Miniproject - Introduction to Schemes

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March 24, 2022

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1 Definition of Proj

Definition 1. A graded ring S is a ring S with a decomposition $S = \bigoplus_{d \in \mathbb{Z}} S_d$ into groups $S_i \subseteq S$ (w.r.t. addition in S) such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{Z}$. If all $S_d = \{0\}$ for d < 0, call S naturally graded ring. Write further $S_+ := \sum_{d \neq 0} S_d$. For a homogeneous $f \in S_d$ say that $\deg(f) := d$ is its degree.

An element $f \in S$ is called *homogeneous* (of degree n), if $f \in S_n$. An ideal $I \leq S$ is called *homogeneous*, if it has a set of homogeneous generators.

Definition 2. For a naturally graded ring S, define the set

$$\operatorname{Proj}(S) := \{ \mathfrak{p} \in \operatorname{Spec}(S) \mid \mathfrak{p} \text{ homogeneous, } S_+ \not\subseteq \mathfrak{p} \}$$

of homogeneous prime ideals not containing S_+ .

This becomes a topological space by endowing it with the Zariski-topology on Proj(S), given by the open sets

$$D_{\mathfrak{a}} := \{ \mathfrak{p} \in \operatorname{Proj}(S) \mid \mathfrak{a} \not\subseteq \mathfrak{p} \}$$

for any homogeneous ideal $\mathfrak{a} \leq S$.

From now on let S be a naturally graded ring.

Proposition 3. The above definition is well-defined, i.e. the sets $D_{\mathfrak{a}}$ indeed form a topology on $\operatorname{Proj}(S)$.

Proof. Clearly $\operatorname{Proj}(S) = D_{\langle 1 \rangle}$ and $\emptyset = D_{\langle 0 \rangle}$ are open. Furthermore, for open sets $D_{\mathfrak{a}}$ and $D_{\mathfrak{b}}$, have that

$$D_{\mathfrak{a}} \cap D_{\mathfrak{b}} = \{ \mathfrak{p} \in \operatorname{Proj}(S) \mid \mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p} \} = \{ \mathfrak{p} \in \operatorname{Proj}(S) \mid \mathfrak{ab} \not\subseteq \mathfrak{p} \} = D_{\mathfrak{ab}}$$

This holds, as $\mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}$ implies that there are $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ with $f, g \notin \mathfrak{p}$. However, then $fg \notin \mathfrak{p}$ as \mathfrak{p} is prime. Obviously \mathfrak{ab} is homogeneous, and so $D_{\mathfrak{a}} \cap D_{\mathfrak{b}}$ is open.

Finally, given a collection \mathcal{A} of homogeneous ideals in S, have that

$$\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}} = \{\mathfrak{p}\in \operatorname{Proj}(S) \mid \exists \mathfrak{a}\in\mathcal{A}: \ \mathfrak{a}\not\subseteq \mathfrak{p}\} = \{\mathfrak{p}\in \operatorname{Proj}(S) \mid \exists \mathfrak{a}\in\mathcal{A}\exists f\in\mathfrak{a}: \ f\notin\mathfrak{p}\} \\
= \left\{\mathfrak{p}\in \operatorname{Proj}(S) \mid \exists f\in\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}: \ f\notin\mathfrak{p}\right\} = D_{\mathfrak{b}} \quad \text{for } \mathfrak{b}=\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}$$

Clearly \mathfrak{b} is again homogeneous, and so $\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}}$ is open.

Proposition 4. The sets $D_f := D_{\langle f \rangle}$ for homogeneous $f \in S$ form a basis of the topology on Proj(S).

Proof. Clearly $\langle f \rangle$ is a homogeneous ideal, so D_f is open. For any homogeneous ideal $\mathfrak{a} = \langle f_i \mid i \in I \rangle$ with $f_i \in S$ homogeneous have

$$D_{\mathfrak{a}} = \bigcup_{i \in I} D_{f_i}$$

as $\mathfrak{a} \not\subseteq \mathfrak{p}$ implies there is some $g = \sum_{i \in I} g_i f_i \notin \mathfrak{p}$, with $g_i \in S$. Hence, at least one $g_j f_j \notin \mathfrak{p}$ and so $f_j \notin \mathfrak{p}$, thus $\mathfrak{p} \in D_{f_j}$. It follows that the D_f generate the topology on Proj(S), so it is left to show that they are a basis.

Consider $\mathfrak{p} \in D_f \cap D_g$, so $f, g \notin \mathfrak{p}$. Since \mathfrak{p} is prime, it follows that $fg \notin \mathfrak{p}$ and so $D_{fg} \subseteq D_f \cap D_g$ is an open neighborhood of \mathfrak{p} .

Lemma 5. Let S be a graded ring (not necessarily naturally graded) and $T \subseteq S$ a multiplicative set consisting of homogeneous elements. Then $T^{-1}S$ becomes a graded ring via

$$\left(T^{-1}S\right)_d = \left\{\frac{g}{h} \in T^{-1}S \ \middle| \ g \ homogeneous \ with \ \deg(g) - \deg(h) = d\right\}$$

Proof. Clearly $(T^{-1}S)_i(T^{-1}S)_j \subseteq (T^{-1}S)_{i+j}$. To see that $(T^{-1}S)_d$ is a subgroup of S, consider $g/f, l/h \in (T^{-1}S)_d$. Now have

$$\frac{g}{f} + \frac{l}{h} = \frac{gh + lf}{hf}$$

and by assumption, find deg(gh) = deg(g) - deg(h) = d + deg(f) - d + deg(l) = deg(lf). So gh + lf is homogeneous and we have

$$\deg(gh+lf) - \deg(fh) = \deg(f) + \deg(l) - \deg(f) - \deg(h) = \deg(l) - \deg(h) = d$$

Thus $g/f + l/h \in (T^{-1}S)_d$.

Finally, we show that $(T^{-1}S)_n \cap (T^{-1}S)_m = \{0\}$ for $i \neq j$. Assume there is $g/f = l/h \in (T^{-1}S)_n \cap (T^{-1}S)_m$ with $\deg(g) - \deg(f) = n$ and $\deg(l) - \deg(h) = m$. Then there exists $t \in T$ such that

$$0 = t(gh - lf) = tgh - tlf \quad \text{with } tgh \in S_{\deg(t) + \deg(g) + \deg(h)}$$
 and
$$tlf \in S_{\deg(t) + \deg(l) + \deg(f)} = S_{\deg(t) + \deg(g) + \deg(h) + (m-n)}$$

If $n \neq m$, then $S_{\deg(t)+\deg(g)+\deg(h)+(m-n)} \cap S_{\deg(t)+\deg(g)+\deg(h)} = \{0\}$ and so tgh = tlf = 0. Thus th(g-0) = 0 and so g/f = 0/1 = 0. Thus $(T^{-1}S)_m \cap (T^{-1}S)_n = \{0\}$.

Lemma 6. Let $\mathfrak{p} \leq S$ be a prime ideal. Then

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \ homogeneous \rangle \leq S$$

 $is\ a\ (homogeneous)\ prime\ ideal.$

Proof. Consider $f, g \in S$ with $fg \in \mathfrak{p}'$ and assume $f, g \notin \mathfrak{p}'$. Write $f = \sum_d f_d$ and $g = \sum_d g_d$ with $f_d, g_d \in S_d$. So

$$\sum_{i,j} f_i g_j = \sum_n \sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

Since \mathfrak{p}' is homogeneous, it follows that

$$\sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

for all $n \in \mathbb{Z}$.

Let now d resp. e be maximal such that $f_d \notin \mathfrak{p}'$ resp. $g_e \notin \mathfrak{p}'$. We have

$$f_d g_e + \sum_{\substack{i+j=d+e\\(i,j)\neq (d,e)}} \underbrace{f_i g_j}_{\in \mathfrak{p}'} = \sum_{\substack{i+j=d+e}} f_i g_j \in \mathfrak{p}'$$

and so $f_d g_e \in \mathfrak{p}' \subseteq \mathfrak{p}$. Since \mathfrak{p} is prime, it follows that $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. However both f_d and g_e are homogeneous, so $f_d \in \mathfrak{p}'$ or $g_e \in \mathfrak{p}'$, a contradiction.

Lemma 7. If $D_g \subseteq D_f$ then there is a homogeneous $h \in S$ such that $g^n = fh$ for some $n \in \mathbb{N}$.

Proof. Assume not, then f is not a unit in S_g . Hence, there is a maximal ideal $\mathfrak{m} \leq S_g$ such that $f \in \mathfrak{m}$. Now let \mathfrak{p} be the preimage of \mathfrak{m} under the localization map $S \to S_g$. Since \mathfrak{m} is maximal, we see that \mathfrak{p} is prime.

Now apply Lemma 6 and see that also

$$\mathfrak{p}' = \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \subset \mathfrak{p}$$

is prime. Furthermore, $g \notin \mathfrak{p}$ and so $g \notin \mathfrak{p}'$. wlog we have that $g \in S_+$, so $S_+ \not\subseteq \mathfrak{p}'$. Since now \mathfrak{p}' is a homogeneous prime ideal, it follows that $\mathfrak{p}' \in \operatorname{Proj}(S)$.

Finally, observe that $f \in \mathfrak{p}'$ as $f \in \mathfrak{p}$ and f is homogeneous. Hence, we have that $\mathfrak{p}' \notin D_f$ and $\mathfrak{p}' \in D_g$, which contradicts the assumption that $D_g \subseteq D_f$.

The next proof works exactly as the corresponding one for Spec in the lecture.

Proposition 8. Let $B = \{D_f \mid f \in S \text{ homogeneous}\}$. The functor

$$\mathcal{F}: \operatorname{Top}(\operatorname{Proj}(S))\big|_{B} \to \operatorname{\textbf{\it Ring}}, \quad D_f \mapsto (S_f)_0$$

$$(D_{fg} \subseteq D_f) \mapsto \left(\cdot\big|_{D_{fg}} : \frac{s}{f^n} \mapsto \frac{sg^n}{(fg)^n}\right)$$

is a B-sheaf on B (here Top(X) is the category given by the open sets of X and their inclusion, as defined in the lecture).

Proof. Clearly, \mathcal{F} is a functor and thus a presheaf. Hence, we have to show the local-to-global property.

Let $D_f = \bigcup_{i \in I} D_{g_i f}$ be a cover and $s_i \in \mathcal{F}(D_{g_i f})$ such that

$$\forall x \in D_{g_if} \cap D_{g_jf} \ \exists V \in B: \ V \subseteq D_{g_if} \cap D_{g_jf}, \ x \in V, \ \frac{s_i}{1} = \frac{s_j}{1} \in \mathcal{F}(V)$$

To show uniqueness, assume there are $\alpha/f^N, \beta/f^N \in D_f$ with

$$\frac{\alpha}{f^N}\Big|_{D_{g_if}} = \frac{\beta}{f^N}\Big|_{D_{g_if}}$$
 for all i

Therefore there is $n_i \in \mathbb{N}$ such that

$$(fg_i)^{n_i}$$

By assumption, have $h_i^{n_i}(\alpha - \beta) = 0$ for each i, and wlog there are only finitely many i. Thus find $N = \max_i n_i \in \mathbb{N}$ and get that $h_i^N(\alpha - \beta) = 0$. Since $\bigcup_i D_{h_i} = \operatorname{Proj}(S)$ it follows that $\bigcup_i D_{h_i^N} = \operatorname{Proj}(S)$ and so $1 \in \langle h_i^N \mid i \rangle$. It follows that

$$\alpha - \beta = 1(\alpha - \beta) \in \langle h_i^N \mid i \rangle (\alpha - \beta) = \langle h_i^N (\alpha - \beta) \mid i \rangle = \{0\}$$

and so $\alpha = \beta$.

Now we show existence. By the uniqueness above, it follows that

$$s_i|_{D_{fg_ig_j}} = s_i|_{D_{fg_i} \cap D_{fg_j}} = s_j|_{D_{fg_i} \cap D_{fg_j}} = s_j|_{D_{fg_ig_j}}$$

wlog have again a finite cover, i.e. only finitely many g_i . Hence find an $N \in \mathbb{N}$ such that each $s_i = s_i'/(fg_i)^N$ with $s_i' \in S$ homogeneous. By possibly replacing N with a bigger N, we can now assume that

$$(f^2 g_i g_j)^N \left(s_i' (f g_j)^N - s_j' (f g_i)^N \right) = 0$$
 as $s_i \big|_{D_{f g_i g_j}} = s_j \big|_{D_{f g_i g_j}}$

Now note that

$$s_i = \frac{a_i}{b_i}$$
 with $a_i = s'_i (fg_i)^N$, $b_i = (fg_i)^{2N}$

and

$$a_i b_j - a_j b_i = s_i'(fg_i)^N (fg_j)^{2N} - s_j'(fg_j)^N (fg_i)^{2N} = \underbrace{(f^2 g_i g_j)^N \left(s_i'(fg_j)^N - s_j'(fg_i)^N\right)}_{\text{O}}$$

Now observe that $D_{b_i} = D_{fg_i}$ and so $f^m \in \langle b_i \mid i \rangle$ for some m. Let $f^m = \sum_i r_i b_i$ and get

$$a_i f^m = \sum_{l} r_l b_l a_i = \sum_{l} r_l a_l b_i = b_i \sum_{l} r_l a_l$$

Note that a_i, b_i, f are homogeneous, and so we can also choose r_i to be homogeneous. Then find that $0 = \deg(s_i) = \deg(a_i) - \deg(b_i) = \deg(\sum r_l a_l) - m \deg(f)$.

Thus

$$s_i = s := \frac{\sum_l r_l a_l}{f^m} \in S_f$$

and since $\deg(\sum r_l a_l) = m \deg(f)$ we find that $s \in (S_f)_0 = \mathcal{F}(D_f)$. Clearly $s|_{D_{fg_i}} = s_i$ and the claim follows.

Corollary 9. Hence we can (uniquely) extend the B-sheaf \mathcal{F} to a sheaf $\mathcal{O}_{\operatorname{Proj}(S)}$ on $\operatorname{Proj}(S)$.

The next to lemmas are based on [Har77, p. II.2.5] and show that $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ is a scheme.

Lemma 10. For $\mathfrak{p} \in \text{Proj}(S)$, the stalk

$$\mathcal{O}_{\operatorname{Proj}(S),\mathfrak{p}} = (T^{-1}S)_0$$

is a local ring, where $T = \{ f \notin \mathfrak{p} \mid f \text{ homogeneous} \}$ contains all homogeneous elements not in \mathfrak{p} .

Proof. Consider the ideal

$$\mathfrak{m} = \left\{ \frac{f}{g} \in (T^{-1}S)_0 \mid f \in \mathfrak{p} \right\}$$

We claim this is the unique maximal ideal of $(T^{-1}S)_0$.

First, note that $1 \notin (T^{-1}S)_0$ as otherwise, there would be $f/g \in T^{-1}S$, $f \in \mathfrak{p}$ with t(f-g)=0 for some $t \in T$. However then $tg=tf \in \mathfrak{p}$, and so $g \in \mathfrak{p}$ (as $t \notin \mathfrak{p}$), contradicting $g \in T$.

Now assume there is any ideal \mathfrak{a} such that $\mathfrak{a} \setminus \mathfrak{m} \neq \emptyset$, i.e. there is $f/g \in \mathfrak{a} \setminus \mathfrak{m}$. Then $f \in T$ as f homogeneous and $f \notin \mathfrak{p}$. Thus $g/f \in (T^{-1}S)_0$ and so $f/g \in (T^{-1}S)_0^*$, which implies $\mathfrak{a} = \langle 1 \rangle$.

Lemma 11. For $f \in S_+$ homogeneous have that $(D_f, \mathcal{O}_{\text{Proj}(S)}|_{D_f})$ is an affine scheme.

Proof. Let $R = (S_f)_0$. Consider the map

$$f: D_f \to \operatorname{Spec}(R), \quad \mathfrak{p} \mapsto \mathfrak{p}S_f \cap (S_f)_0$$

Note that it is continuous, as the preimage of some basic open set D_{g/f^n} is $D_{fg} \subseteq D_f$ open. Furthermore, f has the inverse

$$f^{-1}: \operatorname{Spec}(R) \to D_f, \quad \mathfrak{p} \mapsto \mathfrak{p}S_f$$

which is also continuous, as the preimage of some basic open set D_{fg} is D_{g^n/f^m} where $n \deg(g) = m \deg(f)$. Thus f is a homeomorphism.

Now consider the natural transformation

$$\eta: \mathcal{O}_{\operatorname{Spec} R} \Rightarrow f_* \Big(\mathcal{O}_{\operatorname{Proj}(S)} \Big|_{D_f} \Big)$$

given on basic open sets by

$$\eta_{D_{g/f^n}}: R_{g/f^n} \to (S_{fg})_0, \quad \frac{h/f^m}{(g/f^n)^l} \mapsto \frac{hf^{nl}}{g^lf^m}$$

Clearly this is a ring isomorphism, so η is a natural isomorphism. The claim follows. \square

Corollary 12. $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ is a scheme.

Definition 13. For any ring R and $n \in \mathbb{N}$, define the projective n-space as¹

$$\mathbb{P}_{R}^{n} := \operatorname{Proj}(R[x_{0},...,x_{n}])$$

which naturally becomes a scheme over Spec(R) via the morphism given by

$$f: \operatorname{Proj}(R[x_0, ..., x_n]) \to \operatorname{Spec}(R), \quad \mathfrak{p} \mapsto \mathfrak{p} \cap R$$

$$f_{D_f}^{\#}: R_f \to \mathcal{O}_{\operatorname{Proj}(S)}(D_f) = (R[x_0, ..., x_n]_f)_0, \quad r \mapsto r$$

2 Projective space as a variety

For this section, let R be a ring and $n \in \mathbb{N}$.

Proposition 14. Projective n-space $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ over $\operatorname{Spec}(R)$ is separated.

Proof. By Lemma 11 we find a cover $\operatorname{Proj}(S) = \bigcup_i D_{x_i}$ of affine opens. We use the characterization from the lecture and show that $D_{x_i} \cap D_{x_j}$ is an affine open and the canonical multiplication map

$$\mathcal{O}_{\mathbb{P}^n_R}(D_{x_i}) \times \mathcal{O}_{\mathbb{P}^n_R}(D_{x_j}) \to \mathcal{O}_{\mathbb{P}^n_R}(D_{x_i} \cap D_{x_j})$$

¹Endow $R[x_0,...,x_n]$ with the standard natural grading, i.e. $\deg(x_i)=1$. Note that this makes $R[x_0,...,x_n]_0=R$, so all $r\in R$ are homogeneous of degree 0.

is surjective.

First note that $D_{x_i} \cap D_{x_j} = D_{x_i x_j}$ which is affine open by Lemma 11. By definition, we have $\mathcal{O}_{\mathbb{P}_R^n}(D_f) = (R[x_0, ..., x_n]_f)_0$ and so we have to show that

$$(R[x_0,...,x_n]_{x_i})_0 \times (R[x_0,...,x_n]_{x_i})_0 \to (R[x_0,...,x_n]_{x_ix_i})_0, \quad (a,b) \mapsto ab$$

is surjective.

However, for $f/(x_ix_j)^n \in (R[x_0,...,x_n]_{x_ix_j})_0$ we have that

$$\frac{f}{x_i^{2n}} \cdot \frac{x_i^n}{x_j^n} = \frac{f}{(x_i x_j)^n}$$

and clearly $f/x_i^{2n} \in (R[x_0,...,x_n]_{x_i})_0$ as $\deg(f) = \deg(x_i)n + \deg(x_j)n = 2n$ and $x_i^n/x_j^n \in (R[x_0,...,x_n]_{x_j})_0$. The claim follows

Lemma 15. It holds that $\mathfrak{N}(R)R[x_0,...,x_n] = \mathfrak{N}(R[x_0,...,x_n]).$

Proof. Clearly $\mathfrak{N}(R)R[x_0,...,x_n] \subseteq \mathfrak{N}(R[x_0,...,x_n])$, as for $\sum_{i=1}^m f_i r_i$ with $f_i^n = 0$ have that

$$\left(\sum_{i=1}^m f_i r_i\right)^{mn} = \sum_{\substack{1 \leq i_1, \dots, i_{mn} \leq m \\ \text{as at least one } i \text{ repeats at least } n \text{ times in } i_1, \dots, i_{mn}} \underline{f_{i_1} \dots f_{i_{mn}}} r_{i_1} \dots r_{i_m n}$$

Now note that $\mathfrak{N}(R[x_0,...,x_n]) \subseteq \mathfrak{a}$ for all ideals \mathfrak{a} and the claim follows.

Proposition 16. Assume $\operatorname{Spec}(R)$ is irreducible. Then \mathbb{P}^n_R is an irreducible scheme.

Proof. We show that $D_{x_0} \subseteq \mathbb{P}_R^n$ is dense. The claim then follows, as any two nonempty, disjoint open sets give nonempty, disjoint open sets in D_{x_0} which is isomorphic to the irreducible² affine scheme $\operatorname{Spec}(R[x_0,...,x_n]_{x_0})_0$ by Lemma 11.

So consider any nonempty basic open $D_f \subseteq \mathbb{P}^n_R$ and find that $D_f \cap D_{x_0} = D_{fx_0}$. By assumption, $\mathfrak{N}(R)$ is a prime ideal and so fx_0 is not nilpotent (unless f were nilpotent, but then $D_f = \emptyset$), i.e. the ring $R[x_0, ..., x_n]_{fx_0}$ is not the zero ring. Hence, it has a maximal ideal \mathfrak{m} . Let \mathfrak{p} be the preimage of \mathfrak{m} under the localization map $R[x_0, ..., x_n] \to R[x_0, ..., x_n]_{fx_0}$, which is clearly prime with $fx_0 \notin \mathfrak{p}$. By Lemma 6, we now see that

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \leq R[x_0, ..., x_n]$$

is a homogeneous prime ideal, and since $fx_0 \notin \mathfrak{p} \supseteq \mathfrak{p}'$, we see that $S_+ \not\subseteq \mathfrak{p}'$, i.e. $\mathfrak{p}' \in \operatorname{Proj}(R[x_0,...,x_n])$. Now it follows that $\mathfrak{p}' \in D_{fx_0}$, so $D_{fx_0} \neq \emptyset$.

Proposition 17. Assume R is reduced. Then \mathbb{P}_R^n is a reduced scheme.

Proof. By the lecture, it suffices to show that each $\mathcal{O}_{\mathbb{P}_R^n}(D_{x_i}) \cong \operatorname{Spec}(R[x_0,...,x_n]_{x_i})_0$ is reduced. However this is trivial, as taking the polynomial ring $R[x_0,...,x_n]$ preserves reducedness by Lemma 15, and localizing (and of course taking the subring S_0) also do.

²This is obviously irreducible, as $R[x_0,...,x_n]/\mathfrak{N}(R[x_0,...,x_n]) = (R/\mathfrak{N}(R))[x_0,...,x_n]$ is integral by assumption and Lemma 15.

Corollary 18. Assume R is integral. Then \mathbb{P}_R^n is an integral scheme.

Proposition 19. $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is of finite type.

Proof. First note that \mathbb{P}_{R}^{n} is covered by a finite number of affine opens $D_{x_{i}}$ where the canonical morphisms

$$D_{x_i} \cong \operatorname{Spec}(R[x_0, ..., x_n]_{x_i})_0 \to \operatorname{Spec}(R)$$

are quasi-compact (they are induced by ring homomorphisms $R \to (R[x_0,...,x_n]_{x_i})_0$), hence \mathbb{P}^n_R is quasi-compact.

To see that it is locally of finite type, show that there is the cover by affine opens $\mathbb{P}_{R}^{n} = \bigcup_{i} D_{x_{i}}$ such that the ring homomorphisms $R \to \mathcal{O}_{\mathbb{P}_{R}^{n}}(D_{x_{i}})$ are of finite type. This is clearly the case, as

$$\mathcal{O}_{\mathbb{P}_{p}^{n}}(D_{x_{i}}) = (R[x_{0},...,x_{n}]_{x_{i}})_{0} \cong R[y_{0},...,y_{i-1},y_{i},...,y_{n}]$$

is a finitely generated R-algebra.

Corollary 20. If k is an algebraically closed field, then \mathbb{P}^n_k is a variety.

3 Projective space is proper

We already know that \mathbb{P}_R^n is separated and of finite type over $\operatorname{Spec}(R)$, hence it is left to show that it is universally closed.

Lemma 21. Let X, Y be topological spaces and $f: X \to Y$ a continuous map. Let further $Y = \bigcup_i V_i$ be a cover by open sets. If each of the maps

$$f|_{f^{-1}(V_i)}: f^{-1}(V_i) \to V_i$$

is closed (as a map into V_i), then f is closed.

Proof. Consider some closed $C \subseteq X$ and $x \in Y \setminus f(C)$. Since the V_i cover Y, there is i such that $x \in V_i$. As $f|_{f^{-1}(V_i)}$ is closed, we see that

$$f(C) \cap V_i = f|_{f^{-1}(V_i)}(C)$$

is closed in V_i , hence there is an open neighborhood $U \subseteq V_i$ of x with $U \cap f(C) = \emptyset$. Now note that $V_i \subseteq Y$ is open, and so U is also an open neighborhood of x in Y. This holds for all $x \in Y \setminus f(C)$, hence f(C) is closed.

Corollary 22. A scheme X over $\operatorname{Spec}(R)$ is universally closed, if for all ring homomorphisms $R \to S$ have the following: The map $X \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S) \to \operatorname{Spec}(S)$ that makes the diagram

is closed.

Proof. Consider any scheme Y over $\operatorname{Spec}(R)$. We have to show that the map $\pi: Y \times_{\operatorname{Spec}(R)} X \to Y$ that makes the diagram

$$\begin{array}{ccc} Y \times_{\operatorname{Spec}(R)} X & \longrightarrow & X \\ \pi \downarrow & & \downarrow \\ Y & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

commute, is closed.

Let $Y = \bigcup_i V_i$ be a cover by affine opens. For each i, note that the diagram

$$\pi^{-1}(V_i) \longrightarrow X$$

$$\pi_i := \pi|_{\pi^{-1}(V_i)} \downarrow \qquad \qquad \downarrow$$

$$V_i \longrightarrow \operatorname{Spec}(R)$$

commutes. By the universal property of $V_i \times_{\operatorname{Spec}(R)} X$ we now find a morphism

$$\psi: \pi^{-1}(V_i) \to V_i \times_{\operatorname{Spec}(R)} X$$

of schemes over $Y \times X$, i.e. the inclusion $\pi^{-1}(V_i)$ factors as

$$\pi^{-1}(V_i) \stackrel{\psi}{\to} V_i \times_{\operatorname{Spec}(R)} X \to Y \times_{\operatorname{Spec}(R)} X$$

Considering $V_i \times_{\operatorname{Spec}(R)} X$ as an open subscheme of $Y \times_{\operatorname{Spec}(R)} X$, we thus see that ψ must already be the identity and so $\pi^{-1}(V_i) \subseteq V_i \times_{\operatorname{Spec}(R)} X$. It follows that $\pi^{-1}(V_i) = V_i \times_{\operatorname{Spec}(R)} X$.

By assumption we know that each π_i is closed, as $V_i \cong \operatorname{Spec}(\mathcal{O}_Y(V_i))$ is affine. Now the previous lemma yields that also π is closed.

Remark 23. Note that Proj is not functorial, in the sense that a homomorphism of graded rings $S \to T$ does not induce a natural morphism $\operatorname{Proj}(T) \to \operatorname{Proj}(S)$ (like Spec does). Namely, given $\alpha: S \to T$, we might want to consider

$$\operatorname{Proj}(T) \to \operatorname{Proj}(S), \quad \mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p})$$

However, in general this is not well-defined, as $\alpha^{-1}(\mathfrak{p})$ might contain S_+ .

If we further require the map $\alpha: S \to T$ to fulfill $T_+ \subseteq \alpha(S_+)T$, then this works out and we get a morphism

$$\operatorname{Proj}(\alpha): \operatorname{Proj}(T) \to \operatorname{Proj}(S), \quad \mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p}),$$
$$\operatorname{Proj}(\alpha)_{D_f}^{\#}: \mathcal{O}_{\operatorname{Proj}(S)}(D_f) \to \mathcal{O}_{\operatorname{Proj}(T)}(D_{\alpha(f)}), \quad \frac{x}{y} \mapsto \frac{\alpha(x)}{\alpha(y)}$$

Note that this is indeed a well-defined morphism, as for $\mathfrak{p} \in \operatorname{Proj}(T)$ there is some $f \in T_+ \setminus \mathfrak{p}$ and by assumption, have $g \in S_+, t \in T$ with $\alpha(g)t = f$. Now $g \notin \alpha^{-1}(\mathfrak{p})$, as $g \in \alpha^{-1}(\mathfrak{p})$ would imply $\alpha(g) \in \mathfrak{p}$, thus $f = t\alpha(g) \in \mathfrak{p}$.

In particular, any ring homomorphism $R \to S$ induces a canonical morphism $\mathbb{P}^n_S \to \mathbb{P}^n_R$ of schemes, since the induced ring homomorphism $R[x_0,...,x_n] \to S[x_0,...,x_n]$ satisfies the above additional condition.

Lemma 24. Let S, T be R-algebras via $f: R \to S$ and $g: R \to T$. Let $\mathfrak{p} \leq S, \mathfrak{q} \leq T$ be prime ideals with $\mathfrak{b} = f^{-1}(\mathfrak{p}) = g^{-1}(\mathfrak{q})$. Then have the canonical map

$$S \otimes_R T \to \operatorname{Frac}(S/\mathfrak{p}) \otimes_R \operatorname{Frac}(T/\mathfrak{q}) = \operatorname{Frac}(S/\mathfrak{p}) \otimes_{R/\mathfrak{p}} \operatorname{Frac}(T/\mathfrak{q})$$

Then the preimage $\mathfrak{a} \leq S \otimes_R T$ of a prime ideal in $\operatorname{Frac} S/\mathfrak{p} \otimes_{R/\mathfrak{b}} \operatorname{Frac}(T/\mathfrak{q})$ is a prime ideal such that

$$\iota_S^{-1}(\mathfrak{a}) = \mathfrak{p} \quad and \quad \iota_T^{-1}(\mathfrak{a}) = \mathfrak{q}$$

where $\iota_S: S \to S \otimes_R T$ and $\iota_T: T \to S \otimes_R T$ are the tensor maps.

Lemma 25. Let $R \to S$ be a ring homomorphism. Then the base change of \mathbb{P}^n_R is

$$\mathbb{P}_R^n \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S) = \mathbb{P}_S^n$$

Proof. Consider the canonical map $\alpha: R[x_0,...,x_n] \to S[x_0,...,x_n]$ induced by $R \to S$. Note that

$$S \otimes_R (R[x_0,...,x_n]_f)_0 \cong (S[x_0,...,x_n]_{\alpha(f)})_0 \quad \text{via} \quad \iota_f : s \otimes \frac{r}{f^n} \mapsto \frac{s\alpha(r)}{\alpha(f)^n}$$

for all homogeneous $f \in R[x_0, ..., x_n]$. By Lemma 11, we see that D_f for $f \in S[x_0, ..., x_n]$ homogeneous form an affine open cover of \mathbb{P}^n_S . Furthermore, we have a cover by affine opens $D_f \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$ of $\mathbb{P}^n_R \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$ for $f \in R[x_0, ..., x_n]$ homogeneous. Together, we see that it suffices to show that the isomorphisms (induced by ι_f)

$$\phi_f: D_{\alpha(f)} = \operatorname{Spec}((S[x_0, ..., x_n]_{\alpha(f)})_0) \to D_f \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$$

glue to an isomorphism $\operatorname{Proj}(S[x_0,...,x_n]) \to \mathbb{P}^n_R \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$. By the gluing lemma, it suffices that the compatibility conditions are satisfied, i.e. that for all $f,g \in R[x_0,...,x_n]$ homogeneous the diagram

$$D_{\alpha(f)} \cap D_{\alpha(g)} = \overbrace{D_{\alpha(fg)}}^{\phi_f} D_f \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$$

$$D_{\alpha(f)} \cap D_{\alpha(g)} = \overbrace{D_{\alpha(fg)}}^{p_R} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$$

$$D_{\alpha(g)} \xrightarrow{\phi_g} D_g \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$$

commutes.

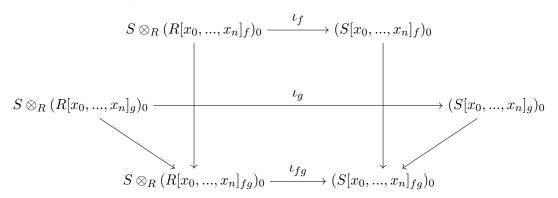
For a prime ideal $\mathfrak{p} \in D_{\alpha(fq)}$ have that

$$\phi_f(\mathfrak{p}) = \iota_f^{-1}(\mathfrak{p} \cap (S[x_0,...,x_n]_f)_0) \in D_f \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$$

and so $g \notin \phi_f(\mathfrak{p})$, i.e. $\phi_f(\mathfrak{p}) \in D_{fg} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$. Hence, we have that both restrictions

$$\phi_f|_{D_{\alpha(fg)}}, \ \phi_g|_{D_{\alpha(fg)}}: D_{\alpha(fg)} \to D_{fg} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$$

map into $D_{fg} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$. Now note that the diagram



commutes, and so the restrictions of ϕ_f resp. ϕ_g to $D_{\alpha(fg)}$ are both induced by the homomorphism ι_{fg} , hence they are equal. The claim follows.

References

[Har77] Robin Hartshorne. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer, 1977.