The University of Oxford

MSc (Mathematics and Foundations of Computer Science)

Combinatorics

Michaelmas Term 2021

The steps of (each) miniproject are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.

Please write or print on one side of the paper only.

A *graded poset* is a poset in which all maximal chains have the same number of elements.

1. Let P be a graded poset in which every maximal chain has $n+1$ elements. Show that there is a unique function $r : P \to \{0, \ldots, n\}$ such that if $x < y$ then $r(x) < r(y)$. We refer to this as the *rank function* for P. We say that x has *rank* i if $r(x) = i$ and we refer to the set $L_i := \{x \in P : r(x) = i\}$ as the *ith level of* P.

   Show that

   (a) Minimal elements have rank 0 and maximal elements have rank n.

   (b) If $x < y$ and $r(x) \neq r(y) - 1$ then there is z such that $x < z < y$.

   (c) If $x < y$ then there is a chain from x to y with $r(y) - r(x) + 1$ elements.

2. Let P be a graded poset with levels $L_0, \ldots, L_n$. Show that the following three statements are equivalent:

   (a) For every antichain A in P, we have

$$\sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} \leq 1. \tag{1}$$

(b) For each $1 \leq i \leq n$ and every subset F of L_i, we have

$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|},$$

where

$$\partial F = \{a \in L_{i-1} : a \leq b \text{ for some } b \in F\}.$$

(c) There is a sequence $\mathcal{C} = (C_1, \ldots, C_t)$ of maximal chains in P such that, for every i, every element of L_i belongs to the same number of chains from $\mathcal{C}$.

[If you cannot prove equivalence, prove as many implications between (a), (b) and (c) as you can.]

3. Let P be a graded poset with levels $L_0, \ldots, L_n$. We say that x *covers* y if $x > y$ and $r(x) = r(y) + 1$. Suppose that, for every i, every element of L_i covers the same number of elements of L_{i-1} and every element of L_i is covered by the same number of elements of L_{i+1}. Prove that P satisfies (1).

4. Let Π_m be the set of partitions of $[m]$, with $\pi \leq \rho$ if π is a refinement of ρ.

(a) Show that Π_m is a graded poset and describe its rank function.

(b) Let $S(m, k)$ be the number of partitions of $[m]$ into k sets. Find expressions for $S(m, 2)$ and $S(m, 3)$.

(c) Let $\mathcal{A}$ be the set of partitions $X \cup Y$ of $[m]$ such that $|X| = |Y| = m/2$. Show that if m is even then

$$|\mathcal{A}| = \frac{1}{2}\binom{m}{m/2}.$$

How big is $\partial\mathcal{A}$?

5. For $k, d \geq 1$, let $\mathcal{P}_{k,d} = \{0, \ldots, k\}^d$, where for $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ we set $a \leq b$ if $a_i \leq b_i$ for every i. (For instance, if $k = 1$ we get a poset isomorphic to $\mathcal{P}(d)$, ordered by set inclusion.)

(a) Show that $\mathcal{P}_{k,d}$ is a graded poset and describe its rank function.

(b) Come up with a sensible definition for a *symmetric chain* in a graded poset. Does $\mathcal{P}_{k,d}$ have a decomposition into symmetric chains?

6. Investigate whether the posets Π_m and $\mathcal{P}_{k,d}$ satisfy an analogue of (1).