Miniproject - Combinatorics

Simon Pohmann

We use the convention that $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.

1 Part I

Proposition 1. Let P be a graded poset in which every maximal chain has length $n+1$. Then the function

$$r : P \to \{0, ..., n\}, \quad x \mapsto \max\{k \in \mathbb{N} \mid \exists a_1, ..., a_k \in P : \ a_1 < a_2 < ... < a_k < x\}$$

is well-defined and the unique function with $x < y$ implies $r(x) < r(y)$ for all $x, y \in P$. We call it the *rank function* of P.

Proof. Clearly r is well-defined, as for every increasing sequence $a_1 < ... < a_k < x$, we have the chain $C := \{a_1, ..., a_k, x\}$ of size $k+1$. Hence by assumption, $k \leq n$ and so $r(x)$ is finite and in $\{0, ..., n\}$.

Further, consider $x < y$ in P. We have a sequence $a_1 < ... < a_{r(x)} < x$ by definition of r. It follows that there is an increasing sequence $a_1 < ... < a_{r(x)} < x < y$ and so $r(y) \geq r(k) + 1 > r(x)$.

Lastly, assume there was another function $r' : P \to \{0, ..., n\}$ with this property. Consider any $x \in P$. By definition of r, there is an increasing sequence $a_1 < ... < a_{r(x)} < x$ in P. Now consider a maximal chain C containing the chain $\{a_1, ..., a_{r(x)}, x\}$.

Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$ and $x = b_j$. Note that we have the increasing sequence $b_1 < ... < b_{j-1} < b_j = x$ and so by the definition of r, find $j \leq r(x) + 1$. On the other hand, have $a_1, ..., a_{r(x)} \in C$ and thus $j = r(x) + 1$, i.e.

$$a_1 = b_1, \ ..., \ a_{r(x)} = b_{r(x)}, \ x = b_{r(x)+1}$$

As $b_i < b_{i+1}$, we know that $r'(b_i) < r'(b_{i+1})$ and inductively, we see $r'(b_i) \geq i - 1$. However, $r'(b_{n+1}) \leq n$ and thus $r'(b_i) = i-1$. Finally it follows that $r'(x) = r'(b_{r(x)+1}) = r(x)$. $\square$

Now we will show some basic properties of the rank function.

Proposition 2. Let P be a graded poset of maximal rank n with rank function r. Then

- $x \in P$ is minimal iff $r(x) = 0$ and maximal iff $r(x) = n$.

- If $x < y$ and $r(x) + 1 \neq r(y)$ then there is $z \in P$ with $x < z < y$.
- If $x < y$ then there is an increasing sequence $x < a_1 < ... < a_{r(y)-r(x)-1} < y$ in P.

Proof. For (i), let $x \in P$ be minimal. Then there is no increasing sequence $a_1 < x$ in P, so $r(x) \leq 0$. Conversely, let $r(x) = 0$. Assume there was $y \in P$ with $y < x$, then $r(y) < r(x) = 0$, a contradiction. The analogous statement for maximal elements is proved in the same way.

For (iii), consider $x < y$ in P. Then the chain $\{x, y\}$ is contained in a maximal chain $C \subseteq P$. Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$.

Hence we find $r(b_i) < r(b_{i+1})$ and so inductively that $r(b_i) \geq i-1$ and $r(b_i) \leq i-1$ since $r(b_1) \geq 0$ and $r(b_{n+1}) \leq n$. It follows that $r(b_i) = i - 1$ and so $x = b_{r(x)+1}, y = r(y) + 1$. Therefore, we have a chain

$$x < b_{r(x)+2} < ... < b_{r(y)} < y$$

of length $r(y) - r(x) - 1$. Statement (ii) follows directly, as in this setting, have $r(y) \geq r(x) + 2$ and so $r(y) - r(x) - 1 \neq 0$. □

2 Part II

Proposition 3. For a graded poset P with layers $L_0, ..., L_n$ the following statements are equivalent:

- For every antichain $A \subseteq P$ have
$$\sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} \leq 1$$
- For each $1 \leq i \leq n$ and $F \subseteq L_i$ have
$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$
where
$$\partial F := \{a \in L_{i-1} \mid \exists b \in F : \; a \leq b\}$$
- There exists a (nonempty) sequence of maximal chains $C_1, ..., C_t$ such that for all $1 \leq i \leq n$ we have:
$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

Proof. We show (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). For convenience of notation, write

$$A_i := \bigcup_{i \leq j \leq n} A \cap L_j$$

(ii) $\Rightarrow$ (i) Define sets

$$G_n := A \cap L_n \quad \text{and} \quad G_i := \partial G_{i+1} \cup (A \cap L_i) \text{ for } 0 \leq i < n$$

We show by induction that for $a \in G_i$ there is some $b \in A_i$ with $a \leq b$ and that we have the inequality

$$\sum_{j=i}^{n} \frac{|A \cap L_j|}{|L_j|} \leq \frac{|G_i|}{|L_i|}$$

The base case is trivial, so let $i < n$. Consider some $a \in G_i$. If $a \in \partial G_{i+1}$, then there is $b \in G_{i+1}$ with $a \leq b$. By induction hypothesis, have $c \in A_{i+1} \supseteq A_i$ with $b \leq c$ and thus $a \leq c$. Otherwise, find $a \in A \cap L_i$ and so $a \leq a$ with $a \in A \cap L_i \subseteq A_i$.

To show the inequality, note that ∂G_{i+1} and $A \cap L_i$ are disjoint. Indeed, if $a \in \partial G_{i+1} \cap A \cap L_i$ then there is $b \in G_{i+1}$ with $a \leq b$, and further by induction hypothesis there is $c \in A_{i+1}$ with $a \leq b \leq c$. However, $a \in L_i$ and so $a \neq c$. So we found $a \leq c$ comparable elements in the antichain A, a contradiction.

So we get

$$\begin{aligned}
\sum_{j=i}^{n} \frac{|A \cap L_j|}{|L_j|} &= \frac{|A \cap L_i|}{|L_i|} + \sum_{j=i+1}^{n} \frac{|A \cap L_j|}{|L_j|} \\
&\leq \frac{|A \cap L_i|}{|L_i|} + \frac{|G_{i+1}|}{|L_{i+1}|} \leq \frac{|A \cap L_i|}{|L_i|} + \frac{|\partial G_{i+1}|}{|L_i|} \\
&= \frac{|A \cap L_i| + |\partial G_{i+1}|}{|L_i|} = \frac{|G_i|}{|L_i|}
\end{aligned}$$

Finally, we have that $G_0 \subseteq L_0$, so

$$\sum_{j=0}^{n} \frac{|A \cap L_j|}{|L_j|} \leq \frac{|G_i|}{|L_i|} \leq 1$$

(i) $\Rightarrow$ (iii) $\square$