Generating random supersingular Elliptic Curves using modular polynomials

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June 6, 2022

Elliptic Curves

Definition

An *Elliptic Curve* is a projective variety with a defining equation of the form
$$y^2z = x^3 + Axz^2 + Bz^3$$

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- One point "at infinity"
- E defined over field k if $A, B \in k$

Elliptic Curves are groups

Proposition

Let E be an Elliptic Curve over k. Then there is $+_E : E \times E \to E$ such that E becomes a group.
Further, $+_E$ is (locally) given by polynomials.

- E is an *algebraic group*

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- E is an *algebraic group*
- For $x_1 \neq x_2$, define $(x_1, y_1) +_E (x_2, y_2)$ to be

$$\left(\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2,\ (2x_1 + x_2)\frac{y_2 - y_1}{x_2 - x_1} - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^3 - y_1\right)$$

Isogenies

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An algebraic map (i.e. morphism) between Elliptic Curves $E \to E'$ is called isogeny, if it maps $\infty \mapsto \infty$.

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- l-isogeny := degree l isogeny

Isogenies (continued)

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$$\Rightarrow \text{have isogeny} \quad [m] : E \to E, \quad P \mapsto \underbrace{P + \ldots + P}_{m \text{ times}}$$

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- If E defined over $\mathbb{F}_q$

$$\Rightarrow \text{have isogeny} \quad \pi : E \to E, \quad (x, y) \mapsto (x^q, y^q)$$

Supersingular and ordinary curves

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Definition

In the first case, E is called *ordinary*, otherwise *supersingular*.

Isogeny graphs

Elliptic Curves up to isomorphism are classified by *j-invariant* $j(E)$

- l-isogeny graph: $V = \{j(E) \mid E \text{ defined over } \mathbb{F}_q\}$
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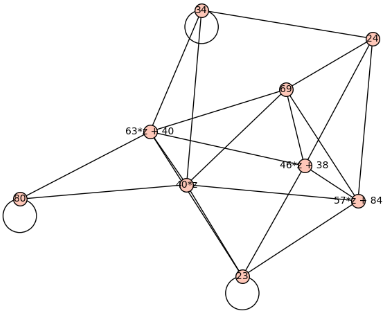

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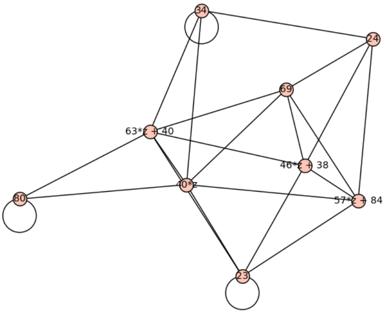

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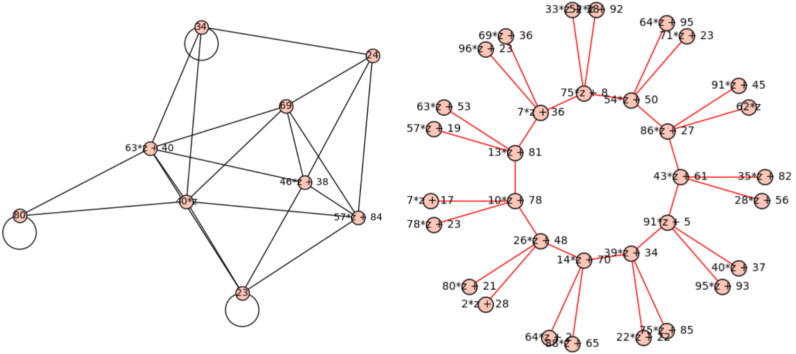
- Ordinary l-isogeny graphs are “vulcanoes”



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How many ordinary resp. supersingular curves with l-isogeny $E \to E^{(p)}$ exist?

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There is a polynomial $\Phi_l(x, y) \in \mathbb{Z}[x, y]$ *such that* $\Phi_l(j(E), j(E')) = 0$ *if and only if there is an* l*-isogeny* $E \to E'$*.*

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- Finding a curve with an l_1-and an l_2 isogeny $E \to E^{(p)}$ corresponds to finding a root of $\gcd(\Phi_{l_1}(x, x^p), \Phi_{l_2}(x, x^p))$

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Is there a way to find a root of $\gcd(\Phi_{l_1}(x, x^p), \Phi_{l_2}(x, x^p))$ *for exponentially large* l_1, l_2 *(and of course* p*)?*

Thank you for your attention!

Jeremy Booher et al. *Failing to hash into supersingular isogeny graphs*. Cryptology ePrint Archive, Report 2022/518. https://ia.cr/2022/518. 2022.