The University of Oxford

MSc (Mathematics and Foundations of Computer Science)

C3.4 Algebraic Geometry

Michaelmas Term 2020

The steps of this mini project are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the mini project, but should make this assumption clear in your presentation.

(1) Let V be an n-dimensional vector space V over a field k. For $0 \leq d \leq n$, recall the d-th exterior product space $\wedge^d V$ of V, as defined for example in the Lecture Notes. Call an element $\omega \in \wedge^d V$ *decomposable*, if there exist $v_1, \dots, v_d$ with

$$\omega = v_1 \wedge v_2 \wedge \dots \wedge v_d \in \wedge^d V.$$

(a) Show that for $n \leq 3$ and any d, every element of $\wedge^d V$ is decomposable.

(b) Show that for $n = 4$, there exist non-decomposable elements in $\wedge^2 V$.

(c) Show that for any n, an element $\omega \in \wedge^2 V$ is decomposable if and only if $\omega \wedge \omega = 0 \in \wedge^4 V$.

(2) Let $\mathrm{Gr}(d, V)$ be the Grassmannian of d-dimensional subspaces of V. Let

$$\phi \colon \mathrm{Gr}(d, V) \to \mathbb{P}(\wedge^d V)$$

be defined by mapping a d-dimensional subspace $U = \langle v_1, \dots, v_d \rangle$ to

$$[v_1 \wedge v_2 \wedge \dots \wedge v_d] \in \mathbb{P}(\wedge^d V).$$

(a) Explain why the map ϕ is well-defined, and show that its image is the (projective) space of decomposable elements of $\wedge^d V$.

(b) By choosing a basis of V, identify the map ϕ with the Plücker map on $\mathrm{Gr}(d, n)$, the set of d-dimensional subspaces of k^n, explained in the Lectures.

(c) Hence or otherwise show that the map ϕ is injective: different subspaces are mapped to different decomposable elements in $\mathbb{P}(\wedge^d V)$.

(3) Restrict now to the case $d = 2$.

(a) Use the result of (1)(c) to show that the image of

$$\mathrm{Gr}(2, V) \hookrightarrow \mathbb{P}(\wedge^2 V)$$

is cut out by quadric equations. Determine these equations in as nice a form as you can.

(b) Show that for $n = 4$, this recovers the description of the image of

$$\mathrm{Gr}(2, 4) \hookrightarrow \mathbb{P}(\wedge^2 k^4) \cong \mathbb{P}^5$$

discussed in the Lectures.

(c) Find the equations describing the image of

$$\mathrm{Gr}(2, 5) \hookrightarrow \mathbb{P}(\wedge^2 k^5) \cong \mathbb{P}^9.$$

(d) Find the number of intersection points of the image of $\mathrm{Gr}(2, 5) \hookrightarrow \mathbb{P}^9$ with some specific 3-dimensional linear subspaces of $\mathbb{P}^9$ of your choice, thereby aiming to verify the degree formula of Lecture 8 for this particular case. If this proves too difficult by hand, try a computer algebra software such as Mathematica, Sage or Macaulay2. You may also wish to consult *Harris, Algebraic Geometry: A First Course*, Example 19.14. (Note that Harris uses the projective notation for the Grassmannian, whereas we use the affine one.)

(e) What can you say about the embedding of $\mathrm{Gr}(2, 6)$ using the map ϕ and its degree?