Miniproject - Algebraic Geometry

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1 Part I

Definition 1. Let V be a vector space. Then define the d-th exterior power as

$$\bigwedge^d(V) := V^{\otimes d} \ / \ \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \mathrm{span}\big\{v \otimes v' + v' \otimes v \ \big| \ v, v' \in V\big\} \otimes V^{\otimes(d-i-1)}$$

Use the notation $v_1 \wedge ... \wedge v_d := [v_1 \otimes ... \otimes v_d] \in \bigwedge^k(V)$.

Lemma 2. Let $v_1, ..., v_d \in V$. Have for $\pi \in S_d$ that

$$v_{\pi(1)} \wedge ... \wedge v_{\pi(k)} = \mathrm{sgn}(\pi)(v_1 \wedge ... \wedge v_d)$$

Furthermore if $v_i = v_j$ for some $i \neq j$, then

$$v_1 \wedge ... \wedge v_d = 0$$

Proof. Note that

$$u \wedge v \wedge v' \wedge w = -(u \wedge v' \wedge v \wedge w)$$

for all $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{(d-i-1)}(V), v, v' \in V$.

Every $\pi \in S_d$ has a decomposition $\pi = \xi_1 ... \xi_n$ into transpositions ξ_i. Applying this inductively, we find that

$$v_1 \wedge ... \wedge v_d = \mathrm{sgn}(\xi_i...\xi_n)(v_{(\xi_i...\xi_n)(1)} \wedge ... v_{(\xi_i...\xi_n)(k)})$$

and so

$$v_1 \wedge ... \wedge v_d = \mathrm{sgn}(\pi)(v_{\pi(1)} \wedge ... \wedge v_{\pi(k)})$$

Furthermore, we find that

$$u \wedge v \wedge v \wedge w = -(u \wedge v \wedge v \wedge w) = 0$$

must be zero. Hence, if $v_1, ..., v_d \in V$ with $v_i = v_j$ for some $i \neq j$, then there is a permutation $\pi \in S_d$ with $\pi(1) = i, \pi(2) = j$ and

$$v_1 \wedge ... \wedge v_d = (\mathrm{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge ... \wedge v_{\pi(k)}) = \mathrm{sgn}(\pi)0 = 0$$

$\square$

Lemma 3 (1a)**.** Let $\dim(V) \leq 3$. Then every element of $\bigwedge^k(V)$ is decomposable.

Proof. Now let v_1, v_2, v_3 be a set of generators of V. Consider $u_1 = \sum \lambda_i v_i, u_2 = \sum_i \mu_i v_i, u_3 = \sum_i \rho_i v_i$. Then by applying Lemma 2, we see that

$$u_1 \wedge u_2 = \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{=\,0 \text{ if } i\,=\,j} = \sum_{i<j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i>j} \lambda_i \mu_j (v_i \wedge v_j)$$

$$= \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)(v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3)$$

$$= \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases}$$

and

$$u_1 \wedge u_2 \wedge u_3 = \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{=\,0 \text{ unless } i,j,l \text{ pairwise distinct}}$$

$$= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)})$$

$$= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \mathrm{sgn}(\pi) (v_1 \wedge v_2 \wedge v_3)$$

$$= (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \mathrm{sgn}(\pi)$$

are decomposable. Further, it is easy to see from Lemma 2 that $\bigwedge^k(V) = \{0\}$ for $k \geq 4$, which is trivially decomposable. $\square$

Example 4 (1b)**.** Consider $V = k^4$. Then the element $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$ is not decomposable.

Proof. Assume it was, then there are $a, b \in k^4$ such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i<j} (a_i b_j - a_j b_i)(e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1, \ a_3 b_4 - a_4 b_3 = 1, \ a_i b_j - a_j b_i = 0 \text{ for all } (i,j) \neq (1,2), (3,4)$$

Clearly $a_1 b_2 \neq 0$ or $a_2 b_1 \neq 0$. Similarly, have $a_3 b_4 \neq 0$ or $a_4 b_3 \neq 0$. As all expressions are symmetric w.r.t swapping a_1, b_2 with a_2, b_1 and a_3, b_4 with a_4, b_3, we may assume wlog that $a_1 b_2, a_3 b_4 \neq 0$.

Have $a_1 b_4 = a_4 b_1$ and $a_2 b_4 = a_4 b_2$. We know that $a_1 b_4 \neq 0$ and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \ \Rightarrow \ a_2 b_1 = a_1 b_2$$

This contradicts $a_1 b_2 - a_2 b_1 = 1$. $\square$

Lemma 5. Let $A = (a_{ij}) \in \mathrm{GL}_d(k)$ and $v_1, ..., v_d \in V$. Then

$$\left(\sum_j a_{1j} v_j \right) \wedge ... \wedge \left(\sum_j a_{dj} v_j \right) = \det(A)(v_1 \wedge ... \wedge v_d)$$

Proof. By a direct computation using Lemma 2, we find

$$\left(\sum_j a_{ij} v_j \right) \wedge ... \wedge \left(\sum_j a_{dj} v_j \right) = \sum_{j_1, ..., j_d} a_{1j_1} ... a_{dj_d}(v_{j_1} \wedge ... \wedge v_{j_d})$$

$$= \sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)}(v_{\pi(1)} \wedge ... \wedge v_{\pi(d)})$$

$$= \sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)} \mathrm{sgn}(\pi)(v_1 \wedge ... \wedge v_d)$$

$$= (v_1 \wedge ... \wedge v_d) \sum_{\pi \in S_d} \mathrm{sgn}(\pi) \prod_{j=1}^{d} a_{j\pi(j)} = \det(A)(v_1 \wedge ... \wedge v_d)$$

where the last equality holds due to the Leibniz determinant formula. $\qquad \square$

Lemma 6. For $v_1, ..., v_d \in V$ have

$$v_1 \wedge ... \wedge v_d = 0 \iff v_1, ..., v_d \text{ linearly dependent}$$

Proof. For the direction $\Leftarrow$, assume that $v_1, ..., v_d$ are not independent. Then there is a nonzero vector $a_1 \in k^d$ with $\sum a_{1i} v_i = 0$. Clearly, we can extend a_1 to a basis $a_1, ..., a_d$ of k^d, which gives a matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$.

However by Lemma 5 we now get

$$0 = 0 \wedge \left(\sum_j a_{2j} v_j \right) \wedge ... \wedge \left(\sum_j a_{dj} v_j \right) = \left(\sum_j a_{1j} v_j \right) \wedge ... \wedge \left(\sum_j a_{dj} v_j \right)$$

$$= \det(A)(v_1 \wedge ... \wedge v_d)$$

and so $v = v_1 \wedge ... \wedge v_d = 0$ as $\det(A) \neq 0$.

Direction $\Rightarrow$ TODO $\qquad \square$

Lemma 7. Let $v \in V$ and $u \in \bigwedge^{d-1} U$ for a linear subspace $U \leq V$. If $v \wedge u \in \bigwedge^d U$ then $v \in U$ or $u = 0$.

Proof. TODO $\qquad \square$

Lemma 8 (1c). Let d be even. An element $\omega \in \bigwedge^d V$ is decomposable if and only if $\omega \wedge \omega \in \bigwedge^{2d} V$ is zero.

Proof. The direction $\Rightarrow$ even holds generally. Assume $\omega = v_1 \wedge ... \wedge v_d$. Then

$$\omega \wedge \omega = v_1 \wedge ... \wedge v_d \wedge v_1 \wedge ... \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let $\omega = v_1 + ... + v_t$ for linearly independent decomposable vectors $v_i \in \bigwedge^2 V$. Then

$$0 = \omega \wedge \omega = \sum_{i,j} v_i \wedge v_j = \sum_{i<j} (v_i \wedge v_j) + (v_j \wedge v_i)$$

$$= \sum_{i<j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \left(\sum_{j>i} v_j \right)$$

Here we used that the permutation $(1 \; 2d)(2 \; (2d-1))...(d \; (d+1)) \in S_{2d}$ has always sign 1 (since d is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \left(\bigwedge^2 \mathrm{span}\{v_2, ..., v_t\} \right) \setminus \{0\}$$

find

$$u_1, u_2 \in \mathrm{span}\{v_2, ..., v_t\}$$

In particular, we know that

$$v_1 \wedge \left(\sum_{j>i} v_j \right) \in \bigwedge^2 \mathrm{span}\{v_2, ..., v_t\}$$

and so $v_1 \in \mathrm{span}\{v_2, ..., v_t\}$ unless $\sum_{j>i} v_j = 0$ by Lemma 7. We assumed that the v_i are linearly independent, so the former would give a contradiction. Hence $\sum_{j>i} v_j = 0$ and thus $t = 1$, i.e. $\omega = v_1$ is decomposable. $\qquad\square$

2 Part II

In this part, we want to consider the connection of external powers to the Grassmanian. First of all, assume there are two bases $v_1, ..., v_d$ and $u_1, ..., u_d$ of a d-dimensional vector space U. Then there exists a basis change matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$ with

$$u_i = \sum_j a_{ij} v_j$$

So by Lemma 5, it follows that

$$u_1 \wedge ... \wedge u_d = \det(A)(v_1 \wedge ... \wedge v_d)$$

As $v_1, ..., v_d$ resp. $u_1, ..., u_d$ are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge ... \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge ... \wedge u_d \neq 0$$

by Lemma 6. Hence they have well-defined images $[v_1 \wedge ... \wedge v_d]$ resp. $[u_1 \wedge ... \wedge u_d]$ in the projective space $\mathbb{P}(\bigwedge^d V)$. By the above, find

$$[v_1 \wedge ... \wedge v_d] = [u_1 \wedge ... \wedge u_d]$$

This allows us to study the Grassmanian $\mathrm{Gr}(d, V)$ of a fixed vector space V.

Definition 9. Define the map

$$\phi : \mathrm{Gr}(d, V) \to \mathbb{P}(\textstyle\bigwedge^d V), \quad \mathrm{span}\{v_1, ..., v_d\} \mapsto [v_1 \wedge ... \wedge v_d]$$

which is well-defined by Lemma 5 as described above.

Lemma 10 (1a)**.** We have

$$\mathrm{im}\phi = D := \{[v] \in \mathbb{P}(\textstyle\bigwedge^d V) \mid v \text{ decomposable}\}$$

Proof. First of all, note that the set D is well-defined, as v is decomposable if and only if λv is decomposable, for all $\lambda \in k^*$.

By definition of ϕ, we can directly observe that $\mathrm{im}\phi \subseteq D$. So consider an element $[v] \in D$. As v is decomposable, it follows that $v = v_1 \wedge ... \wedge v_d$ for $v_i \in V$. Not it suffices to show that the v_i are linearly independent, then clearly $\mathrm{span}\{v_1, ..., v_d\}$ is a well-defined d-dimensional vector subspace of V, thus an element of $\mathrm{Gr}(d, V)$.

This follows directly from Lemma 6. $\qquad\square$

Definition 11. Let $\mathrm{Gr}(d, n) := \mathrm{Gr}(d, k^n)$.

In the lecture, we considered an embedding of $\mathrm{Gr}(d, n)$ into projective space given by minors of the basis matrix. This corresponds to the following definition.

Definition 12. Define the map

$$\rho : \mathrm{Gr}(d, n) \to \mathbb{P}\left(k^{\{1,...,n\}^{(d)}}\right) \cong \mathbb{P}^{\binom{n}{d}-1},$$

$$\mathrm{span}\{v_1, ..., v_d\} \mapsto \left[\det \begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d} \end{pmatrix}\right]_{\{i_1,...,i_d\} \in \{1,...,n\}^{(d)}}$$

where $\{1, ..., n\}^{(d)} := \{I \subset \{1, ..., n\} \mid \#I = d\}$ is the set of all d-element subsets of $\{1, ..., n\}$.

Lemma 13. There is a linear isomorphism

$$f : \textstyle\bigwedge^d k^n \to k^{\{1,...,n\}^{(d)}},$$

$$\sum_j v_1^{(j)} \wedge ... \wedge v_d^{(j)} \mapsto \left(\sum_j \det \begin{pmatrix} v_{1i_1}^{(j)} & \cdots & v_{di_1}^{(j)} \\ \vdots & \ddots & \vdots \\ v_{1i_d}^{(j)} & \cdots & v_{di_d}^{(j)} \end{pmatrix}\right)_{\{i_1,...,i_d\} \in \{1,...,n\}^{(d)}}$$

Proof. For vectors $v_1, ..., v_d$ and $I = \{i_1, ..., i_d\} \in \{1, ..., n\}^{(d)}$ write

$$A_I(v_1, ..., v_d) := \begin{pmatrix} v_{1i_1} & \cdots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \cdots & v_{di_d} \end{pmatrix}$$

5

First of all, we show that f is well-defined. Note that the tensor product can be described as

$$V^{\otimes d} := k^{V \times \ldots \times V} / \mathrm{span}\{ (v_1 \otimes \ldots \otimes v_{i-1} \otimes (v_i + v_i') \otimes v_{i+1} \otimes \ldots \otimes v_d)$$
$$- (v_1 \otimes \ldots \otimes v_d) - (v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \ldots \otimes v_d),$$
$$(v_1 \otimes \ldots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \ldots \otimes v_d)$$
$$- \lambda (v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_d) \mid i \leq d, v_j, v_i' \in V \}$$

where $v_1 \otimes \ldots \otimes v_d := \chi_{(v_1, \ldots, v_d)}$. Hence the external power can be described as

$$\bigwedge\nolimits^d V := k^{V \times \ldots \times V} / \mathrm{span}\Big\{ (v_1 \otimes \ldots \otimes v_{i-1} \otimes (v_i + v_i') \otimes v_{i+1} \otimes \ldots \otimes v_d)$$
$$- (v_1 \otimes \ldots \otimes v_d) - (v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \ldots \otimes v_d),$$
$$(v_1 \otimes \ldots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \ldots \otimes v_d)$$
$$- \lambda (v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_d),$$
$$v_1 \otimes \ldots v_{j-1} \otimes (v_j \otimes v_{j+1} + v_{j+1} \otimes v_j) \otimes v_{j+1} \otimes \ldots \otimes v_d$$
$$\mid i \leq d, \ j \leq d-1, \ v_1, \ldots, v_d, v_i' \in V \Big\}$$

So it suffices to show that for all $I \in \{1, \ldots, n\}^{(d)}$ and vectors $v_1, \ldots, v_d, v_i' \in V$

$$\det(A_I(v_1, \ldots, v_i + v_i', \ldots, v_d)) = \det(A_I(v_1, \ldots, v_d)) + \det(A_I(v_1, \ldots, v_i', \ldots, v_d))$$

and

$$\det(A_I(v_1, \ldots, \lambda v_i, \ldots, v_d)) = \lambda \det(A_I(v_1, \ldots, v_d))$$

and

$$\det(A_I(v_1, \ldots, v_{j+1}, v_j, \ldots, v_d)) = -\det(A_I(v_1, \ldots, v_d))$$

However, these properties follow from the well-known properties of the determinant. In particular, det is linear in each column and swapping columns negates the determinant. It follows that f is indeed well-defined.

It is clear by definition that f is linear, so it is left to show that it is bijective. To show surjectivity, note that the $\pm e_I, I \in \{1, \ldots, n\}^{(d)}$ form a basis of $k^{\{1, \ldots, n\}^{(d)}}$. Clearly for $I = \{i_1, \ldots, i_d\}, J \in \{1, \ldots, n\}^{(d)}$ we have that

$$f(e_{i_1} \wedge \ldots \wedge e_{i_d})_J = \det(A_J(e_{i_1}, \ldots, e_{i_j})) = \begin{cases} 0 & \text{if } J \not\subseteq I \\ \pm 1 & \text{if } J \subseteq I \end{cases}$$

so $f(e_{i_1} \wedge \ldots \wedge e_{i_d}) = e_I$ and we deduce that $\mathrm{im} f = \mathbb{P}^{\{1, \ldots, k\}^{(d)}}$.

Finally, note that

$$e_{i_1} \wedge \ldots \wedge e_{i_d}$$

for $i_1 < ... < i_d$ form a basis of $\bigwedge^d k^n$. Clearly, they span $\bigwedge^d k^n$, and the following argument shows that they are linearly independent. Assume

$$\sum_{i_1 < ... < i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d}) = 0$$

Then

$$0 = e_1 \wedge \Big(\sum_{1 < i_2 < ... < i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge .. \wedge e_{i_d}) \Big) + \sum_{1 < i_1 < ... < i_d} \lambda_{i_1,...,i_d}(e_{i_1} \wedge ... \wedge e_{i_d})$$

Clearly $e_1 \notin \mathrm{span}\{e_2, ..., e_n\}$ and so by Lemma 7 we see that

$$\sum_{1 < i_2 < ... < i_d} \lambda_{1,i_2,...,i_d}(e_{i_2} \wedge ... \wedge e_{i_d}) = 0$$

Repeating this argument inductively shows that $\lambda_{1,2,...,d} = 0$. As k^n is symmetric w.r.t. permuting the e_j, we see that all $\lambda_{i_1,...,i_d} = 0$ are zero.

It follows that $\dim(\bigwedge^d k^n) = \dim(\mathbb{P}^{\{1,...,n\}^{(d)}})$ and we find that f is also injective. $\qquad\square$

Corollary 14 (2b). Let $\bar{f} : \mathbb{P}(\bigwedge^d k^n) \to \mathbb{P}^{\binom{n}{d}-1}$ be the map f from before modulo k^*. Then

$$\rho = \bar{f} \circ \phi$$

and in particular, we see that $\phi(Gr(d,n))$ is a projective variety and isomorphic to $\rho(\mathrm{Gr}(d,n))$.

Proposition 15 (2c). The map ϕ is injective.

Proof. Consider two d-dimensional subspaces U, W of k^n with $\phi(U) = \phi(W)$. Let $u_1, ..., u_l$ be a basis of $U \cap W$ and extend it to bases $u_1, ..., u_d$ of U and $u_1, ..., u_l, w_{l+1}, ..., w_d$ of W. As $\phi(U) = \phi(W)$, we can assume that the u_i, w_i are scaled such that

$$\begin{aligned}0 =& (u_1 \wedge ... \wedge u_d) - (u_1 \wedge ... \wedge u_l \wedge w_{l+1} \wedge ... \wedge w_d) \\ =& u_1 \wedge ... \wedge u_l \wedge \big((u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d) \big)\end{aligned}$$

By Lemma 7 we see that

$$u_2 \wedge ... \wedge u_l \wedge \big((u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d) \big) = 0$$

as $u_1 \notin \mathrm{span}\{u_2, ..., u_d, w_{l+1}, ..., w_d\}$. Inductively, this argument shows that

$$(u_{l+1} \wedge ... \wedge u_d) - (w_{l+1} \wedge ... \wedge w_d) = 0$$

If $l < d$, we can now apply Lemma 7 again to see that

$$u_{l+1} \in \mathrm{span}\{u_{l+2}, ..., u_d, w_{l+1}, ..., w_d\}$$

as $u_{l+2} \wedge ... \wedge u_d \neq 0$ by Lemma 6. However, this contradicts the linear independence of $u_{l+1}, ..., u_d, w_{l+1}, ..., w_d$. Hence it must be $l = d$ and so $U = W$. $\qquad\square$

3 Part III

In this part, we want to investigate the geometric properties of the Grassmanian resp. the image of ϕ. First of all, we introduce coordinates on $\mathbb{P}(\bigwedge^d k^n)$.

Definition 16. Note that in the proof of Lemma 13 it was shown that $v_{i_1} \wedge ... \wedge v_{i_d}$ for $i_1 < ... < i_d$ is a basis of $\bigwedge^d k^n$ if $v_1, ..., v_n$ is a basis of V. We introduce the homogeneous coordinates w.r.t. that basis, namely

$$x : \mathbb{P}(\textstyle\bigwedge^d k^n) \to \mathbb{P}_k^{\{1,...,n\}^{(d)}} \cong \mathbb{P}_k^{\binom{n}{d}-1},$$

$$\Big[\sum_{i_1 < ... < i_d} \lambda_{i_1,...,i_d}(v_{i_1} \wedge ... \wedge v_{i_d}) \Big] \mapsto \big[\lambda_{i_1,...,i_d}\big]_{i_1 < ... < i_d}$$

The individual coordinates will be denoted by x_I for some $I \in \{1, ..., n\}^{(d)}$ or $x_{i_1,...,i_d}$ for $i_1 < ... < i_d$.

Proposition 17 (3a)**.** For the embedding $\phi : \mathrm{Gr}(2, V) \to \bigwedge^2 V$ we have

$$\mathrm{Gr}(2, V) \cong \mathrm{im}\phi = \mathbb{V}(I)$$

where

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v \rangle \leq k\big[\mathbb{P}(\textstyle\bigwedge^d V)\big] = k[x_{i,j} \mid i < j]$$

Proof. By Lemma 14 we have that

$$[\omega] \in \mathrm{im}\phi \iff \omega \text{ decomposable}$$

and so by Lemma 8

$$\omega \in \mathrm{im}\phi \iff \omega \wedge \omega = 0$$

In $\mathbb{P}(\bigwedge^d V)$ we find that

$$\Big(\sum_{i<j} x_{i,j}(e_i \wedge e_j) \Big) \wedge \Big(\sum_{u<v} x_{u,v}(e_u \wedge e_v) \Big) = \sum_{\substack{i<j \\ u<v}} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v)$$

$$= 2 \sum_{i<j<u<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v) + 2 \sum_{i<u<j<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v)$$

$$+ 2 \sum_{u<i<j<v} x_{i,j}x_{u,v}(e_i \wedge e_j \wedge e_u \wedge e_v)$$

$$= 2 \sum_{i<j<u<v} (x_{i,j}x_{u,v} - x_{i,u}x_{j,v} + x_{j,u}x_{i,v})(e_i \wedge e_j \wedge e_u \wedge e_v)$$

As the $e_i \wedge e_j \wedge e_u \wedge e_v$ are linearly independent, we see that for $[\omega] \in \mathbb{P}(\bigwedge^2 V)$ we have

$$[\omega] \in \mathrm{im}\phi \iff \forall i < j < u < v : \ (x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v})(\omega) = 0$$

Hence $\mathrm{im}\phi = \mathbb{V}(I)$. $\qquad\square$

Example 18 (2b). For $n = 4$, Prop. 17 yields that $\mathrm{Gr}(2,4) \cong \mathrm{im}\phi = \mathbb{V}(I)$ where

$$I = \langle x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4} \rangle \in k[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}]$$

Changing the indices used for the coordinates, we find

$$\mathrm{Gr}(2,4) = \mathbb{V}(x_0 x_5 + x_2 x_3 - x_2 x_4)$$

which is exactly what we found in the lecture.

Example 19 (2c). For $n = 5$, Prop. 17 yields that $\mathrm{Gr}(2,5) \cong \mathrm{im}\phi = \mathbb{V}(I)$ where

$$\begin{aligned}
I = \langle &x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4}, \quad x_{1,2}x_{3,5} + x_{1,5}x_{2,3} - x_{1,3}x_{2,5}, \\
&x_{1,2}x_{4,5} + x_{1,5}x_{2,4} - x_{1,4}x_{2,5}, \quad x_{1,3}x_{4,5} + x_{1,5}x_{3,4} - x_{1,4}x_{3,5}, \\
&x_{2,3}x_{4,5} + x_{2,5}x_{3,4} - x_{2,4}x_{3,5} \rangle \le k\left[\mathbb{P}(\textstyle\bigwedge^2 k^5)\right]
\end{aligned}$$

Using the following Sage-code, we can compute the number of intersection points of $\mathrm{Gr}(2,5)$ with 3-dimensional hyperplanes, and find a probable value for its degree.

```python
from itertools import combinations
from math import factorial
import numpy as np

# build up the ring and group the variables nicely
R = PolynomialRing(QQ, [
    "x" + str(i) + str(j)
        for i in range(1, 6) for j in range(i + 1, 6)
])
x12, x13, x14, x15, x23, x24, x25, x34, x35, x45 = R.gens()
_x = [[x12, x13, x14, x15], [x23, x24, x25], [x34, x35], [x45]]
x = lambda i, j: _x[i - 1][j - i - 1]

# construct the ideal describing Gr(2, 5)
polys = []
for seq in combinations([1, 2, 3, 4, 5], 4):
    (i, j, u, v) = sorted(seq)
    p = x(i,j) * x(u,v) + x(i,v) * x(j,u) - x(i,u) * x(j,v)
    polys.append(p)

I = R.ideal(polys)
dimension = I.dimension() - 1
assert dimension == 6
assert I.is_prime()

hyperplane_vectors = [
    np.random.randint(-4, 4, R.ngens(), int)
```

```
            for i in range(dimension + 1)
    ]
    for vecs in combinations(hyperplane_vectors, dimension):
        eqs = [
            sum(map(lambda t: t[0] * t[1], zip(vec, R.gens())))
                for vec in vecs
        ]
        J = I + R.ideal(eqs)
        # the number of intersection points is clearly equal to the
        # dimension of S(X)_d for large enough d
        hp = J.hilbert_polynomial()
        degree = hp.leading_coefficient()
        print(degree) # usually prints 5
```

This shows that the degree of $\mathrm{Gr}(2,5)$ is indeed 5, as expected from the degree formula mentioned in the lecture.

$$\deg(\mathrm{Gr}(d,n)) = (d(n-d))! \frac{1! \cdot 2! \cdot \ldots \cdot (d-1)!}{(n-d)! \cdot (n-d+1)! \cdot \ldots \cdot (n-1)!}$$

which yields

$$\deg(\mathrm{Gr}(2,5)) = 6! \frac{1!}{3! \cdot 4!} = \frac{6 \cdot 5}{3!} = 5$$

To investigate the properties of $\phi(\mathrm{Gr}(2,n))$ for larger n, we use one tool I encountered during an earlier course on Computational Commutative Algebra and Algebraic Geometry.

Proposition 20 (Macaulay Basis Theorem)**.** Let $\preceq$ be a graded monomial ordering on $R = k[x_0, ..., x_n]$. Then for an ideal $I \leq R$ have that the monomials $x_0^{\alpha_0}...x_n^{\alpha_n} \notin \mathrm{lt}(I)$ are a k-vector space basis of R/I.

Here $\mathrm{lt}(I)$ is the leading term ideal of I, i.e. the ideal generated by the leading terms of all $f \in I$, w.r.t. $\preceq$.

Proof. See [KR00]. $\qquad\qquad\square$

Lemma 21. Define the graded reverse monomial ordering $\preceq$ on $R := k[x_{i,j} \mid i < j]$ where the variables $x_{i,j}$ are ordered co-lexicographically w.r.t. (i,j), i.e.

$$x_{i,j} \leq x_{u,v} \ :\Leftrightarrow \ (i,j) \leq_{\mathrm{colex}} (u,v)$$

Moreover, let

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v \rangle \leq R$$

be the ideal defining $\phi(\mathrm{Gr}(d,V))$ that was considered above. Then

$$\mathrm{lt}(I) = J := \langle x_{i,v}x_{j,u} \mid i < j < u < v \rangle \leq R$$

Proof. Note that for $i < j < u < v$ have $x_{u,v}, x_{j,v} \succ x_{i,v} \succ x_{i,j}, x_{i,u}, x_{j,u}$. Thus the leading term of $x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}$ is

$$\mathrm{lt}(x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}) = x_{i,v}x_{j,u}$$

It follows that $J \subseteq \mathrm{lt}(I)$.

For the other direction, we use a quite lengthy degree argument. Sadly, the only argument I came up with is extremely technical. We try to present it as clearly as possible, at the cost of only sketching some parts. In the end, I checked it using Computer Algebra, and everything fits together.

Consider homogeneous polynomials $f_{i,j,u,v} \in R$ and

$$F = \sum_{i<j<u<v} f_{i,j,u,v} \left(x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v} \right)$$

We want to show that $\mathrm{lt}(F) \in J$.

Let

$$f_{i,j,u,v} = \sum_{\alpha \in \mathbb{N}^N} c_\alpha^{(i,j,u,v)} x^\alpha$$

Then

$$F = \sum_{\alpha \in \mathbb{N}^N} \sum_{i<j<u<v} c_\alpha^{(i,j,u,v)} x^\alpha \left(x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v} \right)$$

and so there exists $\alpha \in \mathbb{N}^N$ and $\epsilon \in k^*$ with

$$\mathrm{lt}(F) = \epsilon \cdot \mathrm{lt}\Big(\sum_{i<j<u<v} c_\alpha^{(i,j,u,v)} x^\alpha \left(x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v} \right) \Big)$$

Hence, we may assume wlog that all the $f_{i,j,u,v}$ are scaled monomials.

Now, observe that all monomials in F are of the form

$$\epsilon \, x_{i,v} \, x_{j,u} \, f_{i,j,u,v} \quad \text{or} \quad \epsilon \, x_{i,u} \, x_{j,v} \, f_{i,j,u,v} \quad \text{or} \quad \epsilon \, x_{i,j} \, x_{u,v} \, f_{i,j,u,v}$$

where $\epsilon \in k^*$ and $i < j < u < v$. In particular, this is true for the leading term $\mathrm{lt}(F)$. In the first of those case, clearly $\mathrm{lt}(F) \in J$.

So consider now the second case, i.e. $\mathrm{lt}(F) = \epsilon \, x_{i,u} \, x_{j,v} \, f_{i,j,u,v}$. Since $x_{i,v} \, x_{j,u} \, f_{i,j,u,v} \succ x_{i,u} \, x_{j,v} \, f_{i,j,u,v}$, we see that the term $x_{i,v} \, x_{j,u} \, f_{i,j,u,v}$ cannot occur in F, i.e. must "cancel out". Hence the monomial $\epsilon \, x_{i,v} \, x_{j,u} \, f_{i,j,u,v}$ has a nonzero coefficient in

$$F - f_{i,j,u,v}x_{i,v}x_{j,u} = \sum_{\substack{a<b<c<d \\ (a,b,c,d)\neq(i,j,u,v)}} f_{a,b,c,d} \left(x_{a,d}x_{b,c} - x_{a,c}x_{b,d} + x_{a,b}x_{c,d} \right)$$

and so for $(a,b,c,d) \neq (i,j,u,v), \epsilon' \in k^*$ have that

$$f_{i,j,u,v} \, x_{i,v} \, x_{j,u} = \epsilon' f_{a,b,c,d} \, x_{a,d} \, x_{b,c} \quad \text{or}$$
$$f_{i,j,u,v} \, x_{i,v} \, x_{j,u} = \epsilon' f_{a,b,c,d} \, x_{a,c} \, x_{b,d} \quad \text{or}$$
$$f_{i,j,u,v} \, x_{i,v} \, x_{j,u} = \epsilon' f_{a,b,c,d} \, x_{a,b} \, x_{c,d}$$

However, the second and third case imply that $f_{a,b,c,d}\, x_{a,d}\, x_{b,c} \succ f_{i,j,u,v}\, x_{i,v}\, x_{j,u}$. Hence, the "new" monomial $f_{a,b,c,d}\, x_{a,d}\, x_{b,c}$ also has to "cancel out" in the sum representation of F, as comes after $\mathrm{lt}(F) = \epsilon x_{i,u}\, x_{j,v}\, f_{i,j,u,v}$ in the order $\preceq$. So applying the whole argument inductively (induction on the number of monomials $\succ f_{i,j,u,v}\, x_{i,v}\, x_{j,u}$ that occur in any of the polynomials we work with), we end up in the first case [1].

If $(a,d) = (i,v)$ then $(b,c) \neq (j,u)$ and $x_{b,c} \mid f_{i,j,u,v}$. Thus

$$x_{b,c}\, x_{i,u}\, x_{j,v} \mid \mathrm{lt}(F) = \epsilon\, x_{i,u}\, x_{j,v}\, f_{i,j,u,v}$$

with $i < j < u < v$ and $i < b < c < v$. No matter how j, u, b, c are ordered relatively to each other, we see that in each possible case $x_{b,c}\, x_{i,u}\, x_{j,v} \in J$:

$$
\begin{aligned}
j < u < b < c &\Rightarrow x_{j,v} x_{b,c} \in J \\
j < b < u < c &\Rightarrow x_{j,v} x_{b,c} \in J \\
j < b < c < u &\Rightarrow x_{i,u} x_{b,c} \in J \\
b < j < u < c &\Rightarrow f_{i,b,c,v}\, x_{i,c}\, x_{b,v} \succ f_{i,j,u,v}\, x_{i,u}\, x_{j,v}, \text{ contradiction} \\
b < j < c < u &\Rightarrow x_{i,u} x_{b,c} \in J \\
b < c < j < u &\Rightarrow x_{i,u} x_{b,c} \in J
\end{aligned}
$$

Hence $\mathrm{lt}(F) \in J$.

It is now left to consider the case $(b,c) = (j,u)$ and the case $(a,d) \neq (i,v), (b,c) \neq (j,u)$. The former can be dealt with in exactly the same way, by noting that $x_{a,d} \mid f_{i,j,u,v}$. In the latter case, we even find $x_{a,d}\, x_{b,c} \mid f_{i,j,u,v}$ and a very similar argument works.

Finally, one must also consider the third "big" case, namely that

$$\mathrm{lt}(F) = \epsilon\, x_{i,j}\, x_{u,v}\, f_{i,j,u,v}$$

Again, you can do this similarly as before, but now two monomials "cancel out". We will not present this here as well. $\qquad\square$

References

[KR00] Martin Kreuzer and Lorenzo Robbiano. *Computational Commutative Algebra.* Springer, 2000.

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[1] There is a small argument missing here, namely that we apply the next argument on each step of the induction, to show the claim for $f_{i,j,u,v}\, x_{i,v}\, x_{j,u}$. However, it should be easy to see that this is possible.