Miniproject - Algebraic Geometry

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Definition 1. *Let V be a vector space. Then define the d-th exterior power as*

$$\bigwedge^d(V) := V^{\otimes d} \ / \ \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \left\{ v \otimes v' + v' \otimes v \ \middle| \ v, v' \in V \right\} \otimes V^{\otimes(d-i-1)}$$

Use the notation $v_1 \wedge ... \wedge v_d := [v_1 \otimes ... \otimes v_d] \in \bigwedge^k(V)$.

Lemma 2. *Let $v_1, ..., v_d \in V$. Have for $\pi \in S_d$ that*

$$v_{\pi(1)} \wedge ... \wedge v_{\pi(k)} = \operatorname{sgn}(\pi)(v_1 \wedge ... \wedge v_d)$$

Furthermore if $v_i = v_j$ for some $i \neq j$, then

$$v_1 \wedge ... \wedge v_d = 0$$

Proof. Note that

$$u \otimes v \otimes v' \otimes w = -(u \otimes v' \otimes v \otimes w)$$

for all $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{(d-i-1)}(V), v, v' \in V$.

Every $\pi \in S_d$ has a decomposition $\pi = \xi_1 ... \xi_n$ into transpositions ξ_i. Applying this inductively, we find that

$$v_1 \wedge ... \wedge v_d = \operatorname{sgn}(\xi_i ... \xi_n)(v_{(\xi_i ... \xi_n)(1)} \wedge ... v_{(\xi_i ... \xi_n)(k)})$$

and so

$$v_1 \wedge ... \wedge v_d = \operatorname{sgn}(\pi)(v_{\pi(1)} \wedge ... \wedge v_{\pi(k)})$$

Furthermore, we find that

$$u \otimes v \otimes v \otimes w = -(u \otimes v \otimes v \otimes w) = 0$$

must be zero. Hence, if $v_1, ..., v_d \in V$ with $v_i = v_j$ for some $i \neq j$, then there is a permutation $\pi \in S_d$ with $\pi(1) = i, \pi(2) = j$ and

$$v_1 \wedge ... \wedge v_d = (\operatorname{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge ... \wedge v_{\pi(k)}) = \operatorname{sgn}(\pi) 0 = 0$$

□

Lemma 3 (1a)**.** *Let* $\dim(V) \leq 3$*. Then every element of* $\bigwedge^k(V)$ *is decomposable.*

Proof. Now let v_1, v_2, v_3 be a set of generators of V. Consider $u_1 = \sum \lambda_i v_i, u_2 = \sum_i \mu_i v_i, u_3 = \sum_i \rho_i v_i$. Then by applying Lemma 2, we see that

$$\begin{aligned} u_1 \wedge u_2 =& \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{=0 \text{ if } i=j} = \sum_{i<j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i>j} \lambda_i \mu_j (v_i \wedge v_j) \\ =& \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)(v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3) \\ =& \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} u_1 \wedge u_2 \wedge u_3 =& \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{=0 \text{ unless } i,j,l \text{ pairwise distinct}} \\ =& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)}) \\ =& \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi)(v_1 \wedge v_2 \wedge v_3) \\ =& (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \operatorname{sgn}(\pi) \end{aligned}$$

are decomposable. Further, it is easy to see from Lemma 2 that $\bigwedge^k(V) = \{0\}$ for $k \geq 4$, which is trivially decomposable. □

Example 4 (1b)**.** *Consider* $V = k^4$*. Then the element* $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$ *is not decomposable.*

Proof. Assume it was, then there are $a, b \in k^4$ such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i<j} (a_i b_j - a_j b_i)(e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1,\ a_3 b_4 - a_4 b_3 = 1,\ a_i b_j - a_j b_i = 0 \text{ for all } (i,j) \neq (1,2), (3,4)$$

Clearly $a_1 b_2 \neq 0$ or $a_2 b_1 \neq 0$. Similarly, have $a_3 b_4 \neq 0$ or $a_4 b_3 \neq 0$. As all expressions are symmetric w.r.t swapping a_1, b_2 with a_2, b_1 and a_3, b_4 with a_4, b_3, we may assume wlog that $a_1 b_2, a_3 b_4 \neq 0$.

Have $a_1 b_4 = a_4 b_1$ and $a_2 b_4 = a_4 b_2$. We know that $a_1 b_4 \neq 0$ and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \ \Rightarrow\ a_2 b_1 = a_1 b_2$$

This contradicts $a_1 b_2 - a_2 b_1 = 1$. □

Lemma 5 (1c)**.** *Let d be even. An element $\omega \in \bigwedge^d V$ is decomposable if and only if $\omega \wedge \omega \in \bigwedge^{2d} V$ is zero.*

Proof. The direction $\Rightarrow$ even holds generally. Assume $\omega = v_1 \wedge ... \wedge v_d$. Then

$$\omega \wedge \omega = v_1 \wedge ... \wedge v_d \wedge v_1 \wedge ... \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let $\omega = v_1 + ... + v_t$ for linearly independent decomposable vectors $v_i \in \bigwedge^2 V$. Then

$$\begin{aligned} 0 =& \omega \wedge \omega = \sum_{i,j} v_i \wedge v_j = \sum_{i<j} (v_i \wedge v_j) + (v_j \wedge v_i) \\ =& \sum_{i<j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \Big(\sum_{j>i} v_j\Big) \end{aligned}$$

Here we used that the permutation $\big(1\ 2d\big)\big(2\ (2d-1)\big)...\big(d\ (d+1)\big) \in S_{2d}$ has always sign 1 (since d is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \Big(\bigwedge^2 \mathrm{span}\{v_2, ..., v_t\}\Big) \setminus \{0\}$$

find

$$u_1, u_2 \in \mathrm{span}\{v_2, ..., v_t\}$$

In particular, we know that

$$v_1 \wedge \Big(\sum_{j>i} v_j\Big) \in \bigwedge^2 \mathrm{span}\{v_2, ..., v_t\}$$

and so $v_1 \in \mathrm{span}\{v_2, ..., v_t\}$ unless $\sum_{j>i} v_j = 0$. We assumed that the v_i are linearly independent, so the former would give a contradiction. Hence $\sum_{j>i} v_j = 0$ and thus $t = 1$, i.e. $\omega = v_1$ is decomposable. $\square$

Lemma 6 (2a)**.** *Let $A = (a_{ij}) \in \mathrm{GL}_d(k)$ and $v_1, ..., v_d \in V$. Then*

$$\Big(\sum_j a_{1j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \det(A)(v_1 \wedge ... \wedge v_d)$$

Proof. By a direct computation using Lemma 2, we find

$$\begin{aligned} &\Big(\sum_j a_{ij} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \sum_{j_1, ..., j_d} a_{1j_1} ... a_{dj_d} (v_{j_1} \wedge ... \wedge v_{j_d}) \\ =& \sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)} (v_{\pi(1)} \wedge ... \wedge v_{\pi(d)}) \\ =& \sum_{\pi \in S_d} a_{1\pi(1)} ... a_{d\pi(d)} \mathrm{sgn}(\pi)(v_1 \wedge ... \wedge v_d) \\ =& (v_1 \wedge ... \wedge v_d) \sum_{\pi \in S_d} \mathrm{sgn}(\pi) \prod_{j=1}^d a_{j\pi(j)} = \det(A)(v_1 \wedge ... \wedge v_d) \end{aligned}$$

where the last equality holds due to the Leibniz determinant formula. $\square$

We see that if there are two bases $v_1, ..., v_d$ and $u_1, ..., u_d$ of a d-dimensional vector space U, then there exists a basis change matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$ with

$$u_i = \sum_j a_{ij} v_j$$

So by the Lemma, it follows that

$$u_1 \wedge ... \wedge u_d = \det(A)(v_1 \wedge ... \wedge v_d)$$

As $v_1, ..., v_d$ resp. $u_1, ..., u_d$ are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge ... \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge ... \wedge u_d \neq 0$$

Hence they have well-defined images $[v_1 \wedge ... \wedge v_d]$ resp. $[u_1 \wedge ... \wedge u_d]$ in the projective space $\mathbb{P}(\bigwedge^d V)$. By the above, find

$$[v_1 \wedge ... \wedge v_d] = [u_1 \wedge ... \wedge u_d]$$

This allows us to study the Grassmanian $\mathrm{Gr}(d, V)$ of a fixed vector space V.

Definition 7. *Define the map*

$$\phi : \mathrm{Gr}(d, V) \to \mathbb{P}(\textstyle\bigwedge^d V), \quad \mathrm{span}\{v_1, ..., v_d\} \mapsto [v_1 \wedge ... \wedge v_d]$$

which is well-defined by Lemma 6 as described above.

Lemma 8 (1a)**.** *We have*

$$\mathrm{im}\phi = D := \{[v] \in \mathbb{P}(\textstyle\bigwedge^d V) \mid v \textit{ decomposable}\}$$

Proof. First of all, note that the set D is well-defined, as v is decomposable if and only if λv is decomposable, for all $\lambda \in k^*$.

By definition of ϕ, we can directly observe that $\mathrm{im}\phi \subseteq D$. So consider an element $[v] \in D$. As v is decomposable, it follows that $v = v_1 \wedge ... \wedge v_d$ for $v_i \in V$. Not it suffices to show that the v_i are linearly independent, then clearly $\mathrm{span}\{v_1, ..., v_d\}$ is a well-defined d-dimensional vector subspace of V, thus an element of $\mathrm{Gr}(d, V)$.

Assume not, then there is a nonzero vector $a_1 \in k^d$ with $\sum a_{1i} v_i = 0$. Clearly, we can extend a_1 to a basis $a_1, ..., a_d$ of k^d, which gives a matrix $A = (a_{ij}) \in \mathrm{GL}_d(k)$.

However by Lemma 6 we now get

$$\begin{aligned} 0 =& 0 \wedge \Big(\sum_j a_{2j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) = \Big(\sum_j a_{1j} v_j\Big) \wedge ... \wedge \Big(\sum_j a_{dj} v_j\Big) \\ =& \det(A)(v_1 \wedge ... \wedge v_d) \end{aligned}$$

and so $v = v_1 \wedge ... \wedge v_d = 0$ as $\det(A) \neq 0$. However, v was a representative of a point in $\mathbb{P}(\bigwedge^d V)$, a contradiction. The claim follows. $\square$