Some Notes about the things I encountered

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Notation

If E is an Elliptic Curve defined over a finite field of characteristic p, we write $E^{(p)}$ for the curve defined by the equations of E after replacing all coefficients by their p-th power. Similarly, for an isogeny $\phi : E \to E'$, write $\phi^{(p)} : E^{(p)} \to E'^{(p)}$ for the isogeny defined by the polynomials of ϕ after replacing all coefficients by their p-th power. Furthermore, for a point $P = (x : y : z) \in \mathbb{P}^2$ write $P^{(p)} := (x^p : y^p : z^p)$. Finally, for a set of points or endomorphisms S write $S^{(p)} := \{s^{(p)} \mid s \in S\}$. Note that

$$\cdot^{(p)} : \mathbf{Ell} \to \mathbf{Ell}, \quad E \mapsto E^{(p)}$$
$$\mathrm{Hom}_{\mathbf{Ell}}(E, E') \ni \phi \to \phi^{(p)}$$

is a covariant endofunctor on the category $\mathbf{Ell}$ of Elliptic Curves defined over $\bar{\mathbb{F}}_p$ and their isogenies.

Sometimes, we abuse terminology and speak of Elliptic Curves when we mean isomorphism classes of Elliptic Curves.

Many examples will be over $\mathbb{F}_{101^2}$. Let $p = 101$ and $q = p^2$. We usually use the generator $\alpha \in \mathbb{F}_q$ with minimal polynomial $x^2 + 97x + 2$.

1 Example - The cases I, II and III

1.1 Case I

Finding examples of case I is trivial - just take a curve E with $j(E) \in \mathbb{F}_p$. Then clearly $E^{(p)} = E$ and so also $E_0^{(p)} = E_0$ (since $\cdot^{(p)}$ maps the path $E \to E_0$ to $E = E^{(p)} \to E_0^{(p)}$).

Furthermore, it is easy to see that there are a lot of curve E such that the associated E_0 is defined over $\mathbb{F}_p$ (and we are again in case I).

1.2 Case II

Here I was not quite sure if it even occurs. As it turns out, it does. Consider E with $j(E) = 17\alpha + 45$. Then $[\mathcal{O}_K : \mathbb{Z}[\pi]] = 2^3$ so E lies on the crater of the 3-isogeny graph. However there is a 3-isogeny $E \to E^{(p)}$ since $j(E^{(p)}) = j(E)^p = 84\alpha + 12$. In fact, in

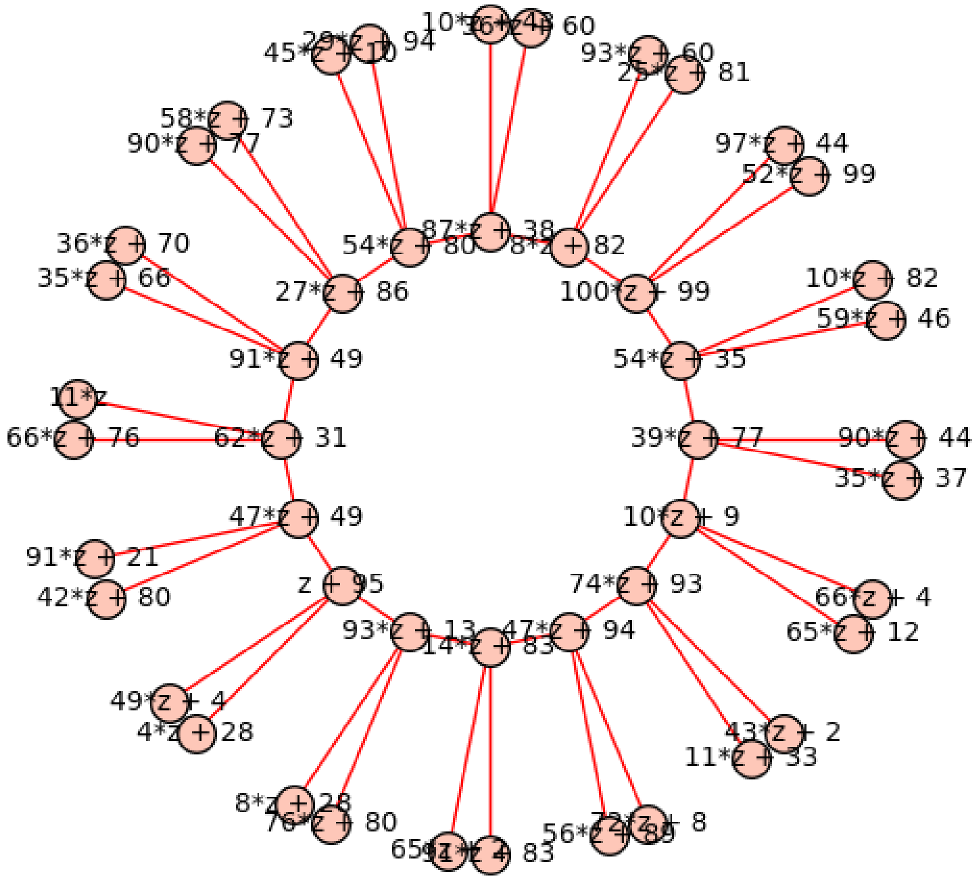

Figure 1: A 3-isogeny vulcano over $\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$ that satisfies case II (in the plot have $z = \alpha$). Note that e.g. $(39\alpha + 77)^{101} = 62\alpha + 31$.

this case, the crater consists only of E and $E^{(p)}$. For a more interesting example, see Figure 1.

Further, when we consider the path $E = E_0 \to ... \to E_n = E^{(p)}$ on the crater, there are more or less two possibilities for the $\cdot^{(p)}$ conjugate path[1].

- It could be that the conjugate of $E_i \to E_{i+1}$ is the dual of $E_{n-i-1} \to E_{n-i}$, hence we just go the path $E \to ... \to E^{(p)}$ backwards.
- It could be that the conjugate of $E_i \to E_{i+1}$ is $E_{n+i} \to E_{n+i+1}$, where

$$E_0, ..., E_n, E_{n+1}, ..., E_{n+m} = E_0$$

is the cycle along the whole crater.

[1]Remember that $\cdot^{(p)}$ is functorial, hence we can also apply to isogenies $E_i \to E_{i+1}$

However, as we will see, the first case is impossible.

Note that we have

Proposition 1.1. *Let* $[\mathfrak{b}] \in \mathrm{Cl}(\mathcal{O})$ *where* $\mathcal{O} = \mathrm{End}(E)$ *for an ordinary Elliptic Curve* $E/\mathbb{F}_{p^2}$ *such that* $[\mathfrak{b}].E = E^{(p)}$. *Then* $[\mathfrak{b}]^2 = [(1)]$.

Proof. I think there is some mistake with my definition of the class group action, see also the next paragraph. With the current (probably wrong) definition, the following works. Otherwise, I suppose that anyway we have $[\mathfrak{b}] = [(p, \pi)]$ and then the claim follows by Lemma 2.5.

We recall the definition of the class group action in the case $[\mathfrak{b}].E^{(p)}$. For an ideal $\mathfrak{b}' \leq \mathrm{End}(E^{(p)})$, have by definition

$$[\mathfrak{b}'].E^{(p)} = E^{(p)}/E^{(p)}[\mathfrak{b}'] = E^{(p)}/\bigcap_{\beta \in \mathfrak{b}'} \ker(\beta)$$

However, $\mathfrak{b}$ is an ideal in $\mathrm{End}(E)$, which is only isomorphic to $\mathrm{End}(E^{(p)})$. Since $\mathrm{End}^0(E)$ is a quadratic imaginary number field, it has one nontrivial field automorphism, and thus the isomorphism $\mathrm{End}(E) \cong \mathrm{End}(E^{(p)})$ is not unique. But there is a unique canonical isomorphism, i.e. an isomorphism that is induced by an (equivalently any) isogeny $\phi : E \to E^{(p)}$ as

$$\Phi_* : \mathrm{End}(E) \to \mathrm{End}(E^{(p)}), \quad \alpha \mapsto \frac{1}{\deg(\phi)}\, \phi \circ \alpha \circ \hat{\phi}$$

This is the isomorphism we use, i.e. we say

$$E^{(p)}[\mathfrak{b}] = E^{(p)}[\Phi_*(\mathfrak{b})] \quad \text{and} \quad [\mathfrak{b}].E^{(p)} = [\Phi_*(\mathfrak{b})].E^{(p)} = E^{(p)}/E^{(p)}[\mathfrak{b}]$$

Now let $\phi : E \to E/E[\mathfrak{b}] = E^{(p)}$ be a separable isogeny with kernel $E[\mathfrak{b}]$ (by choosing the representative $\mathfrak{b}$ of $[\mathfrak{b}] \in \mathrm{Cl}(\mathcal{O})$ correspondingly, we can assume that). We have

$$\ker(\phi^{(p)}) = E[\mathfrak{b}]^{(p)} = \bigcap_{\beta \in \mathfrak{b}} \ker(\beta)^{(p)} = \bigcap_{\beta \in \mathfrak{b}} \ker(\beta^{(p)}) = \bigcap_{\beta \in \mathfrak{b}^{(p)}} \ker(\beta)$$

Now note that the Frobenius isogeny $\pi : E \to E^{(p)},\ P \mapsto P^{(p)}$ induces the canonical isomorphism $\mathrm{End}(E) \to \mathrm{End}(E^{(p)})$ and so the image of $\mathfrak{b}$ under that isomorphism is $\mathfrak{b}' = \mathfrak{b}^{(p)} \leq \mathrm{End}(E^{(p)})$. Thus

$$\bigcap_{\beta \in \mathfrak{b}^{(p)}} \ker(\beta) = \bigcap_{\beta \in \mathfrak{b}'} \ker(\beta) = E^{(p)}[\mathfrak{b}'] = E^{(p)}[\mathfrak{b}]$$

So by the uniqueness of the image curve for an isogeny with fixed kernel yields that $E = \mathrm{im}(\phi^{(p)}) = [\mathfrak{b}].E^{(p)}$. Thus $[\mathfrak{b}]^2.E = [\mathfrak{b}].E^{(p)} = E$ and since the class group action is free, we see that $[\mathfrak{b}]^2 = [(1)]$. $\square$

From this we get the

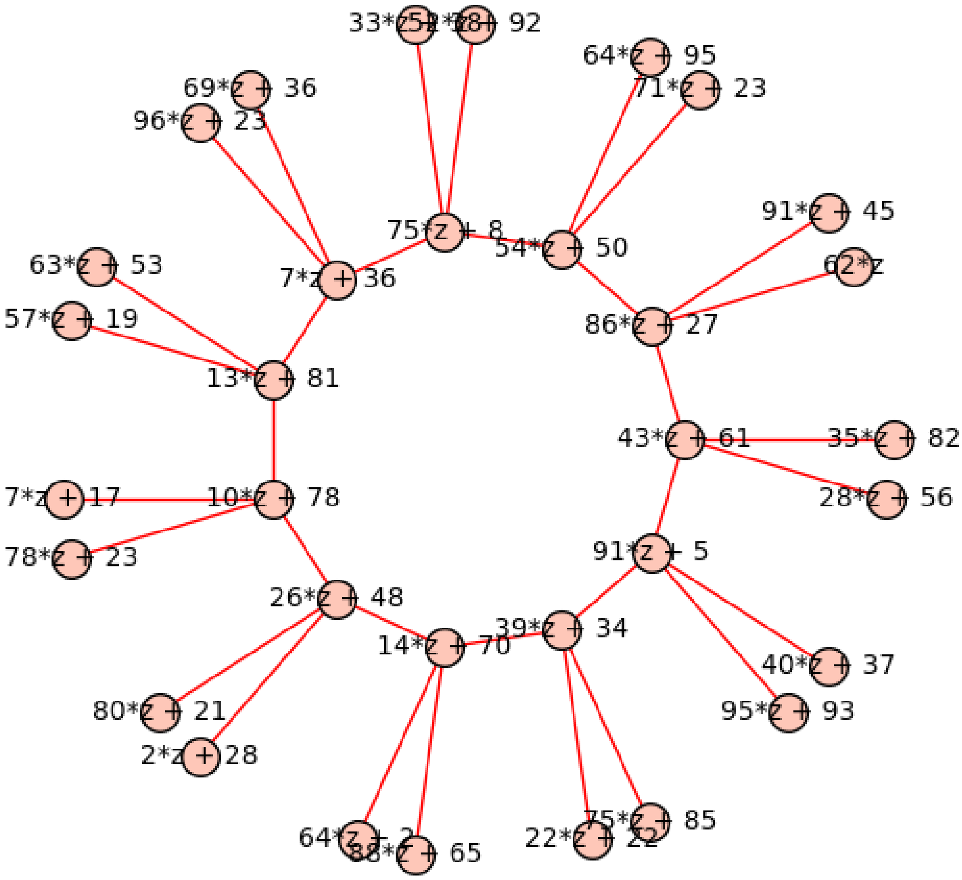

Figure 2: A 3-isogeny vulcano over $\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$ that satisfies case III (in the plot have $z = \alpha$).

Corollary 1.2. *Assume that* $E = E_0 \to E_1 \to ... \to E_n = E$ *is the cycle once around the crater (and* $j(E) \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$*). If* $E^{(p)} = E_i$ *then* n *is even and* $i = n/2$*, i.e.* $E^{(p)}$ *is on the other side of the crater*[2].

Proof. If l does not split in $\mathcal{O}_\mathcal{K}$, then the crater has at most two elements and this is trivial. So assume $(l) = \mathfrak{l}_1\mathfrak{l}_2$. It is known that then the action of $[\mathfrak{l}_1]$ resp. $[\mathfrak{l}_2]$ corresponds to walking around the crater in one direction resp. the other. So wlog $[\mathfrak{l}_1].E_i = E_{i+1}$.

Now assume that $E^{(p)} = E_i$, so $[\mathfrak{b}].E = E_i = [\mathfrak{l}_1]^i.E$. Since the action is free, it follows that $[\mathfrak{b}] = [\mathfrak{l}_1]^i$ By the previous theorem, we have now $[\mathfrak{l}_1]^{2i} = [\mathfrak{b}]^2 = [(1)]$ and so $[\mathfrak{l}_1]^{2i}.E = E_{2i} = E$. Thus $i = n/2$ and the claim follows. $\square$

In particular, the path between E and $E^{(p)}$ is likely to have length $\omega(\log(p))$, since the crater is usually large. This is displayed e.g. Figure 1.

1.3 Case III

We give the example displayed in Figure 3. Consider E with $j(E) = 64\alpha + 5$. Then $j(E^{(p)}) = j(E)^p = 37\alpha + 59$. However, we have that E lies on the crater, together with curve of j-invariants

$$88\alpha + 70,\ 54\alpha + 52,\ 95\alpha + 11$$

Hence there is no 3-isogeny path from E to $E^{(p)}$. Note that $[\mathcal{O}_\mathcal{K} : \mathbb{Z}[\pi]] = 2^2 \cdot 3^2$ but $[\mathcal{O}_\mathcal{K} : \mathrm{End}(E)] = 2^2$, which shows that E lies on the crater.

Now we want to have a closer look onto the class group action in this case. Have $d(\mathrm{End}(E)) = -320$, so $\mathcal{K} = \mathbb{Q}(\sqrt{-5})$ and $d(\mathcal{O}_\mathcal{K}) = -5$. Hence, we have $\mathrm{End}(E) \cong \mathbb{Z}[4\sqrt{-5}]$ and $\mathcal{O}_\mathcal{K} \cong \mathbb{Z}[\sqrt{-5}]$.

Sage tells us that $h(\mathcal{O}_\mathcal{K}) = 2$ and $h(\mathrm{End}(E)) = 8$. With this, we can already see that

$$64\alpha + 5,\ 88\alpha + 70,\ 54\alpha + 52,\ 95\alpha + 11$$

and

$$(64\alpha + 5)^p,\ (88\alpha + 70)^p,\ (54\alpha + 52)^p,\ (95\alpha + 11)^p$$

is the set of j-invariants of all Elliptic Curves with endomorphism ring $\cong \mathrm{End}(E)$. On this set, $\mathrm{Cl}(\mathbb{Z}[4\sqrt{-5}])$ then acts freely and transitively. Now it would be of course interesting to find out how $\mathrm{Cl}(\mathbb{Z}[4\sqrt{-5}])$ really looks like.

2 Properties of the endomorphism ring vs the cases

Proposition 2.1. *Let* E *be an ordinary Elliptic Curve defined over a finite field of characteristic* p*.*

- $\mathrm{End}(E)$ *has an element of norm* p *iff* $j(E) \in \mathbb{F}_p$*.*
- $\mathrm{End}(E)$ *has a nontrivial element (i.e.* $\neq \epsilon p$ *for a unit* ϵ*) of norm* p^2 *iff* $j(E) \in \mathbb{F}_{p^2}$*.*

[2] Note that this does not hold if $E, E^{(p)}$ are not in the same crater, see Figure 2.

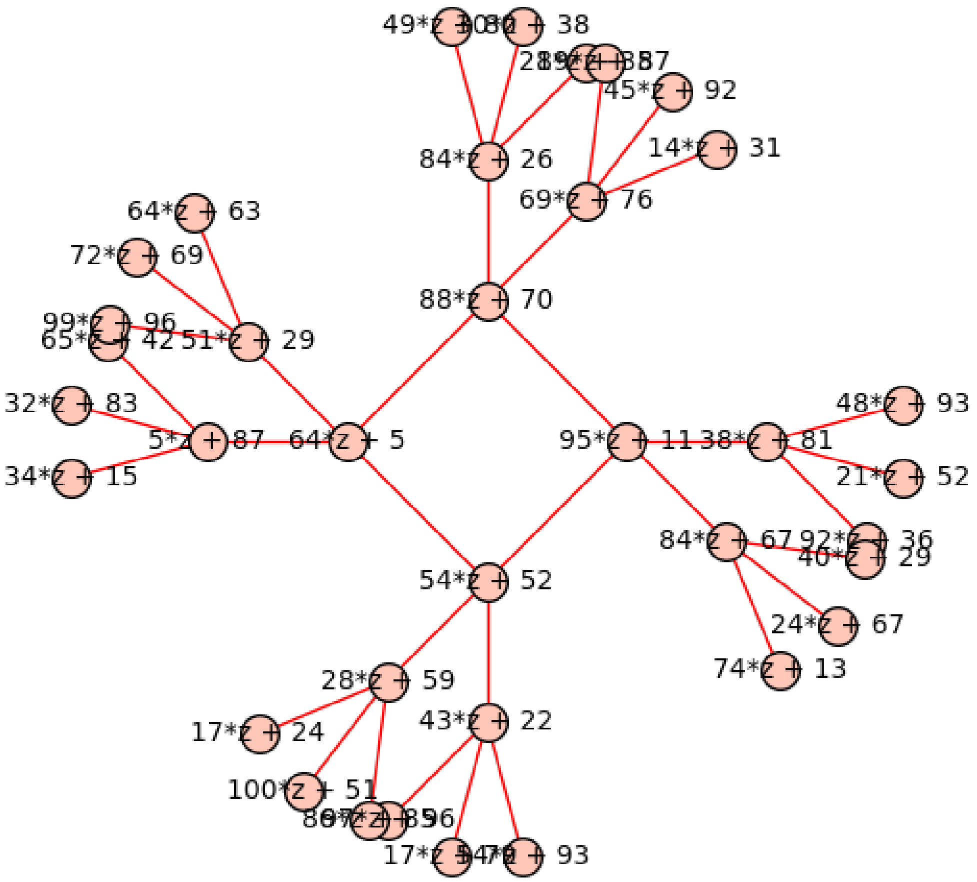

Figure 3: A 3-isogeny vulcano over $\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$ that satisfies case III (in the plot have $z = \alpha$).

Proof. The directions $\Leftarrow$ is clear, as the norm of the q-th power Frobenius endomorphism is q.

For the direction $\Rightarrow$, assume there is an element $\alpha \in \mathrm{End}(E)$ with $N(\alpha) = p$. If α is inseparable (as isogeny), then we have that it factors through the p-th power Frobenius endomorphism π, and thus $\alpha = \lambda \circ \pi$ for an isomorphism $\lambda : E^{(p)} \to E$. Thus $j(E^{(p)}) = j(E)$.

On the other hand, if α is separable, it must have kernel of size p, so $\ker(\alpha) = E[p]$ since $\#E[p] = p$ (E is ordinary). Thus $\ker(\alpha) \subseteq \ker([p])$ and we see that $[p]$ factors through α as $[p] = \psi \circ \alpha$. Now have that $\deg(\psi) = p = p^2/\deg(\alpha)$ and clearly ψ is inseparable. The claim follows as above.

For the second point, assume $\alpha \in \mathrm{End}(E)$ has norm $N(\alpha) = p^2$ and $\alpha \neq \pm p$. If α is purely inseparable, we are done. If α is separable, its kernel must be $E[p^2]$ and so it factors through $[p^2]$. Since $[p^2]$ has inseparability degree p^2, we see that $[p^2] = \pi^2 \circ \alpha$ where π is the p-th power Frobenius morphism. Since α is an endomorphism of E, find $\pi^2 : E \to E$, thus $j(E) \in \mathbb{F}_{p^2}$.

Finally, if α has inseparability degree p, then its kernel must be $E[p]$ and so $\alpha = \beta \circ \pi$ where $\beta : E^{(p)} \to E$ is separable with kernel $E^{(p)}[p]$. However, by the uniqueness of the separable isogeny with kernel $E^{(p)}[p]$, we know that (up to isomorphism) also $[p]$ is $\beta \circ \pi$. This now implies that $\alpha = \epsilon p$ for some unit ϵ. □

Proposition 2.2. *Let $D < 0$. Then the curves E with $\mathrm{End}(E) = \mathbb{Z}[\sqrt{D}]$ have $j(E) \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ if and only if*

$$a^2 - b^2 D = p$$

has no solution $a, b \in \mathbb{Z}$ and

$$a^2 - b^2 D = p$$

has a nontrivial solution $a, b \in \mathbb{Z}$.

Let $E/\mathbb{F}_{p^2}$ be an ordinary Elliptic Curve and write $\mathcal{O} := \mathrm{End}(E), \mathcal{K} := \mathrm{End}(E) \otimes \mathbb{Q}$. Let π be the $q = p^2$-th power Frobenius endomorphism and let t be its trace. Assume $p \neq 2$.

Lemma 2.3. *Have $(p) = (p, \pi)(p, \pi - t)$ in $\mathbb{Z}[\pi]$, $\mathcal{O}$ resp. $\mathcal{O}_{\mathcal{K}}$.*

Proof. Have

$$(p, \pi)(p, \pi - t) = (p^2, p\pi, p\pi - pt, \pi^2 - t\pi) = (p^2, pt, p\pi, -p^2) = (up^2 + vtp, ...) = (p)$$

where $up + vt = 1$ (note that $t \perp p$ since E is ordinary). □

Lemma 2.4. *(p, π) is principal (in $\mathcal{O}$) if and only if $E/\mathbb{F}_p$.*

Proof. If (p, π) is principal, then its generator is an element of norm p, so $E/\mathbb{F}_p$. On the other hand, if $E/\mathbb{F}_p$, then the p-th power Frobenius endomorphism π_p satisfies $p = \pi_p(t_p - \pi_p)$, $\pi = \pi_p^2$ and $\pi_p = u(\pi + p) + v\pi_p p$, where t_p is its trace and $ut + vp = 1$. □

There must be some problem in my definition of the class group action, as it can happen that $[(p,\pi)]$ is not $[(1)]$, but $E[(p,\pi)]$ is clearly trivial, so[3] $(p,\pi).E = E/E[(p,\pi)] = E$. However, this contradicts the freeness of the class group action.

Lemma 2.5. *Assume $j \neq 0, 1728$. $[(p,\pi)]$ has order ≤ 2 in $\mathrm{Cl}(\mathcal{O})$ resp. $\mathrm{Cl}(\mathcal{O}_{\mathcal{K}})$.*

Proof. Since $E/\mathbb{F}_{p^2}$, we know that there is a nontrivial element α of norm p^2. Now have in $\mathcal{O}_{\mathcal{K}}$ that $(\alpha)|(p)^2$ and with $p = (p,\pi)(p,\pi - t)$ have thus $(\alpha) = (p)$ or $(\alpha) = (p,\pi)^2$ or $(\alpha) = (p,\pi - t)^2$. However, by assumption we only have units ± 1 in $\mathcal{O}_{\mathcal{K}}$ resp. $\mathcal{O}$, so the first case is impossible, as it implies $\alpha = \pm p$.

Note that $[(p,\pi)] = [(p,\pi - t)^{-1}]$, so wlog assume $(\alpha) = (p,\pi)^2$. It follows that $(p,\pi)^2$ is principal, so $[(p,\pi)]^2 = [(1)]$. $\square$

3 Ideals in $\mathcal{O}$ resp. $\mathcal{O}_{\mathcal{K}}$

Consider an ordinary Elliptic Curve $E/\mathbb{F}_q$, $\mathcal{O} := \mathrm{End}(E)$, $\mathcal{K} := \mathcal{O} \otimes \mathbb{Q}$ and $\mathcal{O}_{\mathcal{K}}$ the ring of integers in $\mathcal{K}$. Assume that $j(E) \in \mathbb{F}_q$ is not contained in any proper subfield of $\mathbb{F}_q$ and let π be the q-th power Frobenius endomorphism. Let t be its trace.

Proposition 3.1. *$p \nmid d(\mathcal{O})$*

Proof. Have that

$$d(\mathbb{Z}[\pi]) = t^2 - 4q \perp p$$

since $t \perp p$ as E is ordinary. The claim follows since $d(\mathcal{O}) \mid d(\mathcal{O})[\mathcal{O} : \mathbb{Z}[\pi]]^2 = d(\mathbb{Z}[\pi])$. $\square$

Proposition 3.2. *Let $\mathfrak{a} \leq \mathcal{O}$. Then $\mathfrak{a} \cap \mathbb{Z} = (a)$ with $a \mid [\mathcal{O} : \mathfrak{a}] \mid a^2$. Note that if $\mathfrak{a} = \mathfrak{p}$ is prime, then trivially a must be prime.*

Proof. Clearly $[\mathcal{O} : \mathfrak{a}] \in \mathfrak{a}$ as $1 \in \mathcal{O}/\mathfrak{a}$ has order dividing $\#(\mathcal{O}/\mathfrak{a}) = [\mathcal{O} : \mathfrak{a}]$, so $a \mid [\mathcal{O} : \mathfrak{a}]$. On the other hand, have $[\mathcal{O} : \mathfrak{a}] \mid [\mathcal{O} : a\mathcal{O}] = a^2$. $\square$

Lemma 3.3. *Let $\mathfrak{p} \leq \mathcal{O}_{\mathcal{K}}$ be a prime with $\mathfrak{N}(\mathfrak{p}) \perp [\mathcal{O}_{\mathcal{K}} : \mathcal{O}]$. Then $\mathfrak{p}$ has a set of generators in $\mathcal{O}$.*

Proof. Suppose $\mathfrak{p}$ is a prime over p, and let $\mathcal{O} = \mathbb{Z}[\phi]$. If $\mathrm{MiPo}(\phi) = f(X)g(X) \mod p$ splits, then have

$$p\mathcal{O}_{\mathcal{K}} = (p, f(\phi))(p, g(\phi))$$

and so the prime ideals over p are $(p, f(\phi))$ and $(p, g(\phi))$. If $\mathrm{MiPo}(\phi) \mod p$ is irreducible, then have that $p\mathcal{O}_{\mathcal{K}}$ is prime and thus the only prime ideal over p. Hence, all prime ideals over p (including $\mathfrak{p}$) have a set of generators in $\mathcal{O}$. $\square$

Corollary 3.4. *Let $\mathfrak{a} \leq \mathcal{O}_{\mathcal{K}}$ be an ideal with $\mathfrak{N}(\mathfrak{a}) \perp [\mathcal{O}_{\mathcal{K}} : \mathcal{O}]$. Then $\mathfrak{a}$ has a set of generators in $\mathcal{O}$.*

[3]We know that (p,π) is invertible, as $\frac{1}{p}(p,\pi)(p,\pi - t) = (1)$.

Proposition 3.5. *Let $\mathfrak{p} \leq \mathcal{O}$ be a prime and $\mathfrak{p}' = \mathfrak{p}\mathcal{O}_{\mathcal{K}}$. Then $\mathcal{O}_{\mathfrak{p}} = (\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}'}$.*

Proof. We have $\mathcal{O}_{\mathcal{K}} = \mathbb{Z}[\alpha]$ and $\mathcal{O} = \mathbb{Z}[f\alpha]$ where $f = [\mathcal{O}_{\mathcal{K}} : \mathcal{O}]$. Thus $f \notin \mathfrak{p}$ and so $f \in \mathcal{O}_{\mathfrak{p}}^*$. Therefore $\mathcal{O}_{\mathcal{K}} \subseteq \mathcal{O}_{\mathfrak{p}}$ and thus $(\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}'} \subseteq \mathcal{O}_{\mathfrak{p}}$. □

Proposition 3.6. *Let $\mathfrak{I}(\mathcal{O})$ resp. $\mathfrak{I}(\mathcal{O}_{\mathcal{K}})$ denote the set of invertible ideals of norm $\perp [\mathcal{O}_{\mathcal{K}} : \mathcal{O}]$. Then*

$$\mathfrak{I}(\mathcal{O}) \to \mathfrak{I}(\mathcal{O}_{\mathcal{K}}), \quad \mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_{\mathcal{K}}$$

is a monoid isomorphism with inverse

$$\mathfrak{I}(\mathcal{O}_{\mathcal{K}}) \to \mathfrak{I}(\mathcal{O}), \quad \mathfrak{a} \mapsto \mathfrak{a} \cap \mathcal{O}$$

Proof. Clearly, this is a well-defined monoid homomorphism. Hence, we have to show that it is bijective.

By Corollary 3.4, we know that any $\mathfrak{a} \leq \mathcal{O}_{\mathcal{K}}$ with $\mathfrak{N}(\mathfrak{a}) \perp [\mathcal{O}_{\mathcal{K}} : \mathcal{O}]$ has generators in $\mathcal{O}$, thus $(\mathfrak{a} \cap \mathcal{O})\mathcal{O}_{\mathcal{K}} = \mathfrak{a}$. This shows that $\mathfrak{a} \cap \mathcal{O}$ is a preimage of $\mathfrak{a}$, and so the map is surjective.

Assume now $\mathfrak{a}, \mathfrak{b} \leq \mathcal{O}$ with $\mathfrak{a}\mathcal{O}_{\mathcal{K}} = \mathfrak{b}\mathcal{O}_{\mathcal{K}}$ and $\mathfrak{N}(\mathfrak{a}), \mathfrak{N}(\mathfrak{b}) \perp [\mathcal{O}_{\mathcal{K}} : \mathcal{O}]$. We show that $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}$ for all primes $\mathfrak{p} \leq \mathcal{O}$. Note that if $\mathfrak{N}(\mathfrak{p}) \not\perp [\mathcal{O}_{\mathcal{K}} : \mathcal{O}]$, this holds trivially, as $\mathfrak{a}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}$. Otherwise, note that

$$\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}} = \mathfrak{a}(\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}} = \mathfrak{a}\mathcal{O}_{\mathcal{K}}(\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}} = \mathfrak{b}\mathcal{O}_{\mathcal{K}}(\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}(\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}$$

as $\mathcal{O}_{\mathfrak{p}} = (\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}}$. This shows that $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}$ at all primes, so $\mathfrak{a} = \mathfrak{b}$ and our map is injective. Furthermore, since $(\mathfrak{a} \cap \mathcal{O})\mathcal{O}_{\mathcal{K}} = \mathfrak{a}$, we see that it has the inverse

$$\mathfrak{I}(\mathcal{O}_{\mathcal{K}}) \to \mathfrak{I}(\mathcal{O}), \quad \mathfrak{a} \mapsto \mathfrak{a} \cap \mathcal{O}$$

which must then be well-defined. □

Example 3.7. Let $K = \mathbb{Q}(\sqrt{-3})$, then $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ which is a PID. Let further $\mathcal{O} = \mathbb{Z}[2\sqrt{-3}]$, so $f = [\mathcal{O}_K : \mathcal{O}] = 4$.

Proposition 3.8. *Let $\mathfrak{a} \leq R$ be a radical ideal in a commutative unital ring R. If $\alpha \in \mathfrak{p}$ for all primes $\mathfrak{p} \supseteq \mathfrak{a}$ then $\alpha \in \mathfrak{a}$.*

Proof. We have that $\mathfrak{a}_\alpha \neq (1)$ otherwise $\alpha^n \in \mathfrak{a}$, so $\alpha \in \mathfrak{a}$. Thus $\mathfrak{a}_\alpha \subseteq \mathfrak{m}$ for a maximal ideal $\mathfrak{m} \leq R_\alpha$. A preimage under $R \to R_\alpha$ is now a prime $\mathfrak{p}$ with $\mathfrak{a} \subseteq \mathfrak{p}$ and $\alpha \notin \mathfrak{p}$. □

Corollary 3.9. *If $q \perp d(\mathcal{O})$ is an integer and $q \mid \alpha$ in $\mathcal{O}_{\mathcal{K}}$, then also $q \mid \alpha$ in $\mathcal{O}$.*

Proof. It suffices to prove this for primes q. Since $q \nmid d(\mathcal{O})$, we know that (q) is unramified, hence radical. Now observe that $\mathcal{O}_{\mathfrak{p}} = (\mathcal{O}_{\mathcal{K}})_{\mathfrak{p}}$ for primes $\mathfrak{p}$ over q and so $\alpha \in \mathfrak{p}$ for all primes $\mathfrak{p}$ over q. The previous proposition now shows that $\alpha \in (q)$. □

4 Norm equations

Lemma 4.1. *Let $D < 0$ and $\mathcal{O}$ be the imaginary quadratic order of discriminant D. Then $1, \alpha$ with $\alpha = \frac{D+\sqrt{D}}{2}$ is a $\mathbb{Z}$-basis of $\mathcal{O}$ and*

$$N(a + b\alpha) = \left(a + \frac{D}{2}b\right)^2 - \frac{D}{4}b^2$$

Proof.

$$\begin{aligned} N(\alpha) =& a^2 + ab\frac{D+\sqrt{D}}{2} + ab\frac{D-\sqrt{D}}{2} + \frac{D^2-D}{4}b^2 = a^2 + Dab + \frac{D^2-D}{4}b^2 \\ =& \left(a + \frac{D}{2}b\right)^2 - \frac{D}{4}b^2 \end{aligned}$$

□

Corollary 4.2. *Let l be a prime and $D < 0$. Let $\mathcal{O}$ be the quadratic imaginary order of discriminant D. If there exists a nontrivial element $\alpha \in \mathcal{O}$ of norm l^e (i.e. $\alpha \notin \mathbb{Z}$), then*

$$e \geq \log_l(-D) - \log_l(4)$$

Corollary 4.3. *Let $E/\bar{\mathbb{F}}_p$ be an ordinary Elliptic Curve such that* $\mathrm{End}(E)$ *has discriminant D. Suppose that $j(E_0) \notin \mathbb{F}_p$ and l is not ramified in* $\mathrm{End}(E) \otimes \mathbb{Q}$*. Then the shortest l-isogeny path between E and $E^{(p)}$ has length at least*

$$\frac{1}{2}\log_l(-D) - \log_l(2)$$

Proof. First, define E_0 to be the corresponding curve on the crater of the l-isogeny vulcano. Let $v \in \mathfrak{N}$ be maximal with $l^v \mid [\mathrm{End}(E_0) : \mathrm{End}(E)]$, so $\mathrm{End}(E_0)$ has discriminant D/l^v. Now we know that $E_0^{(p)}$ is at the opposite side of the crater, and the size of the crater is the order of $[\mathfrak{l}_1]$ in $\mathrm{Cl}(\mathrm{End}(E_0))$ where $l = \mathfrak{l}_1\mathfrak{l}_2$. If e is this order, then have that $\mathfrak{l}^e = (\alpha)$ is principal. Note that $\alpha \notin \mathbb{Z}$, otherwise we would have $\alpha = \pm l^{e/2}$, but since $(\alpha) = \mathfrak{l}_1^e$, we know that $\mathfrak{l}_2 \nmid (\alpha)$ (here we use that $\mathfrak{l}_1 \neq \mathfrak{l}_2$, i.e. l is unramified).

So

$$e \geq \log_l(-D/l^v) - \log_l(4)$$

Thus, the distance of E_0 and $E_0^{(p)}$ is at least

$$\frac{1}{2}e \geq \frac{1}{2}\log_l(-D/l^v) - \frac{1}{2}\log_l(4) = \frac{1}{2}\log_l(-D) - \frac{1}{2}v - \log_l(2)$$

However, the shortest path from E to E_0 has length v, and similarly for the shortest path from $E^{(p)}$ to $E_0^{(p)}$. Thus we find that the length of the shortest path from E to $E^{(p)}$ is at least

$$\frac{1}{2}\log_l(-D) - \frac{1}{2}v - \log_l(2) + 2v \geq \frac{1}{2}\log_l(-D) - \log_l(2)$$

□

$j(E)$	$h(\mathrm{End}(E))$	$[\mathcal{O}_{\mathcal{K}} : \mathrm{End}(E)]$	$d(\mathrm{End}(E))$	$[\mathrm{End}(E) : \mathbb{Z}[\pi]]$
α	36	3	-36315	1
$4\alpha + 99$	64	1	-40020	1
$61\alpha + 16$	2	1	-24	28
$48\alpha + 73$	64	?	-37440	?
$12\alpha + 79$	12	?	-2548	?
$91\alpha + 34$	24	?	-16468	?
$95\alpha + 20$	64	?	-40548	?
$97\alpha + 12$	48	?	-35475	?
$97\alpha + 8$	48	?	-35620	?
$93\alpha + 8$	24	?	-23643	?
$77\alpha + 16$	16	?	-2340	?
$21\alpha + 48$	30	?	-35179	?
$31\alpha + 59$	48	?	-29355	?
$82\alpha + 39$	24	?	-18603	?
$64\alpha + 38$	36	?	-40075	?
$92\alpha + 74$	32	?	-30195	?
$38\alpha + 18$	16	?	-2340	?
$69\alpha + 25$	40	?	-31588	?
$99\alpha + 64$	32	?	-30195	?
$56\alpha + 4$	32	?	-30195	?
$26\alpha + 90$	12	?	-2548	?
$93\alpha + 49$	48	?	-36708	?
$17\alpha + 16$	32	?	-13908	?
$84\alpha + 67$	4	?	-180	?
$100\alpha + 34$	56	?	-40788	?
$30\alpha + 2$	16	?	-2244	?
$21\alpha + 41$	2	?	-52	?
$24\alpha + 59$	24	?	-26643	?
$67\alpha + 94$	64	?	-37204	?
$88\alpha + 99$	2	?	-88	?
$47\alpha + 26$	48	?	-24420	?
$12\alpha + 7$	16	?	-2520	?
$55\alpha + 77$	24	?	-17395	?
$95\alpha + 92$	8	?	-987	?
$68\alpha + 12$	12	?	-756	?
$82\alpha + 66$	28	?	-4532	?
$91\alpha + 38$	16	?	-6948	?
$99\alpha + 20$	24	?	-18603	?
$52\alpha + 77$	80	?	-40404	?

Table 1: Table of class numbers of $\mathrm{End}(E)$ for Elliptic Curves $E/\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$. Note that the j-values are not uniformly chosen, in particular, j-values that lead to a conductor $[\mathcal{O}_{\mathcal{K}} : \mathbb{Z}[\pi]]$ with "big" prime power divisors have been ignored, as the current implementation of computing the endomorphism ring would take ages for them.

5 Example - j-invariant $61\alpha + 16$

Let E be an Elliptic Curve defined over $\mathbb{F}_{101^2}$ with j-invariant $61\alpha + 16$. Then the q-th power Frobenius π has minimal polynomial

$$X^2 - 190X + 10201$$

Furthermore, we find that $\mathrm{End}(E) = \mathcal{O}_{\mathcal{K}}$ for $\mathcal{K} = \mathbb{Q}(\sqrt{-6})$. So

$$\pi = \frac{190 + 28\sqrt{-6}}{2} = 95 + 14\sqrt{-6}$$

or

$$\sqrt{-6} = \frac{\pi - 95}{14}$$

Note that $\pi - 95$ has norm $2^3 \cdot 3 \cdot 7^2$. The class group of $\mathcal{O}$ has order 2, and a generator is e.g. the coset of $(2, \sqrt{-6})$. Hence, to find $E[(2, \sqrt{-6})]$ we need to find

$$\ker\left(\frac{\pi - 95}{14}\right) \cap E[2] = 14 \ker(\pi - 95) \cap E[2] = 14\big(\ker(\pi - 95) \cap E[28]\big)$$

Now choose a $\mathbb{Z}/4\mathbb{Z}$-basis P_1, P_2 of $E[4]$ and a $\mathbb{Z}/7\mathbb{Z}$-basis Q_1, Q_2 of $E[7]$. Have that w.r.t. these basis, π is given by the matrices

$$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

Since $95 \equiv 3 \mod 4$ and $95 \equiv 4 \mod 7$, we see that $\ker(\pi - 95) \cap E[28]$ projects to

$$\langle P_1, 2P_2 \rangle \subseteq E[4] \quad \text{and} \quad E[7] \subseteq E[7]$$

and thus is $E[14] + \langle P_1 \rangle$. This implies that $E[(2, \sqrt{-6})] = \langle 2P_1 \rangle$. Note that we picked

$$\begin{aligned} P_1 &= (59 + 7\alpha,\ 48 + 75\alpha + (73 + 3\alpha)t), \\ P_2 &= (7 + 17\alpha + 100t,\ 71 + 72\alpha + (31 + 88\alpha)t) \end{aligned}$$

before, where t has minimal polynomial $(24+51\alpha)+(94+84\alpha)T+T^2$. Hence, $\overline{(2, \sqrt{-6})}.E$ is the (isomorphism class of the) image of the 2-isogeny $\phi : E \to E/\langle 2P_1 \rangle$, which is

$$j(E/\langle 2P_1 \rangle) = 40\alpha + 58 = (61\alpha + 16)^{101}$$

The 2-isogeny vulcano containing $61\alpha + 16$ is shown in

To find the whole kernel, pick a $\mathbb{Z}/8\mathbb{Z}$-basis P_1, P_2 of $E[8]$, a $\mathbb{Z}/3\mathbb{Z}$-basis Q_1, Q_2 of $E[3]$ and a $\mathbb{Z}/49\mathbb{Z}$-basis R_1, R_2 of $E[49]$. Find then that modulo 8, 3 resp. 49, π is given by the matrix

$$\begin{pmatrix} 3 & 6 \\ 4 & 3 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 32 & 14 \\ 0 & 11 \end{pmatrix}$$

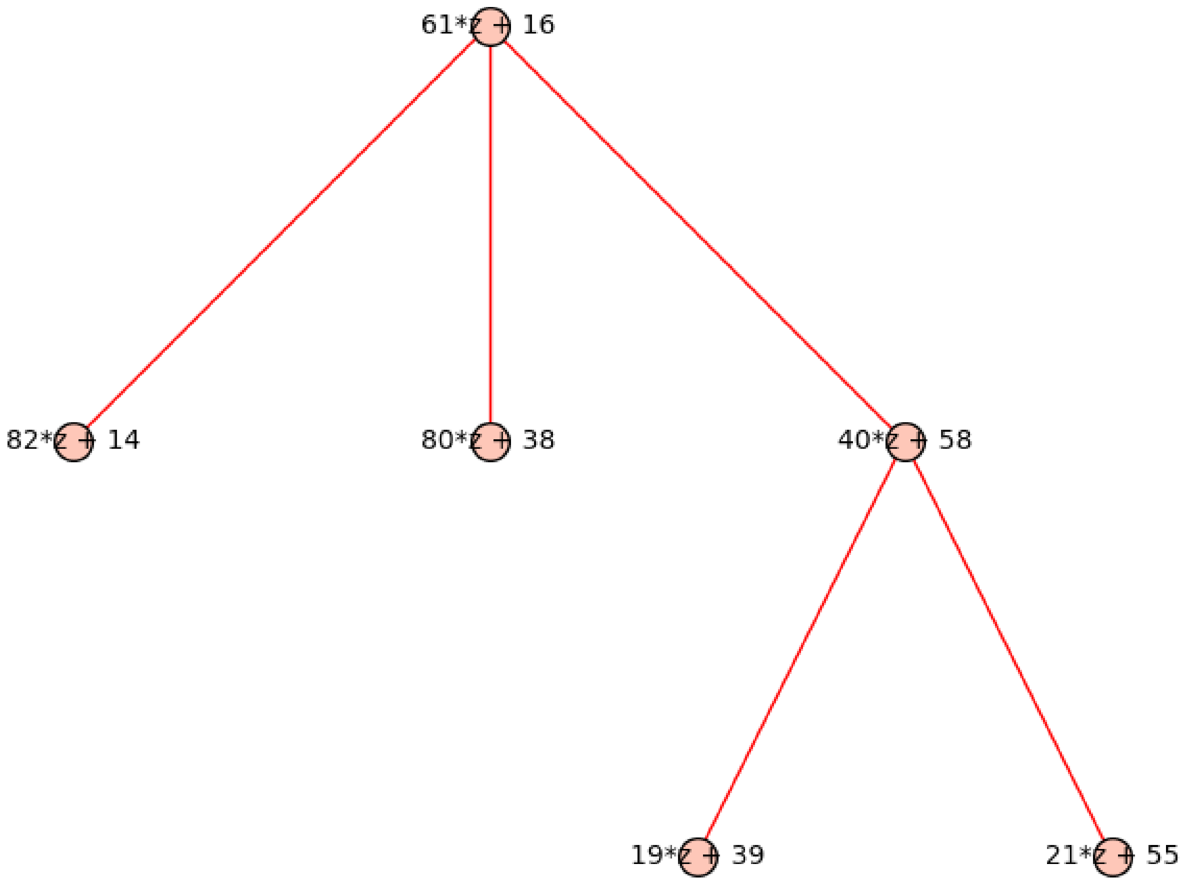

Figure 4: The 2-isogeny vulcano containing $61\alpha+16$. Note that 2 is ramified in $\mathrm{End}(E)\otimes\mathbb{Q}$, thus the crater only has size 2.

6 Example - The ordinary endomorphism ring

The information in this section is all known material - I just wanted to understand properly how one can compute the endomorphism ring, and what problems occur.

Consider the finite field

$$\mathbb{F}_q = \mathbb{F}_{37^2} = \mathbb{F}_{37} + \alpha\mathbb{F}_{37}$$

where $\alpha^2 + 33\alpha + 2 = 0$. Further, consider the Elliptic Curve $E/\mathbb{F}_q$ with j-invariant 3α, given by

$$E : y^2 = x^3 + (15\alpha + 17)x + (5\alpha + 3)$$

Then we find that the q-th power Frobnenius endomorphism π satisfies the minimal equation

$$\pi^2 + 47\pi + 1369$$

and in particular, its trace is -47. Hence, the number field $\mathcal{K} := \mathcal{O} \otimes \mathbb{Q}$ where $\mathcal{O} = \mathrm{End}(E)$ contains $\sqrt{47^2 - 4 \cdot 1369} = \sqrt{-3^3 \cdot 11^2}$. We observe that $\mathcal{K} = \mathbb{Q}(\sqrt{-3})$ and has discriminant -3. Furthermore the ring of integers is $\mathcal{O}_{\mathcal{K}} = \mathbb{Z}[\frac{1}{2}(1 + \sqrt{-3})]$.

Knowing the number field, we want to find the endomorphism ring. First, observe that the Frobenius order $\mathbb{Z}[\pi]$ has conductor 33. Now consider the endomorphism

$$\phi := 2\pi + 47$$

The advantage is that we can evaluate ϕ on points of E, but evaluating $\pi + 47/2$ is not so easy. Clearly $[\mathbb{Z}[\pi] : \mathbb{Z}[\phi]] = 2$ and so $\mathbb{Z}[\phi]$ has conductor 66.

Torsion points

In order to find whether $\phi/n \in \mathcal{O}$, we factor $66 = 2 \cdot 3 \cdot 11$ and compute the corresponding torsion groups. This turns out to be quite difficult.

Assume $\mathbb{F}_{37^{12}} = \mathbb{F}_{37}[\beta]$ with

$$\mathrm{MiPo}_{\mathbb{F}_{37}}(\beta) = x^{12} + 4x^7 + 31x^6 + 10x^5 + 23x^4 + 18x^2 + 33x + 2$$

Then $E[2]$ is generated by

$$P_1 = (11\beta^{11} + 19\beta^{10} + \beta^9 + 27\beta^8 + 8\beta^7 + 16\beta^6 + 17\beta^5 + 32\beta^4 + 12\beta^3 + 14\beta^2 + 24\beta + 32 : 0 : 1)$$
$$Q_1 = (15\beta^{11} + 7\beta^{10} + 33\beta^9 + 11\beta^8 + 6\beta^7 + 12\beta^6 + 26\beta^5 + 7\beta^4 + 33\beta^3 + 25\beta^2 + 8\beta + 19 : 0 : 1)$$

Further $E[3]$ is generated by

$$\begin{aligned} P_2 = (&19\beta^{11} + 34\beta^{10} + 3\beta^9 + 29\beta^8 + 7\beta^7 + 3\beta^6 + 18\beta^5 + 21\beta^4 + 23\beta^3 + 30\beta^2 + 23\beta + 25 \\ &: 6\beta^{11} + 25\beta^{10} + 4\beta^9 + 13\beta^8 + 10\beta^7 + 23\beta^6 + 20\beta^5 + 30\beta^4 + 24\beta^3 + 6\beta^2 + 17\beta + 5 : 1) \\ Q_2 = (&31\beta^{11} + 24\beta^{10} + 35\beta^9 + 32\beta^8 + 2\beta^7 + 10\beta^6 + 23\beta^5 + 35\beta^4 + 22\beta^3 + 13\beta^2 + 12\beta + 12 \\ &: 18\beta^{11} + 2\beta^{10} + 32\beta^9 + 26\beta^8 + 17\beta^7 + 5\beta^6 + 19\beta^5 + 31\beta^4 + 31\beta^3 + \beta^2 + 22\beta + 1 : 1) \end{aligned}$$

For $E[11]$ we must even go to the extension degree 110. So assume $\mathbb{F}_{37^{220}} = \mathbb{F}_{37}[\gamma]$. Then $E[11]$ is generated by P_3 and Q_3. For the values of $\mathrm{MiPo}_{\mathbb{F}_{37}}(\gamma)$ and P_3, Q_3 see Section 8.

Now we can compute $\phi(P_1), \phi(Q_1), \phi(P_2), \phi(Q_2), \phi(P_3), \phi(Q_3)$ and see that none of them is zero. Since $\deg(\phi) = [\mathcal{O} : \mathbb{Z}[\phi]] \mid [\mathcal{O}_{\mathcal{K}} : \mathbb{Z}[\phi]] = 2 \cdot 3 \cdot 11$, we see that the kernel of ϕ is trivial. Thus no ϕ/n is contained in $\mathcal{O}$. Therefore we see that

$$\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z}[\phi]$$

The inclusion $\supseteq$ is clear, and for the other direction, note that $\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z} + t\sqrt{D}\mathbb{Z}$ and $\mathbb{Z}[\phi] = \mathbb{Z} + s\sqrt{D}\mathbb{Z}$. Since $\mathbb{Z}[\phi] \subseteq \mathcal{O} \cap \mathbb{Z}[\phi]$ find thus $t \mid s$. Now observe that by choice of ϕ, have $\phi^2 \in \mathbb{Z}$ and so $\phi = s\sqrt{D}$. However, $\phi/\frac{s}{t} = t\sqrt{D} \in \mathcal{O}$. By the above, it follows that $\frac{s}{t} = 1$, i.e. $s = t$.

The index $[\mathcal{O} : \mathbb{Z}[\phi]]$

From the consideration of the torsion points, we see that $\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z}[\phi]$. However, since $[\mathcal{O}_{\mathcal{K}} : \mathbb{Z}[\sqrt{D}]] \leq 2$, we deduce that $[\mathcal{O} : \mathbb{Z}[\phi]] \leq 2$ and so

$$\mathcal{O} = \mathbb{Z}[\pi]$$

7 More ideas on computing the endomorphism ring

Let $K = \mathbb{Z}[\pi] \otimes \mathbb{Q}$ be the number field, and suppose it has discriminant D. Then 1 and $\alpha := \frac{D+\sqrt{D}}{2}$ form an integral basis of $\mathcal{O}_K$. Now note that

$$\beta := (a + \alpha)(b + c\alpha) = ab - c\underbrace{\frac{D^2 - D}{4}}_{=:\delta} + (ac + b + cD)\alpha$$

If we want this to be a generator of $\mathbb{Z}[\pi]$, we set $ac + b + cD = f$ where $f = [\mathcal{O}_K : \mathbb{Z}[\pi]]$. In other words, have $b = f - c(a + D)$. Thus find

$$\beta = af - a^2c - acD - c\delta + f\alpha$$

To minimize $N(\beta)$, it thus suffices to minimize

$$|af - a^2c - acD - c\delta|, \quad a, c \in \mathbb{Z}$$

Note that once we found a suitable β, have that $\mathrm{End}(E) = \mathbb{Z} + [\mathcal{O}_K : \mathrm{End}(E)]\beta\mathbb{Z}$, so we just have to find the maximal n such that $n \mid \beta$ in $\mathrm{End}(E)$. Note that $\beta = \pi + m$ for some integer m, and so we can do this by computing the kernel of that endomorphism. This is easier the smaller the prime powers dividing $N(\beta)$ are, so we want $N(\beta)$ to be as small as possible.

8 P_3 and Q_3

The minimal polynomial of γ is

```
x^220 + 31*x^219 + 13*x^218 + 21*x^217 + 23*x^216 + 9*x^215
+ 2*x^214 + 35*x^212 + 10*x^211 + 29*x^210 + 25*x^209 + 20*x^208
+ 17*x^207 + 30*x^206 + 5*x^205 + 15*x^204 + 11*x^203 + 10*x^202
+ 11*x^201 + 32*x^200 + 5*x^199 + 28*x^198 + 7*x^197 + 13*x^196
+ 10*x^195 + 32*x^194 + 17*x^193 + 19*x^192 + 36*x^191
+ 17*x^190 + 31*x^189 + 14*x^188 + 6*x^187 + 30*x^186 + 8*x^185
+ 22*x^184 + 2*x^183 + 9*x^182 + 11*x^181 + 6*x^180 + 23*x^179
+ 14*x^178 + 36*x^177 + 16*x^176 + 34*x^175 + 14*x^174
+ 33*x^173 + 14*x^172 + 7*x^171 + 36*x^170 + 18*x^169 + 27*x^168
+ 5*x^167 + 31*x^166 + 6*x^165 + 15*x^164 + 14*x^163 + 17*x^162
+ 7*x^161 + 16*x^160 + 6*x^159 + 29*x^158 + 11*x^157 + 8*x^156
+ 15*x^155 + 20*x^154 + 17*x^153 + 7*x^152 + 8*x^151 + 6*x^150
+ 12*x^149 + 36*x^148 + 7*x^147 + 3*x^146 + 25*x^145 + 13*x^144
+ 6*x^143 + 17*x^142 + 22*x^141 + 9*x^140 + 18*x^139 + 36*x^138
+ x^137 + 6*x^136 + 36*x^135 + 33*x^134 + 32*x^133 + 35*x^132
+ 33*x^131 + 7*x^130 + 3*x^129 + 7*x^128 + 20*x^127 + 31*x^126
+ 26*x^125 + 6*x^124 + 9*x^123 + 10*x^122 + 25*x^121 + 33*x^120
+ 33*x^119 + 30*x^118 + 34*x^117 + 22*x^116 + 8*x^115 + 10*x^114
+ 36*x^113 + 26*x^112 + 8*x^111 + 33*x^110 + 30*x^109 + 11*x^108
+ 14*x^107 + 22*x^106 + 26*x^105 + 11*x^104 + 35*x^103
+ 34*x^102 + 33*x^101 + 27*x^100 + 14*x^99 + 31*x^98 + 24*x^97
+ x^96 + 6*x^95 + 36*x^93 + 32*x^92 + 18*x^91 + 36*x^90 + 3*x^89
+ 22*x^88 + 36*x^87 + 6*x^86 + 20*x^85 + 25*x^84 + 8*x^82
+ 34*x^81 + 7*x^80 + 25*x^79 + 21*x^78 + 17*x^77 + 29*x^76
+ 5*x^75 + 19*x^74 + 19*x^73 + 8*x^72 + 8*x^71 + 26*x^70
+ 7*x^69 + 27*x^68 + 10*x^67 + 31*x^66 + 4*x^65 + 29*x^64
+ 36*x^62 + 3*x^61 + 27*x^60 + 13*x^59 + 23*x^58 + 33*x^57
+ 14*x^56 + 19*x^55 + 12*x^54 + 20*x^53 + 32*x^52 + 18*x^51
+ 20*x^49 + 20*x^48 + x^47 + 17*x^46 + 16*x^45 + 4*x^44
+ 12*x^43 + 7*x^42 + 34*x^41 + 9*x^40 + 16*x^39 + 10*x^38
+ 25*x^37 + 10*x^36 + 10*x^35 + 28*x^34 + 33*x^33 + 22*x^32
+ 24*x^31 + 33*x^30 + 6*x^29 + 8*x^28 + 8*x^27 + 16*x^26
+ 31*x^25 + 7*x^24 + 26*x^23 + 36*x^22 + 29*x^21 + 36*x^20
+ 7*x^19 + x^18 + 26*x^17 + 18*x^16 + 23*x^15 + 10*x^14
+ 4*x^13 + x^12 + 24*x^11 + 25*x^10 + 34*x^9 + 33*x^8
+ 33*x^7 + 8*x^6 + 12*x^5 + x^4 + 15*x^3 + 27*x^2 + 9*x + 2
```

P_3 is given by

```
(23*z220^219 + 5*z220^218 + 26*z220^217 + 27*z220^216
+ 26*z220^215 + 12*z220^214 + 11*z220^213 + 10*z220^212
```

+ 29*z220^211 + 9*z220^210 + 16*z220^209 + 24*z220^208
+ 18*z220^207 + 11*z220^206 + 11*z220^205 + 6*z220^204
+ 24*z220^203 + 3*z220^202 + 34*z220^201 + 18*z220^200
+ 17*z220^199 + 9*z220^198 + 26*z220^197 + 2*z220^196
+ 31*z220^195 + 7*z220^194 + 15*z220^193 + 11*z220^192
+ 15*z220^191 + 28*z220^190 + 13*z220^189 + 6*z220^188
+ 7*z220^187 + 28*z220^186 + 9*z220^185 + 9*z220^184
+ 7*z220^183 + 27*z220^182 + 36*z220^181 + 35*z220^180
+ 30*z220^179 + 32*z220^178 + 16*z220^177 + 15*z220^176
+ 16*z220^175 + 9*z220^174 + 21*z220^173 + 6*z220^172
+ 15*z220^171 + 3*z220^170 + 25*z220^169 + 23*z220^168
+ z220^167 + 8*z220^166 + 34*z220^165 + 14*z220^164
+ 12*z220^163 + 20*z220^162 + 4*z220^161 + 9*z220^160
+ z220^159 + 25*z220^158 + 16*z220^157 + z220^156
+ 21*z220^155 + 10*z220^154 + 7*z220^153 + 13*z220^152
+ 32*z220^151 + 31*z220^150 + 17*z220^148 + 24*z220^147
+ 26*z220^146 + 28*z220^145 + 27*z220^144 + 4*z220^143
+ 5*z220^142 + 14*z220^141 + 26*z220^140 + 10*z220^139
+ 14*z220^138 + 19*z220^137 + 20*z220^136 + 18*z220^135
+ 16*z220^134 + 11*z220^133 + 23*z220^132 + 35*z220^131
+ 22*z220^130 + 31*z220^129 + 34*z220^128 + 17*z220^127
+ z220^126 + 15*z220^125 + 2*z220^124 + 22*z220^123
+ 27*z220^122 + 6*z220^121 + 10*z220^120 + 7*z220^119
+ 4*z220^118 + 26*z220^117 + z220^116 + 32*z220^115
+ 29*z220^114 + 32*z220^113 + 18*z220^112 + 3*z220^111
+ 28*z220^110 + 20*z220^109 + 17*z220^108 + 17*z220^107
+ 32*z220^106 + 32*z220^105 + 26*z220^104 + 24*z220^103
+ 17*z220^102 + 8*z220^101 + 3*z220^100 + 2*z220^99
+ 16*z220^98 + 29*z220^97 + 19*z220^96 + 27*z220^95
+ 4*z220^94 + 29*z220^93 + 24*z220^92 + 19*z220^91
+ 2*z220^90 + 2*z220^89 + 32*z220^88 + 23*z220^87
+ 32*z220^86 + 15*z220^85 + 24*z220^84 + 36*z220^83
+ 29*z220^82 + 18*z220^81 + 2*z220^80 + z220^79
+ 33*z220^78 + 34*z220^77 + 4*z220^76 + 11*z220^75
+ 21*z220^74 + 15*z220^73 + 10*z220^72 + 24*z220^71
+ 22*z220^70 + 22*z220^69 + 31*z220^68 + 32*z220^67
+ 28*z220^66 + z220^65 + 17*z220^64 + 13*z220^63
+ 32*z220^62 + 20*z220^61 + 32*z220^60 + 21*z220^59
+ 34*z220^58 + 11*z220^57 + 29*z220^56 + 12*z220^55
+ 22*z220^54 + 11*z220^53 + 36*z220^52 + 35*z220^51
+ 19*z220^50 + 35*z220^49 + 8*z220^48 + 16*z220^47
+ 16*z220^46 + 27*z220^45 + 32*z220^44 + 12*z220^43
+ 15*z220^42 + 6*z220^41 + 36*z220^40 + 27*z220^39

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+ 17*z220^38 + 20*z220^37 + 33*z220^36 + 34*z220^35
+ 34*z220^34 + 3*z220^33 + 12*z220^32 + 12*z220^31
+ 12*z220^30 + 5*z220^29 + 10*z220^28 + 13*z220^27
+ 36*z220^26 + 16*z220^25 + 16*z220^24 + 15*z220^23
+ 36*z220^22 + 18*z220^21 + 13*z220^20 + 26*z220^19
+ 25*z220^18 + 21*z220^17 + 35*z220^16 + 3*z220^14
+ 31*z220^13 + 8*z220^12 + 7*z220^11 + 10*z220^10
+ 10*z220^9 + 6*z220^8 + 5*z220^7 + 33*z220^6
+ 6*z220^5 + 4*z220^4 + 31*z220^3 + 27*z220^2 + 27*z220 + 14

: 8*z220^219 + 17*z220^218 + 27*z220^217 + 14*z220^216
+ 6*z220^215 + 19*z220^214 + 18*z220^213 + 6*z220^212
+ 30*z220^211 + 24*z220^210 + 33*z220^209 + 19*z220^208
+ 27*z220^207 + 16*z220^206 + 24*z220^205 + 3*z220^204
+ 4*z220^203 + 25*z220^202 + 29*z220^201 + 31*z220^200
+ 23*z220^199 + 7*z220^198 + 28*z220^197 + 4*z220^196
+ 26*z220^195 + 36*z220^194 + 18*z220^193 + 24*z220^192
+ 29*z220^191 + 25*z220^190 + 23*z220^189 + 14*z220^188
+ 33*z220^187 + 19*z220^186 + 14*z220^184 + 21*z220^183
+ 10*z220^182 + 13*z220^181 + 21*z220^180 + 24*z220^179
+ 33*z220^178 + 19*z220^177 + 7*z220^176 + 36*z220^175
+ 30*z220^174 + 34*z220^173 + 27*z220^172 + 3*z220^171
+ 34*z220^170 + 5*z220^169 + 36*z220^168 + 19*z220^167
+ 27*z220^166 + 14*z220^165 + 10*z220^164 + 2*z220^163
+ 31*z220^162 + 22*z220^161 + 7*z220^160 + 14*z220^159
+ 5*z220^158 + 3*z220^157 + 22*z220^156 + 32*z220^155
+ 21*z220^154 + 17*z220^153 + 34*z220^152 + 9*z220^151
+ 33*z220^150 + 32*z220^149 + 24*z220^148 + 16*z220^147
+ 19*z220^146 + 6*z220^145 + 26*z220^144 + 24*z220^143
+ 34*z220^141 + 25*z220^140 + 17*z220^139 + 25*z220^138
+ 19*z220^137 + 36*z220^136 + 7*z220^134 + 32*z220^133
+ 24*z220^132 + 6*z220^131 + 12*z220^130 + 30*z220^129
+ 35*z220^128 + 13*z220^127 + 29*z220^126 + 2*z220^125
+ 24*z220^124 + 36*z220^123 + 34*z220^122 + 2*z220^121
+ 33*z220^120 + 10*z220^119 + 33*z220^118 + 2*z220^117
+ 17*z220^116 + 33*z220^115 + 14*z220^114 + 22*z220^113
+ 27*z220^112 + 20*z220^111 + 23*z220^110 + 34*z220^109
+ 6*z220^108 + 33*z220^107 + 14*z220^106 + 28*z220^105
+ 29*z220^104 + 36*z220^103 + 22*z220^102 + 35*z220^101
+ 8*z220^100 + 10*z220^99 + 10*z220^98 + 16*z220^97
+ 19*z220^96 + 17*z220^95 + 21*z220^94 + 13*z220^93
+ 24*z220^92 + 36*z220^91 + 25*z220^90 + 25*z220^89
+ 22*z220^88 + 27*z220^87 + 28*z220^86 + 11*z220^85
```

```
+ 3*z220^84 + 14*z220^82 + 31*z220^81 + 7*z220^80
+ 33*z220^79 + 33*z220^78 + 2*z220^77 + 15*z220^76
+ 17*z220^75 + 32*z220^74 + 4*z220^73 + 18*z220^72
+ 10*z220^71 + 34*z220^70 + 9*z220^69 + 3*z220^68
+ 20*z220^67 + 33*z220^66 + 23*z220^65 + 5*z220^64
+ 20*z220^63 + 36*z220^62 + 29*z220^61 + 2*z220^60
+ 25*z220^59 + 14*z220^58 + 16*z220^57 + 31*z220^56
+ 22*z220^55 + 31*z220^54 + 33*z220^53 + 19*z220^52
+ 22*z220^51 + 23*z220^50 + 36*z220^49 + 11*z220^48
+ 15*z220^47 + 15*z220^46 + 35*z220^45 + 7*z220^44
+ 27*z220^43 + 28*z220^42 + 15*z220^41 + 31*z220^40
+ 12*z220^39 + 19*z220^38 + 21*z220^37 + 18*z220^36
+ 3*z220^35 + 36*z220^33 + z220^32 + 35*z220^31
+ 21*z220^30 + 2*z220^29 + 13*z220^28 + 19*z220^27
+ 6*z220^26 + 22*z220^24 + 26*z220^23 + 9*z220^22
+ 7*z220^21 + 31*z220^20 + 31*z220^19 + 9*z220^18
+ 23*z220^17 + 23*z220^16 + 6*z220^15 + 27*z220^14
+ 36*z220^13 + 4*z220^12 + 26*z220^11 + 30*z220^10
+ 9*z220^9 + 8*z220^8 + 15*z220^7 + 26*z220^6
+ 17*z220^5 + 29*z220^4 + 24*z220^3 + 8*z220^2
+ 29*z220 : 1)
```

Q_3 is given by

```
(35*z220^219 + 22*z220^218 + 36*z220^216 + 24*z220^215
+ 19*z220^214 + 32*z220^213 + 13*z220^212 + 19*z220^211
+ 3*z220^210 + 36*z220^209 + 29*z220^208 + 35*z220^206
+ 31*z220^205 + 32*z220^204 + 23*z220^203 + 21*z220^202
+ 10*z220^201 + 32*z220^200 + 32*z220^199 + 21*z220^198
+ 16*z220^197 + 23*z220^196 + 32*z220^195 + 12*z220^194
+ 9*z220^193 + 35*z220^192 + 8*z220^191 + 19*z220^190
+ 33*z220^189 + 13*z220^188 + 11*z220^187 + 35*z220^186
+ 25*z220^185 + 28*z220^184 + 5*z220^183 + 7*z220^182
+ 24*z220^181 + 35*z220^180 + 33*z220^179 + 18*z220^178
+ 5*z220^177 + 31*z220^176 + 18*z220^175 + 30*z220^174
+ 27*z220^173 + 3*z220^172 + 8*z220^171 + 24*z220^170
+ 14*z220^169 + 2*z220^168 + 16*z220^167 + 14*z220^166
+ 18*z220^165 + 22*z220^164 + 32*z220^163 + 28*z220^162
+ 7*z220^161 + 19*z220^160 + 3*z220^159 + 14*z220^158
+ 27*z220^157 + 35*z220^156 + 8*z220^155 + 25*z220^154
+ 11*z220^153 + 19*z220^152 + 21*z220^151 + 10*z220^150
+ 2*z220^149 + 4*z220^148 + 4*z220^147 + 31*z220^146
+ 26*z220^145 + 17*z220^143 + 14*z220^142 + 12*z220^141
+ 17*z220^140 + 22*z220^139 + 30*z220^138 + 30*z220^137
+ 15*z220^136 + 16*z220^135 + 25*z220^134 + 8*z220^133
```

+ 28*z220^132 + 5*z220^131 + 14*z220^130 + 26*z220^129
+ 13*z220^128 + 10*z220^127 + 13*z220^126 + 10*z220^125
+ 17*z220^124 + 33*z220^123 + 9*z220^122 + 9*z220^121
+ 10*z220^120 + 12*z220^119 + 4*z220^118 + 6*z220^117
+ 33*z220^116 + 21*z220^115 + 14*z220^114 + 33*z220^113
+ 11*z220^112 + 4*z220^111 + 3*z220^110 + 3*z220^109
+ 3*z220^108 + 3*z220^107 + 27*z220^106 + 8*z220^105
+ 25*z220^104 + 10*z220^103 + 24*z220^102 + 2*z220^101
+ 12*z220^100 + 35*z220^99 + 30*z220^98 + 14*z220^97
+ 8*z220^96 + 16*z220^95 + 24*z220^94 + 23*z220^93
+ 34*z220^91 + 3*z220^90 + 13*z220^89 + 10*z220^88
+ 20*z220^87 + 14*z220^86 + 9*z220^85 + 36*z220^84
+ 33*z220^83 + 12*z220^82 + 20*z220^81 + 5*z220^80
+ 27*z220^79 + 27*z220^78 + 9*z220^77 + 23*z220^76
+ 4*z220^75 + 26*z220^74 + 8*z220^73 + 11*z220^72
+ 25*z220^71 + 35*z220^70 + 19*z220^69 + 36*z220^68
+ 35*z220^67 + 24*z220^66 + 8*z220^65 + 32*z220^64
+ 10*z220^63 + 3*z220^62 + 18*z220^61 + 35*z220^60
+ 17*z220^59 + 30*z220^58 + 2*z220^57 + 25*z220^56
+ 7*z220^55 + 20*z220^54 + 27*z220^53 + z220^52
+ 10*z220^51 + 2*z220^50 + 18*z220^49 + 30*z220^48
+ 32*z220^47 + 20*z220^46 + 4*z220^45 + 16*z220^43
+ 16*z220^42 + 11*z220^41 + 8*z220^40 + 12*z220^39
+ 15*z220^38 + 25*z220^37 + 33*z220^36 + 4*z220^35
+ 11*z220^34 + 6*z220^33 + 7*z220^32 + 32*z220^31
+ 19*z220^30 + 19*z220^29 + 16*z220^28 + 10*z220^27
+ 7*z220^26 + 10*z220^25 + 33*z220^24 + 25*z220^23
+ 21*z220^22 + 35*z220^21 + 15*z220^20 + z220^19
+ 19*z220^18 + 16*z220^17 + 10*z220^16 + 18*z220^15
+ 17*z220^14 + 2*z220^13 + 35*z220^12 + 30*z220^11
+ 17*z220^10 + 30*z220^9 + 26*z220^8 + 9*z220^7
+ 34*z220^6 + 4*z220^5 + 12*z220^4 + 16*z220^3
+ 27*z220^2 + 12*z220 + 36

: 21*z220^219 + 24*z220^218
+ 33*z220^217 + 31*z220^216 + 29*z220^215 + 16*z220^214
+ 26*z220^213 + 7*z220^212 + 15*z220^211 + 9*z220^210
+ 19*z220^209 + 18*z220^208 + 16*z220^207 + 23*z220^206
+ 27*z220^205 + 16*z220^204 + 5*z220^203 + 10*z220^202
+ 2*z220^201 + 19*z220^200 + 19*z220^199 + 8*z220^198
+ 30*z220^197 + 9*z220^196 + 27*z220^195 + 7*z220^194
+ 20*z220^193 + 8*z220^192 + 29*z220^191 + 10*z220^190
+ 32*z220^189 + 9*z220^188 + 4*z220^187 + 31*z220^186

+ 8*z220^185 + 4*z220^184 + 8*z220^183 + 11*z220^182
+ 13*z220^181 + 5*z220^180 + 29*z220^179 + 13*z220^178
+ 20*z220^177 + 9*z220^176 + 3*z220^175 + 32*z220^174
+ 3*z220^173 + 25*z220^172 + 33*z220^171 + 36*z220^170
+ 11*z220^169 + 22*z220^168 + 18*z220^167 + 7*z220^166
+ 4*z220^165 + 9*z220^164 + 33*z220^163 + 33*z220^162
+ 18*z220^161 + 3*z220^160 + 35*z220^159 + 31*z220^158
+ 20*z220^157 + 28*z220^155 + 33*z220^154 + 30*z220^153
+ 28*z220^152 + 18*z220^151 + z220^150 + 34*z220^149
+ 16*z220^148 + 23*z220^147 + 30*z220^146 + 3*z220^144
+ 28*z220^143 + 8*z220^142 + 35*z220^140 + 11*z220^139
+ 16*z220^138 + 20*z220^137 + 31*z220^136 + 11*z220^135
+ 24*z220^134 + 29*z220^133 + 29*z220^132 + 8*z220^131
+ 25*z220^130 + 11*z220^129 + 35*z220^128 + 36*z220^127
+ 33*z220^126 + 18*z220^125 + 8*z220^124 + 9*z220^123
+ 31*z220^122 + 29*z220^121 + 7*z220^120 + 4*z220^119
+ 3*z220^118 + 13*z220^117 + 35*z220^116 + 17*z220^115
+ 6*z220^114 + 3*z220^113 + 13*z220^112 + 5*z220^111
+ 31*z220^110 + 32*z220^109 + 17*z220^108 + 28*z220^107
+ 21*z220^106 + 14*z220^105 + 25*z220^104 + 17*z220^103
+ 33*z220^102 + 19*z220^101 + 4*z220^100 + 2*z220^99
+ 7*z220^98 + 34*z220^97 + 15*z220^96 + 7*z220^95
+ 34*z220^94 + 22*z220^93 + 22*z220^92 + 11*z220^91
+ 33*z220^90 + 32*z220^89 + 19*z220^88 + 21*z220^87
+ 23*z220^86 + 34*z220^85 + 35*z220^84 + 23*z220^83
+ 27*z220^82 + 25*z220^81 + 26*z220^80 + 2*z220^79
+ 33*z220^78 + 32*z220^77 + 8*z220^76 + 32*z220^75
+ 15*z220^74 + 17*z220^73 + 31*z220^72 + 7*z220^71
+ 8*z220^70 + 8*z220^69 + 22*z220^68 + 7*z220^67
+ 14*z220^66 + 15*z220^65 + 26*z220^64 + 26*z220^63
+ 35*z220^62 + 19*z220^61 + 18*z220^60 + 22*z220^59
+ 25*z220^57 + 4*z220^56 + 5*z220^55 + 4*z220^54
+ 20*z220^53 + 32*z220^52 + 17*z220^51 + 14*z220^50
+ 31*z220^49 + 9*z220^48 + 30*z220^47 + 20*z220^46
+ 7*z220^45 + 16*z220^43 + 23*z220^42 + 12*z220^41
+ 21*z220^40 + 14*z220^39 + 8*z220^38 + 14*z220^37
+ 35*z220^36 + 14*z220^35 + 22*z220^34 + 8*z220^33
+ z220^32 + 24*z220^31 + 21*z220^30 + 33*z220^29
+ 21*z220^28 + 22*z220^26 + 33*z220^25 + 13*z220^24
+ 13*z220^23 + 5*z220^22 + 35*z220^21 + 3*z220^20
+ 31*z220^19 + 13*z220^18 + 33*z220^17 + 30*z220^16
+ 16*z220^15 + 30*z220^14 + 16*z220^13 + 11*z220^12
+ 35*z220^11 + 22*z220^10 + 11*z220^9 + 8*z220^8

```
+ z220^7 + 25*z220^6 + 8*z220^5 + 27*z220^4 + z220^3
+ 29*z220^2 + 34*z220 + 29 : 1)
```