Some Notes about the things I encountered

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Notation

If E is an Elliptic Curve defined over a finite field of characteristic p, we write $E^{(p)}$ for the curve defined by the equations of E after replacing all coefficients by their p-th power. Similarly, for an isogeny $\phi : E \to E'$, write $\phi^{(p)} : E^{(p)} \to E'^{(p)}$ for the isogeny defined by the polynomials of ϕ after replacing all coefficients by their p-th power. Furthermore, for a point $P = (x : y : z) \in \mathbb{P}^2$ write $P^{(p)} := (x^p : y^p : z^p)$. Finally, for a set of points or endomorphisms S write $S^{(p)} := \{s^{(p)} \mid s \in S\}$. Note that

$$.^{(p)} : \mathbf{Ell} \to \mathbf{Ell}, \quad E \mapsto E^{(p)}$$
$$\mathrm{Hom}_{\mathbf{Ell}}(E, E') \ni \phi \to \phi^{(p)}$$

is a covariant endofunctor on the category **Ell** of Elliptic Curves defined over $\bar{\mathbb{F}}_p$ and their isogenies.

Sometimes, we abuse terminology and speak of Elliptic Curves when we mean isomorphism classes of Elliptic Curves.

Many examples will be over $\mathbb{F}_{101^2}$. Let $p = 101$ and $q = p^2$. We usually use the generator $\alpha \in \mathbb{F}_q$ with minimal polynomial $x^2 + 97x + 2$.

1 Current status

The main topic of this project is to further analyze and/or modify the method described in [Boo+22, Chapter 3]. As presented there, there are still some open questions and obstacles to making the method practical. Here we list them and mention in what way we addressed those in this project so far.

Can we find a root of $\gcd(\Phi_n(x, x^p), \Phi_m(x, x^p))$ if n resp. m are large?

We did not focus on this question and do not contribute new ideas, except for generalizations/modifications that might circumvent the problem completely.

What proportion of the roots of $\gcd(\Phi_n(x, x^p), \Phi_m(x, x^p))$ give supersingular curves, or curves over $\mathbb{F}_{p^2}$?

Until now, we focus mainly on the case $n = l_1^e$ and $m = l_2^e$ for small primes l_1, l_2 and $e = O(\log(p))$. This case seems a natural candidate, for multiple reasons.

- Since allowing isogenies of multiple fixed degrees makes the ordinary isogeny graph closed to an expander, taking n resp. m to have many prime factors might lead to more found ordinary curves, thus decreasing success.
- If n, m have large prime factors, it seems like it is extremely hard to work with Φ_n resp. Φ_m. On the other hand if n (resp. m) is smooth, the variety defined by $\Phi_n(x, y)$ is the projection of the variety defined by $\Phi_{n_1}(x_1, x_2), ..., \Phi_{n_r}(x_r, y)$ where $n = \prod n_i$. It seems at least possible to efficiently work with the latter, if the n_i are polynomially large.
- Since the supersingular l-isogeny graph is an expander, all supersingular j-invariants are roots of $\Phi_{l_1^e}(x, x^p)$ resp. $\Phi_{l_2^e}(x, x^p)$, thus answering one half of the question positively.

In this case, by the third point, the question is just how many ordinary curves satisfying these equations there are. Using the structure of the isogeny vulcanos and the class group action, we could already partly answer this question. The main goal which we think is reasonable is to prove the following theorem.

Theorem 1.1. *Let l_1, l_2 be small primes of size $O(\log(p))$ and $e = \Theta(\log(p))$. Then the number of roots of $\gcd(\Phi_{l_1^e}(x, x^p), \Phi_{l_2^e}(x, x^p))$ that yield ordinary curves is $\mathcal{O}(p)$.*

In particular, the fraction of supersingular roots is lower bounded by a positive constant.

Generalizations and modifications

We have some ideas how to modify the original method to prevent some of the difficulties mentioned above. However, all our current modifications have different, similarly serious problems and are not practical.

2 The cases I, II and III

For the analysis of these cases, we consider an ordinary Elliptic Curve $E/\mathbb{F}_{p^2}$, a prime l and the curve E_0, defined as the curve on the crater of the l-isogeny vulcano that is the root of the subtree containing E. In other words, E_0 is the curve on the crater that is closest to E, and there is a path of ascending l-isogenies $E \to ... \to E_0$. Finally, if E is already on the crater, then clearly $E = E_0$.

2.1 Case I

We say that we are in *Case I*, if $E_0 = E_0^{(p)}$, i.e. E_0 is defined over $\mathbb{F}_p$.

Finding examples of case I is trivial - just take a curve E with $j(E) \in \mathbb{F}_p$. Then clearly $E^{(p)} = E$ and so also $E_0^{(p)} = E_0$ (since $\cdot^{(p)}$ maps the path $E \to E_0$ to $E = E^{(p)} \to E_0^{(p)}$).

Furthermore, it is easy to see that there are a lot of curve E such that the associated E_0 is defined over $\mathbb{F}_p$ (and we are again in case I).

2.2 Case II

We say that we are in *Case II*, if E_0 and $E_0^{(p)}$ are in the same l-isogeny vulcano, i.e. are connected by an l-isogeny path.

Here I was not quite sure if it even occurs. As it turns out, it does. Consider E with $j(E) = 17\alpha + 45$. Then $[\mathcal{O}_\mathcal{K} : \mathbb{Z}[\pi]] = 2^3$ so E lies on the crater of the 3-isogeny graph. However there is a 3-isogeny $E \to E^{(p)}$ since $j(E^{(p)}) = j(E)^p = 84\alpha + 12$. In fact, in this case, the crater consists only of E and $E^{(p)}$. For a more interesting example, see Figure 1.

Further, when we consider the path $E = E_0 \to ... \to E_n = E^{(p)}$ on the crater, there are more or less two possibilities for the $\cdot^{(p)}$ conjugate path[1].

- It could be that the conjugate of $E_i \to E_{i+1}$ is the dual of $E_{n-i-1} \to E_{n-i}$, hence we just go the path $E \to ... \to E^{(p)}$ backwards.
- It could be that the conjugate of $E_i \to E_{i+1}$ is $E_{n+i} \to E_{n+i+1}$, where

$$E_0, ..., E_n, E_{n+1}, ..., E_{n+m} = E_0$$

  is the cycle along the whole crater.

However, as we will see, the first case is impossible.

Note that we have

Proposition 2.1. *Let* $[\mathfrak{b}] \in \mathrm{Cl}(\mathcal{O})$ *where* $\mathcal{O} = \mathrm{End}(E)$ *for an ordinary Elliptic Curve* $E/\mathbb{F}_{p^2}$ *such that* $[\mathfrak{b}].E = E^{(p)}$. *Then* $[\mathfrak{b}]^2 = [(1)]$.

Proof. I think there is some mistake with my definition of the class group action, see also the next paragraph. With the current (probably wrong) definition, the following works. Otherwise, I suppose that anyway we have $[\mathfrak{b}] = [(p, \pi)]$ and then the claim follows by Lemma 3.5.

We recall the definition of the class group action in the case $[\mathfrak{b}].E^{(p)}$. For an ideal $\mathfrak{b}' \leq \mathrm{End}(E^{(p)})$, have by definition

$$[\mathfrak{b}'].E^{(p)} = E^{(p)}/E^{(p)}[\mathfrak{b}'] = E^{(p)}/\bigcap_{\beta \in \mathfrak{b}'} \ker(\beta)$$

[1] Remember that $\cdot^{(p)}$ is functorial, hence we can also apply to isogenies $E_i \to E_{i+1}$

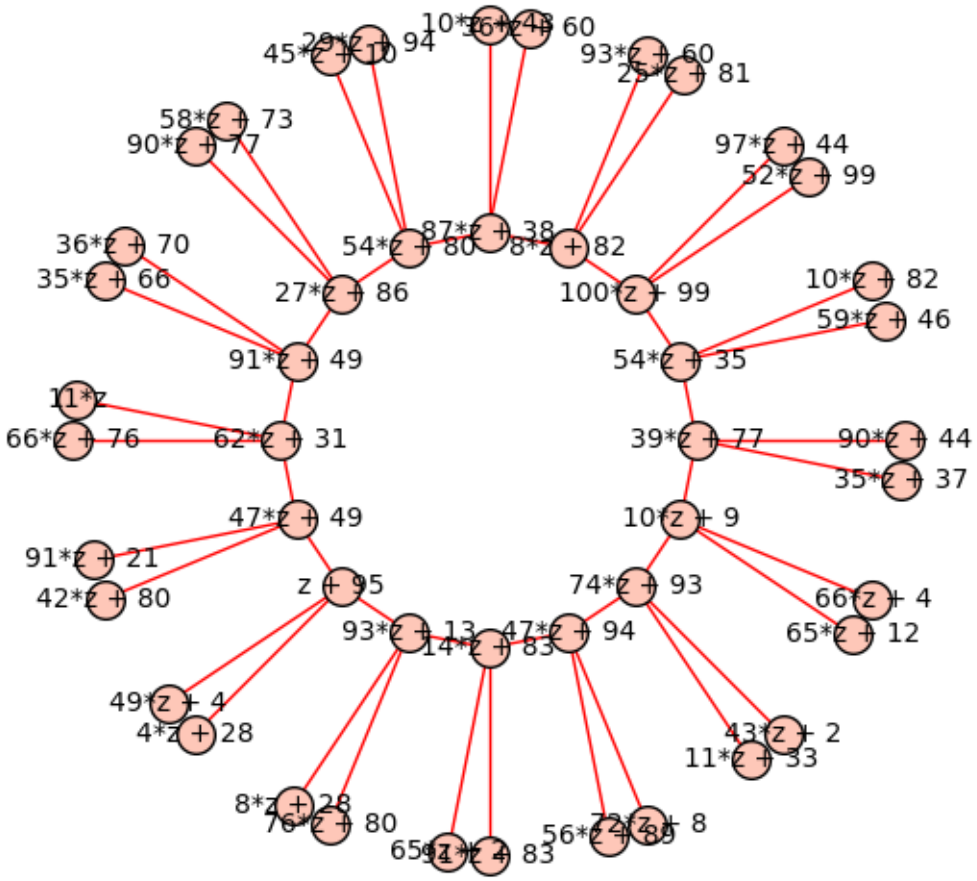


Figure 1: A 3-isogeny vulcano over $\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$ that satisfies case II (in the plot have $z = \alpha$). Note that e.g. $(39\alpha + 77)^{101} = 62\alpha + 31$.

However, $\mathfrak{b}$ is an ideal in $\mathrm{End}(E)$, which is only isomorphic to $\mathrm{End}(E^{(p)})$. Since $\mathrm{End}^0(E)$ is a quadratic imaginary number field, it has one nontrivial field automorphism, and thus the isomorphism $\mathrm{End}(E) \cong \mathrm{End}(E^{(p)})$ is not unique. But there is a unique canonical isomorphism, i.e. an isomorphism that is induced by an (equivalently any) isogeny $\phi : E \to E^{(p)}$ as

$$\Phi_* : \mathrm{End}(E) \to \mathrm{End}(E^{(p)}), \quad \alpha \mapsto \frac{1}{\deg(\phi)}\, \phi \circ \alpha \circ \hat{\phi}$$

This is the isomorphism we use, i.e. we say

$$E^{(p)}[\mathfrak{b}] = E^{(p)}[\Phi_*(\mathfrak{b})] \quad \text{and} \quad [\mathfrak{b}].E^{(p)} = [\Phi_*(\mathfrak{b})].E^{(p)} = E^{(p)}/E^{(p)}[\mathfrak{b}]$$

Now let $\phi : E \to E/E[\mathfrak{b}] = E^{(p)}$ be a separable isogeny with kernel $E[\mathfrak{b}]$ (by choosing the representative $\mathfrak{b}$ of $[\mathfrak{b}] \in \mathrm{Cl}(\mathcal{O})$ correspondingly, we can assume that). We have

$$\ker(\phi^{(p)}) = E[\mathfrak{b}]^{(p)} = \bigcap_{\beta \in \mathfrak{b}} \ker(\beta)^{(p)} = \bigcap_{\beta \in \mathfrak{b}} \ker(\beta^{(p)}) = \bigcap_{\beta \in \mathfrak{b}^{(p)}} \ker(\beta)$$

Now note that the Frobenius isogeny $\pi : E \to E^{(p)},\ P \mapsto P^{(p)}$ induces the canonical isomorphism $\mathrm{End}(E) \to \mathrm{End}(E^{(p)})$ and so the image of $\mathfrak{b}$ under that isomorphism is $\mathfrak{b}' = \mathfrak{b}^{(p)} \leq \mathrm{End}(E^{(p)})$. Thus

$$\bigcap_{\beta \in \mathfrak{b}^{(p)}} \ker(\beta) = \bigcap_{\beta \in \mathfrak{b}'} \ker(\beta) = E^{(p)}[\mathfrak{b}'] = E^{(p)}[\mathfrak{b}]$$

So by the uniqueness of the image curve for an isogeny with fixed kernel yields that $E = \mathrm{im}(\phi^{(p)}) = [\mathfrak{b}].E^{(p)}$. Thus $[\mathfrak{b}]^2.E = [\mathfrak{b}].E^{(p)} = E$ and since the class group action is free, we see that $[\mathfrak{b}]^2 = [(1)]$. □

From this we get the

Corollary 2.2. *Assume that $E = E_0 \to E_1 \to ... \to E_n = E$ is the cycle once around the crater (and $j(E) \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$). If $E^{(p)} = E_i$ then n is even and $i = n/2$, i.e. $E^{(p)}$ is on the other side of the crater[2].*

Proof. If l does not split in $\mathcal{O}_\mathcal{K}$, then the crater has at most two elements and this is trivial. So assume $(l) = \mathfrak{l}_1\mathfrak{l}_2$. It is known that then the action of $[\mathfrak{l}_1]$ resp. $[\mathfrak{l}_2]$ corresponds to walking around the crater in one direction resp. the other. So wlog $[\mathfrak{l}_1].E_i = E_{i+1}$.

Now assume that $E^{(p)} = E_i$, so $[\mathfrak{b}].E = E_i = [\mathfrak{l}_1]^i.E$. Since the action is free, it follows that $[\mathfrak{b}] = [\mathfrak{l}_1]^i$ By the previous theorem, we have now $[\mathfrak{l}_1]^{2i} = [\mathfrak{b}]^2 = [(1)]$ and so $[\mathfrak{l}_1]^{2i}.E = E_{2i} = E$. Thus $i = n/2$ and the claim follows. □

In particular, the path between E and $E^{(p)}$ is likely to have length $\omega(\log(p))$, since the crater is usually large. This is displayed e.g. Figure 1.

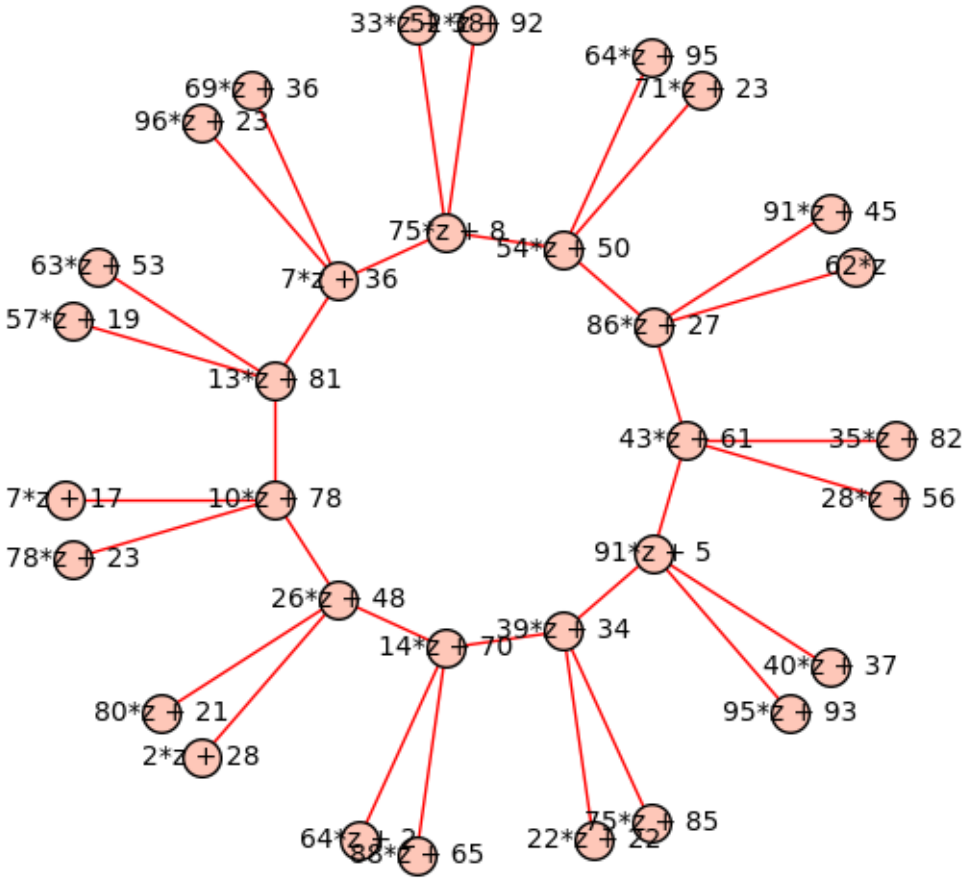


Figure 2: A 3-isogeny vulcano over $\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$ that satisfies case III (in the plot have $z = \alpha$).

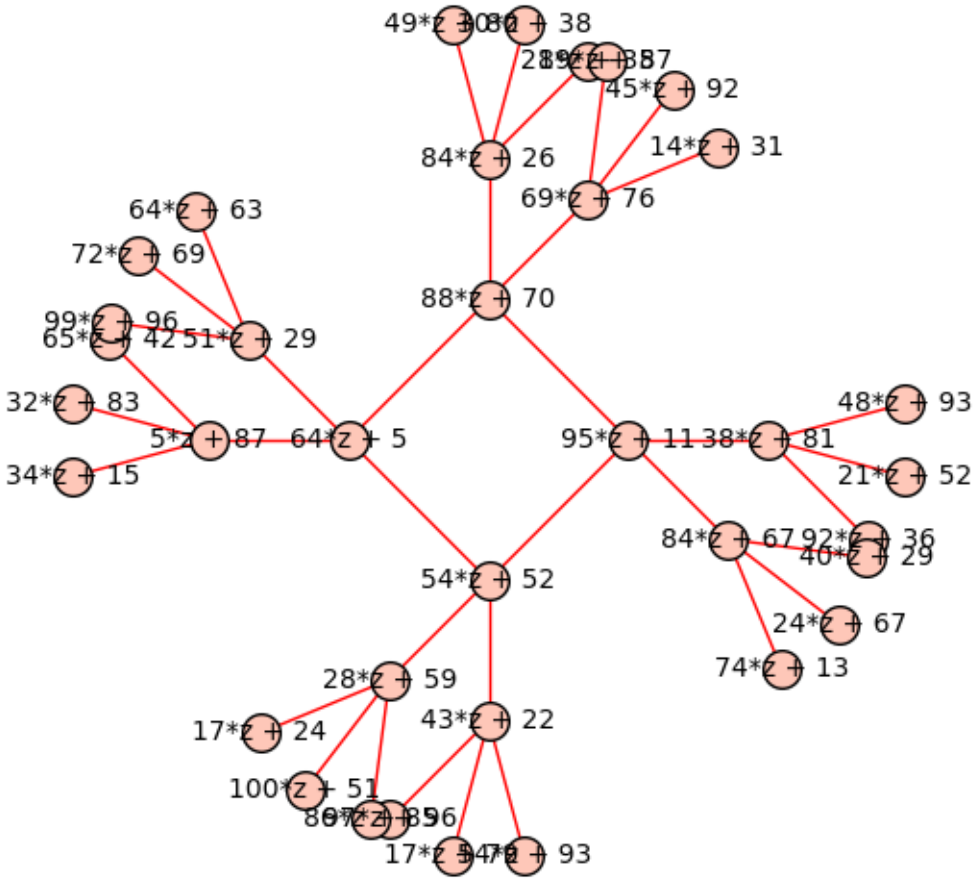


Figure 3: A 3-isogeny volcano over $\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$ that satisfies case III (in the plot have $z = \alpha$).

2.3 Case III

We say we are in *Case III*, if E_0 and $E_0^{(p)}$ are in different l-isogeny volcanos, i.e. there is no l-isogeny path between them.

We give the example displayed in Figure 3. Consider E with $j(E) = 64\alpha + 5$. Then $j(E^{(p)}) = j(E)^p = 37\alpha + 59$. However, we have that E lies on the crater, together with curve of j-invariants

$$88\alpha + 70,\ 54\alpha + 52,\ 95\alpha + 11$$

Hence there is no 3-isogeny path from E to $E^{(p)}$. Note that $[\mathcal{O}_\mathcal{K} : \mathbb{Z}[\pi]] = 2^2 \cdot 3^2$ but $[\mathcal{O}_\mathcal{K} : \mathrm{End}(E)] = 2^2$, which shows that E lies on the crater.

Now we want to have a closer look onto the class group action in this case. Have $d(\mathrm{End}(E)) = -320$, so $\mathcal{K} = \mathbb{Q}(\sqrt{-5})$ and $d(\mathcal{O}_\mathcal{K}) = -5$. Hence, we have $\mathrm{End}(E) \cong \mathbb{Z}[4\sqrt{-5}]$ and $\mathcal{O}_\mathcal{K} \cong \mathbb{Z}[\sqrt{-5}]$.

Sage tells us that $h(\mathcal{O}_\mathcal{K}) = 2$ and $h(\mathrm{End}(E)) = 8$. With this, we can already see that

$$64\alpha + 5,\ 88\alpha + 70,\ 54\alpha + 52,\ 95\alpha + 11$$

and

$$(64\alpha + 5)^p,\ (88\alpha + 70)^p,\ (54\alpha + 52)^p,\ (95\alpha + 11)^p$$

is the set of j-invariants of all Elliptic Curves with endomorphism ring $\cong \mathrm{End}(E)$. On this set, $\mathrm{Cl}(\mathbb{Z}[4\sqrt{-5}])$ then acts freely and transitively. Now it would be of course interesting to find out how $\mathrm{Cl}(\mathbb{Z}[4\sqrt{-5}])$ really looks like.

3 Properties of the endomorphism ring vs the cases

Proposition 3.1. *Let E be an ordinary Elliptic Curve defined over a finite field of characteristic p.*

- $\mathrm{End}(E)$ *has an element of norm p iff $j(E) \in \mathbb{F}_p$.*
- $\mathrm{End}(E)$ *has a nontrivial element (i.e. $\neq \epsilon p$ for a unit ϵ) of norm p^2 iff $j(E) \in \mathbb{F}_{p^2}$.*

Proof. The directions $\Leftarrow$ is clear, as the norm of the q-th power Frobenius endomorphism is q.

For the direction $\Rightarrow$, assume there is an element $\alpha \in \mathrm{End}(E)$ with $N(\alpha) = p$. If α is inseparable (as isogeny), then we have that it factors through the p-th power Frobenius endomorphism π, and thus $\alpha = \lambda \circ \pi$ for an isomorphism $\lambda : E^{(p)} \to E$. Thus $j(E^{(p)}) = j(E)$.

On the other hand, if α is separable, it must have kernel of size p, so $\ker(\alpha) = E[p]$ since $\#E[p] = p$ (E is ordinary). Thus $\ker(\alpha) \subseteq \ker([p])$ and we see that $[p]$ factors through α as $[p] = \psi \circ \alpha$. Now have that $\deg(\psi) = p = p^2/\deg(\alpha)$ and clearly ψ is inseparable. The claim follows as above.

[2] Note that this does not hold if $E, E^{(p)}$ are not in the same crater, see Figure 2.

For the second point, assume $\alpha \in \mathrm{End}(E)$ has norm $N(\alpha) = p^2$ and $\alpha \neq \pm p$. If α is purely inseparable, we are done. If α is separable, its kernel must be $E[p^2]$ and so it factors through $[p^2]$. Since $[p^2]$ has inseparability degree p^2, we see that $[p^2] = \pi^2 \circ \alpha$ where π is the p-th power Frobenius morphism. Since α is an endomorphism of E, find $\pi^2 : E \to E$, thus $j(E) \in \mathbb{F}_{p^2}$.

Finally, if α has inseparability degree p, then its kernel must be $E[p]$ and so $\alpha = \beta \circ \pi$ where $\beta : E^{(p)} \to E$ is separable with kernel $E^{(p)}[p]$. However, by the uniqueness of the separable isogeny with kernel $E^{(p)}[p]$, we know that (up to isomorphism) also $[p]$ is $\beta \circ \pi$. This now implies that $\alpha = \epsilon p$ for some unit ϵ. □

Corollary 3.2. *Let $D < 0$. Then the curves E with $\mathrm{End}(E) = \mathbb{Z}[\sqrt{D}]$ have $j(E) \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ if and only if*

$$a^2 - b^2 D = p$$

has no solution $a, b \in \mathbb{Z}$ and

$$a^2 - b^2 D = p^2$$

has a nontrivial solution $a, b \in \mathbb{Z}$.

Let $E/\mathbb{F}_{p^2}$ be an ordinary Elliptic Curve and write $\mathcal{O} := \mathrm{End}(E), \mathcal{K} := \mathrm{End}(E) \otimes \mathbb{Q}$. Let π be the $q = p^2$-th power Frobenius endomorphism and let t be its trace. Assume $p \neq 2$.

Lemma 3.3. *Have $(p) = (p, \pi)(p, \pi - t)$ in $\mathbb{Z}[\pi]$, $\mathcal{O}$ resp. $\mathcal{O}_\mathcal{K}$.*

Proof. Have

$$(p, \pi)(p, \pi - t) = (p^2, p\pi, p\pi - pt, \pi^2 - t\pi) = (p^2, pt, p\pi, -p^2) = (up^2 + vtp, ...) = (p)$$

where $up + vt = 1$ (note that $t \perp p$ since E is ordinary). □

Lemma 3.4. *(p, π) is principal (in $\mathcal{O}$) if and only if $E/\mathbb{F}_p$.*

Proof. If (p, π) is principal, then its generator is an element of norm p, so $E/\mathbb{F}_p$. On the other hand, if $E/\mathbb{F}_p$, then the p-th power Frobenius endomorphism π_p satisfies $p = \pi_p(t_p - \pi_p)$, $\pi = \pi_p^2$ and $\pi_p = u(\pi + p) + v\pi_p p$, where t_p is its trace and $ut + vp = 1$. □

There must be some problem in my definition of the class group action, as it can happen that $[(p, \pi)]$ is not $[(1)]$, but $E[(p, \pi)]$ is clearly trivial, so[3] $(p, \pi).E = E/E[(p, \pi)] = E$. However, this contradicts the freeness of the class group action.

Lemma 3.5. *Assume $j(E) \neq 0, 1728$ (and E is defined over $\mathbb{F}_{p^2}$). $[(p, \pi)]$ has order ≤ 2 in $\mathrm{Cl}(\mathcal{O})$ resp. $\mathrm{Cl}(\mathcal{O}_\mathcal{K})$.*

Proof. Since $E/\mathbb{F}_{p^2}$, we know that there is a nontrivial element α of norm p^2. Now have in $\mathcal{O}_\mathcal{K}$ that $(\alpha)|(p)^2$ and with $p = (p, \pi)(p, \pi - t)$ have thus $(\alpha) = (p)$ or $(\alpha) = (p, \pi)^2$ or $(\alpha) = (p, \pi - t)^2$. However, by assumption we only have units ± 1 in $\mathcal{O}_\mathcal{K}$ resp. $\mathcal{O}$, so the first case is impossible, as it implies $\alpha = \pm p$.

Note that $[(p, \pi)] = [(p, \pi - t)^{-1}]$, so wlog assume $(\alpha) = (p, \pi)^2$. It follows that $(p, \pi)^2$ is principal, so $[(p, \pi)]^2 = [(1)]$. □

[3] We know that (p, π) is invertible, as $\frac{1}{p}(p, \pi)(p, \pi - t) = (1)$.

4 Ideals in $\mathcal{O}$ resp. $\mathcal{O}_\mathcal{K}$

In this paragraph, I tried to improve my understanding of ideals, invertible ideals and the class group in a nonmaximal order in a quadratic imaginary number field. In particular, I was interested in which properties carry over from $\mathcal{O}_\mathcal{K}$ and which do not hold anymore.

Consider an ordinary Elliptic Curve $E/\mathbb{F}_q$, $\mathcal{O} := \mathrm{End}(E)$, $\mathcal{K} := \mathcal{O} \otimes \mathbb{Q}$ and $\mathcal{O}_\mathcal{K}$ the ring of integers in $\mathcal{K}$. Assume that $j(E) \in \mathbb{F}_q$ is not contained in any proper subfield of $\mathbb{F}_q$ and let π be the q-th power Frobenius endomorphism. Let t be its trace.

Proposition 4.1. $p \nmid d(\mathcal{O})$

Proof. Have that

$$d(\mathbb{Z}[\pi]) = t^2 - 4q \perp p$$

since $t \perp p$ as E is ordinary. The claim follows since $d(\mathcal{O}) \mid d(\mathcal{O})[\mathcal{O} : \mathbb{Z}[\pi]]^2 = d(\mathbb{Z}[\pi])$. □

Proposition 4.2. *Let $\mathfrak{a} \leq \mathcal{O}$. Then $\mathfrak{a} \cap \mathbb{Z} = (a)$ with $a \mid [\mathcal{O} : \mathfrak{a}] \mid a^2$. Note that if $\mathfrak{a} = \mathfrak{p}$ is prime, then trivially a must be prime.*

Proof. Clearly $[\mathcal{O} : \mathfrak{a}] \in \mathfrak{a}$ as $1 \in \mathcal{O}/\mathfrak{a}$ has order dividing $\#(\mathcal{O}/\mathfrak{a}) = [\mathcal{O} : \mathfrak{a}]$, so $a \mid [\mathcal{O} : \mathfrak{a}]$. On the other hand, have $[\mathcal{O} : \mathfrak{a}] \mid [\mathcal{O} : a\mathcal{O}] = a^2$. □

Lemma 4.3. *Let $\mathfrak{p} \leq \mathcal{O}_\mathcal{K}$ be a prime with $\mathfrak{N}(\mathfrak{p}) \perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$. Then $\mathfrak{p}$ has a set of generators in $\mathcal{O}$.*

Proof. Suppose $\mathfrak{p}$ is a prime over p, and let $\mathcal{O} = \mathbb{Z}[\phi]$. We use the decomposition law in Dedekind ring extensions. Since $\mathfrak{N}(\mathfrak{p}) \perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$ are coprime, we can apply it with generator ϕ of $\mathcal{O}$.

If $\mathrm{MiPo}(\phi) = f(X)g(X) \mod p$ splits, then have

$$p\mathcal{O}_\mathcal{K} = (p, f(\phi))(p, g(\phi))$$

and so the prime ideals over p are $(p, f(\phi))$ and $(p, g(\phi))$. If $\mathrm{MiPo}(\phi) \mod p$ is irreducible, then have that $p\mathcal{O}_\mathcal{K}$ is prime and thus the only prime ideal over p. Hence, all prime ideals over p (including $\mathfrak{p}$) have a set of generators in $\mathcal{O}$. □

Corollary 4.4. *Let $\mathfrak{a} \leq \mathcal{O}_\mathcal{K}$ be an ideal with $\mathfrak{N}(\mathfrak{a}) \perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$. Then $\mathfrak{a}$ has a set of generators in $\mathcal{O}$.*

Proposition 4.5. *Let $\mathfrak{p} \leq \mathcal{O}$ be a prime ideal with $\mathfrak{N}(\mathfrak{p}) \perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$ and $\mathfrak{p}' = \mathfrak{p}\mathcal{O}_\mathcal{K}$. Then $\mathcal{O}_\mathfrak{p} = (\mathcal{O}_\mathcal{K})_{\mathfrak{p}'}$.*

Proof. We have $\mathcal{O}_\mathcal{K} = \mathbb{Z}[\alpha]$ and $\mathcal{O} = \mathbb{Z}[f\alpha]$ where $f = [\mathcal{O}_\mathcal{K} : \mathcal{O}]$. Thus $f \notin \mathfrak{p}$ and so $f \in \mathcal{O}_\mathfrak{p}^*$. Therefore $\mathcal{O}_\mathcal{K} \subseteq \mathcal{O}_\mathfrak{p}$ and thus $(\mathcal{O}_\mathcal{K})_{\mathfrak{p}'} \subseteq \mathcal{O}_\mathfrak{p}$. □

Proposition 4.6. *Let $\mathfrak{I}(\mathcal{O})$ resp. $\mathfrak{I}(\mathcal{O}_\mathcal{K})$ denote the set of invertible ideals of norm $\perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$. Then*

$$\mathfrak{I}(\mathcal{O}) \to \mathfrak{I}(\mathcal{O}_\mathcal{K}), \quad \mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_\mathcal{K}$$

is a monoid isomorphism with inverse

$$\mathfrak{I}(\mathcal{O}_\mathcal{K}) \to \mathfrak{I}(\mathcal{O}), \quad \mathfrak{a} \mapsto \mathfrak{a} \cap \mathcal{O}$$

Proof. Clearly, this is a well-defined monoid homomorphism. Hence, we have to show that it is bijective.

By Corollary 4.4, we know that any $\mathfrak{a} \leq \mathcal{O}_\mathcal{K}$ with $\mathfrak{N}(\mathfrak{a}) \perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$ has generators in $\mathcal{O}$, thus $(\mathfrak{a} \cap \mathcal{O})\mathcal{O}_\mathcal{K} = \mathfrak{a}$. This shows that $\mathfrak{a} \cap \mathcal{O}$ is a preimage of $\mathfrak{a}$, and so the map is surjective.

Assume now $\mathfrak{a}, \mathfrak{b} \leq \mathcal{O}$ with $\mathfrak{a}\mathcal{O}_\mathcal{K} = \mathfrak{b}\mathcal{O}_\mathcal{K}$ and $\mathfrak{N}(\mathfrak{a}), \mathfrak{N}(\mathfrak{b}) \perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$. We show that $\mathfrak{a}_\mathfrak{p} = \mathfrak{b}_\mathfrak{p}$ for all primes $\mathfrak{p} \leq \mathcal{O}$. Note that if $\mathfrak{N}(\mathfrak{p}) \not\perp [\mathcal{O}_\mathcal{K} : \mathcal{O}]$, this holds trivially, as $\mathfrak{a}_\mathfrak{p} = \mathcal{O}_\mathfrak{p} = \mathfrak{b}_\mathfrak{p}$. Otherwise, note that

$$\mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}\mathcal{O}_\mathfrak{p} = \mathfrak{a}(\mathcal{O}_\mathcal{K})_\mathfrak{p} = \mathfrak{a}\mathcal{O}_\mathcal{K}(\mathcal{O}_\mathcal{K})_\mathfrak{p} = \mathfrak{b}\mathcal{O}_\mathcal{K}(\mathcal{O}_\mathcal{K})_\mathfrak{p} = \mathfrak{b}_\mathfrak{p}(\mathcal{O}_\mathcal{K})_\mathfrak{p} = \mathfrak{b}_\mathfrak{p}$$

as $\mathcal{O}_\mathfrak{p} = (\mathcal{O}_\mathcal{K})_\mathfrak{p}$. This shows that $\mathfrak{a}_\mathfrak{p} = \mathfrak{b}_\mathfrak{p}$ at all primes, so $\mathfrak{a} = \mathfrak{b}$ and our map is injective. Furthermore, since $(\mathfrak{a} \cap \mathcal{O})\mathcal{O}_\mathcal{K} = \mathfrak{a}$, we see that it has the inverse

$$\mathfrak{I}(\mathcal{O}_\mathcal{K}) \to \mathfrak{I}(\mathcal{O}), \quad \mathfrak{a} \mapsto \mathfrak{a} \cap \mathcal{O}$$

which must then be well-defined. □

Proposition 4.7. *Let $\mathfrak{a} \leq R$ be a radical ideal in a commutative unital ring R. If $\alpha \in \mathfrak{p}$ for all primes $\mathfrak{p} \supseteq \mathfrak{a}$ then $\alpha \in \mathfrak{a}$.*

Proof. We have that $\mathfrak{a}_\alpha \neq (1)$ otherwise $\alpha^n \in \mathfrak{a}$, so $\alpha \in \mathfrak{a}$. Thus $\mathfrak{a}_\alpha \subseteq \mathfrak{m}$ for a maximal ideal $\mathfrak{m} \leq R_\alpha$. A preimage under $R \to R_\alpha$ is now a prime $\mathfrak{p}$ with $\mathfrak{a} \subseteq \mathfrak{p}$ and $\alpha \notin \mathfrak{p}$. □

Corollary 4.8. *If $q \perp d(\mathcal{O})$ is an integer and $q \mid \alpha$ in $\mathcal{O}_\mathcal{K}$, then also $q \mid \alpha$ in $\mathcal{O}$.*

Proof. It suffices to prove this for primes q. Since $q \nmid d(\mathcal{O})$, we know that (q) is unramified, hence radical. Now observe that $\mathcal{O}_\mathfrak{p} = (\mathcal{O}_\mathcal{K})_\mathfrak{p}$ for primes $\mathfrak{p}$ over q and so $\alpha \in \mathfrak{p}$ for all primes $\mathfrak{p}$ over q. The previous proposition now shows that $\alpha \in (q)$. □

5 Norm equations

Lemma 5.1. *Let $D < 0$ and $\mathcal{O}$ be the imaginary quadratic order of discriminant D. Then $1, \alpha$ with $\alpha = \frac{D+\sqrt{D}}{2}$ is a $\mathbb{Z}$-basis of $\mathcal{O}$ and*

$$N(a + b\alpha) = \left(a + \frac{D}{2}b\right)^2 - \frac{D}{4}b^2$$

Proof.

$$\begin{aligned} N(\alpha) =& a^2 + ab\frac{D+\sqrt{D}}{2} + ab\frac{D-\sqrt{D}}{2} + \frac{D^2-D}{4}b^2 = a^2 + Dab + \frac{D^2-D}{4}b^2 \\ =& \Big(a + \frac{D}{2}b\Big)^2 - \frac{D}{4}b^2 \end{aligned}$$

□

Corollary 5.2. *Let l be a prime and $D < 0$. Let $\mathcal{O}$ be the quadratic imaginary order of discriminant D. If there exists a nontrivial element $\alpha \in \mathcal{O}$ of norm l^e (i.e. $\alpha \notin \mathbb{Z}$), then*

$$e \geq \log_l(-D) - \log_l(4)$$

Corollary 5.3. *Let $E/\bar{\mathbb{F}}_p$ be an ordinary Elliptic Curve such that $\mathrm{End}(E)$ has discriminant D. Suppose that $j(E_0) \notin \mathbb{F}_p$ and l is not ramified in $\mathrm{End}(E)\otimes\mathbb{Q}$. Then the shortest l-isogeny path between E and $E^{(p)}$ has length at least*

$$\frac{1}{2}\log_l(-D) - \log_l(2)$$

Proof. First, define E_0 to be the corresponding curve on the crater of the l-isogeny vulcano. Let $v \in \mathfrak{N}$ be maximal with $l^v \mid [\mathrm{End}(E_0) : \mathrm{End}(E)]$, so $\mathrm{End}(E_0)$ has discriminant D/l^v. Now we know that $E_0^{(p)}$ is at the opposite side of the crater, and the size of the crater is the order of $[\mathfrak{l}_1]$ in $\mathrm{Cl}(\mathrm{End}(E_0))$ where $l = \mathfrak{l}_1\mathfrak{l}_2$. If e is this order, then have that $\mathfrak{l}^e = (\alpha)$ is principal. Note that $\alpha \notin \mathbb{Z}$, otherwise we would have $\alpha = \pm l^{e/2}$, but since $(\alpha) = \mathfrak{l}_1^e$, we know that $\mathfrak{l}_2 \nmid (\alpha)$ (here we use that $\mathfrak{l}_1 \neq \mathfrak{l}_2$, i.e. l is unramified).

So

$$e \geq \log_l(-D/l^v) - \log_l(4)$$

Thus, the distance of E_0 and $E_0^{(p)}$ is at least

$$\frac{1}{2}e \geq \frac{1}{2}\log_l(-D/l^v) - \frac{1}{2}\log_l(4) = \frac{1}{2}\log_l(-D) - \frac{1}{2}v - \log_l(2)$$

However, the shortest path from E to E_0 has length v, and similarly for the shortest path from $E^{(p)}$ to $E_0^{(p)}$. Thus we find that the length of the shortest path from E to $E^{(p)}$ is at least

$$\frac{1}{2}\log_l(-D) - \frac{1}{2}v - \log_l(2) + 2v \geq \frac{1}{2}\log_l(-D) - \log_l(2)$$

□

How big is l^e?

First of all, note that the supersingular l-isogeny graph is $(l+1)$-regular and Ramajuan. Hence, the statistical ℓ_1-distance of the random walk distribution after t steps from the uniform distribution is

$$\leq \lambda^t \sqrt{n} = \left(\frac{d}{2\sqrt{d-1}}\right)^t \sqrt{n} \approx \left(\frac{\sqrt{l}}{2}\right)^t \sqrt{n}$$

We look for the minimal number of steps t such that every node has nonzero probability (i.e. there exists a length $\leq t$ path), this is then

$$\log_{\sqrt{l}/2}(n/\sqrt{n}) = \frac{1}{2}\log_{\sqrt{l}/2}(n) = \frac{\ln(n)}{2\ln(\sqrt{l}/2)} = \frac{\ln(n)}{\ln(l) - 2\ln(2)} \approx \log_l(n)$$

Since $n \approx p/12$, assume for now that we consider paths of length $\log_l(p)$.

Hence, we ask how many ordinary vulcanos have l-isogeny crater $\leq 2\log_l(n)$.

A proof boundary

By the Hasse-Minkowski principle, we know that

$$x^2 + Dy^2 = l^e$$

is solvable **with** $x, y \in \mathbb{Q}$ if and only if this is solvable in $\mathbb{R}$ and each $\mathbb{Q}_q$. Since $l^e = \Theta(p^2)$, we can assume that this is solvable in $\mathbb{R}$ if $p \leq D \leq p^2$, the ones we consider anyway. For $q > 2$, for this to be solvable, it suffices if it is solvable in $\mathbb{F}_q$. If $q \perp l, D$, then this is the case by some crazy deep theorem. So we are left with the conditions $(\frac{D}{l}) = 1$ and $(\frac{l}{d}) = 1$ for each prime divisor $d \mid D$.

So assuming that l_1 and l_2 are not $\equiv 3 \mod 4$, we see that for all primes D with $(\frac{D}{l_1}) = (\frac{D}{l_2}) = 1$, both

$$x_1^2 + Dy_1^2 = l_1 \quad \text{and} \quad x_2^2 + Dy_2^2 = l_2$$

are solvable in $\mathbb{Q}$, and thus this is also the case for $* = l_1^{e_i}$ resp. $* = l_2^{e_2}$. In particular, there are $\Omega(p^2/\log(p^2))$ such D.

In particular, this shows that it is not enough to bound the number of possible integers D just by considering things modulo p, as this will never be better than the above bound.

6 Generalizations and Modifications

6.1 Using divisibility constraints on the crater size

Theorem 6.1. *Let $l = \Theta(\log(p))$ be a prime and $e = \lceil \log_l(p) \rceil$. Then the number of $j \in \mathbb{F}_{p^2}$ with*

$$\Phi_{l^e}(j, j^p) = \Phi_{l^{e+1}}(j, j^p) = 0$$

such that the Elliptic Curve E with $j(E) = j$ is ordinary is $O(p^{5/3+\epsilon})$ for any $\epsilon > 0$.

Proof. For the direction $\Rightarrow$, assume that E is ordinary. If we are in case III, then there is no power-of-l isogeny $E \to E^{(p)}$, a contradiction.

Thus, we know that E and $E^{(p)}$ are in the same l-isogeny vulcano. Now let E_0 be the closest curve on the crater to E, i.e. $[\mathrm{End}(E_0) : \mathrm{End}(E)] = l^k$ for some $k \geq 0$ and $l \nmid [\mathcal{O}_{\mathrm{End}(E)\otimes\mathbb{Q}} : \mathrm{End}(E_0)]$.

If $E_0 \neq E_0^{(p)}$ Note that E_0 and $E_0^{(p)}$ are on opposite sides of the crater, and assume the crater has size c. The shortest path from E to E_0 has length k, so the shortest path from E to $E^{(p)}$ has length $2k + c$. Furthermore, all paths from E to $E^{(p)}$ are this path, plus any number of going around the crater, thus they have size $2k+c+2ac \equiv 2k+c \equiv c \mod 2$. In particular, all paths E to $E^{(p)}$ have either odd or even length. However, by assumption there is one path with odd and one path with even length, a contradiction.

If $E_0 = E_0^{(p)}$ Assuming the GRH, the class number of $\mathbb{Q}(\sqrt{-d})$ for $d > 0$ is at least

$$\frac{\sqrt{d}}{\log\log(d)}\Theta(1)$$

Hence, we have

$$\begin{aligned}
&\sum_d h(d) \\
\geq&\Theta(1)\sum_d \frac{\sqrt{d}}{\log\log(d)} \\
\geq&\Theta(1)\sum_{i\leq n} \frac{\sqrt{i}}{\log\log(i)} \\
=&\Theta(1)\int_e^n \frac{\sqrt{x}}{\ln\ln(x)}dx \\
=&\Theta(1)\left(\frac{x^{3/2}}{\ln\ln(x)} - \underbrace{\int_e^n \frac{x^{3/2}}{\ln(x)\ln\ln(x)^2}dx}_{\geq 0}\right) \\
\geq&\Theta(n^{3/2}/\ln\ln(n))
\end{aligned}$$

where the sums over d range over all fundamental discriminants $d < 0$ with $|d| \leq n$.

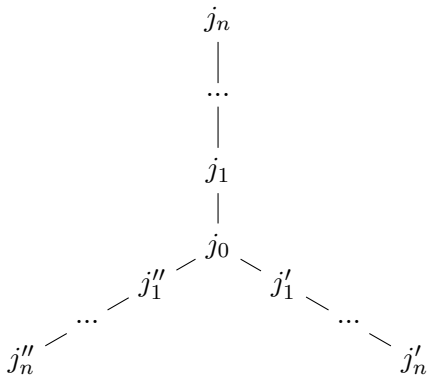
Hence, if there are n different fundamental discriminants d such that curves E with $d(\mathrm{End}(E)) = d$ are defined over $\mathbb{F}_p$, that means there must be

$$\sum_d h(d) = \Theta(n^{3/2}/\ln\ln(n))$$

(isomorphism classes of) curves defined over $\mathbb{F}_p$. Clearly there are at most p of them (since $j(E) \in \mathbb{F}_p$ for those), thus $n^{3/2}/\ln\ln(n) \leq \Theta(p)$. It follows that $n \in O(p^{2/3+\epsilon})$ for any $\epsilon > 0$. □

6.2 Based on Sutherland's supersingularity test

It is a theorem by Sutherland that in an l-isogeny vulcano, at most $O(\log_l(p))$ levels of j-invariants are defined over $\mathbb{F}_{p^2}$. Hence, if we look for subgraphs of the form



where all j-invariants are defined over $\mathbb{F}_{p^2}$, then they will for sure define supersingular curves.

However, it is not as easy as taking a root of the ideal

$$\langle \Phi_2(j_0, j_1), ..., \Phi_2(j_{n-1}, j_n), ..., j_0^{p^2-1} - 1, ... \rangle$$

as this would allow e.g. $j_1 = j_1' = j_1''$.

However, we can use the fact that in general $\Phi_{2^e}(j, j) \neq 0$, even if e is even. In other words, the modular polynomial does not consider trivial endomorphisms. Hence, taking a root of the ideal

$$\begin{aligned}\langle\ & \Phi_2(j_0, j_1), \Phi_2(j_0, j_1').\Phi_2(j_0, j_1''), \\ & \Phi_4(j_1, j_1'), \Phi_4(j_1, j_1''), \Phi_4(j_1', j_1''), \\ & \Phi_2(j_1, j_2), \Phi_4(j_0, j_2),\ \Phi_2(j_2, j_3), \Phi_4(j_1, j_3), ..., \\ & ..., \\ & j_0^{p^2-1} - 1, ...\ \rangle\end{aligned}$$

In other words, to require an edge defined by $\Phi_l(y, z)$ that is part of a path $x \to y \to z$ not to backtrack, add the additional constraint $\Phi_{l^2}(x, z)$.

The only way the above can still yield ordinary curves is if a considered curve has a nontrivial endomorphism of degree l^2, as in this case, one can still have e.g. $j_1 = j_1' = j_1''$. However, similar to the computation done before, there are only very few such curves, as they would have to have a small endomorphism ring. More concretely, there are at most $O(p)$ such curves, and since all supersingular j-invariants are roots of the above ideal, we have a constant nonzero success probability.

Theorem 6.2. *Let l and p be distinct primes, and assume $p \neq 2$. Let $r = \lfloor \log_l(p) \rfloor$. Then there are l-isogeny vulcanos with at least $\lfloor r/2 \rfloor + 1$ levels defined over $\mathbb{F}_{p^2}$.*

Proof. Let $r = \rfloor \log_l(p) \rceil$. Let $D = 4l^r(l^r - p)$. Thus, $D \equiv 0 \mod 4$ and so D is a fundamental discriminant. Further, we have that

$$(p - 2l^r)^2 - D \cdot 1^2 = p^2 - 4pl^r + 4l^{2r} - 4l^{2r} + 4pl^r = p^2$$

Thus, the quadratic imaginary order $\mathcal{O}$ with discriminant D contains an element of norm p^2. Let now $j \in \bar{\mathbb{F}}_p$ be a root of the Hilbert class polynomial $h_{\mathcal{O}}$, i.e. curves E with $j(E) = j$ have endomorphism ring $\mathrm{End}(E) \cong \mathcal{O}$. By Prop. 3.1, we now see that $j \in \mathbb{F}_{p^2}$.

Furthermore, since $l^r \mid d(\mathcal{O})$, we see that $l^{\lfloor r/2 \rfloor} \mid [\mathcal{O}_K : \mathcal{O}]$ where $K = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$. This shows that a path of at least $\lfloor r/2 \rfloor$ ascending l-isogenies is required to reach the crater from j, in other words, j is on level $\lfloor r/2 \rfloor$ in the l-isogeny vulcano. Clearly all j-invariants on higher levels are also defined over $\mathbb{F}_{p^2}$, and thus there are at least $\lfloor r/2 \rfloor + 1$ levels in the vulcano. □

Theorem 6.3. *Let l and p be distinct odd primes. Let $r = \lfloor \log_l(4p) \rfloor$. Then each l-isogeny vulcano has at most $\lfloor r/2 \rfloor + 1$ levels defined over $\mathbb{F}_{p^2}$.*

Proof. Let $k = \lfloor r/2 \rfloor + 1$. Assume there is an l-isogeny vulcano with more than k levels. Let $\mathcal{O}$ be the endomorphism ring of the bottom level, and set $D = d(\mathcal{O})$. Then $l^{2k} \mid D$. Now note that $D = t^2 - 4p^2$ where t is the trace of the p^2-Frobenius. Thus

$$D = t^2 - 4p^2 = (t - 2p)(t + 2p)$$

Since $l \neq 2$, we know that $l \nmid 4p$ and so l^{2k} divides either $t - 2p$ or $t + 2p$. We have $|t| \leq 2p$ and so $l^{2k} \leq 4p$. Thus $2k \leq \log_l(4p)$. However

$$2k = 2\lfloor r/2 \rfloor + 2 \geq r + 1 = \log_l(4p) + 1 > \log_l(4p)$$

which is a contradiction. □

Note that if $l \gg 4$, then it is likely that $\lfloor \log_l(p) \rfloor = \lfloor \log_l(4p) \rfloor$, and so both theorems together give the exact maximum number of levels in l-isogeny vulcanos over $\mathbb{F}_{p^2}$.

Further, note that this bound is better than the one used by Sutherland in his paper, and this should improve the running time of his algorithm by a factor of $\sqrt{n}$.

6.3 Hashing into the supersingular $\mathbb{F}_p$-graph

Proposition 6.4. *Let $j \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ be an ordinary j-invariant and consider the corresponding vertex in the l-isogeny vulcano.*

- *If j is in the crater, then no j-invariant in the vulcano is in $\mathbb{F}_p$*
- *If j is in a lava flow, then none of its children is in $\mathbb{F}_p$*

Proof. Follows directly from Prop. 3.1, since for all vertices j' on the same or a lower level in the l-isogeny vulcano of j have that $\mathrm{End}(j') \subseteq \mathrm{End}(j)$. □

In particular, this shows that the following subgraph never occurs in an ordinary connected component

$$\in \mathbb{F}_p \text{ ——————— } \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p \text{ ——————— } \in \mathbb{F}_p$$

However, experiments have shown that it does indeed occur in the supersingular l-isogeny graph. If it does so frequently enough, it might be used to hash into the supersingular $\mathbb{F}_p$-graph.

Note that if we try to find instances of this subgraph via polynomials, we again have to prevent collapsing, e.g. as

$$\langle \Phi_l(x_1, x_2), \Phi_l(x_2, x_3), \Phi_{l^2}(x_1, x_3) \rangle$$

Here we look for solutions $x_1, x_3 \in \mathbb{F}_p$ and $x_2 \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$.

However, I have not yet analyzed how often this occurs.

7 Example - j-invariant $61\alpha + 16$

Let E be an Elliptic Curve defined over $\mathbb{F}_{101^2}$ with j-invariant $61\alpha + 16$. Then the q-th power Frobenius π has minimal polynomial

$$X^2 - 190X + 10201$$

Furthermore, we find that $\mathrm{End}(E) = \mathcal{O}_\mathcal{K}$ for $\mathcal{K} = \mathbb{Q}(\sqrt{-6})$. So

$$\pi = \frac{190 + 28\sqrt{-6}}{2} = 95 + 14\sqrt{-6}$$

or

$$\sqrt{-6} = \frac{\pi - 95}{14}$$

Note that $\pi - 95$ has norm $2^3 \cdot 3 \cdot 7^2$. The class group of $\mathcal{O}$ has order 2, and a generator is e.g. the coset of $(2, \sqrt{-6})$. Hence, to find $E[(2, \sqrt{-6})]$ we need to find

$$\ker\left(\frac{\pi - 95}{14}\right) \cap E[2] = 14\ker(\pi - 95) \cap E[2] = 14(\ker(\pi - 95) \cap E[28])$$

Now choose a $\mathbb{Z}/4\mathbb{Z}$-basis P_1, P_2 of $E[4]$ and a $\mathbb{Z}/7\mathbb{Z}$-basis Q_1, Q_2 of $E[7]$. Have that w.r.t. these basis, π is given by the matrices

$$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

Since $95 \equiv 3 \mod 4$ and $95 \equiv 4 \mod 7$, we see that $\ker(\pi - 95) \cap E[28]$ projects to

$$\langle P_1, 2P_2 \rangle \subseteq E[4] \quad \text{and} \quad E[7] \subseteq E[7]$$

$j(E)$	$h(\mathrm{End}(E))$	$[\mathcal{O}_\mathcal{K} : \mathrm{End}(E)]$	$d(\mathrm{End}(E))$	$[\mathrm{End}(E) : \mathbb{Z}[\pi]]$
α	36	3	-36315	1
$4\alpha + 99$	64	1	-40020	1
$61\alpha + 16$	2	1	-24	28
$48\alpha + 73$	64	?	-37440	?
$12\alpha + 79$	12	?	-2548	?
$91\alpha + 34$	24	?	-16468	?
$95\alpha + 20$	64	?	-40548	?
$97\alpha + 12$	48	?	-35475	?
$97\alpha + 8$	48	?	-35620	?
$93\alpha + 8$	24	?	-23643	?
$77\alpha + 16$	16	?	-2340	?
$21\alpha + 48$	30	?	-35179	?
$31\alpha + 59$	48	?	-29355	?
$82\alpha + 39$	24	?	-18603	?
$64\alpha + 38$	36	?	-40075	?
$92\alpha + 74$	32	?	-30195	?
$38\alpha + 18$	16	?	-2340	?
$69\alpha + 25$	40	?	-31588	?
$99\alpha + 64$	32	?	-30195	?
$56\alpha + 4$	32	?	-30195	?
$26\alpha + 90$	12	?	-2548	?
$93\alpha + 49$	48	?	-36708	?
$17\alpha + 16$	32	?	-13908	?
$84\alpha + 67$	4	?	-180	?
$100\alpha + 34$	56	?	-40788	?
$30\alpha + 2$	16	?	-2244	?
$21\alpha + 41$	2	?	-52	?
$24\alpha + 59$	24	?	-26643	?
$67\alpha + 94$	64	?	-37204	?
$88\alpha + 99$	2	?	-88	?
$47\alpha + 26$	48	?	-24420	?
$12\alpha + 7$	16	?	-2520	?
$55\alpha + 77$	24	?	-17395	?
$95\alpha + 92$	8	?	-987	?
$68\alpha + 12$	12	?	-756	?
$82\alpha + 66$	28	?	-4532	?
$91\alpha + 38$	16	?	-6948	?
$99\alpha + 20$	24	?	-18603	?
$52\alpha + 77$	80	?	-40404	?

Table 1: Table of class numbers of $\mathrm{End}(E)$ for Elliptic Curves $E/\mathbb{F}_{101^2} = \mathbb{F}_{101}[\alpha]$. Note that the j-values are not uniformly chosen, in particular, j-values that lead to a conductor $[\mathcal{O}_\mathcal{K} : \mathbb{Z}[\pi]]$ with "big" prime power divisors have been ignored, as the current implementation of computing the endomorphism ring would take ages for them.

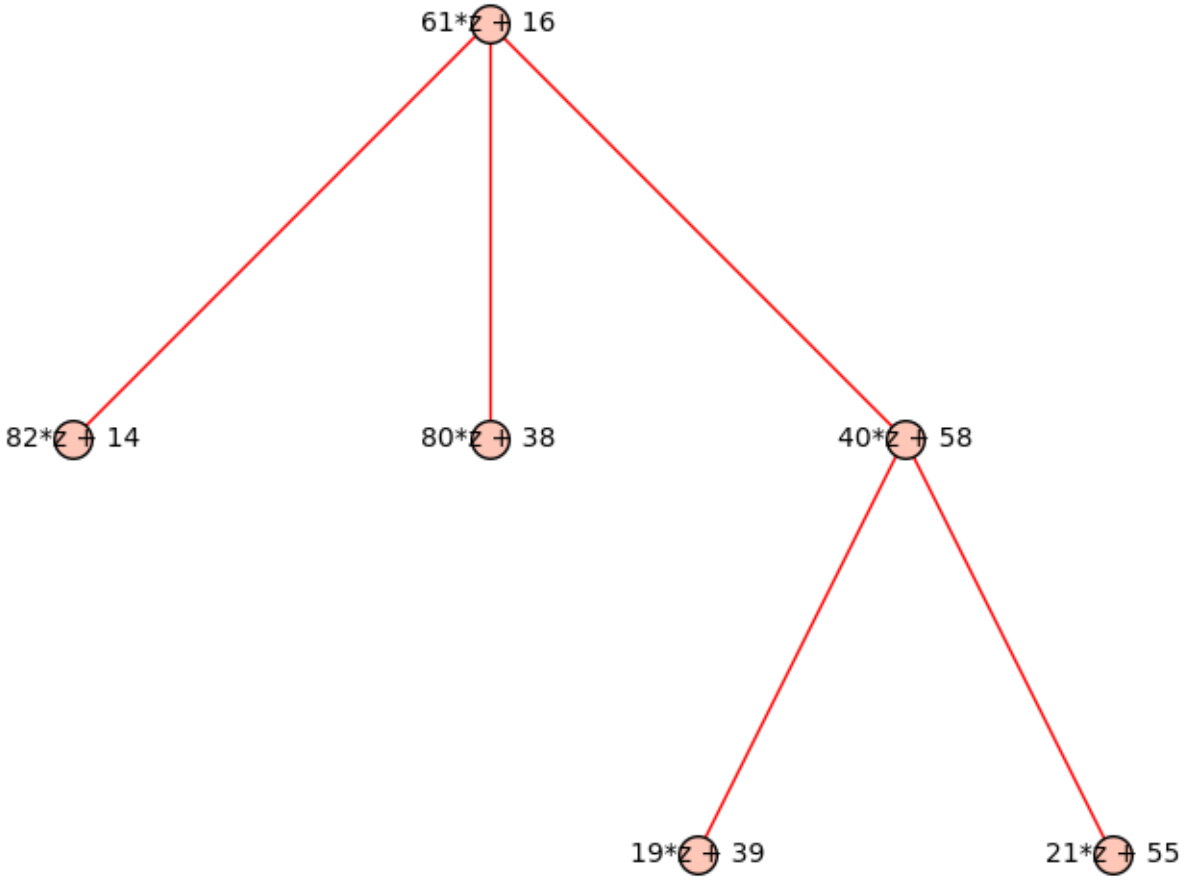


Figure 4: The 2-isogeny vulcano containing $61\alpha+16$. Note that 2 is ramified in $\mathrm{End}(E)\otimes\mathbb{Q}$, thus the crater only has size 2.

and thus is $E[14]+\langle P_1\rangle$. This implies that $E[(2,\sqrt{-6})]=\langle 2P_1\rangle$. Note that we picked

$$P_1 = (59+7\alpha,\ 48+75\alpha+(73+3\alpha)t),$$
$$P_2 = (7+17\alpha+100t,\ 71+72\alpha+(31+88\alpha)t)$$

before, where t has minimal polynomial $(24+51\alpha)+(94+84\alpha)T+T^2$. Hence, $\overline{(2,\sqrt{-6})}.E$ is the (isomorphism class of the) image of the 2-isogeny $\phi: E\to E/\langle 2P_1\rangle$, which is

$$j(E/\langle 2P_1\rangle) = 40\alpha+58 = (61\alpha+16)^{101}$$

The 2-isogeny vulcano containing $61\alpha+16$ is shown in

To find the whole kernel, pick a $\mathbb{Z}/8\mathbb{Z}$-basis P_1, P_2 of $E[8]$, a $\mathbb{Z}/3\mathbb{Z}$-basis Q_1, Q_2 of $E[3]$ and a $\mathbb{Z}/49\mathbb{Z}$-basis R_1, R_2 of $E[49]$. Find then that modulo 8, 3 resp. 49, π is given by the matrix

$$\begin{pmatrix}3&6\\4&3\end{pmatrix}\quad\text{resp.}\quad\begin{pmatrix}2&1\\0&2\end{pmatrix}\quad\text{resp.}\quad\begin{pmatrix}32&14\\0&11\end{pmatrix}$$

8 Example - The ordinary endomorphism ring

The information in this section is all known material - I just wanted to understand properly how one can compute the endomorphism ring, and what problems occur.

Consider the finite field

$$\mathbb{F}_q = \mathbb{F}_{37^2} = \mathbb{F}_{37} + \alpha\mathbb{F}_{37}$$

where $\alpha^2 + 33\alpha + 2 = 0$. Further, consider the Elliptic Curve $E/\mathbb{F}_q$ with j-invariant 3α, given by

$$E : y^2 = x^3 + (15\alpha + 17)x + (5\alpha + 3)$$

Then we find that the q-th power Frobnenius endomorphism π satisfies the minimal equation

$$\pi^2 + 47\pi + 1369$$

and in particular, its trace is -47. Hence, the number field $\mathcal{K} := \mathcal{O} \otimes \mathbb{Q}$ where $\mathcal{O} = \mathrm{End}(E)$ contains $\sqrt{47^2 - 4 \cdot 1369} = \sqrt{-3^3 \cdot 11^2}$. We observe that $\mathcal{K} = \mathbb{Q}(\sqrt{-3})$ and has discriminant -3. Furthermore the ring of integers is $\mathcal{O}_\mathcal{K} = \mathbb{Z}[\frac{1}{2}(1 + \sqrt{-3})]$.

Knowing the number field, we want to find the endomorphism ring. First, observe that the Frobenius order $\mathbb{Z}[\pi]$ has conductor 33. Now consider the endomorphism

$$\phi := 2\pi + 47$$

The advantage is that we can evaluate ϕ on points of E, but evaluating $\pi + 47/2$ is not so easy. Clearly $[\mathbb{Z}[\pi] : \mathbb{Z}[\phi]] = 2$ and so $\mathbb{Z}[\phi]$ has conductor 66.

Torsion points

In order to find whether $\phi/n \in \mathcal{O}$, we factor $66 = 2 \cdot 3 \cdot 11$ and compute the corresponding torsion groups. This turns out to be quite difficult.

Assume $\mathbb{F}_{37^{12}} = \mathbb{F}_{37}[\beta]$ with

$$\mathrm{MiPo}_{\mathbb{F}_{37}}(\beta) = x^{12} + 4x^7 + 31x^6 + 10x^5 + 23x^4 + 18x^2 + 33x + 2$$

Then $E[2]$ is generated by

$$P_1 = (11\beta^{11} + 19\beta^{10} + \beta^9 + 27\beta^8 + 8\beta^7 + 16\beta^6 + 17\beta^5 + 32\beta^4 + 12\beta^3 + 14\beta^2 + 24\beta + 32 : 0 : 1)$$
$$Q_1 = (15\beta^{11} + 7\beta^{10} + 33\beta^9 + 11\beta^8 + 6\beta^7 + 12\beta^6 + 26\beta^5 + 7\beta^4 + 33\beta^3 + 25\beta^2 + 8\beta + 19 : 0 : 1)$$

Further $E[3]$ is generated by

$$P_2 = (19\beta^{11} + 34\beta^{10} + 3\beta^9 + 29\beta^8 + 7\beta^7 + 3\beta^6 + 18\beta^5 + 21\beta^4 + 23\beta^3 + 30\beta^2 + 23\beta + 25$$
$$: 6\beta^{11} + 25\beta^{10} + 4\beta^9 + 13\beta^8 + 10\beta^7 + 23\beta^6 + 20\beta^5 + 30\beta^4 + 24\beta^3 + 6\beta^2 + 17\beta + 5 : 1)$$
$$Q_2 = (31\beta^{11} + 24\beta^{10} + 35\beta^9 + 32\beta^8 + 2\beta^7 + 10\beta^6 + 23\beta^5 + 35\beta^4 + 22\beta^3 + 13\beta^2 + 12\beta + 12$$
$$: 18\beta^{11} + 2\beta^{10} + 32\beta^9 + 26\beta^8 + 17\beta^7 + 5\beta^6 + 19\beta^5 + 31\beta^4 + 31\beta^3 + \beta^2 + 22\beta + 1 : 1)$$

For $E[11]$ we must even go to the extension degree 110. So assume $\mathbb{F}_{37^{220}} = \mathbb{F}_{37}[\gamma]$. Then $E[11]$ is generated by P_3 and Q_3. For the values of $\mathrm{MiPo}_{\mathbb{F}_{37}}(\gamma)$ and P_3, Q_3 see Section 9.

Now we can compute $\phi(P_1), \phi(Q_1), \phi(P_2), \phi(Q_2), \phi(P_3), \phi(Q_3)$ and see that none of them is zero. Since $\deg(\phi) = [\mathcal{O} : \mathbb{Z}[\phi]] \mid [\mathcal{O}_\mathcal{K} : \mathbb{Z}[\phi]] = 2 \cdot 3 \cdot 11$, we see that the kernel of ϕ is trivial. Thus no ϕ/n is contained in $\mathcal{O}$. Therefore we see that

$$\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z}[\phi]$$

The inclusion $\supseteq$ is clear, and for the other direction, note that $\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z} + t\sqrt{D}\mathbb{Z}$ and $\mathbb{Z}[\phi] = \mathbb{Z} + s\sqrt{D}\mathbb{Z}$. Since $\mathbb{Z}[\phi] \subseteq \mathcal{O} \cap \mathbb{Z}[\phi]$ find thus $t \mid s$. Now observe that by choice of ϕ, have $\phi^2 \in \mathbb{Z}$ and so $\phi = s\sqrt{D}$. However, $\phi/\frac{s}{t} = t\sqrt{D} \in \mathcal{O}$. By the above, it follows that $\frac{s}{t} = 1$, i.e. $s = t$.

The index $[\mathcal{O} : \mathbb{Z}[\phi]]$

From the consideration of the torsion points, we see that $\mathcal{O} \cap \mathbb{Z}[\sqrt{D}] = \mathbb{Z}[\phi]$. However, since $[\mathcal{O}_\mathcal{K} : \mathbb{Z}[\sqrt{D}]] \leq 2$, we deduce that $[\mathcal{O} : \mathbb{Z}[\phi]] \leq 2$ and so

$$\mathcal{O} = \mathbb{Z}[\pi]$$

My implementation

I have implemented the above algorithm, with computing the torsion groups via division polynomials in Rust, as the SAGE functions for that are extremely slow (apparently, SAGE is not fast when working with polynomials over $\mathbb{F}_{p^2}$, I assume because elements of $\mathbb{F}_{p^2}$ are considered objects and require dynamic memory allocation etc.). The advantage is that this can be extended to compute kernels of all endomorphisms, and so be used to compute the class group action.

9 P_3 and Q_3

The minimal polynomial of γ is

```
x^220 + 31*x^219 + 13*x^218 + 21*x^217 + 23*x^216 + 9*x^215
+ 2*x^214 + 35*x^212 + 10*x^211 + 29*x^210 + 25*x^209 + 20*x^208
+ 17*x^207 + 30*x^206 + 5*x^205 + 15*x^204 + 11*x^203 + 10*x^202
+ 11*x^201 + 32*x^200 + 5*x^199 + 28*x^198 + 7*x^197 + 13*x^196
+ 10*x^195 + 32*x^194 + 17*x^193 + 19*x^192 + 36*x^191
+ 17*x^190 + 31*x^189 + 14*x^188 + 6*x^187 + 30*x^186 + 8*x^185
+ 22*x^184 + 2*x^183 + 9*x^182 + 11*x^181 + 6*x^180 + 23*x^179
+ 14*x^178 + 36*x^177 + 16*x^176 + 34*x^175 + 14*x^174
+ 33*x^173 + 14*x^172 + 7*x^171 + 36*x^170 + 18*x^169 + 27*x^168
+ 5*x^167 + 31*x^166 + 6*x^165 + 15*x^164 + 14*x^163 + 17*x^162
+ 7*x^161 + 16*x^160 + 6*x^159 + 29*x^158 + 11*x^157 + 8*x^156
```

```
+ 15*x^155 + 20*x^154 + 17*x^153 + 7*x^152 + 8*x^151 + 6*x^150
+ 12*x^149 + 36*x^148 + 7*x^147 + 3*x^146 + 25*x^145 + 13*x^144
+ 6*x^143 + 17*x^142 + 22*x^141 + 9*x^140 + 18*x^139 + 36*x^138
+ x^137 + 6*x^136 + 36*x^135 + 33*x^134 + 32*x^133 + 35*x^132
+ 33*x^131 + 7*x^130 + 3*x^129 + 7*x^128 + 20*x^127 + 31*x^126
+ 26*x^125 + 6*x^124 + 9*x^123 + 10*x^122 + 25*x^121 + 33*x^120
+ 33*x^119 + 30*x^118 + 34*x^117 + 22*x^116 + 8*x^115 + 10*x^114
+ 36*x^113 + 26*x^112 + 8*x^111 + 33*x^110 + 30*x^109 + 11*x^108
+ 14*x^107 + 22*x^106 + 26*x^105 + 11*x^104 + 35*x^103
+ 34*x^102 + 33*x^101 + 27*x^100 + 14*x^99 + 31*x^98 + 24*x^97
+ x^96 + 6*x^95 + 36*x^93 + 32*x^92 + 18*x^91 + 36*x^90 + 3*x^89
+ 22*x^88 + 36*x^87 + 6*x^86 + 20*x^85 + 25*x^84 + 8*x^82
+ 34*x^81 + 7*x^80 + 25*x^79 + 21*x^78 + 17*x^77 + 29*x^76
+ 5*x^75 + 19*x^74 + 19*x^73 + 8*x^72 + 8*x^71 + 26*x^70
+ 7*x^69 + 27*x^68 + 10*x^67 + 31*x^66 + 4*x^65 + 29*x^64
+ 36*x^62 + 3*x^61 + 27*x^60 + 13*x^59 + 23*x^58 + 33*x^57
+ 14*x^56 + 19*x^55 + 12*x^54 + 20*x^53 + 32*x^52 + 18*x^51
+ 20*x^49 + 20*x^48 + x^47 + 17*x^46 + 16*x^45 + 4*x^44
+ 12*x^43 + 7*x^42 + 34*x^41 + 9*x^40 + 16*x^39 + 10*x^38
+ 25*x^37 + 10*x^36 + 10*x^35 + 28*x^34 + 33*x^33 + 22*x^32
+ 24*x^31 + 33*x^30 + 6*x^29 + 8*x^28 + 8*x^27 + 16*x^26
+ 31*x^25 + 7*x^24 + 26*x^23 + 36*x^22 + 29*x^21 + 36*x^20
+ 7*x^19 + x^18 + 26*x^17 + 18*x^16 + 23*x^15 + 10*x^14
+ 4*x^13 + x^12 + 24*x^11 + 25*x^10 + 34*x^9 + 33*x^8
+ 33*x^7 + 8*x^6 + 12*x^5 + x^4 + 15*x^3 + 27*x^2 + 9*x + 2
```

P_3 is given by

```
(23*z220^219 + 5*z220^218 + 26*z220^217 + 27*z220^216
+ 26*z220^215 + 12*z220^214 + 11*z220^213 + 10*z220^212
+ 29*z220^211 + 9*z220^210 + 16*z220^209 + 24*z220^208
+ 18*z220^207 + 11*z220^206 + 11*z220^205 + 6*z220^204
+ 24*z220^203 + 3*z220^202 + 34*z220^201 + 18*z220^200
+ 17*z220^199 + 9*z220^198 + 26*z220^197 + 2*z220^196
+ 31*z220^195 + 7*z220^194 + 15*z220^193 + 11*z220^192
+ 15*z220^191 + 28*z220^190 + 13*z220^189 + 6*z220^188
+ 7*z220^187 + 28*z220^186 + 9*z220^185 + 9*z220^184
+ 7*z220^183 + 27*z220^182 + 36*z220^181 + 35*z220^180
+ 30*z220^179 + 32*z220^178 + 16*z220^177 + 15*z220^176
+ 16*z220^175 + 9*z220^174 + 21*z220^173 + 6*z220^172
+ 15*z220^171 + 3*z220^170 + 25*z220^169 + 23*z220^168
+ z220^167 + 8*z220^166 + 34*z220^165 + 14*z220^164
+ 12*z220^163 + 20*z220^162 + 4*z220^161 + 9*z220^160
+ z220^159 + 25*z220^158 + 16*z220^157 + z220^156
+ 21*z220^155 + 10*z220^154 + 7*z220^153 + 13*z220^152
```

```
+ 32*z220^151 + 31*z220^150 + 17*z220^148 + 24*z220^147
+ 26*z220^146 + 28*z220^145 + 27*z220^144 + 4*z220^143
+ 5*z220^142 + 14*z220^141 + 26*z220^140 + 10*z220^139
+ 14*z220^138 + 19*z220^137 + 20*z220^136 + 18*z220^135
+ 16*z220^134 + 11*z220^133 + 23*z220^132 + 35*z220^131
+ 22*z220^130 + 31*z220^129 + 34*z220^128 + 17*z220^127
+ z220^126 + 15*z220^125 + 2*z220^124 + 22*z220^123
+ 27*z220^122 + 6*z220^121 + 10*z220^120 + 7*z220^119
+ 4*z220^118 + 26*z220^117 + z220^116 + 32*z220^115
+ 29*z220^114 + 32*z220^113 + 18*z220^112 + 3*z220^111
+ 28*z220^110 + 20*z220^109 + 17*z220^108 + 17*z220^107
+ 32*z220^106 + 32*z220^105 + 26*z220^104 + 24*z220^103
+ 17*z220^102 + 8*z220^101 + 3*z220^100 + 2*z220^99
+ 16*z220^98 + 29*z220^97 + 19*z220^96 + 27*z220^95
+ 4*z220^94 + 29*z220^93 + 24*z220^92 + 19*z220^91
+ 2*z220^90 + 2*z220^89 + 32*z220^88 + 23*z220^87
+ 32*z220^86 + 15*z220^85 + 24*z220^84 + 36*z220^83
+ 29*z220^82 + 18*z220^81 + 2*z220^80 + z220^79
+ 33*z220^78 + 34*z220^77 + 4*z220^76 + 11*z220^75
+ 21*z220^74 + 15*z220^73 + 10*z220^72 + 24*z220^71
+ 22*z220^70 + 22*z220^69 + 31*z220^68 + 32*z220^67
+ 28*z220^66 + z220^65 + 17*z220^64 + 13*z220^63
+ 32*z220^62 + 20*z220^61 + 32*z220^60 + 21*z220^59
+ 34*z220^58 + 11*z220^57 + 29*z220^56 + 12*z220^55
+ 22*z220^54 + 11*z220^53 + 36*z220^52 + 35*z220^51
+ 19*z220^50 + 35*z220^49 + 8*z220^48 + 16*z220^47
+ 16*z220^46 + 27*z220^45 + 32*z220^44 + 12*z220^43
+ 15*z220^42 + 6*z220^41 + 36*z220^40 + 27*z220^39
+ 17*z220^38 + 20*z220^37 + 33*z220^36 + 34*z220^35
+ 34*z220^34 + 3*z220^33 + 12*z220^32 + 12*z220^31
+ 12*z220^30 + 5*z220^29 + 10*z220^28 + 13*z220^27
+ 36*z220^26 + 16*z220^25 + 16*z220^24 + 15*z220^23
+ 36*z220^22 + 18*z220^21 + 13*z220^20 + 26*z220^19
+ 25*z220^18 + 21*z220^17 + 35*z220^16 + 3*z220^14
+ 31*z220^13 + 8*z220^12 + 7*z220^11 + 10*z220^10
+ 10*z220^9 + 6*z220^8 + 5*z220^7 + 33*z220^6
+ 6*z220^5 + 4*z220^4 + 31*z220^3 + 27*z220^2 + 27*z220 + 14

: 8*z220^219 + 17*z220^218 + 27*z220^217 + 14*z220^216
+ 6*z220^215 + 19*z220^214 + 18*z220^213 + 6*z220^212
+ 30*z220^211 + 24*z220^210 + 33*z220^209 + 19*z220^208
+ 27*z220^207 + 16*z220^206 + 24*z220^205 + 3*z220^204
+ 4*z220^203 + 25*z220^202 + 29*z220^201 + 31*z220^200
```

+ 23*z220^199 + 7*z220^198 + 28*z220^197 + 4*z220^196
+ 26*z220^195 + 36*z220^194 + 18*z220^193 + 24*z220^192
+ 29*z220^191 + 25*z220^190 + 23*z220^189 + 14*z220^188
+ 33*z220^187 + 19*z220^186 + 14*z220^184 + 21*z220^183
+ 10*z220^182 + 13*z220^181 + 21*z220^180 + 24*z220^179
+ 33*z220^178 + 19*z220^177 + 7*z220^176 + 36*z220^175
+ 30*z220^174 + 34*z220^173 + 27*z220^172 + 3*z220^171
+ 34*z220^170 + 5*z220^169 + 36*z220^168 + 19*z220^167
+ 27*z220^166 + 14*z220^165 + 10*z220^164 + 2*z220^163
+ 31*z220^162 + 22*z220^161 + 7*z220^160 + 14*z220^159
+ 5*z220^158 + 3*z220^157 + 22*z220^156 + 32*z220^155
+ 21*z220^154 + 17*z220^153 + 34*z220^152 + 9*z220^151
+ 33*z220^150 + 32*z220^149 + 24*z220^148 + 16*z220^147
+ 19*z220^146 + 6*z220^145 + 26*z220^144 + 24*z220^143
+ 34*z220^141 + 25*z220^140 + 17*z220^139 + 25*z220^138
+ 19*z220^137 + 36*z220^136 + 7*z220^134 + 32*z220^133
+ 24*z220^132 + 6*z220^131 + 12*z220^130 + 30*z220^129
+ 35*z220^128 + 13*z220^127 + 29*z220^126 + 2*z220^125
+ 24*z220^124 + 36*z220^123 + 34*z220^122 + 2*z220^121
+ 33*z220^120 + 10*z220^119 + 33*z220^118 + 2*z220^117
+ 17*z220^116 + 33*z220^115 + 14*z220^114 + 22*z220^113
+ 27*z220^112 + 20*z220^111 + 23*z220^110 + 34*z220^109
+ 6*z220^108 + 33*z220^107 + 14*z220^106 + 28*z220^105
+ 29*z220^104 + 36*z220^103 + 22*z220^102 + 35*z220^101
+ 8*z220^100 + 10*z220^99 + 10*z220^98 + 16*z220^97
+ 19*z220^96 + 17*z220^95 + 21*z220^94 + 13*z220^93
+ 24*z220^92 + 36*z220^91 + 25*z220^90 + 25*z220^89
+ 22*z220^88 + 27*z220^87 + 28*z220^86 + 11*z220^85
+ 3*z220^84 + 14*z220^82 + 31*z220^81 + 7*z220^80
+ 33*z220^79 + 33*z220^78 + 2*z220^77 + 15*z220^76
+ 17*z220^75 + 32*z220^74 + 4*z220^73 + 18*z220^72
+ 10*z220^71 + 34*z220^70 + 9*z220^69 + 3*z220^68
+ 20*z220^67 + 33*z220^66 + 23*z220^65 + 5*z220^64
+ 20*z220^63 + 36*z220^62 + 29*z220^61 + 2*z220^60
+ 25*z220^59 + 14*z220^58 + 16*z220^57 + 31*z220^56
+ 22*z220^55 + 31*z220^54 + 33*z220^53 + 19*z220^52
+ 22*z220^51 + 23*z220^50 + 36*z220^49 + 11*z220^48
+ 15*z220^47 + 15*z220^46 + 35*z220^45 + 7*z220^44
+ 27*z220^43 + 28*z220^42 + 15*z220^41 + 31*z220^40
+ 12*z220^39 + 19*z220^38 + 21*z220^37 + 18*z220^36
+ 3*z220^35 + 36*z220^33 + z220^32 + 35*z220^31
+ 21*z220^30 + 2*z220^29 + 13*z220^28 + 19*z220^27
+ 6*z220^26 + 22*z220^24 + 26*z220^23 + 9*z220^22

```
+ 7*z220^21 + 31*z220^20 + 31*z220^19 + 9*z220^18
+ 23*z220^17 + 23*z220^16 + 6*z220^15 + 27*z220^14
+ 36*z220^13 + 4*z220^12 + 26*z220^11 + 30*z220^10
+ 9*z220^9 + 8*z220^8 + 15*z220^7 + 26*z220^6
+ 17*z220^5 + 29*z220^4 + 24*z220^3 + 8*z220^2
+ 29*z220 : 1)
```

Q_3 is given by

```
(35*z220^219 + 22*z220^218 + 36*z220^216 + 24*z220^215
+ 19*z220^214 + 32*z220^213 + 13*z220^212 + 19*z220^211
+ 3*z220^210 + 36*z220^209 + 29*z220^208 + 35*z220^206
+ 31*z220^205 + 32*z220^204 + 23*z220^203 + 21*z220^202
+ 10*z220^201 + 32*z220^200 + 32*z220^199 + 21*z220^198
+ 16*z220^197 + 23*z220^196 + 32*z220^195 + 12*z220^194
+ 9*z220^193 + 35*z220^192 + 8*z220^191 + 19*z220^190
+ 33*z220^189 + 13*z220^188 + 11*z220^187 + 35*z220^186
+ 25*z220^185 + 28*z220^184 + 5*z220^183 + 7*z220^182
+ 24*z220^181 + 35*z220^180 + 33*z220^179 + 18*z220^178
+ 5*z220^177 + 31*z220^176 + 18*z220^175 + 30*z220^174
+ 27*z220^173 + 3*z220^172 + 8*z220^171 + 24*z220^170
+ 14*z220^169 + 2*z220^168 + 16*z220^167 + 14*z220^166
+ 18*z220^165 + 22*z220^164 + 32*z220^163 + 28*z220^162
+ 7*z220^161 + 19*z220^160 + 3*z220^159 + 14*z220^158
+ 27*z220^157 + 35*z220^156 + 8*z220^155 + 25*z220^154
+ 11*z220^153 + 19*z220^152 + 21*z220^151 + 10*z220^150
+ 2*z220^149 + 4*z220^148 + 4*z220^147 + 31*z220^146
+ 26*z220^145 + 17*z220^143 + 14*z220^142 + 12*z220^141
+ 17*z220^140 + 22*z220^139 + 30*z220^138 + 30*z220^137
+ 15*z220^136 + 16*z220^135 + 25*z220^134 + 8*z220^133
+ 28*z220^132 + 5*z220^131 + 14*z220^130 + 26*z220^129
+ 13*z220^128 + 10*z220^127 + 13*z220^126 + 10*z220^125
+ 17*z220^124 + 33*z220^123 + 9*z220^122 + 9*z220^121
+ 10*z220^120 + 12*z220^119 + 4*z220^118 + 6*z220^117
+ 33*z220^116 + 21*z220^115 + 14*z220^114 + 33*z220^113
+ 11*z220^112 + 4*z220^111 + 3*z220^110 + 3*z220^109
+ 3*z220^108 + 3*z220^107 + 27*z220^106 + 8*z220^105
+ 25*z220^104 + 10*z220^103 + 24*z220^102 + 2*z220^101
+ 12*z220^100 + 35*z220^99 + 30*z220^98 + 14*z220^97
+ 8*z220^96 + 16*z220^95 + 24*z220^94 + 23*z220^93
+ 34*z220^91 + 3*z220^90 + 13*z220^89 + 10*z220^88
+ 20*z220^87 + 14*z220^86 + 9*z220^85 + 36*z220^84
+ 33*z220^83 + 12*z220^82 + 20*z220^81 + 5*z220^80
+ 27*z220^79 + 27*z220^78 + 9*z220^77 + 23*z220^76
+ 4*z220^75 + 26*z220^74 + 8*z220^73 + 11*z220^72
```

```
+ 25*z220^71 + 35*z220^70 + 19*z220^69 + 36*z220^68
+ 35*z220^67 + 24*z220^66 + 8*z220^65 + 32*z220^64
+ 10*z220^63 + 3*z220^62 + 18*z220^61 + 35*z220^60
+ 17*z220^59 + 30*z220^58 + 2*z220^57 + 25*z220^56
+ 7*z220^55 + 20*z220^54 + 27*z220^53 + z220^52
+ 10*z220^51 + 2*z220^50 + 18*z220^49 + 30*z220^48
+ 32*z220^47 + 20*z220^46 + 4*z220^45 + 16*z220^43
+ 16*z220^42 + 11*z220^41 + 8*z220^40 + 12*z220^39
+ 15*z220^38 + 25*z220^37 + 33*z220^36 + 4*z220^35
+ 11*z220^34 + 6*z220^33 + 7*z220^32 + 32*z220^31
+ 19*z220^30 + 19*z220^29 + 16*z220^28 + 10*z220^27
+ 7*z220^26 + 10*z220^25 + 33*z220^24 + 25*z220^23
+ 21*z220^22 + 35*z220^21 + 15*z220^20 + z220^19
+ 19*z220^18 + 16*z220^17 + 10*z220^16 + 18*z220^15
+ 17*z220^14 + 2*z220^13 + 35*z220^12 + 30*z220^11
+ 17*z220^10 + 30*z220^9 + 26*z220^8 + 9*z220^7
+ 34*z220^6 + 4*z220^5 + 12*z220^4 + 16*z220^3
+ 27*z220^2 + 12*z220 + 36

: 21*z220^219 + 24*z220^218
+ 33*z220^217 + 31*z220^216 + 29*z220^215 + 16*z220^214
+ 26*z220^213 + 7*z220^212 + 15*z220^211 + 9*z220^210
+ 19*z220^209 + 18*z220^208 + 16*z220^207 + 23*z220^206
+ 27*z220^205 + 16*z220^204 + 5*z220^203 + 10*z220^202
+ 2*z220^201 + 19*z220^200 + 19*z220^199 + 8*z220^198
+ 30*z220^197 + 9*z220^196 + 27*z220^195 + 7*z220^194
+ 20*z220^193 + 8*z220^192 + 29*z220^191 + 10*z220^190
+ 32*z220^189 + 9*z220^188 + 4*z220^187 + 31*z220^186
+ 8*z220^185 + 4*z220^184 + 8*z220^183 + 11*z220^182
+ 13*z220^181 + 5*z220^180 + 29*z220^179 + 13*z220^178
+ 20*z220^177 + 9*z220^176 + 3*z220^175 + 32*z220^174
+ 3*z220^173 + 25*z220^172 + 33*z220^171 + 36*z220^170
+ 11*z220^169 + 22*z220^168 + 18*z220^167 + 7*z220^166
+ 4*z220^165 + 9*z220^164 + 33*z220^163 + 33*z220^162
+ 18*z220^161 + 3*z220^160 + 35*z220^159 + 31*z220^158
+ 20*z220^157 + 28*z220^155 + 33*z220^154 + 30*z220^153
+ 28*z220^152 + 18*z220^151 + z220^150 + 34*z220^149
+ 16*z220^148 + 23*z220^147 + 30*z220^146 + 3*z220^144
+ 28*z220^143 + 8*z220^142 + 35*z220^140 + 11*z220^139
+ 16*z220^138 + 20*z220^137 + 31*z220^136 + 11*z220^135
+ 24*z220^134 + 29*z220^133 + 29*z220^132 + 8*z220^131
+ 25*z220^130 + 11*z220^129 + 35*z220^128 + 36*z220^127
+ 33*z220^126 + 18*z220^125 + 8*z220^124 + 9*z220^123
```

```
+ 31*z220^122 + 29*z220^121 + 7*z220^120 + 4*z220^119
+ 3*z220^118 + 13*z220^117 + 35*z220^116 + 17*z220^115
+ 6*z220^114 + 3*z220^113 + 13*z220^112 + 5*z220^111
+ 31*z220^110 + 32*z220^109 + 17*z220^108 + 28*z220^107
+ 21*z220^106 + 14*z220^105 + 25*z220^104 + 17*z220^103
+ 33*z220^102 + 19*z220^101 + 4*z220^100 + 2*z220^99
+ 7*z220^98 + 34*z220^97 + 15*z220^96 + 7*z220^95
+ 34*z220^94 + 22*z220^93 + 22*z220^92 + 11*z220^91
+ 33*z220^90 + 32*z220^89 + 19*z220^88 + 21*z220^87
+ 23*z220^86 + 34*z220^85 + 35*z220^84 + 23*z220^83
+ 27*z220^82 + 25*z220^81 + 26*z220^80 + 2*z220^79
+ 33*z220^78 + 32*z220^77 + 8*z220^76 + 32*z220^75
+ 15*z220^74 + 17*z220^73 + 31*z220^72 + 7*z220^71
+ 8*z220^70 + 8*z220^69 + 22*z220^68 + 7*z220^67
+ 14*z220^66 + 15*z220^65 + 26*z220^64 + 26*z220^63
+ 35*z220^62 + 19*z220^61 + 18*z220^60 + 22*z220^59
+ 25*z220^57 + 4*z220^56 + 5*z220^55 + 4*z220^54
+ 20*z220^53 + 32*z220^52 + 17*z220^51 + 14*z220^50
+ 31*z220^49 + 9*z220^48 + 30*z220^47 + 20*z220^46
+ 7*z220^45 + 16*z220^43 + 23*z220^42 + 12*z220^41
+ 21*z220^40 + 14*z220^39 + 8*z220^38 + 14*z220^37
+ 35*z220^36 + 14*z220^35 + 22*z220^34 + 8*z220^33
+ z220^32 + 24*z220^31 + 21*z220^30 + 33*z220^29
+ 21*z220^28 + 22*z220^26 + 33*z220^25 + 13*z220^24
+ 13*z220^23 + 5*z220^22 + 35*z220^21 + 3*z220^20
+ 31*z220^19 + 13*z220^18 + 33*z220^17 + 30*z220^16
+ 16*z220^15 + 30*z220^14 + 16*z220^13 + 11*z220^12
+ 35*z220^11 + 22*z220^10 + 11*z220^9 + 8*z220^8
+ z220^7 + 25*z220^6 + 8*z220^5 + 27*z220^4 + z220^3
+ 29*z220^2 + 34*z220 + 29 : 1)
```

References

[Boo+22] Jeremy Booher et al. *Failing to hash into supersingular isogeny graphs*. Cryptology ePrint Archive, Report 2022/518. `https://ia.cr/2022/518`. 2022.