

# Miniproject - Algebraic Geometry

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## 1 Part I

**Definition 1.** Let  $V$  be a vector space. Then define the  $d$ -th exterior power as

$$\bigwedge^d(V) := V^{\otimes d} / \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \text{span}\{v \otimes v' + v' \otimes v \mid v, v' \in V\} \otimes V^{\otimes(d-i-1)}$$

Use the notation  $v_1 \wedge \dots \wedge v_d := [v_1 \otimes \dots \otimes v_d] \in \bigwedge^d(V)$ .

**Lemma 2.** Let  $v_1, \dots, v_d \in V$ . Have for  $\pi \in S_d$  that

$$v_{\pi(1)} \wedge \dots \wedge v_{\pi(k)} = \text{sgn}(\pi)(v_1 \wedge \dots \wedge v_d)$$

Furthermore if  $v_i = v_j$  for some  $i \neq j$ , then

$$v_1 \wedge \dots \wedge v_d = 0$$

*Proof.* Note that

$$u \otimes v \otimes v' \otimes w = -(u \otimes v' \otimes v \otimes w)$$

for all  $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{(d-i-1)}(V), v, v' \in V$ .

Every  $\pi \in S_d$  has a decomposition  $\pi = \xi_1 \dots \xi_n$  into transpositions  $\xi_i$ . Applying this inductively, we find that

$$v_1 \wedge \dots \wedge v_d = \text{sgn}(\xi_1 \dots \xi_n)(v_{(\xi_1 \dots \xi_n)(1)} \wedge \dots \wedge v_{(\xi_1 \dots \xi_n)(k)})$$

and so

$$v_1 \wedge \dots \wedge v_d = \text{sgn}(\pi)(v_{\pi(1)} \wedge \dots \wedge v_{\pi(k)})$$

Furthermore, we find that

$$u \otimes v \otimes v \otimes w = -(u \otimes v \otimes v \otimes w) = 0$$

must be zero. Hence, if  $v_1, \dots, v_d \in V$  with  $v_i = v_j$  for some  $i \neq j$ , then there is a permutation  $\pi \in S_d$  with  $\pi(1) = i, \pi(2) = j$  and

$$v_1 \wedge \dots \wedge v_d = (\text{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge \dots \wedge v_{\pi(k)}) = \text{sgn}(\pi)0 = 0$$

□

**Lemma 3** (1a). *Let  $\dim(V) \leq 3$ . Then every element of  $\bigwedge^k(V)$  is decomposable.*

*Proof.* Now let  $v_1, v_2, v_3$  be a set of generators of  $V$ . Consider  $u_1 = \sum \lambda_i v_i, u_2 = \sum \mu_i v_i, u_3 = \sum \rho_i v_i$ . Then by applying Lemma 2, we see that

$$\begin{aligned} u_1 \wedge u_2 &= \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{=0 \text{ if } i=j} = \sum_{i < j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i > j} \lambda_i \mu_j (v_i \wedge v_j) \\ &= \sum_{i < j} (\lambda_i \mu_j - \lambda_j \mu_i) (v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3) \\ &= \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} u_1 \wedge u_2 \wedge u_3 &= \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{=0 \text{ unless } i,j,l \text{ pairwise distinct}} \\ &= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)}) \\ &= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \text{sgn}(\pi) (v_1 \wedge v_2 \wedge v_3) \\ &= (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \text{sgn}(\pi) \end{aligned}$$

are decomposable. Further, it is easy to see from Lemma 2 that  $\bigwedge^k(V) = \{0\}$  for  $k \geq 4$ , which is trivially decomposable.  $\square$

**Example 4** (1b). *Consider  $V = k^4$ . Then the element  $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$  is not decomposable.*

*Proof.* Assume it was, then there are  $a, b \in k^4$  such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i < j} (a_i b_j - a_j b_i) (e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1, \quad a_3 b_4 - a_4 b_3 = 1, \quad a_i b_j - a_j b_i = 0 \text{ for all } (i, j) \neq (1, 2), (3, 4)$$

Clearly  $a_1 b_2 \neq 0$  or  $a_2 b_1 \neq 0$ . Similarly, have  $a_3 b_4 \neq 0$  or  $a_4 b_3 \neq 0$ . As all expressions are symmetric w.r.t swapping  $a_1, b_2$  with  $a_2, b_1$  and  $a_3, b_4$  with  $a_4, b_3$ , we may assume wlog that  $a_1 b_2, a_3 b_4 \neq 0$ .

Have  $a_1 b_4 = a_4 b_1$  and  $a_2 b_4 = a_4 b_2$ . We know that  $a_1 b_4 \neq 0$  and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \Rightarrow a_2 b_1 = a_1 b_2$$

This contradicts  $a_1 b_2 - a_2 b_1 = 1$ .  $\square$

**Lemma 5.** Let  $A = (a_{ij}) \in \text{GL}_d(k)$  and  $v_1, \dots, v_d \in V$ . Then

$$\left(\sum_j a_{1j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) = \det(A)(v_1 \wedge \dots \wedge v_d)$$

*Proof.* By a direct computation using Lemma 2, we find

$$\begin{aligned} & \left(\sum_j a_{1j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) = \sum_{j_1, \dots, j_d} a_{1j_1} \dots a_{dj_d} (v_{j_1} \wedge \dots \wedge v_{j_d}) \\ &= \sum_{\pi \in S_d} a_{1\pi(1)} \dots a_{d\pi(d)} (v_{\pi(1)} \wedge \dots \wedge v_{\pi(d)}) \\ &= \sum_{\pi \in S_d} a_{1\pi(1)} \dots a_{d\pi(d)} \text{sgn}(\pi) (v_1 \wedge \dots \wedge v_d) \\ &= (v_1 \wedge \dots \wedge v_d) \sum_{\pi \in S_d} \text{sgn}(\pi) \prod_{j=1}^d a_{j\pi(j)} = \det(A)(v_1 \wedge \dots \wedge v_d) \end{aligned}$$

where the last equality holds due to the Leibniz determinant formula.  $\square$

**Lemma 6.** For  $v_1, \dots, v_d \in V$  have

$$v_1 \wedge \dots \wedge v_d = 0 \Leftrightarrow v_1, \dots, v_d \text{ linearly dependent}$$

*Proof.* For the direction  $\Rightarrow$ , assume that  $v_1, \dots, v_d$  are not independent. Then there is a nonzero vector  $a_1 \in k^d$  with  $\sum a_{1i}v_i = 0$ . Clearly, we can extend  $a_1$  to a basis  $a_1, \dots, a_d$  of  $k^d$ , which gives a matrix  $A = (a_{ij}) \in \text{GL}_d(k)$ .

However by Lemma 5 we now get

$$\begin{aligned} 0 &= 0 \wedge \left(\sum_j a_{2j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) = \left(\sum_j a_{1j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) \\ &= \det(A)(v_1 \wedge \dots \wedge v_d) \end{aligned}$$

and so  $v = v_1 \wedge \dots \wedge v_d = 0$  as  $\det(A) \neq 0$ .

Direction  $\Leftarrow$  TODO  $\square$

**Lemma 7.** Let  $v \in V$  and  $u \in \bigwedge^{d-1}U$  for a linear subspace  $U \leq V$ . If  $v \wedge u \in \bigwedge^d U$  then  $v \in U$  or  $u = 0$ .

*Proof.* TODO  $\square$

**Lemma 8** (1c). Let  $d$  be even. An element  $\omega \in \bigwedge^d V$  is decomposable if and only if  $\omega \wedge \omega \in \bigwedge^{2d} V$  is zero.

*Proof.* The direction  $\Rightarrow$  even holds generally. Assume  $\omega = v_1 \wedge \dots \wedge v_d$ . Then

$$\omega \wedge \omega = v_1 \wedge \dots \wedge v_d \wedge v_1 \wedge \dots \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let  $\omega = v_1 + \dots + v_t$  for linearly independent decomposable vectors  $v_i \in \bigwedge^2 V$ . Then

$$\begin{aligned} 0 = \omega \wedge \omega &= \sum_{i,j} v_i \wedge v_j = \sum_{i < j} (v_i \wedge v_j) + (v_j \wedge v_i) \\ &= \sum_{i < j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \left( \sum_{j > i} v_j \right) \end{aligned}$$

Here we used that the permutation  $(1 \ 2d)(2 \ (2d-1)) \dots (d \ (d+1)) \in S_{2d}$  has always sign 1 (since  $d$  is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \left( \bigwedge^2 \text{span}\{v_2, \dots, v_t\} \right) \setminus \{0\}$$

find

$$u_1, u_2 \in \text{span}\{v_2, \dots, v_t\}$$

In particular, we know that

$$v_1 \wedge \left( \sum_{j > i} v_j \right) \in \bigwedge^2 \text{span}\{v_2, \dots, v_t\}$$

and so  $v_1 \in \text{span}\{v_2, \dots, v_t\}$  unless  $\sum_{j > i} v_j = 0$  by Lemma 7. We assumed that the  $v_i$  are linearly independent, so the former would give a contradiction. Hence  $\sum_{j > i} v_j = 0$  and thus  $t = 1$ , i.e.  $\omega = v_1$  is decomposable.  $\square$

## 2 Part II

In this part, we want to consider the connection of external powers to the Grassmanian. First of all, assume there are two bases  $v_1, \dots, v_d$  and  $u_1, \dots, u_d$  of a  $d$ -dimensional vector space  $U$ . Then there exists a basis change matrix  $A = (a_{ij}) \in \text{GL}_d(k)$  with

$$u_i = \sum_j a_{ij} v_j$$

So by Lemma 5, it follows that

$$u_1 \wedge \dots \wedge u_d = \det(A)(v_1 \wedge \dots \wedge v_d)$$

As  $v_1, \dots, v_d$  resp.  $u_1, \dots, u_d$  are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge \dots \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge \dots \wedge u_d \neq 0$$

by Lemma 6. Hence they have well-defined images  $[v_1 \wedge \dots \wedge v_d]$  resp.  $[u_1 \wedge \dots \wedge u_d]$  in the projective space  $\mathbb{P}(\bigwedge^d V)$ . By the above, find

$$[v_1 \wedge \dots \wedge v_d] = [u_1 \wedge \dots \wedge u_d]$$

This allows us to study the Grassmanian  $\text{Gr}(d, V)$  of a fixed vector space  $V$ .

**Definition 9.** Define the map

$$\phi : \text{Gr}(d, V) \rightarrow \mathbb{P}(\bigwedge^d V), \quad \text{span}\{v_1, \dots, v_d\} \mapsto [v_1 \wedge \dots \wedge v_d]$$

which is well-defined by Lemma 5 as described above.

**Lemma 10** (1a). We have

$$\text{im}\phi = D := \{[v] \in \mathbb{P}(\bigwedge^d V) \mid v \text{ decomposable}\}$$

*Proof.* First of all, note that the set  $D$  is well-defined, as  $v$  is decomposable if and only if  $\lambda v$  is decomposable, for all  $\lambda \in k^*$ .

By definition of  $\phi$ , we can directly observe that  $\text{im}\phi \subseteq D$ . So consider an element  $[v] \in D$ . As  $v$  is decomposable, it follows that  $v = v_1 \wedge \dots \wedge v_d$  for  $v_i \in V$ . Not it suffices to show that the  $v_i$  are linearly independent, then clearly  $\text{span}\{v_1, \dots, v_d\}$  is a well-defined  $d$ -dimensional vector subspace of  $V$ , thus an element of  $\text{Gr}(d, V)$ .

This follows directly from Lemma 6.  $\square$

**Definition 11.** Let  $\text{Gr}(d, n) := \text{Gr}(d, k^n)$ .

In the lecture, we considered an embedding of  $\text{Gr}(d, n)$  into projective space given by minors of the basis matrix. This corresponds to the following definition.

**Definition 12.** Define the map

$$\begin{aligned} \rho : \text{Gr}(d, n) &\rightarrow \mathbb{P}\left(k^{\{1, \dots, n\}^{(d)}}\right) \cong \mathbb{P}^{\binom{n}{d}-1}, \\ \text{span}\{v_1, \dots, v_d\} &\mapsto \left[ \det \begin{pmatrix} v_{1i_1} & \dots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \dots & v_{di_d} \end{pmatrix} \right]_{\{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}} \end{aligned}$$

where  $\{1, \dots, n\}^{(d)} := \{I \subset \{1, \dots, n\} \mid \#I = d\}$  is the set of all  $d$ -element subsets of  $\{1, \dots, n\}$ .

**Lemma 13.** There is a linear isomorphism

$$\begin{aligned} f : \bigwedge^d k^n &\rightarrow k^{\{1, \dots, n\}^{(d)}}, \\ \sum_j v_1^{(j)} \wedge \dots \wedge v_d^{(j)} &\mapsto \left( \sum_j \det \begin{pmatrix} v_{1i_1}^{(j)} & \dots & v_{di_1}^{(j)} \\ \vdots & \ddots & \vdots \\ v_{1i_d}^{(j)} & \dots & v_{di_d}^{(j)} \end{pmatrix} \right)_{\{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}} \end{aligned}$$

*Proof.* For vectors  $v_1, \dots, v_d$  and  $I = \{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}$  write

$$A_I(v_1, \dots, v_d) := \begin{pmatrix} v_{1i_1} & \dots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \dots & v_{di_d} \end{pmatrix}$$

First of all, we show that  $f$  is well-defined. Note that the tensor product can be described as

$$\begin{aligned} V^{\otimes d} := k^{V \times \dots \times V} / \text{span} \{ & (v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v'_i) \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - (v_1 \otimes \dots \otimes v_d) - (v_1 \otimes \dots \otimes v_{i-1} \otimes v'_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ & (v_1 \otimes \dots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - \lambda (v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \mid i \leq d, v_j, v'_i \in V \} \end{aligned}$$

where  $v_1 \otimes \dots \otimes v_d := \chi_{(v_1, \dots, v_d)}$ . Hence the external power can be described as

$$\begin{aligned} \bigwedge^d V := k^{V \times \dots \times V} / \text{span} \{ & (v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v'_i) \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - (v_1 \otimes \dots \otimes v_d) - (v_1 \otimes \dots \otimes v_{i-1} \otimes v'_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ & (v_1 \otimes \dots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - \lambda (v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ & v_1 \otimes \dots \otimes v_{j-1} \otimes (v_j \otimes v_{j+1} + v_{j+1} \otimes v_j) \otimes v_{j+1} \otimes \dots \otimes v_d \\ & \mid i \leq d, j \leq d-1, v_1, \dots, v_d, v'_i \in V \} \end{aligned}$$

So it suffices to show that for all  $I \in \{1, \dots, n\}^{(d)}$  and vectors  $v_1, \dots, v_d, v'_i \in V$

$$\det(A_I(v_1, \dots, v_i + v'_i, \dots, v_d)) = \det(A_I(v_1, \dots, v_d)) + \det(A_I(v_1, \dots, v'_i, \dots, v_d))$$

and

$$\det(A_I(v_1, \dots, \lambda v_i, \dots, v_d)) = \lambda \det(A_I(v_1, \dots, v_d))$$

and

$$\det(A_I(v_1, \dots, v_{j+1}, v_j, \dots, v_d)) = -\det(A_I(v_1, \dots, v_d))$$

However, these properties follow from the well-known properties of the determinant. In particular,  $\det$  is linear in each column and swapping columns negates the determinant. It follows that  $f$  is indeed well-defined.

It is clear by definition that  $f$  is linear, so it is left to show that it is bijective. To show surjectivity, note that the  $\pm e_I, I \in \{1, \dots, n\}^{(d)}$  form a basis of  $k^{\{1, \dots, n\}^{(d)}}$ . Clearly for  $I = \{i_1, \dots, i_d\}, J \in \{1, \dots, n\}^{(d)}$  we have that

$$f(e_{i_1} \wedge \dots \wedge e_{i_d})_J = \det(A_J(e_{i_1}, \dots, e_{i_d})) = \begin{cases} 0 & \text{if } J \not\subseteq I \\ \pm 1 & \text{if } J \subseteq I \end{cases}$$

so  $f(e_{i_1} \wedge \dots \wedge e_{i_d}) = e_I$  and we deduce that  $\text{im } f = \mathbb{P}^{\{1, \dots, k\}^{(d)}}$ .

Finally, note that

$$e_{i_1} \wedge \dots \wedge e_{i_d}$$

for  $i_1 < \dots < i_d$  form a basis of  $\bigwedge^d k^n$ . Clearly, they span  $\bigwedge^d k^n$ , and the following argument shows that they are linearly independent. Assume

$$\sum_{i_1 < \dots < i_d} \lambda_{i_1, \dots, i_d} (e_{i_1} \wedge \dots \wedge e_{i_d}) = 0$$

Then

$$0 = e_1 \wedge \left( \sum_{1 < i_2 < \dots < i_d} \lambda_{1, i_2, \dots, i_d} (e_{i_2} \wedge \dots \wedge e_{i_d}) \right) + \sum_{1 < i_1 < \dots < i_d} \lambda_{i_1, \dots, i_d} (e_{i_1} \wedge \dots \wedge e_{i_d})$$

Clearly  $e_1 \notin \text{span}\{e_2, \dots, e_n\}$  and so by Lemma 7 we see that

$$\sum_{1 < i_2 < \dots < i_d} \lambda_{1, i_2, \dots, i_d} (e_{i_2} \wedge \dots \wedge e_{i_d}) = 0$$

Repeating this argument inductively shows that  $\lambda_{1, 2, \dots, d} = 0$ . As  $k^n$  is symmetric w.r.t. permuting the  $e_j$ , we see that all  $\lambda_{i_1, \dots, i_d} = 0$  are zero.

It follows that  $\dim(\bigwedge^d k^n) = \dim(\mathbb{P}^{\{1, \dots, n\}^{(d)}})$  and we find that  $f$  is also injective.  $\square$

**Corollary 14** (2b). *Let  $\bar{f} : \mathbb{P}(\bigwedge^d k^n) \rightarrow \mathbb{P}^{\binom{n}{d}-1}$  be the map  $f$  from before modulo  $k^*$ . Then*

$$\rho = f \circ \phi$$

and in particular, we see that  $\phi(\text{Gr}(d, n))$  is a projective variety and isomorphic to  $\rho(\text{Gr}(d, n))$ .

**Proposition 15** (2c). *The map  $\phi$  is injective.*

*Proof.* Consider two  $d$ -dimensional subspaces  $U, W$  of  $k^n$  with  $\phi(U) = \phi(W)$ . Let  $u_1, \dots, u_l$  be a basis of  $U \cap W$  and extend it to bases  $u_1, \dots, u_d$  of  $U$  and  $u_1, \dots, u_l, w_{l+1}, \dots, w_d$  of  $W$ . As  $\phi(U) = \phi(W)$ , we can assume that the  $u_i, w_i$  are scaled such that

$$\begin{aligned} 0 &= (u_1 \wedge \dots \wedge u_d) - (u_1 \wedge \dots \wedge u_l \wedge w_{l+1} \wedge \dots \wedge w_d) \\ &= u_1 \wedge \dots \wedge u_l \wedge ((u_{l+1} \wedge \dots \wedge u_d) - (w_{l+1} \wedge \dots \wedge w_d)) \end{aligned}$$

By Lemma 7 we see that

$$u_2 \wedge \dots \wedge u_l \wedge ((u_{l+1} \wedge \dots \wedge u_d) - (w_{l+1} \wedge \dots \wedge w_d)) = 0$$

as  $u_1 \notin \text{span}\{u_2, \dots, u_d, w_{l+1}, \dots, w_d\}$ . Inductively, this argument shows that

$$(u_{l+1} \wedge \dots \wedge u_d) - (w_{l+1} \wedge \dots \wedge w_d) = 0$$

If  $l < d$ , we can now apply Lemma 7 again to see that

$$u_{l+1} \in \text{span}\{u_{l+2}, \dots, u_d, w_{l+1}, \dots, w_d\}$$

as  $u_{l+2} \wedge \dots \wedge u_d \neq 0$  by Lemma 6. However, this contradicts the linear independence of  $u_{l+1}, \dots, u_d, w_{l+1}, \dots, w_d$ . Hence it must be  $l = d$  and so  $U = W$ .  $\square$

### 3 Part III

In this part, we want to investigate the geometric properties of the Grassmanian resp. the image of  $\phi$ . First of all, we introduce coordinates on  $\mathbb{P}(\bigwedge^d k^n)$ .

**Definition 16.** *Note that in the proof of Lemma 13 it was shown that  $v_{i_1} \wedge \dots \wedge v_{i_d}$  for  $i_1 < \dots < i_d$  is a basis of  $\bigwedge^d k^n$  if  $v_1, \dots, v_n$  is a basis of  $V$ . We introduce the homogeneous coordinates w.r.t. that basis, namely*

$$x : \mathbb{P}(\bigwedge^d k^n) \rightarrow \mathbb{P}_k^{\{1, \dots, n\}^{(d)}} \cong \mathbb{P}_k^{\binom{n}{d}-1},$$

$$\left[ \sum_{i_1 < \dots < i_d} \lambda_{i_1, \dots, i_d} (v_{i_1} \wedge \dots \wedge v_{i_d}) \right] \mapsto [\lambda_{i_1, \dots, i_d}]_{i_1 < \dots < i_d}$$

The individual coordinates will be denoted by  $x_I$  for some  $I \in \{1, \dots, n\}^{(d)}$  or  $x_{i_1, \dots, i_d}$  for  $i_1 < \dots < i_d$ .

**Proposition 17 (3a).** *For the embedding  $\phi : \text{Gr}(2, V) \rightarrow \bigwedge^2 V$  we have*

$$\text{Gr}(2, V) \cong \text{im} \phi = \mathbb{V}(I)$$

where

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v \rangle \leq k[\mathbb{P}(\bigwedge^2 V)] = k[x_{i,j} \mid i < j]$$

*Proof.* By Lemma 14 we have that

$$[\omega] \in \text{im} \phi \Leftrightarrow \omega \text{ decomposable}$$

and so by Lemma 8

$$\omega \in \text{im} \phi \Leftrightarrow \omega \wedge \omega = 0$$

In  $\mathbb{P}(\bigwedge^2 V)$  we find that

$$\begin{aligned} & \left( \sum_{i < j} x_{i,j} (e_i \wedge e_j) \right) \wedge \left( \sum_{u < v} x_{u,v} (e_u \wedge e_v) \right) = \sum_{\substack{i < j \\ u < v}} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) \\ &= 2 \sum_{i < j < u < v} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) + 2 \sum_{i < u < j < v} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) \\ & \quad + 2 \sum_{u < i < j < v} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) \\ &= 2 \sum_{i < j < u < v} (x_{i,j} x_{u,v} - x_{i,u} x_{j,v} + x_{j,u} x_{i,v}) (e_i \wedge e_j \wedge e_u \wedge e_v) \end{aligned}$$

As the  $e_i \wedge e_j \wedge e_u \wedge e_v$  are linearly independent, we see that for  $[\omega] \in \mathbb{P}(\bigwedge^2 V)$  we have

$$[\omega] \in \text{im} \phi \Leftrightarrow \forall i < j < u < v : (x_{i,j} x_{u,v} + x_{i,v} x_{j,u} - x_{i,u} x_{j,v})(\omega) = 0$$

Hence  $\text{im} \phi = \mathbb{V}(I)$ . □