C2.6 Introduction to Schemes

Hilary Term 2022

Title: *The Proj construction in the theory of schemes.*

In the course we constructed the prime spectrum $X = Spec(R)$ of a ring R as a topological space whose underlying set consists of the prime ideals of R, together with a sheaf of rings $\mathcal{O}_X$. The spectrum serves as the local model for any scheme, indeed we showed that schemes can always be obtained by gluing together such affine schemes $Spec(R_i)$ along open subsets. We defined the affine n-space $\mathbb{A}^n = Spec\,\mathbb{Z}[x_1, \ldots, x_n]$ and more generally affine n-space $\mathbb{A}^n_B \to B$ over any scheme B. One can then glue such affine spaces together to define also the projective n-space $\mathbb{P}^n_B$ over B.

The goal of this essay is to consider the projective analogue of the above construction. Namely, given a graded ring $S = \oplus_{n \geq 0} S_n$, one defines $Proj(S)$ to be the collection of *homogeneous* prime ideals in S, but omitting those which include the "irrelevant ideal" $S_+ = \oplus_{n > 0} S_n$ (in classical algebraic geometry over a field k, this is the ideal corresponding to the origin in affine space k^{n+1}, which one excludes when passing to the projective space $\mathbb{P}^n_k = \mathbb{P}(k^{n+1})$). The essay should start with this basic construction, defining the Zariski topology on $Proj(S)$ and the structure sheaf which makes $Proj(S)$ a scheme. In particular, if S is an R-algebra, you should explain how $Proj(S)$ is a separated scheme over $Spec(R)$. Subject to the page constraints mentioned below, you may wish to include a discussion of basic properties, e.g. conditions under which $Proj(S)$ is Noetherian, integral, etc., and properties of $Proj(S) \to Spec(R)$ such as when this is of finite type. You should then use this construction to define $\mathbb{P}^n_R$, i.e. projective n-space over $Spec(R)$. The main theorem of the essay should be a proof that projective space $\mathbb{P}^n_k$ over an algebraically closed field k is a complete variety, meaning: an integral scheme over k which is separated proper and of finite type over $Spec(k)$.

The essay should be around 10 pages (please do not exceed 15 pages).

There are many references for this topic: the ones overleaf are just a suggestion.

References.

Robin Hartshorne, *Algebraic Geometry*, Springer, GTM 52, 1977.

Ravi Vakil, *Foundations of Algebraic Geometry*, 2017
http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf

David Eisenbud and Joe Harris, *The Geometry of Schemes*, Springer
GTM 197, 1999.

The Stacks project
https://stacks.math.columbia.edu/