Definition of Schemes

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1 Category-theoretical stuff

Let $\mathcal{C}$ be a small sub-category of **Set** that has small limits and colimits, and let X be a topological space.

Definition 1.1. A *sheaf* on X is a functor

$$F : \mathrm{Top}(X)^{\mathrm{op}} \to \mathcal{C}$$

that satisfies a local-to-global condition, i.e. for $(U_i)_i$ open and $s_i \in F(U_i)$ such that

$$s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$$

there exists a unique $s \in F(\bigcup_i U_i)$ such that

$$s\big|_{U_i} = s_i$$

Here we write $s\big|_V$ for the image of $s \in F(U)$ under the map $F(U) \to F(V)$ (which we get from the inclusion morphism $V \subseteq U$ in $\mathrm{Top}(X)$).

Definition 1.2. Let F be a sheaf on X. Then the *stalk* of F at some $x \in X$ is the colimit

$$F_x := \operatorname*{colim}_{U \ni x \text{ open}} F(U)$$

Definition 1.3. Let B be a basis of X. A B-sheaf is a functor

$$F : \underbrace{\mathrm{Top}(X)\big|_B{}^{\mathrm{op}}}_{\text{The subcategory of Top}(X)\text{ containing only the objects from }B} \to \mathcal{C}$$

that satisfies a local-to-global condition, i.e. for $(U_i)_i$ in B and $s_i \in F(U_i)$ such that

$$\forall x \in U_i \cap U_j \ \underbrace{\exists x \in V \subseteq U_i \cap U_j}_{\text{Since }B\text{ is a basis, there is always at least one such }V} : \ s_i\big|_V = s_j\big|_V \quad \text{and} \quad U := \bigcup_i U_i \in B$$

there exists a unique $s \in F(U)$ such that

$$s\big|_{U_i} = s_i$$

Theorem 1.4. *Let F be a B-sheaf for some basis B of X. Then there exists a unique (up to unique isomorphism) sheaf $\tilde{F}$ on X that extends F.*

Proof. First, we show Existence. For an open U in X, define

$$\tilde{F}(U) := \lim_{V \subseteq U,\ V \in B} F(V)$$

For an inclusion $U_1 \subseteq U_2$ and $s \in \tilde{F}(U_2)$, define then $\tilde{F}(U_1 \subseteq U_2)$ as the unique map such that

$$\tilde{F}(U_2) \overset{\tilde{F}(U_1 \subseteq U_2)}{\to} \tilde{F}(U_1) \to F(V) \quad \text{is} \quad \tilde{F}(U_1) \to F(V)$$

for all $V \subseteq U_1,\ V \in B$.

Now note that for $U_1 \subseteq U_2 \subseteq U_3$ we also have a map

$$\tilde{F}(U_3) \to \tilde{F}(U_2) \to \tilde{F}(U_1)$$

that is compatible with the maps $F(V_1 \subseteq V_2)$ for $V_1 \subseteq V_2,\ V_1, V_2 \in B$, and so by uniqueness above, we see

$$\tilde{F}(U_1 \subseteq U_3) = \tilde{F}(U_1 \subseteq U_2) \circ \tilde{F}(U_2 \subseteq U_3)$$

Furthermore, clearly $\tilde{F}(\mathrm{id}_U) = \mathrm{id}_{\tilde{F}(U)}$, so $\tilde{F}$ is a presheaf. Note that $\tilde{F}(V) \cong F(V)$ for all $V \in B$ and similar for morphisms, so indeed $\tilde{F}$ extends F.

Now we show the local-to-global condition. Assume we have $(U_i)_i$ open in X and $s_i \in \tilde{F}(U_i)$ such that

$$s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$$

Let $U = \bigcup_i U_i$. By the local-to-global condition of F, for each $V \subseteq U,\ V \in B$ there exists a unique $s_V \in \tilde{F}(V) = F(V)$ with

$$s_V\big|_{V_i} = s_i\big|_{V_i} \quad \text{for all } V_i \subseteq U_i \cap V,\ V_i \in B$$

Since
$$\lim_{V\subseteq U,\ V\in B} F(V) \cong \{(a_V)_V \in \prod_V F(V) \mid F(V_1 \subseteq V_2)(a_{V_2}) = a_{V_1}\}$$
we see that these s_V lift to one (necessarily unique) $s \in \tilde{F}(U)$.

For Uniqueness, assume we have two such sheaves, say G and H. Now note that for all open U in X with $V_i \in B$, we have

$$G(U) \cong \{(a_V)_V \in \prod_{V\subseteq U,\ V\in B} F(V) \mid F(V_1 \subseteq V_2)(a_{V_2}) = a_{V_1}\}$$

where $\supseteq$ follows from general structure and $\subseteq$ from the local-to-global property. The same holds for H, so $G \cong H$. □

2 Schemes

Definition 2.1. A *locally ringed space* is a topological space X with a sheaf of rings $\mathcal{O}_X$ on X such that all stalks $\mathcal{O}_{X,x}$ are local rings.

Definition 2.2. Let R be a ring (commutative, unital). Then let $\mathcal{O}_{\mathrm{Spec}R}$ be the sheaf on $\mathrm{Spec}R$ that results from extending the B-sheaf

$$\mathcal{O}_{\mathrm{Spec}R}(D_f) := R_f, \quad f \in R$$

to a sheaf as in Theorem 1.4. Here the sets

$$D_f = \{\mathfrak{p} \leq R \mid f \notin \mathfrak{p}\} \subseteq \mathrm{Spec}R$$

are the basic open sets and form a basis of $\mathrm{Spec}R$.

Definition 2.3. A morphism between locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a natural transformation $\eta : \mathcal{O}_Y \Rightarrow f_*\mathcal{O}_X$ that satisfies
$$\eta_y : (f_*\mathcal{O}_X)_y \cong \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y} \quad \text{maps} \quad \eta_y(\mathfrak{m}) \subseteq \mathfrak{m}$$
for all $y \in Y$.

Here
$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y), \quad F \mapsto (U \mapsto F(f^{-1}(U)))$$
is the pullback of f.

Definition 2.4. A locally ringed space $(X, \mathcal{O}_X)$ is an *affine scheme*, if there exists a ring R such that $(X, \mathcal{O}_X) \cong (\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$.

Definition 2.5. A locally ringed space $(X, \mathcal{O}_X)$ is a *scheme*, if there exists a covering $X = \bigcup_i U_i$ with open $U_i \subseteq X$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

Definition 2.6. A *morphism* of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ and a morphism of sheaves

$$\phi : \mathcal{O}_Y \to f_* \mathcal{O}_X$$

such that each

$$\phi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a homomorphism of local rings. Here $f_*\mathcal{O}_X$ is the sheaf $(f_*\mathcal{O}_X)(U) := \mathcal{O}_X(f^{-1}(U))$.

Definition 2.7. Denote the category of schemes by **Sch** and the subcategory of affine schemes by **Aff**

Definition 2.8. Let $\phi : R \to S$ be a ring homomorphism. Then this induces a morphism of affine schemes

$$\mathrm{Spec}\phi : \mathrm{Spec}S \to \mathrm{Spec}R$$

given by

$$\mathrm{Spec}\phi : \mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$$

and on basic open sets $D_g,\ g \in R$ by

$$(\mathrm{Spec}\phi)_{D_g} : \mathcal{O}_{\mathrm{Spec}R}(D_g) \to \mathcal{O}_{\mathrm{Spec}S}(D_{\phi(g)}), \quad \frac{x}{g^k} \to \frac{\phi(x)}{\phi(g)^k}$$

Proposition 2.9. *The functor*

$$\mathrm{Spec} : \boldsymbol{Ring}^{\mathrm{op}} \to \boldsymbol{Aff}$$

is an equivalence of categories.

Proposition 2.10. *Let* $(X, \mathcal{O}_X)$ *be a scheme. Then* X *is T0.*

Proof. Assume not, i.e. there are two points $x, y \in X$ such that every open neighborhood of x contains y and vice versa. As $(X, \mathcal{O}_X)$ has a cover by affine opens, consider an affine open U containing x, y. Then $(U, \mathcal{O}_X|_U)$ is isomorphic to $\mathrm{Spec}R$ for some ring R. However, $\mathrm{Spec}R$ is T0, a contradiction. □

3 Coproduct (gluing) in Sch

Proposition 3.1. *For affine schemes* $\mathrm{Spec}R_1$ *and* $\mathrm{Spec}R_2$ *have that the coproduct in* ***Sch*** *is*

$$\mathrm{Spec}R_1 \sqcup \mathrm{Spec}R_2 = \mathrm{Spec}(R_1 \times R_2)$$

Proof. Observe that

$$\mathrm{Spec}(R_1 \times R_2) = \underbrace{\{\mathfrak{p} \times R_2 \mid \mathfrak{p} \in \mathrm{Spec}R_1\}}_{=:V_1} \cup \underbrace{\{R_1 \times \mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}R_2\}}_{=:V_2}$$

First of all, the projection maps $\pi_i : R_1 \times R_2 \to R_i$ give rise to morphisms

$$\phi_i : \mathrm{Spec} R_i \to \mathrm{Spec}(R_1 \times R_2)$$

So we have to show that this cocone is universal.

Let $(W, \mathcal{O}_W)$ be a scheme with morphisms

$$\psi_i : \mathrm{Spec} R_i \to (W, \mathcal{O}_W)$$

Define

$$f : \mathrm{Spec}(R_1 \times R_2) \to W, \quad \begin{array}{ccc} \mathfrak{p} \times R_2 & \mapsto & \psi_1(\mathfrak{p}) \\ R_1 \times \mathfrak{p} & \mapsto & \psi_2(\mathfrak{p}) \end{array}$$

This is well-defined, as $\psi_i(R_i)$ is a point such that $\psi_i(\mathrm{Spec} R_i)$ is contained in every open neighborhood of it. Hence $\psi_1(R_1)$ and $\psi_2(R_2)$ cannot be separated by open sets, and thus must be equal as W is T0.

Furthermore, for any open $U \subseteq W$ note that

$$f^{-1}(U) = \{\mathfrak{p} \times R_2 \mid \mathfrak{p} \in \psi_1^{-1}(U)\} \cup \{R_1 \times \mathfrak{p} \mid \mathfrak{p} \in \psi_2^{-1}(U)\}$$

is a union of open sets, hence open. So f is continuous.

Now note that

$$(V_i, \mathcal{O}_{\mathrm{Spec}(R_1 \times R_2)}\Big|_{V_i}) \cong \mathrm{Spec} R_i$$

where the isomorphisms are natural in V_i. So $s \in \mathcal{O}_W(U)$ yields via $\mathcal{O}_W(U) \to \mathcal{O}_{\mathrm{Spec} R_i}(\psi^{-1}(U)) \cong \mathcal{O}_{\mathrm{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_i)$ elements

$$s_1' \in \mathcal{O}_{\mathrm{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_1), \quad s_2' \in \mathcal{O}_{\mathrm{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_2)$$

These glue together to some $s' \in \mathcal{O}_{\mathrm{Spec}(R_1 \times R_2)}(f^{-1}(U))$. Hence define a morphism of sheaves

$$\eta : \mathcal{O}_W \to f_* \mathcal{O}_{\mathrm{Spec}(R_1 \times R_2)}, \quad \eta_U(s) := s'$$

as it is clearly compatible with restriction maps.

Now, by construction, have that for $\mathfrak{p} \in \mathrm{Spec} R_i$

$$(f \circ \phi_i)(\mathfrak{p}) = f(\pi_i^{-1}(\mathfrak{p})) = \psi_i(\mathfrak{p})$$

and

$$\phi_i \circ \eta = \psi_i$$

The latter is true, as the isomorphism in

$$\mathcal{O}_W(U) \to \mathcal{O}_{\mathrm{Spec} R_i}(\psi_i^{-1}(U)) \cong \mathcal{O}_{\mathrm{Spec}(R_1 \times R_2)}(f^{-1}(U) \cap V_i)$$

in the direction $\leftarrow$ on basic open sets is given by the extension of the projection map $\pi_i : (R_1 \times R_2)_g \to (R_i)_{\pi_i(g)}$ and hence is ϕ_i.

The uniqueness of the morphism (f, η) is clear. $\square$

Proposition 3.2. *Let $(U_i)_i$ be a family of schemes with open subschemes $U_{ij} \subseteq U_i$ and isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$. If*

- $U_{ii} = U_i$ *and* $\phi_{ii} = \mathrm{id}$
- $\phi_{ij}(U_{ij} \cap U_{ik}) \subseteq U_{ji} \cap U_{jk}$
- $\phi_{ik}|_{U_{ij} \cap U_{ik}} = \phi_{jk} \circ \phi_{ij}|_{U_{ij} \cap U_{ik}}$

then there is a scheme X (unique up to unique isomorphism) with an open cover $X = \bigcup X_i$ and isomorphisms of schemes $\psi_i : U_i \to X_i$ such that

$$\psi_j \circ \phi_{ij} = \psi_i|_{U_{ij}}$$

Remark 3.3. The above easily shows that coproducts exist in **Sch** (just take U_i to be the empty scheme $(\emptyset, \emptyset \mapsto \{0\})$). However, **Sch** is not cocomplete[1], so coequalizers do not exist in general.

4 Products in Sch

Proposition 4.1. *Let $f : X \to Z$ and $g : Y \to Z$ be schemes over a scheme Z. The fiber product (or pullback) $X \times_Z Y$ is defined as the limit of*

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

(Finite) fiber products exist in ***Sch****.*

Proof. We show that binary fiber products exist. Let $f : X \to Z$, $g : Y \to Z$ be as above.

Step 1: X, Y, Z affine Assume $X = \mathrm{Spec}R$, $Y = \mathrm{Spec}S$, $Z = \mathrm{Spec}T$ and f, g are induced by ring homomorphisms

$$f^{\#} : T \to R, \quad g^{\#} : T \to S$$

We claim that $\mathrm{Spec}(R \otimes_T S)$ with

$$(\cdot \otimes 1)^{\#} : \mathrm{Spec}(R \otimes_T S) \to \mathrm{Spec}R, \quad (1 \otimes \cdot)^{\#} : \mathrm{Spec}(R \otimes_T S) \to \mathrm{Spec}S$$

works. For this, it suffices to show that $R \otimes_T S$ is the pushout (i.e. co-fiber product) of R, S considered as T-algebras. This follows easily from the universal property of the tensor product.

[1] https://mathoverflow.net/questions/9961/colimits-of-schemes

Step 2: Y, Z **affine** Assume $Y = \mathrm{Spec} S,\ Z = \mathrm{Spec} T$. Let $X = \bigcup X_i$ be a cover of X by affine opens. We want to glue all $X_i \times_Z Y$.

Note that $U_i := X_i \times_Z Y$ comes with a map $\pi_i : U_i \to X_i$. Now we can consider the open subscheme

$$U_{ij} := \left(\pi_i^{-1}(X_i \cap X_j),\ \mathcal{O}_X|_{\pi_i^{-1}(X_i \cap X_j)}\right)$$

Claim: U_{ij} is the fiber product $(X_i \cap X_j) \times_Z Y$. Clearly U_{ij} has maps $U_{ij} \to X_i \cap X_j$ and $U_{ij} \to Y$ inherited from $U_i \to X_i$ and $U_i \to Y$ such that

$$\begin{tikzcd} U_{ij} \arrow[r] \arrow[d] & X_i \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

commutes. It is also not too hard to show that another scheme W with compatible maps $W \to X_i \cap X_j$ and $W \to Y$ factors through U_{ij}.

Hence, there is a unique isomorphism $\phi_{ij} : U_{ij} \to U_{ji}$ by the uniqueness of limits. Claim: The U_i, U_{ij} and ϕ_{ij} satisfy the gluing conditions.

- Clearly $U_{ii} = U_i$ and so $\phi_{ii} = \mathrm{id}$ as there is a unique isomorphism $U_i \to U_i$.
- Note that by choice of ϕ_{ij} we know that the diagram

$$\begin{tikzcd} & X_i & \\ U_{ij} \arrow[ur, "\pi_i"] \arrow[rr, "\phi_{ij}"] & & U_{ji} \arrow[ul, "\pi_j"'] \end{tikzcd}$$

  is commutative. Hence $\phi_{ij}^{-1}(\pi_j^{-1}(X_i \cap X_j)) = \pi_i^{-1}(X_i \cap X_j)$ and so $\phi_{ij}^{-1}(U_{ij}) \subseteq U_{ji}$.
- Let

$$U_{ijk} := \left(\pi_i^{-1}(X_i \cap X_j \cap X_k),\ \mathcal{O}_X|_{\pi_i^{-1}(X_i \cap X_j \cap X_k)}\right) = U_{ij} \cap U_{ik}$$

  A similar argument as above also shows that U_{ijk} is the fiber product $(X_i \cap X_j \cap X_k) \times_Z Y$. Now, by uniqueness of limits, find that there is a unique isomorphism $U_{ijk} \to U_{kij}$. Note that both

$$\phi_{ik}|_{U_{ijk}} \quad \text{and} \quad \phi_{jk}|_{U_{jik}} \circ \phi_{ij}|_{U_{ijk}}$$

  are such isomorphisms, hence

$$\phi_{ik}|_{U_{ijk}} = \phi_{jk}|_{U_{jik}} \circ \phi_{ij}|_{U_{ijk}}$$

Step 3: Z **affine** Exactly as in step 2 (note that we did not use Y affine there).

Step 4: General case Let $Z = \bigcup Z_i$ be a cover of Z by affine opens. $\square$

5 Properties of Schemes and Morphisms

Proposition 5.1. *Let X be a scheme and let P be a property of affine opens (i.e. a class of embeddings $\mathrm{Spec} R \to X$) such that*

- *for all $\alpha : \mathrm{Spec} R \to X$ and $f \in R$ have*
$$\alpha \text{ satisfies } P \quad \Rightarrow \quad \alpha_f \text{ satisfies } P$$
where $\alpha_f : \mathrm{Spec} R_f \to X$.
- *for all $\alpha : \mathrm{Spec} R \to X$ and covers $\mathrm{Spec} R = \bigcup D_{f_i}$ have*
$$\text{all } \alpha_{f_i} \text{ satisfy } P \quad \Rightarrow \quad \alpha \text{ satisfies } P$$

If there is a cover $X = \bigcup X_i$ of affine opens such that all inclusions $X_i \subseteq X$ satisfy P, then all inclusions $U \subseteq X$ of affine opens satisfy P.

Definition 5.2. A scheme X is

- *Noetherian*, if $|X|$ is quasi-compact and for all affine open U have that $\mathcal{O}_X(U)$ is Noetherian.
- *reduced*, if for all open U have that $\mathcal{O}_X(U)$ is reduced.
- *irreducible*, if $|X|$ is irreducible (i.e. not the union of two proper closed subsets).
- *integral*, if for all open U have that $\mathcal{O}_X(U)$ is integral.

Definition 5.3. A morphism of schemes $f : X \to Y$ is

- *affine*, if $f^{-1}(U)$ is affine for all affine open $U \subseteq X$.
- *quasi-compact*, if $f^{-1}(U)$ is quasi-compact for all quasi-compact $U \subseteq X$.
- *locally of finite type*, if for all affine opens $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$ have that $\mathcal{O}_X(U)$ becomes a finitely generated $\mathcal{O}_Y(V)$-algebra when equipped with
$$\mathcal{O}_Y(V) \stackrel{f^\#}{\to} \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_X(U)$$
- *finite type*, if it is quasi-compact and locally of finite type
- a *closed immersion*, if it is an isomorphism onto a closed subscheme of Y.
- an *open immersion*, if it is an isomorphism onto an open subscheme of Y.
- *flat*, if all $\mathcal{O}_{X,x}$ are flat $\mathcal{O}_{Y,f(x)}$-modules, i.e. the functor $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \cdot$ of $\mathcal{O}_{Y,f(x)}$-algebras is exact[2].

Proposition 5.4. *An R-module M is flat if and only if for all injective $\alpha : N_1 \to N_2$ have that* $\mathrm{id} \otimes \alpha : M \otimes_R N_1 \to M \otimes_R N_2$ *is injective.*

[2] here $\mathcal{O}_{X,x}$ becomes a $\mathcal{O}_{Y,f(x)}$-algebra via f_x