Miniproject - Combinatorics

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We use the convention that $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.

1 Part I

Proposition 1. Let P be a graded poset in which every maximal chain has length $n+1$. Then the function

$$r : P \to \{0, ..., n\}, \quad x \mapsto \max\{k \in \mathbb{N} \mid \exists a_1, ..., a_k \in P: \ a_1 < a_2 < ... < a_k < x\}$$

is well-defined and the unique function with $x < y$ implies $r(x) < r(y)$ for all $x, y \in P$. We call it the *rank function* of P.

Proof. Clearly r is well-defined, as for every increasing sequence $a_1 < ... < a_k < x$, we have the chain $C := \{a_1, ..., a_k, x\}$ of size $k+1$. Hence by assumption, $k \leq n$ and so $r(x)$ is finite and in $\{0, ..., n\}$.

Further, consider $x < y$ in P. We have a sequence $a_1 < ... < a_{r(x)} < x$ by definition of r. It follows that there is an increasing sequence $a_1 < ... < a_{r(x)} < x < y$ and so $r(y) \geq r(k) + 1 > r(x)$.

Lastly, assume there was another function $r' : P \to \{0, ..., n\}$ with this property. Consider any $x \in P$. By definition of r, there is an increasing sequence $a_1 < ... < a_{r(x)} < x$ in P. Now consider a maximal chain C containing the chain $\{a_1, ..., a_{r(x)}, x\}$.

Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$ and $x = b_j$. Note that we have the increasing sequence $b_1 < ... < b_{j-1} < b_j = x$ and so by the definition of r, find $j \leq r(x) + 1$. On the other hand, have $a_1, ..., a_{r(x)} \in C$ and thus $j = r(x) + 1$, i.e.

$$a_1 = b_1, \ ..., \ a_{r(x)} = b_{r(x)}, \ x = b_{r(x)+1}$$

As $b_i < b_{i+1}$, we know that $r'(b_i) < r'(b_{i+1})$ and inductively, we see $r'(b_i) \geq i - 1$. However, $r'(b_{n+1}) \leq n$ and thus $r'(b_i) = i-1$. Finally it follows that $r'(x) = r'(b_{r(x)+1}) = r(x)$. $\square$

Now we will show some basic properties of the rank function.

Proposition 2. Let P be a graded poset of maximal rank n with rank function r. Then

- $x \in P$ is minimal iff $r(x) = 0$ and maximal iff $r(x) = n$.

- If $x < y$ and $r(x) + 1 \neq r(y)$ then there is $z \in P$ with $x < z < y$.
- If $x < y$ then there is an increasing sequence $x < a_1 < ... < a_{r(y)-r(x)-1} < y$ in P.

Proof. For (i), let $x \in P$ be minimal. Then there is no increasing sequence $a_1 < x$ in P, so $r(x) \leq 0$. Conversely, let $r(x) = 0$. Assume there was $y \in P$ with $y < x$, then $r(y) < r(x) = 0$, a contradiction. The analogous statement for maximal elements is proved in the same way.

For (iii), consider $x < y$ in P. Then the chain $\{x, y\}$ is contained in a maximal chain $C \subseteq P$. Say $C = \{b_1, ..., b_{n+1}\}$ with $b_1 < ... < b_{n+1}$.

Hence we find $r(b_i) < r(b_{i+1})$ and so inductively that $r(b_i) \geq i-1$ and $r(b_i) \leq i-1$ since $r(b_1) \geq 0$ and $r(b_{n+1}) \leq n$. It follows that $r(b_i) = i - 1$ and so $x = b_{r(x)+1}, y = r(y) + 1$. Therefore, we have a chain

$$x < b_{r(x)+2} < ... < b_{r(y)} < y$$

of length $r(y) - r(x) - 1$. Statement (ii) follows directly, as in this setting, have $r(y) \geq r(x) + 2$ and so $r(y) - r(x) - 1 \neq 0$. □

2 Part II

Proposition 3. For a graded poset P with layers $L_0, ..., L_n$ the following statements are equivalent:

- For every antichain $A \subseteq P$ have
$$\sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} \leq 1$$
- For each $1 < i \leq n$ and $F \subseteq L_i$ have
$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$
where
$$\partial F := \{a \in L_{i-1} \mid \exists b \in F : \ a \leq b\}$$
- There exists a (nonempty) sequence of maximal chains $C_1, ..., C_t$ such that for all $1 \leq i \leq n$ we have:
$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

Proof. Show the directions (ii) ⇒ (iii) ⇒ (i) ⇒ (ii)

(ii) ⇒ (iii) Use induction on n. Again, the base case is trivial, just take chains $\{x\}$ for each $x \in A = L_0$. So assume that $n > 0$. The set

$$A' := \bigcup_{i=0}^{n-1} L_i$$

is a graded poset of maximal rank $n-1$, and hence there are maximal chains $C_1, ..., C_t \subseteq A'$ such that for all $1 \leq i \leq n-1$ have

$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

by induction hypothesis. Let now $V_1, V_2 := [t] \times L_n$ (treat them as disjoint) and consider the bipartite graph $G := (V_1 \sqcup V_2, E)$ where E is defined as follows:

$$\{\underbrace{(i,a)}_{\in V_1}, \underbrace{(j,b)}_{\in V_2}\} \in E \;\Leftrightarrow\; \max C_j < a$$

We use Hall's theorem to show that G has a perfect matching.

Consider any $W = \{(i_1, a_1), ..., (i_w, a_w)\} \subseteq V_1$ and let $F = \{a_1, ..., a_w\}$. Then

$$W \subseteq [t] \times F \;\Rightarrow\; |W| \leq t|F|$$

By choice of $C_1, ..., C_t$, we have that the number of j with $x \in C_j$ is the same for all $x \in L_{n-1}$, say k. Since the C_i are maximal chains, each intersects L_{n-1} in exactly one element. So have bijection

$$\{(x,i) \mid x \in L_{n-1}, 1 \leq i \leq t, x \in C_i\} \to \{i \mid 1 \leq i \leq t\}, \quad (x,i) \mapsto i$$

where the set on the left-hand side has size $k|L_{n-1}|$. It follows that $k|L_{n-1}| = t$.

Since $\max C_j \in L_{n-1}$ for all j, we have

$$\begin{aligned} N(W) =& \{(j,b) \mid b \in L_n, \exists a \in F : \max C_j < a\} \\ =& L_n \times \{j \mid \exists a \in F : \max C_j < a\} \\ =& L_n \times \{j \mid \max C_j \in \partial F\} \end{aligned}$$

and so by the above

$$|N(W)| = |L_n| \sum_{\max C_j \in \partial F} 1 = |L_n| \sum_{x \in \partial F} \sum_{x \in C_j} 1 = |L_n| \sum_{x \in \partial F} k = |L_n||\partial F|k$$

Together with the assumption that $|\partial F||L_n| \geq |F||L_{n-1}|$ we see that

$$|W| \leq t|F| = k|L_{n-1}||F| \leq k|\partial F||L_n| = |N(W)|$$

So Hall's theorem yields a perfect matching $M \subseteq E$ from V_1 to V_2. As $|V_1| = |V_2|$, this is already a 1-to-one correspondence.

Now consider the sets

$$C'_m := C_j \cup \{a\} \text{ where } m = \{\underbrace{(i,a)}_{\in V_1}, \underbrace{(j,b)}_{\in V_2}\} \in M$$

These are chains, as $\max C_j < a$ for each C'_m. Further, for each $x, y \in L_i, i < n$ have that

$$|\{C'_m \mid x \in C'_m\}| = |L_n \times \{j \mid x \in C_j\}| = |L_n \times \{j \mid y \in C_j\}| = |\{C'_m \mid y \in C'_m\}|$$

as M is a matching from V_2 to V_1. Finally, for all $x \in L_n$ we have that

$$|\{C'_m \mid x \in C'_m\}| = |\{m \in M \mid \exists i, v \in V_2 : m = \{(i,x), v\}\}| = |\{(i,x) \mid (i,x) \in V_1\}| = t$$

as M is a matching from V_1 to V_2.

(iii) $\Rightarrow$ (i) Let $C_1, ..., C_t$ be a sequence of maximal chains given by the assumption. For $1 \leq i \leq n$ let k_i be the number of different j such that a fixed element $x \in L_i$ is contained in exactly the C_j. By assumption, this does not depend on x.

As in the direction above, we see that $k_i|L_i| = t$ because each C_j intersects L_i in exactly one element, so there is a bijection

$$\{(x,j) \mid x \in L_i, 1 \leq j \leq t, x \in C_j\} \to \{j \mid 1 \leq j \leq t\}, \quad (x,j) \mapsto j$$

and the set on the left has size $k_i|L_i|$.

Since each C_j is a chain and A is an antichain, we find that A and C_j intersect in at most one element. So

$$\begin{aligned} t \geq& |\{j \mid A \cap C_j \neq \emptyset\}| = \sum_{a \in A} |\{j \mid a \in C_j\}| = \sum_{i=0}^{n} \sum_{a \in A \cap L_i} |\{j \mid a \in C_j\}| \\ =& \sum_{i=0}^{n} \sum_{a \in A \cap L_i} k_i = \sum_{i=0}^{n} k_i |A \cap L_i| = \sum_{i=0}^{n} \frac{|A \cap L_i|}{|L_i|} t \end{aligned}$$

The claim follows by canceling t.

(i) $\Rightarrow$ (ii) Consider $A := F \cup (L_{i-1} \setminus \partial F)$. This is clearly an antichain, as for $x \in F, y \in L_{i-1} \setminus \partial F$ have $y \not< x$. However, $r(y) < r(x)$ so also $x \not< y$, thus x and y are incomparable. Clearly elements from the same layer are incomparable.

Thus, the assumption yields that

$$\frac{|L_{i-1}| - |\partial F|}{|L_{i-1}|} + \frac{|F|}{|L_i|} = \frac{|A \cap L_{i-1}|}{|L_{i-1}|} + \frac{|A \cap L_i|}{|L_i|} = \sum_{j=0}^{n} \frac{|A \cap L_j|}{|L_j|} \leq 1$$

This gives

$$1 + \frac{|F|}{|L_i|} \leq 1 + \frac{|\partial F|}{|L_{i-1}|}$$

and the claim follows. $\square$

3 Part III

We even show the slightly stronger stronger statement that the implication to the second condition of Prop.3 already holds "locally", i.e. for each layer separately.

Proposition 4. Let P be a graded poset with layers $L_0, ..., L_n$ and let $0 < i \leq n$ such that

- Each element $a \in L_i$ covers the same number of elements in L_{i-1}
- Each element $a \in L_{i-1}$ is covered by the same number of elements in L_i

Then for each $F \subseteq L_i$ have

$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$

In particular, if this is true for all i, then all the equivalent conditions from Prop. 3 follow.

Proof. Assume that for all $a \in L_i$

$$|\{b \in L_{i-1} \mid a \text{ covers } b\}| = k \in \mathbb{N}$$

and for all $b \in L_{i-1}$

$$|\{a \in L_i \mid a \text{ covers } b\}| = l \in \mathbb{N}$$

Double-counting pairs $(a, b) \in L_i \times L_{i-1}$ such that a covers b yields $k|L_i| = l|L_{i-1}|$. Now consider $F \subseteq L_i$. Note that by definition of ∂F, we have for all $a \in F$ that

$$\{b \in \partial F \mid a \text{ covers } b\} = \{b \in L_{i-1} \mid a \text{ covers } b\} = \partial\{a\} \subseteq \partial F$$

Again by double-counting pairs $(a, b) \in F \times \partial F$ such that a covers b, we find

$$\begin{aligned} k|F| =& \sum_{a \in F} k = \sum_{a \in F} |\{b \in L_{i-1} \mid a \text{ covers } b\}| = \sum_{a \in F} |\{b \in \partial F \mid a \text{ covers } b\}| \\ =& |\{(a, b) \in F \times \partial F \mid a \text{ covers } b\}| = \sum_{b \in \partial F} |\{a \in F \mid a \text{ covers } b\}| \\ \leq& \sum_{b \in \partial F} |\{a \in L_i \mid a \text{ covers } b\}| = \sum_{b \in \partial F} l = l|\partial F| \end{aligned}$$

Hence

$$\frac{|\partial F|}{|L_{i-1}|} = \frac{l|\partial F|}{l|L_{i-1}|} \leq \frac{k|F|}{k|L_i|} = \frac{|F|}{|L_i|}$$

□

4 Part IV

Example 5. Let $\Pi_m = \{\pi \subseteq \mathfrak{P}(n) \mid \pi \text{ partition}\}$ be the poset of partitions ordered by refinement. Then Π_m is a graded poset with rank function

$$r : \Pi_m \to \{0, ..., n-1\}, \quad \pi \mapsto n - |\pi|$$

Proof. First of all, have $1 \leq |X| \leq n$ for all subsets $X \subseteq \mathfrak{P}(n)$, hence the function r is well-defined. Next we show that for all $x < y$ have $r(x) < r(y)$. However, if x is a proper refinement of y, then clearly $|x| > |y|$, so $r(x) < r(y)$.

Now assume there is a maximal chain $\pi_1 < ... < \pi_t$ in Π_m. As the chain is maximal and Π_m has the largest element $[n]$, we may assume that $\pi_t = [n]$ and thus $r(\pi_t) = n-1$. Similarly, we may assume that $p_1 = \mathfrak{P}(n)$ and so $r(\pi_1) = 0$. We have to show that $t = n$, and so it suffices to show that $r(\pi_i) = 1 + r(\pi_{i-1})$ for all $0 < i \leq t$.

Assume not, i.e. $|\pi_{i-1}| \geq 2 + |\pi_i|$ for some $0 < i \leq t$. Since π_{i-1} is a refinement of π_i, we find that

$$\pi_{i-1} = \{A_{11}, ..., A_{1m_1},\ ...,\ A_{n1}, ..., A_{nm_n}\} \quad \text{with} \quad \pi_i = \left\{ \bigcup_{j \leq m_1} A_{1j}, ..., \bigcup_{j \leq m_n} A_{nj} \right\}$$

where $n = |\pi_i|$ and $\sum_j m_j = |\pi_{i-1}| \geq 2 + n$. Hence, there is one $s \leq n$ with $m_s \geq 3$ or two different $s, r \leq n$ with $m_s, m_r \geq 2$.

In the first case, consider

$$\tilde{\pi} := \{A_{jk} \mid j \neq s\} \cup \left\{ A_{s1}, \bigcup_{2 \leq j \leq m_s} A_{sj} \right\}$$

and in the second case, consider

$$\tilde{\pi} := \{A_{jk} \mid j \neq s\} \cup \left\{ \bigcup_{j \leq m_s} A_{sj} \right\}$$

Then $\pi_{i-1} < \tilde{\pi} < \pi_i$ and we found a longer chain

$$\pi_1 < ... < \pi_{i-1} < \tilde{\pi} < \pi_i < ... < \pi_t$$

which contradicts the assumed maximality. The claim follows. $\square$

Proposition 6. For the number $S(m,k) := |\{\pi \in \Pi_m \mid |\pi| = k\}|$ we have recursion

$$S(m,1) = 1 \quad \text{and} \quad S(m,k) = \sum_{n=1}^{m-1} \binom{m-1}{n} S(n, k-1)$$

where $1 \leq k \leq m$.

Proof. Use induction on k. The base case $k = 1$ is trivial, so let $k > 1$. For any partition $\pi \in \Pi_m$, denote by $A(1, \pi)$ the unique set $A \in \pi$ with $1 \in A$. We find

$$\begin{aligned}
S(m,k) =& |\{\pi \in \Pi_m \mid |\pi| = k\}| = \Big| \bigcup_{\emptyset \neq A \subseteq [m],\ 1 \in A} \{\pi \in \Pi_m \mid |\pi| = k, A(1,\pi) = A\} \Big| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_m \mid |\pi| = k, A(1,\pi) = A\}| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_m \mid |\pi \setminus \{A\}| = k-1, A(1,\pi) = A\}| \\
=& \sum_{\emptyset \neq A \subseteq [m],\ 1 \in A} |\{\pi \in \Pi_{m-|A|} \mid |\pi| = k-1\}| \\
=& \sum_{i=1}^{m} \sum_{A \subseteq [m],\ |A| = i,\ 1 \in A} |\{\pi \in \Pi_{m-i} \mid |\pi| = k-1\} \\
=& \sum_{i=1}^{m} \binom{m-1}{i-1} S(m-i, k-1) = \sum_{i=0}^{m-1} \binom{m-1}{m-i-1} S(i, k-1) \\
=& \sum_{i=1}^{m-1} \binom{m-1}{i} S(i, k-1)
\end{aligned}$$

as for $k > 1$ have that $S(0, k) = 0$. $\square$

Example 7. Have for $m \geq 3$ that

$$S(m, 2) = 2^{m-1} - 1 \quad \text{and} \quad S(m, 3) = \frac{1}{2}(3^{m-1} - 2^m + 1)$$

Note that this is the partition into *exactly* k sets; I am not completely sure what the exercise description tells us to show, but if it is about the number of ways of decomposing $[m]$ into *at most* k sets, then have

$$\Sigma S(m, k)(2) = 2^{m-1} \quad \text{and} \quad \Sigma S(m, k)(3) = 3^{m-1} = \frac{1}{2} 3^{m-1} + \frac{1}{2}$$

for $m \geq 3$. Here $\Sigma S(m, k) = \sum_{l \leq k} S(m, l)$ is the number of partitions of $[m]$ into at most k sets.

Proof. By Prop. 6 have that for $m \geq 1$ we have $S(m, 1) = 1$. Further, for $m \geq 2$ have

$$S(m, 2) = \sum_{n=1}^{m-1} \binom{m-1}{n} 1 = 2^{m-1} - 1$$

Applying this once more yields for $m \geq 3$ that

$$\begin{aligned}
S(m,3) &= \sum_{n=1}^{m-1} \binom{m-1}{n} (2^{n-1}-1) = \frac{1}{2} \sum_{n=1}^{m-1} \binom{m-1}{n} 2^n - \sum_{n=1}^{m-1} \binom{m-1}{n} 1 \\
&= \frac{1}{2} \sum_{n=0}^{m-1} \binom{m-1}{n} 2^n - \frac{1}{2} - \sum_{n=0}^{m-1} \binom{m-1}{n} 1 + 1 \\
&= \frac{1}{2}(1+2)^{m-1} - 2^{m-1} + \frac{1}{2} = \frac{1}{2}(3^{m-1} - 2^m + 1)
\end{aligned}$$

□

Let m be even and consider $\mathcal{A} \subseteq \Pi_m$ of partitions of $[m]$ into two equally sized sets. Then

$$|\mathcal{A}| = \frac{1}{2} \binom{m}{m/2}$$

Proof. Consider the map

$$f : [m]^{(m/2)} \to \mathcal{A}, \quad A \mapsto \{A, [m] \setminus A\}$$

Then for A and $B \neq A, [m] \setminus A$ have that $\{A, [m] \setminus A\} \neq \{B, [m] \setminus B\}$. Conversely, A and $[m] \setminus A$ have the same image under f. This shows that f is 2-to-1 and thus

$$|\mathcal{A}| = \frac{1}{2} \left| [m]^{(m/2)} \right| = \frac{1}{2} \binom{m}{m/2}$$

□

Further have

$$|\partial \mathcal{A}| = \binom{m}{m/2} \left(2^{m/2-1} - 1 \right)$$

Proof. Consider the map

$$g : \underbrace{\{(A,B,C) \mid A \in [m]^{(m/2)}, \{A,B,C\} \in \Pi_m\}}_{=:\mathcal{G}} \to \partial \mathcal{A}, \quad (A,B,C) \mapsto \{A,B,C\}$$

Note that g is well-defined, as for $A \in [m]^{(m/2)}, \{A,B,C\} \in \Pi_m$ have that $\pi := \{A,B,C\}$ is a partition in L_{m-3} that refines the partition $\{A, B \cup C\} \in \mathcal{A} \subseteq L_{m-2}$. Further, it is easy to see that g is surjective, as every partition $\pi \in \partial \mathcal{A}$ satisfies $|\pi| = 3$ and has some $A \in \pi$ with $|A| = m/2$.

To complete the proof, we investigate to what "extend g is injective".

Assume $g(A,B,C) = g(A',B',C')$. As $B, C, B', C' \neq \emptyset$ and $|B \cup C| = |B' \cup C'| = m/2$, we see that $A = A'$. Further, we must then have that $\{B,C\} = \{B',C'\}$, hence $(A,B,C) = (A',B',C')$ or $(A,B,C) = (A',C',B')$ (clearly $B \neq C, B' \neq C'$). This shows that the map g is 2-to-1.

Using this, we find

$$
\begin{aligned}
|\partial\mathcal{A}| =& \frac{1}{2}|\mathcal{G}| = \frac{1}{2}\sum_{A\in[m]^{(m/2)}} |\{(B,C) \mid \{B,C\} \text{ partition of } [m]\setminus A\}| \\
=& \frac{1}{2}\sum_{A\in[m]^{(m/2)}} 2S(m/2,2) = \sum_{A\in[m]^{(m/2)}} 2^{m/2-1}-1 = \binom{m}{m/2}\left(2^{m/2-1}-1\right)
\end{aligned}
$$

□

We can now plug this into the second condition of Prop. 3 to see

$$
\begin{aligned}
\frac{|\partial\mathcal{A}|}{|L_{n-3}|} &= \frac{\binom{m}{m/2}\left(2^{m/2-1}-1\right)}{S(m,3)} = \frac{\binom{m}{m/2}\left(2^{m/2-1}-1\right)}{\frac{1}{2}\left(3^{m-1}-2^m+1\right)} \\
&\sim \frac{2^{m+m/2}}{\sqrt{n}(3^{m-1}-2^m)} \sim \frac{3\left(2^{3/2}\right)^m}{\sqrt{n}3^m} = \frac{3}{\sqrt{n}}c^m
\end{aligned}
$$

and

$$
\frac{|\mathcal{A}|}{|L_{n-2}|} = \frac{\frac{1}{2}\binom{m}{m/2}}{S(m,2)} = \frac{\binom{m}{m/2}}{2^m-2} \sim \frac{2^m}{\sqrt{n}2^m} = \frac{1}{\sqrt{n}}
$$

for some $0<c<1$ (here $\sim$ means asymptotically equivalent as $m\to\infty$). In particular, find for sufficiently large m that

$$
\frac{|\partial\mathcal{A}|}{|L_{n-3}|} < \frac{|\mathcal{A}|}{|L_{n-2}|}
$$

and so the conditions of Prop. 3 are not satisfied.

5 Part V

Example 8. Let $\mathcal{P}_{k,d} := \{0,...,k\}^d$ partially ordered by elementwise ordering. Then $\mathcal{P}_{k,d}$ is a graded poset with rank function

$$
r:\mathcal{P}_{k,d}\to\{0,...,kd\},\quad a\mapsto\sum_i a_i
$$

Proof. First, we show that for $a<b$ have $r(a)<r(b)$. If $a<b$, then we find that $a_j\le b_j$ for all j and $a_i\neq b_i$, so $a_i<b_i$ for some i. Hence

$$
r(a)=\sum_j a_j = a_i+\sum_{j\neq i}a_j \le a_i+\sum_{j\neq i}b_j < b_i+\sum_{j\neq i}b_j = \sum_j b_j = r(b)
$$

Now assume that we have a maximal chain $a_1<...<a_t$ in $\mathcal{P}_{k,d}$. Since $\mathcal{P}_{k,d}$ has the smallest element $0=(0,...,0)$ and the largest element $k=(k,...,k)$, we see that $p_1=0$

and $p_t = k$. We want to show $t = kd + 1$, so it suffices to show that $r(a_i) = r(a_{i-1}) + 1$ for all $0 < i \leq t$.

Assume not, then

$$r(a_i) - r(a_{i-1}) = \sum_j a_{ij} - a_{(i-1)j} \geq 2$$

For all j we have $a_{ij} \geq a_{(i-1)j}$ and so there is $s \leq d$ with $a_{is} - a_{(i-1)s} \geq 2$ or there are different $r, s \leq d$ with $a_{ir} - a_{(i-1)r}, a_{is} - a_{(i-1)s} \geq 1$. In both cases, find

$$\tilde{a} \in \mathcal{P}_{k,d} \text{ defined by } \tilde{a}_j = \begin{cases} a_{(i-1)j} & \text{if } j \neq s \\ a_{(i-1)j} + 1 & \text{otherwise} \end{cases}$$

with $a_{i-1} < \tilde{a} < a_i$. However, this gives a longer chain

$$a_1 < \dots < a_{i-1} < \tilde{a} < a_i < \dots < a_t$$

which contradicts the assumed maximality. The claim follows. $\square$

Definition 9. Let P be a graded poset with layers $L_0, \dots, L_n$. A symmetric chain in P is a chain $C \subseteq P$ such that there is $i \leq n/2$ with

$$\forall 0 \leq j \leq n : \ C \cap L_j \neq \emptyset \ \Leftrightarrow \ i \leq j \leq n - i$$

A decomposition $\mathcal{C}$ of P into chains is called symmetric chain decomposition, if all $C \in \mathcal{C}$ are symmetric.

Note that this definition is compatible with the definition of a symmetric chain for the classical graded poset $\mathfrak{P}(n)$.

Proposition 10. Let $k, d \geq 1$. Then $\mathcal{P}_{k,d}$ has a symmetric chain decomposition.

Proof. Use induction on d. The base case $d = 1$ is trivial, so assume that $d > 1$. By induction hypothesis, have a symmetric chain decomposition $\mathcal{C}$ of $\mathcal{P}_{k,d-1}$. For some $C = \{A_i, A_{i+1}, \dots, A_{d(k-1)-i}\} \in \mathcal{C}, i < \frac{n}{2}$ with $r(A_j) = j$ and $0 \leq l \leq k, n-2i-1$ consider now the chain

$$\tilde{C}_l := \{(A_{i+l}, j) \mid 0 \leq j \leq k - l\} \cup \{(A_j, k - l) \mid i + l < j \leq d(k-1) - (i + l)\}$$

Then $\tilde{C}_l$ is symmetric w.r.t. $i + l$, as for $i + l \leq j \leq dk - (i + l)$ we have

$$\begin{aligned} i + l \leq j \leq k + i \quad &\Rightarrow \quad (A_{i+l}, j - i - l) \in \tilde{C}_l, \\ k + i < j \leq dk - (i + l) \quad &\Rightarrow \quad (A_{j-k}, k - l) \in \tilde{C}_l \end{aligned}$$

Clearly the $\tilde{C}_l$ are chains and disjoint. Hence, we find a symmetric chain decomposition

$$\tilde{\mathcal{C}} := \left\{ \tilde{C}_l \ \middle| \ C \in \mathcal{C} \text{ symmetric chain w.r.t. } i, \ 0 \leq l \leq \min\{k, n - 2i - 1\} \right\}$$

of $\mathcal{P}_{k,d}$. $\square$

Note that for $k = 1$, this is exactly the proof that was done in the lecture to show that $\mathfrak{P}(d)$ has a symmetric chain decomposition.

6 Part VI

First of all, in Example 7 we have already shown that the poset Π_m does not satisfy the conditions from Prop. 3. Now we want to investigate whether $\mathcal{P}_{k,d}$ does.

Lemma 11. Let A be a poset with two layers $L_0 = \{u_1, ..., u_m\}, L_1 = \{v_1, ..., v_n\}$. Then any of the conditions of Prop. 3 is equivalent to the following:

There exists a matrix $A = (a_{ij}) \in \mathbb{Q}_{\geq 0}^{m \times n}$ such that for all i, j

$$e_i^T A 1 = e_j^T A 1 \quad \text{and} \quad 1^T A e_i = 1^T A e_j$$

and $a_{ij} = 0$ whenever $u_i \not< v_j$.

This lemma does not seem super useful at first, but it gives us a way to check in time polynomial in $O(|P|)$ whether a given poset P satisfies the conditions from Prop. 3, by using an LP solver. Note that this is not easily possible when working directly with any of the three conditions defined before.

Using this, I could check that $\mathcal{P}_{k,d}$ indeed satisfies the conditions for reasonably sized $|\mathcal{P}_{k,d}|$ (e.g. for $k, d \approx 6$). Note that already for these “small” numbers, we get quite huge posets $|\mathcal{P}_{k,d}|$. Hence, I believe that this is true for all $\mathcal{P}_{k,d}$.

My successful-looking attempt is based on the observation that elements of $\mathcal{P}_{k,d}$ “behave similarly” (in the sense that there is an element of the automorphism group $\mathrm{Aut}(\mathcal{P}_{k,d})$ mapping one to the other) if the have the same number of equally sized entries. This looks promising, as the proof for $\mathfrak{P}(n)$ and equivalently the proof of Prop. 4 relies on the fact that all elements in one layer of $\mathfrak{P}(n)$ “behave similarly” in this sense.

This is formalized in the next definition.

Definition 12. Let $P = \mathcal{P}_{k,d}$. Then define

$$P(n_0, ..., n_k) := \{x \in P \mid |\{i \mid x(i) = j\}| = n_j\}$$

Further, for any subset $F \subseteq P$ write

$$F(n_0, ..., n_k) := F \cap P(n_0, ..., n_k)$$

Lemma 13. Have

$$|P(n_0, ..., n_k)| = \binom{n_0 + ... + n_k}{n_0} \binom{n_1 + ... + n_k}{n_1} ... \binom{n_{k-1} + n_k}{n_{k-1}} \binom{n_k}{n_k}$$

Further, for all $n_i \in \mathbb{Z}$ with $\sum n_i = d$ and all $a \in P(n_0, ..., n_k), 0 \leq \delta < k$ have

$$|\partial a \cap P(n_0, ..., n_\delta + 1, n_{\delta+1} - 1, ..., n_k)| = n_{\delta+1},$$
$$|\partial^+ a \cap P(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)| = n_\delta$$

even if any of the n_i are negative.

Proposition 14. $\mathcal{P}_{k,d}$ satisfies the conditions of Prop. 3.

Proof Attempt. Fix a layer L_l with $0 < l \leq kd$. Further, consider any subset $F \subseteq L_l$. Now define

$$U := \{(n_0, ..., n_k) \mid n_i \in \mathbb{Z},\ n_0 + ... + n_k = d,\ \sum_i i n_i = l\}$$

and

$$V := \{(n_0, ..., n_k) \mid n_i \in \mathbb{Z},\ n_0 + ... + n_k = d,\ \sum_i i n_i = l - 1\}$$

Note that now

$$L_l = \bigsqcup_{(n_0,...,n_k) \in U} P(n_0, ..., n_k) \quad \text{and} \quad L_{l-1} = \bigsqcup_{(n_0,...,n_k) \in V} P(n_0, ..., n_k)$$

Now make $V \sqcup U$ into a bipartite graph $G = (V \sqcup U, E)$ by defining for $(m_0, ..., m_k) \in V, (n_0, ..., n_k) \in U$

$$\begin{aligned} &\{(m_0, ..., m_k), (n_0, ..., n_k)\} \in E \\ \Leftrightarrow\ &\exists 0 \leq \delta < k :\ (n_0, ..., n_k) = (m_0, ..., m_\delta - 1, m_{\delta+1} + 1, ..., m_k) \end{aligned}$$

Observe that the resulting graph is now $(k-1)$-regular. For $e = \{(n_0, ..., n_k), (n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)\} \in E$ define now

$$\begin{aligned} v_F(e) :=& \frac{|\partial\big(F(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)\big)(n_0, ..., n_k)|}{|P(n_0, ..., n_k)|} \\ &- \frac{|F(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)|}{|P(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)|} \\ \leq& \frac{|\partial F(n_0, ..., n_k)|}{|P(n_0, ..., n_k)|} - \frac{|F(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)|}{|P(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)|} \end{aligned}$$

where we define $|\emptyset|/|\emptyset| := 1$. Now consider the graded poset $A := P(n_0, ..., n_k) \sqcup P(n_0, ..., n_\delta - 1, n_{\delta+1} + 1, ..., n_k)$ with two layers. By Lemma 13 we see that

- Each element in the bottom layer $v \in L_0(A)$ of A is covered by n_δ elements
- Each element in the top layer $u \in L_1(A)$ of A covers $n_{\delta+1} + 1$ elements

Hence we can apply Prop. 4 and see that $v_F(e) \geq 0$. Define

$$p(n_0, ..., n_k) := \frac{|P(n_0, ..., n_k)|}{|L_{\sum i n_i}|}$$

We know that all $v_F(e) \geq 0$ and by using the $(k-1)$-regularity, we see

$$
\begin{aligned}
0 \leq & \sum_{e\in E} v_F(e)p(n_0,...,n_k) \\
\leq & \sum_{e\in E}\left(\frac{|\partial F(n_0,...,n_k)|}{|P(n_0,...,n_k)|} - \frac{|F(n_0,...,n_\delta-1,n_{\delta+1}+1,...,n_k)|}{|P(n_0,...,n_\delta-1,n_{\delta+1}+1,...,n_k)|}\right)p(n_0,...,n_k) \\
=&(k-1)\sum_{(n_0,...,n_k)\in V}\frac{|\partial F(n_0,...,n_k)|}{|P(n_0,...,n_k)|}p(n_0,...,n_k) \\
-&\sum_{(n_0,...,n_k)\in U}\frac{|F(n_0,...,n_k)|}{|P(n_0,...,n_k)|}\sum_{0\leq\delta<k}p(n_0,...,n_\delta+1,n_{\delta+1}-1,...,n_k) \\
=&\frac{(k-1)|\partial F|}{|L_{l-1}|} - \sum_{(n_0,...,n_k)\in U}\frac{|F(n_0,...,n_k)|}{|P(n_0,...,n_k)|}\sum_{0\leq\delta<k}p(n_0,...,n_\delta+1,n_{\delta+1}-1,...,n_k) \\
=&\frac{(k-1)|\partial F|}{|L_{l-1}|} - \sum_{(n_0,...,n_k)\in U}\frac{|F(n_0,...,n_k)|}{|L_l|}\sum_{0\leq\delta<k}\frac{(n_\delta+...+n_k+1)n_{\delta+1}}{(n_\delta+1)(n_{\delta+1}+...+n_k)}
\end{aligned}
$$

However, the sum

$$\sum_{0\leq\delta<k}\frac{(n_\delta+...+n_k+1)n_{\delta+1}}{(n_\delta+1)(n_{\delta+1}+...+n_k)}$$

is not always $k-1$, so this won't work out.

On a higher level, everything I did was to reduce the problem to the case that F is a disjoint union of the $P(n_0,...,n_k)$ which is nice to see, but does not help too much.

Proposition 15. Let $M(k,d,n) := |\{x \in \mathcal{P}_{k,d} \mid r(x) = n\}|$. Then for $k, d \geq 1$ we have the recurrence relation

$$M(k,1,n) = \begin{cases} 1 & \text{if } 0 \leq n \leq k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad M(k,d,n) = \sum_{i=0}^{k} M(k,d-1,n-i)$$

We also have the recurrence relation

$$M(k,d,n) = \sum_{i=0}^{\lfloor n/k \rfloor}\binom{d}{i}M(k-1,d-i,n-ki)$$

Lemma 16. Let R be a ring extension of $\mathbb{Q}$ (commutative with one). For a polynomial $f \in R[X]$ there exists a unique polynomial $\Sigma f \in R[X]$ such that for all $x \in \mathbb{Z}, x \geq 0$ we have

$$\Sigma f(x) = \sum_{i=0}^{x} f(i)$$

Furthermore, Σ is linear and $\deg(\Sigma f) = \deg(f) + 1$ and $\mathrm{lc}(\Sigma f) = \mathrm{lc}(f)/(\deg(f)+1)$.

Proof. Use induction on $\deg(f)$. As everything is linear, it suffices to show the claim for the monomials X^d. Define then

$$\Sigma X^d := \frac{1}{d+1}(X+1)^{d+1} - \frac{1}{d+1}\sum_{k=0}^{d-1}\binom{d+1}{k}\left(\Sigma X^k\right)$$

It follows that for all $x \geq 0$ have

$$\begin{aligned}
&\left(\Sigma X^d\right)(x+1) - \left(\Sigma X^d\right)(x)\\
=&\frac{1}{d+1}\left((x+2)^{d+1} - x^{d+1}\right) - \frac{1}{d+1}\sum_{k=0}^{d-1}\binom{d+1}{k}\left(\left(\Sigma X^k\right)(x+1) - \left(\Sigma X^k\right)(x)\right)\\
=&\frac{1}{d+1}\left(-(x+1)^{d+1} + \sum_{k=0}^{d+1}\binom{d+1}{k}(x+1)^k\right) - \frac{1}{d+1}\sum_{k=0}^{d-1}\binom{d+1}{k}(x+1)^k\\
=&\frac{1}{d+1}\left(\sum_{k=0}^{d}\binom{d+1}{k}(x+1)^k - \sum_{k=0}^{d-1}\binom{d+1}{k}(x+1)^k\right)\\
=&\frac{1}{d+1}\binom{d+1}{d}(x+1)^d = (x+1)^d
\end{aligned}$$

As this holds for infinitely many x, we see that $(\Sigma X^d)(X+1) - \Sigma X^d = (X+1)^d$ and the claim follows. ∎

Example 17. Let $k, d \geq 1$. Then

$$M(k,2,n) := \begin{cases} 0 & \text{if } n > 2k \\ 2k-n+1 & \text{if } k \leq n \leq 2k \\ n+1 & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n < 0 \end{cases}$$

Note that if two cases are applicable, they both yield the same value.