

Miniproject - Combinatorics

Simon Pohmann

We use the convention that $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.

1 Part I

Proposition 1. Let P be a graded poset in which every maximal chain has length $n+1$. Then the function

$$r : P \rightarrow \{0, \dots, n\}, \quad x \mapsto \max\{k \in \mathbb{N} \mid \exists a_1, \dots, a_k \in P : a_1 < a_2 < \dots < a_k < x\}$$

is well-defined and the unique function with $x < y$ implies $r(x) < r(y)$ for all $x, y \in P$. We call it the *rank function* of P .

Proof. Clearly r is well-defined, as for every increasing sequence $a_1 < \dots < a_k < x$, we have the chain $C := \{a_1, \dots, a_k, x\}$ of size $k+1$. Hence by assumption, $k \leq n$ and so $r(x)$ is finite and in $\{0, \dots, n\}$.

Further, consider $x < y$ in P . We have a sequence $a_1 < \dots < a_{r(x)} < x$ by definition of r . It follows that there is an increasing sequence $a_1 < \dots < a_{r(x)} < x < y$ and so $r(y) \geq r(x) + 1 > r(x)$.

Lastly, assume there was another function $r' : P \rightarrow \{0, \dots, n\}$ with this property. Consider any $x \in P$. By definition of r , there is an increasing sequence $a_1 < \dots < a_{r(x)} < x$ in P . Now consider a maximal chain C containing the chain $\{a_1, \dots, a_{r(x)}, x\}$.

Say $C = \{b_1, \dots, b_{n+1}\}$ with $b_1 < \dots < b_{n+1}$ and $x = b_j$. Note that we have the increasing sequence $b_1 < \dots < b_{j-1} < b_j = x$ and so by the definition of r , find $j \leq r(x) + 1$. On the other hand, have $a_1, \dots, a_{r(x)} \in C$ and thus $j = r(x) + 1$, i.e.

$$a_1 = b_1, \dots, a_{r(x)} = b_{r(x)}, \quad x = b_{r(x)+1}$$

As $b_i < b_{i+1}$, we know that $r'(b_i) < r'(b_{i+1})$ and inductively, we see $r'(b_i) \geq i - 1$. However, $r'(b_{n+1}) \leq n$ and thus $r'(b_i) = i - 1$. Finally it follows that $r'(x) = r'(b_{r(x)+1}) = r(x)$. \square

Now we will show some basic properties of the rank function.

Proposition 2. Let P be a graded poset of maximal rank n with rank function r . Then

- $x \in P$ is minimal iff $r(x) = 0$ and maximal iff $r(x) = n$.

- If $x < y$ and $r(x) + 1 \neq r(y)$ then there is $z \in P$ with $x < z < y$.
- If $x < y$ then there is an increasing sequence $x < a_1 < \dots < a_{r(y)-r(x)-1} < y$ in P .

Proof. For (i), let $x \in P$ be minimal. Then there is no increasing sequence $a_1 < x$ in P , so $r(x) \leq 0$. Conversely, let $r(x) = 0$. Assume there was $y \in P$ with $y < x$, then $r(y) < r(x) = 0$, a contradiction. The analogous statement for maximal elements is proved in the same way.

For (iii), consider $x < y$ in P . Then the chain $\{x, y\}$ is contained in a maximal chain $C \subseteq P$. Say $C = \{b_1, \dots, b_{n+1}\}$ with $b_1 < \dots < b_{n+1}$.

Hence we find $r(b_i) < r(b_{i+1})$ and so inductively that $r(b_i) \geq i-1$ and $r(b_i) \leq i-1$ since $r(b_1) \geq 0$ and $r(b_{n+1}) \leq n$. It follows that $r(b_i) = i-1$ and so $x = b_{r(x)+1}, y = b_{r(y)+1}$. Therefore, we have a chain

$$x < b_{r(x)+2} < \dots < b_{r(y)} < y$$

of length $r(y) - r(x) - 1$. Statement (ii) follows directly, as in this setting, have $r(y) \geq r(x) + 2$ and so $r(y) - r(x) - 1 \neq 0$. \square

2 Part II

Proposition 3. For a graded poset P with layers L_0, \dots, L_n the following statements are equivalent:

- For every antichain $A \subseteq P$ have

$$\sum_{i=0}^n \frac{|A \cap L_i|}{|L_i|} \leq 1$$

- For each $1 < i \leq n$ and $F \subseteq L_i$ have

$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$

where

$$\partial F := \{a \in L_{i-1} \mid \exists b \in F : a \leq b\}$$

- There exists a (nonempty) sequence of maximal chains C_1, \dots, C_t such that for all $1 \leq i \leq n$ we have:

$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

Proof. Show the directions (ii) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (ii)

(ii) \Rightarrow (iii) Use induction on n . Again, the base case is trivial, just take chains $\{x\}$ for each $x \in A = L_0$. So assume that $n > 0$. The set

$$A' := \bigcup_{i=0}^{n-1} L_i$$

is a graded poset of maximal rank $n-1$, and hence there are maximal chains $C_1, \dots, C_t \subseteq A'$ such that for all $1 \leq i \leq n-1$ have

$$|\{j \mid x \in C_j\}| = |\{j \mid y \in C_j\}| \text{ for all } x, y \in L_i$$

by induction hypothesis. Let now $V_1, V_2 := [t] \times L_n$ (treat them as disjoint) and consider the bipartite graph $G := (V_1 \sqcup V_2, E)$ where E is defined as follows:

$$\underbrace{\{(i, a)\}}_{\in V_1}, \underbrace{(j, b)\}_{\in V_2}} \in E \Leftrightarrow \max C_j < a$$

We use Hall's theorem to show that G has a perfect matching.

Consider any $W = \{(i_1, a_1), \dots, (i_w, a_w)\} \subseteq V_1$ and let $F = \{a_1, \dots, a_w\}$. Then

$$W \subseteq [t] \times F \Rightarrow |W| \leq t|F|$$

By choice of C_1, \dots, C_t , we have that the number of j with $x \in C_j$ is the same for all $x \in L_{n-1}$, say k . Since the C_i are maximal chains, each intersects L_{n-1} in exactly one element. So have bijection

$$\{(x, i) \mid x \in L_{n-1}, 1 \leq i \leq t, x \in C_i\} \rightarrow \{i \mid 1 \leq i \leq t\}, \quad (x, i) \mapsto i$$

where the set on the left-hand side has size $k|L_{n-1}|$. It follows that $k|L_{n-1}| = t$.

Since $\max C_j \in L_{n-1}$ for all j , we have

$$\begin{aligned} N(W) &= \{(j, b) \mid b \in L_n, \exists a \in F : \max C_j < a\} \\ &= L_n \times \{j \mid \exists a \in F : \max C_j < a\} \\ &= L_n \times \{j \mid \max C_j \in \partial F\} \end{aligned}$$

and so by the above

$$|N(W)| = |L_n| \sum_{\max C_j \in \partial F} 1 = |L_n| \sum_{x \in \partial F} \sum_{x \in C_j} 1 = |L_n| \sum_{x \in \partial F} k = |L_n| |\partial F| k$$

Together with the assumption that $|\partial F| |L_n| \geq |F| |L_{n-1}|$ we see that

$$|W| \leq t|F| = k|L_{n-1}| |F| \leq k|\partial F| |L_n| = |N(W)|$$

So Hall's theorem yields a perfect matching $M \subseteq E$ from V_1 to V_2 . As $|V_1| = |V_2|$, this is already a 1-to-one correspondence.

Now consider the sets

$$C'_m := C_j \cup \{a\} \text{ where } m = \underbrace{\{(i, a)\}}_{\in V_1}, \underbrace{(j, b)\}_{\in V_2}}_{\in M}$$

These are chains, as $\max C_j < a$ for each C'_m . Further, for each $x, y \in L_i, i < n$ have that

$$|\{C'_m \mid x \in C'_m\}| = |L_n \times \{j \mid x \in C_j\}| = |L_n \times \{j \mid y \in C_j\}| = |\{C'_m \mid y \in C'_m\}|$$

as M is a matching from V_2 to V_1 . Finally, for all $x \in L_n$ we have that

$$|\{C'_m \mid x \in C'_m\}| = |\{m \in M \mid \exists i, v \in V_2 : m = \{(i, x), v\}\}| = |\{(i, x) \mid (i, x) \in V_1\}| = t$$

as M is a matching from V_1 to V_2 .

(iii) \Rightarrow (i) Let C_1, \dots, C_t be a sequence of maximal chains given by the assumption. For $1 \leq i \leq n$ let k_i be the number of different j such that a fixed element $x \in L_i$ is contained in exactly the C_j . By assumption, this does not depend on x .

As in the direction above, we see that $k_i |L_i| = t$ because each C_j intersects L_i in exactly one element, so there is a bijection

$$\{(x, j) \mid x \in L_i, 1 \leq j \leq t, x \in C_j\} \rightarrow \{j \mid 1 \leq j \leq t\}, \quad (x, j) \mapsto j$$

and the set on the left has size $k_i |L_i|$.

Since each C_j is a chain and A is an antichain, we find that A and C_j intersect in at most one element. So

$$\begin{aligned} t &\geq |\{j \mid A \cap C_j \neq \emptyset\}| = \sum_{a \in A} |\{j \mid a \in C_j\}| = \sum_{i=0}^n \sum_{a \in A \cap L_i} |\{j \mid a \in C_j\}| \\ &= \sum_{i=0}^n \sum_{a \in A \cap L_i} k_i = \sum_{i=0}^n k_i |A \cap L_i| = \sum_{i=0}^n \frac{|A \cap L_i|}{|L_i|} t \end{aligned}$$

The claim follows by canceling t .

(i) \Rightarrow (ii) Consider $A := F \cup (L_{i-1} \setminus \partial F)$. This is clearly an antichain, as for $x \in F, y \in L_{i-1} \setminus \partial F$ have $y \not\prec x$. However, $r(y) < r(x)$ so also $x \not\prec y$, thus x and y are incomparable. Clearly elements from the same layer are incomparable.

Thus, the assumption yields that

$$\frac{|L_{i-1}| - |\partial F|}{|L_{i-1}|} + \frac{|F|}{|L_i|} = \frac{|A \cap L_{i-1}|}{|L_{i-1}|} + \frac{|A \cap L_i|}{|L_i|} = \sum_{j=0}^n \frac{|A \cap L_j|}{|L_j|} \leq 1$$

This gives

$$1 + \frac{|F|}{|L_i|} \leq 1 + \frac{|\partial F|}{|L_{i-1}|}$$

and the claim follows. \square

3 Part III

We even show the slightly stronger statement that the implication to the second condition of Prop.3 already holds “locally”, i.e. for each layer separately.

Proposition 4. Let P be a graded poset with layers L_0, \dots, L_n and let $0 < i \leq n$ such that

- Each element $a \in L_i$ covers the same number of elements in L_{i-1}
- Each element $a \in L_{i-1}$ is covered by the same number of elements in L_i

Then for each $F \subseteq L_i$ have

$$\frac{|\partial F|}{|L_{i-1}|} \geq \frac{|F|}{|L_i|}$$

In particular, if this is true for all i , then all the equivalent conditions from Prop. 3 follow.

Proof. Assume that for all $a \in L_i$

$$|\{b \in L_{i-1} \mid a \text{ covers } b\}| = k \in \mathbb{N}$$

and for all $b \in L_{i-1}$

$$|\{a \in L_i \mid a \text{ covers } b\}| = l \in \mathbb{N}$$

Double-counting pairs $(a, b) \in L_i \times L_{i-1}$ such that a covers b yields $k|L_i| = l|L_{i-1}|$. Now consider $F \subseteq L_i$. Note that by definition of ∂F , we have for all $a \in F$ that

$$\{b \in \partial F \mid a \text{ covers } b\} = \{b \in L_{i-1} \mid a \text{ covers } b\} = \partial\{a\} \subseteq \partial F$$

Again by double-counting pairs $(a, b) \in F \times \partial F$ such that a covers b , we find

$$\begin{aligned} k|F| &= \sum_{a \in F} k = \sum_{a \in F} |\{b \in L_{i-1} \mid a \text{ covers } b\}| = \sum_{a \in F} |\{b \in \partial F \mid a \text{ covers } b\}| \\ &= |\{(a, b) \in F \times \partial F \mid a \text{ covers } b\}| = \sum_{b \in \partial F} |\{a \in F \mid a \text{ covers } b\}| \\ &\leq \sum_{b \in \partial F} |\{a \in L_i \mid a \text{ covers } b\}| = \sum_{b \in \partial F} l = l|\partial F| \end{aligned}$$

Hence

$$\frac{|\partial F|}{|L_{i-1}|} = \frac{l|\partial F|}{l|L_{i-1}|} \leq \frac{k|F|}{k|L_i|} = \frac{|F|}{|L_i|}$$

□

4 Part IV

Example 5. Let $\Pi = \{\pi \subseteq \mathfrak{P}(n) \mid \pi \text{ partition}\}$ be the poset of partitions ordered by refinement. Then Π is a graded poset with rank function

$$r : \Pi \rightarrow \{0, \dots, n-1\}, \quad \pi \mapsto n - |\pi|$$

Proof. First of all, have $1 \leq |X| \leq n$ for all subsets $X \subseteq \mathfrak{P}(n)$, hence the function r is well-defined. Next we show that for all $x < y$ have $r(x) < r(y)$. However, if x is a proper refinement of y , then clearly $|x| > |y|$, so $r(x) < r(y)$.

Now assume there is a maximal chain $\pi_1 < \dots < \pi_t$ in Π . As the chain is maximal and Π has the maximal element $[n]$, we may assume that $\pi_t = [n]$ and thus $r(\pi_t) = n-1$. Similarly, we may assume that $p_1 = \mathfrak{P}(n)$ and so $r(\pi_1) = 0$. We have to show that $t = n$, and so it suffices to show that $r(\pi_i) = 1 + r(\pi_{i-1})$ for all $0 < i \leq t$.

Assume not, i.e. $|\pi_{i-1}| \geq 2 + |\pi_i|$ for some $0 < i \leq t$. Since π_{i-1} is a refinement of π_i , we find that

$$\pi_{i-1} = \{A_{11}, \dots, A_{1m_1}, \dots, A_{n1}, \dots, A_{nm_n}\} \quad \text{with} \quad \pi_i = \left\{ \bigcup_{j \leq m_1} A_{1j}, \dots, \bigcup_{j \leq m_n} A_{nj} \right\}$$

where $n = |\pi_i|$ and $\sum_j m_j = |\pi_{i-1}| \geq 2 + n$. Hence, for at least two different $s, r \leq n$ we have $m_s, m_r \geq 2$. Now define

$$\tilde{\pi} := \{A_{jk} \mid j \neq s\} \cup \left\{ \bigcup_{j \leq m_s} A_{sj} \right\}$$

Then clearly $\pi_{i-1} < \tilde{\pi} \leq \pi_i$. However, we know that $\bigcup_j A_{rj} \in \pi_i$ but $\bigcup_j A_{rj} \notin \tilde{\pi}$. Thus $\pi_{i-1} < \tilde{\pi} < \pi_i$ and we found a longer chain

$$\pi_1 < \dots < \pi_{i-1} < \tilde{\pi} < \pi_i < \dots < \pi_t$$

which contradicts the assumed maximality. The claim follows. \square

Proposition 6. For the number $S(m, k) := |\{\pi \in \Pi \mid |\pi| = k\}|$ we have recursion

$$S(m, 1) = 1 \quad \text{and} \quad S(m, k) = \sum_{n=1}^{m-1} \binom{m-1}{n} S(n, k-1)$$

where $1 \leq k \leq m$.

Proof. Use induction on k . The base case $k = 1$ is trivial, so let $k > 1$. For any partition

$\pi \in \Pi_m$, denote by $A(1, \pi)$ the unique set $A \in \pi$ with $1 \in A$. We find

$$\begin{aligned}
S(m, k) &= |\{\pi \in \Pi_m \mid |\pi| = k\}| = \left| \bigcup_{\emptyset \neq A \subseteq [m], 1 \in A} \{\pi \in \Pi_m \mid |\pi| = k, A(1, \pi) = A\} \right| \\
&= \sum_{\emptyset \neq A \subseteq [m], 1 \in A} |\{\pi \in \Pi_m \mid |\pi| = k, A(1, \pi) = A\}| \\
&= \sum_{\emptyset \neq A \subseteq [m], 1 \in A} |\{\pi \in \Pi_m \mid |\pi \setminus \{A\}| = k - 1, A(1, \pi) = A\}| \\
&= \sum_{\emptyset \neq A \subseteq [m], 1 \in A} |\{\pi \in \Pi_{m-|A|} \mid |\pi| = k - 1\}| \\
&= \sum_{i=1}^m \sum_{A \subseteq [m], |A|=i, 1 \in A} |\{\pi \in \Pi_{m-i} \mid |\pi| = k - 1\}| \\
&= \sum_{i=1}^m \binom{m-1}{i-1} S(m-i, k-1) = \sum_{i=0}^{m-1} \binom{m-1}{m-i-1} S(i, k-1) \\
&= \sum_{i=1}^{m-1} \binom{m-1}{i} S(i, k-1)
\end{aligned}$$

as for $k > 1$ have that $S(0, k) = 0$. □

Example 7. Have for $m \geq 3$ that

$$S(m, 2) = 2^{m-1} - 1 \quad \text{and} \quad S(m, 3) = \frac{1}{2}(3^{m-1} - 2^m + 1)$$

Note that this is the partition into *exactly* k sets; I am not completely sure what the exercise description tells us to show, but if it is about the number of ways of decomposing $[m]$ into *at most* k sets, then have

$$\Sigma S(m, k)(2) = 2^{m-1} \quad \text{and} \quad \Sigma S(m, k)(3) = 3^{m-1} = \frac{1}{2}3^{m-1} - \frac{1}{2}$$

for $m \geq 3$. Here $\Sigma S(m, k) = \sum_{l \leq k} S(m, l)$ is the number of partitions of $[m]$ into at most k sets.

Proof. By Prop. 6 have that for $m \geq 1$ we have $S(m, 1) = 1$. Further, for $m \geq 2$ have

$$S(m, 2) = \sum_{n=1}^{m-1} \binom{m-1}{n} 1 = 2^{m-1} - 1$$

Applying this once more yields for $m \geq 3$ that

$$\begin{aligned}
S(m, 3) &= \sum_{n=1}^{m-1} \binom{m-1}{n} (2^{n-1} - 1) = \frac{1}{2} \sum_{n=1}^{m-1} \binom{m-1}{n} 2^n - \sum_{n=1}^{m-1} \binom{m-1}{n} 1 \\
&= \frac{1}{2} \sum_{n=0}^{m-1} \binom{m-1}{n} 2^n - \frac{1}{2} - \sum_{n=0}^{m-1} \binom{m-1}{n} 1 + 1 \\
&= \frac{1}{2} (1+2)^{m-1} - 2^{m-1} + \frac{1}{2} = \frac{1}{2} (3^{m-1} - 2^m + 1)
\end{aligned}$$

□