Miniproject - Combinatorics

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We use the convention that $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.

1 Part I

Proposition 1. Let P be a graded poset in which every maximal chain has length $n+1$. Then the function

$$r : P \to \{0, ..., n\}, \quad x \mapsto \max\{k \in \mathbb{N} \mid \exists a_1, ..., a_k \in P : \ a_1 < a_2 < ... < a_k < x\}$$

is well-defined and the unique function with $x < y$ implies $r(x) < r(y)$ for all $x, y \in P$. We call it the *rank function* of P.

Proof. Clearly r is well-defined, as for every increasing sequence $a_1 < ... < a_k < x$, we have the chain $C := \{a_1, ..., a_k, x\}$ of size $k+1$. Hence by assumption, $k \leq n$ and so $r(x)$ is finite and in $\{0, ..., n\}$.

Further, consider $x < y$ in P. We have a sequence $a_1 < ... < a_{r(x)} < x$ by definition of r. It follows that there is an increasing sequence $a_1 < ... < a_{r(x)} < x < y$ and so $r(y) \geq r(k) + 1 > r(x)$.

Lastly, assume there was another function $r' : P \to \{0, ..., n\}$ with this property. Consider any $x \in P$. By definition of r, there is an increasing sequence $a_1 < ... < a_{r(x)} < x$ in P. Now consider a maximal chain C containing the chain $\{a_1, ..., a_{r(x)}, x\}$.

Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$ and $x = b_j$. Note that we have the increasing sequence $b_1 < ... < b_{j-1} < b_j = x$ and so by the definition of r, find $j \leq r(x) + 1$. On the other hand, have $a_1, ..., a_{r(x)} \in C$ and thus $j = r(x) + 1$, i.e.

$$a_1 = b_1, \ ..., \ a_{r(x)} = b_{r(x)}, \ x = b_{r(x)+1}$$

As $b_i < b_{i+1}$, we know that $r'(b_i) < r'(b_{i+1})$ and inductively, we see $r'(b_i) \geq i - 1$. However, $r'(b_{n+1}) \leq n$ and thus $r'(b_i) = i - 1$. Finally it follows that $r'(x) = r'(b_{r(x)+1}) = r(x)$. $\square$

Now we will show some basic properties of the rank function.

Proposition 2. Let P be a graded poset of maximal rank n with rank function r. Then

- $x \in P$ is minimal iff $r(x) = 0$ and maximal iff $r(x) = n$.

- If $x < y$ and $r(x) + 1 \neq r(y)$ then there is $z \in P$ with $x < z < y$.
- If $x < y$ then there is an increasing sequence $x < a_1 < ... < a_{r(y)-r(x)-1} < y$ in P.

Proof. For (i), let $x \in P$ be minimal. Then there is no increasing sequence $a_1 < x$ in P, so $r(x) \leq 0$. Conversely, let $r(x) = 0$. Assume there was $y \in P$ with $y < x$, then $r(y) < r(x) = 0$, a contradiction. The analogous statement for maximal elements is proved in the same way.

For (iii), consider $x < y$ in P. Then the chain $\{x, y\}$ is contained in a maximal chain $C \subseteq P$. Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$.

Hence we find $r(b_i) < r(b_{i+1})$ and so inductively that $r(b_i) \geq i-1$ and $r(b_i) \leq i-1$ since $r(b_1) \geq 0$ and $r(b_{n+1}) \leq n$. It follows that $r(b_i) = i - 1$ and so $x = b_{r(x)+1}, y = r(y) + 1$. Therefore, we have a chain

$$x < b_{r(x)+2} < ... < b_{r(y)} < y$$

of length $r(y) - r(x) - 1$. Statement (ii) follows directly, as in this setting, have $r(y) \geq r(x) + 2$ and so $r(y) - r(x) - 1 \neq 0$. $\square$

2 Part II

Proposition 3. For a graded poset P with layers $L_0, ..., L_n$ the following statements are equivalent:

- For every antichain $A \subseteq P$ have
$$\sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} \leq 1$$
- For each $1 < i \leq n$ and $F \subseteq L_i$ have
$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$
where
$$\partial F := \{a \in L_{i-1} \mid \exists b \in F: \ a \leq b\}$$
- There exists a (nonempty) sequence of maximal chains $C_1, ..., C_t$ such that for all $1 \leq i \leq n$ we have:
$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

Proof. Show the directions (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii) Use induction on n. Again, the base case is trivial, just take chains $\{x\}$ for each $x \in A = L_0$. So assume that $n > 0$. The set

$$A' := \bigcup_{i=0}^{n-1} L_i$$

is a graded poset of maximal rank $n-1$, and hence there are maximal chains $C_1, ..., C_t \subseteq A'$ such that for all $1 \leq i \leq n-1$ have

$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

by induction hypothesis. Let now $V_1, V_2 := [t] \times L_n$ (treat them as disjoint) and consider the bipartite graph $G := (V_1 \sqcup V_2, E)$ where E is defined as follows:

$$\{\underbrace{(i,a)}_{\in V_1}, \underbrace{(j,b)}_{\in V_2}\} \in E \;\Leftrightarrow\; \max C_j < a$$

We use Hall's theorem to show that G has a perfect matching.

Consider any $W = \{(i_1, a_1), ..., (i_w, a_w)\} \subseteq V_1$ and let $F = \{a_1, ..., a_w\}$. Then

$$W \subseteq [t] \times F \;\Rightarrow\; |W| \leq t|F|$$

By choice of $C_1, ..., C_t$, we have that the number of j with $x \in C_j$ is the same for all $x \in L_{n-1}$, say k. Since the C_i are maximal chains, each intersects L_{n-1} in exactly one element. So have bijection

$$\{(x,i) \mid x \in L_{n-1}, 1 \leq i \leq t, x \in C_i\} \to \{i \mid 1 \leq i \leq t\}, \quad (x,i) \mapsto i$$

where the set on the left-hand side has size $k|L_{n-1}|$. It follows that $k|L_{n-1}| = t$.

Since $\max C_j \in L_{n-1}$ for all j, we have

$$\begin{aligned} N(W) =& \{(j,b) \mid b \in L_n, \exists a \in F : \max C_j < a\} \\ =& L_n \times \{j \mid \exists a \in F : \max C_j < a\} \\ =& L_n \times \{j \mid \max C_j \in \partial F\} \end{aligned}$$

and so by the above

$$|N(W)| = |L_n| \sum_{\max C_j \in \partial F} 1 = |L_n| \sum_{x \in \partial F} \sum_{x \in C_j} 1 = |L_n| \sum_{x \in \partial F} k = |L_n||\partial F|k$$

Together with the assumption that $|\partial F||L_n| \geq |F||L_{n-1}|$ we see that

$$|W| \leq t|F| = k|L_{n-1}||F| \leq k|\partial F||L_n| = |N(W)|$$

So Hall's theorem yields a perfect matching $M \subseteq E$ from V_1 to V_2. As $|V_1| = |V_2|$, this is already a 1-to-one correspondence.

Now consider the sets

$$C'_m := C_j \cup \{a\} \text{ where } m = \{\underbrace{(i,a)}_{\in V_1}, \underbrace{(j,b)}_{\in V_2}\} \in M$$

These are chains, as $\max C_j < a$ for each C'_m. Further, for each $x, y \in L_i, i < n$ have that

$$|\{C'_m \mid x \in C'_m\}| = |L_n \times \{j \mid x \in C_j\}| = |L_n \times \{j \mid y \in C_j\}| = |\{C'_m \mid y \in C'_m\}|$$

as M is a matching from V_2 to V_1. Finally, for all $x \in L_n$ we have that

$$|\{C'_m \mid x \in C'_m\}| = |\{m \in M \mid \exists i, v \in V_2 : m = \{(i,x), v\}\}| = |\{(i,x) \mid (i,x) \in V_1\}| = t$$

as M is a matching from V_1 to V_2.

(iii) ⇒ (i) Let $C_1, ..., C_t$ be a sequence of maximal chains given by the assumption. For $1 \leq i \leq n$ let k_i be the number of different j such that a fixed element $x \in L_i$ is contained in exactly the C_j. By assumption, this does not depend on x.

As in the direction above, we see that $k_i|L_i| = t$ because each C_j intersects L_i in exactly one element, so there is a bijection

$$\{(x,j) \mid x \in L_i, 1 \leq j \leq t, x \in C_j\} \to \{j \mid 1 \leq j \leq t\}, \quad (x,j) \mapsto j$$

and the set on the left has size $k_i|L_i|$.

Since each C_j is a chain and A is an antichain, we find that A and C_j intersect in at most one element. So

$$\begin{aligned} t \geq & |\{j \mid A \cap C_j \neq \emptyset\}| = \sum_{a \in A} |\{j \mid a \in C_j\}| = \sum_{i=0}^{n} \sum_{a \in A \cap L_i} |\{j \mid a \in C_j\}| \\ = & \sum_{i=0}^{n} \sum_{a \in A \cap L_i} k_i = \sum_{i=0}^{n} k_i |A \cap L_i| = \sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} t \end{aligned}$$

The claim follows by canceling t.

(i) ⇒ (ii) Consider $A := F \cup (L_{i-1} \setminus \partial F)$. This is clearly an antichain, as for $x \in F, y \in L_{i-1} \setminus \partial F$ have $y \not< x$. However, $r(y) < r(x)$ so also $x \not< y$, thus x and y are incomparable. Clearly elements from the same layer are incomparable.

Thus, the assumption yields that

$$\frac{|L_{i-1}| - |\partial F|}{|L_{i-1}|} + \frac{|F|}{|L_i|} = \frac{|A \cap L_{i-1}|}{|L_{i-1}|} + \frac{|A \cap L_i|}{|L_i|} = \sum_{j=0}^{n} \frac{|A \cap L_j|}{|L_j|} \leq 1$$

This gives

$$1 + \frac{|F|}{|L_i|} \leq 1 + \frac{|\partial F|}{|L_{i-1}|}$$

and the claim follows. □

3 Part III

We even show the slightly stronger stronger statement that the implication to the second condition of Prop.3 already holds "locally", i.e. for each layer separately.

Proposition 4. Let P be a graded poset with layers $L_0, ..., L_n$ and let $0 < i \leq n$ such that

- Each element $a \in L_i$ covers the same number of elements in L_{i-1}
- Each element $a \in L_{i-1}$ is covered by the same number of elements in L_i

Then for each $F \subseteq L_i$ have

$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$

In particular, if this is true for all i, then all the equivalent conditions from Prop. 3 follow.

Proof. Assume that for all $a \in L_i$

$$|\{b \in L_{i-1} \mid a \text{ covers } b\}| = k \in \mathbb{N}$$

and for all $b \in L_{i-1}$

$$|\{a \in L_i \mid a \text{ covers } b\}| = l \in \mathbb{N}$$

Double-counting pairs $(a, b) \in L_i \times L_{i-1}$ such that a covers b yields $k|L_i| = l|L_{i-1}|$. Now consider $F \subseteq L_i$. Note that by definition of ∂F, we have for all $a \in F$ that

$$\{b \in \partial F \mid a \text{ covers } b\} = \{b \in L_{i-1} \mid a \text{ covers } b\} = \partial\{a\} \subseteq \partial F$$

Again by double-counting pairs $(a, b) \in F \times \partial F$ such that a covers b, we find

$$\begin{aligned}
k|F| =& \sum_{a \in F} k = \sum_{a \in F} |\{b \in L_{i-1} \mid a \text{ covers } b\}| = \sum_{a \in F} |\{b \in \partial F \mid a \text{ covers } b\}| \\
=& |\{(a, b) \in F \times \partial F \mid a \text{ covers } b\}| = \sum_{b \in \partial F} |\{a \in F \mid a \text{ covers } b\}| \\
\leq& \sum_{b \in \partial F} |\{a \in L_i \mid a \text{ covers } b\}| = \sum_{b \in \partial F} l = l|\partial F|
\end{aligned}$$

Hence

$$\frac{|\partial F|}{|L_{i-1}|} = \frac{l|\partial F|}{l|L_{i-1}|} \leq \frac{k|F|}{k|L_i|} = \frac{|F|}{|L_i|}$$

$\square$

4 Part IV

Example 5. Let $\Pi_m = \{\pi \subseteq \mathfrak{P}(n) \mid \pi \text{ partition}\}$ be the poset of partitions ordered by refinement. Then Π_m is a graded poset with rank function

$$r : \Pi_m \to \{0, ..., n-1\}, \quad \pi \mapsto n - |\pi|$$

Proof. First of all, have $1 \leq |X| \leq n$ for all subsets $X \subseteq \mathfrak{P}(n)$, hence the function r is well-defined. Next we show that for all $x < y$ have $r(x) < r(y)$. However, if x is a proper refinement of y, then clearly $|x| > |y|$, so $r(x) < r(y)$.

Now assume there is a maximal chain $\pi_1 < ... < \pi_t$ in Π_m. As the chain is maximal and Π_m has the largest element $[n]$, we may assume that $\pi_t = [n]$ and thus $r(\pi_t) = n-1$. Similarly, we may assume that $p_1 = \mathfrak{P}(n)$ and so $r(\pi_1) = 0$. We have to show that $t = n$, and so it suffices to show that $r(\pi_i) = 1 + r(\pi_{i-1})$ for all $0 < i \leq t$.

Assume not, i.e. $|\pi_{i-1}| \geq 2 + |\pi_i|$ for some $0 < i \leq t$. Since π_{i-1} is a refinement of π_i, we find that

$$\pi_{i-1} = \{A_{11}, ..., A_{1m_1},\ ...,\ A_{n1}, ..., A_{nm_n}\} \quad \text{with} \quad \pi_i = \Big\{ \bigcup_{j \leq m_1} A_{1j}, ..., \bigcup_{j \leq m_n} A_{nj} \Big\}$$

where $n = |\pi_i|$ and $\sum_j m_j = |\pi_{i-1}| \geq 2 + n$. Hence, there is one $s \leq n$ with $m_s \geq 3$ or two different $s, r \leq n$ with $m_s, m_r \geq 2$.

In the first case, consider

$$\tilde{\pi} := \{A_{jk} \mid j \neq s\} \cup \Big\{A_{s1}, \bigcup_{2 \leq j \leq m_s} A_{sj}\Big\}$$

and in the second case, consider

$$\tilde{\pi} := \{A_{jk} \mid j \neq s\} \cup \Big\{ \bigcup_{j \leq m_s} A_{sj} \Big\}$$

Then $\pi_{i-1} < \tilde{\pi} < \pi_i$ and we found a longer chain

$$\pi_1 < ... < \pi_{i-1} < \tilde{\pi} < \pi_i < ... < \pi_t$$

which contradicts the assumed maximality. The claim follows. $\square$

Proposition 6. For the number $S(m,k) := |\{\pi \in \Pi_m \mid |\pi| = k\}|$ we have recursion

$$S(m, 1) = 1 \quad \text{and} \quad S(m,k) = \sum_{n=1}^{m-1} \binom{m-1}{n} S(n, k-1)$$

where $1 \leq k \leq m$.

Proof. Use induction on k. The base case $k = 1$ is trivial, so let $k > 1$. For any partition $\pi \in \Pi_m$, denote by $A(1, \pi)$ the unique set $A \in \pi$ with $1 \in A$. We find

$$\begin{aligned}
S(m,k) =& |\{\pi \in \Pi_m \mid |\pi| = k\}| = \Big| \bigcup_{\emptyset \neq A \subseteq [m],\ 1 \in A} \{\pi \in \Pi_m \mid |\pi| = k, A(1,\pi) = A\} \Big| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_m \mid |\pi| = k, A(1,\pi) = A\}| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_m \mid |\pi \setminus \{A\}| = k-1, A(1,\pi) = A\}| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_{m-|A|} \mid |\pi| = k-1\}| \\
=& \sum_{i=1}^{m} \sum_{A \subseteq [m],\ |A|=i,\ 1 \in A} |\{\pi \in \Pi_{m-i} \mid |\pi| = k-1\} \\
=& \sum_{i=1}^{m} \binom{m-1}{i-1} S(m-i, k-1) = \sum_{i=0}^{m-1} \binom{m-1}{m-i-1} S(i, k-1) \\
=& \sum_{i=1}^{m-1} \binom{m-1}{i} S(i, k-1)
\end{aligned}$$

as for $k > 1$ have that $S(0, k) = 0$. □

Example 7. Have for $m \geq 3$ that

$$S(m,2) = 2^{m-1} - 1 \quad \text{and} \quad S(m,3) = \frac{1}{2}(3^{m-1} - 2^m + 1)$$

Note that this is the partition into *exactly* k sets; I am not completely sure what the exercise description tells us to show, but if it is about the number of ways of decomposing $[m]$ into *at most* k sets, then have

$$\Sigma S(m,k)(2) = 2^{m-1} \quad \text{and} \quad \Sigma S(m,k)(3) = 3^{m-1} = \frac{1}{2}3^{m-1} + \frac{1}{2}$$

for $m \geq 3$. Here $\Sigma S(m,k) = \sum_{l \leq k} S(m,l)$ is the number of partitions of $[m]$ into at most k sets.

Proof. By Prop. 6 have that for $m \geq 1$ we have $S(m,1) = 1$. Further, for $m \geq 2$ have

$$S(m,2) = \sum_{n=1}^{m-1} \binom{m-1}{n} 1 = 2^{m-1} - 1$$

Applying this once more yields for $m \geq 3$ that

$$
\begin{aligned}
S(m,3) = \sum_{n=1}^{m-1} \binom{m-1}{n} (2^{n-1} - 1) = \frac{1}{2} \sum_{n=1}^{m-1} \binom{m-1}{n} 2^n - \sum_{n=1}^{m-1} \binom{m-1}{n} 1 \\
= \frac{1}{2} \sum_{n=0}^{m-1} \binom{m-1}{n} 2^n - \frac{1}{2} - \sum_{n=0}^{m-1} \binom{m-1}{n} 1 + 1 \\
= \frac{1}{2}(1+2)^{m-1} - 2^{m-1} + \frac{1}{2} = \frac{1}{2}(3^{m-1} - 2^m + 1)
\end{aligned}
$$

□

Let m be even and consider $\mathcal{A} \subseteq \Pi_m$ of partitions of $[m]$ into two equally sized sets. Then

$$
|\mathcal{A}| = \frac{1}{2} \binom{m}{m/2}
$$

Proof. Consider the map

$$
f : [m]^{(m/2)} \to \mathcal{A}, \quad A \mapsto \{A, [m] \setminus A\}
$$

Then for A and $B \neq A, [m] \setminus A$ have that $\{A, [m] \setminus A\} \neq \{B, [m] \setminus B\}$. Conversely, A and $[m] \setminus A$ have the same image under f. This shows that f is 2-to-1 and thus

$$
|\mathcal{A}| = \frac{1}{2} \left| [m]^{(m/2)} \right| = \frac{1}{2} \binom{m}{m/2}
$$

□

Further have

$$
|\partial \mathcal{A}| = \binom{m}{m/2} \left(2^{m/2-1} - 1 \right)
$$

Proof. Consider the map

$$
g : \underbrace{\{(A,B,C) \mid A \in [m]^{(m/2)}, \{A,B,C\} \in \Pi_m\}}_{=:\mathcal{G}} \to \partial \mathcal{A}, \quad (A,B,C) \mapsto \{A,B,C\}
$$

Note that g is well-defined, as for $A \in [m]^{(m/2)}, \{A,B,C\} \in \Pi_m$ have that $\pi := \{A,B,C\}$ is a partition in L_{m-3} that refines the partition $\{A, B \cup C\} \in \mathcal{A} \subseteq L_{m-2}$. Further, it is easy to see that g is surjective, as every partition $\pi \in \partial \mathcal{A}$ satisfies $|\pi| = 3$ and has some $A \in \pi$ with $|A| = m/2$.

To complete the proof, we investigate to what "extend g is injective".

Assume $g(A,B,C) = g(A',B',C')$. As $B, C, B', C' \neq \emptyset$ and $|B \cup C| = |B' \cup C'| = m/2$, we see that $A = A'$. Further, we must then have that $\{B,C\} = \{B',C'\}$, hence $(A,B,C) = (A',B',C')$ or $(A,B,C) = (A',C',B')$ (clearly $B \neq C, B' \neq C'$). This shows that the map g is 2-to-1.

Using this, we find

$$
\begin{aligned}
|\partial\mathcal{A}| =& \frac{1}{2}|\mathcal{G}| = \frac{1}{2}\sum_{A\in[m]^{(m/2)}} |\{(B,C) \mid \{B,C\} \text{ partition of } [m]\setminus A\}| \\
=& \frac{1}{2}\sum_{A\in[m]^{(m/2)}} 2S(m/2,2) = \sum_{A\in[m]^{(m/2)}} 2^{m/2-1}-1 = \binom{m}{m/2}\left(2^{m/2-1}-1\right)
\end{aligned}
$$

$\square$

We can now plug this into the second condition of Prop. 3 to see

$$
\begin{aligned}
\frac{|\partial\mathcal{A}|}{|L_{n-3}|} &= \frac{\binom{m}{m/2}\left(2^{m/2-1}-1\right)}{S(m,3)} = \frac{\binom{m}{m/2}\left(2^{m/2-1}-1\right)}{\frac{1}{2}\left(3^{m-1}-2^m+1\right)} \\
&\sim \frac{2^{m+m/2}}{\sqrt{n}(3^{m-1}-2^m)} \sim \frac{3\left(2^{3/2}\right)^m}{\sqrt{n}3^m} = \frac{3}{\sqrt{n}}c^m
\end{aligned}
$$

and

$$
\frac{|\mathcal{A}|}{|L_{n-2}|} = \frac{\frac{1}{2}\binom{m}{m/2}}{S(m,2)} = \frac{\binom{m}{m/2}}{2^m-2} \sim \frac{2^m}{\sqrt{n}2^m} = \frac{1}{\sqrt{n}}
$$

for some $0 < c < 1$ (here $\sim$ means asymptotically equivalent as $m \to \infty$). In particular, find for sufficiently large m that

$$
\frac{|\partial\mathcal{A}|}{|L_{n-3}|} < \frac{|\mathcal{A}|}{|L_{n-2}|}
$$

and so the conditions of Prop. 3 are not satisfied.

5 Part V

Example 8. Let $\mathcal{P}_{k,d} := \{0,...,k\}^d$ partially ordered by elementwise ordering. Then $\mathcal{P}_{k,d}$ is a graded poset with rank function

$$
r : \mathcal{P}_{k,d} \to \{0,...,kd\}, \quad a \mapsto \sum_i a_i
$$

Proof. First, we show that for $a < b$ have $r(a) < r(b)$. If $a < b$, then we find that $a_j \leq b_j$ for all j and $a_i \neq b_i$, so $a_i < b_i$ for some i. Hence

$$
r(a) = \sum_j a_j = a_i + \sum_{j\neq i} a_j \leq a_i + \sum_{j\neq i} b_j < b_i + \sum_{j\neq i} b_j = \sum_j b_j = r(b)
$$

Now assume that we have a maximal chain $a_1 < ... < a_t$ in $\mathcal{P}_{k,d}$. Since $\mathcal{P}_{k,d}$ has the smallest element $0 = (0,...,0)$ and the largest element $k = (k,...,k)$, we see that $p_1 = 0$

and $p_t = k$. We want to show $t = kd + 1$, so it suffices to show that $r(a_i) = r(a_{i-1}) + 1$ for all $0 < i \leq t$.

Assume not, then

$$r(a_i) - r(a_{i-1}) = \sum_j a_{ij} - a_{(i-1)j} \geq 2$$

For all j we have $a_{ij} \geq a_{(i-1)j}$ and so there is $s \leq d$ with $a_{is} - a_{(i-1)s} \geq 2$ or there are different $r, s \leq d$ with $a_{ir} - a_{(i-1)r}, a_{is} - a_{(i-1)s} \geq 1$. In both cases, find

$$\tilde{a} \in \mathcal{P}_{k,d} \text{ defined by } \tilde{a}_j = \begin{cases} a_{(i-1)j} & \text{if } j \neq s \\ a_{(i-1)j} + 1 & \text{otherwise} \end{cases}$$

with $a_{i-1} < \tilde{a} < a_i$. However, this gives a longer chain

$$a_1 < ... < a_{i-1} < \tilde{a} < a_i < ... < a_t$$

which contradicts the assumed maximality. The claim follows. $\square$

Definition 9. Let P be a graded poset with layers $L_0, ..., L_n$. A symmetric chain in P is a chain $C \subseteq P$ such that there is $i \leq n/2$ with

$$\forall 0 \leq j \leq n : \; C \cap L_j \neq \emptyset \;\Leftrightarrow\; i \leq j \leq n - i$$

A decomposition $\mathcal{C}$ of P into chains is called symmetric chain decomposition, if all $C \in \mathcal{C}$ are symmetric.

Note that this definition is compatible with the definition of a symmetric chain for the classical graded poset $\mathfrak{P}(n)$.

Proposition 10. Let $k, d \geq 1$. Then $\mathcal{P}_{k,d}$ has a symmetric chain decomposition.

Proof. $\square$

Proposition 11. Let $M(k,d,n) := |\{x \in \mathcal{P}_{k,d} \mid r(x) = n\}|$. Then for $k, d \geq 1$ we have the recurrence relation

$$M(k,1,n) = \begin{cases} 1 & \text{if } 0 \leq n \leq k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad M(k,d,n) = \sum_{i=0}^{k} M(k,d-1,n-i)$$

Corollary 12. Let $k, d \geq 1$. Then

$$M(k,2,n) := \begin{cases} 0 & \text{if } n > 2k \\ k+1 & \text{if } k < n \leq 2k \\ n+1 & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n < 0 \end{cases}$$

and

$$M(k,3,n) := \begin{cases} 0 & \text{if } n > 3k \\ (k+1)^2 & \text{if } 2k < n \leq 3k \\ (k+1)(2n+2-k)/2 & \text{if } k < n \leq 2k \\ \binom{n+1}{2} & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n < 0 \end{cases}$$

Proof. Just use the recurrence relation twice. The only nontrivial computation is if $k < n \leq 2k$ and $d = 3$; We find

$$\begin{aligned} &\sum_{i=0}^{k} M(k,2,n) = \sum_{i=0}^{n} M(k,2,n) = \sum_{i=0}^{n-2k-1} 0 + \sum_{i=n-2k}^{n-k-1} (k+1) + \sum_{i=n-k}^{n} (i+1) \\ =&(k+1)(n-k-1-(n-2k)+1) + \frac{(n+1)(n+2)}{2} - \frac{(n-k)(n-k+1)}{2} \\ =&k(k+1) + kn + n + 1 - \frac{k^2}{2} + \frac{k}{2} = \frac{(k+1)(2n+2-k)}{2} \end{aligned}$$

$\square$