Miniproject - Introduction to Schemes

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1 Definition of Proj

Definition 1. A *graded ring* S is a ring S with a decomposition $S = \oplus_{d \in \mathbb{Z}} S_d$ into groups $S_i \subseteq S$ (w.r.t. addition in S) such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{Z}$. If all $S_d = \{0\}$ for $d < 0$, call S *naturally graded ring*. Write further $S_+ := \sum_{d \neq 0} S_d$. For a homogeneous $f \in S_d$ say that $\deg(f) := d$ is its *degree*.

An element $f \in S$ is called *homogeneous* (of degree n), if $f \in S_n$. An ideal $I \leq S$ is called *homogeneous*, if it has a set of homogeneous generators.

Definition 2. For a naturally graded ring S, define the set

$$\mathrm{Proj}(S) := \{\mathfrak{p} \in \mathrm{Spec}(S) \mid \mathfrak{p} \text{ homogeneous}, \ S_+ \not\subseteq \mathfrak{p}\}$$

of homogeneous prime ideals not containing S_+.

This becomes a topological space by endowing it with the *Zariski-topology* on $\mathrm{Proj}(S)$, given by the open sets

$$D_\mathfrak{a} := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a} \not\subseteq \mathfrak{p}\}$$

for any homogeneous ideal $\mathfrak{a} \leq S$.

From now on let S be a naturally graded ring.

Proposition 3. *The above definition is well-defined, i.e. the sets $D_\mathfrak{a}$ indeed form a topology on* $\mathrm{Proj}(S)$.

Proof. Clearly $\mathrm{Proj}(S) = D_{\langle 1\rangle}$ and $\emptyset = D_{\langle 0\rangle}$ are open. Furthermore, for open sets $D_{\mathfrak{a}}$ and $D_{\mathfrak{b}}$, have that

$$D_{\mathfrak{a}} \cap D_{\mathfrak{b}} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}\} = D_{\mathfrak{a}\mathfrak{b}}$$

This holds, as $\mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}$ implies that there are $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ with $f, g \notin \mathfrak{p}$. However, then $fg \notin \mathfrak{p}$ as $\mathfrak{p}$ is prime. Obviously $\mathfrak{a}\mathfrak{b}$ is homogeneous, and so $D_{\mathfrak{a}} \cap D_{\mathfrak{b}}$ is open.

Finally, given a collection $\mathcal{A}$ of homogeneous ideals in S, have that

$$\begin{aligned}\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}} =& \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \exists \mathfrak{a} \in \mathcal{A} :\ \mathfrak{a} \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \exists \mathfrak{a} \in \mathcal{A} \exists f \in \mathfrak{a} :\ f \notin \mathfrak{p}\} \\ =& \left\{\mathfrak{p} \in \mathrm{Proj}(S) \;\middle|\; \exists f \in \sum_{\mathfrak{a}\in\mathcal{A}} \mathfrak{a} :\ f \notin \mathfrak{p}\right\} = D_{\mathfrak{b}} \quad \text{for } \mathfrak{b} = \sum_{\mathfrak{a}\in\mathcal{A}} \mathfrak{a}\end{aligned}$$

Clearly $\mathfrak{b}$ is again homogeneous, and so $\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}}$ is open. ☐

Proposition 4. *The sets $D_f := D_{\langle f\rangle}$ for homogeneous $f \in S$ form a basis of the topology on* $\mathrm{Proj}(S)$.

Proof. Clearly $\langle f\rangle$ is a homogenous ideal, so D_f is open. For any homogeneous ideal $\mathfrak{a} = \langle f_i \mid i \in I\rangle$ with $f_i \in S$ homogeneous have

$$D_{\mathfrak{a}} = \bigcup_{i\in I} D_{f_i}$$

as $\mathfrak{a} \not\subseteq \mathfrak{p}$ implies there is some $g = \sum_{i\in I} g_i f_i \notin \mathfrak{p}$, with $g_i \in S$. Hence, at least one $g_j f_j \notin \mathfrak{p}$ and so $f_j \notin \mathfrak{p}$, thus $\mathfrak{p} \in D_{f_j}$. It follows that the D_f generate the topology on $\mathrm{Proj}(S)$, so it is left to show that they are a basis.

Consider $\mathfrak{p} \in D_f \cap D_g$, so $f, g \notin \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $fg \notin \mathfrak{p}$ and so $D_{fg} \subseteq D_f \cap D_g$ is an open neighborhood of $\mathfrak{p}$. ☐

Lemma 5. *Let S be a graded ring (not necessarily naturally graded) and $T \subseteq S$ a multiplicative set consisting of homogeneous elements. Then $T^{-1}S$ becomes a graded ring via*

$$\left(T^{-1}S\right)_d = \left\{\frac{g}{h} \in T^{-1}S \;\middle|\; g \text{ homogeneous with } \deg(g) - \deg(h) = d\right\}$$

Proof. Clearly $(T^{-1}S)_i(T^{-1}S)_j \subseteq (T^{-1}S)_{i+j}$. To see that $(T^{-1}S)_d$ is a subgroup of S, consider $g/f, l/h \in (T^{-1}S)_d$. Now have

$$\frac{g}{f} + \frac{l}{h} = \frac{gh + lf}{hf}$$

and by assumption, find $\deg(gh) = \deg(g) - \deg(h) = d + \deg(f) - d + \deg(l) = \deg(lf)$. So $gh + lf$ is homogeneous and we have

$$\deg(gh + lf) - \deg(fh) = \deg(f) + \deg(l) - \deg(f) - \deg(h) = \deg(l) - \deg(h) = d$$

Thus $g/f + l/h \in (T^{-1}S)_d$.

Finally, we show that $(T^{-1}S)_n \cap (T^{-1}S)_m = \{0\}$ for $i \neq j$. Assume there is $g/f = l/h \in (T^{-1}S)_n \cap (T^{-1}S)_m$ with $\deg(g) - \deg(f) = n$ and $\deg(l) - \deg(h) = m$. Then there exists $t \in T$ such that

$$\begin{aligned} 0 = t(gh - lf) =& tgh - tlf \quad \text{with } tgh \in S_{\deg(t)+\deg(g)+\deg(h)} \\ & \text{and } tlf \in S_{\deg(t)+\deg(l)+\deg(f)} = S_{\deg(t)+\deg(g)+\deg(h)+(m-n)} \end{aligned}$$

If $n \neq m$, then $S_{\deg(t)+\deg(g)+\deg(h)+(m-n)} \cap S_{\deg(t)+\deg(g)+\deg(h)} = \{0\}$ and so $tgh = tlf = 0$. Thus $th(g - 0) = 0$ and so $g/f = 0/1 = 0$. Thus $(T^{-1}S)_m \cap (T^{-1}S)_n = \{0\}$. □

Lemma 6. *Let $\mathfrak{p} \leq S$ be a prime ideal. Then*

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \textit{ homogeneous} \rangle \leq S$$

is a (homogeneous) prime ideal.

Proof. Consider $f, g \in S$ with $fg \in \mathfrak{p}'$ and assume $f, g \notin \mathfrak{p}'$. Write $f = \sum_d f_d$ and $g = \sum_d g_d$ with $f_d, g_d \in S_d$. So

$$\sum_{i,j} f_i g_j = \sum_n \sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

Since $\mathfrak{p}'$ is homogeneous, it follows that

$$\sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

for all $n \in \mathbb{Z}$.

Let now d resp. e be maximal such that $f_d \notin \mathfrak{p}'$ resp. $g_e \notin \mathfrak{p}'$. We have

$$f_d g_e + \sum_{\substack{i+j=d+e \\ (i,j)\neq(d,e)}} \underbrace{f_i g_j}_{\in \mathfrak{p}'} = \sum_{i+j=d+e} f_i g_j \in \mathfrak{p}'$$

and so $f_d g_e \in \mathfrak{p}' \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. However both f_d and g_e are homogeneous, so $f_d \in \mathfrak{p}'$ or $g_e \in \mathfrak{p}'$, a contradiction. □

Lemma 7. *If $D_g \subseteq D_f$ then there is a homogeneous $h \in S$ such that $g^n = fh$ for some $n \in \mathbb{N}$.*

Proof. Assume not, then f is not a unit in S_g. Hence, there is a maximal ideal $\mathfrak{m} \leq S_g$ such that $f \in \mathfrak{m}$. Now let $\mathfrak{p}$ be the preimage of $\mathfrak{m}$ under the localization map $S \to S_g$. Since $\mathfrak{m}$ is maximal, we see that $\mathfrak{p}$ is prime.

Now apply Lemma 6 and see that also

$$\mathfrak{p}' = \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \subseteq \mathfrak{p}$$

is prime. Furthermore, $g \notin \mathfrak{p}$ and so $g \notin \mathfrak{p}'$. wlog we have that $g \in S_+$, so $S_+ \not\subseteq \mathfrak{p}'$. Since now $\mathfrak{p}'$ is a homogeneous prime ideal, it follows that $\mathfrak{p}' \in \mathrm{Proj}(S)$.

Finally, observe that $f \in \mathfrak{p}'$ as $f \in \mathfrak{p}$ and f is homogeneous. Hence, we have that $\mathfrak{p}' \notin D_f$ and $\mathfrak{p}' \in D_g$, which contradicts the assumption that $D_g \subseteq D_f$. □

The next proof works exactly as the corresponding one for Spec in the lecture.

Proposition 8. *Let $B = \{D_f \mid f \in S$ homogeneous$\}$. The functor*

$$\mathcal{F} : \mathrm{Top}(\mathrm{Proj}(S))\big|_B \to \textbf{\textit{Ring}}, \quad D_f \mapsto (S_f)_0$$
$$(D_{fg} \subseteq D_f) \mapsto \left(\cdot|_{D_{fg}} : \frac{s}{f^n} \mapsto \frac{sg^n}{(fg)^n}\right)$$

is a B-sheaf on B (here $\mathrm{Top}(X)$ is the category given by the open sets of X and their inclusion, as defined in the lecture).

Proof. Clearly, $\mathcal{F}$ is a functor and thus a presheaf. Hence, we have to show the local-to-global property.

Let $D_f = \bigcup_{i \in I} D_{g_i f}$ be a cover and $s_i \in \mathcal{F}(D_{g_i f})$ such that

$$\forall x \in D_{g_i f} \cap D_{g_j f} \ \exists V \in B : \ V \subseteq D_{g_i f} \cap D_{g_j f}, \ x \in V, \ \frac{s_i}{1} = \frac{s_j}{1} \in \mathcal{F}(V)$$

To show uniqueness, assume there are $\alpha/f^N, \beta/f^N \in D_f$ with

$$\frac{\alpha}{f^N}\bigg|_{D_{g_i f}} = \frac{\beta}{f^N}\bigg|_{D_{g_i f}} \quad \text{for all } i$$

Therefore there is $n_i \in \mathbb{N}$ such that

$$(fg_i)^{n_i}$$

By assumption, have $h_i^{n_i}(\alpha - \beta) = 0$ for each i, and wlog there are only finitely many i. Thus find $N = \max_i n_i \in \mathbb{N}$ and get that $h_i^N(\alpha - \beta) = 0$. Since $\bigcup_i D_{h_i} = \mathrm{Proj}(S)$ it follows that $\bigcup_i D_{h_i^N} = \mathrm{Proj}(S)$ and so $1 \in \langle h_i^N \mid i \rangle$. It follows that

$$\alpha - \beta = 1(\alpha - \beta) \in \langle h_i^N \mid i \rangle (\alpha - \beta) = \langle h_i^N(\alpha - \beta) \mid i \rangle = \{0\}$$

and so $\alpha = \beta$.

Now we show existence. By the uniqueness above, it follows that

$$s_i\big|_{D_{fg_ig_j}} = s_i\big|_{D_{fg_i} \cap D_{fg_j}} = s_j\big|_{D_{fg_i} \cap D_{fg_j}} = s_j\big|_{D_{fg_ig_j}}$$

wlog have again a finite cover, i.e. only finitely many g_i. Hence find an $N \in \mathbb{N}$ such that each $s_i = s_i'/(fg_i)^N$ with $s_i' \in S$ homogeneous. By possibly replacing N with a bigger N, we can now assume that

$$(f^2 g_i g_j)^N \left(s_i'(fg_j)^N - s_j'(fg_i)^N\right) = 0 \quad \text{as } s_i\big|_{D_{fg_ig_j}} = s_j\big|_{D_{fg_ig_j}}$$

Now note that

$$s_i = \frac{a_i}{b_i} \quad \text{with } a_i = s_i'(fg_i)^N, \ b_i = (fg_i)^{2N}$$

and

$$a_ib_j - a_jb_i = s_i'(fg_i)^N(fg_j)^{2N} - s_j'(fg_j)^N(fg_i)^{2N} = \underbrace{(f^2g_ig_j)^N\left(s_i'(fg_j)^N - s_j'(fg_i)^N\right)}_{=0}$$

Now observe that $D_{b_i} = D_{fg_i}$ and so $f^m \in \langle b_i \mid i\rangle$ for some m. Let $f^m = \sum_i r_ib_i$ and get

$$a_if^m = \sum_l r_lb_la_i = \sum_l r_la_lb_i = b_i\sum_l r_la_l$$

Note that a_i, b_i, f are homogeneous, and so we can also choose r_i to be homogeneous. Then find that $0 = \deg(s_i) = \deg(a_i) - \deg(b_i) = \deg(\sum r_la_l) - m\deg(f)$.

Thus

$$s_i = s := \frac{\sum_l r_la_l}{f^m} \in S_f$$

and since $\deg(\sum r_la_l) = m\deg(f)$ we find that $s \in (S_f)_0 = \mathcal{F}(D_f)$. Clearly $s\big|_{D_{fg_i}} = s_i$ and the claim follows. ☐

Corollary 9. *Hence we can (uniquely) extend the B-sheaf $\mathcal{F}$ to a sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$ on* $\mathrm{Proj}(S)$.

The next to lemmas are based on [Har77, p. II.2.5] and show that $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a scheme.

Lemma 10. *For $\mathfrak{p} \in \mathrm{Proj}(S)$, the stalk*

$$\mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}} = (T^{-1}S)_0$$

is a local ring, where $T = \{f \notin \mathfrak{p} \mid f \text{ homogeneous}\}$ contains all homogeneous elements not in $\mathfrak{p}$.

Proof. Consider the ideal

$$\mathfrak{m} = \left\{\frac{f}{g} \in (T^{-1}S)_0 \;\middle|\; f \in \mathfrak{p}\right\}$$

We claim this is the unique maximal ideal of $(T^{-1}S)_0$.

First, note that $1 \notin (T^{-1}S)_0$ as otherwise, there would be $f/g \in T^{-1}S, f \in \mathfrak{p}$ with $t(f-g) = 0$ for some $t \in T$. However then $tg = tf \in \mathfrak{p}$, and so $g \in \mathfrak{p}$ (as $t \notin \mathfrak{p}$), contradicting $g \in T$.

Now assume there is any ideal $\mathfrak{a}$ such that $\mathfrak{a} \setminus \mathfrak{m} \neq \emptyset$, i.e. there is $f/g \in \mathfrak{a} \setminus \mathfrak{m}$. Then $f \in T$ as f homogeneous and $f \notin \mathfrak{p}$. Thus $g/f \in (T^{-1}S)_0$ and so $f/g \in (T^{-1}S)_0^*$, which implies $\mathfrak{a} = \langle 1\rangle$. ☐

Lemma 11. *For $f \in S_+$ homogeneous have that $(D_f, \mathcal{O}_{\mathrm{Proj}(S)}\big|_{D_f})$ is an affine scheme.*

Proof. Let $R = (S_f)_0$. Consider the map

$$f : D_f \to \operatorname{Spec}(R), \quad \mathfrak{p} \mapsto \mathfrak{p}S_f \cap (S_f)_0$$

Note that it is continuous, as the preimage of some basic open set D_{g/f^n} is $D_{fg} \subseteq D_f$ open. Furthermore, f has the inverse

$$f^{-1} : \operatorname{Spec}(R) \to D_f, \quad \mathfrak{p} \mapsto \mathfrak{p}S_f$$

which is also continuous, as the preimage of some basic open set D_{fg} is D_{g^n/f^m} where $n\deg(g) = m\deg(f)$. Thus f is a homeomorphism.

Now consider the natural transformation

$$\eta : \mathcal{O}_{\operatorname{Spec}R} \Rightarrow f_*\Big(\mathcal{O}_{\operatorname{Proj}(S)}\Big|_{D_f}\Big)$$

given on basic open sets by

$$\eta_{D_{g/f^n}} : R_{g/f^n} \to (S_{fg})_0, \quad \frac{h/f^m}{(g/f^n)^l} \mapsto \frac{hf^{nl}}{g^l f^m}$$

Clearly this is a ring isomorphism, so η is a natural isomorphism. The claim follows. □

Corollary 12. $(\operatorname{Proj}(S), \mathcal{O}_{\operatorname{Proj}(S)})$ *is a scheme.*

Definition 13. For any ring R and $n \in \mathbb{N}$, define the projective n-space as[1]

$$\mathbb{P}^n_R := \operatorname{Proj}(R[x_0, ..., x_n])$$

which naturally becomes a scheme over $\operatorname{Spec}(R)$ via the morphism given by

$$f : \operatorname{Proj}(R[x_0, ..., x_n]) \to \operatorname{Spec}(R), \quad \mathfrak{p} \mapsto \mathfrak{p} \cap R$$
$$f^{\#}_{D_f} : R_f \to \mathcal{O}_{\operatorname{Proj}(R[x_0,...,x_n])}(D_f) = (R[x_0, ..., x_n]_f)_0 = R_f, \quad r \mapsto r$$

2 Projective space as a variety

For this section, let R be a ring and $n \in \mathbb{N}$.

Proposition 14. *Projective n-space $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ over $\operatorname{Spec}(R)$ is separated.*

Proof. By Lemma 11 we find a cover $\operatorname{Proj}(S) = \bigcup_i D_{x_i}$ of affine opens. We use the characterization from the lecture and show that $D_{x_i} \cap D_{x_j}$ is an affine open and the canonical multiplication map

$$\mathcal{O}_{\mathbb{P}^n_R}(D_{x_i}) \times \mathcal{O}_{\mathbb{P}^n_R}(D_{x_j}) \to \mathcal{O}_{\mathbb{P}^n_R}(D_{x_i} \cap D_{x_j})$$

[1] Endow $R[x_0, ..., x_n]$ with the standard natural grading, i.e. $\deg(x_i) = 1$. Note that this makes $R[x_0, ..., x_n]_0 = R$, so all $r \in R$ are homogeneous of degree 0.

is surjective.

First note that $D_{x_i} \cap D_{x_j} = D_{x_i x_j}$ which is affine open by Lemma 11. By definition, we have $\mathcal{O}_{\mathbb{P}^n_R}(D_f) = (R[x_0, ..., x_n]_f)_0$ and so we have to show that

$$(R[x_0, ..., x_n]_{x_i})_0 \times (R[x_0, ..., x_n]_{x_j})_0 \to (R[x_0, ..., x_n]_{x_i x_j})_0, \quad (a, b) \mapsto ab$$

is surjective.

However, for $f/(x_i x_j)^n \in (R[x_0, ..., x_n]_{x_i x_j})_0$ we have that

$$\frac{f}{x_i^{2n}} \cdot \frac{x_i^n}{x_j^n} = \frac{f}{(x_i x_j)^n}$$

and clearly $f/x_i^{2n} \in (R[x_0, ..., x_n]_{x_i})_0$ as $\deg(f) = \deg(x_i)n + \deg(x_j)n = 2n$ and $x_i^n/x_j^n \in (R[x_0, ..., x_n]_{x_j})_0$. The claim follows $\square$

Lemma 15. *It holds that* $\mathfrak{N}(R)R[x_0, ..., x_n] = \mathfrak{N}(R[x_0, ..., x_n])$.

Proof. Clearly $\mathfrak{N}(R)R[x_0, ..., x_n] \subseteq \mathfrak{N}(R[x_0, ..., x_n])$, as for $\sum_{i=1}^m f_i r_i$ with $f_i^n = 0$ have that

$$\Big(\sum_{i=1}^m f_i r_i\Big)^{mn} = \sum_{1 \leq i_1, ..., i_{mn} \leq m} \underbrace{f_{i_1} ... f_{i_{mn}}}_{\substack{=0 \\ \text{as at least one } i \text{ repeats at least } n \text{ times in } i_1, ..., i_{mn}}} r_{i_1} ... r_{i_{mn}}$$

Now note that $\mathfrak{N}(R[x_0, ..., x_n]) \subseteq \mathfrak{a}$ for all ideals $\mathfrak{a}$ and the claim follows. $\square$

Proposition 16. *Assume* $\mathrm{Spec}(R)$ *is irreducible. Then* $\mathbb{P}^n_R$ *is an irreducible scheme.*

Proof. We show that $D_{x_0} \subseteq \mathbb{P}^n_R$ is dense. The claim then follows, as any two nonempty, disjoint open sets give nonempty, disjoint open sets in D_{x_0} which is isomorphic to the irreducible[2] affine scheme $\mathrm{Spec}(R[x_0, ..., x_n]_{x_0})_0$ by Lemma 11.

So consider any nonempty basic open $D_f \subseteq \mathbb{P}^n_R$ and find that $D_f \cap D_{x_0} = D_{fx_0}$. By assumption, $\mathfrak{N}(R)$ is a prime ideal and so fx_0 is not nilpotent (unless f were nilpotent, but then $D_f = \emptyset$), i.e. the ring $R[x_0, ..., x_n]_{fx_0}$ is not the zero ring. Hence, it has a maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ be the preimage of $\mathfrak{m}$ under the localization map $R[x_0, ..., x_n] \to R[x_0, ..., x_n]_{fx_0}$, which is clearly prime with $fx_0 \notin \mathfrak{p}$. By Lemma 6, we now see that

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \leq R[x_0, ..., x_n]$$

is a homogeneous prime ideal, and since $fx_0 \notin \mathfrak{p} \supseteq \mathfrak{p}'$, we see that $S_+ \not\subseteq \mathfrak{p}'$, i.e. $\mathfrak{p}' \in \mathrm{Proj}(R[x_0, ..., x_n])$. Now it follows that $\mathfrak{p}' \in D_{fx_0}$, so $D_{fx_0} \neq \emptyset$. $\square$

Proposition 17. *Assume* R *is reduced. Then* $\mathbb{P}^n_R$ *is a reduced scheme.*

Proof. By the lecture, it suffices to show that each $\mathcal{O}_{\mathbb{P}^n_R}(D_{x_i}) \cong \mathrm{Spec}(R[x_0, ..., x_n]_{x_i})_0$ is reduced. However this is trivial, as taking the polynomial ring $R[x_0, ..., x_n]$ preserves reducedness by Lemma 15, and localizing (and of course taking the subring S_0) also do. $\square$

[2]This is obviously irreducible, as $R[x_0, ..., x_n]/\mathfrak{N}(R[x_0, ..., x_n]) = (R/\mathfrak{N}(R))[x_0, ..., x_n]$ is integral by assumption and Lemma 15.

Corollary 18. *Assume R is integral. Then $\mathbb{P}^n_R$ is an integral scheme.*

Proposition 19. $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ *is of finite type.*

Proof. First note that $\mathbb{P}^n_R$ is covered by a finite number of affine opens D_{x_i} where the canonical morphisms

$$D_{x_i} \cong \mathrm{Spec}(R[x_0, ..., x_n]_{x_i})_0 \to \mathrm{Spec}(R)$$

are quasi-compact (they are induced by ring homomorphisms $R \to (R[x_0, ..., x_n]_{x_i})_0)$, hence $\mathbb{P}^n_R$ is quasi-compact.

To see that it is locally of finite type, show that there is the cover by affine opens $\mathbb{P}^n_R = \bigcup_i D_{x_i}$ such that the ring homomorphisms $R \to \mathcal{O}_{\mathbb{P}^n_R}(D_{x_i})$ are of finite type. This is clearly the case, as

$$\mathcal{O}_{\mathbb{P}^n_R}(D_{x_i}) = (R[x_0, ..., x_n]_{x_i})_0 \cong R[y_0, ..., y_{i-1}, y_i, ..., y_n]$$

is a finitely generated R-algebra. $\square$

Corollary 20. *If k is an algebraically closed field, then $\mathbb{P}^n_k$ is a variety.*

3 Projective space is proper

We already know that $\mathbb{P}^n_R$ is separated and of finite type over $\mathrm{Spec}(R)$, hence it is left to show that it is universally closed.

Lemma 21. *Let X, Y be topological spaces and $f : X \to Y$ a continuous map. Let further $Y = \bigcup_i V_i$ be a cover by open sets. If each of the maps*

$$f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$$

is closed (as a map into V_i), then f is closed.

Proof. Consider some closed $C \subseteq X$ and $x \in Y \setminus f(C)$. Since the V_i cover Y, there is i such that $x \in V_i$. As $f|_{f^{-1}(V_i)}$ is closed, we see that

$$f(C) \cap V_i = f|_{f^{-1}(V_i)}(C)$$

is closed in V_i, hence there is an open neighborhood $U \subseteq V_i$ of x with $U \cap f(C) = \emptyset$. Now note that $V_i \subseteq Y$ is open, and so U is also an open neighborhood of x in Y. This holds for all $x \in Y \setminus f(C)$, hence $f(C)$ is closed. $\square$

[Stacks, p. 66.9.5.]

Corollary 22. *A scheme X over* $\mathrm{Spec}(R)$ *is universally closed, if for all ring homomorphisms $R \to S$ have the following: The base change map $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \to \mathrm{Spec}(S)$, i.e. the map that makes the diagram*

$$\begin{tikzcd}
\mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(S) \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

commute, is closed.

Proof. Consider any scheme Y over $\mathrm{Spec}(R)$. We have to show that the map $\pi : Y \times_{\mathrm{Spec}(R)} X \to Y$ that makes the diagram

$$\begin{tikzcd}
Y \times_{\mathrm{Spec}(R)} X \arrow[r] \arrow[d, "\pi"'] & X \arrow[d] \\
Y \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

commute, is closed.

Let $Y = \bigcup_i V_i$ be a cover by affine opens. For each i, note that the diagram

$$\begin{tikzcd}
\pi^{-1}(V_i) \arrow[r] \arrow[d, "\pi_i := \pi|_{\pi^{-1}(V_i)}"'] & X \arrow[d] \\
V_i \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

commutes. By the universal property of $V_i \times_{\mathrm{Spec}(R)} X$ we now find a morphism

$$\psi : \pi^{-1}(V_i) \to V_i \times_{\mathrm{Spec}(R)} X$$

of schemes over $Y \times X$, i.e. the inclusion $\pi^{-1}(V_i)$ factors as

$$\pi^{-1}(V_i) \stackrel{\psi}{\to} V_i \times_{\mathrm{Spec}(R)} X \to Y \times_{\mathrm{Spec}(R)} X$$

Considering $V_i \times_{\mathrm{Spec}(R)} X$ as an open subscheme of $Y \times_{\mathrm{Spec}(R)} X$, we thus see that ψ must already be the identity and so $\pi^{-1}(V_i) \subseteq V_i \times_{\mathrm{Spec}(R)} X$. It follows that $\pi^{-1}(V_i) = V_i \times_{\mathrm{Spec}(R)} X$.

By assumption we know that each π_i is closed, as $V_i \cong \mathrm{Spec}(\mathcal{O}_Y(V_i))$ is affine. Now the previous lemma yields that also π is closed. □

Remark 23. Note that Proj is not functorial, in the sense that a homomorphism of graded rings $S \to T$ does not induce a natural morphism $\mathrm{Proj}(T) \to \mathrm{Proj}(S)$ (like Spec does). Namely, given $\alpha : S \to T$, we might want to consider

$$\mathrm{Proj}(T) \to \mathrm{Proj}(S), \quad \mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p})$$

However, in general this is not well-defined, as $\alpha^{-1}(\mathfrak{p})$ might contain S_+.

If we further require the map $\alpha : S \to T$ to fulfill $T_+ \subseteq \alpha(S_+)T$, then this works out and we get a morphism

$$\mathrm{Proj}(\alpha) : \mathrm{Proj}(T) \to \mathrm{Proj}(S), \quad \mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p}),$$
$$\mathrm{Proj}(\alpha)^{\#}_{D_f} : \mathcal{O}_{\mathrm{Proj}(S)}(D_f) \to \mathcal{O}_{\mathrm{Proj}(T)}(D_{\alpha(f)}), \quad \frac{x}{y} \mapsto \frac{\alpha(x)}{\alpha(y)}$$

Note that this is indeed a well-defined morphism, as for $\mathfrak{p} \in \mathrm{Proj}(T)$ there is some $f \in T_+ \setminus \mathfrak{p}$ and by assumption, have $g \in S_+, t \in T$ with $\alpha(g)t = f$. Now $g \notin \alpha^{-1}(\mathfrak{p})$, as $g \in \alpha^{-1}(\mathfrak{p})$ would imply $\alpha(g) \in \mathfrak{p}$, thus $f = t\alpha(g) \in \mathfrak{p}$.

In particular, any ring homomorphism $R \to S$ induces a canonical morphism $\mathbb{P}^n_S \to \mathbb{P}^n_R$ of schemes, since the induced ring homomorphism $R[x_0, ..., x_n] \to S[x_0, ..., x_n]$ satisfies the above additional condition.

Lemma 24. *Let $R \to S$ be a ring homomorphism. Then the base change of $\mathbb{P}^n_R$ is*

$$\mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \cong \mathbb{P}^n_S$$

as schemes over $\mathrm{Spec}(S)$.

Proof. Consider the canonical map $\alpha : R[x_0, ..., x_n] \to S[x_0, ..., x_n]$ induced by $R \to S$. Note that

$$S \otimes_R (R[x_0, ..., x_n]_f)_0 \cong (S[x_0, ..., x_n]_{\alpha(f)})_0 \quad \text{via} \quad \iota_f : s \otimes \frac{r}{f^n} \mapsto \frac{s\alpha(r)}{\alpha(f)^n}$$

for all homogeneous $f \in R[x_0, ..., x_n]$. By Lemma 11, we see that D_f for $f \in S[x_0, ..., x_n]$ homogeneous form an affine open cover of $\mathbb{P}^n_S$. Furthermore, we have a cover by affine opens $D_f \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ of $\mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ for $f \in R[x_0, ..., x_n]$ homogeneous. Together, we see that it suffices to show that the isomorphisms (induced by ι_f)

$$\phi_f : D_{\alpha(f)} = \mathrm{Spec}((S[x_0, ..., x_n]_{\alpha(f)})_0) \to D_f \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$$

glue to an isomorphism $\mathrm{Proj}(S[x_0, ..., x_n]) \to \mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$. By the gluing lemma, it suffices that the compatibility conditions are satisfied, i.e. that for all $f, g \in R[x_0, ..., x_n]$ homogeneous the diagram

$$\begin{tikzcd}
 & D_{\alpha(f)} \arrow[r, "\phi_f"] & D_f \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \arrow[dr] & \\
D_{\alpha(f)} \cap D_{\alpha(g)} = D_{\alpha(fg)} \arrow[ur] \arrow[dr] & & & \mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \\
 & D_{\alpha(g)} \arrow[r, "\phi_g"] & D_g \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \arrow[ur] &
\end{tikzcd}$$

commutes.

For a prime ideal $\mathfrak{p} \in D_{\alpha(fg)}$ have that

$$\phi_f(\mathfrak{p}) = \iota_f^{-1}(\mathfrak{p} \cap (S[x_0, ..., x_n]_f)_0) \in D_f \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$$

and so $g \notin \phi_f(\mathfrak{p})$, i.e. $\phi_f(\mathfrak{p}) \in D_{fg} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$. Hence, we have that both restrictions

$$\phi_f|_{D_{\alpha(fg)}},\ \phi_g|_{D_{\alpha(fg)}} : D_{\alpha(fg)} \to D_{fg} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$$

map into $D_{fg} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$. Now note that the diagram

$$\begin{tikzcd}
S \otimes_R (R[x_0, ..., x_n]_f)_0 \arrow[rrr, "\iota_f"] \arrow[ddd] & & & (S[x_0, ..., x_n]_f)_0 \arrow[ddd] \\
 & S \otimes_R (R[x_0, ..., x_n]_g)_0 \arrow[r, "\iota_g"] \arrow[ddl] & (S[x_0, ..., x_n]_g)_0 \arrow[ddr] & \\
 & & & \\
S \otimes_R (R[x_0, ..., x_n]_{fg})_0 \arrow[rrr, "\iota_{fg}"] & & & (S[x_0, ..., x_n]_{fg})_0
\end{tikzcd}$$

commutes, and so the restrictions of ϕ_f resp. ϕ_g to $D_{\alpha(fg)}$ are both induced by the homomorphism ι_{fg}, hence they are equal. The claim follows. □

The next lemma is directly from [Har77, p. II.4.5].

Lemma 25. *Let $f : X \to Y$ be a quasi-compact morphism. Then $f(|X|)$ is closed if and only if it is stable under specialization.*

Lemma 26. *Let $\mathfrak{p} \leq R[x_0, ..., x_n]$ be a homogeneous prime ideal with $R[x_0, ..., x_n]_+ \not\subseteq \mathfrak{p}$. Further let $\mathfrak{q} \leq R$ be a prime ideal such that $\mathfrak{p} \cap R \subseteq \mathfrak{q}$. Then there some i such that no αx_i^m is in $\mathfrak{b} := \mathfrak{p} + \mathfrak{q}R[x_0, ..., x_n]$, for any $\alpha \in R \setminus \mathfrak{b}$ and any $m \in \mathbb{N}$.*

Proof. Assume there is $m \in \mathbb{N}$ such that $\alpha_i x_i^m \in \mathfrak{b}$ for all i and some $\alpha_i \in \mathfrak{b} \setminus R$. Since $\mathfrak{b} \cap R = \mathfrak{q}$ is prime, have that $\alpha := \prod_i \alpha_i \notin \mathfrak{b}$. Further we have that $\alpha x_i^m \in \mathfrak{b}$ for all i.

Consider now the vector $(w_i)_{i<N}$ containing all monomials of degree $(n+1)m$. By assumption, all $\alpha w_i \in \mathfrak{b}$. Since $\mathfrak{p}$ is homogeneous, observe that now there are $r_{ij} \in \mathfrak{q}$ such that

$$\alpha w_i - \sum_{j<N} r_{ij} w_j \in \mathfrak{p}$$

Working modulo $\mathfrak{p}$, we see that $Aw = \alpha w$ where $A \in (\mathfrak{b}/\mathfrak{p})^{N \times N}$.

Now observe that $\alpha \notin \mathfrak{b}/\mathfrak{p}$ since $\alpha \notin \mathfrak{b}$. Hence there exists a prime ideal in $R/\mathfrak{p}$ containing $\mathfrak{b}/\mathfrak{p}$ and not containing α. Localizing at that prime ideal gives a local ring S with maximal ideal $\mathfrak{l}$. Now assume that $w \in S^n$ and $A/\alpha \in \mathfrak{l}^{n \times n}$. Note that we still have $(A/\alpha)w = w$.

wlog S is noetherian, otherwise continue with the ring

$$\tilde{S} := \begin{cases} \mathbb{Z}[A/\alpha, x] & \text{if } \mathrm{char}(S) = 0 \\ (\mathbb{Z}/\mathrm{char}(R)\mathbb{Z})[A/\alpha, x] & \text{if } \mathrm{char}(S) \neq 0 \end{cases}$$

generated by the coefficients of A/α and w (this is noetherian, as it is a quotient of a polynomial ring in finitely many variables).

Now equip $S_{\mathfrak{l}}$ with the $\mathfrak{l}$-adic topology. Note that S is noetherian and local, thus the Krull intersection theorem shows that the $\mathfrak{l}$-adic topology is Hausdorff and we find

$$w = \lim_{i\to\infty} w = \lim_{i\to\infty} \frac{A^i}{\alpha^i} w = \Big(\lim_{i\to\infty} \frac{A^i}{\alpha^i}\Big) w = 0w = 0$$

since $(A/\alpha)^i$ has coefficients in $\mathfrak{l}^i$, thus converges to 0 as $i \to \infty$.

Thus $w \equiv 0 \mod \mathfrak{p}$ and so $x_i^N \equiv 0 \mod \mathfrak{p}$, i.e. $x_i^N \in \mathfrak{p}$ for all i. However, since $\mathfrak{p}$ is prime, this implies $R[x_0, ..., x_n]_+ = \langle x_0, ..., x_n \rangle \subseteq \mathfrak{p}$, contradicting the assumption. □

Proposition 27. *The morphism* $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ *is closed.*

Proof. Denote $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ by ϕ. Consider a closed set $\mathbb{V}(\mathfrak{a}) = \mathbb{P}^n_R \setminus D_{\mathfrak{a}}$ given by a homogeneous ideal $\mathfrak{a} \leq R[x_0, ..., x_n]$. By Lemma 25, it suffices to show that $\phi(\mathbb{V}(\mathfrak{a}))$ is closed under specialization (note that $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is quasi-compact, e.g. by Proposition 19).

Consider $\mathfrak{q} \in \phi(\mathbb{V}(\mathfrak{a})) \subseteq \mathrm{Spec}(R)$ and a prime ideal $\mathfrak{q}'$ that specializes $\mathfrak{q}$, i.e. $\mathfrak{q}' \supseteq \mathfrak{q}$. Then there is a $\mathfrak{p} \in \mathbb{V}(\mathfrak{a})$ with $\mathfrak{p} \cap R = \mathfrak{q}$, in particular $\mathfrak{a} \subseteq \mathfrak{p}$. We want to show that $\mathfrak{q}' \in \phi(\mathbb{V}(\mathfrak{a}))$.

Note that by Lemma 26, there is i such that the multiplicative set

$$T := \{\alpha x_i^m \mid \alpha \in R \setminus \mathfrak{q}',\ m \in \mathbb{N}\}$$

has empty intersection with the ideal $\mathfrak{b} := \mathfrak{p} + \mathfrak{q}' R[x_0, ..., x_n]$.

Hence, the ring $T^{-1}(R[x_0, ..., x_n]/\mathfrak{b})$ is nonzero, thus has a prime. Taking its preimage under

$$R[x_0, ..., x_n] \to R[x_0, ..., x_n]/\mathfrak{b} \to T^{-1}(R[x_0, ..., x_n]/\mathfrak{b})$$

yields a prime ideal $\mathfrak{p}'$ such that $x_i \notin \mathfrak{p}'$ and $\mathfrak{b} \subseteq \mathfrak{p}'$. Clearly have that $\mathfrak{q}' \subseteq \mathfrak{p}'$ and since $\mathfrak{p}' \cap T = \emptyset$, find that $\mathfrak{p}' \cap R = \mathfrak{q}'$. By Lemma 6, assume wlog that $\mathfrak{p}'$ is homogeneous. Now have a homogeneous prime ideal $\mathfrak{p}'$ with $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$, i.e. $\mathfrak{p}' \in \mathbb{V}(\mathfrak{a})$ and $\phi(\mathfrak{p}') = \mathfrak{p}' \cap R = \mathfrak{q}'$. Thus $\mathfrak{q} \in \phi(\mathbb{V}(\mathfrak{a}))$ and the claim follows. □

Corollary 28. *The morphism* $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ *is universally closed.*

Proof. Use Lemma 22, so consider an affine base change $f_{\mathrm{Spec}(S)} : \mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \to \mathrm{Spec}(S)$. By Lemma 24, have that $\mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \cong \mathbb{P}^n_S$ as schemes over S, hence $f_{\mathrm{Spec}(S)}$ is isomorphic to $\mathbb{P}^n_S \to \mathrm{Spec}(S)$. This morphism is closed by Proposition 27 and the claim follows. □

Corollary 29. *Projective space* $\mathbb{P}^n_R$ *over* $\mathrm{Spec}(R)$ *is proper.*

Corollary 30. *Let k be an algebraically closed field. Then projective space* $\mathbb{P}^n_k$ *over* $\mathrm{Spec}(k)$ *is a complete variety.*

4 Valuative criterion of properness

Proposition 31. *Let X be noetherian and $f : X \to Y$ a morphism of finite type. The following are equivalent*

- *f is proper*
- *For every valuation ring R with field of fractions $K = \mathrm{Frac}(R)$ and all morphisms $\mathrm{Spec}(R) \to Y$, $\mathrm{Spec}(K) \to X$ that make the diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] & Y \end{tikzcd}$$

  commute, there exists a unique compatible morphism $\mathrm{Spec}(R) \to X$.

References

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