

Miniproject - Algebraic Geometry

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Definition 1. Let V be a vector space. Then define the d -th exterior power as

$$\bigwedge^d(V) := V^{\otimes d} / \sum_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \{v \otimes v' + v' \otimes v \mid v, v' \in V\} \otimes V^{\otimes(d-i-1)}$$

Use the notation $v_1 \wedge \dots \wedge v_d := [v_1 \otimes \dots \otimes v_d] \in \bigwedge^d(V)$.

Lemma 2. Let $v_1, \dots, v_d \in V$. Have for $\pi \in S_d$ that

$$v_{\pi(1)} \wedge \dots \wedge v_{\pi(d)} = \text{sgn}(\pi)(v_1 \wedge \dots \wedge v_d)$$

Furthermore if $v_i = v_j$ for some $i \neq j$, then

$$v_1 \wedge \dots \wedge v_d = 0$$

Proof. Note that

$$u \otimes v \otimes v' \otimes w = -(u \otimes v' \otimes v \otimes w)$$

for all $u \in \bigwedge^{i-1}(V), w \in \bigwedge^{(d-i-1)}(V), v, v' \in V$.

Every $\pi \in S_d$ has a decomposition $\pi = \xi_1 \dots \xi_n$ into transpositions ξ_i . Applying this inductively, we find that

$$v_1 \wedge \dots \wedge v_d = \text{sgn}(\xi_1 \dots \xi_n)(v_{(\xi_1 \dots \xi_n)(1)} \wedge \dots \wedge v_{(\xi_1 \dots \xi_n)(d)})$$

and so

$$v_1 \wedge \dots \wedge v_d = \text{sgn}(\pi)(v_{\pi(1)} \wedge \dots \wedge v_{\pi(d)})$$

Furthermore, we find that

$$u \otimes v \otimes v \otimes w = -(u \otimes v \otimes v \otimes w) = 0$$

must be zero. Hence, if $v_1, \dots, v_d \in V$ with $v_i = v_j$ for some $i \neq j$, then there is a permutation $\pi \in S_d$ with $\pi(1) = i, \pi(2) = j$ and

$$v_1 \wedge \dots \wedge v_d = (\text{sgn}(\pi))(v_i \wedge v_j \wedge v_{\pi(3)} \wedge \dots \wedge v_{\pi(d)}) = \text{sgn}(\pi)0 = 0$$

□

Lemma 3 (1a). *Let $\dim(V) \leq 3$. Then every element of $\bigwedge^k(V)$ is decomposable.*

Proof. Now let v_1, v_2, v_3 be a set of generators of V . Consider $u_1 = \sum \lambda_i v_i, u_2 = \sum \mu_i v_i, u_3 = \sum \rho_i v_i$. Then by applying Lemma 2, we see that

$$\begin{aligned} u_1 \wedge u_2 &= \sum_{i,j} \lambda_i \mu_j \underbrace{(v_i \wedge v_j)}_{=0 \text{ if } i=j} = \sum_{i < j} \lambda_i \mu_j (v_i \wedge v_j) - \sum_{i > j} \lambda_i \mu_j (v_i \wedge v_j) \\ &= \sum_{i < j} (\lambda_i \mu_j - \lambda_j \mu_i) (v_i \wedge v_j) = \alpha(v_1 \wedge v_2) + \beta(v_1 \wedge v_3) + \gamma(v_2 \wedge v_3) \\ &= \begin{cases} \beta v_1 + \gamma v_2 \wedge \frac{\alpha}{\beta} v_2 + v_3 & \text{if } \beta \neq 0 \\ \alpha v_1 - \gamma v_3 \wedge v_2 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} u_1 \wedge u_2 \wedge u_3 &= \sum_{i,j,l} \lambda_i \mu_j \rho_l \underbrace{(v_i \wedge v_j \wedge v_l)}_{=0 \text{ unless } i,j,l \text{ pairwise distinct}} \\ &= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} (v_{\pi(1)} \wedge v_{\pi(2)} \wedge v_{\pi(3)}) \\ &= \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \text{sgn}(\pi) (v_1 \wedge v_2 \wedge v_3) \\ &= (v_1 \wedge v_2 \wedge v_3) \sum_{\pi \in S_3} \lambda_{\pi(1)} \mu_{\pi(2)} \rho_{\pi(3)} \text{sgn}(\pi) \end{aligned}$$

are decomposable. Further, it is easy to see from Lemma 2 that $\bigwedge^k(V) = \{0\}$ for $k \geq 4$, which is trivially decomposable. \square

Example 4 (1b). *Consider $V = k^4$. Then the element $w := (e_1 \wedge e_2) + (e_3 \wedge e_4) \in \bigwedge^2(V)$ is not decomposable.*

Proof. Assume it was, then there are $a, b \in k^4$ such that

$$w = \sum_i a_i e_i \wedge \sum_j b_j e_j = \sum_{i < j} (a_i b_j - a_j b_i) (e_i \wedge e_j)$$

In other words

$$a_1 b_2 - a_2 b_1 = 1, \quad a_3 b_4 - a_4 b_3 = 1, \quad a_i b_j - a_j b_i = 0 \text{ for all } (i, j) \neq (1, 2), (3, 4)$$

Clearly $a_1 b_2 \neq 0$ or $a_2 b_1 \neq 0$. Similarly, have $a_3 b_4 \neq 0$ or $a_4 b_3 \neq 0$. As all expressions are symmetric w.r.t swapping a_1, b_2 with a_2, b_1 and a_3, b_4 with a_4, b_3 , we may assume wlog that $a_1 b_2, a_3 b_4 \neq 0$.

Have $a_1 b_4 = a_4 b_1$ and $a_2 b_4 = a_4 b_2$. We know that $a_1 b_4 \neq 0$ and so

$$\frac{a_2}{a_1} = \frac{a_2 b_4}{a_1 b_4} = \frac{a_4 b_2}{a_4 b_1} = \frac{b_2}{b_1} \Rightarrow a_2 b_1 = a_1 b_2$$

This contradicts $a_1 b_2 - a_2 b_1 = 1$. \square

Lemma 5 (1c). *Let d be even. An element $\omega \in \bigwedge^d V$ is decomposable if and only if $\omega \wedge \omega \in \bigwedge^{2d} V$ is zero.*

Proof. The direction \Rightarrow even holds generally. Assume $\omega = v_1 \wedge \dots \wedge v_d$. Then

$$\omega \wedge \omega = v_1 \wedge \dots \wedge v_d \wedge v_1 \wedge \dots \wedge v_d = 0$$

by Lemma 2. The other direction is more interesting.

Let $\omega = v_1 + \dots + v_t$ for linearly independent decomposable vectors $v_i \in \bigwedge^2 V$. Then

$$\begin{aligned} 0 = \omega \wedge \omega &= \sum_{i,j} v_i \wedge v_j = \sum_{i < j} (v_i \wedge v_j) + (v_j \wedge v_i) \\ &= \sum_{i < j} 2(v_i \wedge v_j) = 2 \sum_i v_i \wedge \left(\sum_{j > i} v_j \right) \end{aligned}$$

Here we used that the permutation $(1 \ 2d)(2 \ (2d-1)) \dots (d \ (d+1)) \in S_{2d}$ has always sign 1 (since d is even).

Note that for any nonzero decomposable vector

$$u_1 \wedge u_2 \in \left(\bigwedge^2 \text{span}\{v_2, \dots, v_t\} \right) \setminus \{0\}$$

find

$$u_1, u_2 \in \text{span}\{v_2, \dots, v_t\}$$

In particular, we know that

$$v_1 \wedge \left(\sum_{j > i} v_j \right) \in \bigwedge^2 \text{span}\{v_2, \dots, v_t\}$$

and so $v_1 \in \text{span}\{v_2, \dots, v_t\}$ unless $\sum_{j > i} v_j = 0$. We assumed that the v_i are linearly independent, so the former would give a contradiction. Hence $\sum_{j > i} v_j = 0$ and thus $t = 1$, i.e. $\omega = v_1$ is decomposable. \square

Lemma 6 (2a). *Let $A = (a_{ij}) \in \text{GL}_d(k)$ and $v_1, \dots, v_d \in V$. Then*

$$\left(\sum_j a_{1j} v_j \right) \wedge \dots \wedge \left(\sum_j a_{dj} v_j \right) = \det(A) (v_1 \wedge \dots \wedge v_d)$$

Proof. By a direct computation using Lemma 2, we find

$$\begin{aligned} \left(\sum_j a_{1j} v_j \right) \wedge \dots \wedge \left(\sum_j a_{dj} v_j \right) &= \sum_{j_1, \dots, j_d} a_{1j_1} \dots a_{dj_d} (v_{j_1} \wedge \dots \wedge v_{j_d}) \\ &= \sum_{\pi \in S_d} a_{1\pi(1)} \dots a_{d\pi(d)} (v_{\pi(1)} \wedge \dots \wedge v_{\pi(d)}) \\ &= \sum_{\pi \in S_d} a_{1\pi(1)} \dots a_{d\pi(d)} \text{sgn}(\pi) (v_1 \wedge \dots \wedge v_d) \\ &= (v_1 \wedge \dots \wedge v_d) \sum_{\pi \in S_d} \text{sgn}(\pi) \prod_{j=1}^d a_{j\pi(j)} = \det(A) (v_1 \wedge \dots \wedge v_d) \end{aligned}$$

where the last equality holds due to the Leibniz determinant formula. \square

We see that if there are two bases v_1, \dots, v_d and u_1, \dots, u_d of a d -dimensional vector space U , then there exists a basis change matrix $A = (a_{ij}) \in \text{GL}_d(k)$ with

$$u_i = \sum_j a_{ij} v_j$$

So by the Lemma, it follows that

$$u_1 \wedge \dots \wedge u_d = \det(A)(v_1 \wedge \dots \wedge v_d)$$

As v_1, \dots, v_d resp. u_1, \dots, u_d are bases, they are linearly independent and in particular, we see that

$$v_1 \wedge \dots \wedge v_d \neq 0 \quad \text{and} \quad u_1 \wedge \dots \wedge u_d \neq 0$$

Hence they have well-defined images $[v_1 \wedge \dots \wedge v_d]$ resp. $[u_1 \wedge \dots \wedge u_d]$ in the projective space $\mathbb{P}(\bigwedge^d V)$. By the above, find

$$[v_1 \wedge \dots \wedge v_d] = [u_1 \wedge \dots \wedge u_d]$$

This allows us to study the Grassmanian $\text{Gr}(d, V)$ of a fixed vector space V .

Definition 7. *Define the map*

$$\phi : \text{Gr}(d, V) \rightarrow \mathbb{P}(\bigwedge^d V), \quad \text{span}\{v_1, \dots, v_d\} \mapsto [v_1 \wedge \dots \wedge v_d]$$

which is well-defined by Lemma 6 as described above.

Lemma 8 (1a). *We have*

$$\text{im}\phi = D := \{[v] \in \mathbb{P}(\bigwedge^d V) \mid v \text{ decomposable}\}$$

Proof. First of all, note that the set D is well-defined, as v is decomposable if and only if λv is decomposable, for all $\lambda \in k^*$.

By definition of ϕ , we can directly observe that $\text{im}\phi \subseteq D$. So consider an element $[v] \in D$. As v is decomposable, it follows that $v = v_1 \wedge \dots \wedge v_d$ for $v_i \in V$. Not it suffices to show that the v_i are linearly independent, then clearly $\text{span}\{v_1, \dots, v_d\}$ is a well-defined d -dimensional vector subspace of V , thus an element of $\text{Gr}(d, V)$.

Assume not, then there is a nonzero vector $a_1 \in k^d$ with $\sum a_{1i} v_i = 0$. Clearly, we can extend a_1 to a basis a_1, \dots, a_d of k^d , which gives a matrix $A = (a_{ij}) \in \text{GL}_d(k)$.

However by Lemma 6 we now get

$$\begin{aligned} 0 &= 0 \wedge \left(\sum_j a_{2j} v_j \right) \wedge \dots \wedge \left(\sum_j a_{dj} v_j \right) = \left(\sum_j a_{1j} v_j \right) \wedge \dots \wedge \left(\sum_j a_{dj} v_j \right) \\ &= \det(A)(v_1 \wedge \dots \wedge v_d) \end{aligned}$$

and so $v = v_1 \wedge \dots \wedge v_d = 0$ as $\det(A) \neq 0$. However, v was a representative of a point in $\mathbb{P}(\bigwedge^d V)$, a contradiction. The claim follows. \square

Definition 9. Let $\text{Gr}(d, n) := \text{Gr}(d, k^n)$.

In the lecture, we considered an embedding of $\text{Gr}(d, n)$ into projective space given by minors of the basis matrix. This corresponds to the following definition.

Definition 10. Define the map

$$\rho : \text{Gr}(d, n) \rightarrow \mathbb{P}\left(k^{\{1, \dots, n\}^{(d)}}\right) \cong \mathbb{P}^{\binom{n}{d}-1},$$

$$\text{span}\{v_1, \dots, v_d\} \mapsto \left[\det \begin{pmatrix} v_{1i_1} & \dots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \dots & v_{di_d} \end{pmatrix} \right]_{\{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}}$$

where $\{1, \dots, n\}^{(d)} := \{I \subset \{1, \dots, n\} \mid \#I = d\}$ is the set of all d -element subsets of $\{1, \dots, n\}$.

Lemma 11. There is a linear isomorphism

$$f : \bigwedge^d k^n \rightarrow k^{\{1, \dots, n\}^{(d)}},$$

$$\sum_j v_1^{(j)} \wedge \dots \wedge v_d^{(j)} \mapsto \left(\sum_j \det \begin{pmatrix} v_{1i_1}^{(j)} & \dots & v_{di_1}^{(j)} \\ \vdots & \ddots & \vdots \\ v_{1i_d}^{(j)} & \dots & v_{di_d}^{(j)} \end{pmatrix} \right)_{\{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}}$$

Proof. For vectors v_1, \dots, v_d and $I = \{i_1, \dots, i_d\} \in \{1, \dots, n\}^{(d)}$ write

$$A_I(v_1, \dots, v_d) := \begin{pmatrix} v_{1i_1} & \dots & v_{di_1} \\ \vdots & \ddots & \vdots \\ v_{1i_d} & \dots & v_{di_d} \end{pmatrix}$$

First of all, we show that f is well-defined. Note that the tensor product can be described as

$$\begin{aligned} V^{\otimes d} := k^{V \times \dots \times V} / \text{span}\{ & (v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v'_i) \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - (v_1 \otimes \dots \otimes v_d) - (v_1 \otimes \dots \otimes v_{i-1} \otimes v'_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ & (v_1 \otimes \dots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - \lambda(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \mid i \leq d, v_j, v'_i \in V \} \end{aligned}$$

where $v_1 \otimes \dots \otimes v_d := \mathcal{K}_{(v_1, \dots, v_d)}$. Hence the external power can be described as

$$\begin{aligned} \bigwedge^d V := k^{V \times \dots \times V} / \text{span}\{ & (v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v'_i) \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - (v_1 \otimes \dots \otimes v_d) - (v_1 \otimes \dots \otimes v_{i-1} \otimes v'_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ & (v_1 \otimes \dots \otimes v_{i-1} \otimes \lambda v_i \otimes v_{i+1} \otimes \dots \otimes v_d) \\ & - \lambda(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d), \\ & v_1 \otimes \dots \otimes v_{j-1} \otimes (v_j \otimes v_{j+1} + v_{j+1} \otimes v_j) \otimes v_{j+1} \otimes \dots \otimes v_d \\ & \mid i \leq d, j \leq d-1, v_1, \dots, v_d, v'_i \in V \} \end{aligned}$$

So it suffices to show that for all $I \in \{1, \dots, n\}^{(d)}$ and vectors $v_1, \dots, v_d, v'_i \in V$

$$\det(A_I(v_1, \dots, v_i + v'_i, \dots, v_d)) = \det(A_I(v_1, \dots, v_d)) + \det(A_I(v_1, \dots, v'_i, \dots, v_d))$$

and

$$\det(A_I(v_1, \dots, \lambda v_i, \dots, v_d)) = \lambda \det(A_I(v_1, \dots, v_d))$$

and

$$\det(A_I(v_1, \dots, v_{j+1}, v_j, \dots, v_d)) = -\det(A_I(v_1, \dots, v_d))$$

However, these properties follow from the well-known properties of the determinant. In particular, \det is linear in each column and swapping columns negates the determinant. It follows that f is indeed well-defined.

It is clear by definition that f is linear, so it is left to show that it is bijective. \square