

Some Notes about the things I encountered

Simon Pohmann

May 13, 2022

1 Example - The cases I, II and III

Consider the prime $p = 101$ and \mathbb{F}_q where $q = p^2$. Let $\alpha \in \mathbb{F}_q$ with minimal polynomial $x^2 + 97x + 2$.

1.1 Case I

Finding examples of case I is trivial - just take a curve E with $j(E) \in \mathbb{F}_p$. Then clearly $E^{(p)} = E$ and so also $E_0^{(p)} = E_0$ (since $\cdot^{(p)}$ maps the path $E \rightarrow E_0$ to $E = E^{(p)} \rightarrow E_0^{(p)}$).

Furthermore, it is easy to see that there are a lot of curve E such that the associated E_0 is defined over \mathbb{F}_p (and we are again in case I).

1.2 Case II

Here I was not quite sure if it even occurs. As it turns out, it does. Consider E with $j(E) = 17\alpha + 45$. Then $[\mathcal{O}_K : \mathbb{Z}[\pi]] = 2^3$ so E lies on the crater of the 3-isogeny graph. However there is a 3-isogeny $E \rightarrow E^{(p)}$ since $j(E^{(p)}) = j(E)^p = 84\alpha + 12$. In fact, in this case, the crater consists only of E and $E^{(p)}$.

1.3 Case III

Consider E with $j(E) = 64\alpha + 5$. Then $j(E^{(p)}) = j(E)^p = 37\alpha + 59$. However, we have that E lies on the crater, together with curve of j-invariants

$$88\alpha + 70, 54\alpha + 52, 95\alpha + 11$$

Hence there is no 3-isogeny path from E to $E^{(p)}$. Note that $[\mathcal{O}_K : \mathbb{Z}[\pi]] = 2^2 \cdot 3^2$ but $[\mathcal{O}_K : \text{End}(E)] = 2^2$, which shows that E lies on the crater.

Now we want to have a closer look onto the class group action in this case. Have $d(\text{End}(E)) = -320$, so $K = \mathbb{Q}(\sqrt{-5})$ and $d(\mathcal{O}_K) = -5$. Hence, we have $\text{End}(E) \cong \mathbb{Z}[4\sqrt{-5}]$ and $\mathcal{O}_K \cong \mathbb{Z}[\sqrt{-5}]$.

Sage tells us that $h(\mathcal{O}_K) = 2$ and $h(\text{End}(E)) = 8$. With this, we can already see that

$$64\alpha + 5, 88\alpha + 70, 54\alpha + 52, 95\alpha + 11$$

and

$$(64\alpha + 5)^p, (88\alpha + 70)^p, (54\alpha + 52)^p, (95\alpha + 11)^p$$

is the set of j -invariants of all Elliptic Curves with endomorphism ring $\cong \text{End}(E)$. On this set, $\text{Cl}(\mathbb{Z}[4\sqrt{-5}])$ then acts freely and transitively. Now it would be of course interesting to find out how $\text{Cl}(\mathbb{Z}[4\sqrt{-5}])$ really looks like.

2 Example - The ordinary endomorphism ring

The information in this section is all known material - I just wanted to understand properly how one can compute the endomorphism ring, and what problems occur.

Consider the finite field

$$\mathbb{F}_q = \mathbb{F}_{37^2} = \mathbb{F}_{37} + \alpha\mathbb{F}_{37}$$

where $\alpha^2 + 33\alpha + 2 = 0$. Further, consider the Elliptic Curve E/\mathbb{F}_q with j -invariant 3α , given by

$$E : y^2 = x^3 + (15\alpha + 17)x + (5\alpha + 3)$$

Then we find that the q -th power Frobenius endomorphism π satisfies the minimal equation

$$\pi^2 + 47\pi + 1369$$

and in particular, its trace is -47 . Hence, the number field $\mathcal{K} := \mathcal{O} \otimes \mathbb{Q}$ where $\mathcal{O} = \text{End}(E)$ contains $\sqrt{47^2 - 4 \cdot 1369} = \sqrt{-3^3 \cdot 11^2}$. We observe that $\mathcal{K} = \mathbb{Q}(\sqrt{-3})$ and has discriminant -3 . Furthermore the ring of integers is $\mathcal{O}_{\mathcal{K}} = \mathbb{Z}[\frac{1}{2}(1 + \sqrt{-3})]$.

Knowing the number field, we want to find the endomorphism ring. First, observe that the Frobenius order $\mathbb{Z}[\pi]$ has conductor 33. Now consider the endomorphism

$$\phi := 2\pi + 47$$

The advantage is that we can evaluate ϕ on points of E , but evaluating $\pi + 47/2$ is not so easy. Clearly $[\mathbb{Z}[\pi] : \mathbb{Z}[\phi]] = 2$ and so $\mathbb{Z}[\phi]$ has conductor 66.

Torsion points

In order to find whether $\phi/n \in \mathcal{O}$, we factor $66 = 2 \cdot 3 \cdot 11$ and compute the corresponding torsion groups. This turns out to be quite difficult.

Assume $\mathbb{F}_{37^{12}} = \mathbb{F}_{37}[\beta]$ with

$$\text{MiPo}_{\mathbb{F}_{37}}(\beta) = x^{12} + 4x^7 + 31x^6 + 10x^5 + 23x^4 + 18x^2 + 33x + 2$$

Then $E[2]$ is generated by

$$\begin{aligned} P_1 &= (11\beta^{11} + 19\beta^{10} + \beta^9 + 27\beta^8 + 8\beta^7 + 16\beta^6 + 17\beta^5 + 32\beta^4 + 12\beta^3 + 14\beta^2 + 24\beta + 32 : 0 : 1) \\ Q_1 &= (15\beta^{11} + 7\beta^{10} + 33\beta^9 + 11\beta^8 + 6\beta^7 + 12\beta^6 + 26\beta^5 + 7\beta^4 + 33\beta^3 + 25\beta^2 + 8\beta + 19 : 0 : 1) \end{aligned}$$

Further $E[3]$ is generated by

$$\begin{aligned} P_2 &= (19\beta^{11} + 34\beta^{10} + 3\beta^9 + 29\beta^8 + 7\beta^7 + 3\beta^6 + 18\beta^5 + 21\beta^4 + 23\beta^3 + 30\beta^2 + 23\beta + 25 \\ &\quad : 6\beta^{11} + 25\beta^{10} + 4\beta^9 + 13\beta^8 + 10\beta^7 + 23\beta^6 + 20\beta^5 + 30\beta^4 + 24\beta^3 + 6\beta^2 + 17\beta + 5 : 1) \\ Q_2 &= (31\beta^{11} + 24\beta^{10} + 35\beta^9 + 32\beta^8 + 2\beta^7 + 10\beta^6 + 23\beta^5 + 35\beta^4 + 22\beta^3 + 13\beta^2 + 12\beta + 12 \\ &\quad : 18\beta^{11} + 2\beta^{10} + 32\beta^9 + 26\beta^8 + 17\beta^7 + 5\beta^6 + 19\beta^5 + 31\beta^4 + 31\beta^3 + \beta^2 + 22\beta + 1 : 1) \end{aligned}$$

For $E[11]$ we must even go to the extension degree 110. So assume $\mathbb{F}_{37^{220}} = \mathbb{F}_{37}[\gamma]$. Then $E[11]$ is generated by P_3 and Q_3 . For the values of $\text{MiPo}_{\mathbb{F}_{37}}(\gamma)$ and P_3, Q_3 see Section 3.

Now we can compute $\phi(P_1), \phi(Q_1), \phi(P_2), \phi(Q_2), \phi(P_3), \phi(Q_3)$ and see that none of them is zero. Since $\deg(\phi) = [\mathcal{O} : \mathbb{Z}[\phi]] \mid [\mathcal{O}_K : \mathbb{Z}[\phi]] = 2 \cdot 3 \cdot 11$, we see that the kernel of ϕ is trivial. Thus no ϕ/n is contained in \mathcal{O} . Therefore we see that

$$\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z}[\phi]$$

The inclusion \supseteq is clear, and for the other direction, note that $\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z} + t\sqrt{D}\mathbb{Z}$ and $\mathbb{Z}[\phi] = \mathbb{Z} + s\sqrt{D}\mathbb{Z}$. Since $\mathbb{Z}[\phi] \subseteq \mathcal{O} \cap \mathbb{Z}[\phi]$ find thus $t \mid s$. Now observe that by choice of ϕ , have $\phi^2 \in \mathbb{Z}$ and so $\phi = s\sqrt{D}$. However, $\phi/\frac{s}{t} = t\sqrt{D} \in \mathcal{O}$. By the above, it follows that $\frac{s}{t} = 1$, i.e. $s = t$.

The index $[\mathcal{O} : \mathbb{Z}[\phi]]$

From the consideration of the torsion points, we see that $\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z}[\phi]$. However, since $[\mathcal{O}_K : \mathbb{Z}[\sqrt{D}]] \leq 2$, we deduce that $[\mathcal{O} : \mathbb{Z}[\phi]] \leq 2$ and so

$$\mathcal{O} = \mathbb{Z}[\pi]$$

3 P_3 and Q_3

The minimal polynomial of γ is

$$\begin{aligned} &x^{220} + 31x^{219} + 13x^{218} + 21x^{217} + 23x^{216} + 9x^{215} \\ &+ 2x^{214} + 35x^{212} + 10x^{211} + 29x^{210} + 25x^{209} + 20x^{208} \\ &+ 17x^{207} + 30x^{206} + 5x^{205} + 15x^{204} + 11x^{203} + 10x^{202} \\ &+ 11x^{201} + 32x^{200} + 5x^{199} + 28x^{198} + 7x^{197} + 13x^{196} \\ &+ 10x^{195} + 32x^{194} + 17x^{193} + 19x^{192} + 36x^{191} \\ &+ 17x^{190} + 31x^{189} + 14x^{188} + 6x^{187} + 30x^{186} + 8x^{185} \\ &+ 22x^{184} + 2x^{183} + 9x^{182} + 11x^{181} + 6x^{180} + 23x^{179} \\ &+ 14x^{178} + 36x^{177} + 16x^{176} + 34x^{175} + 14x^{174} \\ &+ 33x^{173} + 14x^{172} + 7x^{171} + 36x^{170} + 18x^{169} + 27x^{168} \\ &+ 5x^{167} + 31x^{166} + 6x^{165} + 15x^{164} + 14x^{163} + 17x^{162} \\ &+ 7x^{161} + 16x^{160} + 6x^{159} + 29x^{158} + 11x^{157} + 8x^{156} \\ &+ 15x^{155} + 20x^{154} + 17x^{153} + 7x^{152} + 8x^{151} + 6x^{150} \end{aligned}$$

$$\begin{aligned}
& + 12*x^{149} + 36*x^{148} + 7*x^{147} + 3*x^{146} + 25*x^{145} + 13*x^{144} \\
& + 6*x^{143} + 17*x^{142} + 22*x^{141} + 9*x^{140} + 18*x^{139} + 36*x^{138} \\
& + x^{137} + 6*x^{136} + 36*x^{135} + 33*x^{134} + 32*x^{133} + 35*x^{132} \\
& + 33*x^{131} + 7*x^{130} + 3*x^{129} + 7*x^{128} + 20*x^{127} + 31*x^{126} \\
& + 26*x^{125} + 6*x^{124} + 9*x^{123} + 10*x^{122} + 25*x^{121} + 33*x^{120} \\
& + 33*x^{119} + 30*x^{118} + 34*x^{117} + 22*x^{116} + 8*x^{115} + 10*x^{114} \\
& + 36*x^{113} + 26*x^{112} + 8*x^{111} + 33*x^{110} + 30*x^{109} + 11*x^{108} \\
& + 14*x^{107} + 22*x^{106} + 26*x^{105} + 11*x^{104} + 35*x^{103} \\
& + 34*x^{102} + 33*x^{101} + 27*x^{100} + 14*x^{99} + 31*x^{98} + 24*x^{97} \\
& + x^{96} + 6*x^{95} + 36*x^{93} + 32*x^{92} + 18*x^{91} + 36*x^{90} + 3*x^{89} \\
& + 22*x^{88} + 36*x^{87} + 6*x^{86} + 20*x^{85} + 25*x^{84} + 8*x^{82} \\
& + 34*x^{81} + 7*x^{80} + 25*x^{79} + 21*x^{78} + 17*x^{77} + 29*x^{76} \\
& + 5*x^{75} + 19*x^{74} + 19*x^{73} + 8*x^{72} + 8*x^{71} + 26*x^{70} \\
& + 7*x^{69} + 27*x^{68} + 10*x^{67} + 31*x^{66} + 4*x^{65} + 29*x^{64} \\
& + 36*x^{62} + 3*x^{61} + 27*x^{60} + 13*x^{59} + 23*x^{58} + 33*x^{57} \\
& + 14*x^{56} + 19*x^{55} + 12*x^{54} + 20*x^{53} + 32*x^{52} + 18*x^{51} \\
& + 20*x^{49} + 20*x^{48} + x^{47} + 17*x^{46} + 16*x^{45} + 4*x^{44} \\
& + 12*x^{43} + 7*x^{42} + 34*x^{41} + 9*x^{40} + 16*x^{39} + 10*x^{38} \\
& + 25*x^{37} + 10*x^{36} + 10*x^{35} + 28*x^{34} + 33*x^{33} + 22*x^{32} \\
& + 24*x^{31} + 33*x^{30} + 6*x^{29} + 8*x^{28} + 8*x^{27} + 16*x^{26} \\
& + 31*x^{25} + 7*x^{24} + 26*x^{23} + 36*x^{22} + 29*x^{21} + 36*x^{20} \\
& + 7*x^{19} + x^{18} + 26*x^{17} + 18*x^{16} + 23*x^{15} + 10*x^{14} \\
& + 4*x^{13} + x^{12} + 24*x^{11} + 25*x^{10} + 34*x^9 + 33*x^8 \\
& + 33*x^7 + 8*x^6 + 12*x^5 + x^4 + 15*x^3 + 27*x^2 + 9*x + 2
\end{aligned}$$

P_3 is given by

$$\begin{aligned}
& (23*z^{220}^{219} + 5*z^{220}^{218} + 26*z^{220}^{217} + 27*z^{220}^{216} \\
& + 26*z^{220}^{215} + 12*z^{220}^{214} + 11*z^{220}^{213} + 10*z^{220}^{212} \\
& + 29*z^{220}^{211} + 9*z^{220}^{210} + 16*z^{220}^{209} + 24*z^{220}^{208} \\
& + 18*z^{220}^{207} + 11*z^{220}^{206} + 11*z^{220}^{205} + 6*z^{220}^{204} \\
& + 24*z^{220}^{203} + 3*z^{220}^{202} + 34*z^{220}^{201} + 18*z^{220}^{200} \\
& + 17*z^{220}^{199} + 9*z^{220}^{198} + 26*z^{220}^{197} + 2*z^{220}^{196} \\
& + 31*z^{220}^{195} + 7*z^{220}^{194} + 15*z^{220}^{193} + 11*z^{220}^{192} \\
& + 15*z^{220}^{191} + 28*z^{220}^{190} + 13*z^{220}^{189} + 6*z^{220}^{188} \\
& + 7*z^{220}^{187} + 28*z^{220}^{186} + 9*z^{220}^{185} + 9*z^{220}^{184} \\
& + 7*z^{220}^{183} + 27*z^{220}^{182} + 36*z^{220}^{181} + 35*z^{220}^{180} \\
& + 30*z^{220}^{179} + 32*z^{220}^{178} + 16*z^{220}^{177} + 15*z^{220}^{176} \\
& + 16*z^{220}^{175} + 9*z^{220}^{174} + 21*z^{220}^{173} + 6*z^{220}^{172} \\
& + 15*z^{220}^{171} + 3*z^{220}^{170} + 25*z^{220}^{169} + 23*z^{220}^{168} \\
& + z^{220}^{167} + 8*z^{220}^{166} + 34*z^{220}^{165} + 14*z^{220}^{164} \\
& + 12*z^{220}^{163} + 20*z^{220}^{162} + 4*z^{220}^{161} + 9*z^{220}^{160} \\
& + z^{220}^{159} + 25*z^{220}^{158} + 16*z^{220}^{157} + z^{220}^{156} \\
& + 21*z^{220}^{155} + 10*z^{220}^{154} + 7*z^{220}^{153} + 13*z^{220}^{152} \\
& + 32*z^{220}^{151} + 31*z^{220}^{150} + 17*z^{220}^{148} + 24*z^{220}^{147}
\end{aligned}$$

$$\begin{aligned}
& + 26*z^{220}_{146} + 28*z^{220}_{145} + 27*z^{220}_{144} + 4*z^{220}_{143} \\
& + 5*z^{220}_{142} + 14*z^{220}_{141} + 26*z^{220}_{140} + 10*z^{220}_{139} \\
& + 14*z^{220}_{138} + 19*z^{220}_{137} + 20*z^{220}_{136} + 18*z^{220}_{135} \\
& + 16*z^{220}_{134} + 11*z^{220}_{133} + 23*z^{220}_{132} + 35*z^{220}_{131} \\
& + 22*z^{220}_{130} + 31*z^{220}_{129} + 34*z^{220}_{128} + 17*z^{220}_{127} \\
& + z^{220}_{126} + 15*z^{220}_{125} + 2*z^{220}_{124} + 22*z^{220}_{123} \\
& + 27*z^{220}_{122} + 6*z^{220}_{121} + 10*z^{220}_{120} + 7*z^{220}_{119} \\
& + 4*z^{220}_{118} + 26*z^{220}_{117} + z^{220}_{116} + 32*z^{220}_{115} \\
& + 29*z^{220}_{114} + 32*z^{220}_{113} + 18*z^{220}_{112} + 3*z^{220}_{111} \\
& + 28*z^{220}_{110} + 20*z^{220}_{109} + 17*z^{220}_{108} + 17*z^{220}_{107} \\
& + 32*z^{220}_{106} + 32*z^{220}_{105} + 26*z^{220}_{104} + 24*z^{220}_{103} \\
& + 17*z^{220}_{102} + 8*z^{220}_{101} + 3*z^{220}_{100} + 2*z^{220}_{99} \\
& + 16*z^{220}_{98} + 29*z^{220}_{97} + 19*z^{220}_{96} + 27*z^{220}_{95} \\
& + 4*z^{220}_{94} + 29*z^{220}_{93} + 24*z^{220}_{92} + 19*z^{220}_{91} \\
& + 2*z^{220}_{90} + 2*z^{220}_{89} + 32*z^{220}_{88} + 23*z^{220}_{87} \\
& + 32*z^{220}_{86} + 15*z^{220}_{85} + 24*z^{220}_{84} + 36*z^{220}_{83} \\
& + 29*z^{220}_{82} + 18*z^{220}_{81} + 2*z^{220}_{80} + z^{220}_{79} \\
& + 33*z^{220}_{78} + 34*z^{220}_{77} + 4*z^{220}_{76} + 11*z^{220}_{75} \\
& + 21*z^{220}_{74} + 15*z^{220}_{73} + 10*z^{220}_{72} + 24*z^{220}_{71} \\
& + 22*z^{220}_{70} + 22*z^{220}_{69} + 31*z^{220}_{68} + 32*z^{220}_{67} \\
& + 28*z^{220}_{66} + z^{220}_{65} + 17*z^{220}_{64} + 13*z^{220}_{63} \\
& + 32*z^{220}_{62} + 20*z^{220}_{61} + 32*z^{220}_{60} + 21*z^{220}_{59} \\
& + 34*z^{220}_{58} + 11*z^{220}_{57} + 29*z^{220}_{56} + 12*z^{220}_{55} \\
& + 22*z^{220}_{54} + 11*z^{220}_{53} + 36*z^{220}_{52} + 35*z^{220}_{51} \\
& + 19*z^{220}_{50} + 35*z^{220}_{49} + 8*z^{220}_{48} + 16*z^{220}_{47} \\
& + 16*z^{220}_{46} + 27*z^{220}_{45} + 32*z^{220}_{44} + 12*z^{220}_{43} \\
& + 15*z^{220}_{42} + 6*z^{220}_{41} + 36*z^{220}_{40} + 27*z^{220}_{39} \\
& + 17*z^{220}_{38} + 20*z^{220}_{37} + 33*z^{220}_{36} + 34*z^{220}_{35} \\
& + 34*z^{220}_{34} + 3*z^{220}_{33} + 12*z^{220}_{32} + 12*z^{220}_{31} \\
& + 12*z^{220}_{30} + 5*z^{220}_{29} + 10*z^{220}_{28} + 13*z^{220}_{27} \\
& + 36*z^{220}_{26} + 16*z^{220}_{25} + 16*z^{220}_{24} + 15*z^{220}_{23} \\
& + 36*z^{220}_{22} + 18*z^{220}_{21} + 13*z^{220}_{20} + 26*z^{220}_{19} \\
& + 25*z^{220}_{18} + 21*z^{220}_{17} + 35*z^{220}_{16} + 3*z^{220}_{14} \\
& + 31*z^{220}_{13} + 8*z^{220}_{12} + 7*z^{220}_{11} + 10*z^{220}_{10} \\
& + 10*z^{220}_9 + 6*z^{220}_8 + 5*z^{220}_7 + 33*z^{220}_6 \\
& + 6*z^{220}_5 + 4*z^{220}_4 + 31*z^{220}_3 + 27*z^{220}_2 + 27*z^{220} + 14 \\
\\
& : 8*z^{220}_{219} + 17*z^{220}_{218} + 27*z^{220}_{217} + 14*z^{220}_{216} \\
& + 6*z^{220}_{215} + 19*z^{220}_{214} + 18*z^{220}_{213} + 6*z^{220}_{212} \\
& + 30*z^{220}_{211} + 24*z^{220}_{210} + 33*z^{220}_{209} + 19*z^{220}_{208} \\
& + 27*z^{220}_{207} + 16*z^{220}_{206} + 24*z^{220}_{205} + 3*z^{220}_{204} \\
& + 4*z^{220}_{203} + 25*z^{220}_{202} + 29*z^{220}_{201} + 31*z^{220}_{200} \\
& + 23*z^{220}_{199} + 7*z^{220}_{198} + 28*z^{220}_{197} + 4*z^{220}_{196}
\end{aligned}$$

$$\begin{aligned}
& + 26*z^{220}_{195} + 36*z^{220}_{194} + 18*z^{220}_{193} + 24*z^{220}_{192} \\
& + 29*z^{220}_{191} + 25*z^{220}_{190} + 23*z^{220}_{189} + 14*z^{220}_{188} \\
& + 33*z^{220}_{187} + 19*z^{220}_{186} + 14*z^{220}_{184} + 21*z^{220}_{183} \\
& + 10*z^{220}_{182} + 13*z^{220}_{181} + 21*z^{220}_{180} + 24*z^{220}_{179} \\
& + 33*z^{220}_{178} + 19*z^{220}_{177} + 7*z^{220}_{176} + 36*z^{220}_{175} \\
& + 30*z^{220}_{174} + 34*z^{220}_{173} + 27*z^{220}_{172} + 3*z^{220}_{171} \\
& + 34*z^{220}_{170} + 5*z^{220}_{169} + 36*z^{220}_{168} + 19*z^{220}_{167} \\
& + 27*z^{220}_{166} + 14*z^{220}_{165} + 10*z^{220}_{164} + 2*z^{220}_{163} \\
& + 31*z^{220}_{162} + 22*z^{220}_{161} + 7*z^{220}_{160} + 14*z^{220}_{159} \\
& + 5*z^{220}_{158} + 3*z^{220}_{157} + 22*z^{220}_{156} + 32*z^{220}_{155} \\
& + 21*z^{220}_{154} + 17*z^{220}_{153} + 34*z^{220}_{152} + 9*z^{220}_{151} \\
& + 33*z^{220}_{150} + 32*z^{220}_{149} + 24*z^{220}_{148} + 16*z^{220}_{147} \\
& + 19*z^{220}_{146} + 6*z^{220}_{145} + 26*z^{220}_{144} + 24*z^{220}_{143} \\
& + 34*z^{220}_{141} + 25*z^{220}_{140} + 17*z^{220}_{139} + 25*z^{220}_{138} \\
& + 19*z^{220}_{137} + 36*z^{220}_{136} + 7*z^{220}_{134} + 32*z^{220}_{133} \\
& + 24*z^{220}_{132} + 6*z^{220}_{131} + 12*z^{220}_{130} + 30*z^{220}_{129} \\
& + 35*z^{220}_{128} + 13*z^{220}_{127} + 29*z^{220}_{126} + 2*z^{220}_{125} \\
& + 24*z^{220}_{124} + 36*z^{220}_{123} + 34*z^{220}_{122} + 2*z^{220}_{121} \\
& + 33*z^{220}_{120} + 10*z^{220}_{119} + 33*z^{220}_{118} + 2*z^{220}_{117} \\
& + 17*z^{220}_{116} + 33*z^{220}_{115} + 14*z^{220}_{114} + 22*z^{220}_{113} \\
& + 27*z^{220}_{112} + 20*z^{220}_{111} + 23*z^{220}_{110} + 34*z^{220}_{109} \\
& + 6*z^{220}_{108} + 33*z^{220}_{107} + 14*z^{220}_{106} + 28*z^{220}_{105} \\
& + 29*z^{220}_{104} + 36*z^{220}_{103} + 22*z^{220}_{102} + 35*z^{220}_{101} \\
& + 8*z^{220}_{100} + 10*z^{220}_{99} + 10*z^{220}_{98} + 16*z^{220}_{97} \\
& + 19*z^{220}_{96} + 17*z^{220}_{95} + 21*z^{220}_{94} + 13*z^{220}_{93} \\
& + 24*z^{220}_{92} + 36*z^{220}_{91} + 25*z^{220}_{90} + 25*z^{220}_{89} \\
& + 22*z^{220}_{88} + 27*z^{220}_{87} + 28*z^{220}_{86} + 11*z^{220}_{85} \\
& + 3*z^{220}_{84} + 14*z^{220}_{82} + 31*z^{220}_{81} + 7*z^{220}_{80} \\
& + 33*z^{220}_{79} + 33*z^{220}_{78} + 2*z^{220}_{77} + 15*z^{220}_{76} \\
& + 17*z^{220}_{75} + 32*z^{220}_{74} + 4*z^{220}_{73} + 18*z^{220}_{72} \\
& + 10*z^{220}_{71} + 34*z^{220}_{70} + 9*z^{220}_{69} + 3*z^{220}_{68} \\
& + 20*z^{220}_{67} + 33*z^{220}_{66} + 23*z^{220}_{65} + 5*z^{220}_{64} \\
& + 20*z^{220}_{63} + 36*z^{220}_{62} + 29*z^{220}_{61} + 2*z^{220}_{60} \\
& + 25*z^{220}_{59} + 14*z^{220}_{58} + 16*z^{220}_{57} + 31*z^{220}_{56} \\
& + 22*z^{220}_{55} + 31*z^{220}_{54} + 33*z^{220}_{53} + 19*z^{220}_{52} \\
& + 22*z^{220}_{51} + 23*z^{220}_{50} + 36*z^{220}_{49} + 11*z^{220}_{48} \\
& + 15*z^{220}_{47} + 15*z^{220}_{46} + 35*z^{220}_{45} + 7*z^{220}_{44} \\
& + 27*z^{220}_{43} + 28*z^{220}_{42} + 15*z^{220}_{41} + 31*z^{220}_{40} \\
& + 12*z^{220}_{39} + 19*z^{220}_{38} + 21*z^{220}_{37} + 18*z^{220}_{36} \\
& + 3*z^{220}_{35} + 36*z^{220}_{33} + z^{220}_{32} + 35*z^{220}_{31} \\
& + 21*z^{220}_{30} + 2*z^{220}_{29} + 13*z^{220}_{28} + 19*z^{220}_{27} \\
& + 6*z^{220}_{26} + 22*z^{220}_{24} + 26*z^{220}_{23} + 9*z^{220}_{22} \\
& + 7*z^{220}_{21} + 31*z^{220}_{20} + 31*z^{220}_{19} + 9*z^{220}_{18}
\end{aligned}$$

$$\begin{aligned}
& + 23*z^{220}_{17} + 23*z^{220}_{16} + 6*z^{220}_{15} + 27*z^{220}_{14} \\
& + 36*z^{220}_{13} + 4*z^{220}_{12} + 26*z^{220}_{11} + 30*z^{220}_{10} \\
& + 9*z^{220}_9 + 8*z^{220}_8 + 15*z^{220}_7 + 26*z^{220}_6 \\
& + 17*z^{220}_5 + 29*z^{220}_4 + 24*z^{220}_3 + 8*z^{220}_2 \\
& + 29*z^{220} : 1)
\end{aligned}$$

Q_3 is given by

$$\begin{aligned}
& (35*z^{220}_{219} + 22*z^{220}_{218} + 36*z^{220}_{216} + 24*z^{220}_{215} \\
& + 19*z^{220}_{214} + 32*z^{220}_{213} + 13*z^{220}_{212} + 19*z^{220}_{211} \\
& + 3*z^{220}_{210} + 36*z^{220}_{209} + 29*z^{220}_{208} + 35*z^{220}_{206} \\
& + 31*z^{220}_{205} + 32*z^{220}_{204} + 23*z^{220}_{203} + 21*z^{220}_{202} \\
& + 10*z^{220}_{201} + 32*z^{220}_{200} + 32*z^{220}_{199} + 21*z^{220}_{198} \\
& + 16*z^{220}_{197} + 23*z^{220}_{196} + 32*z^{220}_{195} + 12*z^{220}_{194} \\
& + 9*z^{220}_{193} + 35*z^{220}_{192} + 8*z^{220}_{191} + 19*z^{220}_{190} \\
& + 33*z^{220}_{189} + 13*z^{220}_{188} + 11*z^{220}_{187} + 35*z^{220}_{186} \\
& + 25*z^{220}_{185} + 28*z^{220}_{184} + 5*z^{220}_{183} + 7*z^{220}_{182} \\
& + 24*z^{220}_{181} + 35*z^{220}_{180} + 33*z^{220}_{179} + 18*z^{220}_{178} \\
& + 5*z^{220}_{177} + 31*z^{220}_{176} + 18*z^{220}_{175} + 30*z^{220}_{174} \\
& + 27*z^{220}_{173} + 3*z^{220}_{172} + 8*z^{220}_{171} + 24*z^{220}_{170} \\
& + 14*z^{220}_{169} + 2*z^{220}_{168} + 16*z^{220}_{167} + 14*z^{220}_{166} \\
& + 18*z^{220}_{165} + 22*z^{220}_{164} + 32*z^{220}_{163} + 28*z^{220}_{162} \\
& + 7*z^{220}_{161} + 19*z^{220}_{160} + 3*z^{220}_{159} + 14*z^{220}_{158} \\
& + 27*z^{220}_{157} + 35*z^{220}_{156} + 8*z^{220}_{155} + 25*z^{220}_{154} \\
& + 11*z^{220}_{153} + 19*z^{220}_{152} + 21*z^{220}_{151} + 10*z^{220}_{150} \\
& + 2*z^{220}_{149} + 4*z^{220}_{148} + 4*z^{220}_{147} + 31*z^{220}_{146} \\
& + 26*z^{220}_{145} + 17*z^{220}_{143} + 14*z^{220}_{142} + 12*z^{220}_{141} \\
& + 17*z^{220}_{140} + 22*z^{220}_{139} + 30*z^{220}_{138} + 30*z^{220}_{137} \\
& + 15*z^{220}_{136} + 16*z^{220}_{135} + 25*z^{220}_{134} + 8*z^{220}_{133} \\
& + 28*z^{220}_{132} + 5*z^{220}_{131} + 14*z^{220}_{130} + 26*z^{220}_{129} \\
& + 13*z^{220}_{128} + 10*z^{220}_{127} + 13*z^{220}_{126} + 10*z^{220}_{125} \\
& + 17*z^{220}_{124} + 33*z^{220}_{123} + 9*z^{220}_{122} + 9*z^{220}_{121} \\
& + 10*z^{220}_{120} + 12*z^{220}_{119} + 4*z^{220}_{118} + 6*z^{220}_{117} \\
& + 33*z^{220}_{116} + 21*z^{220}_{115} + 14*z^{220}_{114} + 33*z^{220}_{113} \\
& + 11*z^{220}_{112} + 4*z^{220}_{111} + 3*z^{220}_{110} + 3*z^{220}_{109} \\
& + 3*z^{220}_{108} + 3*z^{220}_{107} + 27*z^{220}_{106} + 8*z^{220}_{105} \\
& + 25*z^{220}_{104} + 10*z^{220}_{103} + 24*z^{220}_{102} + 2*z^{220}_{101} \\
& + 12*z^{220}_{100} + 35*z^{220}_{99} + 30*z^{220}_{98} + 14*z^{220}_{97} \\
& + 8*z^{220}_{96} + 16*z^{220}_{95} + 24*z^{220}_{94} + 23*z^{220}_{93} \\
& + 34*z^{220}_{91} + 3*z^{220}_{90} + 13*z^{220}_{89} + 10*z^{220}_{88} \\
& + 20*z^{220}_{87} + 14*z^{220}_{86} + 9*z^{220}_{85} + 36*z^{220}_{84} \\
& + 33*z^{220}_{83} + 12*z^{220}_{82} + 20*z^{220}_{81} + 5*z^{220}_{80} \\
& + 27*z^{220}_{79} + 27*z^{220}_{78} + 9*z^{220}_{77} + 23*z^{220}_{76} \\
& + 4*z^{220}_{75} + 26*z^{220}_{74} + 8*z^{220}_{73} + 11*z^{220}_{72} \\
& + 25*z^{220}_{71} + 35*z^{220}_{70} + 19*z^{220}_{69} + 36*z^{220}_{68}
\end{aligned}$$

$$\begin{aligned}
& + 35*z^{220^67} + 24*z^{220^66} + 8*z^{220^65} + 32*z^{220^64} \\
& + 10*z^{220^63} + 3*z^{220^62} + 18*z^{220^61} + 35*z^{220^60} \\
& + 17*z^{220^59} + 30*z^{220^58} + 2*z^{220^57} + 25*z^{220^56} \\
& + 7*z^{220^55} + 20*z^{220^54} + 27*z^{220^53} + z^{220^52} \\
& + 10*z^{220^51} + 2*z^{220^50} + 18*z^{220^49} + 30*z^{220^48} \\
& + 32*z^{220^47} + 20*z^{220^46} + 4*z^{220^45} + 16*z^{220^43} \\
& + 16*z^{220^42} + 11*z^{220^41} + 8*z^{220^40} + 12*z^{220^39} \\
& + 15*z^{220^38} + 25*z^{220^37} + 33*z^{220^36} + 4*z^{220^35} \\
& + 11*z^{220^34} + 6*z^{220^33} + 7*z^{220^32} + 32*z^{220^31} \\
& + 19*z^{220^30} + 19*z^{220^29} + 16*z^{220^28} + 10*z^{220^27} \\
& + 7*z^{220^26} + 10*z^{220^25} + 33*z^{220^24} + 25*z^{220^23} \\
& + 21*z^{220^22} + 35*z^{220^21} + 15*z^{220^20} + z^{220^19} \\
& + 19*z^{220^18} + 16*z^{220^17} + 10*z^{220^16} + 18*z^{220^15} \\
& + 17*z^{220^14} + 2*z^{220^13} + 35*z^{220^12} + 30*z^{220^11} \\
& + 17*z^{220^10} + 30*z^{220^9} + 26*z^{220^8} + 9*z^{220^7} \\
& + 34*z^{220^6} + 4*z^{220^5} + 12*z^{220^4} + 16*z^{220^3} \\
& + 27*z^{220^2} + 12*z^{220} + 36 \\
& : 21*z^{220^219} + 24*z^{220^218} \\
& + 33*z^{220^217} + 31*z^{220^216} + 29*z^{220^215} + 16*z^{220^214} \\
& + 26*z^{220^213} + 7*z^{220^212} + 15*z^{220^211} + 9*z^{220^210} \\
& + 19*z^{220^209} + 18*z^{220^208} + 16*z^{220^207} + 23*z^{220^206} \\
& + 27*z^{220^205} + 16*z^{220^204} + 5*z^{220^203} + 10*z^{220^202} \\
& + 2*z^{220^201} + 19*z^{220^200} + 19*z^{220^199} + 8*z^{220^198} \\
& + 30*z^{220^197} + 9*z^{220^196} + 27*z^{220^195} + 7*z^{220^194} \\
& + 20*z^{220^193} + 8*z^{220^192} + 29*z^{220^191} + 10*z^{220^190} \\
& + 32*z^{220^189} + 9*z^{220^188} + 4*z^{220^187} + 31*z^{220^186} \\
& + 8*z^{220^185} + 4*z^{220^184} + 8*z^{220^183} + 11*z^{220^182} \\
& + 13*z^{220^181} + 5*z^{220^180} + 29*z^{220^179} + 13*z^{220^178} \\
& + 20*z^{220^177} + 9*z^{220^176} + 3*z^{220^175} + 32*z^{220^174} \\
& + 3*z^{220^173} + 25*z^{220^172} + 33*z^{220^171} + 36*z^{220^170} \\
& + 11*z^{220^169} + 22*z^{220^168} + 18*z^{220^167} + 7*z^{220^166} \\
& + 4*z^{220^165} + 9*z^{220^164} + 33*z^{220^163} + 33*z^{220^162} \\
& + 18*z^{220^161} + 3*z^{220^160} + 35*z^{220^159} + 31*z^{220^158} \\
& + 20*z^{220^157} + 28*z^{220^155} + 33*z^{220^154} + 30*z^{220^153} \\
& + 28*z^{220^152} + 18*z^{220^151} + z^{220^150} + 34*z^{220^149} \\
& + 16*z^{220^148} + 23*z^{220^147} + 30*z^{220^146} + 3*z^{220^144} \\
& + 28*z^{220^143} + 8*z^{220^142} + 35*z^{220^140} + 11*z^{220^139} \\
& + 16*z^{220^138} + 20*z^{220^137} + 31*z^{220^136} + 11*z^{220^135} \\
& + 24*z^{220^134} + 29*z^{220^133} + 29*z^{220^132} + 8*z^{220^131} \\
& + 25*z^{220^130} + 11*z^{220^129} + 35*z^{220^128} + 36*z^{220^127} \\
& + 33*z^{220^126} + 18*z^{220^125} + 8*z^{220^124} + 9*z^{220^123} \\
& + 31*z^{220^122} + 29*z^{220^121} + 7*z^{220^120} + 4*z^{220^119}
\end{aligned}$$

$$\begin{aligned}
& + 3*z^{220^{118}} + 13*z^{220^{117}} + 35*z^{220^{116}} + 17*z^{220^{115}} \\
& + 6*z^{220^{114}} + 3*z^{220^{113}} + 13*z^{220^{112}} + 5*z^{220^{111}} \\
& + 31*z^{220^{110}} + 32*z^{220^{109}} + 17*z^{220^{108}} + 28*z^{220^{107}} \\
& + 21*z^{220^{106}} + 14*z^{220^{105}} + 25*z^{220^{104}} + 17*z^{220^{103}} \\
& + 33*z^{220^{102}} + 19*z^{220^{101}} + 4*z^{220^{100}} + 2*z^{220^{99}} \\
& + 7*z^{220^{98}} + 34*z^{220^{97}} + 15*z^{220^{96}} + 7*z^{220^{95}} \\
& + 34*z^{220^{94}} + 22*z^{220^{93}} + 22*z^{220^{92}} + 11*z^{220^{91}} \\
& + 33*z^{220^{90}} + 32*z^{220^{89}} + 19*z^{220^{88}} + 21*z^{220^{87}} \\
& + 23*z^{220^{86}} + 34*z^{220^{85}} + 35*z^{220^{84}} + 23*z^{220^{83}} \\
& + 27*z^{220^{82}} + 25*z^{220^{81}} + 26*z^{220^{80}} + 2*z^{220^{79}} \\
& + 33*z^{220^{78}} + 32*z^{220^{77}} + 8*z^{220^{76}} + 32*z^{220^{75}} \\
& + 15*z^{220^{74}} + 17*z^{220^{73}} + 31*z^{220^{72}} + 7*z^{220^{71}} \\
& + 8*z^{220^{70}} + 8*z^{220^{69}} + 22*z^{220^{68}} + 7*z^{220^{67}} \\
& + 14*z^{220^{66}} + 15*z^{220^{65}} + 26*z^{220^{64}} + 26*z^{220^{63}} \\
& + 35*z^{220^{62}} + 19*z^{220^{61}} + 18*z^{220^{60}} + 22*z^{220^{59}} \\
& + 25*z^{220^{57}} + 4*z^{220^{56}} + 5*z^{220^{55}} + 4*z^{220^{54}} \\
& + 20*z^{220^{53}} + 32*z^{220^{52}} + 17*z^{220^{51}} + 14*z^{220^{50}} \\
& + 31*z^{220^{49}} + 9*z^{220^{48}} + 30*z^{220^{47}} + 20*z^{220^{46}} \\
& + 7*z^{220^{45}} + 16*z^{220^{43}} + 23*z^{220^{42}} + 12*z^{220^{41}} \\
& + 21*z^{220^{40}} + 14*z^{220^{39}} + 8*z^{220^{38}} + 14*z^{220^{37}} \\
& + 35*z^{220^{36}} + 14*z^{220^{35}} + 22*z^{220^{34}} + 8*z^{220^{33}} \\
& + z^{220^{32}} + 24*z^{220^{31}} + 21*z^{220^{30}} + 33*z^{220^{29}} \\
& + 21*z^{220^{28}} + 22*z^{220^{26}} + 33*z^{220^{25}} + 13*z^{220^{24}} \\
& + 13*z^{220^{23}} + 5*z^{220^{22}} + 35*z^{220^{21}} + 3*z^{220^{20}} \\
& + 31*z^{220^{19}} + 13*z^{220^{18}} + 33*z^{220^{17}} + 30*z^{220^{16}} \\
& + 16*z^{220^{15}} + 30*z^{220^{14}} + 16*z^{220^{13}} + 11*z^{220^{12}} \\
& + 35*z^{220^{11}} + 22*z^{220^{10}} + 11*z^{220^9} + 8*z^{220^8} \\
& + z^{220^7} + 25*z^{220^6} + 8*z^{220^5} + 27*z^{220^4} + z^{220^3} \\
& + 29*z^{220^2} + 34*z^{220} + 29 : 1)
\end{aligned}$$