Miniproject - Algebraic Geometry

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1 Part I

Definition 1. Let V be a vector space. Then define the d-th exterior power as

$$\bigwedge^d(V) := V^{\otimes d} \ / \ \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \mathrm{span}\{v \otimes v' + v' \otimes v \ \mid \ v, v' \in V\} \otimes V^{\otimes(d-i-1)}$$

Use the notation $v_1 \wedge ... \wedge v_d := [v_1 \otimes ... \otimes v_d] \in \bigwedge^k(V)$.

Lemma 2. Let $v_1, ..., v_d \in V$. Have for $\pi \in S_d$ that

$$v_{\pi(1)} \wedge ... \wedge v_{\pi(k)} = \mathrm{sgn}(\pi)(v_1 \wedge ... \wedge v_d)$$

Furthermore if $v_i = v_j$ for some $i \neq j$, then

$$v_1 \wedge ... \wedge v_d = 0$$

Proof. Note that

$$u \wedge v \wedge v' \wedge w = -(u \wedge v' \wedge v \wedge w)$$

for all $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{(d-i-1)}(V), v, v' \in V$.

Every $\pi \in S_d$ has a decomposition $\pi = \xi_1 ... \xi_n$ into transpositions ξ_i. Applying this inductively, we find that

$$v_1 \wedge ... \wedge v_d = \mathrm{sgn}(\xi_i ... \xi_n)(v_{(\xi_i ... \xi_n)(1)} \wedge ... v_{(\xi_i ... \xi_n)(k)})$$

and so

$$v_1 \wedge ... \wedge v_d = \mathrm{sgn}(\pi)(v_{\pi(1)} \wedge ... \wedge v_{\pi(k)})$$

Furthermore, we find that

$$u \wedge v \wedge v \wedge w = -(u \wedge v \wedge v \wedge w) = 0$$

must be zero. Hence, if $v_1, ..., v_d \in V$ with $v_i = v_j$ for some $i \neq j$, then there is a permutation $\pi \in S_d$ with $\pi(1) = i, \pi(2) = j$ and

$$v_1 \wedge ... \wedge v_d = (\mathrm{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge ... \wedge v_{\pi(k)}) = \mathrm{sgn}(\pi)0 = 0$$

∎

Lemma 3 (1a)**.** Let $\dim(V) \leq 3$. Then every element of $\bigwedge^k(V)$ is decomposable.

Proof. Now let v_1, v_2, v_3 be a set of generators of V. Consider $u_1 = \sum \lambda_i v_i, u_2 = \sum_i \mu_i v_i, u_3 = \sum_i \rho_i v_i$. Then by applying Lemma 2, we see that

$$\begin{aligned}
u_1 \wedge u_2 =& \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{= 0 \text{ if } i = j} = \sum_{i<j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i>j} \lambda_i \mu_j (v_i \wedge v_j) \\
=& \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)(v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3) \\
=& \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
u_1 \wedge u_2 \wedge u_3 =& \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{= 0 \text{ unless } i,j,l \text{ pairwise distinct}} \\
=& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)}) \\
=& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi)(v_1 \wedge v_2 \wedge v_3) \\
=& (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi)
\end{aligned}$$

are decomposable. Further, it is easy to see from Lemma 2 that $\bigwedge^k(V) = \{0\}$ for $k \geq 4$, which is trivially decomposable. □

Example 4 (1b)**.** Consider $V = k^4$. Then the element $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$ is not decomposable.

Proof. Assume it was, then there are $a, b \in k^4$ such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i<j} (a_i b_j - a_j b_i)(e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1, \ a_3 b_4 - a_4 b_3 = 1, \ a_i b_j - a_j b_i = 0 \text{ for all } (i,j) \neq (1,2), (3,4)$$

Clearly $a_1 b_2 \neq 0$ or $a_2 b_1 \neq 0$. Similarly, have $a_3 b_4 \neq 0$ or $a_4 b_3 \neq 0$. As all expressions are symmetric w.r.t swapping a_1, b_2 with a_2, b_1 and a_3, b_4 with a_4, b_3, we may assume wlog that $a_1 b_2, a_3 b_4 \neq 0$.

Have $a_1 b_4 = a_4 b_1$ and $a_2 b_4 = a_4 b_2$. We know that $a_1 b_4 \neq 0$ and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \ \Rightarrow \ a_2 b_1 = a_1 b_2$$

This contradicts $a_1 b_2 - a_2 b_1 = 1$. □

Lemma 5. Let $A = (a_{ij}) \in \mathrm{GL}_d(k)$ and $v_1, ..., v_d \in V$. Then

$$\Big(\sum_j a_{1j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \det(A)(v_1 \wedge ... \wedge v_d)$$

Proof. By a direct computation using Lemma 2, we find

$$\begin{aligned}
&\Big(\sum_j a_{ij} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \sum_{j_1,...,j_d} a_{1j_1} ... a_{dj_d} (v_{j_1} \wedge ... \wedge v_{j_d}) \\
= &\sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)} (v_{\pi(1)} \wedge ... \wedge v_{\pi(d)}) \\
= &\sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)} \mathrm{sgn}(\pi)(v_1 \wedge ... \wedge v_d) \\
= &(v_1 \wedge ... \wedge v_d) \sum_{\pi \in S_d} \mathrm{sgn}(\pi) \prod_{j=1}^d a_{j\pi(j)} = \det(A)(v_1 \wedge ... \wedge v_d)
\end{aligned}$$

where the last equality holds due to the Leibniz determinant formula. □

Lemma 6. For $v_1, ..., v_d \in V$ have

$$v_1 \wedge ... \wedge v_d = 0 \;\Leftrightarrow\; v_1, ..., v_d \text{ linearly dependent}$$

Proof. For the direction $\Leftarrow$, assume that $v_1, ..., v_d$ are not independent. Then there is a nonzero vector $a_1 \in k^d$ with $\sum a_{1i} v_i = 0$. Clearly, we can extend a_1 to a basis $a_1, ..., a_d$ of k^d, which gives a matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$.

However by Lemma 5 we now get

$$\begin{aligned}
0 = &0 \wedge \Big(\sum_j a_{2j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \Big(\sum_j a_{1j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) \\
= &\det(A)(v_1 \wedge ... \wedge v_d)
\end{aligned}$$

and so $v = v_1 \wedge ... \wedge v_d = 0$ as $\det(A) \neq 0$.

Direction $\Rightarrow$ TODO □

Lemma 7. Let $v \in V$ and $u \in \bigwedge^{d-1} U$ for a linear subspace $U \leq V$. If $v \wedge u \in \bigwedge^d U$ then $v \in U$ or $u = 0$.

Proof. TODO □

Lemma 8 (1c)**.** Let d be even. An element $\omega \in \bigwedge^d V$ is decomposable if and only if $\omega \wedge \omega \in \bigwedge^{2d} V$ is zero.

Proof. The direction $\Rightarrow$ even holds generally. Assume $\omega = v_1 \wedge ... \wedge v_d$. Then

$$\omega \wedge \omega = v_1 \wedge ... \wedge v_d \wedge v_1 \wedge ... \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let $\omega = v_1 + ... + v_t$ for linearly independent decomposable vectors $v_i \in \bigwedge^2 V$. Then

$$\begin{aligned} 0 =& \omega \wedge \omega = \sum_{i,j} v_i \wedge v_j = \sum_{i<j} (v_i \wedge v_j) + (v_j \wedge v_i) \\ =& \sum_{i<j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \Big(\sum_{j>i} v_j \Big) \end{aligned}$$

Here we used that the permutation $\big(1\ 2d\big)\big(2\ (2d-1)\big)...\big(d\ (d+1)\big) \in S_{2d}$ has always sign 1 (since d is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \Big(\bigwedge^2 \mathrm{span}\{v_2, ..., v_t\} \Big) \setminus \{0\}$$

find

$$u_1, u_2 \in \mathrm{span}\{v_2, ..., v_t\}$$

In particular, we know that

$$v_1 \wedge \Big(\sum_{j>i} v_j \Big) \in \bigwedge^2 \mathrm{span}\{v_2, ..., v_t\}$$

and so $v_1 \in \mathrm{span}\{v_2, ..., v_t\}$ unless $\sum_{j>i} v_j = 0$ by Lemma 7. We assumed that the v_i are linearly independent, so the former would give a contradiction. Hence $\sum_{j>i} v_j = 0$ and thus $t = 1$, i.e. $\omega = v_1$ is decomposable. □

2 Part II

In this part, we want to consider the connection of external powers to the Grassmanian. First of all, assume there are two bases $v_1, ..., v_d$ and $u_1, ..., u_d$ of a d-dimensional vector space U. Then there exists a basis change matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$ with

$$u_i = \sum_j a_{ij} v_j$$

So by Lemma 5, it follows that

$$u_1 \wedge ... \wedge u_d = \det(A)(v_1 \wedge ... \wedge v_d)$$

As $v_1, ..., v_d$ resp. $u_1, ..., u_d$ are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge ... \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge ... \wedge u_d \neq 0$$

by Lemma 6. Hence they have well-defined images $[v_1 \wedge ... \wedge v_d]$ resp. $[u_1 \wedge ... \wedge u_d]$ in the projective space $\mathbb{P}(\bigwedge^d V)$. By the above, find

$$[v_1 \wedge ... \wedge v_d] = [u_1 \wedge ... \wedge u_d]$$

This allows us to study the Grassmanian $\mathrm{Gr}(d, V)$ of a fixed vector space V.

Definition 9. Define the map

$$\phi : \mathrm{Gr}(d,V) \to \mathbb{P}(\textstyle\bigwedge^d V), \quad \mathrm{span}\{v_1,...,v_d\} \mapsto [v_1 \wedge ... \wedge v_d]$$

which is well-defined by Lemma 5 as described above.

Lemma 10 (1a)**.** We have

$$\mathrm{im}\phi = D := \{[v] \in \mathbb{P}(\textstyle\bigwedge^d V) \mid v \text{ decomposable}\}$$

Proof. First of all, note that the set D is well-defined, as v is decomposable if and only if λv is decomposable, for all $\lambda \in k^*$.

By definition of ϕ, we can directly observe that $\mathrm{im}\phi \subseteq D$. So consider an element $[v] \in D$. As v is decomposable, it follows that $v = v_1 \wedge ... \wedge v_d$ for $v_i \in V$. Not it suffices to show that the v_i are linearly independent, then clearly $\mathrm{span}\{v_1,...,v_d\}$ is a well-defined d-dimensional vector subspace of V, thus an element of $\mathrm{Gr}(d,V)$.

This follows directly from Lemma 6. □

Definition 11. Let $\mathrm{Gr}(d,n) := \mathrm{Gr}(d,k^n)$.

In the lecture, we considered an embedding of $\mathrm{Gr}(d,n)$ into projective space given by minors of the basis matrix. This corresponds to the following definition.

Definition 12. Define the map

$$\rho : \mathrm{Gr}(d,n) \to \mathbb{P}\left(k^{\{1,...,n\}^{(d)}}\right) \cong \mathbb{P}^{\binom{n}{d}-1},$$

$$\mathrm{span}\{v_1,...,v_d\} \mapsto \left[\det \begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d} \end{pmatrix}\right]_{\{i_1,...,i_d\}\in\{1,...,n\}^{(d)}}$$

where $\{1,...,n\}^{(d)} := \{I \subset \{1,...,n\} \mid \#I = d\}$ is the set of all d-element subsets of $\{1,...,n\}$.

Lemma 13. There is a linear isomorphism

$$f : \textstyle\bigwedge^d k^n \to k^{\{1,...,n\}^{(d)}},$$

$$\sum_j v_1^{(j)} \wedge ... \wedge v_d^{(j)} \mapsto \left(\sum_j \det \begin{pmatrix} v_{1i_1}^{(j)} & \cdots & v_{di_1}^{(j)} \\ \vdots & \ddots & \vdots \\ v_{1i_d}^{(j)} & \cdots & v_{di_d}^{(j)} \end{pmatrix}\right)_{\{i_1,...,i_d\}\in\{1,...,n\}^{(d)}}$$

Proof. For vectors $v_1,...,v_d$ and $I = \{i_1,...,i_d\} \in \{1,...,n\}^{(d)}$ write

$$A_I(v_1,...,v_d) := \begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d} \end{pmatrix}$$

First of all, we show that f is well-defined. Note that the tensor product can be described as

$$\begin{aligned} V^{\otimes d} := k^{V\times\ldots\times V}/\mathrm{span}\{&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\big(v_i+v_i'\big)\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\big(v_1\otimes\ldots\otimes v_d\big)-\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i'\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\ &\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\lambda v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\lambda\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\mid i\le d, v_j, v_i'\in V\}\end{aligned}$$

where $v_1\otimes\ldots\otimes v_d := \chi_{(v_1,\ldots,v_d)}$. Hence the external power can be described as

$$\begin{aligned} \textstyle\bigwedge^d V := k^{V\times\ldots\times V}/\mathrm{span}\Big\{&\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\big(v_i+v_i'\big)\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\big(v_1\otimes\ldots\otimes v_d\big)-\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i'\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\ &\big(v_1\otimes\ldots\otimes v_{i-1}\otimes\lambda v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big)\\ &-\lambda\big(v_1\otimes\ldots\otimes v_{i-1}\otimes v_i\otimes v_{i+1}\otimes\ldots\otimes v_d\big),\\ &v_1\otimes\ldots v_{j-1}\otimes(v_j\otimes v_{j+1}+v_{j+1}\otimes v_j)\otimes v_{j+1}\otimes\ldots\otimes v_d\\ &\mid i\le d,\ j\le d-1,\ v_1,\ldots,v_d,v_i'\in V\Big\}\end{aligned}$$

So it suffices to show that for all $I\in\{1,\ldots,n\}^{(d)}$ and vectors $v_1,\ldots,v_d,v_i'\in V$

$$\det(A_I(v_1,\ldots,v_i+v_i',\ldots,v_d)) = \det(A_I(v_1,\ldots,v_d)) + \det(A_I(v_1,\ldots,v_i',\ldots,v_d))$$

and

$$\det(A_I(v_1,\ldots,\lambda v_i,\ldots,v_d)) = \lambda\det(A_I(v_1,\ldots,v_d))$$

and

$$\det(A_I(v_1,\ldots,v_{j+1},v_j,\ldots,v_d)) = -\det(A_I(v_1,\ldots,v_d))$$

However, these properties follow from the well-known properties of the determinant. In particular, det is linear in each column and swapping columns negates the determinant. It follows that f is indeed well-defined.

It is clear by definition that f is linear, so it is left to show that it is bijective. To show surjectivity, note that the $\pm e_I, I\in\{1,\ldots,n\}^{(d)}$ form a basis of $k^{\{1,\ldots,n\}^{(d)}}$. Clearly for $I=\{i_1,\ldots,i_d\}, J\in\{1,\ldots,n\}^{(d)}$ we have that

$$f(e_{i_1}\wedge\ldots\wedge e_{i_d})_J = \det(A_J(e_{i_1},\ldots,e_{i_j})) = \begin{cases}0 & \text{if } J\not\subseteq I\\ \pm1 & \text{if } J\subseteq I\end{cases}$$

so $f(e_{i_1}\wedge\ldots\wedge e_{i_d}) = e_I$ and we deduce that $\mathrm{im} f = \mathbb{P}^{\{1,\ldots,k\}^{(d)}}$.

Finally, note that

$$e_{i_1}\wedge\ldots\wedge e_{i_d}$$

for $i_1 < ... < i_d$ form a basis of $\bigwedge^d k^n$. Clearly, they span $\bigwedge^d k^n$, and the following argument shows that they are linearly independent. Assume

$$\sum_{i_1<...<i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d}) = 0$$

Then

$$0 = e_1 \wedge \Big(\sum_{1<i_2<...<i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge .. \wedge e_{i_d})\Big) + \sum_{1<i_1<...<i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d})$$

Clearly $e_1 \notin \operatorname{span}\{e_2, ..., e_n\}$ and so by Lemma 7 we see that

$$\sum_{1<i_2<...<i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge ... \wedge e_{i_d}) = 0$$

Repeating this argument inductively shows that $\lambda_{1,2,...,d} = 0$. As k^n is symmetric w.r.t. permuting the e_j, we see that all $\lambda_{i_1,...,i_d} = 0$ are zero.

It follows that $\dim(\bigwedge^d k^n) = \dim(\mathbb{P}^{\{1,...,n\}^{(d)}})$ and we find that f is also injective. □

Corollary 14 (2b)**.** Let $\bar{f} : \mathbb{P}(\bigwedge^d k^n) \to \mathbb{P}^{\binom{n}{d}-1}$ be the map f from before modulo k^*. Then

$$\rho = \bar{f} \circ \phi$$

and in particular, we see that $\phi(Gr(d,n))$ is a projective variety and isomorphic to $\rho(\mathrm{Gr}(d,n))$.

Proposition 15 (2c)**.** The map ϕ is injective.

Proof. Consider two d-dimensional subspaces U, W of k^n with $\phi(U) = \phi(W)$. Let $u_1, ..., u_l$ be a basis of $U \cap W$ and extend it to bases $u_1, ..., u_d$ of U and $u_1, ..., u_l, w_{l+1}, ..., w_d$ of W. As $\phi(U) = \phi(W)$, we can assume that the u_i, w_i are scaled such that

$$\begin{aligned} 0 =&(u_1 \wedge ... \wedge u_d) - (u_1 \wedge ... \wedge u_l \wedge w_{l+1} \wedge ... \wedge w_d) \\ =&u_1 \wedge ... \wedge u_l \wedge \big((u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d)\big) \end{aligned}$$

By Lemma 7 we see that

$$u_2 \wedge ... \wedge u_l \wedge \big((u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d)\big) = 0$$

as $u_1 \notin \operatorname{span}\{u_2, ..., u_d, w_{l+1}, ..., w_d\}$. Inductively, this argument shows that

$$(u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d) = 0$$

If $l < d$, we can now apply Lemma 7 again to see that

$$u_{l+1} \in \operatorname{span}\{u_{l+2}, ..., u_d, w_{l+1}, ..., w_d\}$$

as $u_{l+2} \wedge ... \wedge u_d \neq 0$ by Lemma 6. However, this contradicts the linear independence of $u_{l+1}, ..., u_d, w_{l+1}, ..., w_d$. Hence it must be $l = d$ and so $U = W$. □

3 Part III

In this part, we want to investigate the geometric properties of the Grassmanian resp. the image of ϕ. First of all, we introduce coordinates on $\mathbb{P}(\bigwedge^d k^n)$.

Definition 16. Note that in the proof of Lemma 13 it was shown that $v_{i_1} \wedge ... \wedge v_{i_d}$ for $i_1 < ... < i_d$ is a basis of $\bigwedge^d k^n$ if $v_1, ..., v_n$ is a basis of V. We introduce the homogeneous coordinates w.r.t. that basis, namely

$$x : \mathbb{P}(\bigwedge^d k^n) \to \mathbb{P}_k^{\{1,...,n\}^{(d)}} \cong \mathbb{P}_k^{\binom{n}{d}-1},$$
$$\Big[\sum_{i_1<...<i_d} \lambda_{i_1,...,i_d}(v_{i_1} \wedge ... \wedge v_{i_d})\Big] \mapsto \big[\lambda_{i_1,...,i_d}\big]_{i_1<...<i_d}$$

The individual coordinates will be denoted by x_I for some $I \in \{1, ..., n\}^{(d)}$ or $x_{i_1,...,i_d}$ for $i_1 < ... < i_d$.

Proposition 17 (3a)**.** For the embedding $\phi : \mathrm{Gr}(2, V) \to \bigwedge^2 V$ we have

$$\mathrm{Gr}(2, V) \cong \mathrm{im}\phi = \mathbb{V}(I)$$

where

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v\rangle \leq k\big[\mathbb{P}(\bigwedge^d V)\big] = k[x_{i,j} \mid i < j]$$

Proof. By Lemma 14 we have that

$$[\omega] \in \mathrm{im}\phi \;\Leftrightarrow\; \omega \text{ decomposable}$$

and so by Lemma 8

$$\omega \in \mathrm{im}\phi \;\Leftrightarrow\; \omega \wedge \omega = 0$$

In $\mathbb{P}(\bigwedge^d V)$ we find that

$$\begin{aligned}
&\Big(\sum_{i<j} x_{i,j}(e_i \wedge e_j)\Big) \wedge \Big(\sum_{u<v} x_{u,v}(e_u \wedge e_v)\Big) = \sum_{\substack{i<j\\u<v}} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v) \\
&= 2\sum_{i<j<u<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v) + 2\sum_{i<u<j<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v) \\
&\quad + 2\sum_{u<i<j<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v) \\
&=2\sum_{i<j<u<v} (x_{i,j}x_{u,v} - x_{i,u}x_{j,v} + x_{j,u}x_{i,v})(e_i \wedge e_j \wedge e_u \wedge e_v)
\end{aligned}$$

As the $e_i \wedge e_j \wedge e_u \wedge e_v$ are linearly independent, we see that for $[\omega] \in \mathbb{P}(\bigwedge^2 V)$ we have

$$[\omega] \in \mathrm{im}\phi \;\Leftrightarrow\; \forall i < j < u < v : \; (x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v})(\omega) = 0$$

Hence $\mathrm{im}\phi = \mathbb{V}(I)$. $\square$

Example 18 (2b)**.** For $n = 4$, Prop. 17 yields that $\mathrm{Gr}(2, 4) \cong \mathrm{im}\phi = \mathbb{V}(I)$ where

$$I = \langle x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4} \rangle \in k[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}]$$

Changing the indices used for the coordinates, we find

$$\mathrm{Gr}(2, 4) = \mathbb{V}(x_0x_5 + x_2x_3 - x_2x_4)$$

which is exactly what we found in the lecture.

Example 19 (2c)**.** For $n = 5$, Prop. 17 yields that $\mathrm{Gr}(2, 5) \cong \mathrm{im}\phi = \mathbb{V}(I)$ where

$$\begin{aligned} I = \langle x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4}, &\quad x_{1,2}x_{3,5} + x_{1,5}x_{2,3} - x_{1,3}x_{2,5}, \\ x_{1,2}x_{4,5} + x_{1,5}x_{2,4} - x_{1,4}x_{2,5}, &\quad x_{1,3}x_{4,5} + x_{1,5}x_{3,4} - x_{1,4}x_{3,5}, \\ x_{2,3}x_{4,5} + x_{2,5}x_{3,4} - x_{2,4}x_{3,5} \rangle &\leq k\big[\mathbb{P}(\textstyle\bigwedge^2 k^5)\big] \end{aligned}$$

Using the following Sage-code, we can compute the number of intersection points of $\mathrm{Gr}(2, 5)$ with 3-dimensional hyperplanes, and find a probable value for its degree.

```
from itertools import combinations
from math import factorial
import numpy as np

# build up the ring and group the variables nicely
R = PolynomialRing(QQ, [
    "x" + str(i) + str(j)
        for i in range(1, 6) for j in range(i + 1, 6)
])
x12, x13, x14, x15, x23, x24, x25, x34, x35, x45 = R.gens()
_x = [[x12, x13, x14, x15], [x23, x24, x25], [x34, x35], [x45]]
x = lambda i, j: _x[i - 1][j - i - 1]

# construct the ideal describing Gr(2, 5)
polys = []
for seq in combinations([1, 2, 3, 4, 5], 4):
    (i, j, u, v) = sorted(seq)
    p = x(i,j) * x(u,v) + x(i,v) * x(j,u) - x(i,u) * x(j,v)
    polys.append(p)

I = R.ideal(polys)
dimension = I.dimension() - 1
assert dimension == 6
assert I.is_prime()

hyperplane_vectors = [
    np.random.randint(-4, 4, R.ngens(), int)
```

```
        for i in range(dimension + 1)
]
for vecs in combinations(hyperplane_vectors, dimension):
    eqs = [
        sum(map(lambda t: t[0] * t[1], zip(vec, R.gens())))
            for vec in vecs
    ]
    J = I + R.ideal(eqs)
    # the number of intersection points is clearly equal to the
    # dimension of S(X)_d for large enough d
    hp = J.hilbert_polynomial()
    degree = hp.leading_coefficient()
    print(degree) # usually prints 5
```

This shows that the degree of $\mathrm{Gr}(2, 5)$ is indeed 5, as expected from the degree formula mentioned in the lecture.

$$\deg(\mathrm{Gr}(d, n)) = (d(n-d))! \frac{1! \cdot 2! \cdot ... \cdot (d-1)!}{(n-d)! \cdot (n-d+1)! \cdot ... \cdot (n-1)!}$$

which yields

$$\deg(\mathrm{Gr}(2, 5)) = 6! \frac{1!}{3! \cdot 4!} = \frac{6 \cdot 5}{3!} = 5$$

To investigate the properties of $\phi(\mathrm{Gr}(2, n))$ for larger n, we use one tool I encountered during an earlier course on Computational Commutative Algebra and Algebraic Geometry.

Proposition 20 (Macaulay Basis Theorem)**.** Let $\preceq$ be a graded monomial ordering on $R = k[x_0, ..., x_n]$. Then for an ideal $I \leq R$ have that the monomials $x_0^{\alpha_0} ... x_n^{\alpha_n} \notin \mathrm{lt}(I)$ are a k-vector space basis of R/I.

Here $\mathrm{lt}(I)$ is the leading term ideal of I, i.e. the ideal generated by the leading terms of all $f \in I$, w.r.t. $\preceq$.

Proof. See [KR00]. □

Lemma 21. Define the graded reverse monomial ordering $\preceq$ on $R := k[x_{i,j} \mid i < j]$ where the variables $x_{i,j}$ are ordered co-lexicographically w.r.t. (i, j), i.e.

$$x_{i,j} \leq x_{u,v} \;:\Leftrightarrow\; (i, j) \leq_{\mathrm{colex}} (u, v)$$

Moreover, let

$$I := \langle x_{i,j} x_{u,v} + x_{i,v} x_{j,u} - x_{i,u} x_{j,v} \mid i < j < u < v \rangle \leq R$$

be the ideal defining $\phi(\mathrm{Gr}(d, V))$ that was considered above. Then

$$\mathrm{lt}(I) = J := \langle x_{i,v} x_{j,u} \mid i < j < u < v \rangle \leq R$$

Proof. Note that for $i < j < u < v$ have $x_{u,v}, x_{j,v} \succ x_{i,v} \succ x_{i,j}, x_{i,u}, x_{j,u}$. Thus the leading term of $x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}$ is

$$\operatorname{lt}(x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}) = x_{i,v}x_{j,u}$$

It follows that $J \subseteq \operatorname{lt}(I)$.

For the other direction, we use a quite lengthy degree argument. Sadly, the only argument I came up with is extremely technical. We try to present it as clearly as possible, at the cost of only sketching some parts. In the end, I checked it using Computer Algebra, and everything fits together.

Consider homogeneous polynomials $f_{i,j,u,v} \in R$ and

$$F = \sum_{i<j<u<v} f_{i,j,u,v}\ (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})$$

We want to show that $\operatorname{lt}(F) \in J$.

Let

$$f_{i,j,u,v} = \sum_{\alpha \in \mathbb{N}^N} c_\alpha^{(i,j,u,v)} x^\alpha$$

Then

$$F = \sum_{\alpha \in \mathbb{N}^N} \sum_{i<j<u<v} c_\alpha^{(i,j,u,v)} x^\alpha\ (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})$$

and so there exists $\alpha \in \mathbb{N}^N$ and $\epsilon \in k^*$ with

$$\operatorname{lt}(F) = \epsilon \cdot \operatorname{lt}\Big(\sum_{i<j<u<v} c_\alpha^{(i,j,u,v)} x^\alpha\ (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})\Big)$$

Hence, we may assume wlog that all the $f_{i,j,u,v}$ are scaled monomials.

Now, observe that all monomials in F are of the form

$$\epsilon\ x_{i,v}\ x_{j,u}\ f_{i,j,u,v} \quad \text{or} \quad \epsilon\ x_{i,u}\ x_{j,v}\ f_{i,j,u,v} \quad \text{or} \quad \epsilon\ x_{i,j}\ x_{u,v}\ f_{i,j,u,v}$$

where $\epsilon \in k^*$ and $i < j < u < v$. In particular, this is true for the leading term $\operatorname{lt}(F)$. In the first of those case, clearly $\operatorname{lt}(F) \in J$.

So consider now the second case, i.e. $\operatorname{lt}(F) = \epsilon\ x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$. Since $x_{i,v}\ x_{j,u}\ f_{i,j,u,v} \succ x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$, we see that the term $x_{i,v}\ x_{j,u}\ f_{i,j,u,v}$ cannot occur in F, i.e. must "cancel out". Hence the monomial $\epsilon\ x_{i,v}\ x_{j,u}\ f_{i,j,u,v}$ has a nonzero coefficient in

$$F - f_{i,j,u,v}x_{i,v}x_{j,u} = \sum_{\substack{a<b<c<d \\ (a,b,c,d) \neq (i,j,u,v)}} f_{a,b,c,d}\ (x_{a,d}x_{b,c} - x_{a,c}x_{b,d} + x_{a,b}x_{c,d})$$

and so for $(a, b, c, d) \neq (i, j, u, v)$, $\epsilon' \in k^*$ have that

$$\begin{aligned} f_{i,j,u,v}\ x_{i,v}\ x_{j,u} &= \epsilon' f_{a,b,c,d}\ x_{a,d}\ x_{b,c} \quad \text{or} \\ f_{i,j,u,v}\ x_{i,v}\ x_{j,u} &= \epsilon' f_{a,b,c,d}\ x_{a,c}\ x_{b,d} \quad \text{or} \\ f_{i,j,u,v}\ x_{i,v}\ x_{j,u} &= \epsilon' f_{a,b,c,d}\ x_{a,b}\ x_{c,d} \end{aligned}$$

However, the second and third case imply that $f_{a,b,c,d}\ x_{a,d}\ x_{b,c} \succ f_{i,j,u,v}\ x_{i,v}\ x_{j,u}$. Hence, the "new" monomial $f_{a,b,c,d}\ x_{a,d}\ x_{b,c}$ also has to "cancel out" in the sum representation of F, as comes after $\mathrm{lt}(F) = \epsilon x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$ in the order $\preceq$. So applying the whole argument inductively (induction on the number of monomials $\succ f_{i,j,u,v}\ x_{i,v}\ x_{j,u}$ that occur in any of the polynomials we work with), we end up in the first case [1].

If $(a,d) = (i,v)$ then $(b,c) \neq (j,u)$ and $x_{b,c} \mid f_{i,j,u,v}$. Thus

$$x_{b,c}\ x_{i,u}\ x_{j,v} \mid \mathrm{lt}(F) = \epsilon\ x_{i,u}\ x_{j,v}\ f_{i,j,u,v}$$

with $i < j < u < v$ and $i < b < c < v$. No matter how j,u,b,c are ordered relatively to each other, we see that in each possible case $x_{b,c}\ x_{i,u}\ x_{j,v} \in J$:

$$\begin{aligned}
j < u < b < c &\Rightarrow x_{j,v}x_{b,c} \in J \\
j < b < u < c &\Rightarrow x_{j,v}x_{b,c} \in J \\
j < b < c < u &\Rightarrow x_{i,u}x_{b,c} \in J \\
b < j < u < c &\Rightarrow f_{i,b,c,v}\ x_{i,c}\ x_{b,v} \succ f_{i,j,u,v}\ x_{i,u}\ x_{j,v},\ \text{contradiction} \\
b < j < c < u &\Rightarrow x_{i,u}x_{b,c} \in J \\
b < c < j < u &\Rightarrow x_{i,u}x_{b,c} \in J
\end{aligned}$$

Hence $\mathrm{lt}(F) \in J$.

It is now left to consider the case $(b,c) = (j,u)$ and the case $(a,d) \neq (i,v), (b,c) \neq (j,u)$. The former can be dealt with in exactly the same way, by noting that $x_{a,d} \mid f_{i,j,u,v}$. In the latter case, we even find $x_{a,d}\ x_{b,c} \mid f_{i,j,u,v}$ and a very similar argument works.

Finally, one must also consider the third "big" case, namely that

$$\mathrm{lt}(F) = \epsilon\ x_{i,j}\ x_{u,v}\ f_{i,j,u,v}$$

Again, you can do this similarly as before, but now two monomials "cancel out". We will not present this here as well. □

Definition 22. Consider the graph $G_n = (V_n, E_n)$ where $V_n = \{(i,j) \mid 0 \leq i < j \leq n\}$ and

$$E_n = \{\{(i,v),(j,u)\} \mid 0 \leq i < j < u < v \leq n\}$$

Let further

$$s_n(d) := |\{I \subseteq V_n \mid I \text{ independent set of size } d\}|$$

Proposition 23. Let d be the largest integer such that $s_n(d) \neq 0$, i.e. the size of the largest independent set in G_n. Then

$$\dim(\mathrm{Gr}(2,n)) = d - 1$$

and

$$\deg(\mathrm{Gr}(2,n)) = s_n(d)$$

[1] There is a small argument missing here, namely that we apply the next argument on each step of the induction, to show the claim for $f_{i,j,u,v}\ x_{i,v}\ x_{j,u}$. However, it should be easy to see that this is possible.

Proof. By Lemma 21 we know

$$J = \operatorname{lt}(\mathbb{I}(\operatorname{Gr}(2,n))) = \langle x_{i,v}x_{j,u} \mid i < j < u < v \rangle \leq R := k[x_{i,j} \mid i < j]$$

Let $N = \binom{n}{2}$ be the number of variables in R, i.e. $\{x_1, ..., x_N\} = \{x_{i,j} \mid i < j\}$. Now we find for sufficiently large m that

$$\begin{aligned}
&|\{x^\alpha \text{ monomial in } R \mid \deg(x^\alpha) = m,\ \forall i < j < u < v : x_{iv}x_{ju} \nmid x^\alpha\}| \\
=\ &|\{x^\alpha \text{ monomial in } R \mid \deg(x^\alpha) = m,\ \forall i < j < u < v : x_{iv}x_{ju} \nmid \operatorname{sqfr}(x^\alpha)\}| \\
=\ &\Big| \bigcup_{\substack{\alpha \in \{0,1\}^N \\ x_{iv}x_ju \nmid x^\alpha}} \{x^\alpha x^\beta \mid \deg(x^\beta) = m - \deg(x^\alpha),\ \forall i : \alpha_i = 0 \Rightarrow \beta_i = 0\} \Big| \\
=\ &\sum_{\substack{\alpha \in \{0,1\}^N \\ x_{iv}x_ju \nmid x^\alpha}} \left(\!\!\binom{\deg(x^\alpha)}{m - \deg(x^\alpha)}\!\!\right) = \sum_{l=0}^{N} \sum_{\substack{\alpha \in \{0,1\}^N \\ \sum_j \alpha_j = l \\ x_{i,v}x_{j,u} \nmid x^\alpha}} \left(\!\!\binom{l}{m-l}\!\!\right) \\
=\ &\sum_{l=0}^{N} s_n(l) \left(\!\!\binom{l}{m-l}\!\!\right) = \sum_{l=0}^{N} s_n(l) \binom{m-1}{l-1}
\end{aligned}$$

where $\operatorname{sqrf}(f)$ denotes the square-free part of f. This holds, as by definition of s_n and E_n we find

$$s_n(l) = |\big\{ I \subseteq \{x_{i,j}\} \;\big|\; \forall i < j < u < v : x_{i,v}x_{j,u} \nmid xy \text{ for all } x, y \in I \big\}|$$

Now Macaulay's basis theorem 20 yields that for sufficiently large m have

$$\begin{aligned}
\dim_k(R/\mathbb{I}(\operatorname{Gr}(2,n))) =& |\big\{ x^\alpha \text{ monomial in } R \;\big|\; \deg(x^\alpha) = m,\ x^\alpha \notin \operatorname{lt}(\mathbb{I}(\operatorname{Gr}(2,n))) \big\}| \\
=& \sum_{l=0}^{N} s_n(l) \binom{m-1}{l-1}
\end{aligned}$$

Hence we find for the Hilbert polynomial that

$$p_{\operatorname{Gr}(2,n)} = \sum_{l=0}^{N} s_n(l) \binom{m-1}{l-1}$$

and in particular, it has the leading term $s_n(d)\binom{m-1}{d-1}$. The claim follows by the characterization of degree and dimension using the Hilbert polynomial that we did in the lecture. $\square$

Lemma 24. The largest independent set in G_n is of size $2n - 4$.

References

[KR00] Martin Kreuzer and Lorenzo Robbiano. *Computational Commutative Algebra*. Springer, 2000.