Miniproject - Introduction to Schemes

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1 Definition of Proj

This section is based on [Har77, p. II.2]

Definition 1. A *graded ring* S is a ring S with a decomposition $S = \oplus_{d \in \mathbb{Z}} S_d$ into groups $S_i \subseteq S$ (w.r.t. addition in S) such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{Z}$. If all $S_d = \{0\}$ for $d < 0$, call S *naturally graded ring*. Write further $S_+ := \sum_{d \neq 0} S_d$. For a homogeneous $f \in S_d$ say that $\deg(f) := d$ is its *degree*.

An element $f \in S$ is called *homogeneous* (of degree n), if $f \in S_n$. An ideal $I \leq S$ is called *homogeneous*, if it has a set of homogeneous generators.

Definition 2. For a naturally graded ring S, define the set

$$\mathrm{Proj}(S) := \{\mathfrak{p} \in \mathrm{Spec}(S) \mid \mathfrak{p} \text{ homogeneous}, \ S_+ \not\subseteq \mathfrak{p}\}$$

of homogeneous prime ideals not containing S_+.

This becomes a topological space by endowing it with the *Zariski-topology* on $\mathrm{Proj}(S)$, given by the open sets

$$D_{\mathfrak{a}} := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a} \not\subseteq \mathfrak{p}\}$$

for any homogeneous ideal $\mathfrak{a} \leq S$.

From now on let S be a naturally graded ring.

Proposition 3. *The above definition is well-defined, i.e. the sets $D_{\mathfrak{a}}$ indeed form a topology on* $\mathrm{Proj}(S)$.

Proof. Clearly $\mathrm{Proj}(S) = D_{\langle 1 \rangle}$ and $\emptyset = D_{\langle 0 \rangle}$ are open. Furthermore, for open sets $D_{\mathfrak{a}}$ and $D_{\mathfrak{b}}$, have that

$$D_{\mathfrak{a}} \cap D_{\mathfrak{b}} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}\} = D_{\mathfrak{a}\mathfrak{b}}$$

This holds, as $\mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}$ implies that there are $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ with $f, g \notin \mathfrak{p}$. However, then $fg \notin \mathfrak{p}$ as $\mathfrak{p}$ is prime. Obviously $\mathfrak{a}\mathfrak{b}$ is homogeneous, and so $D_{\mathfrak{a}} \cap D_{\mathfrak{b}}$ is open.

Finally, given a collection $\mathcal{A}$ of homogeneous ideals in S, have that

$$\begin{aligned}\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}} &=\{\mathfrak{p}\in\operatorname{Proj}(S)\mid \exists\mathfrak{a}\in\mathcal{A}:\ \mathfrak{a}\not\subseteq\mathfrak{p}\}=\{\mathfrak{p}\in\operatorname{Proj}(S)\mid \exists\mathfrak{a}\in\mathcal{A}\exists f\in\mathfrak{a}:\ f\notin\mathfrak{p}\}\\ &=\Big\{\mathfrak{p}\in\operatorname{Proj}(S)\ \Big|\ \exists f\in\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}:\ f\notin\mathfrak{p}\Big\}=D_{\mathfrak{b}}\quad\text{for }\mathfrak{b}=\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}\end{aligned}$$

Clearly $\mathfrak{b}$ is again homogeneous, and so $\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}}$ is open. □

Proposition 4. *The sets $D_f := D_{\langle f\rangle}$ for homogeneous $f\in S$ form a basis of the topology on* $\operatorname{Proj}(S)$.

Proof. Clearly $\langle f\rangle$ is a homogenous ideal, so D_f is open. For any homogeneous ideal $\mathfrak{a}=\langle f_i\mid i\in I\rangle$ with $f_i\in S$ homogeneous have

$$\mathfrak{a}=\bigcup_{i\in I} D_{f_i}$$

as $\mathfrak{a}\not\subseteq\mathfrak{p}$ implies there is some $g=\sum_{i\in I} g_i f_i\notin\mathfrak{p}$, with $g_i\in S$. Hence, at least one $g_j f_j\notin\mathfrak{p}$ and so $f_j\notin\mathfrak{p}$, thus $\mathfrak{p}\in D_{f_j}$. It follows that the D_f generate the topology on $\operatorname{Proj}(S)$, so it is left to show that they are a basis.

Consider $\mathfrak{p}\in D_f\cap D_g$, so $f,g\notin\mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $fg\notin\mathfrak{p}$ and so $D_{fg}\subseteq D_f\cap D_g$ is an open neighborhood of $\mathfrak{p}$. □

Lemma 5. *Let S be a graded ring (not necessarily naturally graded) and $f\in S$ be homogeneous. Then S_f becomes a graded ring via*

$$(S_f)_d=\{\frac{g}{f^n}\mid g\in S_{d+n}\}$$

Proof. Clearly $(S_f)_i(S_f)_j\subseteq(S_f)_{i+j}$. To see that $(S_f)_d$ is a subgroup of S, consider $g/f^n, h/f^m\in(S_f)_d$. wlog $n=m$ as we can replace g by gf^{m-n} resp. h by hf^{n-m}. Now have that $\deg(g)=\deg(h)=d+n$ and so $g+h\in S_{d+n}$. Thus

$$\frac{g}{f^n}+\frac{h}{f^n}=\frac{g+h}{f^n}\in(S_f)_d$$

Finally, we show that $(S_f)_i\cap(S_f)_j=\{0\}$ for $i\neq j$. Assume there is $g/f^i=h/f^j\in(S_f)_n\cap(S_f)_m$. Then there exists $t=f^d$ such that

$$0=t(gf^j-hf^i)=\underbrace{gf^{j+d}}_{\in S_{(j+d)\deg(f)+\deg(g)}}-\underbrace{hf^{i+n}}_{\in S_{(i+d)\deg(f)+\deg(h)}}$$

Thus either $(j+d)\deg(f)+\deg(g)=(i+d)\deg(f)+\deg(h)$ or $gf^{j+d}=hf^{i+d}=0$. However have that $\deg(g)=n-\deg(f)i$ and $\deg(h)=m-\deg(f)j$, so

$$(j+d-i)\deg(f)+n\neq(i+d-j)\deg(f)+n$$

as $i\neq j$. So $gf^{j+d}=hf^{i+d}=0$ and hence $g/f^i=hf^j=0$. Therefore we have $(S_f)_i\cap(S_f)_j=\{0\}$ as claimed. □

Lemma 6. *Let $\mathfrak{p} \leq S$ be a prime ideal. Then*

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \leq S$$

is a (homogeneous) prime ideal.

Proof. Consider $f, g \in S$ with $fg \in \mathfrak{p}'$ and assume $f, g \notin \mathfrak{p}'$. Write $f = \sum_d f_d$ and $g = \sum_d g_d$ with $f_d, g_d \in S_d$. So

$$\sum_{i,j} f_i g_j = \sum_n \sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

Since $\mathfrak{p}'$ is homogeneous, it follows that

$$\sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

for all $n \in \mathbb{Z}$.

Let now d resp. e be maximal such that $f_d \notin \mathfrak{p}'$ resp. $g_e \notin \mathfrak{p}'$. We have

$$f_d g_e + \sum_{\substack{i+j=d+e \\ (i,j)\neq(d,e)}} \underbrace{f_i g_j}_{\in \mathfrak{p}'} = \sum_{i+j=d+e} f_i g_j \in \mathfrak{p}'$$

and so $f_d g_e \in \mathfrak{p}' \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. However both f_d and g_e are homogeneous, so $f_d \in \mathfrak{p}'$ or $g_e \in \mathfrak{p}'$, a contradiction. □

Lemma 7. *If $D_g \subseteq D_f$ then there is a homogeneous $h \in S$ such that $g^n = fh$ for some $n \in \mathbb{N}$.*

Proof. Assume not, then f is not a unit in S_g. Hence, there is a maximal ideal $\mathfrak{m} \leq S_g$ such that $f \in \mathfrak{m}$. Now let $\mathfrak{p}$ be the preimage of $\mathfrak{m}$ under the localization map $S \to S_g$. Since $\mathfrak{m}$ is maximal, we see that $\mathfrak{p}$ is prime.

Now apply Lemma 6 and see that also

$$\mathfrak{p}' = \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \subseteq \mathfrak{p}$$

is prime. Furthermore, $g \notin \mathfrak{p}$ and so $g \notin \mathfrak{p}'$. wlog we have that $g \in S_+$, so $S_+ \not\subseteq \mathfrak{p}'$. Since now $\mathfrak{p}'$ is a homogeneous prime ideal, it follows that $\mathfrak{p}' \in \mathrm{Proj}(S)$.

Finally, observe that $f \in \mathfrak{p}'$ as $f \in \mathfrak{p}$ and f is homogeneous. Hence, we have that $\mathfrak{p}' \notin D_f$ and $\mathfrak{p}' \in D_g$, which contradicts the assumption that $D_g \subseteq D_f$. □

Proposition 8. *Let $B = \{D_f \mid f \in S \text{ homogeneous}\}$. The functor*

$$\begin{aligned} \mathcal{F} : \mathrm{Top}(\mathrm{Proj}(S))\big|_B \to \textbf{\textit{Ring}}, \quad & D_f \mapsto (S_f)_0 \\ & (D_{fg} \subseteq D_f) \mapsto \left(\frac{s}{f^n} \mapsto \frac{sg^n}{(fg)^n} \right) \end{aligned}$$

is a B-sheaf on B (here $\mathrm{Top}(X)$ is the category given by the open sets of X and their inclusion, as defined in the lecture).

Proof. Clearly, $\mathcal{F}$ is a functor and thus a presheaf. Hence, we have to show the local-to-global property.

Let $D_f = \bigcup_{i \in I} D_{g_i f}$ be a cover and $s_i \in \mathcal{F}(D_{g_i f})$ such that

$$\forall x \in D_{g_i f} \cap D_{g_j f} \; \exists V \in B : \; V \subseteq D_{g_i f} \cap D_{g_j f}, \; x \in V, \; \frac{s_i}{1} = \frac{s_j}{1} \in \mathcal{F}(V)$$

□

References

[Har77] Robin Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer, 1977.