Miniproject - Combinatorics

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We use the convention that $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.

1 Part I

First of all, we introduce the notion of a rank function on a graded poset.

Proposition 1. Let P be a graded poset in which every maximal chain has length $n+1$. Then the function

$$r : P \to \{0, ..., n\}, \quad x \mapsto \max\{k \in \mathbb{N} \mid \exists a_1, ..., a_k \in P : \ a_1 < a_2 < ... < a_k < x\}$$

is well-defined and the unique function such that $x < y$ implies $r(x) < r(y)$ for all $x, y \in P$. We call it the *rank function* of P.

Proof. Clearly r is well-defined, as for every increasing sequence $a_1 < ... < a_k < x$, we have the chain $C := \{a_1, ..., a_k, x\}$ of size $k+1$. Hence by assumption, $k \leq n$ and so $r(x)$ is finite and in $\{0, ..., n\}$.

Further, consider $x < y$ in P. We have a sequence $a_1 < ... < a_{r(x)} < x$ by definition of r. It follows that there is an increasing sequence $a_1 < ... < a_{r(x)} < x < y$ and so $r(y) \geq r(k) + 1 > r(x)$.

Lastly, assume there was another function $r' : P \to \{0, ..., n\}$ with this property. Consider any $x \in P$. By definition of r, there is an increasing sequence $a_1 < ... < a_{r(x)} < x$ in P. Now consider a maximal chain C containing the chain $\{a_1, ..., a_{r(x)}, x\}$.

Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$ and $x = b_j$. Note that we have the increasing sequence $b_1 < ... < b_{j-1} < b_j = x$ and so by the definition of r, find $j \leq r(x) + 1$. On the other hand, have $a_1, ..., a_{r(x)} \in C$ and thus $j = r(x) + 1$, i.e.

$$a_1 = b_1, \ ..., \ a_{r(x)} = b_{r(x)}, \ x = b_{r(x)+1}$$

As $b_i < b_{i+1}$, we know that $r'(b_i) < r'(b_{i+1})$ and inductively, we see $r'(b_i) \geq i - 1$. However, $r'(b_{n+1}) \leq n$ and thus $r'(b_i) = i-1$. Finally it follows that $r'(x) = r'(b_{r(x)+1}) = r(x)$. $\square$

Now we will show some basic properties of the rank function.

Proposition 2. Let P be a graded poset of maximal rank n with rank function r. Then

- $x \in P$ is minimal iff $r(x) = 0$ and maximal iff $r(x) = n$.
- If $x < y$ and $r(x) + 1 \neq r(y)$ then there is $z \in P$ with $x < z < y$.
- If $x < y$ then there is an increasing sequence $x < a_1 < ... < a_{r(y)-r(x)-1} < y$ in P.

Proof. For (i), let $x \in P$ be minimal. Then there is no increasing sequence $a_1 < x$ in P, so $r(x) \leq 0$. Conversely, let $r(x) = 0$. Assume there was $y \in P$ with $y < x$, then $r(y) < r(x) = 0$, a contradiction. The analogous statement for maximal elements is proved in the same way.

For (iii), consider $x < y$ in P. Then the chain $\{x, y\}$ is contained in a maximal chain $C \subseteq P$. Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$.

Hence we find $r(b_i) < r(b_{i+1})$ and so inductively that $r(b_i) \geq i-1$ and $r(b_i) \leq i-1$ since $r(b_1) \geq 0$ and $r(b_{n+1}) \leq n$. It follows that $r(b_i) = i - 1$ and so $x = b_{r(x)+1}, y = b_{r(y)+1}$. Therefore, we have a chain

$$x < b_{r(x)+2} < ... < b_{r(y)} < y$$

of length $r(y) - r(x) - 1$. Statement (ii) follows directly, as in this setting, have $r(y) \geq r(x) + 2$ and so $r(y) - r(x) - 1 \neq 0$. □

2 Part II

In this part, we want to study whether graded posets satisfy an analogue of the LYM inequality.

Proposition 3. For a graded poset P with layers $L_0, ..., L_n$ the following statements are equivalent:

- For every antichain $A \subseteq P$ have
$$\sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} \leq 1$$
- For each $1 < i \leq n$ and $F \subseteq L_i$ have
$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$
where
$$\partial F := \{a \in L_{i-1} \mid \exists b \in F : \; a \leq b\}$$
- There exists a (nonempty) sequence of maximal chains $C_1, ..., C_t$ such that for all $1 \leq i \leq n$ we have:
$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

Proof. Show the directions (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii)

(ii) ⇒ (iii) Use induction on n. The base case is trivial, just take chains $\{x\}$ for each $x \in A = L_0$. So assume that $n > 0$. The set

$$A' := \bigcup_{i=0}^{n-1} L_i$$

is a graded poset of maximal rank $n-1$, and hence there are maximal chains $C_1, ..., C_t \subseteq A'$ such that for all $1 \le i \le n-1$ have

$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

by induction hypothesis. Let now $V_1 := \{1\} \times [t] \times L_n$ and $V_2 := \{2\} \times [t] \times L_n$ and consider the bipartite graph $G := (V_1 \sqcup V_2, E)$ where E is defined as follows:

$$\{\underbrace{(1,i,a)}_{\in V_1}, \underbrace{(2,j,b)}_{\in V_2}\} \in E \;\Leftrightarrow\; \max C_j < a$$

We use Hall's theorem to show that G has a matching.

Consider any $W = \{(1, i_1, a_1), ..., (1, i_w, a_w)\} \subseteq V_1$ and let $F = \{a_1, ..., a_w\}$. Then

$$W \subseteq [t] \times F \;\Rightarrow\; |W| \le t|F|$$

By choice of $C_1, ..., C_t$, we have that the number of j with $x \in C_j$ is the same for all $x \in L_{n-1}$, say k. Since the C_i are maximal chains, each intersects L_{n-1} in exactly one element. So have bijection

$$\{(x,i) \mid x \in L_{n-1}, 1 \le i \le t, x \in C_i\} \to \{i \mid 1 \le i \le t\}, \quad (x,i) \mapsto i$$

where the set on the left-hand side has size $k|L_{n-1}|$. It follows that $k|L_{n-1}| = t$.

Since $\max C_j \in L_{n-1}$ for all j, we have

$$\begin{aligned} N(W) =& \{(2,j,b) \mid b \in L_n, \exists a \in F : \max C_j < a\} \\ =& \{2\} \times \{j \mid \exists a \in F : \max C_j < a\} \times L_n \\ =& \{2\} \times \{j \mid \max C_j \in \partial F\} \times L_n \end{aligned}$$

and so by the above

$$|N(W)| = |L_n| \sum_{\max C_j \in \partial F} 1 = |L_n| \sum_{x \in \partial F} \sum_{x \in C_j} 1 = |L_n| \sum_{x \in \partial F} k = |L_n||\partial F|k$$

Together with the assumption that $|\partial F||L_n| \ge |F||L_{n-1}|$ we see that

$$|W| \le t|F| = k|L_{n-1}||F| \le k|\partial F||L_n| = |N(W)|$$

So Hall's theorem yields a matching $M \subseteq E$ from V_1 to V_2. As $|V_1| = |V_2|$, this is already a 1-to-one correspondence.

Now consider the sets

$$C'_m := C_j \cup \{a\} \text{ for each } m = \{\underbrace{(1,i,a)}_{\in V_1}, \underbrace{(2,j,b)}_{\in V_2}\} \in M$$

These are chains, as $\max C_j < a$ for each C'_m. Further, for each $x, y \in L_i, i < n$ have that

$$|\{m \mid x \in C'_m\}| = |L_n \times \{j \mid x \in C_j\}| = |L_n \times \{j \mid y \in C_j\}| = |\{C'_m \mid y \in C'_m\}|$$

as M is a matching from V_2 to V_1. Finally, for all $x \in L_n$ we have that

$$|\{C'_m \mid x \in C'_m\}| = |\{m \in M \mid \exists i, v \in V_2 : m = \{(i,x), v\}\}| = |\{(i,x) \mid (i,x) \in V_1\}| = t$$

as M is a matching from V_1 to V_2.

(iii) $\Rightarrow$ (i) Let $C_1, ..., C_t$ be a sequence of maximal chains given by the assumption. For $1 \leq i \leq n$ let k_i be the number of different j such that a fixed element $x \in L_i$ is contained in exactly the C_j. By assumption, this does not depend on x.

As in the direction above, we see that $k_i|L_i| = t$ because each C_j intersects L_i in exactly one element, so there is a bijection

$$\{(x,j) \mid x \in L_i, 1 \leq j \leq t, x \in C_j\} \to \{j \mid 1 \leq j \leq t\}, \quad (x,j) \mapsto j$$

and the set on the left has size $k_i|L_i|$.

Since each C_j is a chain and A is an antichain, we find that A and C_j intersect in at most one element. So

$$\begin{aligned} t \geq & |\{j \mid A \cap C_j \neq \emptyset\}| = \sum_{a \in A} |\{j \mid a \in C_j\}| = \sum_{i=0}^{n} \sum_{a \in A \cap L_i} |\{j \mid a \in C_j\}| \\ = & \sum_{i=0}^{n} \sum_{a \in A \cap L_i} k_i = \sum_{i=0}^{n} k_i |A \cap L_i| = \sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} t \end{aligned}$$

The claim follows by canceling t.

(i) $\Rightarrow$ (ii) Consider $A := F \cup (L_{i-1} \setminus \partial F)$. This is clearly an antichain, as for $x \in F, y \in L_{i-1} \setminus \partial F$ have $y \not< x$. However, $r(y) < r(x)$ so also $x \not< y$, thus x and y are incomparable. Clearly elements from the same layer are incomparable.

Thus, the assumption yields that

$$\frac{|L_{i-1}| - |\partial F|}{|L_{i-1}|} + \frac{|F|}{|L_i|} = \frac{|A \cap L_{i-1}|}{|L_{i-1}|} + \frac{|A \cap L_i|}{|L_i|} = \sum_{j=0}^{n} \frac{|A \cap L_j|}{|L_j|} \leq 1$$

This gives

$$1 + \frac{|F|}{|L_i|} \leq 1 + \frac{|\partial F|}{|L_{i-1}|}$$

and the claim follows. □

3 Part III

In this part, we show that sufficiently symmetric posets satisfy the conditions introduced in the previous part. We even show a slightly stronger stronger statement than required in the task, namely that the implication to the second condition of Prop.3 already holds "locally", i.e. for each layer separately.

Proposition 4. Let P be a graded poset with layers $L_0, ..., L_n$ and let $0 < i \leq n$ such that

- Each element $a \in L_i$ covers the same number of elements in L_{i-1}
- Each element $a \in L_{i-1}$ is covered by the same number of elements in L_i

Then for each $F \subseteq L_i$ have

$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$

In particular, if this is true for all i, then all the equivalent conditions from Prop. 3 follow.

Proof. Assume that for all $a \in L_i$

$$|\{b \in L_{i-1} \mid a \text{ covers } b\}| = k \in \mathbb{N}$$

and for all $b \in L_{i-1}$

$$|\{a \in L_i \mid a \text{ covers } b\}| = l \in \mathbb{N}$$

Double-counting pairs $(a, b) \in L_i \times L_{i-1}$ such that a covers b yields $k|L_i| = l|L_{i-1}|$. Now consider $F \subseteq L_i$. Note that by definition of ∂F, we have for all $a \in F$ that

$$\{b \in \partial F \mid a \text{ covers } b\} = \{b \in L_{i-1} \mid a \text{ covers } b\} = \partial\{a\} \subseteq \partial F$$

Again by double-counting pairs $(a, b) \in F \times \partial F$ such that a covers b, we find

$$\begin{aligned} k|F| = \sum_{a \in F} k = \sum_{a \in F} |\{b \in L_{i-1} \mid a \text{ covers } b\}| = \sum_{a \in F} |\{b \in \partial F \mid a \text{ covers } b\}| \\ = |\{(a, b) \in F \times \partial F \mid a \text{ covers } b\}| = \sum_{b \in \partial F} |\{a \in F \mid a \text{ covers } b\}| \\ \leq \sum_{b \in \partial F} |\{a \in L_i \mid a \text{ covers } b\}| = \sum_{b \in \partial F} l = l|\partial F| \end{aligned}$$

Hence

$$\frac{|\partial F|}{|L_{i-1}|} = \frac{l|\partial F|}{l|L_{i-1}|} \leq \frac{k|F|}{k|L_i|} = \frac{|F|}{|L_i|}$$

□

4 Part IV

Example 5. Let $\Pi_m = \{\pi \subseteq \mathfrak{P}(m) \mid \pi \text{ partition}\}$ be the poset of partitions ordered by refinement. Then Π_m is a graded poset with rank function

$$r : \Pi_m \to \{0, ..., m-1\}, \quad \pi \mapsto m - |\pi|$$

Proof. First of all, have $1 \leq |X| \leq n$ for all subsets $X \subseteq \mathfrak{P}(n)$, hence the function r is well-defined. Next we show that for all $x < y$ have $r(x) < r(y)$. However, if x is a proper refinement of y, then clearly $|x| > |y|$, so $r(x) < r(y)$.

Now assume there is a maximal chain $\pi_1 < ... < \pi_t$ in Π_m, and want to show that $t = m$. As the chain is maximal and Π_m has the largest element $[m]$, we may assume that $\pi_t = [m]$ and thus $r(\pi_t) = m - 1$. Similarly, we may assume that $p_1 = \{\{a\} \mid a \in [m]\}$ and so $r(\pi_1) = 0$. So it suffices to show that $r(\pi_i) = 1 + r(\pi_{i-1})$ for all $0 < i \leq t$.

Assume not, i.e. $|\pi_{i-1}| \geq 2 + |\pi_i|$ for some $0 < i \leq t$. Since π_{i-1} is a refinement of π_i, we find that

$$\pi_{i-1} = \{A_{11}, ..., A_{1m_1},\ ...,\ A_{n1}, ..., A_{nm_n}\} \quad \text{with} \quad \pi_i = \left\{\bigcup_{j \leq m_1} A_{1j}, ..., \bigcup_{j \leq m_n} A_{nj}\right\}$$

where $n = |\pi_i|$ and $\sum_j m_j = |\pi_{i-1}| \geq 2 + n$. Hence, there is one $s \leq n$ with $m_s \geq 3$ or two different $s, r \leq n$ with $m_s, m_r \geq 2$.

In the first case, consider

$$\tilde{\pi} := \{A_{jk} \mid j \neq s\} \cup \left\{A_{s1}, \bigcup_{2 \leq j \leq m_s} A_{sj}\right\}$$

and in the second case, consider

$$\tilde{\pi} := \{A_{jk} \mid j \neq s\} \cup \left\{\bigcup_{j \leq m_s} A_{sj}\right\}$$

Then $\pi_{i-1} < \tilde{\pi} < \pi_i$ and we found a longer chain

$$\pi_1 < ... < \pi_{i-1} < \tilde{\pi} < \pi_i < ... < \pi_t$$

which contradicts the assumed maximality. The claim follows. $\square$

Proposition 6. For the number $S(m,k) := |\{\pi \in \Pi_m \mid |\pi| = k\}|$ we have recursion

$$S(m,1) = 1 \quad \text{and} \quad S(m,k) = \sum_{n=1}^{m-1} \binom{m-1}{n} S(n, k-1)$$

where $1 \leq k \leq m$.

Proof. Use induction on k. The base case $k = 1$ is trivial, so let $k > 1$. For any partition $\pi \in \Pi_m$, denote by $A(1, \pi)$ the unique set $A \in \pi$ with $1 \in A$. We find

$$\begin{aligned}
S(m,k) =& |\{\pi \in \Pi_m \mid |\pi| = k\}| = \Big| \bigcup_{\emptyset \neq A \subseteq [m],\ 1 \in A} \{\pi \in \Pi_m \mid |\pi| = k, A(1,\pi) = A\} \Big| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_m \mid |\pi| = k, A(1,\pi) = A\}| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_m \mid |\pi \setminus \{A\}| = k-1, A(1,\pi) = A\}| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_{m-|A|} \mid |\pi| = k-1\}| \\
=& \sum_{i=1}^{m} \sum_{A \subseteq [m],\ |A|=i,\ 1 \in A} |\{\pi \in \Pi_{m-i} \mid |\pi| = k-1\} \\
=& \sum_{i=1}^{m} \binom{m-1}{i-1} S(m-i, k-1) = \sum_{i=0}^{m-1} \binom{m-1}{m-i-1} S(i, k-1) \\
=& \sum_{i=1}^{m-1} \binom{m-1}{i} S(i, k-1)
\end{aligned}$$

as for $k > 1$ have that $S(0, k) = 0$. $\square$

Example 7. Have for $m \geq 3$ that

$$S(m,2) = 2^{m-1} - 1 \quad \text{and} \quad S(m,3) = \frac{1}{2}(3^{m-1} - 2^m + 1)$$

Note that this is the partition into *exactly* k sets; I am not completely sure what the exercise description tells us to show, but if it is about the number of ways of decomposing $[m]$ into *at most* k sets, then have

$$\Sigma S(m,k)(2) = 2^{m-1} \quad \text{and} \quad \Sigma S(m,k)(3) = 3^{m-1} = \frac{1}{2} 3^{m-1} + \frac{1}{2}$$

for $m \geq 3$. Here $\Sigma S(m,k) = \sum_{l \leq k} S(m,l)$ is the number of partitions of $[m]$ into at most k sets.

Proof. By Prop. 6 have that for $m \geq 1$ we have $S(m,1) = 1$. Further, for $m \geq 2$ have

$$S(m,2) = \sum_{n=1}^{m-1} \binom{m-1}{n} 1 = 2^{m-1} - 1$$

Applying this once more yields for $m \geq 3$ that

$$\begin{aligned}
S(m,3) = \sum_{n=1}^{m-1} \binom{m-1}{n}(2^{n-1}-1) &= \frac{1}{2}\sum_{n=1}^{m-1}\binom{m-1}{n}2^n - \sum_{n=1}^{m-1}\binom{m-1}{n}1 \\
&= \frac{1}{2}\sum_{n=0}^{m-1}\binom{m-1}{n}2^n - \frac{1}{2} - \sum_{n=0}^{m-1}\binom{m-1}{n}1 + 1 \\
&= \frac{1}{2}(1+2)^{m-1} - 2^{m-1} + \frac{1}{2} = \frac{1}{2}(3^{m-1} - 2^m + 1)
\end{aligned}$$

□

Let m be even and consider $\mathcal{A} \subseteq \Pi_m$ of partitions of $[m]$ into two equally sized sets. Then

$$|\mathcal{A}| = \frac{1}{2}\binom{m}{m/2}$$

Proof. Consider the map

$$f : [m]^{(m/2)} \to \mathcal{A}, \quad A \mapsto \{A, [m] \setminus A\}$$

Then for A and $B \neq A, [m] \setminus A$ have that $\{A, [m] \setminus A\} \neq \{B, [m] \setminus B\}$. Conversely, A and $[m] \setminus A$ have the same image under f. This shows that f is 2-to-1 and thus

$$|\mathcal{A}| = \frac{1}{2}\left|[m]^{(m/2)}\right| = \frac{1}{2}\binom{m}{m/2}$$

□

Further have

$$|\partial\mathcal{A}| = \binom{m}{m/2}\left(2^{m/2-1} - 1\right)$$

Proof. Consider the map

$$g : \underbrace{\{(A,B,C) \mid A \in [m]^{(m/2)}, \{A,B,C\} \in \Pi_m\}}_{=:\mathcal{G}} \to \partial\mathcal{A}, \quad (A,B,C) \mapsto \{A,B,C\}$$

Note that g is well-defined, as for $A \in [m]^{(m/2)}, \{A,B,C\} \in \Pi_m$ have that $\pi := \{A,B,C\}$ is a partition in L_{m-3} that refines the partition $\{A, B \cup C\} \in \mathcal{A} \subseteq L_{m-2}$. Further, it is easy to see that g is surjective, as every partition $\pi \in \partial\mathcal{A}$ satisfies $|\pi| = 3$ and has some $A \in \pi$ with $|A| = m/2$.

To complete the proof, we investigate to what "extend g is injective".

Assume $g(A,B,C) = g(A',B',C')$. As $B, C, B', C' \neq \emptyset$ and $|B \cup C| = |B' \cup C'| = m/2$, we see that $A = A'$. Further, we must then have that $\{B,C\} = \{B',C'\}$, hence $(A,B,C) = (A',B',C')$ or $(A,B,C) = (A',C',B')$ (clearly $B \neq C, B' \neq C'$). This shows that the map g is 2-to-1.

Using this, we find

$$
\begin{aligned}
|\partial\mathcal{A}| =& \frac{1}{2}|\mathcal{G}| = \frac{1}{2}\sum_{A\in[m]^{(m/2)}} |\{(B,C) \mid \{B,C\} \text{ partition of } [m]\setminus A\}| \\
=& \frac{1}{2}\sum_{A\in[m]^{(m/2)}} 2S(m/2,2) = \sum_{A\in[m]^{(m/2)}} 2^{m/2-1}-1 = \binom{m}{m/2}\left(2^{m/2-1}-1\right)
\end{aligned}
$$

□

We can now plug this into the second condition of Prop. 3 to see

$$
\begin{aligned}
\frac{|\partial\mathcal{A}|}{|L_{m-3}|} &= \frac{\binom{m}{m/2}\left(2^{m/2-1}-1\right)}{S(m,3)} = \frac{\binom{m}{m/2}\left(2^{m/2-1}-1\right)}{\frac{1}{2}\left(3^{m-1}-2^m+1\right)} \\
&\sim \frac{2^{m+m/2}}{\sqrt{m}(3^{m-1}-2^m)} \sim \frac{3\left(2^{3/2}\right)^m}{\sqrt{m}3^m} = \frac{3}{\sqrt{m}}c^m
\end{aligned}
$$

and

$$
\frac{|\mathcal{A}|}{|L_{m-2}|} = \frac{\frac{1}{2}\binom{m}{m/2}}{S(m,2)} = \frac{\binom{m}{m/2}}{2^m-2} \sim \frac{2^m}{\sqrt{m}2^m} = \frac{1}{\sqrt{m}}
$$

for some $0 < c < 1$ (here $\sim$ means asymptotically equivalent as $m \to \infty$). In particular, find for all $\epsilon > 0$ and sufficiently large m that

$$
\frac{|\partial\mathcal{A}|}{|L_{m-3}|} < \epsilon\frac{|\mathcal{A}|}{|L_{m-2}|}
$$

and so the conditions of Prop. 3 are not satisfied.

5 Part V

Example 8. Let $\mathcal{P}_{k,d} := \{0,...,k\}^d$ partially ordered by elementwise ordering. Then $\mathcal{P}_{k,d}$ is a graded poset with rank function

$$
r : \mathcal{P}_{k,d} \to \{0,...,kd\}, \quad a \mapsto \sum_i a_i
$$

Proof. First, we show that for $a < b$ have $r(a) < r(b)$. If $a < b$, then we find that $a_j \leq b_j$ for all j and $a_i \neq b_i$, so $a_i < b_i$ for some i. Hence

$$
r(a) = \sum_j a_j = a_i + \sum_{j\neq i} a_j \leq a_i + \sum_{j\neq i} b_j < b_i + \sum_{j\neq i} b_j = \sum_j b_j = r(b)
$$

Now assume that we have a maximal chain $a_1 < ... < a_t$ in $\mathcal{P}_{k,d}$. Since $\mathcal{P}_{k,d}$ has the smallest element $0 = (0,...,0)$ and the largest element $k = (k,...,k)$, we see that $p_1 = 0$

and $p_t = k$. We want to show $t = kd + 1$, so it suffices to show that $r(a_i) = r(a_{i-1}) + 1$ for all $0 < i \le t$.

Assume not, then

$$r(a_i) - r(a_{i-1}) = \sum_j a_{ij} - a_{(i-1)j} \ge 2$$

For all j we have $a_{ij} \ge a_{(i-1)j}$ and so there is $s \le d$ with $a_{is} - a_{(i-1)s} \ge 2$ or there are different $r, s \le d$ with $a_{ir} - a_{(i-1)r}, a_{is} - a_{(i-1)s} \ge 1$. In both cases, find

$$\tilde{a} \in \mathcal{P}_{k,d} \text{ defined by } \tilde{a}_j = \begin{cases} a_{(i-1)j} & \text{if } j \neq s \\ a_{(i-1)j} + 1 & \text{otherwise} \end{cases}$$

with $a_{i-1} < \tilde{a} < a_i$. However, this gives a longer chain

$$a_1 < ... < a_{i-1} < \tilde{a} < a_i < ... < a_t$$

which contradicts the assumed maximality. The claim follows. $\square$

Definition 9. Let P be a graded poset with layers $L_0, ..., L_n$. A symmetric chain in P is a chain $C \subseteq P$ such that there is $i \le n/2$ with

$$\forall 0 \le j \le n: \ C \cap L_j \neq \emptyset \ \Leftrightarrow \ i \le j \le n - i$$

A decomposition $\mathcal{C}$ of P into chains is called symmetric chain decomposition, if all $C \in \mathcal{C}$ are symmetric.

Note that this definition is compatible with the definition of a symmetric chain for the classical graded poset $\mathfrak{P}(n)$.

Proposition 10. Let $k, d \ge 1$. Then $\mathcal{P}_{k,d}$ has a symmetric chain decomposition.

Proof. Use induction on d. The base case $d = 1$ is trivial, so assume that $d > 1$. By induction hypothesis, have a symmetric chain decomposition $\mathcal{C}$ of $\mathcal{P}_{k,d-1}$. Now consider some $C = \{A_i, A_{i+1}, ..., A_{d(k-1)-i}\} \in \mathcal{C}, i < \frac{n}{2}$ with $r(A_j) = j$. For each $0 \le l \le k, n - 2i - 1$ define the chain

$$\tilde{C_l} := \{\underbrace{(A_{i+l}, j)}_{:=(a_1,...,a_{d-1},j) \text{ if } A_{i+l}=(a_1,...,a_{d-1})} \mid 0 \le j \le k - l\} \cup \{(A_j, k - l) \mid i + l < j \le d(k-1) - (i + l)\}$$

Then $\tilde{C_l}$ is symmetric w.r.t. $i + l$, as for $i + l \le j \le dk - (i + l)$ we have

$$\begin{aligned} i + l \le j \le k + i \quad &\Rightarrow \quad (A_{i+l}, j - i - l) \in \tilde{C_l}, \\ k + i < j \le dk - (i + l) \quad &\Rightarrow \quad (A_{j-k}, k - l) \in \tilde{C_l} \end{aligned}$$

Clearly the $\tilde{C_l}$ are chains and disjoint. Hence, we find a symmetric chain decomposition

$$\tilde{\mathcal{C}} := \left\{ \tilde{C_l} \;\middle|\; C \in \mathcal{C} \text{ symmetric chain w.r.t. } i, \ 0 \le l \le \min\{k, n - 2i - 1\} \right\}$$

of $\mathcal{P}_{k,d}$. $\square$

Note that for $k = 1$, this is exactly the proof that was done in the lecture to show that $\mathfrak{P}(d)$ has a symmetric chain decomposition.

6 Part VI

First of all, in Example 7 we have already shown that the poset Π_m does not satisfy the conditions from Prop. 3. Now we want to investigate whether $\mathcal{P}_{k,d}$ does.

Lemma 11. Let P be a poset with layers $L_0, ..., L_d$. Fix $0 < l \leq d$ and let $L_{l-1} = \{u_1, ..., u_m\}, L_l = \{v_1, ..., v_n\}$.

Assume there exists a matrix $A = (a_{ij}) \in \mathbb{Q}_{\geq 0}^{m \times n}$ such that [1]

$$1^T A 1 = 1$$

and for all i, j

$$e_i^T A 1 = \frac{1}{|L_0|} \quad \text{and} \quad 1^T A e_j = \frac{1}{|L_1|}$$

and $a_{ij} = 0$ whenever $u_i \not< v_j$ holds.

Then the conditions of Prop. 3 hold for the poset $L_{l-1} \cup L_l$. In particular, if the assumption holds for all $0 < l \leq d$, then the conditions of Prop. 3 hold for P.

Proof. Let $A = (a_{ij}) \in \mathbb{Q}_{\geq 0}^{m \times n}$ be this matrix. Note that a maximal chain in $L_{l-1} \cup L_l$ corresponds to a pair $u_i < v_j$.

Now consider a (finite) sequence of maximal chains $(C_k)_{k \in I}$ such that the chain corresponding to $u_i < v_j$ occurs $a_{ij}|I|$ times. By the assumption, we know that for each i have

$$|\{k \mid u_i \in C_k\}| = \sum_{u_i < v_j} a_{ij}|I| = \frac{1}{|L_0|}|I|$$

and so each element $u_i \in L_{l-1}$ is contained in the same number $|I|/|L_0|$ of chains C_k.

Similarly, for each J have

$$|\{k \mid v_j \in C_k\}| = \sum_{v_j > u_i} a_{ij}|I| = \frac{1}{|L_1|}|I|$$

The claim follows. □

This lemma does not seem super useful at first, but it gives us a way to check in time polynomial in $O(|P|)$ whether a given poset P satisfies the conditions from Prop. 3, by using an LP solver. Note that this is not easily possible when working directly with any of the three conditions defined before.

Using this, I could check that $\mathcal{P}_{k,d}$ indeed satisfies the conditions for reasonably sized $|\mathcal{P}_{k,d}|$ (up to $k = d = 6$). Note that already for these "small" numbers, we get quite huge posets $|\mathcal{P}_{k,d}|$. Hence, I believe that this is true for all $\mathcal{P}_{k,d}$.

My successful-looking attempt at proving this is based on the observation that elements of $\mathcal{P}_{k,d}$ "behave similarly" (in the sense that there is an element of the automorphism group $\mathrm{Aut}(\mathcal{P}_{k,d})$ mapping one to the other) if the have the same number of equally sized entries. This looks promising, as the proof for $\mathfrak{P}(n)$ relies on the fact that all elements in

[1] We use $1 = (1, ..., 1)^T$ as the vector consisting of 1 only here.

one layer of $\mathfrak{P}(n)$ "behave similarly" in this sense. In particular, note that Prop. 4 would already be applicable in such a case.

This is formalized in the next definition.

Definition 12. Let $P = \mathcal{P}_{k,d}$. Then define

$$P(n_0, ..., n_k) := \{x \in P \mid |\{i \mid x(i) = j\}| = n_j\}$$

Further, for any subset $F \subseteq P$ write

$$F(n_0, ..., n_k) := F \cap P(n_0, ..., n_k)$$

Lemma 13. The size of $|P(n_0, ..., n_k)|$ is given by the multinomial coefficient

$$|P(n_0, ..., n_k)| = \binom{d}{n_0, ..., n_k} = \frac{d!}{n_0!...n_k!}$$

if $n_0 + ... + n_k = d$ and $n_i \geq 0$ (otherwise, it is clearly zero).

Further, for all $n_0, ..., n_k \in \mathbb{Z}$ with $\sum n_i = d$ and all $a \in P(n_0, ..., n_k), 0 \leq \delta < k$ have

$$|\partial a \cap P(n_0, ..., n_\delta + 1, n_{\delta+1} - 1, ..., n_k)| = n_{\delta+1},$$
$$|\partial^+ a \cap P(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)| = n_\delta$$

even if any of the n_i are negative.

Proof. For $n_0 + ... + n_k = d$ and $n_i \geq 0$ we have

$$\begin{aligned}
&|P(n_0, ..., n_k)| = |\{x \in P \mid |\{i \mid x(i) = j\}| = n_j\}| \\
=& \sum_{A_0 \in [d]^{(n_0)}} |\{x \in \{0, ..., k\}^{[d] \setminus A_0} \mid |\{i \mid x(i) = j\}| = n_j\}| \\
=& \sum_{A_0 \in [d]^{(n_0)}} \sum_{A_1 \in ([d] \setminus A_0)^{(n_1)}} |\{x \in \{0, ..., k\}^{[d] \setminus A_0 \setminus A_1} \mid |\{i \mid x(i) = j\}| = n_j\}| \\
=& ... \\
=& \sum_{A_0 \in [d]^{(n_0)}} ... \sum_{A_k \in ([d] \setminus A_0 \setminus ... \setminus A_{k-1})^{(n_k)}} 1 \\
=& \sum_{A_0 \in [d]^{(n_0)}} ... \sum_{A_{k-1} \in ([d] \setminus A_0 \setminus ... \setminus A_{k-2})^{(n_k)}} \binom{d - n_0 - ... - n_{k-1}}{n_k} \\
=& ... \\
=& \binom{d}{n_0} \binom{d - n_0}{n_1} ... \binom{d - n_0 - ... - n_{k-1}}{n_k} = \prod_i \frac{(d - n_0 - ... - n_{i-1})!}{n_i!(d - n_0 - ... - n_i!)} \\
=& \frac{d!}{n_0!...n_k!} \prod_i \frac{(d - n_0 - ... - n_i)!}{(d - n_0 - ... - n_i)!} = \frac{d!}{n_0!...n_k!}
\end{aligned}$$

Further have for all $n_0 + ... + n_k = d$ and $a \in P(n_0, ..., n_k)$ that there is a bijection

$$\{i \in [d] \mid a(i) = \delta + 1\} \to \partial a \cap P(n_0, ..., n_\delta + 1, n_{\delta+1} - 1, ..., n_k),$$
$$i \mapsto f_{a,i} : j \mapsto \begin{cases} a(j) & \text{if } j \neq i \\ a(j) - 1 & \text{otherwise} \end{cases}$$

It is easy to see that the image of some i is indeed an element of ∂a and of $P(n_0, ..., n_\delta + 1, n_{\delta+1} - 1, ..., n_k)$. Clearly the function is also injective, and surjectivity can be shown by considering some $b \in \partial a \cap P(n_0, ..., n_\delta + 1, n_{\delta+1} - 1, ..., n_k)$ and showing by induction that

$$\{i \mid b(i) = j\} = \{i \mid a(i) = j\}$$

for all $j \neq \delta, \delta + 1$. Note that the set $\{i \in [d] \mid a(i) = \delta + 1\}$ has size exactly $n_{\delta+1}$ and so the claim follows.

The analogue statement for the upper shadow of a can be proven in the same way. □

Definition 14. Define the *core* of $\mathcal{P}_{k,d}$ as the poset

$$\mathcal{C}_{k,d} := \{(n_0, ..., n_k) \mid n_0 + ... + n_k = d\}$$

whose partial order is the transitive closure of the relation $\prec$ defined by

$$(n_0, ..., n_k) \prec (m_0, ..., m_k) \;:\Leftrightarrow\; \exists 0 \leq \delta < k : m_\delta = n_\delta - 1,\; m_{\delta+1} = n_{\delta+1} - 1,$$
$$\forall i \neq \delta, \delta + 1 : m_i = n_i$$

Lemma 15. The layers of $\mathcal{C}_{k,d}$ are

$$L_i = \{(n_0, ..., n_k) \mid \sum_j j n_j = i\}$$

Further, we have a surjective map

$$R : \mathcal{P}_{k,d} \to \mathcal{C}_{k,d}, \quad A \mapsto \big(|\{j \mid A(j) = i\}|\big)_{0 \leq i \leq k}$$

that preserves the ordering and the fibre of a point $(n_0, ..., n_k)$ is exactly $P(n_0, ..., n_k)$.

Proof. All of these statements are very easy to check. □

Proposition 16. Assume that $\mathcal{C}_{k,d}$ satisfies a "scaled variant" of condition (3) of Prop. 3, namely: There exists a (finite) sequence $C'_1, ..., C'_t$ of maximal chains such that for a chain C' chosen uniformly at random among them, we have

$$\Pr[(n_0, ..., n_k) \in C'] = \frac{|P(n_0, ..., n_k)|}{|L_{r(n_0, ..., n_k)}|}$$

Then $\mathcal{P}_{k,d}$ satisfies the conditions of Prop. 3.

Proof. Denote the layers of $\mathcal{P}_{k,d}$ by L_k and the layers of $\mathcal{C}_{k,d}$ by L'_k.

Now consider a chain C'_i from the assumption. Note that by Lemma 13, we see that for $(n_0, ..., n_k) \in C'_i$ have

- Every element in $P(n_0, ..., n_k)$ is covered by n_δ elements in P'_i
- Every element in $P(n_0, ..., n_k)$ covers $n_{\delta+1}$ elements in P'_i

where $P'_i \subseteq P$ is the graded poset that is given by the preimage of C'_i under the canonical map $\mathcal{P}_{k,d} \to \mathcal{C}_{k,d}$, i.e.

$$P'_i := \bigcup_{c \in C'_i} P(c) = R^{-1}(C'_i)$$

The $P(n_0, ..., n_k)$ are exactly the layers of the poset P'_i. Hence Prop. 4 yields that there is a sequence of maximal chains

$$C_{i,1}, ..., C_{i,t_i}$$

in the poset P'_i such that each element in $P(n_0, ..., n_k)$ is contained in $t_i/|P(n_0, ..., n_k)|$ of them. Note that $R(C_{i,j}) := \{R(x) \mid x \in C_{i,j}\} = C'_i$ holds.

Now consider a sequence $(C_{i_h,j_h})_{h \in I}$ of maximal chains in the poset A' that consists of the $C_{i,j}$ such that one $C_{i,j}$ occurs $\frac{1}{tt_i}|I|$ times. Let H be a random variable distributed uniformly on I. Observe that then i_H is a random variable distributed uniformly on $1, ..., t$.

Let $x \in L_k$ with $x \in P(n_0, ..., n_k)$. Then

$$\begin{aligned}
\Pr[x \in C_{i_H,j_H}] &= \sum_{1 \le i \le t} \Pr[x \in C_{i_H,j_H} \mid i_H = i] \ \Pr[i_H = i] \\
&= \sum_{1 \le i \le t} \underbrace{\Pr[x \in C_{i_H,j_H} \mid i_H = i]}_{=1/|P(n_0,...,n_k)| \text{ if } (n_0,...,n_k) \in C'_i} \frac{1}{t} \\
&= \sum_{\substack{1 \le i \le t \\ (n_0,...,n_k) \in C'_i}} \frac{1}{|P(n_0, ..., n_k)|t} \\
&= \frac{t}{|P(n_0, ..., n_k)|t} \underbrace{\Pr[(n_0, ..., n_k) \in C'_{i_H}]}_{=|P(n_0,...,n_k)|/|L_k| \text{ by assumption}} = \frac{1}{|L_k|}
\end{aligned}$$

Clearly, this shows that condition (3) of Prop. 3 is satisfied by $(C_{i_h,j_h})_{h \in I}$ for $\mathcal{P}_{k,d}$. $\square$

After some reflection, one can observe that we did not really use any properties of $\mathcal{P}_{k,d}$. The fact that all elements in $P(n_0, ..., n_k)$ cover resp. are covered by the same number of elements requires only that $P(n_0, ..., n_k)$ is an orbit under the action of $\mathrm{Aut}(P)$, i.e. $P(n_0, ..., n_k) = \mathrm{Aut}(P).a$ for any $a \in P(n_0, ..., n_k)$ (where the dot . denotes the group action of $\mathrm{Aut}(P)$ on P).

Hence, we get the following statement.

Proposition 17. Let P be a graded poset. Define the *core* $\mathcal{C}(P)$ of P as the set of orbits under the action of $\mathrm{Aut}(P)$

$$\mathcal{C}(P) := \{\mathrm{Aut}(P).x \mid x \in P\}$$

with the induced partial ordering from the elements of each orbit, i.e.

$$A < B \;:\Leftrightarrow\; \exists a \in A, b \in B : a < b$$

for $A, B \in \mathcal{C}(P)$. Note that then $\mathcal{C}(P)$ is a graded poset.

If there exists a sequence of maximal chains $C_1', ..., C_t'$ in $\mathcal{C}(P)$ such that for a chain C chosen uniformly at random from among them, have

$$\Pr[A \in C] = \frac{|A|}{|L_{r(A)}|}$$

for all $A \in \mathcal{C}(P)$, then P satisfies the conditions of Prop. 3.

Proof. Exactly as in Prop. 16. □

I still do not know how to use this to prove that $\mathcal{P}_{k,d}$ does indeed satisfy Prop. 3, but at least it provides a way for an even faster algorithm to test it for specific k, d. The idea is just that we can adjust the LP algorithm from Lemma 11 to check the scaled variant, and so we get a polynomial time algorithm in $|\mathcal{C}_{k,d}|$. In particular, we have the following.

Lemma 18. Let $L_{l-1} = \{u_1, ..., u_m\}$ and $L_l = \{v_1, ..., v_n\}$ be two consecutive layers of $\mathcal{C}_{k,d}$. Assume there exists a matrix $A = (a_{ij}) \in \mathbb{Q}_{\geq 0}^{m\times n}$ such that

$$1^T A 1 = 1$$

and for all i, j

$$e_i^T A 1 = \frac{|P(u_i)|}{\sum_{u \in L_{l-1}} |P(u)|} \quad \text{and} \quad 1^T A e_j = \frac{|P(v_j)|}{\sum_{v \in L_l} |P(v)|}$$

and $a_{ij} = 0$ whenever $u_i \not< v_j$ holds.

Then the assumption of Prop. 16 is fulfilled for $L_{l-1} \cup L_l$. In particular, if this holds for all $0 < l \leq kd$ then the assumption is fulfilled for the whole of $\mathcal{C}_{k,d}$ and it follows that $\mathcal{P}_{k,d}$ satisfies the conditions of Prop. 3.

Proof. Exactly as in Lemma 11. □

Using a simple implementation of this approach, I could check that for some arbitrarily chosen values of k, d up to $(k, d) = (8, 9)$, the poset $\mathcal{P}_{k,d}$ indeed satisfies the conditions of Prop. 3. Note that for this choice of k, d we already have that $|\mathcal{P}_{8,9}| = 134217728$ which is absolutely infeasible for more naive approaches.

After quite a long time trying to prove that this holds generally, I finally gave up and searched the literature. There I found that $\mathcal{P}_{k,d}$ is also called divisor lattice and indeed we have the following statement.

Proposition 19. $\mathcal{P}_{k,d}$ satisfies the conditions of Prop. 3, i.e. an LYM-inequality.

Proof. The proof is similar to [And87, Theorem 4.2.3]. However, I really like the proof and so we present it here in a somewhat modified form.

We show condition (2) of Prop. 3, so consider some pair of consecutive layers L_{i-1}, L_i. Let further $F \subseteq L_i$. We want to show that

$$|L_{i-1} \times F| \leq |L_i \times \partial F|$$

Let now $P' := \{0, ..., 2k\}^d$ and note that we have the decompositions

$$\begin{aligned} L_{i-1} \times F &= \bigcup_{M \in P', r(M) = 2i-1} \{(u, a) \in L_{i-1} \times F \mid u + a = M\} \quad \text{and} \\ L_i \times \partial F &= \bigcup_{M \in P', r(M) = 2i-1} \{(v, b) \in L_i \times \partial F \mid v + b = M\} \end{aligned}$$

Hence it suffices to show that

$$|\{(u, a) \in L_{i-1} \times F \mid u + a = M\}| \leq |\{(v, b) \in L_i \times \partial F \mid v + b = M\}|$$

for each $M \in P', r(M) = 2i - 1$.

To do so, we show the following, stronger claim by induction. □

Lemma 20. Let $k_1, ..., k_d > 0$ and consider the poset

$$P = \bigtimes_{i=1}^{d} \{0, ..., k_d\}$$

Let further $M = (M_1, ..., M_d)$ with $M_i \leq 2k_i$ of rank $\sum_i M_i = 2l - 1$ and $F \subseteq L_l$ such that M is an upper bound for F. Then

$$|\{(u, a) \in L_{l-1} \times F \mid u + a = M\}| \leq |\{(v, b) \in L_l \times \partial F \mid v + b = M\}|$$

Proof. Use induction on $\sum_i k_i$. The base case is trivial, so assume $\sum_i k_i > 1$.

Case 1 If there exists j such that $M_j < k_j$ then clearly

$$F, \partial F \in P' := \{0, ..., k_1\} \times ... \times \{0, ..., k_j - 1\} \times ... \times \{0, ..., k_d\}$$

and also $u \in P'$ for all $u \in L_{l-1}$ with $u + a = M$ for some $a \in F$. Hence the induction hypothesis yields

$$|\{(u, a) \in L_{l-1} \times F \mid u + a = M\}| \leq |\{(v, b) \in L_l \times \partial F \mid v + b = M\}|$$

Case 2 If there exists j such that $M_j > k_j \geq 1$, then consider

$$\begin{aligned} F' :=& \{a - e_j \mid a \in F, a(j) \neq 0\} \in \\ & P' := \{0, ..., k_1\} \times ... \times \{0, ..., k_j - 1\} \times ... \times \{0, ..., k_d\} \end{aligned}$$

Clearly $M - 2e_j \geq (k_1, ..., k_j - 1, ..., k_d) \geq a - e_j$ for all $a \in F$, and so $M - 2e_j$ is an upper bound for F'. By induction hypothesis, we now get that

$$|\{(u, a) \in L_{l-2} \times F' \mid u + a = M - 2e_j\}| \leq |\{(v, b) \in L_{l-1} \times \partial F' \mid v + b = M - 2e_j\}|$$

However, observe that the map

$$\begin{aligned} \{(u, a) \in L_{l-2} \times F' \mid u + a = M - 2e_j\} &\to \{(u, a) \in L_{l-1} \times F \mid u + a = M\} \\ (u, a) &\mapsto (u + e_j, a + e_j) \end{aligned}$$

is a bijection. The only nontrivial part is to see that it is surjective, so consider some $(u, a) \in L_{l-1} \times F$ with $u + a = M$. Then $u(j), a(j) \leq k_j < M(j)$ and so $a(j) = M(j) - u(j) \geq 1$, hence $a - e_j \in F'$. Clearly $u - e_j \in L_{l-2}$ and $u - e_j + a - e_j = M - 2e_j$. So (u, a) has the preimage $(u - e_j, a - e_j)$.

Similarly, see that the map

$$\begin{aligned} \{(u, a) \in L_{l-1} \times \partial F' \mid u + a = M - 2e_j\} &\to \{(u, a) \in L_l \times \partial F \mid u + a = M\} \\ (u, a) &\mapsto (u + e_j, a + e_j) \end{aligned}$$

is a bijection. Thus

$$|\{(u, a) \in L_{l-1} \times F \mid u + a = M\}| \leq |\{(v, b) \in L_l \times \partial F \mid v + b = M\}|$$

Case 3 If $M = (k_1, ..., k_d)$, see that $2l - 1 = \sum_i k_i$ and so the layers L_{l-1}, L_l must be the two unique middle layers. Now note that $|L_l| = |L_{l-1}|$ and so it suffices to show that $|\partial F| \geq |F|$. However, this follows directly by the symmetric chain decomposition of the poset P. We have proven the symmetric chain decomposition only for $k_1 = ... = k_d$, but the general version can be found in [And87]. □

There are two parts I find interesting in this proof.

- First of all, the idea to consider all $\bigtimes_i \{0, ..., k_i\}$ with possibly distinct k_i makes it easy to use induction, as it suffices to decrease just one coordinate to use the induction hypothesis.
- The decomposition

  $$L_{i-1} \times F = \bigcup_{M \in P', r(M) = 2i-1} \{(u, a) \in L_{i-1} \times F \mid u + a = M\}$$

  is just very clever, as it really transforms the set of tuples (u, a) into multiple sets that are more or less $\{a \in F \mid a \leq M\}$ and much easier to handle.

Interestingly, I have also found a variant of the LYM inequality ("pseudo-LYM inequality") for the poset Π_m that we studied earlier.

Proposition 21. Let $\mathcal{A} \subseteq \Pi_m$ be a nonempty antichain. Then define

$$a(n_1, ..., n_k) := |\{A \in \mathcal{A} \mid |A| = k,\ |\{S \in A \mid |S| = j\}| = n_j\}|$$

the number of partitions $A \in \mathcal{A}$ such that the classes in A have sizes $n_1, ..., n_k$. Then

$$\sum_{k \geq 1} \sum_{n_1, ..., n_k} \frac{a(n_1, ..., n_k)k!}{\binom{m}{n_1, ..., n_k}\binom{m-1}{k-1}} \leq 1$$

Proof. See [ES97]. □

Note that the total number of partitions π such that the classes in π have sizes $n_1, ..., n_k$ is given by an expression involving the multinomial coefficient, namely

$$|\{\pi \in \Pi_m \mid |\pi| = k, |\{S \in \pi \mid |S| = j\}| = n_j\}| = \binom{m}{n_1, ..., n_k} \underbrace{\prod_i \frac{1}{|\{j \mid n_j = i\}|!}}_{=:\gamma(n_1, ..., n_k)}$$

Intuitively, the reason is that there are $m!$ ways to order the elements $[m]$, and after splitting such a sequence in chunks of sizes $n_1, ..., n_k$, there are $n_1! \cdot ... \cdot n_k!$ ways to order the element within the chunks. Hence the multinomial coefficient

$$\binom{m}{n_1, ..., n_k}$$

gives the number of ways to distribute the elements of $[m]$ into ordered buckets of sizes $n_1, ..., n_k$. A partition corresponds to unordered buckets, i.e. swapping chunks of equal size does not change the partition. There are $1/\gamma(n_1, ..., n_k)$ ways to do so, and hence we get the expression mentioned above.

Furthermore, the number $a(n_1, ..., n_k)$ and the underlying set do not depend on the order of $n_1, ..., n_k$. Hence, the partitions $A \in \mathcal{A}$ such that the classes of A have sizes $n_1, ..., n_k$ might be counted multiple times. Note that the number of different orderings of $n_1, ..., n_k$ is given by $k!/\gamma(n_1, ..., n_k)$. Hence

$$\sum_{n_1, ..., n_k} \frac{1}{k!}\binom{m}{n_1, ..., n_k} = |L_{m-k}|$$

just counts all the partitions in one layer of Π_m.

Now note also that

$$\binom{m-1}{k-1}$$

is exactly the number of different sequences $n_1 \leq ... \leq n_k$ summing to m by the "stars and bars" method. Then one can see that the expression

$$\sum_{k \geq 1} \sum_{n_1, ..., n_k} \frac{a(n_1, ..., n_k)k!}{\binom{m}{n_1, ..., n_k}\binom{m-1}{k-1}} = \sum_{k \geq 1} \frac{1}{\binom{m-1}{k-1}} \sum_{n_1, ..., n_k} \frac{a(n_1, ..., n_k)}{\frac{1}{k!}\binom{m}{n_1, ..., n_k}}$$

is very similar to the LYM inequality. The only difference is that elements elements corresponding to different class sizes $n_1, ..., n_k$ might receive slightly different weight, depending on how many duplicate n_i occur in that sequence.

Finally, there is still one interesting thought I had while thinking about the above inequality. Note that the decomposition

$$\Pi_m = \bigcup_{k \geq 1,\ n_1 \leq ... \leq n_k} \{\pi \in \Pi_m \mid |\pi| = k, |\{S \in \pi \mid |S| = j\}| = n_j\}$$

corresponds exactly to the core $\mathcal{C}(\Pi_m)$ of Π_m as discussed above. In particular, the decomposition is exactly

$$\Pi_m = \bigcup \{\mathrm{Aut}(\Pi_m).\pi \mid \pi \in \Pi_m\}$$

7 Appendix (Code described in Part VI)

To find the computational results mentioned above, I have implemented the described algorithm in Rust. For this, I use my own library containing a collection of math functions as well as Mosek LP solver [ApS19] and its rust binding library. After installing the Mosek binaries manually, it therefore suffices to add

```
feanor_la = {
    git = "https://github.com/feanorTheElf/FeanorLA.git"
}
mosek = "0.2"
```

to the dependencies of your `Cargo.toml` file.

After all dependencies are present, the following code will yield the result.

```
extern crate feanor_la;
extern crate mosek;

use feanor_la::combinatorics::iters::*;
use feanor_la::combinatorics::bitset::*;
use feanor_la::la::mat::*;
use feanor_la::algebra::rat::*;

///
/// Uses the LP solver to find x >= 0
/// with Ax = b or states that this does
/// not exist.
///
fn find_feasible_solution<M, V>(
    A: Matrix<M, r64>,
    b: Vector<V, r64>
) -> Option<Vector<VectorOwned<f64>, f64>>
```

```
    where M: MatrixView<r64>, V: VectorView<r64>
{
    let cast = |x: r64| (x.num() as f64) / (x.den() as f64);

    let mut env = mosek::Env::new().unwrap();
    let mut task = env.task().unwrap();
    task.append_vars(A.col_count() as i32).unwrap();
    task.append_cons(A.row_count() as i32).unwrap();

    // initialize variable bounds and target function
    for j in 0..A.col_count() {
        task.put_c_j(j as i32, 1.).unwrap();
        task.put_var_bound(
            j as i32, mosek::MSK_BK_LO, 0.0, f64::INFINITY
        ).unwrap();
    }
    // initialize conditions
    for i in 0..A.row_count() {
        // we need sparse format
        let indices = A.row(i).iter().enumerate()
            .filter(|(_, x)| **x != r64::ZERO)
            .map(|(i, _)| i as i32)
            .collect::<Vec<_>>();

        let values = A.row(i).iter()
            .filter(|x| **x != r64::ZERO)
            .map(|x| cast(*x)).collect::<Vec<_>>();

        task.put_a_row(i as i32, &indices, &values).unwrap();
        task.put_con_bound(
            i as i32, mosek::MSK_BK_FX,
            cast(*b.at(i)),
            cast(*b.at(i))
        ).unwrap();
    }
    task.optimize().unwrap();
    let state = task.get_sol_sta(0).unwrap();
    if state == mosek::MSK_SOL_STA_PRIM_INFEAS_CER {
        // problem is infeasible
        return None;
    };
    // parse the result
    let mut result = (0..A.col_count())
        .map(|_| 0.0)
```

```
        .collect::<Vec<_>>();
    task.get_xx(0, &mut result).unwrap();
    return Some(Vector::from_fn(result.len(), |i| result[i]));
}

///
/// Returns a matrix defining conditions on a
/// vector indexed by u < v, u in L, v in U
/// where U, V are two consecutive layers in
/// a graded poset.
///
/// Builds the matrix (b | A) with an mxn matrix A
/// (m is the number of pairs u < v) representing
/// the conditions:
///  - sum over all u < v is 1
///  - for u in L the sum over all u < v is
///    constant
///  - for v in U the sum over all u < v is
///    constant
///
fn build_matrix<T>(
    L: &Vec<T>,
    U: &Vec<T>,
    edges: Vec<(usize, usize)>
) -> Matrix<MatrixOwned<r64>, r64> {
    let e = edges.len();
    let mut A: Matrix<_, r64> = Matrix::zero(
        L.len() + U.len() - 1, e + 1
    ).into_owned();
    {
        // add the conditions for the lower layer L
        let mut L_conds = A.submatrix_mut(..(L.len() - 1), 1..);
        for (j, (u, _)) in edges.iter().enumerate() {
            if *u == 0 {
                L_conds.col_mut(j).assign(
                    Vector::constant(L.len() - 1, -r64::ONE)
                );
            } else {
                *L_conds.at_mut(u - 1, j) = r64::ONE;
            }
        }
    }
    {
        // add the conditions for the upper layer U
```

```
            let mut U_conds = A.submatrix_mut(
                (L.len() - 1)..(A.row_count() - 1), 1..
            );
            for (j, (_, v)) in edges.iter().enumerate() {
                if *v == 0 {
                    U_conds.col_mut(j).assign(
                        Vector::constant(U.len() - 1, -r64::ONE)
                    );
                } else {
                    *U_conds.at_mut(v - 1, j) = r64::ONE;
                }
            }
        }
        A.row_mut(A.row_count() - 1).assign(
            Vector::constant(e + 1, r64::ONE)
        );
        return A;
    }

    fn simple_lp_alg(k: usize, d: usize) {

        // a function to find the edges between two layers
        let edges = |L: &[Box<[usize]>], U: &[Box<[usize]>]|
            cartesian_product(0..L.len(), 0..U.len())
            .filter(|&(a, b)| {
                (0..L[a].len()).all(|j| L[a][j] <= U[b][j])
            }).collect::<Vec<_>>();

        let superset = (0..d).map(|_| k).collect::<Vec<_>>();
        let layers: Vec<_> = (0..(k * d)).map(|i|
            multiset_combinations(&superset, i, clone_slice)
            .collect::<Vec<_>>()
        ).collect();

        for i in 1..=(k * d) {
            let edges = edges(&layers[i - 1], &layers[i]);
            let A = build_matrix(&layers[i - 1], &layers[i], edges);
            let solution = find_feasible_solution(
                A.submatrix(.., 1..),
                A.col(0)
            );
            println!(
                "(k, d) = ({}, {}), i = {}:        sol: {}",
                k, d, i, solution.is_some()
```

```
        );
    }
}

///
/// Returns a matrix defining conditions on a
/// vector indexed by u < v, u in L, v in U
/// where U, V are two consecutive layers in
/// a graded poset.
///
/// Builds the matrix (b | A) with an mxn matrix A
/// (m is the number of pairs u < v) representing
/// the conditions:
///  - sum over all pairs u < v is 1
///  - for u in L, the sum over all u < v
///    is fraction(u)
///  - for v in U, the sum over all u < v
///    is fraction(v)
///
fn build_symmetry_using_matrix<T, F>(
    L: &Vec<T>,
    U: &Vec<T>,
    edges: Vec<(usize, usize)>,
    mut fraction: F
) -> Matrix<MatrixOwned<r64>, r64>
    where F: FnMut(&T) -> r64
{
    let e = edges.len();
    let mut A: Matrix<_, r64> = Matrix::zero(
        L.len() + U.len() + 1, e + 1
    ).into_owned();
    {
        for u in 0..L.len() {
            *A.at_mut(u, 0) = fraction(&L[u]);
        }
        let mut L_conds = A.submatrix_mut(..L.len(), 1..);
        for (j, (u, _)) in edges.iter().enumerate() {
            *L_conds.at_mut(*u, j) = r64::ONE;
        }
    }
    {
        for v in 0..U.len() {
            *A.at_mut(L.len() + v, 0) = fraction(&U[v]);
        }
```

```
        let mut U_conds = A.submatrix_mut(
            L.len()..(A.row_count() - 1), 1..
        );
        for (j, (_, v)) in edges.iter().enumerate() {
            *U_conds.at_mut(*v, j) = r64::ONE;
        }
    }
    A.row_mut(A.row_count() - 1).assign(
        Vector::constant(e + 1, r64::ONE)
    );
    return A;
}

fn factorial(x: u128) -> u128 {
    if x == 0 { 1 } else { x * factorial(x - 1) }
}

fn multinomial_coefficient(d: usize, n: &[usize]) -> usize {
    (
        factorial(d as u128) /
        n.iter()
            .map(|x| factorial(*x as u128))
            .product::<u128>()
    ) as usize
}

fn symmetry_using_lp_alg(k: usize, d: usize) {
    let bar_positions = (0..=d).map(|_| k).collect::<Vec<_>>();

    // this generates all (n_0, ..., n_k) with sum(n_i) = d
    let core_poset = multiset_combinations(
        &bar_positions[..], k,
        |pos: &[usize]| {
            let mut result = pos.iter().enumerate()
                .flat_map(|(i, c)| (0..*c).map(move |_| i))
                .chain(std::iter::once(d))
                .collect::<Vec<_>>();
            for i in (1..=k).rev() {
                result[i] -= result[i - 1];
            }
            return result;
        }
    );
```

```rust
// group the elements of C_{k, d} into layers
let mut layers = (0..=(k * d))
    .map(|_| Vec::new())
    .collect::<Vec<_>>();

for p in core_poset {
    let layer = p.iter().enumerate()
        .map(|(i, x)| i * *x)
        .sum::<usize>();

    layers[layer].push(p);
}

// a function to find the edges between two layers
let edges = |L: &Vec<Vec<usize>>, U: &Vec<Vec<usize>>|
    cartesian_product(0..L.len(), 0..U.len())
    .filter(|&(a, b)| {
        (0..k).any(|delta|
            U[b][delta] + 1 == L[a][delta] &&
            U[b][delta + 1] == L[a][delta + 1] + 1
        )
    }).collect::<Vec<_>>();

for i in 1..=(k * d) {
    let edges = edges(&layers[i - 1], &layers[i]);
    let A = build_symmetry_using_matrix(
        &layers[i - 1], &layers[i], edges, |element| {
            let layer = element.iter().enumerate()
                .map(|(i, x)| i * *x)
                .sum::<usize>();
            let total_size = layers[layer].iter()
                .map(|e| multinomial_coefficient(d, e))
                .sum::<usize>();
            let size = multinomial_coefficient(d, element);
            return r64::new(size as i64, total_size as i64);
        }
    );
    let solution = find_feasible_solution(
        A.submatrix(.., 1..), A.col(0)
    );
    println!(
        "(k, d) = ({}, {}), i = {}:         sol: {}",
        k, d, i, solution.is_some()
    );
```

```
    }
}

fn main() {
    symmetry_using_lp_alg(8, 9);
}
```

References

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[ES97] Péter L. Erdős and László A. Székely. "Pseudo-LYM Inequalities and AZ Identities". In: *Advances in Applied Mathematics* 19.4 (1997), pp. 431–443.