Miniproject - Introduction to Schemes

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March 20, 2022

1 Definition of Proj

Definition 1. A *graded ring* S is a ring S with a decomposition $S = \oplus_{d \in \mathbb{Z}} S_d$ into groups $S_i \subseteq S$ (w.r.t. addition in S) such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{Z}$. If all $S_d = \{0\}$ for $d < 0$, call S *naturally graded ring.* Write further $S_+ := \sum_{d \neq 0} S_d$. For a homogeneous $f \in S_d$ say that $\deg(f) := d$ is its *degree*.

An element $f \in S$ is called *homogeneous* (of degree n), if $f \in S_n$. An ideal $I \leq S$ is called *homogeneous*, if it has a set of homogeneous generators.

Definition 2. For a naturally graded ring S, define the set

$$\mathrm{Proj}(S) := \{\mathfrak{p} \in \mathrm{Spec}(S) \mid \mathfrak{p} \text{ homogeneous},\ S_+ \not\subseteq \mathfrak{p}\}$$

of homogeneous prime ideals not containing S_+.

This becomes a topological space by endowing it with the *Zariski-topology* on $\mathrm{Proj}(S)$, given by the open sets

$$D_\mathfrak{a} := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a} \not\subseteq \mathfrak{p}\}$$

for any homogeneous ideal $\mathfrak{a} \leq S$.

From now on let S be a naturally graded ring.

Proposition 3. *The above definition is well-defined, i.e. the sets $D_\mathfrak{a}$ indeed form a topology on* $\mathrm{Proj}(S)$.

Proof. Clearly $\mathrm{Proj}(S) = D_{\langle 1 \rangle}$ and $\emptyset = D_{\langle 0 \rangle}$ are open. Furthermore, for open sets $D_\mathfrak{a}$ and $D_\mathfrak{b}$, have that

$$D_\mathfrak{a} \cap D_\mathfrak{b} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}\} = D_{\mathfrak{a}\mathfrak{b}}$$

This holds, as $\mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}$ implies that there are $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ with $f, g \notin \mathfrak{p}$. However, then $fg \notin \mathfrak{p}$ as $\mathfrak{p}$ is prime. Obviously $\mathfrak{a}\mathfrak{b}$ is homogeneous, and so $D_\mathfrak{a} \cap D_\mathfrak{b}$ is open.

Finally, given a collection $\mathcal{A}$ of homogeneous ideals in S, have that

$$\begin{aligned}\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}} =& \{\mathfrak{p}\in\operatorname{Proj}(S) \mid \exists\mathfrak{a}\in\mathcal{A}:\ \mathfrak{a}\not\subseteq\mathfrak{p}\} = \{\mathfrak{p}\in\operatorname{Proj}(S) \mid \exists\mathfrak{a}\in\mathcal{A}\exists f\in\mathfrak{a}:\ f\notin\mathfrak{p}\}\\ =& \Big\{\mathfrak{p}\in\operatorname{Proj}(S) \;\Big|\; \exists f\in\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}:\ f\notin\mathfrak{p}\Big\} = D_{\mathfrak{b}} \quad \text{for } \mathfrak{b}=\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}\end{aligned}$$

Clearly $\mathfrak{b}$ is again homogeneous, and so $\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}}$ is open. □

Proposition 4. *The sets $D_f := D_{\langle f\rangle}$ for homogeneous $f\in S$ form a basis of the topology on* $\operatorname{Proj}(S)$.

Proof. Clearly $\langle f\rangle$ is a homogenous ideal, so D_f is open. For any homogeneous ideal $\mathfrak{a}=\langle f_i \mid i\in I\rangle$ with $f_i\in S$ homogeneous have

$$\mathfrak{a}=\bigcup_{i\in I} D_{f_i}$$

as $\mathfrak{a}\not\subseteq\mathfrak{p}$ implies there is some $g=\sum_{i\in I} g_i f_i\notin\mathfrak{p}$, with $g_i\in S$. Hence, at least one $g_j f_j\notin\mathfrak{p}$ and so $f_j\notin\mathfrak{p}$, thus $\mathfrak{p}\in D_{f_j}$. It follows that the D_f generate the topology on $\operatorname{Proj}(S)$, so it is left to show that they are a basis.

Consider $\mathfrak{p}\in D_f\cap D_g$, so $f,g\notin\mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $fg\notin\mathfrak{p}$ and so $D_{fg}\subseteq D_f\cap D_g$ is an open neighborhood of $\mathfrak{p}$. □

Lemma 5. *Let S be a graded ring (not necessarily naturally graded) and $T\subseteq S$ a multiplicative set consisting of homogeneous elements. Then $T^{-1}S$ becomes a graded ring via*

$$\left(T^{-1}S\right)_d = \left\{\frac{g}{h}\in T^{-1}S \;\middle|\; g \textit{ homogeneous with } \deg(g)-\deg(h)=d\right\}$$

Proof. Clearly $(T^{-1}S)_i(T^{-1}S)_j\subseteq(T^{-1}S)_{i+j}$. To see that $(T^{-1}S)_d$ is a subgroup of S, consider $g/f, l/h\in(T^{-1}S)_d$. Now have

$$\frac{g}{f}+\frac{l}{h}=\frac{gh+lf}{hf}$$

and by assumption, find $\deg(gh)=\deg(g)-\deg(h)=d+\deg(f)-d+\deg(l)=\deg(lf)$. So $gh+lf$ is homogeneous and we have

$$\deg(gh+lf)-\deg(fh)=\deg(f)+\deg(l)-\deg(f)-\deg(h)=\deg(l)-\deg(h)=d$$

Thus $g/f+l/h\in(T^{-1}S)_d$.

Finally, we show that $(T^{-1}S)_n\cap(T^{-1}S)_m=\{0\}$ for $i\neq j$. Assume there is $g/f=l/h\in(T^{-1}S)_n\cap(T^{-1}S)_m$ with $\deg(g)-\deg(f)=n$ and $\deg(l)-\deg(h)=m$. Then there exists $t\in T$ such that

$$\begin{aligned}0=t(gh-lf) =& tgh-tlf \quad \text{with } tgh\in S_{\deg(t)+\deg(g)+\deg(h)}\\ & \text{and } tlf\in S_{\deg(t)+\deg(l)+\deg(f)}=S_{\deg(t)+\deg(g)+\deg(h)+(m-n)}\end{aligned}$$

If $n\neq m$, then $S_{\deg(t)+\deg(g)+\deg(h)+(m-n)}\cap S_{\deg(t)+\deg(g)+\deg(h)}=\{0\}$ and so $tgh=tlf=0$. Thus $th(g-0)=0$ and so $g/f=0/1=0$. Thus $(T^{-1}S)_m\cap(T^{-1}S)_n=\{0\}$. □

Lemma 6. *Let $\mathfrak{p} \leq S$ be a prime ideal. Then*

$$\mathfrak{p}' := \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \leq S$$

is a (homogeneous) prime ideal.

Proof. Consider $f, g \in S$ with $fg \in \mathfrak{p}'$ and assume $f, g \notin \mathfrak{p}'$. Write $f = \sum_d f_d$ and $g = \sum_d g_d$ with $f_d, g_d \in S_d$. So

$$\sum_{i,j} f_i g_j = \sum_n \sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

Since $\mathfrak{p}'$ is homogeneous, it follows that

$$\sum_{i+j=n} f_i g_j \in \mathfrak{p}'$$

for all $n \in \mathbb{Z}$.

Let now d resp. e be maximal such that $f_d \notin \mathfrak{p}'$ resp. $g_e \notin \mathfrak{p}'$. We have

$$f_d g_e + \sum_{\substack{i+j=d+e \\ (i,j)\neq(d,e)}} \underbrace{f_i g_j}_{\in \mathfrak{p}'} = \sum_{i+j=d+e} f_i g_j \in \mathfrak{p}'$$

and so $f_d g_e \in \mathfrak{p}' \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. However both f_d and g_e are homogeneous, so $f_d \in \mathfrak{p}'$ or $g_e \in \mathfrak{p}'$, a contradiction. □

Lemma 7. *If $D_g \subseteq D_f$ then there is a homogeneous $h \in S$ such that $g^n = fh$ for some $n \in \mathbb{N}$.*

Proof. Assume not, then f is not a unit in S_g. Hence, there is a maximal ideal $\mathfrak{m} \leq S_g$ such that $f \in \mathfrak{m}$. Now let $\mathfrak{p}$ be the preimage of $\mathfrak{m}$ under the localization map $S \to S_g$. Since $\mathfrak{m}$ is maximal, we see that $\mathfrak{p}$ is prime.

Now apply Lemma 6 and see that also

$$\mathfrak{p}' = \langle f \in \mathfrak{p} \mid f \text{ homogeneous} \rangle \subseteq \mathfrak{p}$$

is prime. Furthermore, $g \notin \mathfrak{p}$ and so $g \notin \mathfrak{p}'$. wlog we have that $g \in S_+$, so $S_+ \not\subseteq \mathfrak{p}'$. Since now $\mathfrak{p}'$ is a homogeneous prime ideal, it follows that $\mathfrak{p}' \in \operatorname{Proj}(S)$.

Finally, observe that $f \in \mathfrak{p}'$ as $f \in \mathfrak{p}$ and f is homogeneous. Hence, we have that $\mathfrak{p}' \notin D_f$ and $\mathfrak{p}' \in D_g$, which contradicts the assumption that $D_g \subseteq D_f$. □

The next proof works exactly as the corresponding one for Spec in the lecture.

Proposition 8. *Let $B = \{D_f \mid f \in S \text{ homogeneous}\}$. The functor*

$$\mathcal{F} : \operatorname{Top}(\operatorname{Proj}(S))\big|_B \to \textbf{\textit{Ring}}, \quad D_f \mapsto (S_f)_0$$
$$(D_{fg} \subseteq D_f) \mapsto \left(\cdot\big|_{D_{fg}} : \frac{s}{f^n} \mapsto \frac{sg^n}{(fg)^n} \right)$$

is a B-sheaf on B (here $\operatorname{Top}(X)$ is the category given by the open sets of X and their inclusion, as defined in the lecture).

Proof. Clearly, $\mathcal{F}$ is a functor and thus a presheaf. Hence, we have to show the local-to-global property.

Let $D_f = \bigcup_{i\in I} D_{g_i f}$ be a cover and $s_i \in \mathcal{F}(D_{g_i f})$ such that

$$\forall x \in D_{g_i f} \cap D_{g_j f}\ \exists V \in B :\ V \subseteq D_{g_i f} \cap D_{g_j f},\ x \in V,\ \frac{s_i}{1} = \frac{s_j}{1} \in \mathcal{F}(V)$$

To show uniqueness, it suffices to show that if

$$\alpha\big|_{D_{h_i}} = \beta\big|_{D_{h_i}} \quad \text{for any } h_i \text{ with } \operatorname{Proj}(S) = \bigcup_i D_{h_i}$$

then $\alpha = \beta$.

By assumption, have $h_i^{n_i}(\alpha - \beta) = 0$ for each i, and wlog there are only finitely many i. Thus find $N = \max_i n_i \in \mathbb{N}$ and get that $h_i^N(\alpha - \beta) = 0$. Since $\bigcup_i D_{h_i} = \operatorname{Proj}(S)$ it follows that $\bigcup_i D_{h_i^N} = \operatorname{Proj}(S)$ and so $1 \in \langle h_i^N \mid i\rangle$. It follows that

$$\alpha - \beta = 1(\alpha - \beta) \in \langle h_i^N \mid i\rangle(\alpha - \beta) = \langle h_i^N(\alpha - \beta) \mid i\rangle = \{0\}$$

and so $\alpha = \beta$.

Now we show existence. By the uniqueness above, it follows that

$$s_i\big|_{D_{fg_ig_j}} = s_i\big|_{D_{fg_i}\cap D_{fg_j}} = s_j\big|_{D_{fg_i}\cap D_{fg_j}} = s_j\big|_{D_{fg_ig_j}}$$

wlog have again a finite cover, i.e. only finitely many g_i. Hence find an $N \in \mathbb{N}$ such that each $s_i = s_i'/(fg_i)^N$ with $s_i' \in S$ homogeneous. By possibly replacing N with a bigger N, we can now assume that

$$(f^2 g_i g_j)^N \left(s_i'(fg_j)^N - s_j'(fg_i)^N\right) = 0 \quad \text{as } s_i\big|_{D_{fg_ig_j}} = s_j\big|_{D_{fg_ig_j}}$$

Now note that

$$s_i = \frac{a_i}{b_i} \quad \text{with } a_i = s_i'(fg_i)^N,\ b_i = (fg_i)^{2N}$$

and

$$a_i b_j - a_j b_i = s_i'(fg_i)^N (fg_j)^{2N} - s_j'(fg_j)^N (fg_i)^{2N} = \underbrace{(f^2 g_i g_j)^N \left(s_i'(fg_j)^N - s_j'(fg_i)^N\right)}_{=0}$$

Now observe that $D_{b_i} = D_{fg_i}$ and so $f^m \in \langle b_i \mid i\rangle$ for some m. Let $1 = \sum_i r_i b_i$ and get

$$a_i f^m = \sum_l r_l b_l a_i = \sum_l r_l a_l b_i = b_i \sum_l r_l a_l$$

Note that a_i, b_i, f are homogeneous, and so we can also choose r_i to be homogeneous. Then find that $0 = \deg(s_i) = \deg(a_i) - \deg(b_i) = \deg(\sum r_l a_l) - m\deg(f)$.

Thus

$$s_i = s := \frac{\sum_l r_l a_l}{f^m} \in S_f$$

and since $\deg(\sum r_l a_l) = m\deg(f)$ we find that $s \in (S_f)_0 = \mathcal{F}(D_f)$. Clearly $s\big|_{D_{fg_i}} = s_i$ and the claim follows. $\square$

Corollary 9. *Hence we can (uniquely) extend the B-sheaf $\mathcal{F}$ to a sheaf $\mathcal{O}_{\operatorname{Proj}(S)}$ on* $\operatorname{Proj}(S)$.

The next to lemmas are based on [Har77, p. II.2.5] and show that $(\operatorname{Proj}(S), \mathcal{O}_{\operatorname{Proj}(S)})$ is a scheme.

Lemma 10. *For $\mathfrak{p} \in \operatorname{Proj}(S)$, the stalk*

$$\mathcal{O}_{\operatorname{Proj}(S),\mathfrak{p}} = (T^{-1}S)_0$$

is a local ring, where $T = \{f \notin \mathfrak{p} \mid f \text{ homogeneous}\}$ contains all homogeneous elements not in $\mathfrak{p}$.

Proof. Consider the ideal

$$\mathfrak{m} = \left\{ \frac{f}{g} \in (T^{-1}S)_0 \;\middle|\; f \in \mathfrak{p} \right\}$$

We claim this is the unique maximal ideal of $(T^{-1}S)_0$.

First, note that $1 \notin (T^{-1}S)_0$ as otherwise, there would be $f/g \in T^{-1}S, f \in \mathfrak{p}$ with $t(f-g) = 0$ for some $t \in T$. However then $tg = tf \in \mathfrak{p}$, and so $g \in \mathfrak{p}$ (as $t \notin \mathfrak{p}$), contradicting $g \in T$.

Now assume there is any ideal $\mathfrak{a}$ such that $\mathfrak{a} \setminus \mathfrak{m} \neq \emptyset$, i.e. there is $f/g \in \mathfrak{a} \setminus \mathfrak{m}$. Then $f \in T$ as f homogeneous and $f \notin \mathfrak{p}$. Thus $g/f \in (T^{-1}S)_0$ and so $f/g \in (T^{-1}S)_0^*$, which implies $\mathfrak{a} = \langle 1 \rangle$. $\square$

Lemma 11. *For $f \in S_+$ homogeneous have that $(D_f, \mathcal{O}_{\operatorname{Proj}(S)}\big|_{D_f})$ is an affine scheme.*

Proof. Let $R = (S_f)_0$. Consider the map

$$f : D_f \to \operatorname{Spec}(R), \quad \mathfrak{p} \mapsto \mathfrak{p}S_f \cap (S_f)_0$$

Note that it is continuous, as the preimage of some basic open set D_{g/f^n} is $D_{fg} \subseteq D_f$ open. Furthermore, f has the inverse

$$f^{-1} : \operatorname{Spec}(R) \to D_f, \quad \mathfrak{p} \mapsto \mathfrak{p}S_f$$

which is also continuous, as the preimage of some basic open set D_{fg} is D_{g^n/f^m} where $n \deg(g) = m \deg(f)$. Thus f is a homeomorphism.

Now consider the natural transformation

$$\eta : \mathcal{O}_{\operatorname{Spec} R} \Rightarrow f_*\Big(\mathcal{O}_{\operatorname{Proj}(S)}\Big|_{D_f}\Big)$$

given on basic open sets by

$$\eta_{D_{g/f^n}} : R_{g/f^n} \to (S_{fg})_0, \quad \frac{h/f^m}{g/f^n} \mapsto \frac{hf^n}{gf^m}$$

Clearly this is a ring isomorphism, so η is a natural isomorphism. The claim follows. $\square$

Corollary 12. $(\operatorname{Proj}(S), \mathcal{O}_{\operatorname{Proj}(S)})$ *is a scheme.*

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer, 1977.