# Miniproject - Introduction to Schemes

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## Contents



## 1 Definition of Proj

First of all, we will start with a reminder on graded rings, as they are the fundamental object used in the Proj-construction.

**Definition 1.** A *graded ring* $S$ is a ring $S$ with a decomposition $S = \oplus_{d \in \mathbb{N}} S_d$ into groups $S_i \subseteq S$ (w.r.t. addition in $S$) such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{Z}$. Write further $S_+ := \sum_{d \neq 0} S_d$. For a homogeneous $f \in S_d$ say that $\deg(f) := d$ is its *degree*.

An element $f \in S$ is called *homogeneous* (of degree $n$), if $f \in S_n$. An ideal $I \leq S$ is called *homogeneous*, if it has a set of homogeneous generators.

From now on let $S$ be a graded ring.

**Definition 2.** Define the set

$$\mathrm{Proj}(S) := \{\mathfrak{p} \in \mathrm{Spec}(S) \mid \mathfrak{p} \text{ homogeneous}, \ S_+ \not\subseteq \mathfrak{p}\}$$

of homogeneous prime ideals not containing $S_+$.

This becomes a topological space by endowing it with the *Zariski-topology* on $\mathrm{Proj}(S)$, given by the open sets

$$\mathrm{D}_+(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a} \not\subseteq \mathfrak{p}\}$$

for any homogeneous ideal $\mathfrak{a} \leq S$.

**Proposition 3.** *The above definition is well-defined, i.e. the sets* $\mathrm{D}_+(\mathfrak{a})$ *indeed form a topology on* $\mathrm{Proj}(S)$.

*Proof.* Clearly $\mathrm{Proj}(S) = \mathrm{D}_+(\langle 1 \rangle)$ and $\emptyset = \mathrm{D}_+(\langle 0 \rangle)$ are open. Furthermore, for open sets $\mathrm{D}_+(\mathfrak{a})$ and $\mathrm{D}_+(\mathfrak{b})$, have that

$$\mathrm{D}_+(\mathfrak{a}) \cap \mathrm{D}_+(\mathfrak{b}) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}\} = \mathrm{D}_+(\mathfrak{a}\mathfrak{b})$$

This holds, as $\mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}$ implies that there are $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ with $f, g \notin \mathfrak{p}$. However, then $fg \notin \mathfrak{p}$ as $\mathfrak{p}$ is prime. Obviously $\mathfrak{a}\mathfrak{b}$ is homogeneous, and so $\mathrm{D}_+(\mathfrak{a}) \cap \mathrm{D}_+(\mathfrak{b})$ is open.

Finally, given a collection $\mathcal{A}$ of homogeneous ideals in $S$, have that

$$\begin{aligned}\bigcup_{\mathfrak{a} \in \mathcal{A}} \mathrm{D}_+(\mathfrak{a}) =& \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \exists \mathfrak{a} \in \mathcal{A} : \ \mathfrak{a} \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \exists \mathfrak{a} \in \mathcal{A} \exists f \in \mathfrak{a} : \ f \notin \mathfrak{p}\} \\ =& \Big\{\mathfrak{p} \in \mathrm{Proj}(S) \ \Big| \ \exists f \in \sum_{\mathfrak{a} \in \mathcal{A}} \mathfrak{a} : \ f \notin \mathfrak{p}\Big\} = \mathrm{D}_+(\mathfrak{b}) \quad \text{for } \mathfrak{b} = \sum_{\mathfrak{a} \in \mathcal{A}} \mathfrak{a}\end{aligned}$$

Clearly $\mathfrak{b}$ is again homogeneous, and so $\bigcup_{\mathfrak{a} \in \mathcal{A}} \mathrm{D}_+(\mathfrak{a})$ is open. ☐

As opposed to Hartshorne [Har77], we follow an approach similar to the construction of $\mathcal{O}_{\mathrm{Spec}(R)}$ to construct a sheaf of rings on $\mathrm{Proj}(S)$. In particular, we will define $\mathcal{O}_{\mathrm{Proj}(S)}$ on a basis of the topology, and extend it to a sheaf. This will be somewhat more tedious, but gives a nice idea of how $\mathcal{O}_{\mathrm{Proj}(S)}(U)$ looks like.

**Lemma 4.** *For a homogeneous ideal* $\mathfrak{a}$ *have* $\mathrm{D}_+(\mathfrak{a}) = \mathrm{D}_+(\mathfrak{a}S_+)$.

*Proof.* The inclusion $\supseteq$ is clear, as $\mathfrak{a}S_+ \subseteq \mathfrak{a}$. So assume there is a $\mathfrak{p} \in \mathrm{D}_+(\mathfrak{a})$. Since $\mathfrak{p} \in \mathrm{Proj}(S)$ find $S_+ \not\subseteq \mathfrak{p}$ and by assumption, $\mathfrak{a} \not\subseteq \mathfrak{p}$. Thus there are $s \in S_+$ and $a \in \mathfrak{a}$ with $a, s \notin \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $as \notin \mathfrak{p}$ and so $\mathfrak{a}S_+ \not\subseteq \mathfrak{p}$. ☐

**Proposition 5.** *The sets* $\mathrm{D}_+(f) := \mathrm{D}_+(\langle f \rangle)$ *for homogeneous* $f \in S_+$ *form a basis of the topology on* $\mathrm{Proj}(S)$.

*Proof.* Clearly $\langle f \rangle$ is a homogenous ideal, so $\mathrm{D}_+(f)$ is open. For any homogeneous ideal $\mathfrak{a} = \langle f_i \mid i \in I \rangle$ with $f_i \in S$ homogeneous have that $\mathrm{D}_+(\mathfrak{a}) = \mathrm{D}_+(\mathfrak{a}S_+)$ by Lemma 4, so we can assume wlog that all $f_i \in S_+$. Now have

$$\mathrm{D}_+(\mathfrak{a}) = \bigcup_{i \in I} \mathrm{D}_+(f_i)$$

as $\mathfrak{a} \not\subseteq \mathfrak{p}$ implies there is some $g = \sum_{i \in I} g_i f_i \notin \mathfrak{p}$, with $g_i \in S$. Hence, at least one $g_j f_j \notin \mathfrak{p}$ and so $f_j \notin \mathfrak{p}$, thus $\mathfrak{p} \in \mathrm{D}_+(f_j)$. It follows that the $\mathrm{D}_+(f)$ generate the topology on $\mathrm{Proj}(S)$, so it is left to show that they are a basis.

Consider $\mathfrak{p} \in \mathrm{D}_+(f) \cap \mathrm{D}_+(g)$, so $f, g \notin \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $fg \notin \mathfrak{p}$ and so $\mathrm{D}_+(fg) \subseteq \mathrm{D}_+(f) \cap \mathrm{D}_+(g)$ is an open neighborhood of $\mathfrak{p}$. ☐

**Lemma 6.** *Let $\mathfrak{p} \leq S$ be a prime ideal. Then*

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \leq S$$

*is a (homogeneous) prime ideal.*

*Proof.* Consider $f, g \in S$ with $fg \in \mathfrak{p}'$ and assume $f, g \notin \mathfrak{p}'$. Write $f = \sum_d f_d$ and $g = \sum_d g_d$ with $f_d, g_d \in S_d$. So

$$\sum_{i,j} f_i g_j = \sum_n \sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

Since $\mathfrak{p}'$ is homogeneous, it follows that

$$\sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

for all $n \in \mathbb{Z}$.

Let now $d$ resp. $e$ be maximal such that $f_d \notin \mathfrak{p}'$ resp. $g_e \notin \mathfrak{p}'$. We have

$$f_d g_e + \sum_{\substack{i+j=d+e \\ (i,j) \neq (d,e)}} \underbrace{f_i g_j}_{\in \mathfrak{p}'} = \sum_{i+j=d+e} f_i g_j \in \mathfrak{p}'$$

and so $f_d g_e \in \mathfrak{p}' \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. However both $f_d$ and $g_e$ are homogeneous, so $f_d \in \mathfrak{p}'$ or $g_e \in \mathfrak{p}'$, a contradiction. $\square$

**Lemma 7** ("Covering Trick")**.** *Let $f, f_i \in S_+$ be homogeneous. Then $\mathrm{D}_+(f) \subseteq \bigcup_i \mathrm{D}_+(f_i)$ if and only if $f^n \in \langle f_i \mid i \rangle$ for some $n \in \mathbb{N}$*

*Proof.* The direction $\Leftarrow$ is clear. So assume $\mathrm{D}_+(f) \subseteq \bigcup_i \mathrm{D}_+(f_i)$. If no $f^n$ is in $\mathfrak{a} := \langle f_i \mid i \rangle$, then $f$ is not nilpotent in $S/\mathfrak{a}$ and so $S/\mathfrak{a}$ is not the zero ring. Hence, $(S/\mathfrak{a})_f$ has a prime ideal, and its preimage under $S \to S/\mathfrak{a} \to (S/\mathfrak{a})_f$ yields a prime ideal $\mathfrak{p}'$ containing $\mathfrak{a}$ and not containing $f$. Since $f \in S_+$ note that $S_+ \not\subseteq \mathfrak{p}'$ and as both $f$ and $\mathfrak{a}$ are homogeneous, Lemma 6 yields now that $\langle g \in \mathfrak{p}' \mid g \text{ homogeneous} \rangle$ is a prime ideal in $\mathrm{D}_+(f)$ but not in $\bigcup_i \mathrm{D}_+(f_i)$, contradicting the assumption. $\square$

**Corollary 8.** *If $\mathrm{D}_+(g) \subseteq \mathrm{D}_+(f)$ with $f, g \in S_+$, then there is a homogeneous $h \in S$ such that $g^n = fh$ for some $n \in \mathbb{N}$.*

*Proof.* By assumption, $\mathrm{D}_+(f)$ covers $\mathrm{D}_+(g)$, thus by the Covering Trick (Lemma 7), observe that $g^n \in \langle f \rangle$ for some $n \in \mathbb{N}$. The claim follows. $\square$

The next proof works exactly as the corresponding one for Spec in the lecture.

**Proposition 9.** *Let $B = \{\mathrm{D}_+(f) \mid f \in S_+ \text{ homogeneous}\}$. The functor*[1]

$$\mathcal{F} : \mathrm{Top}(\mathrm{Proj}(S))\big|_B \to \textbf{\textit{Ring}}, \quad \mathrm{D}_+(f) \mapsto (S_f)_0$$
$$(\mathrm{D}_+(fg) \subseteq \mathrm{D}_+(f)) \mapsto \left( \cdot|_{\mathrm{D}_+(fg)} : \frac{s}{f^n} \mapsto \frac{sg^n}{(fg)^n} \right)$$

[1]The map on arrows is well-defined by Corollary 8.

*is a B-sheaf on B (here* $\mathrm{Top}(X)$ *is the category given by the open sets of* $X$ *and their inclusion, as defined in the lecture).*

*Proof.* Clearly, $\mathcal{F}$ is a functor and thus a presheaf. Hence, we have to show the local-to-global property.

Let $\mathrm{D}_+(f) = \bigcup_{i\in I} \mathrm{D}_+(g_i f)$ be a cover and $s_i \in \mathcal{F}(\mathrm{D}_+(g_i f))$ such that

$$\forall x \in \mathrm{D}_+(g_i f) \cap \mathrm{D}_+(g_j f)\ \exists V \in B:\ V \subseteq \mathrm{D}_+(g_i f) \cap \mathrm{D}_+(g_j f),\ x \in V,\ \frac{s_i}{1} = \frac{s_j}{1} \in \mathcal{F}(V)$$

To show uniqueness, assume there are $\alpha/f^N, \beta/f^N \in \mathrm{D}_+(f)$ with

$$\left.\frac{\alpha}{f^N}\right|_{\mathrm{D}_+(g_i f)} = \left.\frac{\beta}{f^N}\right|_{\mathrm{D}_+(g_i f)} \quad \text{for all } i$$

Therefore there is $n_i \in \mathbb{N}$ such that

$$(fg_i)^{n_i}(\alpha f^N + \beta f^N) = 0$$

wlog there are only finitely many $i$ (by the Covering Trick, Lemma 7).

wlog $N$ is sufficiently large (we can always increase $N$), and so $(fg_i)^N(\alpha - \beta) = 0$.

Since $\bigcup_i \mathrm{D}_+((fg_i)^N) = \bigcup_i \mathrm{D}_+(fg_i) = \mathrm{D}_+(f)$ it follows that $f^n \in \langle (fg_i)^N \mid i\rangle$ for some $n \in \mathbb{N}$ (again the Covering Trick, Lemma 7). Therefore we find

$$f^n(\alpha - \beta) \in \langle (fg_i)^N \mid i\rangle(\alpha - \beta) = \langle (fg_i)^N(\alpha - \beta) \mid i\rangle = \{0\}$$

and so $\alpha = \beta \in S_f$.

Now we show existence. By the uniqueness above, it follows that

$$s_i|_{\mathrm{D}_+(fg_ig_j)} = s_i|_{\mathrm{D}_+(fg_i)\cap\mathrm{D}_+(fg_j)} = s_j|_{\mathrm{D}_+(fg_i)\cap\mathrm{D}_+(fg_j)} = s_j|_{\mathrm{D}_+(fg_ig_j)}$$

wlog have again a finite cover, i.e. only finitely many $g_i$. Hence find an $N \in \mathbb{N}$ such that each $s_i = s_i'/(fg_i)^N$ with $s_i' \in S$ homogeneous. By possibly replacing $N$ with a bigger $N$, we can now assume that

$$(f^2 g_i g_j)^N\left(s_i'(fg_j)^N - s_j'(fg_i)^N\right) = 0 \quad \text{as } s_i|_{\mathrm{D}_+(fg_ig_j)} = s_j|_{\mathrm{D}_+(fg_ig_j)}$$

Now note that

$$s_i = \frac{a_i}{b_i} \quad \text{with } a_i = s_i'(fg_i)^N,\ b_i = (fg_i)^{2N}$$

and

$$a_ib_j - a_jb_i = s_i'(fg_i)^N(fg_j)^{2N} - s_j'(fg_j)^N(fg_i)^{2N} = \underbrace{(f^2g_ig_j)^N\left(s_i'(fg_j)^N - s_j'(fg_i)^N\right)}_{=0}$$

Now observe that $\mathrm{D}_+(b_i) = \mathrm{D}_+(fg_i)$ and so $f^n \in \langle b_i \mid i\rangle$ for some $n$ (once again the Covering Trick, Lemma 7). Let $f^n = \sum_i r_ib_i$ and get

$$a_if^n = \sum_l r_lb_la_i = \sum_l r_la_lb_i = b_i\sum_l r_la_l$$

Note that $a_i, b_i, f$ are homogeneous, and so we can also choose $r_i$ to be homogeneous. Then find that $0 = \deg(s_i) = \deg(a_i) - \deg(b_i) = \deg(\sum r_l a_l) - n\deg(f)$.

Thus

$$s_i = s := \frac{\sum_l r_l a_l}{fn} \in S_f$$

and since $\deg(\sum r_l a_l) = n\deg(f)$ we find that $s \in (S_f)_0 = \mathcal{F}(\mathrm{D}_+(f))$. Clearly $s\big|_{\mathrm{D}_+(fg_i)} = s_i$ and the claim follows. $\square$

Now we the B-sheaf extension theorem from the lecture (from Section 1.11) yields the following corollary.

**Corollary 10.** *The B-sheaf $\mathcal{F}$ can be (uniquely, up to isomorphism) extended to a sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$ on* $\mathrm{Proj}(S)$*.*

The next to lemmas are based on [Har77, p. II.2.5] and show that $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a scheme.

**Lemma 11.** *For $\mathfrak{p} \in \mathrm{Proj}(S)$, the stalk*

$$\mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}} = (T^{-1}S)_0$$

*is a local ring, where $T = \{f \notin \mathfrak{p} \mid f \text{ homogeneous}\}$ contains all homogeneous elements not in $\mathfrak{p}$.*

*Proof.* Consider the ideal

$$\mathfrak{m} = \left\{ \frac{f}{g} \in (T^{-1}S)_0 \;\middle|\; f \in \mathfrak{p} \right\}$$

We claim this is the unique maximal ideal of $(T^{-1}S)_0$.

First, note that $1 \notin \mathfrak{m}$ as otherwise, there would be $f/g \in T^{-1}S$, $f \in \mathfrak{p}$ with $t(f-g) = 0$ for some $t \in T$. However then $tg = tf \in \mathfrak{p}$, and so $g \in \mathfrak{p}$ (as $t \notin \mathfrak{p}$), contradicting $g \in T$.

Now assume there is any ideal $\mathfrak{a}$ such that $\mathfrak{a} \setminus \mathfrak{m} \neq \emptyset$, i.e. there is $f/g \in \mathfrak{a} \setminus \mathfrak{m}$. Then $f \in T$ as $f$ homogeneous and $f \notin \mathfrak{p}$. Thus $g/f \in (T^{-1}S)_0$ and so $f/g \in (T^{-1}S)_0^*$, which implies $\mathfrak{a} = \langle 1 \rangle$. $\square$

**Lemma 12.** *For $f \in S_+$ homogeneous have that $(\mathrm{D}_+(f), \mathcal{O}_{\mathrm{Proj}(S)}\big|_{\mathrm{D}_+(f)})$ is an affine scheme.*

*Proof.* Let $R = (S_f)_0$. Consider the map

$$\phi : \mathrm{D}_+(f) \to \mathrm{Spec}(R), \quad \mathfrak{p} \mapsto \mathfrak{p}S_f \cap R$$

Note that it is continuous, as the preimage of some basic open set $D_{g/f^n}$ is $\mathrm{D}_+(fg) \subseteq \mathrm{D}_+(f)$ open. Furthermore, have the map

$$\phi' : \mathrm{Spec}(R) \to \mathrm{D}_+(f), \quad \mathfrak{p} \mapsto \mathfrak{p}S_f \cap S$$

which is also continuous, as the preimage of some basic open set $\mathrm{D}_+(fg)$ is $D_{g^n/f^m}$ where $n\deg(g) = m\deg(f)$. Note that $\phi \circ \phi' = \mathrm{id}_{\mathrm{Spec}(R)}$ because $\mathfrak{p} \subseteq (\mathfrak{p}S_f \cap S)S_f \cap R$ and

$$(\mathfrak{p}S_f \cap S)S_f \cap R \subseteq \mathfrak{p}S_f \cap S_f \cap R = \mathfrak{p}S_f \cap (S_f)_0 = \mathfrak{p}$$

where the last equality holds, since $\mathfrak{p} \leq (S_f)_0$ is homogeneous.

To see that $\phi' \circ \phi = \mathrm{id}_{\mathrm{D}_+(f)}$, it suffices now to show that $\phi$ is injective ($\phi$ has a right-inverse, hence it is then bijective, and the one-sided inverse $\phi'$ is already the inverse). Let $\mathfrak{p}, \mathfrak{q} \leq S$ homogeneous primes with $f \notin \mathfrak{p}, \mathfrak{q}$. Assume there is $g \in \mathfrak{p} \setminus \mathfrak{q}$. Then

$$g^{\deg(f)} \in \mathfrak{p}S_f \setminus \mathfrak{q}S_f \;\Rightarrow\; \frac{g^{\deg(f)}}{f^{\deg(g)}} \in (\mathfrak{p}S_f \setminus \mathfrak{q}S_f) \cap R = (\mathfrak{p}S_f \cap R) \setminus (\mathfrak{q}S_f \cap R)$$

and so $\phi(\mathfrak{p}) \neq \phi(\mathfrak{q})$. Thus $\phi$ is a homeomorphism.

Now consider the natural transformation

$$\eta : \mathcal{O}_{\mathrm{Spec}R} \Rightarrow \phi_*\Big(\mathcal{O}_{\mathrm{Proj}(S)}\Big|_{\mathrm{D}_+(f)}\Big)$$

given on basic open sets by

$$\eta_{D_{g/f^n}} : R_{g/f^n} \to (S_{fg})_0, \quad \frac{h/f^m}{(g/f^n)^l} \mapsto \frac{hf^{nl}}{g^l f^m}$$

Clearly this is a ring isomorphism, so $\eta$ is a natural isomorphism. The claim follows. □

**Corollary 13.** $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ *is a scheme.*

**Definition 14.** For any ring $R$ and $n \in \mathbb{N}$, define the *projective $n$-space* as[2]

$$\mathbb{P}^n_R := \mathrm{Proj}(R[x_0, ..., x_n])$$

which naturally becomes a scheme over $\mathrm{Spec}(R)$ via the morphism given by

$$\begin{aligned} &f : \mathrm{Proj}(R[x_0, ..., x_n]) \to \mathrm{Spec}(R), \quad \mathfrak{p} \mapsto \mathfrak{p} \cap R \\ &f^{\#}_{D_f} : R_f \to \mathcal{O}_{\mathrm{Proj}(R[x_0,...,x_n])}(\mathrm{D}_+(f)) = R_f[x_0, ..., x_n], \quad r \mapsto r \end{aligned}$$

## 2 Projective space as a variety

For this section, let $R$ be a ring and $n \in \mathbb{N}$.

First, we will study the property of being separated.

**Proposition 15.** *The morphism $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is separated.*

[2]Endow $R[x_0, ..., x_n]$ with the standard natural grading, i.e. $\deg(x_i) = 1$. Note that this makes $R[x_0, ..., x_n]_0 = R$, so all $r \in R$ are homogeneous of degree 0.

*Proof.* By Lemma 12 we find a cover $\text{Proj}(S) = \bigcup_i \text{D}_+(x_i)$ of affine opens. We use the characterization from the lecture and show that $\text{D}_+(x_i) \cap \text{D}_+(x_j)$ is an affine open and the canonical multiplication map

$$\mathcal{O}_{\mathbb{P}^n_R}(\text{D}_+(x_i)) \times \mathcal{O}_{\mathbb{P}^n_R}(\text{D}_+(x_j)) \to \mathcal{O}_{\mathbb{P}^n_R}(\text{D}_+(x_i) \cap \text{D}_+(x_j))$$

is surjective.

First note that $\text{D}_+(x_i) \cap \text{D}_+(x_j) = \text{D}_+(x_i x_j)$ which is affine open by Lemma 12. By definition, we have $\mathcal{O}_{\mathbb{P}^n_R}(\text{D}_+(f)) = (R[x_0, ..., x_n]_f)_0$ and so we have to show that

$$(R[x_0, ..., x_n]_{x_i})_0 \times (R[x_0, ..., x_n]_{x_j})_0 \to (R[x_0, ..., x_n]_{x_i x_j})_0, \quad (a, b) \mapsto ab$$

is surjective.

However, for $f/(x_i x_j)^n \in (R[x_0, ..., x_n]_{x_i x_j})_0$ we have that

$$\frac{f}{x_i^{2n}} \cdot \frac{x_i^n}{x_j^n} = \frac{f}{(x_i x_j)^n}$$

and clearly $f/x_i^{2n} \in (R[x_0, ..., x_n]_{x_i})_0$ as $\deg(f) = \deg(x_i)n + \deg(x_j)n = 2n$ and $x_i^n/x_j^n \in (R[x_0, ..., x_n]_{x_j})_0$. The claim follows. $\square$

To study integrality, it is useful to have the next small lemma on the nilradical.

**Lemma 16.** *It holds that* $\mathfrak{N}(R)R[x_0, ..., x_n] = \mathfrak{N}(R[x_0, ..., x_n])$.

*Proof.* Clearly $\mathfrak{N}(R)R[x_0, ..., x_n] \subseteq \mathfrak{N}(R[x_0, ..., x_n])$, as for $\sum_{i=1}^m f_i r_i$ with $f_i^n = 0$ have that

$$\Big(\sum_{i=1}^m f_i r_i\Big)^{mn} = \sum_{1 \le i_1, ..., i_{mn} \le m} \underbrace{f_{i_1} \cdots f_{i_{mn}}}_{\substack{=0 \\ \text{as at least one } i \text{ repeats at least } n \text{ times in } i_1, ..., i_{mn}}} r_{i_1} \cdots r_{i_{mn}}$$

Now note that $\mathfrak{N}(R[x_0, ..., x_n]) \subseteq \mathfrak{a}$ for all ideals $\mathfrak{a}$ and the claim follows. $\square$

**Proposition 17.** *Assume* $\text{Spec}(R)$ *is irreducible. Then* $\mathbb{P}^n_R$ *is an irreducible scheme.*

*Proof.* We show that $\text{D}_+(x_0) \subseteq \mathbb{P}^n_R$ is dense. The claim then follows, as any two nonempty, disjoint open sets give nonempty, disjoint open sets in $D_{x_0}$ which is isomorphic to the irreducible[3] affine scheme $\text{Spec}(R[x_0, ..., x_n]_{x_0})_0$ by Lemma 12.

So consider any nonempty basic open $\text{D}_+(f) \subseteq \mathbb{P}^n_R$ and find that $\text{D}_+(f) \cap \text{D}_+(x_0) = \text{D}_+(f x_0)$. By assumption, $\mathfrak{N}(R)$ is a prime ideal and so $f x_0$ is not nilpotent (unless $f$ were nilpotent, but then $\text{D}_+(f) = \emptyset$), i.e. the ring $R[x_0, ..., x_n]_{f x_0}$ is not the zero ring. Hence, it has a maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ be the preimage of $\mathfrak{m}$ under the localization map $R[x_0, ..., x_n] \to R[x_0, ..., x_n]_{f x_0}$, which is clearly prime with $f x_0 \notin \mathfrak{p}$. By Lemma 6, we now see that

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \le R[x_0, ..., x_n]$$

is a homogeneous prime ideal, and since $f x_0 \notin \mathfrak{p} \supseteq \mathfrak{p}'$, we see that $S_+ \not\subseteq \mathfrak{p}'$, i.e. $\mathfrak{p}' \in \text{Proj}(R[x_0, ..., x_n])$. Now it follows that $\mathfrak{p}' \in \text{D}_+(f x_0)$, so $\text{D}_+(f x_0) \neq \emptyset$. $\square$

[3]This is obviously irreducible, as $R[x_0, ..., x_n]/\mathfrak{N}(R[x_0, ..., x_n]) = (R/\mathfrak{N}(R))[x_0, ..., x_n]$ is integral by assumption and Lemma 16.

**Proposition 18.** *Assume $R$ is reduced. Then $\mathbb{P}^n_R$ is a reduced scheme.*

*Proof.* By the lecture, it suffices to show that each $\mathcal{O}_{\mathbb{P}^n_R}(\mathrm{D}_+(x_i)) \cong \mathrm{Spec}(R[x_0, ..., x_n]_{x_i})_0$ is reduced. However this is trivial, as taking the polynomial ring $R[x_0, ..., x_n]$ preserves reducedness by Lemma 16, and localizing (and of course taking the subring $S_0$) also do. $\square$

**Corollary 19.** *Assume $R$ is integral. Then $\mathbb{P}^n_R$ is an integral scheme.*

**Proposition 20.** *The morphism $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is of finite type.*

*Proof.* First note that $\mathbb{P}^n_R$ is covered by a finite number of affine opens $\mathrm{D}_+(x_i)$ where the canonical morphisms

$$\mathrm{D}_+(x_i) \cong \mathrm{Spec}(R[x_0, ..., x_n]_{x_i})_0 \to \mathrm{Spec}(R)$$

are quasi-compact (they are induced by ring homomorphisms $R \to (R[x_0, ..., x_n]_{x_i})_0$), hence $\mathbb{P}^n_R$ is quasi-compact.

To see that it is locally of finite type, show that there is the cover by affine opens $\mathbb{P}^n_R = \bigcup_i \mathrm{D}_+(x_i)$ such that the ring homomorphisms $R \to \mathcal{O}_{\mathbb{P}^n_R}(\mathrm{D}_+(x_i))$ are of finite type. This is clearly the case, as

$$\mathcal{O}_{\mathbb{P}^n_R}(\mathrm{D}_+(x_i)) = (R[x_0, ..., x_n]_{x_i})_0 \cong R[y_0, ..., y_{i-1}, y_i, ..., y_n]$$

is a finitely generated $R$-algebra. $\square$

**Corollary 21.** *If $k$ is an algebraically closed field, then $\mathbb{P}^n_k$ is a variety.*

# 3 Projective space is proper

We already know that $\mathbb{P}^n_R$ is separated and of finite type over $\mathrm{Spec}(R)$, now we want to show that it is proper. Both in Hartshorne [Har77] and the Stacks project [Stacks], and in every other source I could find, this is done using the “valuative criterion for properness” (see also Section 33). However, its proof is apparently quite lengthy and requires a lot of abstract algebra. In the following, I tried to find a simpler and shorter proof. That was only partially successful, as the proof I came up with is shorter, but definitely not simpler. However, it gives a slightly different perspective onto the issue, which is quite nice.

**Lemma 22.** *Let $X, Y$ be topological spaces and $f : X \to Y$ a continuous map. Let further $Y = \bigcup_i V_i$ be a cover by open sets. If each of the maps*

$$f\big|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$$

*is closed (as a map into $V_i$), then $f$ is closed.*

*Proof.* Consider some closed $C \subseteq X$ and $x \in Y \setminus f(C)$. Since the $V_i$ cover $Y$, there is $i$ such that $x \in V_i$. As $f|_{f^{-1}(V_i)}$ is closed, we see that

$$f(C) \cap V_i = f|_{f^{-1}(V_i)}(C)$$

is closed in $V_i$, hence there is an open neighborhood $U \subseteq V_i$ of $x$ with $U \cap f(C) = \emptyset$. Now note that $V_i \subseteq Y$ is open, and so $U$ is also an open neighborhood of $x$ in $Y$. This holds for all $x \in Y \setminus f(C)$, hence $f(C)$ is closed. $\square$

I am slightly surprised that the following lemma was to be found nowhere, except in [Stacks, p. 66.9.5.] (for general algebraic spaces). It seems to be the kind of characterization I would expect to find in most textbooks.

**Corollary 23.** *A scheme $X$ over $\mathrm{Spec}(R)$ is universally closed, if for all ring homomorphisms $R \to S$ satisfy the following: The base change map $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \to \mathrm{Spec}(S)$, i.e. the map that makes the diagram*

$$\begin{tikzcd}
\mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(S) \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

*commute, is closed.*

*Proof.* Consider any scheme $Y$ over $\mathrm{Spec}(R)$. We have to show that the map $\pi : Y \times_{\mathrm{Spec}(R)} X \to Y$ that makes the diagram

$$\begin{tikzcd}
Y \times_{\mathrm{Spec}(R)} X \arrow[r] \arrow[d, "\pi"'] & X \arrow[d] \\
Y \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

commute, is closed.

Let $Y = \bigcup_i V_i$ be a cover by affine opens. For each $i$, note that the diagram

$$\begin{tikzcd}
\pi^{-1}(V_i) \arrow[r] \arrow[d, "\pi_i := \pi|_{\pi^{-1}(V_i)}"'] & X \arrow[d] \\
V_i \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

commutes. By the universal property of $V_i \times_{\mathrm{Spec}(R)} X$ we now find a morphism

$$\psi : \pi^{-1}(V_i) \to V_i \times_{\mathrm{Spec}(R)} X$$

of schemes over $Y \times X$, i.e. the inclusion $\pi^{-1}(V_i)$ factors as

$$\pi^{-1}(V_i) \overset{\psi}{\to} V_i \times_{\mathrm{Spec}(R)} X \to Y \times_{\mathrm{Spec}(R)} X$$

Considering $V_i \times_{\mathrm{Spec}(R)} X$ as an open subscheme of $Y \times_{\mathrm{Spec}(R)} X$, we thus see that $\psi$ must already be the identity and so $\pi^{-1}(V_i) \subseteq V_i \times_{\mathrm{Spec}(R)} X$. It follows that $\pi^{-1}(V_i) = V_i \times_{\mathrm{Spec}(R)} X$.

By assumption we know that each $\pi_i$ is closed, as $V_i \cong \mathrm{Spec}(\mathcal{O}_Y(V_i))$ is affine. Now the previous lemma yields that also $\pi$ is closed. □

**Remark 24.** Note that Proj is not functorial, in the sense that a homomorphism of graded rings $S \to T$ does not induce a natural morphism $\mathrm{Proj}(T) \to \mathrm{Proj}(S)$ (like Spec does). Namely, given $\alpha : S \to T$, we might want to consider

$$\mathrm{Proj}(T) \to \mathrm{Proj}(S), \quad \mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p})$$

However, in general this is not well-defined, as $\alpha^{-1}(\mathfrak{p})$ might contain $S_+$[4].

If we further require the map $\alpha : S \to T$ to fulfill $T_+ \subseteq \alpha(S_+)T$, then this works out and we get a morphism

$$\mathrm{Proj}(\alpha) : \mathrm{Proj}(T) \to \mathrm{Proj}(S), \quad \mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p}),$$
$$\mathrm{Proj}(\alpha)^{\#}_{\mathrm{D}_+(f)} : \mathcal{O}_{\mathrm{Proj}(S)}(\mathrm{D}_+(f)) \to \mathcal{O}_{\mathrm{Proj}(T)}(\mathrm{D}_+(\alpha(f))), \quad \frac{x}{y} \mapsto \frac{\alpha(x)}{\alpha(y)}$$

Note that this is indeed a well-defined morphism, as for $\mathfrak{p} \in \mathrm{Proj}(T)$ there is some $f \in T_+ \setminus \mathfrak{p}$ and by assumption, have $g \in S_+, t \in T$ with $\alpha(g)t = f$. Now $g \notin \alpha^{-1}(\mathfrak{p})$, as $g \in \alpha^{-1}(\mathfrak{p})$ would imply $\alpha(g) \in \mathfrak{p}$, thus $f = t\alpha(g) \in \mathfrak{p}$.

In particular, any ring homomorphism $R \to S$ induces a canonical morphism $\mathbb{P}^n_S \to \mathbb{P}^n_R$ of schemes, since the induced ring homomorphism $R[x_0, ..., x_n] \to S[x_0, ..., x_n]$ satisfies the above additional condition.

**Lemma 25.** *Let $R \to S$ be a ring homomorphism. Then the base change of $\mathbb{P}^n_R$ is*

$$\mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \cong \mathbb{P}^n_S$$

*as schemes over* $\mathrm{Spec}(S)$.

*Proof.* Consider the canonical map $\alpha : R[x_0, ..., x_n] \to S[x_0, ..., x_n]$ induced by $R \to S$. Note that

$$S \otimes_R (R[x_0, ..., x_n]_f)_0 \cong (S[x_0, ..., x_n]_{\alpha(f)})_0 \quad \text{via} \quad \iota_f : s \otimes \frac{r}{f^n} \mapsto \frac{s\alpha(r)}{\alpha(f)^n}$$

for all homogeneous $f \in R[x_0, ..., x_n]$. By Lemma 12, we see that $\mathrm{D}_+(f)$ for $f \in S[x_0, ..., x_n]$ homogeneous form an affine open cover of $\mathbb{P}^n_S$. Furthermore, we have a

[4] Just consider the inclusion $R[x] \to R[x, y]$ and the ideal $\langle x \rangle \in \mathrm{Proj}(R[x, y])$.

cover by affine opens $D_+(f) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ of $\mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ for $f \in R[x_0, ..., x_n]$ homogeneous. Together, we see that it suffices to show that the isomorphisms (induced by $\iota_f$)

$$\phi_f : D_+(\alpha(f)) = \mathrm{Spec}((S[x_0, ..., x_n]_{\alpha(f)})_0) \to D_+(f) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$$

glue to an isomorphism $\mathrm{Proj}(S[x_0, ..., x_n]) \to \mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$. By the gluing lemma, it suffices that the compatibility conditions are satisfied, i.e. that for all $f, g \in R[x_0, ..., x_n]$ homogeneous the diagram

$$\begin{tikzcd}
 & D_+(\alpha(f)) \arrow[r, "\phi_f"] & D_+(f) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \arrow[dr] & \\
D_+(\alpha(f)) \cap D_+(\alpha(g)) = D_+(\alpha(fg)) \arrow[ur] \arrow[dr] & & & \mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \\
 & D_+(\alpha(g)) \arrow[r, "\phi_g"] & D_+(g) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \arrow[ur] &
\end{tikzcd}$$

commutes.

For a prime ideal $\mathfrak{p} \in D_+(\alpha(fg))$ have that

$$\phi_f(\mathfrak{p}) = \iota_f^{-1}(\mathfrak{p} \cap (S[x_0, ..., x_n]_f)_0) \in D_+(f) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$$

and so $g \notin \phi_f(\mathfrak{p})$, i.e. $\phi_f(\mathfrak{p}) \in D_+(fg) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$. Hence, we have that both restrictions

$$\phi_f|_{D_+(\alpha(fg))},\ \phi_g|_{D_+(\alpha(fg))} : D_+(\alpha(fg)) \to D_+(fg) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$$

map into $D_+(fg) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$. Now note that the diagram

$$\begin{tikzcd}
S \otimes_R (R[x_0, ..., x_n]_f)_0 \arrow[rrr, "\iota_f"] \arrow[ddd] & & & (S[x_0, ..., x_n]_f)_0 \arrow[ddd] \\
 & S \otimes_R (R[x_0, ..., x_n]_g)_0 \arrow[r, "\iota_g"] \arrow[ddl] & (S[x_0, ..., x_n]_g)_0 \arrow[ddr] & \\
 & & & \\
S \otimes_R (R[x_0, ..., x_n]_{fg})_0 \arrow[rrr, "\iota_{fg}"] & & & (S[x_0, ..., x_n]_{fg})_0
\end{tikzcd}$$

commutes, and so the restrictions of $\phi_f$ resp. $\phi_g$ to $D_+(\alpha(fg))$ are both induced by the homomorphism $\iota_{fg}$, hence they are equal. The claim follows. $\square$

We now come close to the core of the proof. However, there is one further notion we need.

**Definition 26.** Let $x \in X$ be a point in a scheme (or more generally, a topological space). A point $x' \in X$ is said to *specialize* $x$, if it is contained in the closure $x' \in \overline{\{x\}}$.

The next lemma is directly from [Har77, p. II.4.5].

**Lemma 27.** *Let $f : X \to Y$ be a quasi-compact morphism. Then $f(|X|)$ is closed if and only if it is stable under specialization.*

*Proof.* The implication $\Leftarrow$ is clear. It suffices to show that $f(|X|) \cap V \subseteq V$ is closed for all affine opens $V \subseteq Y$.

Note that the preimage $f^{-1}(V)$ is quasi-compact, hence has a finite cover of affine opens $U_i \subseteq X$. Now let $V_i = \overline{f(|U_i|)}$ be the closed subscheme of $V$ with the canonical (reduced) scheme structure[5]. Since $V$ is affine, also $V_i$ is affine. Note that now it suffices to show that $f(|U_i|) \subseteq V_i$ is closed.

In other words, we have reduced the situation to the case that

$$f' = f\big|_{U_i} : U_i = \operatorname{Spec}(A) \to V_i = \operatorname{Spec}(B)$$

is a morphism between affine schemes with dense image. We want to show that $f'(|U_i|)$ is already $V_i$.

Consider some $\mathfrak{p} \in V_i$. By Zorn's Lemma, there is a minimal prime ideal $\mathfrak{p}' \subseteq \mathfrak{p}$ in the ring $B$. Note that $\mathfrak{p}$ is a specialization of $\mathfrak{p}'$. Hence, it suffices to show that $\mathfrak{p}' \in f'(|U_i|)$. Note that $B_{\mathfrak{p}'}$ must be a field, otherwise we could take a proper nonzero prime ideal of $B_{\mathfrak{p}'}$ and its preimage under $B \to B_{\mathfrak{p}'}$ would be a smaller prime ideal in $B$.

Since $f' : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ has dense image, we find that $B \to A$ is injective. Consider $A$ as a $B$-algebra. Let $\mathfrak{q}$ be a prime ideal of $A \otimes_B B_{\mathfrak{p}'}$ (this is not the zero ring, as $B_{\mathfrak{p}'}$ is a flat $B$-module and so the inclusion $B \to B_{\mathfrak{p}'}$ induces an inclusion $B_{\mathfrak{p}'} \to A \otimes_B B_{\mathfrak{p}'}$).

Taking the preimage of $\mathfrak{q}$ under the map $A \to A \otimes_B B_{\mathfrak{p}'}$ now gives a prime ideal $\mathfrak{q}' \leq A$. Since $\mathfrak{q} \cap B_{\mathfrak{p}'} = \{0\}$ (here we use that $B_{\mathfrak{p}'}$ is a field), find $\mathfrak{q}' \cap B = \mathfrak{p}'$, thus $\mathfrak{q}'$ maps to $\mathfrak{p}'$ under $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$. This shows $\mathfrak{p}' \in f'(|U_i|)$ and the claim follows. $\square$

The next lemma is now the decisive statement that will allow us to show that $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is closed. It is also the place where things get somewhat ugly, and we require a good deal of commutative algebra to prove it.

**Lemma 28.** *Let $\mathfrak{p} \leq R[x_0, ..., x_n]$ be a homogeneous prime ideal with $R[x_0, ..., x_n]_+ \not\subseteq \mathfrak{p}$. Further let $\mathfrak{q} \leq R$ be a prime ideal such that $\mathfrak{p} \cap R \subseteq \mathfrak{q}$. Then there some $i$ such that no $\alpha x_i^m$ is in $\mathfrak{b} := \mathfrak{p} + \mathfrak{q}R[x_0, ..., x_n]$, for any $\alpha \in R \setminus \mathfrak{b}$ and any $m \in \mathbb{N}$.*

*Proof.* Assume there is $m \in \mathbb{N}$ such that $\alpha_i x_i^m \in \mathfrak{b}$ for all $i$ and some $\alpha_i \in \mathfrak{b} \setminus R$. Since $\mathfrak{b} \cap R = \mathfrak{q}$ is prime, have that $\alpha := \prod_i \alpha_i \notin \mathfrak{b}$. Further we have that $\alpha x_i^m \in \mathfrak{b}$ for all $i$.

Consider now the vector $(w_i)_{i<N}$ containing all monomials of degree $(n+1)m$. By assumption, all $\alpha w_i \in \mathfrak{b}$. Since $\mathfrak{p}$ is homogeneous, observe that now there are $r_{ij} \in \mathfrak{q}$ such that

$$\alpha w_i - \sum_{j<N} r_{ij} w_j \in \mathfrak{p}$$

[5] As in the Lecture Notes, Section 5.6.

Working modulo $\mathfrak{p}$, we see that $Aw = \alpha w$ where $A \in (\mathfrak{b}/\mathfrak{p})^{N \times N}$.

Now observe that $\alpha \notin \mathfrak{b}/\mathfrak{p}$ since $\alpha \notin \mathfrak{b}$. Hence there exists a prime ideal in $R/\mathfrak{p}$ containing $\mathfrak{b}/\mathfrak{p}$ and not containing $\alpha$. Localizing at that prime ideal gives a local ring $S$ with maximal ideal $\mathfrak{l}$. Now assume that $w \in S^n$ and $A/\alpha \in \mathfrak{l}^{n \times n}$. Note that we still have $(A/\alpha)w = w$.

wlog $S$ is noetherian, otherwise continue with the ring

$$\tilde{S} := (\mathbb{Z}/\mathrm{char}(S)\mathbb{Z})\,[A/\alpha, x]$$

generated by the coefficients of $A/\alpha$ and $w$ (this is noetherian, as it is a quotient of a polynomial ring in finitely many variables).

Now equip $S_{\mathfrak{l}}$ with the $\mathfrak{l}$-adic topology. Note that $S$ is noetherian and local, thus the Krull intersection theorem [Cla15, p. 8.41] shows that the $\mathfrak{l}$-adic topology is Hausdorff and we find

$$w = \lim_{i \to \infty} w = \lim_{i \to \infty} \frac{A^i}{\alpha^i} w = \Big(\lim_{i \to \infty} \frac{A^i}{\alpha^i}\Big) w = 0w = 0$$

since $(A/\alpha)^i$ has coefficients in $\mathfrak{l}^i$, thus converges to 0 as $i \to \infty$.

Thus $w \equiv 0 \mod \mathfrak{p}$ and so $x_i^N \equiv 0 \mod \mathfrak{p}$, i.e. $x_i^N \in \mathfrak{p}$ for all $i$. However, since $\mathfrak{p}$ is prime, this implies $R[x_0, ..., x_n]_+ = \langle x_0, ..., x_n \rangle \subseteq \mathfrak{p}$, contradicting the assumption. □

As we will see in the next proof, this lemma shows that if $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ maps a point $\mathfrak{p}$ to a point $\mathfrak{q}$, then it also maps a specialization of $\mathfrak{p}$ to any specialization of $\mathfrak{q}$. From this, we can derive the next proposition.

**Proposition 29.** *The morphism $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is closed.*

*Proof.* Denote $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ by $\phi$. Consider a closed set $\mathrm{V}_+(()\mathfrak{a}) = \mathbb{P}^n_R \setminus D_{\mathfrak{a}}$ given by a homogeneous ideal $\mathfrak{a} \leq R[x_0, ..., x_n]$. By Lemma 27, it suffices to show that $\phi(\mathrm{V}_+(()\mathfrak{a}))$ is closed under specialization (note that $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is quasi-compact, e.g. by Proposition 20).

Consider $\mathfrak{q} \in \phi(\mathrm{V}_+(()\mathfrak{a})) \subseteq \mathrm{Spec}(R)$ and a prime ideal $\mathfrak{q}'$ that specializes $\mathfrak{q}$, i.e. $\mathfrak{q}' \supseteq \mathfrak{q}$. Then there is a $\mathfrak{p} \in \mathrm{V}_+(()\mathfrak{a})$ with $\mathfrak{p} \cap R = \mathfrak{q}$, in particular $\mathfrak{a} \subseteq \mathfrak{p}$. We want to show that $\mathfrak{q}' \in \phi(\mathrm{V}_+(()\mathfrak{a}))$.

Note that by Lemma 28, there is $i$ such that the multiplicative set

$$T := \{\alpha x_i^m \mid \alpha \in R \setminus \mathfrak{q}',\ m \in \mathbb{N}\}$$

has empty intersection with the ideal $\mathfrak{b} := \mathfrak{p} + \mathfrak{q}' R[x_0, ..., x_n]$.

Hence, the ring $T^{-1}(R[x_0, ..., x_n]/\mathfrak{b})$ is nonzero, thus has a prime. Taking its preimage under

$$R[x_0, ..., x_n] \to R[x_0, ..., x_n]/\mathfrak{b} \to T^{-1}(R[x_0, ..., x_n]/\mathfrak{b})$$

yields a prime ideal $\mathfrak{p}'$ such that $x_i \notin \mathfrak{p}'$ and $\mathfrak{b} \subseteq \mathfrak{p}'$. Clearly have that $\mathfrak{q}' \subseteq \mathfrak{p}'$ and since $\mathfrak{p}' \cap T = \emptyset$, find that $\mathfrak{p}' \cap R = \mathfrak{q}'$. By Lemma 6, assume wlog that $\mathfrak{p}'$ is homogeneous. Now have a homogeneous prime ideal $\mathfrak{p}'$ with $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$, i.e. $\mathfrak{p}' \in \mathrm{V}_+(()\mathfrak{a})$ and $\phi(\mathfrak{p}') = \mathfrak{p}' \cap R = \mathfrak{q}'$. Thus $\mathfrak{q} \in \phi(\mathrm{V}_+(()\mathfrak{a}))$ and the claim follows. □

**Corollary 30.** *The morphism $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is universally closed.*

*Proof.* Use Lemma 23, so consider an affine base change $f_{\operatorname{Spec}(S)} : \mathbb{P}^n_R \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S) \to \operatorname{Spec}(S)$. By Lemma 25, have that $\mathbb{P}^n_R \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S) \cong \mathbb{P}^n_S$ as schemes over $S$, hence $f_{\operatorname{Spec}(S)}$ is isomorphic to $\mathbb{P}^n_S \to \operatorname{Spec}(S)$. This morphism is closed by Proposition 29 and the claim follows. □

**Corollary 31.** *Projective space $\mathbb{P}^n_R$ over $\operatorname{Spec}(R)$ is proper.*

**Corollary 32.** *Let $k$ be an algebraically closed field. Then projective space $\mathbb{P}^n_k$ over $\operatorname{Spec}(k)$ is a complete variety.*

## 4 Valuative criterion of properness

Instead of using some magical ad-hoc argument as before, there is also the following nice characterization of the notion of properness. We will omit the proof here, but present it as it can give more geometric intuition on properness and why $\mathbb{P}^n_R$ is proper.

**Proposition 33.** *Let $X$ be noetherian and $f : X \to Y$ a morphism of finite type. The following are equivalent*

- *$f$ is proper*
- *For every valuation ring $R$ with field of fractions $K = \operatorname{Frac}(R)$ and all morphisms $\operatorname{Spec}(R) \to Y$, $\operatorname{Spec}(K) \to X$ that make the diagram*

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r] & Y \end{tikzcd}$$

  *commute, there exists a unique compatible morphism $\operatorname{Spec}(R) \to X$.*

*Proof.* Omitted, see e.g. [Har77, p. II.4.7] □

**Proposition 34.** *The morphism $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is proper.*

*Alternative Proof.* Since properness is preserved under base change, by Lemma 25 it suffices to show the statement for $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$.

Apply Proposition 33. Let $R$ be a valuation ring, $K = \operatorname{Frac}(R)$ and assume we have morphisms $\operatorname{Spec}(R) \to Y$, $\operatorname{Spec}(K) \to X$ such that the diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r] & Y \end{tikzcd}$$

commutes. Note that $\operatorname{Spec}(K) = \{\langle 0\rangle\}$, so let $\xi_1 \in X$ be the image of $\langle 0\rangle$ under $\operatorname{Spec}(K) \to X$. Since $\langle x_0, ..., x_n\rangle \not\subseteq \xi$, there is $x_i \notin \xi$. Now let

$$f_j := \begin{cases} x_j & \text{if } x_j \notin \xi_1 \\ x_j - x_i & \text{if } x_j \in \xi_1 \end{cases}$$

Thus all $f_j$ are homogeneous of degree 1 with $\langle f_j \mid j\rangle = \langle x_0, ..., x_n\rangle$. It follows that $X = \bigcup_j \mathrm{D}_+(f_j)$ is a cover by affine opens such that $\xi_1 \in \bigcap_j \mathrm{D}_+(f_j)$. Now possibly apply a linear automorphism $f_j \to x_j$ and thus assume wlog that $f_j = x_j$. In particular, find $x_j \in \mathcal{O}^*_{X,\xi_1}$.

The morphism $\operatorname{Spec}(K) \to X$ now gives an inclusion $k(\xi_1) \subseteq K$. Since $x_j \in \mathcal{O}^*_{X,\xi_1}$, observe that $g_{ij} := x_i/x_j \in K$ is well-defined and nonzero with $g_{ij}g_{jk} = g_{ik}$.

As $R$ is a valuation ring, we find $k$ such that $g_{k0}$ is the smallest element (w.r.t. divisibility by $R$) among $g_{00}, ..., g_{n0}$. So we see that $g_{ik} = g_{i0}/g_{k0} \in R$ for all $i$. In particular, we find a ring homomorphism

$$\mathbb{Z}\left[\frac{x_0}{x_k}, ..., \frac{x_n}{x_k}\right] \to R, \quad \frac{x_j}{x_k} \mapsto g_{ik}$$

that is compatible with the field inclusion $k(\xi_1) \subseteq K$.

The induced morphism

$$\operatorname{Spec}(R) \to \operatorname{Spec}(\mathbb{Z}\Big[\frac{x_0}{x_k}, ..., \frac{x_n}{x_k}\Big]) = \mathrm{D}_+(x_j) \subseteq X$$

is now a compatible morphism $\operatorname{Spec}(R) \to X$.

Also it is not too hard to show that the morphism is unique, and the claim follows. $\square$

Note that in some way, this is the same proof as before. The core idea - namely the choice of the smallest element $g_{k0}$ (w.r.t. divisibility by $R$) - more or less achieves the same effect as the more complicated ideal argument from Lemma 28.

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