

Miniproject - Analytic Number Theory

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We use the convention that $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Further, we write $a \mid b$ if a divides b and $a \perp b$ if a and b are coprime. Finally, let \mathbb{P} be the set of prime numbers in \mathbb{N} .

1 Part I

For convenience, we include the definition of a Dirichlet character from the task description first.

Definition 1. Let $q \geq 2$, then a *Dirichlet character (mod q)* is a function $\chi : \mathbb{N} \rightarrow \mathbb{C}$ such that

- χ is completely multiplicative, so $\chi(a)\chi(b) = \chi(ab)$
- χ is periodic modulo q , so $\chi(n + q) = \chi(n)$
- $\chi(n) \neq 0$ if and only if $n \perp q$

First, we will give another characterization of Dirichlet characters.

Lemma 2 (Characterization of Dirichlet characters). We have a one-to-one correspondence between Dirichlet characters mod q and group homomorphisms $(\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ via

$$\{f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \mid f \text{ group hom}\} \rightarrow \{\chi : \mathbb{N} \rightarrow \mathbb{C} \mid \chi \text{ Dirichlet character mod } q\}$$
$$f \mapsto \chi_f := \left(\mathbb{N} \rightarrow \mathbb{C}, n \mapsto \begin{cases} f([n]) & \text{if } n \perp q \\ 0 & \text{otherwise} \end{cases} \right)$$

Proof. First of all, we show that the map is well-defined. Let $f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be a (multiplicative) group homomorphism, and we show that χ_f is a Dirichlet character.

Note that property (ii) and (iii) directly follow from the definition, as $\chi_f(n)$ only depends on the value of $n \pmod q$. So consider some $a, b \in \mathbb{N}$. If both $a \perp q$ and $b \perp q$ then

$$\chi_f(a)\chi_f(b) = \chi([a])\chi([b]) = \chi([ab]) = \chi_f(ab)$$

as also $ab \perp q$.

On the other hand, if $a \not\perp q$ or $b \not\perp q$ have $\chi_f(a) = 0$ resp. $\chi_f(b) = 0$. We also have in this case that $ab \not\perp q$ and so

$$\chi_f(a)\chi_f(b) = 0 = \chi_f(ab)$$

Now it is left to show that the correspondence is a bijection. Clearly, if $f \neq g$ then $f(x) \neq g(x)$ for some $x \in (\mathbb{Z}/q\mathbb{Z})^\times$ and so $\chi_f(n) \neq \chi_g(n)$ for some representative $n \in \mathbb{N}$ of x .

To show surjectivity, consider some Dirichlet character $\chi : \mathbb{N} \rightarrow \mathbb{C}$ and construct a group homomorphism $f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$. For each $x \in (\mathbb{Z}/q\mathbb{Z})^\times$, there is a representative $n \in \mathbb{N}$ of x and as $\chi(n)$ does not depend on the choice of n , we may define $f(x) := \chi(n)$. Note that as $x \in (\mathbb{Z}/q\mathbb{Z})^\times$, we find $n \perp q$ and so $\chi(n) \neq 0$, i.e. $\chi(n) \in \mathbb{C}^*$. Then clearly for $a, b \in (\mathbb{Z}/q\mathbb{Z})^*$ with representatives $n, m \in \mathbb{N}$ have

$$f(ab) = \chi(nm) = \chi(n)\chi(m) = f(a)f(b)$$

So f is a well-defined group homomorphism and we obviously have $\chi_f = \chi$. \square

For simplicity of notation we sometimes will identify a Dirichlet character and its group homomorphism if it is always clear which one is meant.

Example 3 (Ex (i)). The function

$$f : \mathbb{N} \rightarrow \mathbb{C}, \quad n \mapsto \begin{cases} 0 & \text{if } n \equiv 0, 2 \pmod{4} \\ 1 & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

is a Dirichlet character.

Proof. This follows directly from Lemma 2, as $f = \chi_g$ for the group homomorphism

$$g : (\mathbb{Z}/4\mathbb{Z})^\times = \{1, 3\} \rightarrow \mathbb{C}^*, \quad 1 \mapsto 1, \quad 3 \mapsto -1$$

(this is a group homomorphism, as $3^2 = 9 \equiv 1 \pmod{4}$) \square

Now we want to define Dirichlet series of Dirichlet characters.

Proposition 4. For a Dirichlet character $\chi : \mathbb{N} \rightarrow \mathbb{C}$ and some $\epsilon > 0$, the series

$$L(s, \chi) := \sum_{n \geq 1} \chi(n)n^{-s}$$

converges uniformly on $\Re(s) \geq 1 + \epsilon$. We will call it the Dirichlet series of χ .

Proof. By Lemma 2, we know that χ corresponds to a group homomorphism $f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ such that $\chi(\mathbb{N}) = f((\mathbb{Z}/q\mathbb{Z})^*) \cup \{0\} \subseteq \mathbb{C}$ is a finite subset of \mathbb{C} . Hence, there is $C > 0$ with $|\chi(n)| \leq C$ for all $n \in \mathbb{N}$, and it follows that

$$\sum_{1 \leq n \leq X} |\chi(n)n^{-s}| \leq \sum_{1 \leq n \leq X} C |n^{-s}| \leq C \sum_{1 \leq n \leq X} n^{-1-\epsilon} \leq C \sum_{n \geq 1} n^{-1-\epsilon}$$

which is finite. \square

Proposition 5. Let $f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be a group homomorphism. Then for the associated Dirichlet character $\chi = \chi_f$ we have that

$$\lim_{s \rightarrow 1^+} L(s, \chi) \text{ exists} \Leftrightarrow \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^\times} f(x) = 0$$

If this is the case, then

$$\lim_{s \rightarrow 1^+} L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n}$$

where the right-hand side converges (but not absolutely).

Proof. Let $c = \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^\times} f(x)$. For the direction \Rightarrow assume that $c \neq 0$. Then have for $\Re(s) > 1$ that

$$\begin{aligned} \operatorname{sgn}(c) \sum_{n \geq 1} \chi(n) n^{-s} &= \sum_{n \geq 0} \sum_{0 < k \leq q} \operatorname{sgn}(c) \chi(qn + k) (qn + k)^{-s} \\ &\geq \sum_{n \geq 0} \sum_{0 < k \leq q} \operatorname{sgn}(c) \chi(qn + k) (qn + n)^{-s} \\ &= \sum_{n \geq 0} \operatorname{sgn}(c) (qn + n)^{-s} \underbrace{\sum_{0 < k \leq q} \chi(qn + k)}_{=c} \\ &\geq \frac{|c|}{(q+1)^s} \sum_{n \geq 1} n^{-s} = \frac{|c|}{(q+1)^s} \zeta(s) \end{aligned}$$

which clearly has a pole at $s = 1$. Hence $\lim_{s \rightarrow 1^+} L(s, \chi_f)$ cannot exist.

For the other direction, assume that $c = 0$. We will only consider real s now. Observe that by Bernoulli's inequality, have for $0 < s \leq 1$ that

$$\begin{aligned} (qn)^{-s} - (qn + k)^{-s} &= \frac{(qn + k)^s - (qn)^s}{(q^2 n^2 + qnk)^s} = (qn)^s \frac{(1 + k(qn)^{-1})^s - 1}{(q^2 n^2 + qnk)^s} \\ &\leq (qn)^s \frac{sk(qn)^{-1}}{(q^2 n^2 + qnk)^s} = \frac{sk}{qn(qn + k)^s} = O(sn^{-s-1}) \end{aligned}$$

If $s > 1$ and $k \leq q$, then also $(qn)^{-s} - (qn + k)^{-s} = O(sn^{-(1+\epsilon)})$ for some small enough $0 < \epsilon < 1$. As $\chi((\mathbb{Z}/q\mathbb{Z})^\times) \subseteq \mathbb{C}$ is finite, find $C > 0$ with $|\chi_f(n)| \leq C$ for all $n \in \mathbb{N}$.

Then for all $s \geq \epsilon$ and $X \leq Y$ it holds

$$\begin{aligned}
& \sum_{X \leq n \leq Y} \chi(n) n^{-s} \\
&= O(qCX^{-s} + qCY^{-s}) + \sum_{X/q \leq n \leq Y/q} \sum_{0 < k \leq q} \chi(qn + k) \left((qn)^{-s} + \underbrace{(qn + k)^{-s} - (qn)^{-s}}_{=O(sn^{-(1+\epsilon)})} \right) \\
&= O(qCX^{-s}) + \sum_{X/q \leq n \leq Y/q} \left((qn)^{-s} c + \sum_{0 < k \leq q} O(Csn^{-(1+\epsilon)}) \right) = \\
&= O(qCX^{-s}) + 0 + O\left(Cqs \sum_{X/q \leq n \leq Y/q} n^{-(1+\epsilon)}\right) \\
&= O(qCX^{-s}) + O\left(Cqs \sum_{X/q \leq n} n^{-(1+\epsilon)}\right)
\end{aligned}$$

which is well-defined and finite. Further, the expression converges uniformly (as a function in s on $[\epsilon, \infty[$) to 0 as $X \rightarrow \infty$. So

$$\sum_{n < X} \chi(n) n^{-s} \text{ converges uniformly to } \sum_{n \geq 1} \chi(n) n^{-s}$$

as $X \rightarrow \infty$ (on $[\epsilon, \infty[$). Thus the limit is continuous and extends $L(s, \chi_f)$ defined on $]1, \infty[$. It follows that $\lim_{t \rightarrow s^+} L(t, \chi_f)$ exists and is equal to $\sum_{n \geq 1} \chi(n) n^{-s}$. \square

Applied to our example, we find

Example 6 (Ex (ii)). Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be the Dirichlet character from Example 3 with corresponding group homomorphism $g : (\mathbb{Z}/4\mathbb{Z})^\times \rightarrow \mathbb{C}$. Then

$$\sum_{x \in (\mathbb{Z}/4\mathbb{Z})^*} g(x) = g(1) + g(3) = 1 - 1 = 0$$

and so by Lemma 5 the limit $\lim_{s \rightarrow 1^+} L(s, f)$ exists. The lemma further yields that

$$\begin{aligned}
\lim_{s \rightarrow 1} L(s, f) &= \sum_{n \geq 1} f(n) n^{-1} = \sum_{n \geq 0} \frac{f(4n+1)}{4n+1} + \frac{f(4n+3)}{4n+3} = \sum_{n \geq 0} \frac{1}{4n+1} - \frac{1}{4n+3} \\
&= 2 \sum_{n \geq 0} \frac{1}{(4n+1)(4n+3)} > 0
\end{aligned}$$

is positive. Wolfram Alpha [Wol] can give an explicit value to this sum, using the digamma function ψ . Namely

$$\sum_{x \in (\mathbb{Z}/4\mathbb{Z})^\times} f(x) x^{-1} = \frac{1}{4} (\psi(\frac{7}{4}) - \psi(\frac{5}{4}))$$

which seems to be $\frac{1}{4}$.

Interestingly, we can also study the series

$$\sum_p f(p)p^{-s}$$

using Euler Products.

Example 7 (Ex (ii)). Have for $\Re(s) > 1$ that

$$\sum_p \frac{f(p)}{p^s} = \log(L(f, s)) + O(\zeta(2\Re(s)))$$

In particular, we see that

$$\lim_{s \rightarrow 1^+} \sum_p \frac{f(p)}{p^s}$$

exists.

Proof. By Taylor series expansion, see that

$$\log(1 - x) = -x + O(|x|^2)$$

Hence for $\Re(s) > 1$ we can consider the logarithm of the Euler Product to find

$$\begin{aligned} \log(L(f, s)) &= \log\left(\prod_p \sum_{k \geq 0} \frac{f(p^k)}{p^{ks}}\right) = \log\left(\prod_p \frac{1}{1 - f(p)/p^s}\right) \\ &= \sum_p -\log\left(1 - \frac{f(p)}{p^s}\right) = \sum_p \frac{f(p)}{p^s} + O\left(\frac{f(p)^2}{p^{2\Re(s)}}\right) \\ &= \sum_p \frac{f(p)}{p^s} + O\left(\underbrace{\sum_p \frac{f(p)}{p^{2\Re(s)}}}_{\leq \zeta(2\Re(s))}\right) = \sum_p \frac{f(p)}{p^s} + O(\zeta(2\Re(s))) \end{aligned}$$

□

Lemma 8. Let $n \equiv 3 \pmod{4}$. Then n has a prime factor $p \equiv 3 \pmod{4}$.

Proof. Use induction on n . If $n = 3$, the claim is trivial. So let $n > 3$. If n is prime, the claim again follows. Otherwise, have $n = ab$ with nontrivial divisors a, b . However, $3 \equiv n$ is not a square modulo 4, so find that $a \not\equiv b \pmod{4}$. As both a and b must be odd, we see that either $a \equiv 3 \pmod{4}$ or $b \equiv 3 \pmod{4}$ and the claim follows by the induction hypothesis. □

Corollary 9 (Ex (iii)). There are infinitely many primes p with $p \equiv 3 \pmod{4}$.

Proof. Assume there were only finitely many, say p_1, \dots, p_N . Let $P := p_1 \dots p_N$ if N is even and $P := p_1^2 p_2 \dots p_N$ if N is odd. Then

$$P \equiv 3^{2^{\lceil \frac{N}{2} \rceil}} \equiv 1^{\lceil \frac{N}{2} \rceil} = 1 \pmod{4}$$

Thus, by Lemma 8, $P+2$ has a prime factor $q \equiv 3 \pmod{4}$. However, $q \neq p_i$ as $p_i \nmid P+2$ for all i (if $p_i \mid P+2$, then $p_i \mid P+2-P=2$, a contradiction). This contradicts our assumption. \square

For the case of primes $\equiv 1 \pmod{4}$, I have remembered the two-square theorem and its connection to primes in the ring $\mathbb{Z}[i]$ of Gaussian integers, and somehow my train of thoughts went into Algebraic Number Theory. After some research, I have found an exercise in [Neu92, Chapter I, §10] that requires the reader to prove the following proposition.

Proposition 10. Let $q \geq 3$ be an integer. Then there are infinitely many primes p with $p \equiv 1 \pmod{q}$.

Proof. Assume there were only finitely many such primes p_i , then we have their product $P = \prod_i p_i \in \mathbb{Z}$. Consider now the q -th cyclotomic polynomial Φ_q . Clearly $\Phi_q(qPX) - 1 \in \mathbb{Q}[X]$ has at most $\phi(q)$ zeros, so there exists some $x \in \mathbb{Z}$ with $\Phi_q(qPx) \neq 1$ (this “Ansatz” was given as a hint).

Let now $K = \mathbb{Q}(\omega_q)$ be the q -th cyclotomic number field with a primitive q -th root of unity ω_q (i.e. $\Phi_q(\omega_q) = 0$). Let further $\mathcal{O} \subseteq K$ be the ring of integral elements over \mathbb{Z} in K . The prime decomposition law for Dedekind ring extension [Neu92, Chapter I, Prop 8.3] tells us that for a prime p , the ideal (p) is reducible in \mathcal{O} if and only if $\Phi_q \pmod{p}$ is reducible. As $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic of order $p-1$, this is the case if and only if $q \mid p-1$, i.e. $p \equiv 1 \pmod{q}$.

Now consider the element $\alpha = \omega_q - xqP \in \mathcal{O}$. Then

$$\begin{aligned} N_{K/\mathbb{Q}}(\alpha) &= \prod_{\sigma: K \rightarrow \mathbb{C} \text{ } \mathbb{Q}\text{-field homomorphism}} \sigma(\omega_q - xqP) \\ &= \prod_{\sigma} (\sigma(\omega_q) - xqP) = \text{MiPo}_{\mathbb{Q}}(\omega_q)(xqP) = \Phi_q(xqP) \neq 1 \end{aligned}$$

as $\text{MiPo}_{\mathbb{Q}}(\omega_q) = \prod_{\sigma} (\sigma(\omega_q) - X)$. Hence, α is not a unit in \mathcal{O} . On the other hand, (α) is coprime to (p_i) for each p_i , as

$$\omega_q = \alpha - xqP \in (\alpha) + (p_i) \quad \text{and} \quad \omega_q \in \mathcal{O}^\times$$

By our assumption, the only prime ideals in \mathcal{O} are the prime ideal factors of (p_i) and (p) for $p \neq p_i$. Thus, the prime ideal factorization of (α) consists only of prime ideals (p) , $p \neq p_i$ and it follows that $(\alpha) = (n)$ for some integer $n \geq 2$. As ω_q and $xqP \in \mathbb{Z}$ are \mathbb{Q} -linearly independent, we see that $n \mid \omega_q$ and $n \mid xqP$. However, the former is a contradiction, as $\omega_q \in \mathcal{O}^\times$ is a unit and no $n \geq 2$ is a unit. \square

The book also mentions that the general case can be proven by using L-series in algebraic number fields.

Corollary 11 (Ex (iii)). There are infinitely many primes p with $p \equiv 1 \pmod{4}$.

Proof. This is just a special case of Prop. 10. □

Example 12 (Ex (iii)). Using a computer, we can also study the actual frequency of prime numbers $\equiv 1, 3 \pmod{4}$ among e.g. the first 10^8 integers. This seems to indicate that both numbers are asymptotically equal, which seems natural, given the result of Example 7. For example, there are 332180 primes $\equiv 1 \pmod{4}$ and 332398 primes $\equiv 3 \pmod{4}$ smaller than 10^8 . To find these numbers, the following python code was used.

```
import itertools
import math

def primes():
    yield 2
    found_primes = [2]
    for n in itertools.count(3):
        for p in found_primes:
            if n % p == 0:
                break
            elif p >= math.sqrt(n):
                yield n
                found_primes.append(n)
                break

def primes_leq(n):
    return itertools.takewhile(lambda p: p <= n, primes())

for i in range(1, 8):
    print("Consider interval [1, 10**" + str(i) + "]")
    print("__Number of primes ≡ 1 mod 4 is __" + str(
        sum(1 for p in primes_leq(10**i) if (p - 1) % 4 == 0)
    ))
    print("__Number of primes ≡ 3 mod 4 is __" + str(
        sum(1 for p in primes_leq(10**i) if (p - 3) % 4 == 0)
    ))
    print()
```

2 Part II

We have already shown that Dirichlet characters are, in principle, group homomorphisms $(\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$. If we now assume q to be prime, we get an even nicer characterization.

So for the whole section, assume that $q \geq 3$ is a prime.

Corollary 13 (Ex (i)). Let $\chi, \chi' : \mathbb{N} \rightarrow \mathbb{C}$ be Dirichlet characters mod q and r a representative of a primitive root modulo q . If $\chi(r) = \chi'(r)$, then $\chi = \chi'$. Further, have that $\chi(n)^{q-1} = 1$ for all $n \in \mathbb{N}$ with $n \perp q$.

Proof. The properties follow directly from Lemma 2. Let $f, f' : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be the associated group homomorphisms of χ, χ' as in Lemma 2. If $f([r]) = \chi(r) = \chi'(r) = f'([r])$ then clearly $f = f'$, as these are group homomorphisms and $\langle [r] \rangle = (\mathbb{Z}/q\mathbb{Z})^\times$. Hence $\chi = \chi'$.

Further, have for $n \in \mathbb{N}$ with $n \perp q$ that $[n] \in (\mathbb{Z}/q\mathbb{Z})^\times$ and thus

$$[n]^{q-1} = [n]^{\phi(q)} = [n]^{|(\mathbb{Z}/q\mathbb{Z})^\times|} = 1$$

As f is a group homomorphism, find

$$\chi(n)^{q-1} = f([n])^{q-1} = f([n]^{q-1}) = f(1) = 1$$

□

This correspondence also works in the other direction.

Corollary 14 (Ex (ii)). Let $\omega \in \mathbb{C}$ be a $(q-1)$ -th root of unity, i.e. $\omega^{q-1} = 1$ and let $r \in \mathbb{Z}$ be a representative of a primitive root modulo q . Then

$$g : \mathbb{N} \rightarrow \mathbb{C}, \quad n \mapsto \begin{cases} \omega^{\log_r n} & \text{if } n \perp q \\ 0 & \text{otherwise} \end{cases}$$

is a well-defined Dirichlet character.

Proof. Follows again directly from Lemma 2, as $[r] \mapsto \omega$ induces a unique group homomorphism $(\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$. The associated Dirichlet character is obviously g . □

Note that the image of a group homomorphism $f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ is a subgroup of \mathbb{C}^\times . Using Corollary 13, we can describe it quite concretely.

Proposition 15. Let $\chi : \mathbb{N} \rightarrow \mathbb{C}$ be a Dirichlet character with group homomorphism $f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$. Then $\text{im } f \leq S$ is a subgroup where $S_q := \{\omega_q^k \mid k \in \mathbb{Z}\}$ is the group of q -th roots of unity.

It is a fact from Algebra that $S_q \cong (\mathbb{Z}/q\mathbb{Z})^\times$, hence Dirichlet characters modulo a prime q are in 1-to-1 correspondence with the endomorphisms $\text{End}((\mathbb{Z}/q\mathbb{Z})^\times)$ of $(\mathbb{Z}/q\mathbb{Z})^\times$.

Proof. We have that $S_q = \{x \in \mathbb{C}^\times \mid x^{q-1} = 1\}$ and the claim directly follows from Corollary 13. □

Note that the endomorphism monoid $\text{End}((\mathbb{Z}/q\mathbb{Z})^\times)$ is not a group, except in the trivial case $q = 2$. The reason is that e.g. the trivial group homomorphism $r \mapsto 1$ is not surjective and thus not invertible.

Definition 16. Denote by $\text{Dir}(q)$ the set of Dirichlet characters modulo q .

By Corollary 14 each group endomorphism $f \in \text{End}((\mathbb{Z}/q\mathbb{Z})^\times)$ is determined by its value at a primitive root of unity $r \in (\mathbb{Z}/q\mathbb{Z})^\times$, hence

$$|\text{Dir}(q)| = |\text{End}((\mathbb{Z}/q\mathbb{Z})^\times)| = |(\mathbb{Z}/q\mathbb{Z})^\times| = q - 1$$

It follows that there are exactly $q - 1$ distinct Dirichlet characters modulo a prime q .

Remark 17. It is again a fact that $(\mathbb{Z}/p^k\mathbb{Z})^\times$ is cyclic for an odd prime p and $k \geq 1$. Hence, everything up to now can also be done for odd prime powers, if we replace $q - 1$ by $\phi(q)$.

Because of Lemma 5 it might seem like a good idea to study in which cases the value $\sum_{x \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(x)$ is zero.

Proposition 18 (Ex (iii)). Let χ_0 be the trivial Dirichlet character given by $r \mapsto 1$. Then

$$\begin{aligned} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) &= \begin{cases} q - 1 & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases}, \\ \sum_{\chi \in \text{Dir}(q)} \chi(a) &= \begin{cases} q - 1 & \text{if } a \equiv 1 \pmod{q} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Furthermore, for $b \not\equiv 1 \pmod{q}$ have

$$\sum_{\chi \in \text{Dir}(q)} \chi(a) \overline{\chi(b)} = \begin{cases} q - 1 & \text{if } a \equiv b \pmod{q} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Clearly

$$\sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi_0(a) = q - 1 \quad \text{and} \quad \sum_{\chi \in \text{Dir}(q)} \chi(1) = \sum_{\chi \in \text{Dir}(q)} 1 = q - 1$$

So it is left to show that we get zero in the other cases.

Consider a Dirichlet character $\chi \neq \chi_0$ given by $r \mapsto \xi$ for a q -th root of unity $\xi \neq 1$. Then

$$\sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) = \sum_{k=0}^{q-2} \chi(r^k) = \sum_{k=0}^{q-2} \xi^k = \frac{1 - \xi^{q-1}}{1 - \xi} = 0$$

By using the earlier results on the structure of $\text{Dir}(q)$ we see that for $a \equiv r^k \not\equiv 1 \pmod{q}$, have

$$\begin{aligned} \sum_{\chi \in \text{Dir}(q)} \chi(a) &= \sum_{\chi \in \text{Dir}(q)} \chi(r)^k \\ &= \sum_{\xi \text{ } q\text{-th root of unity}} \xi^k = \sum_{l=0}^{q-2} \omega^{kl} = \frac{1 - (\omega^{q-1})^k}{1 - \omega^k} = 0 \end{aligned}$$

where $\omega \in \mathbb{C}$ is a primitive q -th root of unity and $r \in \mathbb{Z}$ is a primitive root modulo q .

For the last part, note that for any q -th root of unity ξ , we have $\xi\bar{\xi} \in \mathbb{R}$ with $\xi\bar{\xi} = |\xi|^2 > 0$. Furthermore, $\bar{\xi}$ is also a q -th root of unity, and so we see that $\xi\bar{\xi} = 1$ (the only real, positive root of unity is 1). It follows that for any Dirichlet character χ have $\overline{\chi([a])} = \chi([a]^{-1})$. Thus

$$\sum_{\chi \in \text{Dir}(q)} \chi(a)\overline{\chi(b)} = \sum_{\chi \in \text{Dir}(q)} \chi([a][b]^{-1}) = \begin{cases} q-1 & \text{if } [a][b]^{-1} = 1 \in (\mathbb{Z}/q\mathbb{Z})^\times \\ 0 & \text{otherwise} \end{cases}$$

The condition $ab^{-1} = 1$ is equivalent to $a \equiv b \pmod{q}$, so the claim follows. \square

Using these basic results, we can now prove facts on the Dirichlet series of characters.

Proposition 19 (Ex (iv)). Let $a \perp q$. Then for $\Re(s) > 1$ have

$$\sum_{n \equiv a \pmod{q}} \frac{\Lambda(n)}{n^s} = \frac{1}{q-1} \sum_{\chi \in \text{Dir}(q)} \overline{\chi(a)} \sum_{n \geq 1} \frac{\Lambda(n)\chi(n)}{n^s}$$

(All those series obviously converge absolutely since $\Re(s) > 1$)

Proof. By Prop. 18 we have for all $n \in \mathbb{N}$ that

$$\frac{1}{q-1} \sum_{\chi \in \text{Dir}(q)} \overline{\chi(a)}\chi(n) = \begin{cases} 1 & \text{if } a \equiv n \pmod{q} \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$\begin{aligned} \sum_{n \equiv a \pmod{q}} \Lambda(n)n^{-s} &= \sum_{n \geq 1} \Lambda(n)n^{-s} \begin{cases} 1 & \text{if } a \equiv n \pmod{q} \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{n \geq 1} \Lambda(n)n^{-s} \frac{1}{q-1} \sum_{\chi \in \text{Dir}(q)} \overline{\chi(a)}\chi(n) \\ &= \frac{1}{q-1} \sum_{\chi \in \text{Dir}(q)} \overline{\chi(a)} \sum_{n \geq 1} \Lambda(n)\chi(n)n^{-s} \end{aligned}$$

as infinite summation clearly commutes with finite sums. \square

Example 20 (Ex (v)). We consider the Dirichlet characters mod 5. A (primitive) 5-th root of unity $\omega_5 \in \mathbb{C}$ is given by $\omega_5 = \exp(2\pi i/5)$. On the other hand, a primitive root modulo 5 is e.g. given by $r = 2$ since $2^2 \equiv -1 \pmod{5}$. Thus we have the trivial Dirichlet character χ_0 and $5-1=4$ nontrivial Dirichlet characters mod 5, namely those given by

$$\begin{aligned} \chi_1 : 1 \mapsto 1, 2 \mapsto \omega_5 &= \exp(2\pi i/5), 3 \mapsto \omega_5^3 = \exp(6\pi i/5), 4 \mapsto \omega_5^2 = \exp(4\pi i/5), \\ \chi_2 : 1 \mapsto 1, 2 \mapsto \omega_5^2 &= \exp(4\pi i/5), 3 \mapsto \omega_5 = \exp(2\pi i/5), 4 \mapsto \omega_5^4 = \exp(8\pi i/5), \\ \chi_3 : 1 \mapsto 1, 2 \mapsto \omega_5^3 &= \exp(6\pi i/5), 3 \mapsto \omega_5^4 = \exp(8\pi i/5), 4 \mapsto \omega_5^1 = \exp(2\pi i/5), \\ \chi_4 : 1 \mapsto 1, 2 \mapsto \omega_5^4 &= \exp(8\pi i/5), 3 \mapsto \omega_5^2 = \exp(4\pi i/5), 4 \mapsto \omega_5^3 = \exp(6\pi i/5), \end{aligned}$$

3 Part III

Again, let $q \geq 3$ be a prime. Let further χ be a Dirichlet character mod q .

Proposition 21 (Ex (i)). For $\Re(s) > 1$ have that

$$\frac{L(s, \chi)'}{L(s, \chi)} = \sum_{n \geq 1} \frac{\Lambda(n) \chi(n)}{n^s}$$

Proof. Consider any $\epsilon > 0$. The series

$$\sum_{n \geq 1} \frac{d}{ds} \chi(n) n^{-s} = \sum_{n \geq 1} \chi(n) \log(n) n^{-s}$$

converges uniformly on $\Re(s) \geq 1 + \epsilon$, as $|\chi(n)| \leq C$ for some $C > 0$ and all $n \in \mathbb{N}$ (by the lecture, we know that $\sum_n \log(n) n^{-s}$ converges uniformly on $\Re(s) \geq 1 + \epsilon$). Hence, we may interchange summation and differentiation to get

$$L(s, \chi)' = \sum_{n \geq 1} \chi(n) \frac{d}{ds} n^{-s} = \sum_{n \geq 1} \chi(n) \log(n) n^{-s}$$

for $\Re(s) \geq 1 + \epsilon$. As $\epsilon > 0$ was arbitrary, we get

$$L(s, \chi)' = \sum_{n \geq 1} \chi(n) \log(n) n^{-s}$$

for all $\Re(s) > 1$.

Furthermore, χ and μ are multiplicative, and hence so is $(\chi\mu)(n) := \chi(n)\mu(n)$. Thus we have the Euler products

$$\sum_{n \geq 1} \mu(n) \chi(n) n^{-s} = \prod_{p \in \mathbb{P}} \sum_{k \in \mathbb{N}} \mu(p^k) \chi(p^k) p^{-sk} = \prod_{p \in \mathbb{P}} (1 - \chi(p) p^{-s})$$

and

$$\sum_{n \geq 1} \chi(n) n^{-s} = \prod_{p \in \mathbb{P}} \sum_{k \in \mathbb{N}} \chi(p^k) p^{-sk} = \prod_{p \in \mathbb{P}} \frac{1}{1 - \chi(p) p^{-s}}$$

Everything converges absolutely for $\Re(s) > 1$, and so it follows

$$\frac{1}{L(s, \chi)} = \sum_{n \geq 1} (\chi\mu)(n) n^{-s}$$

By the compatibility of Dirichlet convolution and Dirichlet summation, we now find

$$L(s, \chi)' \frac{1}{L(s, \chi)} = \left(\sum_{n \geq 1} \chi(n) \log(n) n^{-s} \right) \left(\sum_{n \geq 1} \chi(n) \mu(n) n^{-s} \right) = \sum_{n \geq 1} (\chi \log * \chi\mu)(n) n^{-s}$$

and so it is left to show that $\chi \log * \chi\mu = \chi\Lambda$.

This is true, as for all $n \in \mathbb{N}$ it holds

$$\begin{aligned} (\chi \log * \chi \mu)(n) &= \sum_{ab=n} \chi(a) \chi(b) \log(a) \mu(b) \\ &= \sum_{ab=n} \chi(ab) \log(a) \mu(b) = \chi(n) \sum_{ab=n} \log(a) \mu(b) \\ &= \chi(n) (\log * \mu)(n) = (\chi \Lambda)(n) \end{aligned}$$

□

Now we want to find an analytic continuation of $L(s, \chi)$ to $\Re(s) > 0$. First of all, we consider χ_0 .

Proposition 22 (Ex (ii)). For $\Re(s) > 1$ we have

$$L(s, \chi_0) = (1 - q^{-s}) \zeta(s)$$

In particular, $L(s, \chi_0)$ has a meromorphic continuation to $\Re(s) > 0$ with only one simple pole at $s = 1$.

Proof. As χ_0 is fully multiplicative, we have the Euler product

$$\begin{aligned} L(s, \chi_0) &= \sum_{n \geq 1} \chi_0(n) n^{-s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - \chi_0(p) p^{-s}} = \prod_{p \neq q} \frac{1}{1 - p^{-s}} \\ &= (1 - q^{-s}) \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}} = (1 - q^{-s}) \zeta(s) \end{aligned}$$

as all products converge absolutely. □

For the other Dirichlet characters, the situation is slightly more complicated. First, we will bound the value of the partial sums of the Dirichlet series for $0 < \Re(s) \leq 1$.

Lemma 23. Let $\chi \neq \chi_0$ be a Dirichlet character mod q and consider the sum function

$$A(n) := \sum_{1 \leq k \leq n} \chi(k)$$

Then $|A(n)| \leq q$ for all $n \in \mathbb{N}$.

Proof. Have

$$\begin{aligned} |A(n)| &= \left| \sum_{1 \leq k \leq n} \chi(k) \right| \leq \left| \sum_{q \lfloor n/q \rfloor < l \leq n} \chi(l) \right| + \underbrace{\left| \sum_{0 \leq k < \lfloor n/q \rfloor} \sum_{0 < l \leq q} \chi(kq + l) \right|}_{=0 \text{ by Prop. 18}} \\ &= \left| \sum_{q \lfloor n/q \rfloor < l \leq n} \chi(l) \right| \leq \sum_{q \lfloor n/q \rfloor < l \leq n} |\chi(l)| = \sum_{q \lfloor n/q \rfloor < l \leq n} 1 \\ &= q(n/q - \lfloor n/q \rfloor) \leq q \end{aligned}$$

for all $n \in \mathbb{N}$. □

Lemma 24 (Ex (iii)). Let $\chi \neq \chi_0$ be a Dirichlet character. Then for $\Re(s) > 0$ have

$$\left| \sum_{1 \leq n \leq X} \chi(n)n^{-s} \right| \leq q + \frac{q|s|}{\Re(s)} = O\left(\frac{q|s|}{\Re(s)}\right)$$

and in particular, this does not depend on X .

Proof. Using partial summation yields

$$\begin{aligned} \sum_{1 \leq n \leq X} \chi(n)n^{-s} &= \sum_{1-\epsilon < n \leq X} \chi(n)n^{-s} \\ &= \underbrace{A(1-\epsilon)(1-\epsilon)^{-s}}_{=0} - A(X)X^{-s} + s \int_{1-\epsilon}^X A(t)t^{-(s+1)} dt \\ &= -A(X)X^{-s} + s \int_1^X A(t)t^{-(s+1)} dt \end{aligned}$$

So

$$\begin{aligned} \left| \sum_{1 \leq n \leq X} \chi(n)n^{-s} \right| &\leq |A(X)X^{-s}| + |s| \int_1^X |A(t)t^{-(s+1)}| dt \\ &\leq qX^{-\Re(s)} + |s| \int_1^X qt^{-\Re(s)-1} dt \\ &= qX^{-\Re(s)} + q|s| \left(\frac{1}{\Re(s)} - \frac{1}{X^{\Re(s)}\Re(s)} \right) \\ &\leq q + q\frac{|s|}{\Re(s)} = O\left(\frac{q|s|}{\Re(s)}\right) \end{aligned}$$

for $\Re(s) > 0$. □

Now we can show the analytic continuation of $L(s, \chi)$ to $\Re(s) > 0$.

Proposition 25 (Ex (iv)). Let $\chi \neq \chi_0$ be a Dirichlet character mod q . Then

$$L(s, \chi) = s \int_1^\infty A(t)t^{-(s+1)} dt$$

for $\Re(s) > 1$. Further, the right-hand side is a holomorphic function on $\Re(s) > 0$ and thus provides an analytic continuation of $L(s, \chi)$ to $\Re(s) > 0$.

Proof. Similar to the proof of Lemma 24, partial summation yields

$$\sum_{n \geq 1} \chi(n)n^{-s} = s \int_1^\infty A(t)t^{-(s+1)} dt$$

By Lemma 24, this is bounded and hence well-defined and finite for $\Re(s) > 0$. Further, the integral converges absolutely by Lemma 23, and thus is holomorphic on $\Re(s) > 0$. □

Corollary 26 (Ex (iv)). The function $L(s, \chi)' / L(s, \chi)$ is bounded on a neighborhood of 1, provided that $L(1, \chi) \neq 0$.

Proof. As $L(s, \chi)' / L(s, \chi)$ is meromorphic on $\Re(s) > 0$, we know that it is holomorphic on some neighborhood of 1 unless it has a pole at $s = 1$. In the third exercise class of ANT, it was shown that this would imply $L(1, \chi) = 0$ or $L(s, \chi)'$ has a pole at $s = 1$.

However, the derivative of a holomorphic function is again holomorphic, so $L(s, \chi)'$ has no pole at $s = 1$. Provided that $L(1, \chi) \neq 0$, it follows that $L(s, \chi)' / L(s, \chi)$ is holomorphic on a compact neighborhood of 1, so bounded. \square

Now we can show the main result of this miniproject. We will prove two auxiliary lemmas before.

Lemma 27. For $a \perp q$, the function

$$\rho_a(s) := \frac{1}{q-1} \sum_{\chi \in \text{Dir}(q)} \overline{\chi(a)} \sum_{n \geq 1} \frac{\Lambda(n) \chi(n)}{n^s}$$

is a meromorphic function on $\Re(s) > 0$ with a simple poles at 1 (and possibly other poles on $\Re(s) > 0$).

Proof. By Prop. 21, we have for $\chi \in \text{Dir}(q)$ that

$$\frac{\overline{\chi(a)}}{q-1} \sum_{n \geq 1} \frac{\Lambda(n) \chi(n)}{n^s} = \frac{\overline{\chi(a)}}{q-1} \frac{L(s, \chi)'}{L(s, \chi)}$$

If $\chi \neq \chi_0$, then Corollary 26 shows that this function has no pole at $s = 1$.

If $\chi = \chi_0$ on the other hand, Prop. 22 shows that

$$L(s, \chi_0) = (1 - q^{-s}) \zeta(s)$$

Hence

$$\frac{L(s, \chi_0)'}{L(s, \chi_0)} = \frac{\log(q) q^{-s} \zeta(s) + (1 - q^{-s}) \zeta'(s)}{(1 - q^{-s}) \zeta(s)} = \frac{\log(q)}{q^s - 1} + \frac{\zeta'(s)}{\zeta(s)}$$

has a simple pole at $s = 1$. Since $a \perp q$, we see that $\chi_0(a) = 1$ and thus also

$$\frac{\overline{\chi_0(a)}}{q-1} \sum_{n \geq 1} \frac{\Lambda(n) \chi_0(n)}{n^s} = \frac{\overline{\chi_0(a)}}{q-1} \frac{L(s, \chi_0)'}{L(s, \chi_0)}$$

has a simple pole at $s = 1$.

Together, this yields that the sum of those functions

$$\rho_a(s) = \frac{1}{q-1} \sum_{\chi \in \text{Dir}(q)} \overline{\chi(a)} \sum_{n \geq 1} \frac{\Lambda(n) \chi(n)}{n^s}$$

is a meromorphic function with a simple pole at $s = 1$. \square

Lemma 28. Let $a \perp q$ and define

$$\Psi_a(x) := \sum_{n < x, n \equiv a \pmod{q}} \Lambda(n)$$

and

$$\theta_a(x) := \sum_{p < x, p \equiv a \pmod{q}} \log(p)$$

Then

$$\Psi_a(x) - \theta_a(x) = O(x^{1/2} \log(x))$$

Proof. Have

$$\begin{aligned} \Psi_a(x) - \theta_a(x) &= \sum_{p^k < x, p^k \equiv a \pmod{q}} \log(p) - \sum_{p < x, p \equiv a \pmod{q}} \log(p) \\ &= \sum_{p^k < x, k \geq 2, p^k \equiv a \pmod{q}} \log(p) \leq \sum_{p^k < x, k \geq 2} \log(p) \\ &= \Psi(x) - \theta(x) = O(x^{1/2} \log(x)) \end{aligned}$$

where the last equality was proven in the lecture. \square

Proposition 29 (Ex (v)). Assume that $L(1, \chi) \neq 0$ for all $\chi \in \text{Dir}(q) \setminus \{\chi_0\}$. Then for all $a \perp q$ there are infinitely many primes $p \equiv a \pmod{q}$.

Proof. Assume not, then $\theta_a(x)$ is bounded, i.e. $\theta_a(x) = O(1)$. With Lemma 28 it follows that $\Psi_a(x) = O(x^{1/2} \log x)$.

By Prop. 19 we have that for $\Re(s) > 1$

$$\rho_a(s) = \sum_{n \equiv a \pmod{q}} \Lambda(n) n^{-s}$$

Partial summation yields that for $\Re(s) > 1$ have

$$\begin{aligned} \rho_a(s) &= \sum_{n \equiv a \pmod{q}} \Lambda(n) n^{-s} \\ &= \lim_{t \rightarrow \infty} \left(\underbrace{t^{-s} \sum_{n < t, n \equiv a \pmod{q}} \Lambda(n)}_{= t^{-s} \Psi_a(t) = t^{-s} O(t \log t) = o(1)} \right) + s \int_1^\infty \left(\underbrace{\sum_{n < t, n \equiv a \pmod{q}} \Lambda(n)}_{= \Psi_a(t)} \right) t^{-(s+1)} dt \\ &= s \int_1^\infty \Psi_a(t) t^{-s-1} dt = s \int_1^\infty O(t^{1/2} \log t) t^{-s-1} dt \\ &= O \left(|s| \int_1^\infty \log(t) t^{-\Re(s)-1/2} dt \right) \\ &= O \left(|s| \left(\frac{1}{1/2 - \Re(s)} 1^{1/2 - \Re(s)} + \frac{1}{1/2 - \Re(s)} \int_1^\infty t^{-\Re(s)-1/2} dt \right) \right) \\ &= O \left(|s| \frac{1}{(1/2 - \Re(s))^2} \right) \end{aligned}$$

However this function has no pole at $s = 1$, a contradiction to Lemma 27. \square

Example 30 (Ex (vi)). For all the nontrivial Dirichlet characters χ_1, \dots, χ_4 defined in Example 20, we have

$$L(1, \chi_i) \neq 0$$

It follows that there are infinitely many primes $\equiv a \pmod{5}$, for all $a \perp 5$.

Proof. By Prop. 5 we know that

$$L(1, \chi_i) = \sum_{n \geq 1} \chi(n) n^{-s}$$

For the fifth root of unity $\omega \in \mathbb{C}$ such that $\chi_i(1) = \omega$, we thus find

$$\begin{aligned} L(1, \chi_i) &= \sum_{k \geq 0} \sum_{1 \leq n \leq 4} \frac{\chi(5k+n)}{(5k+n)^s} = \sum_{k \geq 0} \frac{\omega}{(5k+1)^s} + \frac{\omega^2}{(5k+2)^s} + \frac{\omega^3}{(5k+3)^s} + \frac{\omega^4}{(5k+4)^s} \\ &= \sum_{k \geq 0} \omega \left(\underbrace{\frac{1}{(5k+1)^s} - \frac{1}{(5k+4)^s}}_{>0} \right) + \omega^2 \left(\underbrace{\frac{1}{(5k+2)^s} - \frac{1}{(5k+3)^s}}_{>0} \right) \end{aligned}$$

Hence we have positive coefficients $a_k, b_k > 0$ with

$$L(1, \chi_i) - \sum_{k \geq 0} a_k \omega + b_k \omega^2 = \omega \sum_{k \geq 0} a_k + b_k \omega$$

In particular,

$$\Im \left(\frac{L(1, \chi_i)}{\omega} \right) = \sum_{k \geq 0} b_k \Im(\omega) = \Im(\omega) \underbrace{\sum_{k \geq 0} b_k}_{>0}$$

Since χ_i is a nontrivial Dirichlet character, we see that $\Im(\omega) \neq 0$ and so $\Im(L(1, \chi_i)/\omega) \neq 0$, thus $L(1, \chi_i) \neq 0$. \square

Remark 31 (Ex (vii)). TODO

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