Miniproject - Algebraic Geometry

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1 Part I

Definition 1. *Let V be a vector space. Then define the d-th exterior power as*

$$\bigwedge^d(V) := V^{\otimes d} \ / \ \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \operatorname{span}\{v \otimes v' + v' \otimes v \ \mid \ v, v' \in V\} \otimes V^{\otimes(d-i-1)}$$

Use the notation $v_1 \wedge ... \wedge v_d := [v_1 \otimes ... \otimes v_d] \in \bigwedge^k(V)$.

Lemma 2. *Let $v_1, ..., v_d \in V$. Have for $\pi \in S_d$ that*

$$v_{\pi(1)} \wedge ... \wedge v_{\pi(k)} = \operatorname{sgn}(\pi)(v_1 \wedge ... \wedge v_d)$$

Furthermore if $v_i = v_j$ for some $i \neq j$, then

$$v_1 \wedge ... \wedge v_d = 0$$

Proof. Note that

$$u \otimes v \otimes v' \otimes w = -(u \otimes v' \otimes v \otimes w)$$

for all $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{(d-i-1)}(V), v, v' \in V$.

Every $\pi \in S_d$ has a decomposition $\pi = \xi_1 ... \xi_n$ into transpositions ξ_i. Applying this inductively, we find that

$$v_1 \wedge ... \wedge v_d = \operatorname{sgn}(\xi_i ... \xi_n)(v_{(\xi_i ... \xi_n)(1)} \wedge ... v_{(\xi_i ... \xi_n)(k)})$$

and so

$$v_1 \wedge ... \wedge v_d = \operatorname{sgn}(\pi)(v_{\pi(1)} \wedge ... \wedge v_{\pi(k)})$$

Furthermore, we find that

$$u \otimes v \otimes v \otimes w = -(u \otimes v \otimes v \otimes w) = 0$$

must be zero. Hence, if $v_1, ..., v_d \in V$ with $v_i = v_j$ for some $i \neq j$, then there is a permutation $\pi \in S_d$ with $\pi(1) = i, \pi(2) = j$ and

$$v_1 \wedge ... \wedge v_d = (\operatorname{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge ... \wedge v_{\pi(k)}) = \operatorname{sgn}(\pi)0 = 0$$

□

Lemma 3 (1a)**.** *Let* $\dim(V) \leq 3$. *Then every element of* $\bigwedge^k(V)$ *is decomposable.*

Proof. Now let v_1, v_2, v_3 be a set of generators of V. Consider $u_1 = \sum \lambda_i v_i, u_2 = \sum_i \mu_i v_i, u_3 = \sum_i \rho_i v_i$. Then by applying Lemma 2, we see that

$$\begin{aligned} u_1 \wedge u_2 =& \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{=0 \text{ if } i=j} = \sum_{i<j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i>j} \lambda_i \mu_j (v_i \wedge v_j) \\ =& \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)(v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3) \\ =& \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} u_1 \wedge u_2 \wedge u_3 =& \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{=0 \text{ unless } i,j,l \text{ pairwise distinct}} \\ =& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)}) \\ =& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi)(v_1 \wedge v_2 \wedge v_3) \\ =& (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi) \end{aligned}$$

are decomposable. Further, it is easy to see from Lemma 2 that $\bigwedge^k(V) = \{0\}$ for $k \geq 4$, which is trivially decomposable. □

Example 4 (1b)**.** *Consider* $V = k^4$. *Then the element* $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$ *is not decomposable.*

Proof. Assume it was, then there are $a, b \in k^4$ such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i<j} (a_i b_j - a_j b_i)(e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1, \;\; a_3 b_4 - a_4 b_3 = 1, \;\; a_i b_j - a_j b_i = 0 \text{ for all } (i,j) \neq (1,2), (3,4)$$

Clearly $a_1 b_2 \neq 0$ or $a_2 b_1 \neq 0$. Similarly, have $a_3 b_4 \neq 0$ or $a_4 b_3 \neq 0$. As all expressions are symmetric w.r.t swapping a_1, b_2 with a_2, b_1 and a_3, b_4 with a_4, b_3, we may assume wlog that $a_1 b_2, a_3 b_4 \neq 0$.

Have $a_1 b_4 = a_4 b_1$ and $a_2 b_4 = a_4 b_2$. We know that $a_1 b_4 \neq 0$ and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \;\; \Rightarrow \;\; a_2 b_1 = a_1 b_2$$

This contradicts $a_1 b_2 - a_2 b_1 = 1$. □

Lemma 5. *Let $A = (a_{ij}) \in \mathrm{GL}_d(k)$ and $v_1, ..., v_d \in V$. Then*

$$\Big(\sum_j a_{1j}v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj}v_j\Big) = \det(A)(v_1 \wedge ... \wedge v_d)$$

Proof. By a direct computation using Lemma 2, we find

$$\begin{aligned}
&\Big(\sum_j a_{ij}v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj}v_j\Big) = \sum_{j_1,...,j_d} a_{1j_1}...a_{dj_d}(v_{j_1} \wedge ... \wedge v_{j_d}) \\
=& \sum_{\pi \in S_d} a_{1\pi(1)}...a_{d\pi(d)}(v_{\pi(1)} \wedge ... \wedge v_{\pi(d)}) \\
=& \sum_{\pi \in S_d} a_{1\pi(1)}...a_{d\pi(d)}\mathrm{sgn}(\pi)(v_1 \wedge ... \wedge v_d) \\
=&(v_1 \wedge ... \wedge v_d) \sum_{\pi \in S_d} \mathrm{sgn}(\pi) \prod_{j=1}^{d} a_{j\pi(j)} = \det(A)(v_1 \wedge ... \wedge v_d)
\end{aligned}$$

where the last equality holds due to the Leibniz determinant formula. □

Lemma 6. *For $v_1, ..., v_d \in V$ have*

$$v_1 \wedge ... \wedge v_d = 0 \;\Leftrightarrow\; v_1, ..., v_d \textit{ linearly dependent}$$

Proof. For the direction $\Rightarrow$, assume that $v_1, ..., v_d$ are not independent. Then there is a nonzero vector $a_1 \in k^d$ with $\sum a_{1i}v_i = 0$. Clearly, we can extend a_1 to a basis $a_1, ..., a_d$ of k^d, which gives a matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$.

However by Lemma 5 we now get

$$\begin{aligned}
0 =&0 \wedge \Big(\sum_j a_{2j}v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj}v_j\Big) = \Big(\sum_j a_{1j}v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj}v_j\Big) \\
=& \det(A)(v_1 \wedge ... \wedge v_d)
\end{aligned}$$

and so $v = v_1 \wedge ... \wedge v_d = 0$ as $\det(A) \neq 0$.

Direction $\Leftarrow$ TODO □

Lemma 7. *Let $v \in V$ and $u \in \bigwedge^{d-1} U$ for a linear subspace $U \leq V$. If $u \neq 0$ then $v \in U$.*

Proof. TODO □

Lemma 8 (1c)**.** *Let d be even. An element $\omega \in \bigwedge^d V$ is decomposable if and only if $\omega \wedge \omega \in \bigwedge^{2d} V$ is zero.*

Proof. The direction $\Rightarrow$ even holds generally. Assume $\omega = v_1 \wedge ... \wedge v_d$. Then

$$\omega \wedge \omega = v_1 \wedge ... \wedge v_d \wedge v_1 \wedge ... \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let $\omega = v_1 + ... + v_t$ for linearly independent decomposable vectors $v_i \in \bigwedge^2 V$. Then

$$\begin{aligned} 0 =& \omega \wedge \omega = \sum_{i,j} v_i \wedge v_j = \sum_{i<j} (v_i \wedge v_j) + (v_j \wedge v_i) \\ =& \sum_{i<j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \Big(\sum_{j>i} v_j \Big) \end{aligned}$$

Here we used that the permutation $\big(1\ 2d\big)\big(2\ (2d-1)\big)...\big(d\ (d+1)\big) \in S_{2d}$ has always sign 1 (since d is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \Big(\bigwedge^2 \mathrm{span}\{v_2, ..., v_t\} \Big) \setminus \{0\}$$

find

$$u_1, u_2 \in \mathrm{span}\{v_2, ..., v_t\}$$

In particular, we know that

$$v_1 \wedge \Big(\sum_{j>i} v_j \Big) \in \bigwedge^2 \mathrm{span}\{v_2, ..., v_t\}$$

and so $v_1 \in \mathrm{span}\{v_2, ..., v_t\}$ unless $\sum_{j>i} v_j = 0$ by Lemma 7. We assumed that the v_i are linearly independent, so the former would give a contradiction. Hence $\sum_{j>i} v_j = 0$ and thus $t = 1$, i.e. $\omega = v_1$ is decomposable. □

2 Part II

In this part, we want to consider the connection of external powers to the Grassmanian. First of all, assume there are two bases $v_1, ..., v_d$ and $u_1, ..., u_d$ of a d-dimensional vector space U. Then there exists a basis change matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$ with

$$u_i = \sum_j a_{ij} v_j$$

So by Lemma 5, it follows that

$$u_1 \wedge ... \wedge u_d = \det(A)(v_1 \wedge ... \wedge v_d)$$

As $v_1, ..., v_d$ resp. $u_1, ..., u_d$ are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge ... \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge ... \wedge u_d \neq 0$$

by Lemma 6. Hence they have well-defined images $[v_1 \wedge ... \wedge v_d]$ resp. $[u_1 \wedge ... \wedge u_d]$ in the projective space $\mathbb{P}(\bigwedge^d V)$. By the above, find

$$[v_1 \wedge ... \wedge v_d] = [u_1 \wedge ... \wedge u_d]$$

This allows us to study the Grassmanian $\mathrm{Gr}(d, V)$ of a fixed vector space V.

Definition 9. *Define the map*

$$\phi : \mathrm{Gr}(d, V) \to \mathbb{P}(\textstyle\bigwedge^d V), \quad \mathrm{span}\{v_1, ..., v_d\} \mapsto [v_1 \wedge ... \wedge v_d]$$

which is well-defined by Lemma 5 as described above.

Lemma 10 (1a)**.** *We have*

$$\mathrm{im}\phi = D := \{[v] \in \mathbb{P}(\textstyle\bigwedge^d V) \mid v \text{ decomposable}\}$$

Proof. First of all, note that the set D is well-defined, as v is decomposable if and only if λv is decomposable, for all $\lambda \in k^*$.

By definition of ϕ, we can directly observe that $\mathrm{im}\phi \subseteq D$. So consider an element $[v] \in D$. As v is decomposable, it follows that $v = v_1 \wedge ... \wedge v_d$ for $v_i \in V$. Not it suffices to show that the v_i are linearly independent, then clearly $\mathrm{span}\{v_1, ..., v_d\}$ is a well-defined d-dimensional vector subspace of V, thus an element of $\mathrm{Gr}(d, V)$.

This follows directly from Lemma 6. □

Definition 11. *Let* $\mathrm{Gr}(d, n) := \mathrm{Gr}(d, k^n)$.

In the lecture, we considered an embedding of $\mathrm{Gr}(d, n)$ into projective space given by minors of the basis matrix. This corresponds to the following definition.

Definition 12. *Define the map*

$$\rho : \mathrm{Gr}(d, n) \to \mathbb{P}\left(k^{\{1,...,n\}^{(d)}}\right) \cong \mathbb{P}^{\binom{n}{d}-1},$$

$$\mathrm{span}\{v_1, ..., v_d\} \mapsto \left[\det \begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d} \end{pmatrix}\right]_{\{i_1,...,i_d\} \in \{1,...,n\}^{(d)}}$$

where $\{1, ..., n\}^{(d)} := \{I \subset \{1, ..., n\} \mid \#I = d\}$ *is the set of all d-element subsets of* $\{1, ..., n\}$.

Lemma 13. *There is a linear isomorphism*

$$f : \textstyle\bigwedge^d k^n \to k^{\{1,...,n\}^{(d)}},$$

$$\sum_j v_1^{(j)} \wedge ... \wedge v_d^{(j)} \mapsto \left(\sum_j \det \begin{pmatrix} v_{1i_1}^{(j)} & \cdots & v_{di_1}^{(j)} \\ \vdots & \ddots & \vdots \\ v_{1i_d}^{(j)} & \cdots & v_{di_d}^{(j)} \end{pmatrix}\right)_{\{i_1,...,i_d\} \in \{1,...,n\}^{(d)}}$$

Proof. For vectors $v_1, ..., v_d$ and $I = \{i_1, ..., i_d\} \in \{1, ..., n\}^{(d)}$ write

$$A_I(v_1, ..., v_d) := \begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d} \end{pmatrix}$$

First of all, we show that f is well-defined. Note that the tensor product can be described as

$$\begin{aligned}V^{\otimes d} := k^{V\times\ldots\times V}/\text{span}\{&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\big(v_i+v_i'\big)\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\&-\big(v_1\otimes\ldots\otimes v_d\big)-\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i'\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\lambda v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\&-\lambda\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\mid i\le d, v_j, v_i'\in V\}\end{aligned}$$

where $v_1\otimes\ldots\otimes v_d := \chi_{(v_1,\ldots,v_d)}$. Hence the external power can be described as

$$\begin{aligned}\textstyle\bigwedge^d V := k^{V\times\ldots\times V}/\text{span}\Big\{&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\big(v_i+v_i'\big)\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\&-\big(v_1\otimes\ldots\otimes v_d\big)-\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i'\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\lambda v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\&-\lambda\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\&v_1\otimes\ldots v_{j-1}\otimes(v_j\otimes v_{j+1}+v_{j+1}\otimes v_j)\otimes v_{j+1}\otimes\ldots\otimes v_d\\&\mid i\le d,\ j\le d-1,\ v_1,\ldots,v_d,v_i'\in V\Big\}\end{aligned}$$

So it suffices to show that for all $I\in\{1,\ldots,n\}^{(d)}$ and vectors $v_1,\ldots,v_d,v_i'\in V$

$$\det(A_I(v_1,\ldots,v_i+v_i',\ldots,v_d)) = \det(A_I(v_1,\ldots,v_d)) + \det(A_I(v_1,\ldots,v_i',\ldots,v_d))$$

and

$$\det(A_I(v_1,\ldots,\lambda v_i,\ldots,v_d)) = \lambda\det(A_I(v_1,\ldots,v_d))$$

and

$$\det(A_I(v_1,\ldots,v_{j+1},v_j,\ldots,v_d)) = -\det(A_I(v_1,\ldots,v_d))$$

However, these properties follow from the well-known properties of the determinant. In particular, det is linear in each column and swapping columns negates the determinant. It follows that f is indeed well-defined.

It is clear by definition that f is linear, so it is left to show that it is bijective. To show surjectivity, note that the $\pm e_I, I\in\{1,\ldots,n\}^{(d)}$ form a basis of $k^{\{1,\ldots,n\}^{(d)}}$. Clearly for $I=\{i_1,\ldots,i_d\}, J\in\{1,\ldots,n\}^{(d)}$ we have that

$$f(e_{i_1}\wedge\ldots\wedge e_{i_d})_J = \det(A_J(e_{i_1},\ldots,e_{i_j})) = \begin{cases}0 & \text{if } J\not\subseteq I\\ \pm1 & \text{if } J\subseteq I\end{cases}$$

so $f(e_{i_1}\wedge\ldots\wedge e_{i_d}) = e_I$ and we deduce that $\text{im}f = \mathbb{P}^{\{1,\ldots,k\}^{(d)}}$.

Finally, note that

$$e_{i_1}\wedge\ldots\wedge e_{i_d}$$

for $i_1 < ... < i_d$ form a basis of $\bigwedge^d k^n$. Clearly, they span $\bigwedge^d k^n$, and the following argument shows that they are linearly independent. Assume

$$\sum_{i_1<...<i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d}) = 0$$

Then

$$0 = e_1 \wedge \Big(\sum_{1<i_2<...<i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge .. \wedge e_{i_d}) \Big) + \sum_{1<i_1<...<i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d})$$

Clearly $e_1 \notin \mathrm{span}\{e_2, ..., e_n\}$ and so by Lemma 7 we see that

$$\sum_{1<i_2<...<i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge ... \wedge e_{i_d}) = 0$$

Repeating this argument inductively shows that $\lambda_{1,2,...,d} = 0$. As k^n is symmetric w.r.t. permuting the e_j, we see that all $\lambda_{i_1,...,i_d} = 0$ are zero.

It follows that $\dim(\bigwedge^d k^n) = \dim(\mathbb{P}^{\{1,...,n\}^{(d)}})$ and we find that f is also injective. $\square$

Corollary 14 (2b)**.** *Let $\bar{f} : \mathbb{P}(\bigwedge^d k^n) \to \mathbb{P}^{\binom{n}{d}-1}$ be the map f from before modulo k^*. Then*

$$\rho = f \circ \phi$$

and in particular, we see that $\phi(Gr(d, n))$ is a projective variety and isomorphic to $\rho(\mathrm{Gr}(d, n))$.

Proposition 15 (2c)**.** *The map ϕ is injective.*

Proof. Consider two subspaces $U = \mathrm{span}\{u_1, ..., u_d\}$ and $W = \mathrm{span}\{w_1, ..., w_d\}$. Assume that $\phi(U) = \phi(W)$. $\square$