tilde{F}(U_1 \subseteq U_2)} \tilde{F}(U_1) \rightarrow F(V) \quad \text{is} \quad \tilde{F}(U_1) \rightarrow F(V)$$

for all $V \subseteq U_1, V \in B$.

Now note that for $U_1 \subseteq U_2 \subseteq U_3$ we also have a map

$$\tilde{F}(U_3) \rightarrow \tilde{F}(U_2) \rightarrow \tilde{F}(U_1)$$

that is compatible with the maps $F(V_1 \subseteq V_2)$ for $V_1 \subseteq V_2, V_1, V_2 \in B$, and so by uniqueness above, we see

$$\tilde{F}(U_1 \subseteq U_3) = \tilde{F}(U_1 \subseteq U_2) \circ \tilde{F}(U_2 \subseteq U_3)$$

Furthermore, clearly $\tilde{F}(\text{id}_U) = \text{id}_{\tilde{F}(U)}$, so \tilde{F} is a presheaf. Note that $\tilde{F}(V) \cong F(V)$ for all $V \in B$ and similar for morphisms, so indeed \tilde{F} extends F .

Now we show the local-to-global condition. Assume we have $(U_i)_i$ open in X and $s_i \in \tilde{F}(U_i)$ such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

Let $U = \bigcup_i U_i$. By the local-to-global condition of F , for each $V \subseteq U, V \in B$ there exists a unique $s_V \in \tilde{F}(V) = F(V)$ with

$$s_V|_{V_i} = s_i|_{V_i} \quad \text{for all } V_i \subseteq U_i \cap V, V_i \in B$$

Since

$$\lim_{V \subseteq U, V \in B} F(V) \cong \{(a_V)_V \in \prod_V F(V) \mid F(V_1 \subseteq V_2)(a_{V_2}) = a_{V_1}\}$$

we see that these s_V lift to one (necessarily unique) $s \in \tilde{F}(U)$.

For Uniqueness, assume we have two such sheaves, say G and H . Now note that for all open U in X with $V_i \in B$, we have

$$G(U) \cong \{(a_V)_V \in \prod_{V \subseteq U, V \in B} F(V) \mid F(V_1 \subseteq V_2)(a_{V_2}) = a_{V_1}\}$$

where \supseteq follows from general structure and \subseteq from the local-to-global property. The same holds for H , so $G \cong H$. \square

2 Schemes

Definition 2.1. A *locally ringed space* is a topological space X with a sheaf of rings \mathcal{O}_X on X such that all stalks $\mathcal{O}_{X,x}$ are local rings.

Definition 2.2. Let R be a ring (commutative, unital). Then let $\mathcal{O}_{\text{Spec}R}$ be the sheaf on $\text{Spec}R$ that results from extending the B-sheaf

$$\mathcal{O}_{\text{Spec}R}(D_f) := R_f, \quad f \in R$$

to a sheaf as in Theorem 1.4. Here the sets

$$D_f = \{\mathfrak{p} \leq R \mid f \notin \mathfrak{p}\} \subseteq \text{Spec}R$$

are the basic open sets and form a basis of $\text{Spec}R$.

Definition 2.3. A morphism between locally ringed spaces (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) is a continuous map $f : X \rightarrow Y$ together with a natural transformation $\eta : \mathcal{O}_Y \Rightarrow f_*\mathcal{O}_X$ that satisfies

$$\eta_y : (f_*\mathcal{O}_X)_y \cong \mathcal{O}_{X,f(y)} \rightarrow \mathcal{O}_{Y,y} \quad \text{maps} \quad \eta_y(\mathfrak{m}) \subseteq \mathfrak{m}$$

for all $y \in Y$.

Here

$$f_* : \text{Sh}(X) \rightarrow \text{Sh}(Y), \quad F \mapsto (U \mapsto F(f^{-1}(U)))$$

is the pullback of f .

Definition 2.4. A locally ringed space (X, \mathcal{O}_X) is an *affine scheme*, if there exists a ring R such that $(X, \mathcal{O}_X) \cong (\text{Spec}R, \mathcal{O}_{\text{Spec}R})$.

Definition 2.5. A locally ringed space (X, \mathcal{O}_X) is a *scheme*, if there exists a covering $X = \bigcup_i U_i$ with open $U_i \subseteq X$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. Such U are also called *affine opens*.

Definition 2.6. Denote the category of schemes by **Sch** and the subcategory of affine schemes by **Aff**

Definition 2.7. Let $\phi : R \rightarrow S$ be a ring homomorphism. Then this induces a morphism of affine schemes

$$\mathrm{Spec}\phi : \mathrm{Spec}S \rightarrow \mathrm{Spec}R$$

given by

$$\mathrm{Spec}\phi : \mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$$

and on basic open sets D_g , $g \in R$ by

$$(\mathrm{Spec}\phi)_{D_g} : \mathcal{O}_{\mathrm{Spec}R}(D_g) \rightarrow \mathcal{O}_{\mathrm{Spec}S}(D_{\phi(g)}), \quad \frac{x}{g^k} \mapsto \frac{\phi(x)}{\phi(g)^k}$$

Proposition 2.8. *The functor*

$$\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \rightarrow \mathbf{Aff}$$

is an equivalence of categories.

Proposition 2.9. *Let (X, \mathcal{O}_X) be a scheme. Then X is T0.*

Proof. Assume not, i.e. there are two points $x, y \in X$ such that every open neighborhood of x contains y and vice versa. As (X, \mathcal{O}_X) has a cover by affine opens, consider an affine open U containing x, y . Then $(U, \mathcal{O}_X|_U)$ is isomorphic to $\mathrm{Spec}R$ for some ring R . However, $\mathrm{Spec}R$ is T0, a contradiction. \square

Definition 2.10. An open subscheme of a scheme (X, \mathcal{O}_X) is $(U, \mathcal{O}_X|_U)$ for an open subset $U \subseteq X$.

Remark 2.11. The notion of a general subscheme is more tricky. We more or less want to have some subset $U \subseteq X$ and a corresponding sheaf \mathcal{O}_U that should be derived from \mathcal{O}_X . However, we obviously cannot take $\mathcal{O}_X|_U$, and in fact, it is not easy to find a suitable one. Hence, for now, we will remain with defining open subschemes.

3 Coproduct (gluing) in Sch

Proposition 3.1. *For affine schemes $\mathrm{Spec}R_1$ and $\mathrm{Spec}R_2$ have that the coproduct in **Sch** is*

$$\mathrm{Spec}R_1 \sqcup \mathrm{Spec}R_2 = \mathrm{Spec}(R_1 \times R_2)$$

Proof. Observe that

$$\mathrm{Spec}(R_1 \times R_2) = \underbrace{\{\mathfrak{p} \times R_2 \mid \mathfrak{p} \in \mathrm{Spec}R_1\}}_{=:V_1} \cup \underbrace{\{R_1 \times \mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}R_2\}}_{=:V_2}$$

First of all, the projection maps $\pi_i : R_1 \times R_2 \rightarrow R_i$ give rise to morphisms

$$\phi_i : \text{Spec} R_i \rightarrow \text{Spec}(R_1 \times R_2)$$

So we have to show that this cocone is universal.

Let (W, \mathcal{O}_W) be a scheme with morphisms

$$\psi_i : \text{Spec} R_i \rightarrow (W, \mathcal{O}_W)$$

Define

$$f : \text{Spec}(R_1 \times R_2) \rightarrow W, \quad \begin{array}{ll} \mathfrak{p} \times R_2 & \mapsto \psi_1(\mathfrak{p}) \\ R_1 \times \mathfrak{p} & \mapsto \psi_2(\mathfrak{p}) \end{array}$$

This is well-defined, as $\psi_i(R_i)$ is a point such that $\psi_i(\text{Spec} R_i)$ is contained in every open neighborhood of it. Hence $\psi_1(R_1)$ and $\psi_2(R_2)$ cannot be separated by open sets, and thus must be equal as W is T0.

Furthermore, for any open $U \subseteq W$ note that

$$f^{-1}(U) = \{\mathfrak{p} \times R_2 \mid \mathfrak{p} \in \psi_1^{-1}(U)\} \cup \{R_1 \times \mathfrak{p} \mid \mathfrak{p} \in \psi_2^{-1}(U)\}$$

is a union of open sets, hence open. So f is continuous.

Now note that

$$(V_i, \mathcal{O}_{\text{Spec}(R_1 \times R_2)}|_{V_i}) \cong \text{Spec} R_i$$

where the isomorphisms are natural in V_i . So $s \in \mathcal{O}_W(U)$ yields via $\mathcal{O}_W(U) \rightarrow \mathcal{O}_{\text{Spec} R_i}(\psi_i^{-1}(U)) \cong \mathcal{O}_{\text{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_i)$ elements

$$s'_1 \in \mathcal{O}_{\text{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_1), \quad s'_2 \in \mathcal{O}_{\text{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_2)$$

These glue together to some $s' \in \mathcal{O}_{\text{Spec}(R_1 \times R_2)}(f^{-1}(U))$. Hence define a morphism of sheaves

$$\eta : \mathcal{O}_W \rightarrow f_* \mathcal{O}_{\text{Spec}(R_1 \times R_2)}, \quad \eta_U(s) := s'$$

as it is clearly compatible with restriction maps.

Now, by construction, have that for $\mathfrak{p} \in \text{Spec} R_i$

$$(f \circ \phi_i)(\mathfrak{p}) = f(\pi_i^{-1}(\mathfrak{p})) = \psi_i(\mathfrak{p})$$

and

$$\phi_i \circ \eta = \psi_i$$

The latter is true, as the isomorphism in

$$\mathcal{O}_W(U) \rightarrow \mathcal{O}_{\text{Spec} R_i}(\psi_i^{-1}(U)) \cong \mathcal{O}_{\text{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_i)$$

in the direction \leftarrow on basic open sets is given by the extension of the projection map $\pi_i : (R_1 \times R_2)_g \rightarrow (R_i)_{\pi_i(g)}$ and hence is ϕ_i .

The uniqueness of the morphism (f, η) is clear. \square

Proposition 3.2. *Let $(U_i)_i$ be a family of schemes with open subschemes $U_{ij} \subseteq U_i$ and isomorphisms $\phi_{ij} : U_{ij} \rightarrow U_{ji}$. If*

- $U_{ii} = U_i$ and $\phi_{ii} = \text{id}$
- $\phi_{ij}(U_{ij} \cap U_{ik}) \subseteq U_{ji} \cap U_{jk}$
- $\phi_{ik}|_{U_{ij} \cap U_{ik}} = \phi_{jk} \circ \phi_{ij}|_{U_{ij} \cap U_{ik}}$

then there is a scheme X (unique up to unique isomorphism) with an open cover $X = \bigcup X_i$ and isomorphisms of schemes $\psi_i : U_i \rightarrow X_i$ such that

$$\psi_j \circ \phi_{ij} = \psi_i|_{U_{ij}}$$

Remark 3.3. The above easily shows that coproducts exist in **Sch** (just take U_i to be the empty scheme $(\emptyset, \emptyset \mapsto \{0\})$). However, **Sch** is not cocomplete¹, so coequalizers do not exist in general.

4 Products in Sch

Proposition 4.1. *Let $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ be schemes over a scheme Z . The fiber product (or pullback) $X \times_Z Y$ is defined as the limit of*

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

*(Finite) fiber products exist in **Sch**.*

Proof. We show that binary fiber products exist. Let $f : X \rightarrow Z$, $g : Y \rightarrow Z$ be as above.

Step 1: X, Y, Z **affine** Assume $X = \text{Spec} R$, $Y = \text{Spec} S$, $Z = \text{Spec} T$ and f, g are induced by ring homomorphisms

$$f^\# : T \rightarrow R, \quad g^\# : T \rightarrow S$$

We claim that $\text{Spec}(R \otimes_T S)$ with

$$(\cdot \otimes 1)^\# : \text{Spec}(R \otimes_T S) \rightarrow \text{Spec} R, \quad (1 \otimes \cdot)^\# : \text{Spec}(R \otimes_T S) \rightarrow \text{Spec} S$$

works. For this, it suffices to show that $R \otimes_T S$ is the pushout (i.e. co-fiber product) of R, S considered as T -algebras. This follows easily from the universal property of the tensor product.

¹<https://mathoverflow.net/questions/9961/colimits-of-schemes>

Step 2: Y, Z affine Assume $Y = \text{Spec} S$, $Z = \text{Spec} T$. Let $X = \bigcup X_i$ be a cover of X by affine opens. We want to glue all $X_i \times_Z Y$.

Note that $U_i := X_i \times_Z Y$ comes with a map $\pi_i : U_i \rightarrow X_i$. Now we can consider the open subscheme

$$U_{ij} := \left(\pi_i^{-1}(X_i \cap X_j), \mathcal{O}_X|_{\pi_i^{-1}(X_i \cap X_j)} \right)$$

Claim: U_{ij} is the fiber product $(X_i \cap X_j) \times_Z Y$. Clearly U_{ij} has maps $U_{ij} \rightarrow X_i \cap X_j$ and $U_{ij} \rightarrow Y$ inherited from $U_i \rightarrow X_i$ and $U_i \rightarrow Y$ such that

$$\begin{array}{ccc} U_{ij} & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

commutes. It is also not too hard to show that another scheme W with compatible maps $W \rightarrow X_i \cap X_j$ and $W \rightarrow Y$ factors through U_{ij} .

Hence, there is a unique isomorphism $\phi_{ij} : U_{ij} \rightarrow U_{ji}$ by the uniqueness of limits.

Claim: The U_i , U_{ij} and ϕ_{ij} satisfy the gluing conditions.

- Clearly $U_{ii} = U_i$ and so $\phi_{ii} = \text{id}$ as there is a unique isomorphism $U_i \rightarrow U_i$.
- Note that by choice of ϕ_{ij} we know that the diagram

$$\begin{array}{ccc} & X_i & \\ \pi_i \nearrow & & \nwarrow \pi_j \\ U_{ij} & \xrightarrow{\phi_{ij}} & U_{ji} \end{array}$$

is commutative. Hence $\phi_{ij}^{-1}(\pi_j^{-1}(X_i \cap X_j)) = \pi_i^{-1}(X_i \cap X_j)$ and so $\phi_{ij}^{-1}(U_{ij}) \subseteq U_{ji}$.

- Let

$$U_{ijk} := \left(\pi_i^{-1}(X_i \cap X_j \cap X_k), \mathcal{O}_X|_{\pi_i^{-1}(X_i \cap X_j \cap X_k)} \right) = U_{ij} \cap U_{ik}$$

A similar argument as above also shows that U_{ijk} is the fiber product $(X_i \cap X_j \cap X_k) \times_Z Y$. Now, by uniqueness of limits, find that there is a unique isomorphism $U_{ijk} \rightarrow U_{kij}$. Note that both

$$\phi_{ik}|_{U_{ijk}} \quad \text{and} \quad \phi_{jk}|_{U_{jik}} \circ \phi_{ij}|_{U_{ijk}}$$

are such isomorphisms, hence

$$\phi_{ik}|_{U_{ijk}} = \phi_{jk}|_{U_{jik}} \circ \phi_{ij}|_{U_{ijk}}$$

Step 3: Z affine Exactly as in step 2 (note that we did not use Y affine there).

Step 4: General case Let $Z = \bigcup Z_i$ be a cover of Z by affine opens. □

5 Properties of Schemes and Morphisms

Proposition 5.1. *Let X be a scheme and let P be a property of affine opens (i.e. a class of embeddings $\text{Spec} R \rightarrow X$) such that*

- *for all $\alpha : \text{Spec} R \rightarrow X$ and $f \in R$ have*

$$\alpha \text{ satisfies } P \quad \Rightarrow \quad \alpha_f \text{ satisfies } P$$

where $\alpha_f : \text{Spec} R_f \rightarrow X$.

- *for all $\alpha : \text{Spec} R \rightarrow X$ and covers $\text{Spec} R = \bigcup D_{f_i}$ have*

$$\text{all } \alpha_{f_i} \text{ satisfy } P \quad \Rightarrow \quad \alpha \text{ satisfies } P$$

If there is a cover $X = \bigcup X_i$ of affine opens such that all inclusions $X_i \subseteq X$ satisfy P , then all inclusions $U \subseteq X$ of affine opens satisfy P .

Definition 5.2. A scheme X is

- *Noetherian*, if $|X|$ is quasi-compact and for all affine open U have that $\mathcal{O}_X(U)$ is Noetherian.
- *reduced*, if for all open U have that $\mathcal{O}_X(U)$ is reduced.
- *irreducible*, if $|X|$ is irreducible (i.e. not the union of two proper closed subsets).
- *integral*, if for all open U have that $\mathcal{O}_X(U)$ is integral.

Note that as defined before, the notion of an open subscheme is relatively easy. Now we will also say what a closed subscheme is.

Definition 5.3. Let (X, \mathcal{O}_X) be a scheme. A scheme (C, \mathcal{O}_C) is a closed subscheme of (X, \mathcal{O}_X) if $C \subseteq X$ is closed and there exists a quasi-coherent sheaf of ideals \mathcal{I} on \mathcal{O}_X such that

$$i_* \mathcal{O}_C \cong \mathcal{O}_X / \mathcal{I}$$

where $i : C \rightarrow X$ is the inclusion map and $i_* : \text{Sh}(C) \rightarrow \text{Sh}(X)$ its pullback.

Here a sheaf of ideals \mathcal{I} on \mathcal{O}_X is called *quasi-coherent*, if for all affine opens $U \subseteq X$, there exists an ideal $I \subseteq \mathcal{O}_X(U)$ such that

$$\forall f \in \mathcal{O}_X(U) : \mathcal{I}(D_f) = I \mathcal{O}_X(U)_f$$

Definition 5.4. A morphism of schemes $f : X \rightarrow Y$ is

- *affine*, if $f^{-1}(U)$ is affine for all affine open $U \subseteq Y$.

- *quasi-compact*, if $f^{-1}(U)$ is quasi-compact for all quasi-compact and open $U \subseteq X$.
- *locally of finite type*, if for all affine opens $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$ have that $\mathcal{O}_X(U)$ becomes a finitely generated $\mathcal{O}_Y(V)$ -algebra when equipped with

$$\mathcal{O}_Y(V) \xrightarrow{f^\#} \mathcal{O}_X(f^{-1}(V)) \rightarrow \mathcal{O}_X(U)$$

- *finite type*, if it is quasi-compact and locally of finite type
- a *closed immersion*, if it is an isomorphism onto a closed subscheme of Y .
- an *open immersion*, if it is an isomorphism onto an open subscheme of Y .
- *flat*, if all $\mathcal{O}_{X,x}$ are flat $\mathcal{O}_{Y,f(x)}$ -modules, i.e. the functor $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \cdot$ of $\mathcal{O}_{Y,f(x)}$ -algebras is exact².
- *separated*, if $\Delta : X \rightarrow X \times_B X$ is a closed immersion. Here Δ is the unique morphism $X \rightarrow X \times_B X$ such that $\pi_i \circ \Delta = \text{id}_X$, where $\pi_i : X \times_B X \rightarrow X$.
- *universally closed*, if for each $h : Z \rightarrow Y$ the unique morphism $X \times_Y Z \rightarrow Z$ that makes the diagram

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

commute is closed.

- *proper*, if it is of finite type, separated and universally closed.

We say a scheme X is separated, if the canonical morphism $X \rightarrow \text{Spec} \mathbb{Z}$ is.

Definition 5.5. A scheme X is called *variety*, if it is of finite type, integral and separated.

Proposition 5.6 (Characterization Noetherian). *A quasi-compact scheme X is Noetherian, if and only if there is a cover $X = \bigcup_i U_i$ of affine opens such that all $\mathcal{O}_X(U_i)$ are Noetherian.*

Proposition 5.7 (Characterization reduced). *A scheme X is reduced, if and only if all stalks $\mathcal{O}_{X,x}$ are reduced.*

Proposition 5.8 (Characterization irreducible). *A scheme X is irreducible, if and only if all affine opens $U \subseteq X$ are irreducible³.*

²here $\mathcal{O}_{X,x}$ becomes a $\mathcal{O}_{Y,f(x)}$ -algebra via f_x

³Irreducibility is not affine-local, and it does not suffice to have a cover of irreducible affine opens.

Proposition 5.9 (Characterization integral). *A scheme X is integral, if and only if it is irreducible and reduced.*

Proposition 5.10 (Characterization flat). *An R -module M is flat if and only if for all injective $\alpha : N_1 \rightarrow N_2$ have that $\text{id} \otimes \alpha : M \otimes_R N_1 \rightarrow M \otimes_R N_2$ is injective.*