Structure-Zoo

Inhaltsverzeichnis

1 Topological spaces

1.1 Euclidean space

Definition	$(\mathbb{R}^n, \tau_{\text{Eucl}})$
type	Normed space via $\|\cdot\|_2$
separation	T4
compact	no, $(ne_1)_{n\in\mathbb{N}}$ has no convergent subnet
Baire	yes, as completly metrizable
connected	connected
countability	second countable via $\{B_q(r) \mid q, r \in \mathbb{Q},\ q > 0\}$; separable

Note that this is homeomorphic to each open ball.

1.2 Compact Euclidean spaces

1.3 Euclidean space

Definition	$([0,1]^n, \tau_{\text{Eucl}}\|_{[0,1]^n})$
type	Normed space via $\\| \cdot \\|_2$
separation	T4
compact	yes, by Heine-Borel
Baire	yes, as completly metrizable
connected	connected
countability	second countable; separable (see 1.1)

1.4 Co-finite topology

Let X be infinite.

Definition	$(X, \tau_{\text{co-finite}} := \{U \subseteq X \mid \text{card}(U^c) < \aleph_0\} \cup \emptyset)$
type	not metric, as not Hausdorff
separation	T1, as there are no disjoint open sets except $\emptyset$
compact	yes, as a chain of finite sets contains a smallest element
Baire	iff X is uncountable
connected	connected, as there are no disjoint, open, nonempty sets
countability	separable; if X is uncountable then not first countable

Note that each sequence converges to each point. If X is uncountable, the intersection of countably many open sets is co-countable, so nonempty (this proves the non-first-countability).

Also, each infinite set is dense. If X is countable, therefore all $\{x\}^c$ are open dense but have empty intersection. On the other hand, if X is uncountable, the intersection of countably many nonempty open sets has at most countable complement, so is dense.

1.5 Co-countable topology

Let X be uncountable.

Definition	$(X, \tau_{\text{co-countable}} := \{U \subseteq X \mid \text{card}(U^c) \leq \aleph_0\} \cup \emptyset)$
type	not metric, as not Hausdorff
separation	T1, as there are no disjoint open sets except $\emptyset$
compact	no, as $(\{n, n+1, ...\})_{n \in \mathbb{N}}$ are closed, nonempty with empty intersection
Baire	yes
connected	connected, as there are no disjoint, open, nonempty sets
countability	not separable; not first countable (see 1.4)

The intersection of countably many nonempty open sets has countable complement, so is open and dense.

1.6 Discrete topology

Let X be a set of at least two elements.

Definition	$(X, 2^X)$
type	metric by $d(x,y) = 1$ if $x \neq y$
separation	T4
compact	iff X is finite
Baire	yes, as the only dense set is X
connected	totally disconnected
countability	first countable; not separable resp. second countable if X is infinite

1.7 Indiscrete topology

Let X be a set of at least two elements.

Definition	$(X, \{\emptyset, X\})$
type	not metric, as not Hausdorff
separation	none
compact	yes
Baire	yes, as the only open, dense set is X
connected	connected, as there are no disjoint, open, nonempty sets
countability	second countable; separable

1.8 Order topology

Let $(X, \leq)$ be a totally ordered set.

Definition	$(X, \tau_{\text{Ord}}$ generated by $X_{<x}$ and $X_{>x}$ for each $x \in X$
type	in general not metric, as not first countable
separation	T4
compact	iff $\leq$ is order complete (i.e. sup and inf exist for all subsets)
Baire	not in general, see $\mathbb{Q}$
connected	iff $(X, \leq)$ is dense and conditionally order complete
countability	in general neither separable nor first countable, see e.g. $\aleph_1 + 1$

For $x \notin C$ and $C \subseteq X$ closed have $x \in]y,z[$ and $]y,z[$ disjoint to C by closedness of C. Then one easily sees that x can be separated from C by distinguishing the cases $y < u < x$ or $]y,x[= \emptyset$ and similarly for z (this shows T3).

For T4 see math SE.

For the characterizations of compactness and connectedness see topology exercise sheets.

1.8.1 Order topology $\aleph_1 + 1$

Definition	$(\aleph_1 + 1, \tau_{\text{Ord}}$
type	not metric, as not first countable
separation	T4
compact	yes, as order complete
Baire	yes, as compact Hausdorff space
connected	not connected, as $\leq$ not dense
countability	not first countable ($\aleph_1$ has no countable neighborhood basis), not separable (countable union of countable sets is countable)

1.8.2 Order topology $\mathbb{Q}$

Definition	$(\mathbb{Q}, \tau_{\text{Ord}}$
type	metric, as subspace topology of τ_{Eucl}
separation	T4
compact	no, as not Baire
Baire	no, as there is no open singleton and $\mathbb{Q}$ is countable
connected	totally disconnected, as $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$
countability	second countable, as $\mathbb{Q}$ is countable

This is equal to the euclidean subspace topology on $\mathbb{Q}$

1.8.3 Order topology $\aleph_\alpha \times \mathbb{R}$ (lexiographic order)

Definition	$(\text{card}(2^{\mathbb{R}}) \times \mathbb{R}, \tau_{\text{Ord}}$
type	metric, via $d((\mu, x), (\mu, y)) = \frac{d(x,y)}{1+d(x,y)}$ and $d(u, v) = 1$ otherwise
separation	T4
compact	no, as $\mathbb{R}$ is not compact
Baire	yes because it is metric and complete
connected	not connected, as $(X, \leq)$ not conditionally order complete, e.g. $\{0\} \times \mathbb{R}$ has no supremum
countability	first countable (for (μ, x) take $(\mu,]x - \frac{1}{n}, x + \frac{1}{n}[)$); second countable and separable iff α is countable

1.9 Infinite-dimensional hypercubes

Let α be a ordinal.

Definition	$([0,1]^{\aleph_\alpha}, \tau)$ with product topology τ of $\tau_{\text{Eucl}}\|_{[0,1]}$
type	metric iff $\alpha = 0$, so α_α is countable (with $d(x,y) = \sum_n 2^{-n}\|x_n - y_n\|$)
separation	T4
compact	yes by Tychonoffs theorem
Baire	yes, as compact Hausdorff space
connected	connected
countability	first countable iff $\alpha = 0$; In this case, also second countable (see e.g. exercise problem 33); Separable iff $\aleph_\alpha \leq \text{card}(\mathbb{R})$.

To prove non-first-countability for $\alpha > 0$ use that the sets $]\frac{1}{3}, \frac{2}{3}[\times \prod_{\xi \neq \chi}[0,1]$ are open and that the intersection of countably many open sets contains $\prod_{\xi \in A} S_\xi \times \prod_{\xi \notin A}[0,1]$ for a countable A and any $S_\xi \subseteq [0,1]$.

For the T4 property, consider $C_0, C_1 \subseteq [0,1]^{\aleph_\alpha}$ closed. Then as above $C_i \subseteq A_i \times \prod_{\alpha \notin \mathcal{F}_i}[0,1]$ for $\mathcal{F}_i$ finite. Therefore, separating C_0, C_1 at the coordinates $\mathcal{F}_0 \cup \mathcal{F}_1$ is sufficient (this is possible, as the set is finite).

For separability see some blog.

By Urysohn's Lemma, each compact T4 space is homeomorphic to a subspace of $[0,1]^\kappa$, where κ is a big enough cardinal number. It suffices to take $\kappa = \text{card}(X^2)$, but it even suffices if there is a base of cardinality κ. Note that because of this, every second countable compact T4 space is metrizable.

For connectedness, assume there was non-constant $\phi : [0,1]^{\aleph_\alpha} \to \{0,1\}$ with $\phi(x) \neq \phi(y)$. Consider the net $((z_{\beta,\gamma})_\beta)_\gamma$ with $z_{\beta,\gamma} = x_\beta$ if $\beta \leq \gamma$ and $z_{\beta,\gamma} = y_\beta$ otherwise. By transfinite induction, it is easy to see that $\phi(x) = \phi(z_\gamma)$ for each γ (in the limit ordinal case, use that the net up to now converges and $\phi^{-1}(\{\phi(x)\})$ is closed). This yields a contradiction as $\phi(x) = \phi(z_{\aleph_\alpha}) = \phi(y)$.

1.10 Sorgenfrey topology

Definition	$(\mathbb{R}, \tau_{\text{Sorg}})$ where τ_{Sorg} is generated by $[a,b[$
type	not metric, as separable but not first countable
separation	T4
compact	no, even the subspace $[0,1]$ is not compact, as the cover $[1,2[$ and $[1-\frac{1}{n}, 1-\frac{1}{n+1}[$ has no finite subcover
Baire	yes
connected	totally disconnected
countability	first countable, separable; not second countable

Note that the Sorgenfrey is not connected. Assume there is a countable basis. Then there is some x such that x is not the infimum of any basis set. Then there is no basis set B with $x \in B \subseteq [x, x+\epsilon]$.

To see that it is a Baire space, consider $(U_n)_{n\in\mathbb{N}}$ open, dense and construct decreasing $[x_n, x_n + \epsilon_n[$ in $U_n \cap N$ for some open N. Then the x_n have a supremum x, and it is in $\bigcap_{n\in\mathbb{N}} U_n$.

1.11 The co-less-continuum topology

Let X be a set with $\mathrm{card}(X) \geq \mathrm{card}(\mathbb{R})$.

Definition	$(X, \tau = \{U \subseteq X \mid \mathrm{card}(U^c) < \mathrm{card}(\mathbb{R})\}) \cup \{\emptyset\}$
type	not metric, as not Hausdorff
separation	T1
compact	no
Baire	no
connected	connected, as there are no disjoint, open, nonempty sets
countability	not separable; not first countable

Note that the quotient topology of the euclidean space with the following, evil equivalence relation is of this form: Let $x \sim y :\Leftrightarrow x = y \vee x = x_\xi, y = y_\xi$ and $(x_\xi)_\xi, (y_\xi)_\xi$ are all distinct and constructed by transfinite induction such that for every $U \subseteq \mathbb{R}, U^\circ \neq \emptyset, \mathrm{card}(U) = \mathrm{card}(\mathbb{R})$ have ξ with $x_\xi \in U, y_\xi \notin U$ (using a countable basis of τ_{Eucl}). This is the case, since for all preimages $U \subseteq \mathbb{R}$ under the projection, have that if U is open, it is empty or its complement is of cardinality $< \mathrm{card}(\mathbb{R})$ (otherwise, it would contain x_ξ but not y_ξ, a contradiction to being a preimage).

Assuming the continuum hypothesis, this is the co-countable topology 1.5. In any case, its properties can be shown exactly the same way.

1.12 Axiomatization topology

Let $\mathcal{L}$ be a formal language and $X(\mathcal{L})$ a representant system of all $\mathcal{L}$-structures modulo $\equiv$ (elementary equivalence). Assume $\kappa(\mathcal{L}) = \aleph_0$.

Definition	Consider a topology on $X(\mathcal{L})$ where a set is closed iff it is axiomatizable by some $\Sigma \subseteq \mathrm{Sen}(\mathcal{L})$
type	metric (? iff $\kappa(\mathcal{L}) = \aleph_0$?)
separation	T4
compact	yes, by the compactness theorem
Baire	yes
connected	not connected, as $W(\gamma)$ and $W(\neg\gamma)$ is a disjoint, open, nontrivial cover of $X(\mathcal{L})$ for a suitable $\gamma \in \mathrm{Sen}(\mathcal{L})$
countability	second countable, separable (? iff $\kappa(\mathcal{L}) = \aleph_0$?)

Denote by $W(\Sigma) \subseteq X(\mathcal{L})$ the set of models of Σ.

T4 follows, as the space is clearly T2 (two non-elementarily-equivalent models can be separated by a single sentence, so by open sets) and compact.

By the separation lemma, closed disjoint sets can even be separated by the open sets $W(\gamma)$ and $W(\neg\gamma)$ for a single $\mathcal{L}$-sentence γ. Therefore, the $W(\gamma)^c$ for each $\gamma \in \mathrm{Sen}(\mathcal{L})$ for a countabale base (as by assumption, $\kappa(\mathcal{L}) = \aleph_0$).