Miniproject - Algebraic Geometry

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1 Part I

Definition 1. Let V be a vector space. Then define the d-th exterior power as

$$\bigwedge^d(V) := V^{\otimes d} \ / \ \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \operatorname{span}\{v \otimes v' + v' \otimes v \ \mid \ v, v' \in V\} \otimes V^{\otimes(d-i-1)}$$

Use the notation $v_1 \wedge ... \wedge v_d := [v_1 \otimes ... \otimes v_d] \in \bigwedge^k(V)$.

Lemma 2. Let $v_1, ..., v_d \in V$. Have for $\pi \in S_d$ that

$$v_{\pi(1)} \wedge ... \wedge v_{\pi(k)} = \operatorname{sgn}(\pi)(v_1 \wedge ... \wedge v_d)$$

Furthermore if $v_i = v_j$ for some $i \neq j$, then

$$v_1 \wedge ... \wedge v_d = 0$$

Proof. Note that

$$u \wedge v \wedge v' \wedge w = -(u \wedge v' \wedge v \wedge w)$$

for all $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{(d-i-1)}(V), v, v' \in V$.

Every $\pi \in S_d$ has a decomposition $\pi = \xi_1 ... \xi_n$ into transpositions ξ_i. Applying this inductively, we find that

$$v_1 \wedge ... \wedge v_d = \operatorname{sgn}(\xi_i ... \xi_n)(v_{(\xi_i ... \xi_n)(1)} \wedge ... v_{(\xi_i ... \xi_n)(k)})$$

and so

$$v_1 \wedge ... \wedge v_d = \operatorname{sgn}(\pi)(v_{\pi(1)} \wedge ... \wedge v_{\pi(k)})$$

Furthermore, we find that

$$u \wedge v \wedge v \wedge w = -(u \wedge v \wedge v \wedge w) = 0$$

must be zero. Hence, if $v_1, ..., v_d \in V$ with $v_i = v_j$ for some $i \neq j$, then there is a permutation $\pi \in S_d$ with $\pi(1) = i, \pi(2) = j$ and

$$v_1 \wedge ... \wedge v_d = (\operatorname{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge ... \wedge v_{\pi(k)}) = \operatorname{sgn}(\pi)0 = 0$$

□

Lemma 3 (1a)**.** Let $\dim(V) \leq 3$. Then every element of $\bigwedge^k(V)$ is decomposable.

Proof. Now let v_1, v_2, v_3 be a set of generators of V. Consider $u_1 = \sum \lambda_i v_i, u_2 = \sum_i \mu_i v_i, u_3 = \sum_i \rho_i v_i$. Then by applying Lemma 2, we see that

$$\begin{aligned} u_1 \wedge u_2 =& \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{=0 \text{ if } i=j} = \sum_{i<j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i>j} \lambda_i \mu_j (v_i \wedge v_j) \\ =& \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)(v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3) \\ =& \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} u_1 \wedge u_2 \wedge u_3 =& \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{=0 \text{ unless } i,j,l \text{ pairwise distinct}} \\ =& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)}) \\ =& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi)(v_1 \wedge v_2 \wedge v_3) \\ =& (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi) \end{aligned}$$

are decomposable. Further, it is easy to see from Lemma 2 that $\bigwedge^k(V) = \{0\}$ for $k \geq 4$, which is trivially decomposable. $\square$

Example 4 (1b)**.** Consider $V = k^4$. Then the element $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$ is not decomposable.

Proof. Assume it was, then there are $a, b \in k^4$ such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i<j} (a_i b_j - a_j b_i)(e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1, \;\; a_3 b_4 - a_4 b_3 = 1, \;\; a_i b_j - a_j b_i = 0 \text{ for all } (i,j) \neq (1,2), (3,4)$$

Clearly $a_1 b_2 \neq 0$ or $a_2 b_1 \neq 0$. Similarly, have $a_3 b_4 \neq 0$ or $a_4 b_3 \neq 0$. As all expressions are symmetric w.r.t swapping a_1, b_2 with a_2, b_1 and a_3, b_4 with a_4, b_3, we may assume wlog that $a_1 b_2, a_3 b_4 \neq 0$.

Have $a_1 b_4 = a_4 b_1$ and $a_2 b_4 = a_4 b_2$. We know that $a_1 b_4 \neq 0$ and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \;\; \Rightarrow \;\; a_2 b_1 = a_1 b_2$$

This contradicts $a_1 b_2 - a_2 b_1 = 1$. $\square$

Lemma 5. Let $A = (a_{ij}) \in \mathrm{GL}_d(k)$ and $v_1, ..., v_d \in V$. Then

$$\Big(\sum_j a_{1j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \det(A)(v_1 \wedge ... \wedge v_d)$$

Proof. By a direct computation using Lemma 2, we find

$$\begin{aligned}
&\Big(\sum_j a_{ij} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \sum_{j_1,...,j_d} a_{1j_1} ... a_{dj_d} (v_{j_1} \wedge ... \wedge v_{j_d}) \\
=& \sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)} (v_{\pi(1)} \wedge ... \wedge v_{\pi(d)}) \\
=& \sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)} \mathrm{sgn}(\pi)(v_1 \wedge ... \wedge v_d) \\
=& (v_1 \wedge ... \wedge v_d) \sum_{\pi \in S_d} \mathrm{sgn}(\pi) \prod_{j=1}^{d} a_{j\pi(j)} = \det(A)(v_1 \wedge ... \wedge v_d)
\end{aligned}$$

where the last equality holds due to the Leibniz determinant formula. □

Lemma 6. For $v_1, ..., v_d \in V$ have

$$v_1 \wedge ... \wedge v_d = 0 \;\Leftrightarrow\; v_1, ..., v_d \text{ linearly dependent}$$

Proof. For the direction $\Leftarrow$, assume that $v_1, ..., v_d$ are not independent. Then there is a nonzero vector $a_1 \in k^d$ with $\sum a_{1i} v_i = 0$. Clearly, we can extend a_1 to a basis $a_1, ..., a_d$ of k^d, which gives a matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$.

However by Lemma 5 we now get

$$\begin{aligned}
0 =& 0 \wedge \Big(\sum_j a_{2j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \Big(\sum_j a_{1j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) \\
=& \det(A)(v_1 \wedge ... \wedge v_d)
\end{aligned}$$

and so $v = v_1 \wedge ... \wedge v_d = 0$ as $\det(A) \neq 0$.

Direction $\Rightarrow$ TODO □

Lemma 7. Let $v \in V$ and $u \in \bigwedge^{d-1} U$ for a linear subspace $U \leq V$. If $v \wedge u \in \bigwedge^d U$ then $v \in U$ or $u = 0$.

Proof. TODO □

Lemma 8 (1c)**.** Let d be even. An element $\omega \in \bigwedge^d V$ is decomposable if and only if $\omega \wedge \omega \in \bigwedge^{2d} V$ is zero.

Proof. The direction $\Rightarrow$ even holds generally. Assume $\omega = v_1 \wedge ... \wedge v_d$. Then

$$\omega \wedge \omega = v_1 \wedge ... \wedge v_d \wedge v_1 \wedge ... \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let $\omega = v_1 + ... + v_t$ for linearly independent decomposable vectors $v_i \in \bigwedge^2 V$. Then

$$\begin{aligned} 0 =& \omega \wedge \omega = \sum_{i,j} v_i \wedge v_j = \sum_{i<j} (v_i \wedge v_j) + (v_j \wedge v_i) \\ =& \sum_{i<j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \Big(\sum_{j>i} v_j \Big) \end{aligned}$$

Here we used that the permutation $\big(1\ 2d\big)\big(2\ (2d-1)\big)...\big(d\ (d+1)\big) \in S_{2d}$ has always sign 1 (since d is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \Big(\bigwedge^2 \mathrm{span}\{v_2, ..., v_t\}\Big) \setminus \{0\}$$

find

$$u_1, u_2 \in \mathrm{span}\{v_2, ..., v_t\}$$

In particular, we know that

$$v_1 \wedge \Big(\sum_{j>i} v_j\Big) \in \bigwedge^2 \mathrm{span}\{v_2, ..., v_t\}$$

and so $v_1 \in \mathrm{span}\{v_2, ..., v_t\}$ unless $\sum_{j>i} v_j = 0$ by Lemma 7. We assumed that the v_i are linearly independent, so the former would give a contradiction. Hence $\sum_{j>i} v_j = 0$ and thus $t = 1$, i.e. $\omega = v_1$ is decomposable. $\square$

2 Part II

In this part, we want to consider the connection of external powers to the Grassmanian. First of all, assume there are two bases $v_1, ..., v_d$ and $u_1, ..., u_d$ of a d-dimensional vector space U. Then there exists a basis change matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$ with

$$u_i = \sum_j a_{ij} v_j$$

So by Lemma 5, it follows that

$$u_1 \wedge ... \wedge u_d = \det(A)(v_1 \wedge ... \wedge v_d)$$

As $v_1, ..., v_d$ resp. $u_1, ..., u_d$ are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge ... \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge ... \wedge u_d \neq 0$$

by Lemma 6. Hence they have well-defined images $[v_1 \wedge ... \wedge v_d]$ resp. $[u_1 \wedge ... \wedge u_d]$ in the projective space $\mathbb{P}(\bigwedge^d V)$. By the above, find

$$[v_1 \wedge ... \wedge v_d] = [u_1 \wedge ... \wedge u_d]$$

This allows us to study the Grassmanian $\mathrm{Gr}(d, V)$ of a fixed vector space V.

Definition 9. Define the map

$$\phi : \mathrm{Gr}(d,V) \to \mathbb{P}(\textstyle\bigwedge^d V), \quad \mathrm{span}\{v_1,...,v_d\} \mapsto [v_1 \wedge ... \wedge v_d]$$

which is well-defined by Lemma 5 as described above.

Lemma 10 (1a)**.** We have

$$\mathrm{im}\phi = D := \{[v] \in \mathbb{P}(\textstyle\bigwedge^d V) \mid v \text{ decomposable}\}$$

Proof. First of all, note that the set D is well-defined, as v is decomposable if and only if λv is decomposable, for all $\lambda \in k^*$.

By definition of ϕ, we can directly observe that $\mathrm{im}\phi \subseteq D$. So consider an element $[v] \in D$. As v is decomposable, it follows that $v = v_1 \wedge ... \wedge v_d$ for $v_i \in V$. Not it suffices to show that the v_i are linearly independent, then clearly $\mathrm{span}\{v_1,...,v_d\}$ is a well-defined d-dimensional vector subspace of V, thus an element of $\mathrm{Gr}(d,V)$.

This follows directly from Lemma 6. □

Definition 11. Let $\mathrm{Gr}(d,n) := \mathrm{Gr}(d,k^n)$.

In the lecture, we considered an embedding of $\mathrm{Gr}(d,n)$ into projective space given by minors of the basis matrix. This corresponds to the following definition.

Definition 12. Define the map

$$\rho : \mathrm{Gr}(d,n) \to \mathbb{P}\left(k^{\{1,...,n\}^{(d)}}\right) \cong \mathbb{P}^{\binom{n}{d}-1},$$
$$\mathrm{span}\{v_1,...,v_d\} \mapsto \left[\det\begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d}\end{pmatrix}\right]_{\{i_1,...,i_d\}\in\{1,...,n\}^{(d)}}$$

where $\{1,...,n\}^{(d)} := \{I \subset \{1,...,n\} \mid \#I = d\}$ is the set of all d-element subsets of $\{1,...,n\}$.

Lemma 13. There is a linear isomorphism

$$f : \textstyle\bigwedge^d k^n \to k^{\{1,...,n\}^{(d)}},$$
$$\sum_j v_1^{(j)} \wedge ... \wedge v_d^{(j)} \mapsto \left(\sum_j \det\begin{pmatrix} v_{1i_1}^{(j)} & \cdots & v_{di_1}^{(j)} \\ \vdots & \ddots & \vdots \\ v_{1i_d}^{(j)} & \cdots & v_{di_d}^{(j)}\end{pmatrix}\right)_{\{i_1,...,i_d\}\in\{1,...,n\}^{(d)}}$$

Proof. For vectors $v_1,...,v_d$ and $I = \{i_1,...,i_d\} \in \{1,...,n\}^{(d)}$ write

$$A_I(v_1,...,v_d) := \begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d}\end{pmatrix}$$

First of all, we show that f is well-defined. Note that the tensor product can be described as

$$\begin{aligned}V^{\otimes d} := k^{V\times\ldots\times V}/\mathrm{span}\{&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\big(v_i+v_i'\big)\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\big(v_1\otimes\ldots\otimes v_d\big)-\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i'\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\ &\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\lambda v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\lambda\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\mid i\le d, v_j, v_i'\in V\}\end{aligned}$$

where $v_1\otimes\ldots\otimes v_d := \chi_{(v_1,\ldots,v_d)}$. Hence the external power can be described as

$$\begin{aligned}\textstyle\bigwedge^d V := k^{V\times\ldots\times V}/\mathrm{span}\Big\{&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\big(v_i+v_i'\big)\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\big(v_1\otimes\ldots\otimes v_d\big)-\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i'\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\ &\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\lambda v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\lambda\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\ &v_1\otimes\ldots v_{j-1}\otimes(v_j\otimes v_{j+1}+v_{j+1}\otimes v_j)\otimes v_{j+1}\otimes\ldots\otimes v_d\\ &\quad\mid i\le d,\ j\le d-1,\ v_1,\ldots,v_d,v_i'\in V\Big\}\end{aligned}$$

So it suffices to show that for all $I\in\{1,\ldots,n\}^{(d)}$ and vectors $v_1,\ldots,v_d,v_i'\in V$

$$\det(A_I(v_1,\ldots,v_i+v_i',\ldots,v_d)) = \det(A_I(v_1,\ldots,v_d)) + \det(A_I(v_1,\ldots,v_i',\ldots,v_d))$$

and

$$\det(A_I(v_1,\ldots,\lambda v_i,\ldots,v_d)) = \lambda\det(A_I(v_1,\ldots,v_d))$$

and

$$\det(A_I(v_1,\ldots,v_{j+1},v_j,\ldots,v_d)) = -\det(A_I(v_1,\ldots,v_d))$$

However, these properties follow from the well-known properties of the determinant. In particular, det is linear in each column and swapping columns negates the determinant. It follows that f is indeed well-defined.

It is clear by definition that f is linear, so it is left to show that it is bijective. To show surjectivity, note that the $\pm e_I, I\in\{1,\ldots,n\}^{(d)}$ form a basis of $k^{\{1,\ldots,n\}^{(d)}}$. Clearly for $I=\{i_1,\ldots,i_d\}, J\in\{1,\ldots,n\}^{(d)}$ we have that

$$f(e_{i_1}\wedge\ldots\wedge e_{i_d})_J = \det(A_J(e_{i_1},\ldots,e_{i_j})) = \begin{cases}0 & \text{if } J\not\subseteq I\\ \pm1 & \text{if } J\subseteq I\end{cases}$$

so $f(e_{i_1}\wedge\ldots\wedge e_{i_d}) = e_I$ and we deduce that $\mathrm{im}f = \mathbb{P}^{\{1,\ldots,k\}^{(d)}}$.

Finally, note that

$$e_{i_1}\wedge\ldots\wedge e_{i_d}$$

for $i_1 < ... < i_d$ form a basis of $\bigwedge^d k^n$. Clearly, they span $\bigwedge^d k^n$, and the following argument shows that they are linearly independent. Assume

$$\sum_{i_1<...<i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d}) = 0$$

Then

$$0 = e_1 \wedge \Big(\sum_{1<i_2<...<i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge .. \wedge e_{i_d}) \Big) + \sum_{1<i_1<...<i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d})$$

Clearly $e_1 \notin \operatorname{span}\{e_2, ..., e_n\}$ and so by Lemma 7 we see that

$$\sum_{1<i_2<...<i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge ... \wedge e_{i_d}) = 0$$

Repeating this argument inductively shows that $\lambda_{1,2,...,d} = 0$. As k^n is symmetric w.r.t. permuting the e_j, we see that all $\lambda_{i_1,...,i_d} = 0$ are zero.

It follows that $\dim(\bigwedge^d k^n) = \dim(\mathbb{P}^{\{1,...,n\}^{(d)}})$ and we find that f is also injective. $\square$

Corollary 14 (2b)**.** Let $\bar{f} : \mathbb{P}(\bigwedge^d k^n) \to \mathbb{P}^{\binom{n}{d}-1}$ be the map f from before modulo k^*. Then

$$\rho = \bar{f} \circ \phi$$

and in particular, we see that $\phi(Gr(d, n))$ is a projective variety and isomorphic to $\rho(\mathrm{Gr}(d, n))$.

Proposition 15 (2c)**.** The map ϕ is injective.

Proof. Consider two d-dimensional subspaces U, W of k^n with $\phi(U) = \phi(W)$. Let $u_1, ..., u_l$ be a basis of $U \cap W$ and extend it to bases $u_1, ..., u_d$ of U and $u_1, ..., u_l, w_{l+1}, ..., w_d$ of W. As $\phi(U) = \phi(W)$, we can assume that the u_i, w_i are scaled such that

$$\begin{aligned} 0 =&(u_1 \wedge ... \wedge u_d) - (u_1 \wedge ... \wedge u_l \wedge w_{l+1} \wedge ... \wedge w_d) \\ =&u_1 \wedge ... \wedge u_l \wedge \big((u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d)\big) \end{aligned}$$

By Lemma 7 we see that

$$u_2 \wedge ... \wedge u_l \wedge \big((u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d)\big) = 0$$

as $u_1 \notin \operatorname{span}\{u_2, ..., u_d, w_{l+1}, ..., w_d\}$. Inductively, this argument shows that

$$(u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d) = 0$$

If $l < d$, we can now apply Lemma 7 again to see that

$$u_{l+1} \in \operatorname{span}\{u_{l+2}, ..., u_d, w_{l+1}, ..., w_d\}$$

as $u_{l+2} \wedge ... \wedge u_d \neq 0$ by Lemma 6. However, this contradicts the linear independence of $u_{l+1}, ..., u_d, w_{l+1}, ..., w_d$. Hence it must be $l = d$ and so $U = W$. $\square$

3 Part III

In this part, we want to investigate the geometric properties of the Grassmanian resp. the image of ϕ. First of all, we introduce coordinates on $\mathbb{P}(\bigwedge^d k^n)$.

Definition 16. Note that in the proof of Lemma 13 it was shown that $v_{i_1} \wedge ... \wedge v_{i_d}$ for $i_1 < ... < i_d$ is a basis of $\bigwedge^d k^n$ if $v_1, ..., v_n$ is a basis of V. We introduce the homogeneous coordinates w.r.t. that basis, namely

$$x : \mathbb{P}(\bigwedge^d k^n) \to \mathbb{P}_k^{\{1,...,n\}^{(d)}} \cong \mathbb{P}_k^{\binom{n}{d}-1},$$
$$\Big[\sum_{i_1<...<i_d} \lambda_{i_1,...,i_d}(v_{i_1} \wedge ... \wedge v_{i_d})\Big] \mapsto \big[\lambda_{i_1,...,i_d}\big]_{i_1<...<i_d}$$

The individual coordinates will be denoted by x_I for some $I \in \{1, ..., n\}^{(d)}$ or $x_{i_1,...,i_d}$ for $i_1 < ... < i_d$.

Proposition 17 (3a)**.** For the embedding $\phi : \mathrm{Gr}(2, V) \to \bigwedge^2 V$ we have

$$\mathrm{Gr}(2, V) \cong \mathrm{im}\phi = \mathbb{V}(I)$$

where

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v\rangle \leq k\big[\mathbb{P}(\bigwedge^d V)\big] = k[x_{i,j} \mid i < j]$$

Proof. By Lemma 14 we have that

$$[\omega] \in \mathrm{im}\phi \ \Leftrightarrow\ \omega \text{ decomposable}$$

and so by Lemma 8

$$\omega \in \mathrm{im}\phi \ \Leftrightarrow\ \omega \wedge \omega = 0$$

In $\mathbb{P}(\bigwedge^d V)$ we find that

$$\Big(\sum_{i<j} x_{i,j}(e_i \wedge e_j)\Big) \wedge \Big(\sum_{u<v} x_{u,v}(e_u \wedge e_v)\Big) = \sum_{\substack{i<j\\u<v}} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v)$$
$$= 2\sum_{i<j<u<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v) + 2\sum_{i<u<j<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v)$$
$$+ 2\sum_{u<i<j<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v)$$
$$=2\sum_{i<j<u<v} (x_{i,j}x_{u,v} - x_{i,u}x_{j,v} + x_{j,u}x_{i,v})(e_i \wedge e_j \wedge e_u \wedge e_v)$$

As the $e_i \wedge e_j \wedge e_u \wedge e_v$ are linearly independent, we see that for $[\omega] \in \mathbb{P}(\bigwedge^2 V)$ we have

$$[\omega] \in \mathrm{im}\phi \ \Leftrightarrow\ \forall i < j < u < v : \ (x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v})(\omega) = 0$$

Hence $\mathrm{im}\phi = \mathbb{V}(I)$. ∎

Example 18 (2b)**.** For $n = 4$, Prop. 17 yields that $\mathrm{Gr}(2, 4) \cong \mathrm{im}\phi = \mathbb{V}(I)$ where

$$I = \langle x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4} \rangle \in k[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}]$$

Changing the indices used for the coordinates, we find

$$\mathrm{Gr}(2, 4) = \mathbb{V}(x_0x_5 + x_2x_3 - x_2x_4)$$

which is exactly what we found in the lecture.

Example 19 (2c)**.** For $n = 5$, Prop. 17 yields that $\mathrm{Gr}(2, 5) \cong \mathrm{im}\phi = \mathbb{V}(I)$ where

$$\begin{aligned} I = \langle & x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4}, \quad x_{1,2}x_{3,5} + x_{1,5}x_{2,3} - x_{1,3}x_{2,5}, \\ & x_{1,2}x_{4,5} + x_{1,5}x_{2,4} - x_{1,4}x_{2,5}, \quad x_{1,3}x_{4,5} + x_{1,5}x_{3,4} - x_{1,4}x_{3,5}, \\ & x_{2,3}x_{4,5} + x_{2,5}x_{3,4} - x_{2,4}x_{3,5} \rangle \leq k\big[\mathbb{P}(\textstyle\bigwedge^2 k^5)\big] \end{aligned}$$

Using the following Sage-code, we can compute the number of intersection points of $\mathrm{Gr}(2, 5)$ with 3-dimensional hyperplanes, and find a probable value for its degree.

```
from itertools import combinations
from math import factorial
import numpy as np

# build up the ring and group the variables nicely
R = PolynomialRing(QQ, [
    "x" + str(i) + str(j)
        for i in range(1, 6) for j in range(i + 1, 6)
])
x12, x13, x14, x15, x23, x24, x25, x34, x35, x45 = R.gens()
_x = [[x12, x13, x14, x15], [x23, x24, x25], [x34, x35], [x45]]
x = lambda i, j: _x[i - 1][j - i - 1]

# construct the ideal describing Gr(2, 5)
polys = []
for seq in combinations([1, 2, 3, 4, 5], 4):
    (i, j, u, v) = sorted(seq)
    p = x(i,j) * x(u,v) + x(i,v) * x(j,u) - x(i,u) * x(j,v)
    polys.append(p)

I = R.ideal(polys)
dimension = I.dimension() - 1
assert dimension == 6
assert I.is_prime()

hyperplane_vectors = [
    np.random.randint(-4, 4, R.ngens(), int)
```

```
        for i in range(dimension + 1)
]
for vecs in combinations(hyperplane_vectors, dimension):
    eqs = [
        sum(map(lambda t: t[0] * t[1], zip(vec, R.gens())))
            for vec in vecs
    ]
    J = I + R.ideal(eqs)
    # the number of intersection points is clearly equal to the
    # dimension of S(X)_d for large enough d
    hp = J.hilbert_polynomial()
    degree = hp.leading_coefficient()
    print(degree) # usually prints 5
```

This shows that the degree of $\mathrm{Gr}(2,5)$ is indeed 5, as expected from the degree formula mentioned in the lecture.

$$\deg(\mathrm{Gr}(d,n)) = (d(n-d))!\frac{1!\cdot 2!\cdot ...\cdot (d-1)!}{(n-d)!\cdot (n-d+1)!\cdot ...\cdot (n-1)!}$$

which yields

$$\deg(\mathrm{Gr}(2,5)) = 6!\frac{1!}{3!\cdot 4!} = \frac{6\cdot 5}{3!} = 5$$

To investigate the properties of $\phi(\mathrm{Gr}(2,n))$ for larger n, we use one tool I encountered during an earlier course on Computational Commutative Algebra and Algebraic Geometry.

Proposition 20 (Macaulay Basis Theorem)**.** Let $\preceq$ be a graded monomial ordering on $R = k[x_0,...,x_n]$. Then for an ideal $I \leq R$ have that the monomials $x_0^{\alpha_0}...x_n^{\alpha_n} \notin \mathrm{lt}(I)$ are a k-vector space basis of R/I.

Here $\mathrm{lt}(I)$ is the leading term ideal of I, i.e. the ideal generated by the leading terms of all $f \in I$, w.r.t. $\preceq$.

Proof. See [KR00]. □

To apply this, first of all we have to collect information about the leading term ideal of $\mathbb{I}(\mathrm{Gr}(2,n))$. This is done in the following lemma.

Lemma 21. Define the graded reverse monomial ordering $\preceq$ on $R := k[x_{i,j} \mid i < j]$ where the variables $x_{i,j}$ are ordered co-lexicographically w.r.t. (i,j), i.e.

$$x_{i,j} \leq x_{u,v} \;:\Leftrightarrow\; (i,j) \leq_{\mathrm{colex}} (u,v)$$

Moreover, let

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v\rangle \leq R$$

be the ideal defining $\phi(\mathrm{Gr}(d,V))$ that was considered above. Then

$$\mathrm{lt}(I) = J := \langle x_{i,v}x_{j,u} \mid i < j < u < v\rangle \leq R$$

Proof. Note that for $i < j < u < v$ have $x_{u,v}, x_{j,v} \succ x_{i,v} \succ x_{i,j}, x_{i,u}, x_{j,u}$. Thus the leading term of $x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}$ is

$$\mathrm{lt}(x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}) = x_{i,v}x_{j,u}$$

It follows that $J \subseteq \mathrm{lt}(I)$.

For the other direction, we use a quite lengthy degree argument. Sadly, the only argument I came up with is extremely technical. We try to present it as clearly as possible, at the cost of only sketching some parts. In the end, I checked it using Computer Algebra, and everything fits together.

Consider homogeneous polynomials $f_{i,j,u,v} \in R$ and

$$F = \sum_{i<j<u<v} f_{i,j,u,v}\ (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})$$

We want to show that $\mathrm{lt}(F) \in J$.

Let

$$f_{i,j,u,v} = \sum_{\alpha\in\mathbb{N}^N} c_\alpha^{(i,j,u,v)} x^\alpha$$

Then

$$F = \sum_{\alpha\in\mathbb{N}^N} \sum_{i<j<u<v} c_\alpha^{(i,j,u,v)} x^\alpha\ (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})$$

and so there exists $\alpha \in \mathbb{N}^N$ and $\epsilon \in k^*$ with

$$\mathrm{lt}(F) = \epsilon \cdot \mathrm{lt}\Big(\sum_{i<j<u<v} c_\alpha^{(i,j,u,v)} x^\alpha\ (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})\Big)$$

Hence, we may assume wlog that all the $f_{i,j,u,v}$ are scaled monomials.

Now, observe that all monomials in F are of the form

$$\epsilon\ x_{i,v}\ x_{j,u}\ f_{i,j,u,v} \quad \text{or} \quad \epsilon\ x_{i,u}\ x_{j,v}\ f_{i,j,u,v} \quad \text{or} \quad \epsilon\ x_{i,j}\ x_{u,v}\ f_{i,j,u,v}$$

where $\epsilon \in k^*$ and $i < j < u < v$. In particular, this is true for the leading term $\mathrm{lt}(F)$. In the first of those case, clearly $\mathrm{lt}(F) \in J$.

So consider now the second case, i.e. $\mathrm{lt}(F) = \epsilon\ x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$. Since $x_{i,v}\ x_{j,u}\ f_{i,j,u,v} \succ x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$, we see that the term $x_{i,v}\ x_{j,u}\ f_{i,j,u,v}$ cannot occur in F, i.e. must "cancel out". Hence the monomial $\epsilon\ x_{i,v}\ x_{j,u}\ f_{i,j,u,v}$ has a nonzero coefficient in

$$F - f_{i,j,u,v}x_{i,v}x_{j,u} = \sum_{\substack{a<b<c<d\\(a,b,c,d)\neq(i,j,u,v)}} f_{a,b,c,d}\ (x_{a,d}x_{b,c} - x_{a,c}x_{b,d} + x_{a,b}x_{c,d})$$

and so for $(a,b,c,d) \neq (i,j,u,v), \epsilon' \in k^*$ have that

$$\begin{aligned} f_{i,j,u,v}\ x_{i,v}\ x_{j,u} &= \epsilon' f_{a,b,c,d}\ x_{a,d}\ x_{b,c} \quad \text{or} \\ f_{i,j,u,v}\ x_{i,v}\ x_{j,u} &= \epsilon' f_{a,b,c,d}\ x_{a,c}\ x_{b,d} \quad \text{or} \\ f_{i,j,u,v}\ x_{i,v}\ x_{j,u} &= \epsilon' f_{a,b,c,d}\ x_{a,b}\ x_{c,d} \end{aligned}$$

However, the second and third case imply that $f_{a,b,c,d}\ x_{a,d}\ x_{b,c} \succ f_{i,j,u,v}\ x_{i,v}\ x_{j,u}$. Hence, the "new" monomial $f_{a,b,c,d}\ x_{a,d}\ x_{b,c}$ also has to "cancel out" in the sum representation of F, as comes after $\mathrm{lt}(F) = \epsilon x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$ in the order $\preceq$. So applying the whole argument inductively (induction on the number of monomials $\succ f_{i,j,u,v}\ x_{i,v}\ x_{j,u}$ that occur in any of the polynomials we work with), we end up in the first case [1].

If $(a,d) = (i,v)$ then $(b,c) \neq (j,u)$ and $x_{b,c} \mid f_{i,j,u,v}$. Thus

$$x_{b,c}\ x_{i,u}\ x_{j,v} \mid \mathrm{lt}(F) = \epsilon\ x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$$

with $i < j < u < v$ and $i < b < c < v$. No matter how j, u, b, c are ordered relatively to each other, we see that in each possible case $x_{b,c}\ x_{i,u}\ x_{j,v} \in J$:

$$\begin{aligned}
j < u < b < c &\Rightarrow x_{j,v}x_{b,c} \in J \\
j < b < u < c &\Rightarrow x_{j,v}x_{b,c} \in J \\
j < b < c < u &\Rightarrow x_{i,u}x_{b,c} \in J \\
b < j < u < c &\Rightarrow f_{i,b,c,v}\ x_{i,c}\ x_{b,v} \succ f_{i,j,u,v}\ x_{i,u}\ x_{j,v}, \text{ contradiction} \\
b < j < c < u &\Rightarrow x_{i,u}x_{b,c} \in J \\
b < c < j < u &\Rightarrow x_{i,u}x_{b,c} \in J
\end{aligned}$$

Hence $\mathrm{lt}(F) \in J$.

It is now left to consider the case $(b,c) = (j,u)$ and the case $(a,d) \neq (i,v), (b,c) \neq (j,u)$. The former can be dealt with in exactly the same way, by noting that $x_{a,d} \mid f_{i,j,u,v}$. In the latter case, we even find $x_{a,d}\ x_{b,c} \mid f_{i,j,u,v}$ and a very similar argument works.

Finally, one must also consider the third "big" case, namely that

$$\mathrm{lt}(F) = \epsilon\ x_{i,j}\ x_{u,v}\ f_{i,j,u,v}$$

Again, you can do this similarly as before, but now two monomials "cancel out". We will not present this here as well. □

Definition 22. Consider the graph $G_n = (V_n, E_n)$ where $V_n = \{(i,j) \mid 0 \leq i < j \leq n\}$ and

$$E_n = \{\{(i,v),(j,u)\} \mid 0 \leq i < j < u < v \leq n\}$$

Let further

$$s_n(d) := |\{I \subseteq V_n \mid I \text{ independent set of size } d\}|$$

Proposition 23. Let d be the largest integer such that $s_n(d) \neq 0$, i.e. the size of the largest independent set in G_n. Then

$$\dim(\mathrm{Gr}(2,n)) = d - 1$$

and

$$\deg(\mathrm{Gr}(2,n)) = s_n(d)$$

[1] There is a small argument missing here, namely that we apply the next argument on each step of the induction, to show the claim for $f_{i,j,u,v}\ x_{i,v}\ x_{j,u}$. However, it should be easy to see that this is possible.

Proof. The idea is to apply Macaulay's basis theorem to get information about the Hilbert function of $\mathrm{Gr}(2,n)$. To this end, observe that all the monomials generating $\mathrm{lt}(\mathbb{I}(\mathrm{Gr}(2,n)))$ are square-free. Hence, we can partition all monomials not in $\mathrm{lt}(\mathbb{I}(\mathrm{Gr}(2,n)))$ into sets depending on which variables occur in them. Doing this shows that leading coefficient of the Hilbert polynomial depends only on the groups with the maximal count of monomials. These monomials then form a maximal independent set in G_n, which allows us to relate degree and dimension of $\mathrm{Gr}(2,n)$ to $s_n(d)$.

By Lemma 21 we know

$$J = \mathrm{lt}(\mathbb{I}(\mathrm{Gr}(2,n))) = \langle x_{i,v}x_{j,u} \mid i < j < u < v \rangle \leq R := k[x_{i,j} \mid i < j]$$

Let $N = \binom{n}{2}$ be the number of variables in R, i.e. $\{x_1, ..., x_N\} = \{x_{i,j} \mid i < j\}$. Now we find for sufficiently large m that

$$\begin{aligned}
&|\{x^\alpha \text{ monomial in } R \mid \deg(x^\alpha) = m,\ \forall i < j < u < v : x_{iv}x_{ju} \nmid x^\alpha\}| \\
&= |\{x^\alpha \text{ monomial in } R \mid \deg(x^\alpha) = m,\ \forall i < j < u < v : x_{iv}x_{ju} \nmid \mathrm{sqfr}(x^\alpha)\}| \\
&= \Bigg| \bigcup_{\substack{\alpha\in\{0,1\}^N \\ x_{iv}x_ju \nmid x^\alpha}} \{x^\alpha x^\beta \mid \deg(x^\beta) = m - \deg(x^\alpha),\ \forall i : \alpha_i = 0 \Rightarrow \beta_i = 0\}\Bigg| \\
&= \sum_{\substack{\alpha\in\{0,1\}^N \\ x_{iv}x_ju \nmid x^\alpha}} \left(\!\!\binom{\deg(x^\alpha)}{m - \deg(x^\alpha)}\!\!\right) = \sum_{l=0}^{N} \sum_{\substack{\alpha\in\{0,1\}^N \\ \sum_j \alpha_j = l \\ x_{i,v}x_{j,u} \nmid x^\alpha}} \left(\!\!\binom{l}{m-l}\!\!\right) \\
&= \sum_{l=0}^{N} s_n(l) \left(\!\!\binom{l}{m-l}\!\!\right) = \sum_{l=0}^{N} s_n(l) \binom{m-1}{l-1}
\end{aligned}$$

where $\mathrm{sqrf}(f)$ denotes the square-free part of f. This holds, as by definition of s_n and E_n we find

$$s_n(l) = |\big\{I \subseteq \{x_{i,j}\} \bigm| \forall i < j < u < v : x_{i,v}x_{j,u} \nmid xy \text{ for all } x, y \in I\big\}|$$

Now Macaulay's basis theorem 20 yields that for sufficiently large m have

$$\begin{aligned}
\dim_k(R/\mathbb{I}(\mathrm{Gr}(2,n))) &= |\big\{x^\alpha \text{ monomial in } R \bigm| \deg(x^\alpha) = m,\ x^\alpha \notin \mathrm{lt}(\mathbb{I}(\mathrm{Gr}(2,n)))\big\}| \\
&= \sum_{l=0}^{N} s_n(l) \binom{m-1}{l-1}
\end{aligned}$$

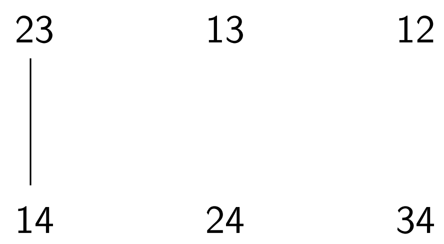
Hence we find for the Hilbert polynomial that

$$p_{\mathrm{Gr}(2,n)} = \sum_{l=0}^{N} s_n(l) \binom{m-1}{l-1}$$

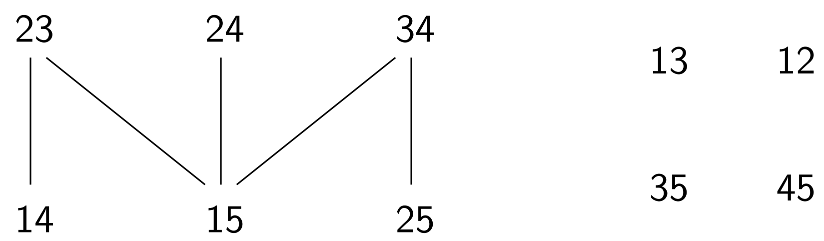
and in particular, it has the leading term $s_n(d)\binom{m-1}{d-1}$. The claim follows by the characterization of degree and dimension using the Hilbert polynomial that we did in the lecture. $\square$

Now we want to study how the independent sets in G_n look like.

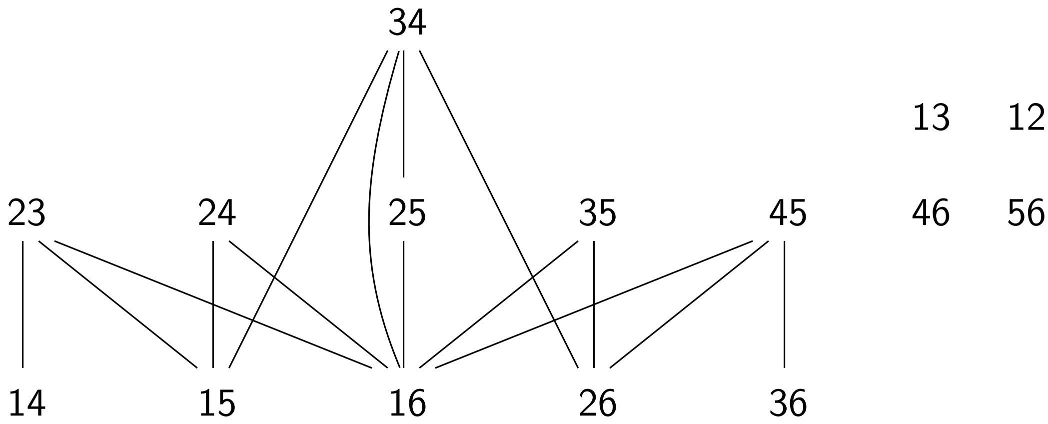
Example 24. The graph G_4 is the following



The graph G_5 is the following



The graph G_6 is the following



Lemma 25. The largest independent set in G_n is of size $2n - 4$.

Proof. The idea is to write the graph in layers (or as a kind of "pyramid") as displayed in Example 24. Then we can "push" an independent set onto the bottom layer, by repeatedly taking vertices in the set that are maximally high up and on the outside, and replacing them by their "child". This shows the claim, as the bottom layer (plus the 4 unconnected vertices) has size $2n - 4$. □

References

[KR00] Martin Kreuzer and Lorenzo Robbiano. *Computational Commutative Algebra*. Springer, 2000.