

Miniproject - Algebraic Geometry

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1 Part I

Definition 1. Let V be a vector space. Then define the d -th exterior power as

$$\bigwedge^d(V) := V^{\otimes d} / \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \text{span}\{v \otimes v' + v' \otimes v \mid v, v' \in V\} \otimes V^{\otimes(d-i-1)}$$

Use the notation $v_1 \wedge \dots \wedge v_d := [v_1 \otimes \dots \otimes v_d] \in \bigwedge^d(V)$.

Lemma 2. Let $v_1, \dots, v_d \in V$. Have for $\pi \in S_d$ that

$$v_{\pi(1)} \wedge \dots \wedge v_{\pi(d)} = \text{sgn}(\pi)(v_1 \wedge \dots \wedge v_d)$$

Furthermore if $v_i = v_j$ for some $i \neq j$, then

$$v_1 \wedge \dots \wedge v_d = 0$$

Proof. Note that

$$u \wedge v \wedge v' \wedge w = -(u \wedge v' \wedge v \wedge w)$$

for all $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{d-i-1}(V), v, v' \in V$.

Every $\pi \in S_d$ has a decomposition $\pi = \xi_1 \dots \xi_n$ into transpositions ξ_i . Applying this inductively, we find that

$$v_1 \wedge \dots \wedge v_d = \text{sgn}(\xi_1 \dots \xi_n)(v_{(\xi_1 \dots \xi_n)(1)} \wedge \dots \wedge v_{(\xi_1 \dots \xi_n)(d)})$$

and so

$$v_1 \wedge \dots \wedge v_d = \text{sgn}(\pi)(v_{\pi(1)} \wedge \dots \wedge v_{\pi(d)})$$

Furthermore, we find that

$$u \wedge v \wedge v \wedge w = -(u \wedge v \wedge v \wedge w) = 0$$

must be zero. Hence, if $v_1, \dots, v_d \in V$ with $v_i = v_j$ for some $i \neq j$, then there is a permutation $\pi \in S_d$ with $\pi(1) = i, \pi(2) = j$ and

$$v_1 \wedge \dots \wedge v_d = (\text{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge \dots \wedge v_{\pi(d)}) = \text{sgn}(\pi)0 = 0$$

□

Lemma 3 (1a). Let $\dim(V) \leq 3$. Then every element of $\bigwedge^k(V)$ is decomposable.

Proof. Now let v_1, v_2, v_3 be a set of generators of V . Consider $u_1 = \sum \lambda_i v_i, u_2 = \sum \mu_i v_i, u_3 = \sum \rho_i v_i$. Then by applying Lemma 2, we see that

$$\begin{aligned} u_1 \wedge u_2 &= \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{=0 \text{ if } i=j} = \sum_{i < j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i > j} \lambda_i \mu_j (v_i \wedge v_j) \\ &= \sum_{i < j} (\lambda_i \mu_j - \lambda_j \mu_i) (v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3) \\ &= \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} u_1 \wedge u_2 \wedge u_3 &= \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{=0 \text{ unless } i,j,l \text{ pairwise distinct}} \\ &= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)}) \\ &= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \text{sgn}(\pi) (v_1 \wedge v_2 \wedge v_3) \\ &= (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \text{sgn}(\pi) \end{aligned}$$

are decomposable. Further, it is easy to see from Lemma 2 that $\bigwedge^k(V) = \{0\}$ for $k \geq 4$, which is trivially decomposable. \square

Example 4 (1b). Consider $V = k^4$. Then the element $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$ is not decomposable.

Proof. Assume it was, then there are $a, b \in k^4$ such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i < j} (a_i b_j - a_j b_i) (e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1, \quad a_3 b_4 - a_4 b_3 = 1, \quad a_i b_j - a_j b_i = 0 \text{ for all } (i, j) \neq (1, 2), (3, 4)$$

Clearly $a_1 b_2 \neq 0$ or $a_2 b_1 \neq 0$. Similarly, have $a_3 b_4 \neq 0$ or $a_4 b_3 \neq 0$. As all expressions are symmetric w.r.t swapping a_1, b_2 with a_2, b_1 and a_3, b_4 with a_4, b_3 , we may assume wlog that $a_1 b_2, a_3 b_4 \neq 0$.

Have $a_1 b_4 = a_4 b_1$ and $a_2 b_4 = a_4 b_2$. We know that $a_1 b_4 \neq 0$ and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \Rightarrow a_2 b_1 = a_1 b_2$$

This contradicts $a_1 b_2 - a_2 b_1 = 1$. \square

Lemma 5. Let $A = (a_{ij}) \in \text{GL}_d(k)$ and $v_1, \dots, v_d \in V$. Then

$$\left(\sum_j a_{1j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) = \det(A)(v_1 \wedge \dots \wedge v_d)$$

Proof. By a direct computation using Lemma 2, we find

$$\begin{aligned} & \left(\sum_j a_{1j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) = \sum_{j_1, \dots, j_d} a_{1j_1} \dots a_{dj_d} (v_{j_1} \wedge \dots \wedge v_{j_d}) \\ &= \sum_{\pi \in S_d} a_{1\pi(1)} \dots a_{d\pi(d)} (v_{\pi(1)} \wedge \dots \wedge v_{\pi(d)}) \\ &= \sum_{\pi \in S_d} a_{1\pi(1)} \dots a_{d\pi(d)} \text{sgn}(\pi) (v_1 \wedge \dots \wedge v_d) \\ &= (v_1 \wedge \dots \wedge v_d) \sum_{\pi \in S_d} \text{sgn}(\pi) \prod_{j=1}^d a_{j\pi(j)} = \det(A)(v_1 \wedge \dots \wedge v_d) \end{aligned}$$

where the last equality holds due to the Leibniz determinant formula. \square

Lemma 6. For $v_1, \dots, v_d \in V$ have

$$v_1 \wedge \dots \wedge v_d = 0 \Leftrightarrow v_1, \dots, v_d \text{ linearly dependent}$$

Proof. For the direction \Leftarrow , assume that v_1, \dots, v_d are not independent. Then there is a nonzero vector $a_1 \in k^d$ with $\sum a_{1i}v_i = 0$. Clearly, we can extend a_1 to a basis a_1, \dots, a_d of k^d , which gives a matrix $A = (a_{ij}) \in \text{GL}_d(k)$.

However by Lemma 5 we now get

$$\begin{aligned} 0 &= 0 \wedge \left(\sum_j a_{2j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) = \left(\sum_j a_{1j}v_j\right) \wedge \dots \wedge \left(\sum_j a_{dj}v_j\right) \\ &= \det(A)(v_1 \wedge \dots \wedge v_d) \end{aligned}$$

and so $v = v_1 \wedge \dots \wedge v_d = 0$ as $\det(A) \neq 0$.

Direction \Rightarrow TODO \square

Lemma 7. Let $v \in V$ and $u \in \bigwedge^{d-1}U$ for a linear subspace $U \leq V$. If $v \wedge u \in \bigwedge^d U$ then $v \in U$ or $u = 0$.

Proof. TODO \square

Lemma 8 (1c). Let d be even. An element $\omega \in \bigwedge^d V$ is decomposable if and only if $\omega \wedge \omega \in \bigwedge^{2d} V$ is zero.

Proof. The direction \Rightarrow even holds generally. Assume $\omega = v_1 \wedge \dots \wedge v_d$. Then

$$\omega \wedge \omega = v_1 \wedge \dots \wedge v_d \wedge v_1 \wedge \dots \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let $\omega = v_1 + \dots + v_t$ for linearly independent decomposable vectors $v_i \in \bigwedge^2 V$. Then

$$\begin{aligned} 0 = \omega \wedge \omega &= \sum_{i,j} v_i \wedge v_j = \sum_{i < j} (v_i \wedge v_j) + (v_j \wedge v_i) \\ &= \sum_{i < j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \left(\sum_{j > i} v_j \right) \end{aligned}$$

Here we used that the permutation $(1 \ 2d)(2 \ (2d-1)) \dots (d \ (d+1)) \in S_{2d}$ has always sign 1 (since d is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \left(\bigwedge^2 \text{span}\{v_2, \dots, v_t\} \right) \setminus \{0\}$$

find

$$u_1, u_2 \in \text{span}\{v_2, \dots, v_t\}$$

In particular, we know that

$$v_1 \wedge \left(\sum_{j > i} v_j \right) \in \bigwedge^2 \text{span}\{v_2, \dots, v_t\}$$

and so $v_1 \in \text{span}\{v_2, \dots, v_t\}$ unless $\sum_{j > i} v_j = 0$ by Lemma 7. We assumed that the v_i are linearly independent, so the former would give a contradiction. Hence $\sum_{j > i} v_j = 0$ and thus $t = 1$, i.e. $\omega = v_1$ is decomposable. \square

2 Part II

In this part, we want to consider the connection of external powers to the Grassmanian. First of all, assume there are two bases v_1, \dots, v_d and u_1, \dots, u_d of a d -dimensional vector space U . Then there exists a basis change matrix $A = (a_{ij}) \in \text{GL}_d(k)$ with

$$u_i = \sum_j a_{ij} v_j$$

So by Lemma 5, it follows that

$$u_1 \wedge \dots \wedge u_d = \det(A)(v_1 \wedge \dots \wedge v_d)$$

As v_1, \dots, v_d resp. u_1, \dots, u_d are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge \dots \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge \dots \wedge u_d \neq 0$$

by Lemma 6. Hence they have well-defined images $[v_1 \wedge \dots \wedge v_d]$ resp. $[u_1 \wedge \dots \wedge u_d]$ in the projective space $\mathbb{P}(\bigwedge^d V)$. By the above, find

$$[v_1 \wedge \dots \wedge v_d] = [u_1 \wedge \dots \wedge u_d]$$

This allows us to study the Grassmanian $\text{Gr}(d, V)$ of a fixed vector space V .

Definition 9. Define the map

$$\phi : \text{Gr}(d, V) \rightarrow \mathbb{P}(\bigwedge^d V), \quad \text{span}\{v_1, \dots, v_d\} \mapsto [v_1 \wedge \dots \wedge v_d]$$

which is well-defined by Lemma 5 as described above.

Lemma 10 (1a). We have

$$\text{im}\phi = D := \{[v] \in \mathbb{P}(\bigwedge^d V) \mid v \text{ decomposable}\}$$

Proof. First of all, note that the set D is well-defined, as v is decomposable if and only if λv is decomposable, for all $\lambda \in k^*$.

By definition of ϕ , we can directly observe that $\text{im}\phi \subseteq D$. So consider an element $[v] \in D$. As v is decomposable, it follows that $v = v_1 \wedge \dots \wedge v_d$ for $v_i \in V$. Not it suffices to show that the v_i are linearly independent, then clearly $\text{span}\{v_1, \dots, v_d\}$ is a well-defined d -dimensional vector subspace of V , thus an element of $\text{Gr}(d, V)$.

This follows directly from Lemma 6. \square

Definition 11. Let $\text{Gr}(d, n) := \text{Gr}(d, k^n)$.

In the lecture, we considered an embedding of $\text{Gr}(d, n)$ into projective space given by minors of the basis matrix. This corresponds to the following definition.

Definition 12. Define the map

$$\begin{aligned} \rho : \text{Gr}(d, n) &\rightarrow \mathbb{P}\left(k^{\{1, \dots, n\}^{(d)}}\right) \cong \mathbb{P}^{\binom{n}{d}-1}, \\ \text{span}\{v_1, \dots, v_d\} &\mapsto \left[\det \begin{pmatrix} v_{1i_1} & \dots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \dots & v_{di_d} \end{pmatrix} \right]_{\{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}} \end{aligned}$$

where $\{1, \dots, n\}^{(d)} := \{I \subset \{1, \dots, n\} \mid \#I = d\}$ is the set of all d -element subsets of $\{1, \dots, n\}$.

Lemma 13. There is a linear isomorphism

$$\begin{aligned} f : \bigwedge^d k^n &\rightarrow k^{\{1, \dots, n\}^{(d)}}, \\ \sum_j v_1^{(j)} \wedge \dots \wedge v_d^{(j)} &\mapsto \left(\sum_j \det \begin{pmatrix} v_{1i_1}^{(j)} & \dots & v_{di_1}^{(j)} \\ \vdots & \ddots & \vdots \\ v_{1i_d}^{(j)} & \dots & v_{di_d}^{(j)} \end{pmatrix} \right)_{\{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}} \end{aligned}$$

Proof. For vectors v_1, \dots, v_d and $I = \{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}$ write

$$A_I(v_1, \dots, v_d) := \begin{pmatrix} v_{1i_1} & \dots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \dots & v_{di_d} \end{pmatrix}$$

First of all, we show that f is well-defined. Note that the tensor product can be described as

$$V^{\otimes d} := k^{V \times \dots \times V} / \text{span}\{ (v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v'_i) \otimes v_{i+1} \otimes \dots \otimes v_d) \\ - (v_1 \otimes \dots \otimes v_d) - (v_1 \otimes \dots \otimes v_{i-1} \otimes v'_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ (v_1 \otimes \dots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \\ - \lambda(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \mid i \leq d, v_j, v'_i \in V \}$$

where $v_1 \otimes \dots \otimes v_d := \chi_{(v_1, \dots, v_d)}$. Hence the external power can be described as

$$\bigwedge^d V := k^{V \times \dots \times V} / \text{span}\{ (v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v'_i) \otimes v_{i+1} \otimes \dots \otimes v_d) \\ - (v_1 \otimes \dots \otimes v_d) - (v_1 \otimes \dots \otimes v_{i-1} \otimes v'_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ (v_1 \otimes \dots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \\ - \lambda(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ v_1 \otimes \dots \otimes v_{j-1} \otimes (v_j \otimes v_{j+1} + v_{j+1} \otimes v_j) \otimes v_{j+1} \otimes \dots \otimes v_d \\ \mid i \leq d, j \leq d-1, v_1, \dots, v_d, v'_i \in V \}$$

So it suffices to show that for all $I \in \{1, \dots, n\}^{(d)}$ and vectors $v_1, \dots, v_d, v'_i \in V$

$$\det(A_I(v_1, \dots, v_i + v'_i, \dots, v_d)) = \det(A_I(v_1, \dots, v_d)) + \det(A_I(v_1, \dots, v'_i, \dots, v_d))$$

and

$$\det(A_I(v_1, \dots, \lambda v_i, \dots, v_d)) = \lambda \det(A_I(v_1, \dots, v_d))$$

and

$$\det(A_I(v_1, \dots, v_{j+1}, v_j, \dots, v_d)) = -\det(A_I(v_1, \dots, v_d))$$

However, these properties follow from the well-known properties of the determinant. In particular, \det is linear in each column and swapping columns negates the determinant. It follows that f is indeed well-defined.

It is clear by definition that f is linear, so it is left to show that it is bijective. To show surjectivity, note that the $\pm e_I, I \in \{1, \dots, n\}^{(d)}$ form a basis of $k^{\{1, \dots, n\}^{(d)}}$. Clearly for $I = \{i_1, \dots, i_d\}, J \in \{1, \dots, n\}^{(d)}$ we have that

$$f(e_{i_1} \wedge \dots \wedge e_{i_d})_J = \det(A_J(e_{i_1}, \dots, e_{i_d})) = \begin{cases} 0 & \text{if } J \not\subseteq I \\ \pm 1 & \text{if } J \subseteq I \end{cases}$$

so $f(e_{i_1} \wedge \dots \wedge e_{i_d}) = e_I$ and we deduce that $\text{im } f = \mathbb{P}^{\{1, \dots, k\}^{(d)}}$.

Finally, note that

$$e_{i_1} \wedge \dots \wedge e_{i_d}$$

for $i_1 < \dots < i_d$ form a basis of $\bigwedge^d k^n$. Clearly, they span $\bigwedge^d k^n$, and the following argument shows that they are linearly independent. Assume

$$\sum_{i_1 < \dots < i_d} \lambda_{i_1, \dots, i_d} (e_{i_1} \wedge \dots \wedge e_{i_d}) = 0$$

Then

$$0 = e_1 \wedge \left(\sum_{1 < i_2 < \dots < i_d} \lambda_{1, i_2, \dots, i_d} (e_{i_2} \wedge \dots \wedge e_{i_d}) \right) + \sum_{1 < i_1 < \dots < i_d} \lambda_{i_1, \dots, i_d} (e_{i_1} \wedge \dots \wedge e_{i_d})$$

Clearly $e_1 \notin \text{span}\{e_2, \dots, e_n\}$ and so by Lemma 7 we see that

$$\sum_{1 < i_2 < \dots < i_d} \lambda_{1, i_2, \dots, i_d} (e_{i_2} \wedge \dots \wedge e_{i_d}) = 0$$

Repeating this argument inductively shows that $\lambda_{1, 2, \dots, d} = 0$. As k^n is symmetric w.r.t. permuting the e_j , we see that all $\lambda_{i_1, \dots, i_d} = 0$ are zero.

It follows that $\dim(\bigwedge^d k^n) = \dim(\mathbb{P}^{\{1, \dots, n\}^{(d)}})$ and we find that f is also injective. \square

Corollary 14 (2b). Let $\bar{f} : \mathbb{P}(\bigwedge^d k^n) \rightarrow \mathbb{P}^{\binom{n}{d}-1}$ be the map f from before modulo k^* . Then

$$\rho = \bar{f} \circ \phi$$

and in particular, we see that $\phi(\text{Gr}(d, n))$ is a projective variety and isomorphic to $\rho(\text{Gr}(d, n))$.

Proposition 15 (2c). The map ϕ is injective.

Proof. Consider two d -dimensional subspaces U, W of k^n with $\phi(U) = \phi(W)$. Let u_1, \dots, u_l be a basis of $U \cap W$ and extend it to bases u_1, \dots, u_d of U and $u_1, \dots, u_l, w_{l+1}, \dots, w_d$ of W . As $\phi(U) = \phi(W)$, we can assume that the u_i, w_i are scaled such that

$$\begin{aligned} 0 &= (u_1 \wedge \dots \wedge u_d) - (u_1 \wedge \dots \wedge u_l \wedge w_{l+1} \wedge \dots \wedge w_d) \\ &= u_1 \wedge \dots \wedge u_l \wedge ((u_{l+1} \wedge \dots \wedge u_d) - (w_{l+1} \wedge \dots \wedge w_d)) \end{aligned}$$

By Lemma 7 we see that

$$u_2 \wedge \dots \wedge u_l \wedge ((u_{l+1} \wedge \dots \wedge u_d) - (w_{l+1} \wedge \dots \wedge w_d)) = 0$$

as $u_1 \notin \text{span}\{u_2, \dots, u_d, w_{l+1}, \dots, w_d\}$. Inductively, this argument shows that

$$(u_{l+1} \wedge \dots \wedge u_d) - (w_{l+1} \wedge \dots \wedge w_d) = 0$$

If $l < d$, we can now apply Lemma 7 again to see that

$$u_{l+1} \in \text{span}\{u_{l+2}, \dots, u_d, w_{l+1}, \dots, w_d\}$$

as $u_{l+2} \wedge \dots \wedge u_d \neq 0$ by Lemma 6. However, this contradicts the linear independence of $u_{l+1}, \dots, u_d, w_{l+1}, \dots, w_d$. Hence it must be $l = d$ and so $U = W$. \square

3 Part III

In this part, we want to investigate the geometric properties of the Grassmanian resp. the image of ϕ . First of all, we introduce coordinates on $\mathbb{P}(\bigwedge^d k^n)$.

Definition 16. Note that in the proof of Lemma 13 it was shown that $v_{i_1} \wedge \dots \wedge v_{i_d}$ for $i_1 < \dots < i_d$ is a basis of $\bigwedge^d k^n$ if v_1, \dots, v_n is a basis of V . We introduce the homogeneous coordinates w.r.t. that basis, namely

$$x : \mathbb{P}(\bigwedge^d k^n) \rightarrow \mathbb{P}_k^{\{1, \dots, n\}^{(d)}} \cong \mathbb{P}_k^{\binom{n}{d}-1},$$

$$\left[\sum_{i_1 < \dots < i_d} \lambda_{i_1, \dots, i_d} (v_{i_1} \wedge \dots \wedge v_{i_d}) \right] \mapsto [\lambda_{i_1, \dots, i_d}]_{i_1 < \dots < i_d}$$

The individual coordinates will be denoted by x_I for some $I \in \{1, \dots, n\}^{(d)}$ or x_{i_1, \dots, i_d} for $i_1 < \dots < i_d$.

Proposition 17 (3a). For the embedding $\phi : \text{Gr}(2, V) \rightarrow \bigwedge^2 V$ we have

$$\text{Gr}(2, V) \cong \text{im} \phi = \mathbb{V}(I)$$

where

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v \rangle \leq k[\mathbb{P}(\bigwedge^2 V)] = k[x_{i,j} \mid i < j]$$

Proof. By Lemma 14 we have that

$$[\omega] \in \text{im} \phi \Leftrightarrow \omega \text{ decomposable}$$

and so by Lemma 8

$$\omega \in \text{im} \phi \Leftrightarrow \omega \wedge \omega = 0$$

In $\mathbb{P}(\bigwedge^2 V)$ we find that

$$\begin{aligned} & \left(\sum_{i < j} x_{i,j} (e_i \wedge e_j) \right) \wedge \left(\sum_{u < v} x_{u,v} (e_u \wedge e_v) \right) = \sum_{\substack{i < j \\ u < v}} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) \\ &= 2 \sum_{i < j < u < v} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) + 2 \sum_{i < u < j < v} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) \\ & \quad + 2 \sum_{u < i < j < v} x_{i,j} x_{u,v} (e_i \wedge e_j \wedge e_u \wedge e_v) \\ &= 2 \sum_{i < j < u < v} (x_{i,j} x_{u,v} - x_{i,u} x_{j,v} + x_{j,u} x_{i,v}) (e_i \wedge e_j \wedge e_u \wedge e_v) \end{aligned}$$

As the $e_i \wedge e_j \wedge e_u \wedge e_v$ are linearly independent, we see that for $[\omega] \in \mathbb{P}(\bigwedge^2 V)$ we have

$$[\omega] \in \text{im} \phi \Leftrightarrow \forall i < j < u < v : (x_{i,j} x_{u,v} + x_{i,v} x_{j,u} - x_{i,u} x_{j,v})(\omega) = 0$$

Hence $\text{im} \phi = \mathbb{V}(I)$. □

Example 18 (2b). For $n = 4$, Prop. 17 yields that $\text{Gr}(2, 4) \cong \text{im}\phi = \mathbb{V}(I)$ where

$$I = \langle x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4} \rangle \in k[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}]$$

Changing the indices used for the coordinates, we find

$$\text{Gr}(2, 4) = \mathbb{V}(x_0x_5 + x_2x_3 - x_2x_4)$$

which is exactly what we found in the lecture.

Example 19 (2c). For $n = 5$, Prop. 17 yields that $\text{Gr}(2, 5) \cong \text{im}\phi = \mathbb{V}(I)$ where

$$I = \langle x_{1,2}x_{3,4} + x_{1,4}x_{2,3} - x_{1,3}x_{2,4}, \quad x_{1,2}x_{3,5} + x_{1,5}x_{2,3} - x_{1,3}x_{2,5}, \\ x_{1,2}x_{4,5} + x_{1,5}x_{2,4} - x_{1,4}x_{2,5}, \quad x_{1,3}x_{4,5} + x_{1,5}x_{3,4} - x_{1,4}x_{3,5}, \\ x_{2,3}x_{4,5} + x_{2,5}x_{3,4} - x_{2,4}x_{3,5} \rangle \leq k[\mathbb{P}(\bigwedge^2 k^5)]$$

Using the following Sage-code, we can compute the number of intersection points of $\text{Gr}(2, 5)$ with 3-dimensional hyperplanes, and find a probable value for its degree.

```

from itertools import combinations
from math import factorial
import numpy as np

# build up the ring and group the variables nicely
R = PolynomialRing(QQ, [
    "x" + str(i) + str(j)
    for i in range(1, 6) for j in range(i + 1, 6)
])
x12, x13, x14, x15, x23, x24, x25, x34, x35, x45 = R.gens()
_x = [[x12, x13, x14, x15], [x23, x24, x25], [x34, x35], [x45]]
x = lambda i, j: _x[i - 1][j - i - 1]

# construct the ideal describing Gr(2, 5)
polys = []
for seq in combinations([1, 2, 3, 4, 5], 4):
    (i, j, u, v) = sorted(seq)
    p = x(i, j) * x(u, v) + x(i, v) * x(j, u) - x(i, u) * x(j, v)
    polys.append(p)

I = R.ideal(polys)
dimension = I.dimension() - 1
assert dimension == 6
assert I.is_prime()

hyperplane_vectors = [
    np.random.randint(-4, 4, R.ngens(), int)

```

```

        for i in range(dimension + 1)
    ]
    for vecs in combinations(hyperplane_vectors, dimension):
        eqs = [
            sum(map(lambda t: t[0] * t[1], zip(vec, R.gens()))))
            for vec in vecs
        ]
    J = I + R.ideal(eqs)
    # the number of intersection points is clearly equal to the
    # dimension of S(X)_d for large enough d
    hp = J.hilbert_polynomial()
    degree = hp.leading_coefficient()
    print(degree) # usually prints 5

```

This shows that the degree of $\text{Gr}(2, 5)$ is indeed 5, as expected.

To investigate the properties of $\phi(\text{Gr}(2, n))$ for larger n , we use one tool I encountered during an earlier course on Computational Commutative Algebra and Algebraic Geometry.

Proposition 20 (Macaulay Basis Theorem). Let \preceq be a graded monomial ordering on $R = k[x_0, \dots, x_n]$. Then for an ideal $I \leq R$ have that the monomials $x_0^{\alpha_0} \dots x_n^{\alpha_n} \notin \text{lt}(I)$ are a k -vector space basis of R/I .

Here $\text{lt}(I)$ is the leading term ideal of I , i.e. the ideal generated by the leading terms of all $f \in I$, w.r.t. \preceq .

Proof. See [KR00]. □

Lemma 21. Define the graded reverse monomial ordering \preceq on $R := k[x_{i,j} \mid i < j]$ where the variables $x_{i,j}$ are ordered co-lexicographically w.r.t. (i, j) , i.e.

$$x_{i,j} \leq x_{u,v} \iff (i, j) \leq_{\text{colex}} (u, v)$$

Moreover, let

$$I := \langle x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v} \mid i < j < u < v \rangle \leq R$$

be the ideal defining $\phi(\text{Gr}(d, V))$ that was considered above. Then

$$\text{lt}(I) = J := \langle x_{i,v}x_{j,u} \mid i < j < u < v \rangle \leq R$$

Proof. Note that for $i < j < u < v$ have $x_{u,v}, x_{j,v} \succ x_{i,v} \succ x_{i,j}, x_{i,u}, x_{j,u}$. Thus the leading term of $x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}$ is

$$\text{lt}(x_{i,j}x_{u,v} + x_{i,v}x_{j,u} - x_{i,u}x_{j,v}) = x_{i,v}x_{j,u}$$

It follows that $J \subseteq \text{lt}(I)$.

For the other direction, we use a quite lengthy degree argument. Sadly, the only argument I came up with is extremely technical. We try to present it as clearly as possible, at the cost of only sketching some parts. In the end, I checked it using Computer Algebra, and everything fits together.

Consider homogeneous polynomials $f_{i,j,u,v} \in R$ and

$$F = \sum_{i < j < u < v} f_{i,j,u,v} (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})$$

We want to show that $\text{lt}(F) \in J$.

Let

$$f_{i,j,u,v} = \sum_{\alpha \in \mathbb{N}^N} c_{\alpha}^{(i,j,u,v)} x^{\alpha}$$

Then

$$F = \sum_{\alpha \in \mathbb{N}^N} \sum_{i < j < u < v} c_{\alpha}^{(i,j,u,v)} x^{\alpha} (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v})$$

and so there exists $\alpha \in \mathbb{N}^N$ and $\epsilon \in k^*$ with

$$\text{lt}(F) = \epsilon \cdot \text{lt} \left(\sum_{i < j < u < v} c_{\alpha}^{(i,j,u,v)} x^{\alpha} (x_{i,v}x_{j,u} - x_{i,u}x_{j,v} + x_{i,j}x_{u,v}) \right)$$

Hence, we may assume wlog that all the $f_{i,j,u,v}$ are scaled monomials.

Now, observe that all monomials in F are of the form

$$\epsilon x_{i,v} x_{j,u} f_{i,j,u,v} \quad \text{or} \quad \epsilon x_{i,u} x_{j,v} f_{i,j,u,v} \quad \text{or} \quad \epsilon x_{i,j} x_{u,v} f_{i,j,u,v}$$

where $\epsilon \in k^*$ and $i < j < u < v$. In particular, this is true for the leading term $\text{lt}(F)$. In the first of those case, clearly $\text{lt}(F) \in J$.

So consider now the second case, i.e. $\text{lt}(F) = \epsilon x_{i,u} x_{j,v} f_{i,j,u,v}$. Since $x_{i,v} x_{j,u} f_{i,j,u,v} \succ x_{i,u} x_{j,v} f_{i,j,u,v}$, we see that the term $x_{i,v} x_{j,u} f_{i,j,u,v}$ cannot occur in F , i.e. must “cancel out”. Hence the monomial $\epsilon x_{i,v} x_{j,u} f_{i,j,u,v}$ has a nonzero coefficient in

$$F - f_{i,j,u,v} x_{i,v} x_{j,u} = \sum_{\substack{a < b < c < d \\ (a,b,c,d) \neq (i,j,u,v)}} f_{a,b,c,d} (x_{a,d}x_{b,c} - x_{a,c}x_{b,d} + x_{a,b}x_{c,d})$$

and so for $(a,b,c,d) \neq (i,j,u,v)$, $\epsilon' \in k^*$ have that

$$\begin{aligned} f_{i,j,u,v} x_{i,v} x_{j,u} &= \epsilon' f_{a,b,c,d} x_{a,d} x_{b,c} \quad \text{or} \\ f_{i,j,u,v} x_{i,v} x_{j,u} &= \epsilon' f_{a,b,c,d} x_{a,c} x_{b,d} \quad \text{or} \\ f_{i,j,u,v} x_{i,v} x_{j,u} &= \epsilon' f_{a,b,c,d} x_{a,b} x_{c,d} \end{aligned}$$

However, the second and third case imply that $f_{a,b,c,d} x_{a,d} x_{b,c} \succ f_{i,j,u,v} x_{i,v} x_{j,u}$. Hence, the “new” monomial $f_{a,b,c,d} x_{a,d} x_{b,c}$ also has to “cancel out” in the sum representation of F , as comes after $\text{lt}(F) = \epsilon x_{i,u} x_{j,v} f_{i,j,u,v}$ in the order \preceq . So applying the whole

argument inductively (induction on the number of monomials $\succ f_{i,j,u,v} x_{i,v} x_{j,u}$ that occur in any of the polynomials we work with), we end up in the first case ¹.

If $(a, d) = (i, v)$ then $(b, c) \neq (j, u)$ and $x_{b,c} \mid f_{i,j,u,v}$. Thus

$$x_{b,c} x_{i,u} x_{j,v} \mid \text{lt}(F) = \epsilon x_{i,u} x_{j,v} f_{i,j,u,v}$$

with $i < j < u < v$ and $i < b < c < v$. No matter how j, u, b, c are ordered relatively to each other, we see that in each possible case $x_{b,c} x_{i,u} x_{j,v} \in J$:

$$\begin{aligned} j < u < b < c &\Rightarrow x_{j,v} x_{b,c} \in J \\ j < b < u < c &\Rightarrow x_{j,v} x_{b,c} \in J \\ j < b < c < u &\Rightarrow x_{i,u} x_{b,c} \in J \\ b < j < u < c &\Rightarrow f_{i,b,c,v} x_{i,c} x_{b,v} \succ f_{i,j,u,v} x_{i,u} x_{j,v}, \text{ contradiction} \\ b < j < c < u &\Rightarrow x_{i,u} x_{b,c} \in J \\ b < c < j < u &\Rightarrow x_{i,u} x_{b,c} \in J \end{aligned}$$

Hence $\text{lt}(F) \in J$.

It is now left to consider the case $(b, c) = (j, u)$ and the case $(a, d) \neq (i, v), (b, c) \neq (j, u)$. The former can be dealt with in exactly the same way, by noting that $x_{a,d} \mid f_{i,j,u,v}$. In the latter case, we even find $x_{a,d} x_{b,c} \mid f_{i,j,u,v}$ and a very similar argument works.

Finally, one must also consider the third “big” case, namely that

$$\text{lt}(F) = \epsilon x_{i,j} x_{u,v} f_{i,j,u,v}$$

Again, you can do this similarly as before, but now two monomials “cancel out”. We will not present this here as well. \square

Lemma 22. Let $n \geq 4$ and

$$J := \{\{A, B\} \mid A, B \in X, A \cap B = \emptyset\} \subseteq X^{(2)}$$

be a set, where

$$X := \{1, \dots, n\}^{(2)}$$

Then for $k \geq 3$ have

$$|\{A \in X^{(k)} \mid J \subseteq A\}| =$$

Proof. Use induction on n . In the base case $n = 4$ have that \square

References

[KR00] Martin Kreuzer and Lorenzo Robbiano. *Computational Commutative Algebra*. Springer, 2000.

¹There is a small argument missing here, namely that we apply the next argument on each step of the induction, to show the claim for $f_{i,j,u,v} x_{i,v} x_{j,u}$. However, it should be easy to see that this is possible.