Miniproject - Introduction to Schemes

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1 Definition of Proj

This section is based on [Har77, p. II.2]

Reminder 1. A *graded ring* S is a ring S with a decomposition $S = \oplus_{d \in \mathbb{N}} S_d$ into groups $S_i \subseteq S$ (w.r.t. addition in S) such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{N}$. Write further $S_+ := \sum_{d>0} S_d$.

An element $f \in S$ is called *homogeneous* (of degree n), if $f \in S_n$. An ideal $I \leq S$ is called *homogeneous*, if it has a set of homogeneous generators.

Definition 2. For a graded ring S, define the set

$$\mathrm{Proj}(S) := \{\mathfrak{p} \in \mathrm{Spec}(S) \mid \mathfrak{p} \text{ homogeneous}, \ S_+ \not\subseteq \mathfrak{p}\}$$

of homogeneous prime ideals not containing S_+.

This becomes a topological space by endowing it with the *Zariski-topology* on $\mathrm{Proj}(S)$, given by the open sets

$$D_{\mathfrak{a}} := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a} \not\subseteq \mathfrak{p}\}$$

for any homogeneous ideal $\mathfrak{a} \leq S$.

From now on let S be a graded ring.

Proposition 3. *The above definition is well-defined, i.e. the sets $D_{\mathfrak{a}}$ indeed form a topology on* $\mathrm{Proj}(S)$.

Proof. Clearly $\mathrm{Proj}(S) = D_{\langle 1 \rangle}$ and $\emptyset = D_{\langle 0 \rangle}$ are open. Furthermore, for open sets $D_{\mathfrak{a}}$ and $D_{\mathfrak{b}}$, have that

$$D_{\mathfrak{a}} \cap D_{\mathfrak{b}} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}\} = D_{\mathfrak{a}\mathfrak{b}}$$

This holds, as $\mathfrak{a}, \mathfrak{b} \not\subseteq \mathfrak{p}$ implies that there are $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ with $f, g \notin \mathfrak{p}$. However, then $fg \notin \mathfrak{p}$ as $\mathfrak{p}$ is prime. Obviously $\mathfrak{a}\mathfrak{b}$ is homogeneous, and so $D_{\mathfrak{a}} \cap D_{\mathfrak{b}}$ is open.

Finally, given a collection $\mathcal{A}$ of homogeneous ideals in S, have that

$$\begin{aligned}\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}} =& \{\mathfrak{p}\in\operatorname{Proj}(S) \mid \exists\mathfrak{a}\in\mathcal{A}:\ \mathfrak{a}\not\subseteq\mathfrak{p}\} = \{\mathfrak{p}\in\operatorname{Proj}(S) \mid \exists\mathfrak{a}\in\mathcal{A}\exists f\in\mathfrak{a}:\ f\notin\mathfrak{p}\} \\ =& \Big\{\mathfrak{p}\in\operatorname{Proj}(S) \ \Big|\ \exists f\in\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}:\ f\notin\mathfrak{p}\Big\} = D_{\mathfrak{b}} \quad \text{for } \mathfrak{b}=\sum_{\mathfrak{a}\in\mathcal{A}}\mathfrak{a}\end{aligned}$$

Clearly $\mathfrak{b}$ is again homogeneous, and so $\bigcup_{\mathfrak{a}\in\mathcal{A}} D_{\mathfrak{a}}$ is open. □

Proposition 4. *The sets $D_f := D_{\langle f\rangle}$ for homogeneous $f\in S$ form a basis of the topology on* $\operatorname{Proj}(S)$.

Proof. Clearly $\langle f\rangle$ is a homogenous ideal, so D_f is open. For any homogeneous ideal $\mathfrak{a}=\langle f_i \mid i\in I\rangle$ with $f_i\in S$ homogeneous have

$$\mathfrak{a}=\bigcup_{i\in I} D_{f_i}$$

as $\mathfrak{a}\not\subseteq\mathfrak{p}$ implies there is some $g=\sum_{i\in I} g_i f_i\notin\mathfrak{p}$, with $g_i\in S$. Hence, at least one $g_j f_j\notin\mathfrak{p}$ and so $f_j\notin\mathfrak{p}$, thus $\mathfrak{p}\in D_{f_j}$. It follows that the D_f generate the topology on $\operatorname{Proj}(S)$, so it is left to show that they are a basis.

Consider $\mathfrak{p}\in D_f\cap D_g$, so $f,g\notin\mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $fg\notin\mathfrak{p}$ and so $D_{fg}\subseteq D_f\cap D_g$ is an open neighborhood of $\mathfrak{p}$. □

Lemma 5. *If $D_g\subseteq D_f$ then there is a homogeneous $h\in S$ such that $g^n=fh$ for some $n\in\mathbb{N}$.*

Proof. Assume not, then f is not a unit in S_g. Thus $\langle f\rangle\subseteq\mathfrak{m}$ for a maximal ideal $\mathfrak{m}\le S_g$. Now let $\mathfrak{p}$ be the preimage of $\mathfrak{m}$ under the localization map $S\to S_g$, so $g\in\mathfrak{p}$ and $f\notin\mathfrak{p}$. □

Proposition 6. *Let $B=\{D_f \mid f\in S \text{ homogeneous}\}$. The functor*

$$\begin{aligned}\mathcal{F}:\operatorname{Top}(\operatorname{Proj}(S))\big|_B \to \textbf{\textit{Ring}},\quad & D_f\mapsto S_f \\ & (D_{fg}\subseteq D_f)\mapsto\Big(\frac{s}{f^n}\mapsto\frac{sg^n}{(fg)^n}\Big)\end{aligned}$$

is a B-sheaf on B (here $\operatorname{Top}(X)$ *is the category given by the open sets of X and their inclusion, as defined in the lecture).*

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer, 1977.