

Miniproject - Elliptic Curves

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1 Question 1

Example 1 (1(i)). Have

$$|162^2 + 6|_5 = |26250|_5 = |5^4 \cdot 7 \cdot 2 \cdot 3|_5 = 5^{-4} < 5^{-3}$$

Example 2 (1(ii)). Let

$$\alpha = 5^{-1} + 2 \cdot 5^0 + 5(1 + 4 \cdot 5) \sum_{n \geq 0} 5^{2n} \in \mathbb{Q}_5$$

Note that in \mathbb{Q}_5 we have

$$\sum_{n \geq 0} 5^{2n} = \sum_{n \geq 0} 25^n = \frac{1}{1 - 25} = -\frac{1}{24}$$

So

$$\alpha = \frac{1}{5} + 2 + 5(21) \frac{1}{24} = \frac{263}{40}$$

For the next exercises, we will slightly abuse notation and write

$$E(R) := \{(x, y) \in E \mid x, y \in R\}$$

for an Elliptic Curve E defined over k and any ring R contained in some extension field of k .

Proposition 3 (1(iii)). *Consider the Elliptic Curve $E : y^2 = x^3 + 2x + 2$ defined over \mathbb{Z} . Then $E(\mathbb{Z}) = \{\mathcal{O}\}$ and*

$$E(\mathbb{Z}_p) \neq \{\mathcal{O}\} \Leftrightarrow p \neq 3$$

Proof. First show that there exists some $(x, y) \in \tilde{E}(\mathbb{F}_p)$ with $y \neq 0$ for all primes $p \neq 3$.

If $p \equiv 1, 5 \pmod{8}$, then -1 is a square in \mathbb{F}_p , thus there is $\alpha \in \mathbb{F}_p$ with $\alpha^2 = -1$ and so $(-1, \alpha) \in \tilde{E}(\mathbb{F}_p)$. If $p \equiv 7 \pmod{8}$, then (by Quadratic Reciprocity) it follows that 2 is a square in \mathbb{F}_p . Thus there is $\alpha \in \mathbb{F}_p$ with $\alpha^2 = 2$ and so $(0, \alpha) \in \tilde{E}(\mathbb{F}_p)$.

Hence, consider now the case $p \equiv 3 \pmod{8}$. Note that

$$\Delta(E) = 4 \cdot 2^3 + 27 \cdot 2^2 = 140 = 2^2 \cdot 5 \cdot 7$$

Hence we see that $p \nmid \Delta(E)$ and so \tilde{E} is an Elliptic Curve defined over \mathbb{F}_p . Now the Hasse bound shows that

$$\#\tilde{E}(\mathbb{F}_p) \in [p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$$

Note that for $p > 9$ have $\sqrt{p} < p/3$ and thus

$$p + 1 - 2\sqrt{p} > 4$$

Thus $\#\tilde{E}(\mathbb{F}_p) \geq 5$ and so there must be $(x, y) \in \tilde{E}(\mathbb{F}_p)$ with $y \neq 0$, as there are at most four points on $\tilde{E}(\mathbb{F}_p)$ that do not satisfy this (\mathcal{O} and possibly $(\alpha_i, 0)$ with α_i a root of $x^3 + 2x + 2$).

Now consider any prime $p \neq 2, 3$ and $(\tilde{x}, \tilde{y}) \in \tilde{E}(\mathbb{F}_p), x, y \in \mathbb{Z}$. Let $f(t) := t^2 - x^3 - 2x - 2$. Then

$$|f(y)|_p \leq p^{-1} \quad \text{and} \quad |f'(y)|_p = |y|_p = 1$$

Thus $|f(y)|_p < |f'(y)|_p^2$ and Hensel's Lemma yields a root $\gamma \in \mathbb{Z}_p$ with $(x, \gamma) \in E(\mathbb{Z}_p)$.

In the case $p = 2$, note that $f(t) := t^2 - 5^3 - 2 \cdot 5 - 2 = t^2 - 137$ satisfies

$$|f(1)|_2 = |-136|_2 = |-17 \cdot 2^3|_2 = 2^{-3} < (2^{-1})^2 = |2|_2^2 = |f'(1)|_2^2$$

and so Hensel's Lemma yields a point $(5, \gamma) \in E(\mathbb{Z}_2)$.

The only remaining case is $p = 3$, and a trying all 9 points in \mathbb{F}_3^2 shows that $\tilde{E}(\mathbb{F}_3) = \{\mathcal{O}\}$. This clearly shows that $E(\mathbb{Z}_3) = \{\mathcal{O}\}$ and so $E(\mathbb{Z}) = \{\mathcal{O}\}$. \square

For the next exercise, we first summarize the techniques we have used above.

Proposition 4. *Let $E : y^2 = x^3 + a_2x^2 + a_4x + a_6$ be an Elliptic Curve defined over \mathbb{Z} . Let p be a prime. Then*

- *If $E(\mathbb{Z}_p) \neq \{\mathcal{O}\}$ then $\tilde{E}(\mathbb{F}_p) \neq \{\mathcal{O}\}$.*
- *Suppose $p \neq 2$. There is $(x, y) \in \tilde{E}(\mathbb{F}_p)$ with $y \neq 0$ if and only if there is $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$.*
- *Suppose $p \neq 2$. If $\#\tilde{E}(\mathbb{F}_p) \geq 5$ then there is $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$.*
- *Suppose $p \geq 11$ and $p \nmid \Delta(E)$. Then there is $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$.*

Proof. The first part is trivial and follows from the fact that any $(x, y) \in E(\mathbb{Z}_p)$ yields $(\tilde{x}, \tilde{y}) \in \tilde{E}(\mathbb{F}_p)$.

For the second part, note that by assumption, there is $(\tilde{x}, \tilde{y}) \in \tilde{E}(\mathbb{F}_p), x, y \in \mathbb{Z}$ with $|y|_p = 1$ and so

$$|y^2 - x^3 - a_2x^2 - a_4x - a_6|_p \leq p^{-1} < 1 = 1^2 = |2y|_p$$

Hensel's Lemma now shows that there is $\gamma \in \mathbb{Z}_p$ such that $\gamma^2 = x^3 + a_2x^2 + a_4x + a_6$ and so $(x, \gamma) \in E(\mathbb{Z}_p)$. Since $|y|_p = 1$ clearly also $|\gamma|_p = 1$. The other direction is obvious and follows directly by taking the reduction modulo p .

For the third part, notice that there are at most three different points $(x, y) \in \tilde{E}(\mathbb{F}_p)$ with $y = 0$, as in this case x is a root of the cubic $t^3 + a_2t^2 + a_4t + a_6$. Thus, if $\#\tilde{E}(\mathbb{F}_p) \geq 5$, there must be $(x, y) \in \tilde{E}(\mathbb{F}_p)$ with $y \neq 0$ and so the claim follows by the second part.

For the fourth part, note that as above, $p > 9$ implies $\sqrt{p} < p/3$ and so the Hasse bound yields (since \tilde{E} is an Elliptic Curve by assumption, as $p \nmid \Delta(E)$)

$$\#\tilde{E}(\mathbb{F}_p) \geq p + 1 - 2\sqrt{p} > 4$$

thus $\#\tilde{E}(\mathbb{F}_p) \geq 5$. The claim now follows by the third part. \square

This already shows that we do not have to worry too much about the condition $E(\mathbb{Z}_p) \neq \{\mathcal{O}\}$ for $p \neq 2, 3, 5, 7$ prime, as we expect that it is fulfilled quite often. This gives the following condition.

Proposition 5. *Let $f_0, f_1, f_2 \in \mathbb{Z}$ and consider the Elliptic Curve $E : y^2 = x^3 + f_2x^2 + f_1x + f_0$. Let $p \in \{3, 5, 7\}$. Then there is no $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$ if and only if*

$$\begin{aligned} p = 3 &\Rightarrow n^3 + f_2n^2 + f_1n + f_0 \equiv 0, 2 \pmod{3} \\ p = 5 &\Rightarrow n^3 + f_2n^2 + f_1n + f_0 \equiv 0, 2, 3 \pmod{5} \\ p = 7 &\Rightarrow n^3 + f_2n^2 + f_1n + f_0 \equiv 0, 3, 5, 6 \pmod{7} \end{aligned}$$

for all $n \in \mathbb{Z}$.

In particular, this is necessary for E to satisfy the desired properties, i.e. there is $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$ if and only if $p \neq 3, 5, 7$ ¹.

Proof. Let $p \in \{3, 5, 7\}$. Assume there is some $(\tilde{x}, \tilde{y}) \in \tilde{E}(\mathbb{F}_p)$, $x, y \in \mathbb{Z}$ with $\tilde{y} \neq 0$. Then have

$$y^2 \equiv x^3 + f_2x^2 + f_1x + f_0 \pmod{p}$$

and so $x^3 + f_2x^2 + f_1x + f_0$ is a quadratic residue modulo p .

By checking all elements in $\mathbb{F}_3, \mathbb{F}_5$ and \mathbb{F}_7 , one finds

$$\begin{aligned} n \text{ quadratic residue modulo } 3 &\Leftrightarrow n \equiv 0, 1 \pmod{3} \\ n \text{ quadratic residue modulo } 5 &\Leftrightarrow n \equiv 0, 1, 4 \pmod{5} \\ n \text{ quadratic residue modulo } 7 &\Leftrightarrow n \equiv 0, 1, 4, 2 \pmod{7} \end{aligned}$$

Except for $n \equiv 0 \pmod{p}$, these cases have been excluded by assumption. However we assumed that $y \not\equiv 0 \pmod{p}$, so $y^2 \not\equiv 0 \pmod{p}$ and the claim follows.

The other direction follows by reversing the above computation. The claim now follows from Proposition 4. \square

¹I understood the task here to be “if and only if”. As mentioned later, this interpretation is probably wrong, and I will discuss the other case next.

However, there is one problem here. Using a computer, one can easily find (e.g. by trying all possibilities) that there are no $f_0, f_1, f_2 \in \mathbb{Z}$ such that the above conditions are (simultaneously) fulfilled for 3, 5 and 7. This seems to indicate that I have indeed misunderstood the task, and we only look for Elliptic Curves $E : y^2 = x^3 + f_2x^2 + f_1x + f_0$ such that there is $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$ for every $p \neq 3, 5, 7$, and do not require further properties for $E(\mathbb{Z}_3), E(\mathbb{Z}_5)$ and $E(\mathbb{Z}_7)$.

So instead consider a strengthening of the last part of Proposition 4.

Proposition 6. *Let $p \geq 11$ be a prime and $E : y^2 = x^3 + f_2x^2 + f_1x + f_0$ an Elliptic Curve with $f_0, f_1, f_2 \in \mathbb{Z}$. Then there is $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$.*

Proof. If $p \geq 11$ and $p \nmid \Delta(E)$ then \tilde{E} is an Elliptic Curve over \mathbb{F}_p and the claim follows from Proposition 4.

So assume now that $p \mid \Delta(E)$, hence $x^3 + f_2x^2 + f_1x + f_0$ factors as

$$x^3 + \tilde{f}_2x^2 + \tilde{f}_1x + \tilde{f}_0 \equiv (x - \alpha)^2(x - \beta)$$

with $\alpha, \beta \in \bar{\mathbb{F}}_p$. However, note that \mathbb{F}_p is perfect, so $(x - \alpha)^2(x - \beta)$ cannot be irreducible over \mathbb{F}_p , otherwise $\mathbb{F}_p[x]/\langle (x - \alpha)^2(x - \beta) \rangle$ would be a non-separable field extension of \mathbb{F}_p . Thus, either $\alpha \in \mathbb{F}_p$ or $\beta \in \mathbb{F}_p$. If $\alpha \in \mathbb{F}_p$, then clearly also $\beta = -2\alpha - \tilde{f}_2 \in \mathbb{F}_p$. If $\beta \in \mathbb{F}_p$, then also $(x - \alpha)^2 \in \mathbb{F}_p[x]$ and again by perfectness of \mathbb{F}_p , note that $\alpha \in \mathbb{F}_p$. So $\alpha, \beta \in \mathbb{F}_p$.

Now note that for $t \in \mathbb{F}_p$ have

$$(t^2 + \beta, t(t^2 + \beta - \alpha)) \in \tilde{E}$$

Hence, we find a function

$$\phi : \mathbb{F}_p \rightarrow \tilde{E}(\mathbb{F}_p) \setminus \{\mathcal{O}\}, \quad t \mapsto (t^2 + \beta, t(t^2 + \beta - \alpha))$$

If there is $\gamma \in \mathbb{F}_p$ with $\gamma^2 = \alpha - \beta$, then

$$\phi|_{\mathbb{F}_p \setminus \{-\gamma\}} : \mathbb{F}_p \setminus \{-\gamma\} \rightarrow \tilde{E}(\mathbb{F}_p)$$

is injective, otherwise ϕ is injective. Hence, we see that $\#(\tilde{E}(\mathbb{F}_p) \setminus \{\mathcal{O}\}) \geq \#\mathbb{F}_p - 1 \geq 4$ and so $\#\tilde{E}(\mathbb{F}_p) \geq 5$. It follows that there is $(\tilde{x}, \tilde{y}) \in \tilde{E}(\mathbb{F}_p)$ with $\tilde{y} \neq 0$. By a Hensel-lifting argument as in Proposition 4, we now see that there is $\gamma \in \mathbb{Z}_p$ with $(x, \gamma) \in E(\mathbb{Z}_p)$ and $|\gamma|_p = 1$. \square

The above proposition shows that constructing Elliptic Curves $E : y^2 = x^3 + f_2x^2 + f_1x + f_0$ such that there is $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$ for all primes $p \neq 3, 5, 7$ is indeed quite simple, as almost all curves satisfy this. This only case that can fail is $p = 2$, but here, the condition is fulfilled quite often, so we can just try different choices.

Example 7. Let

$$\begin{aligned} E_1 : y^2 &= x^3 + 2x \\ E_2 : y^2 &= x^3 + 2x^2 + 6x + 5 \\ E_3 : y^2 &= x^3 + 6x + 1 \end{aligned}$$

Note that

$$\begin{aligned} 1^2 &\equiv 3^3 + 2 \cdot 3 = 33 \pmod{8} \\ 1^2 &\equiv 2^3 + 2 \cdot 2^2 + 6 \cdot 2 + 5 = 33 \pmod{8} \\ 1^2 &\equiv 4^3 + 6 \cdot 4 + 1 = 89 \pmod{8} \end{aligned}$$

so Hensel's Lemma yields points $(x, y) \in E_i(\mathbb{Z}_2)$ with $|y|_2 = 1$ for $i \in \{1, 2, 3\}$. By Proposition 6, we have points $(x, y) \in E(\mathbb{Z}_p)$ with $|y|_p = 1$ for all $p \geq 11$.

Finally, note that trying all points shows

$$\begin{aligned} \tilde{E}_1(\mathbb{F}_3) &= \{(0, 0), (1, 0), (2, 0), \mathcal{O}\} \\ \tilde{E}_1(\mathbb{F}_5) &= \{(0, 0), \mathcal{O}\} \\ \tilde{E}_2(\mathbb{F}_7) &= \{(1, 0), (5, 0), (6, 0), \mathcal{O}\} \end{aligned}$$

so there is no point $(x, y) \in E_i(\mathbb{Z}_p)$ with $|y|_p = 1$ for $p \in \{3, 5, 7\}$ and a suitable i .

2 Question 2

Example 8 (2(i)). Consider the Elliptic Curve $E : y^2 = x(x+1)(x+4)$ defined over \mathbb{Q} . Have

$$x(x+1)(x+4) = x^3 + 5x^2 + 4x = \left(x + \frac{5}{3}\right)^3 - \frac{13}{3} \left(x + \frac{5}{3}\right) + \frac{70}{27}$$

So E is isomorphic to $E' : y^2 = x^3 - \frac{13}{3}x + \frac{70}{27}$ via

$$\psi : E \rightarrow E', \quad (x, y) \mapsto \left(x + \frac{5}{3}, y\right)$$

This isomorphism preserves the group structure², and so $E'_{\text{tors}} \cong E_{\text{tors}}$. Have

$$\Delta(E') = 4 \left(-\frac{13}{3}\right)^3 + 27 \left(\frac{70}{27}\right)^2 = -144 = \Delta(E)$$

Consider now any prime $p \neq 2, 3$. Then the isomorphism $\psi : E \rightarrow E'$ induces an isomorphism $\tilde{\psi} : \tilde{E} \rightarrow \tilde{E}'$ between the reductions modulo p . Note that \tilde{E}, \tilde{E}' are still Elliptic Curves.

²It is a general result that all isogenies are group homomorphisms, but in this case, it is also directly obvious as the isomorphism is linear, hence maps lines to lines.

Hence, we can use results from the lecture on the reduction modulo $p \neq 2, 3$ for the curve E , even though it is not given by an equation of the form $y^2 = x^3 + Ax + B$, as all these statements are preserved under isomorphism. In particular, we see that the kernel of reduction $E_1(\mathbb{Q}_5)$ has no torsion and so there is an embedding

$$E_{\text{tors}}(\mathbb{Q}) \hookrightarrow \tilde{E}(\mathbb{F}_5)$$

Note that

$$\tilde{E}(\mathbb{F}_5) = \{(0, 0), (1, 0), (2, 1), (2, 4), (3, 2), (3, 3), (4, 0), \mathcal{O}\}$$

has order 8. Clearly

$$(0, 0), (-1, 0), (-4, 0), \mathcal{O} \in E_{\text{tors}}(\mathbb{Q})$$

So the only remaining question is whether this is all the torsion (i.e. $\#E_{\text{tors}}(\mathbb{Q}) = 4$) or there are more points (i.e. $\#E_{\text{tors}}(\mathbb{Q}) = 8$).

Consider now $P = (-2, 2) \in E$. The tangent at P is given by $y = -x$ and the third point of intersection with E is thus $(0, 0)$. Hence $P + P = (0, 0)$ and so $[4]P = \mathcal{O}$. It follows that $\#E_{\text{tors}}(\mathbb{Q}) = 8$ and furthermore that

$$E_{\text{tors}}(E) = \langle P, (-1, 0) \rangle \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$