Inspire…Educate…Transform.

Statistics and Probability in Decision Modeling

Regularization, Time Series Forecasting, Naïve Bayes

Dr. Sridhar Pappu
Executive VP – Academics, INSOFE

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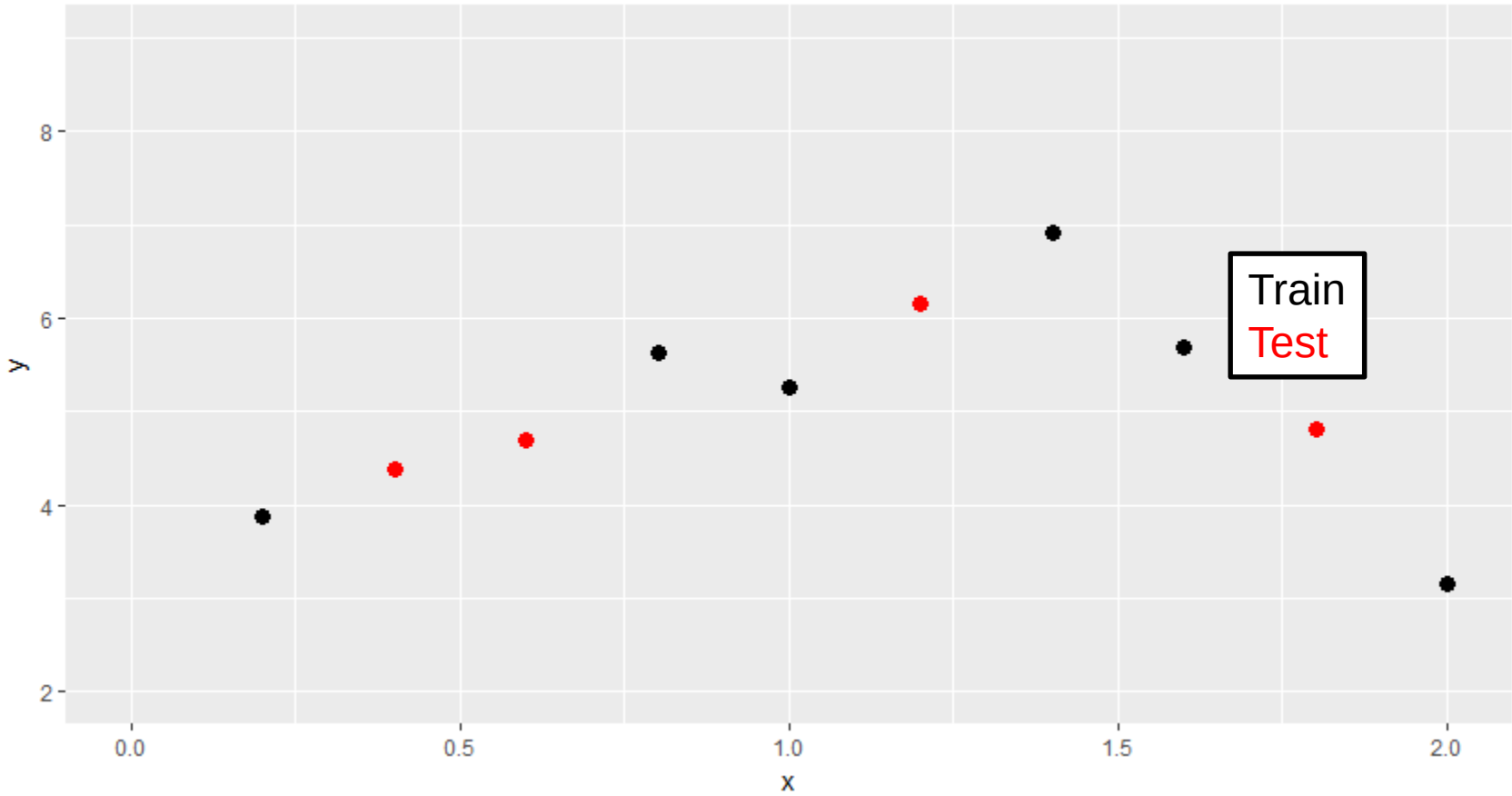
Regularization and Naïve Bayes slides courtesy Dr. Anand Jayaraman

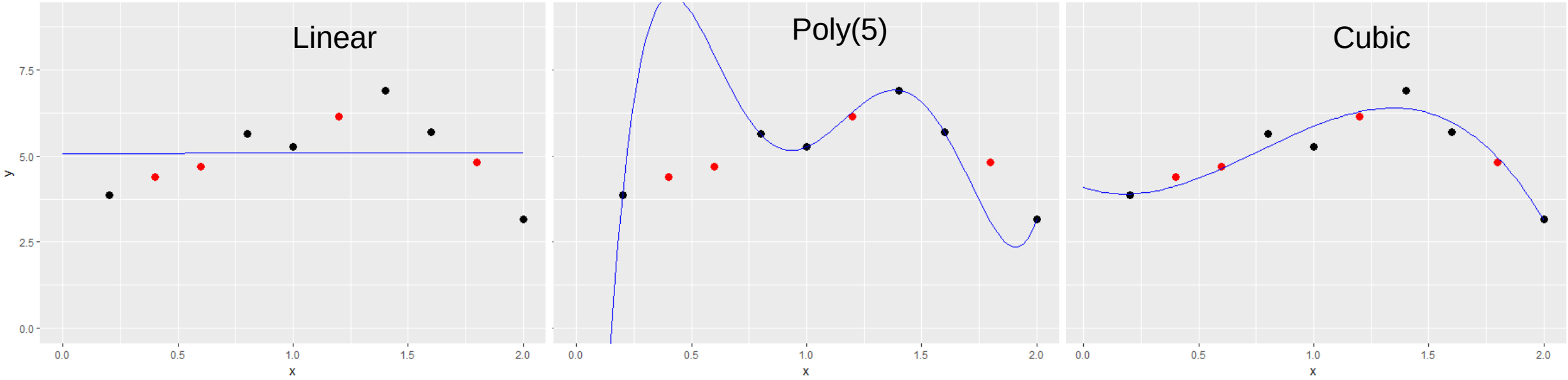
Advanced Regression Methods: Ridge Regression, LASSO Regression, Elastic Nets

REGULARIZATION



Decision point: What order polynomial should we use for the fit?

Problem with Overfitting



Underfit
TrainErr=1.23
TestErr=0.7

Overfit
TrainErr=0
TestErr=3.2

Right fit
TrainErr=0.38
TestErr=0.17

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Overfitting Problem

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \cdots$$

How many β_i do we need for a reasonable fit, while at the same time minimizing the risk of overfitting?

In general, given a regression problem with several features, is there a way to determine how many β_i do we need for a reasonable fit, while at the same time minimizing the risk of overfitting?

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \cdots$$

Regularization

- Standard Least Squares Regression involves minimizing SSE

$$\text{SSE} = \sum_{j=1}^{N} (y_j - \sum_{i=0}^{d} \beta_i \cdot x_i)^2 = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^{\mathrm{T}} \cdot (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$$

$$\mathbf{y} = \text{vector of all training responses } y_j$$

$$\mathbf{X} = \text{matrix of all training samples } \mathbf{x}_j$$

- We can reduce overfitting by penalizing large coefficients (**must standardize** in order to be able to compare/add βs)

Ridge Regression

$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\sum_{i=1}^{d}\beta_i^{\,2}\right)$$

- Penalization of large coefficients through L2 norm (another name for Least Squares)
- Regularization parameter λ controls the extent of penalization
- $\lambda = 0$ corresponds to Least Squares Regression

Ridge Regression

- Occam's Razor

$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\sum_{i=1}^{d}\beta_i^{\,2}\right)$$

- Least Squares Regression
  - Fails when the number of features *p* is larger than number of data points *n*, e.g., in genetics studies. For example, there are infinite solutions for $5=\beta_1*3+\beta_2*2$ (two unknowns and one equation)

Ridge Regression

$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\sum_{i=1}^{d}\beta_i^{\,2}\right)$$

- Least Squares Regression
  - Fails when we have perfect multicollinearity

y	x1	x2
0	0	0
2	1	1
4	2	2
6	3	3

$y = x_1 + x_2$ or $y = 0.5x_1 + 1.5x_2$ or $y = 2x_1$ and so on

- Ridge Regression allows for a unique solution even in above situation. Which equation will you choose if you add the constraint to minimize $\sum_{i=1}^{d} b_i^{\,2}$ as well?

Ridge Regularization

The penalty term “regularizes” the problem when there is perfect multicollinearity

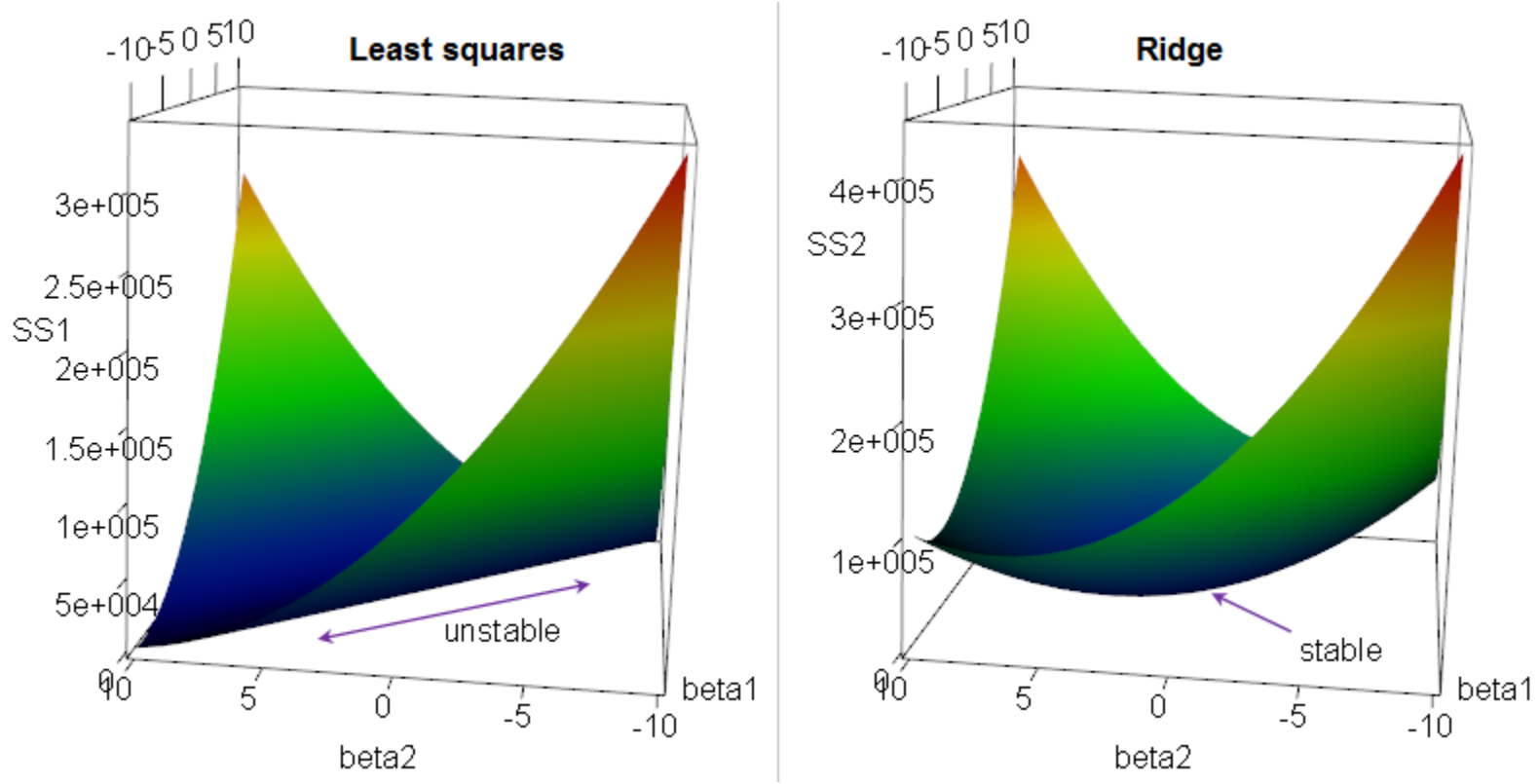
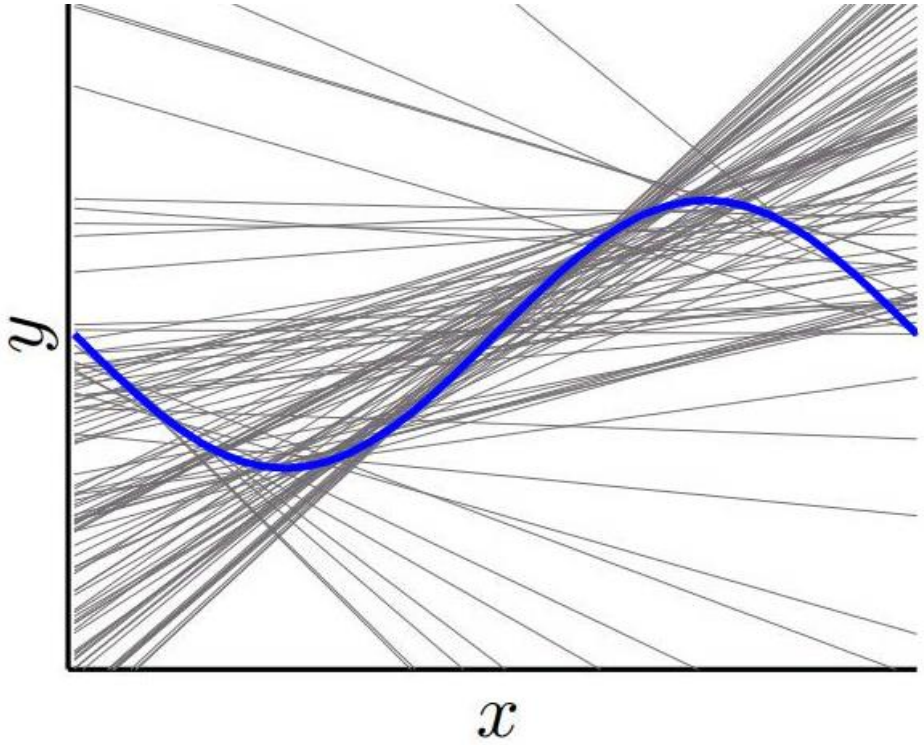
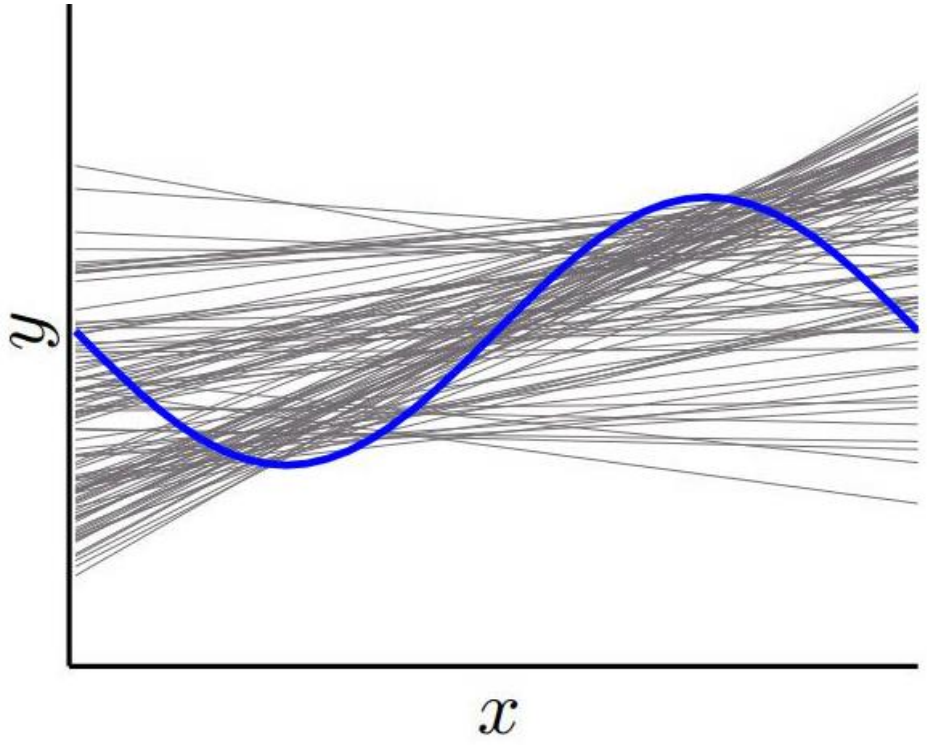


Image Source: https://stats.stackexchange.com/questions/151304/why-is-ridge-regression-called-ridge-why-is-it-needed-and-what-happens-when

Effect of Regularization



without regularization

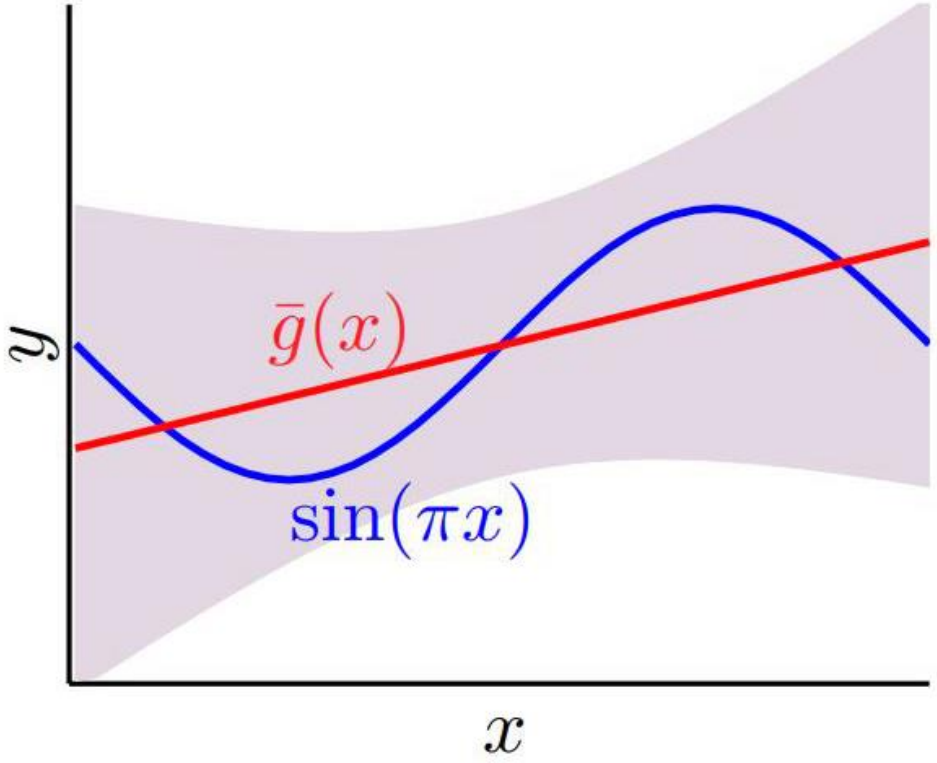


with regularization

Source: http://work.caltech.edu/slides/slides12.pdf

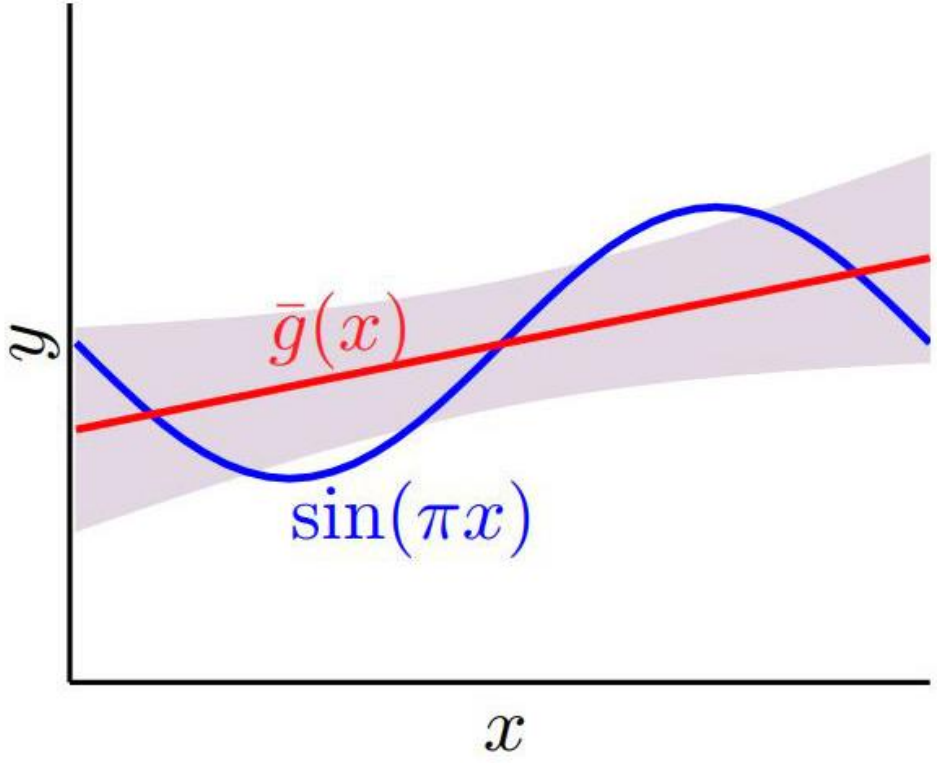
Effect of Regularization

without regularization



bias $= \mathbf{0.21}$ var $= \mathbf{1.69}$

with regularization



bias $= \mathbf{0.23}$ var $= \mathbf{0.33}$

Source: http://work.caltech.edu/slides/slides12.pdf

LASSO Regularization

$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\sum_{i=1}^{d}|\beta_i|\right)$$

- Penalization of large coefficients through L1 norm (another name for Least Absolute Error)
- Least Absolute Shrinkage and Selection Operator (LASSO)

LASSO Regularization

$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\sum_{i=1}^{d}|\beta_i|\right)$$

y	x1	x2
0	0	0
2	1	1
4	2	2
6	3	3

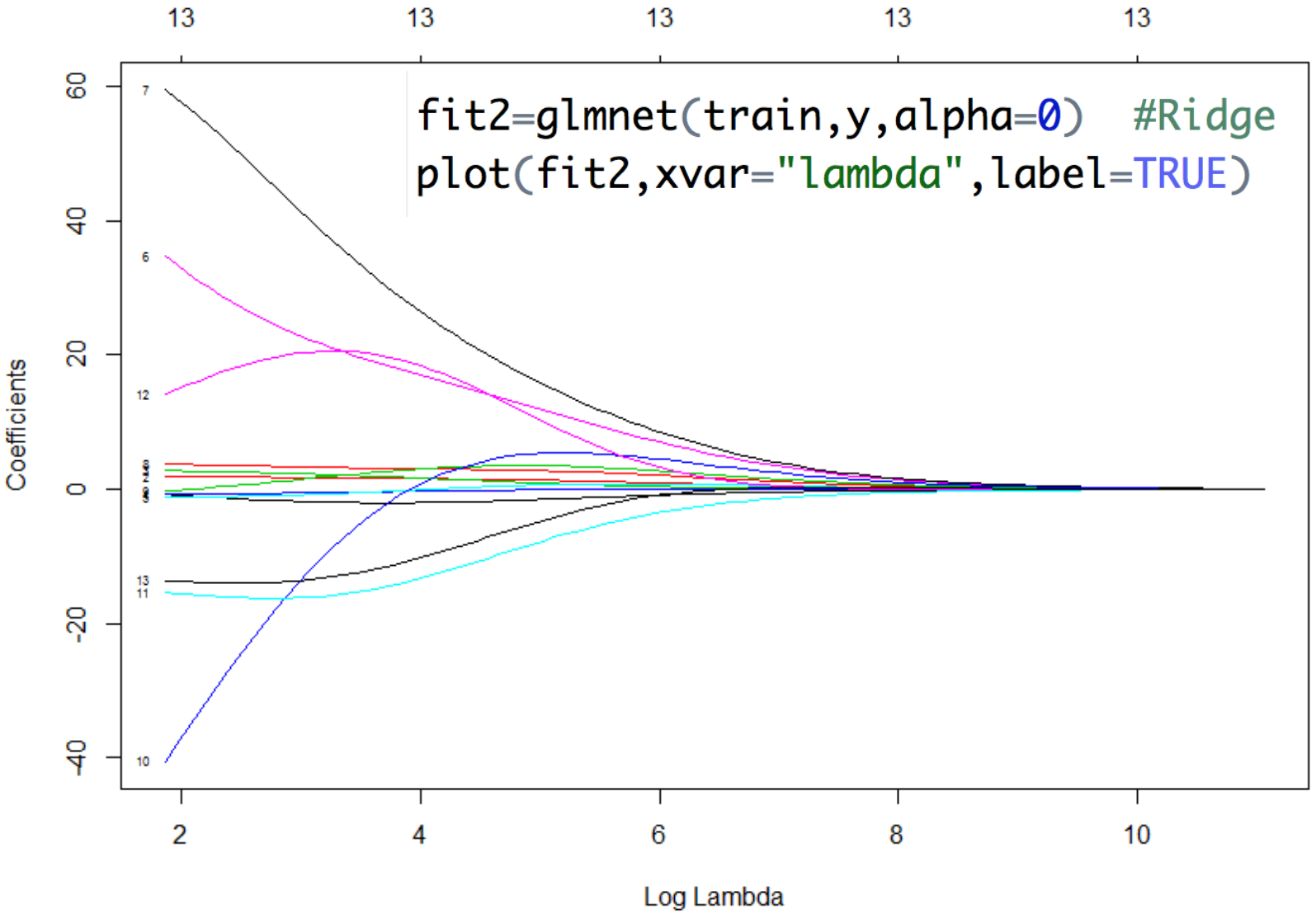
Not useful when there is strong multicollinearity.

$y = x_1 + x_2$ or $y = 0.5x_1 + 1.5x_2$ or $y = 2x_1$ and so on

Regularization – Toy Purchase Data

CustomerID	City	NoOfChildre	MinAgeOfCh	MaxAgeOfCh	Tenure	FrquncyOfPu	NoOfUnitsPu	FrequencyOF	NoOfGames	NoOfGames	FavoriteChar	FavoriteGam	TotalRevenueGenerated
1001	1	2	3	8	210	11	11	2344	108	10	Uniform	Uniform	107.51
1002	1	2	3	6	442	20	20	245	22	7	Favorite	Uniform	382.4
1003	1	4	3	5	424	18	18	1059	130	18	Favorite	Uniform	135.01
1004	1	1	6	6	261	11	9	365	34	11	Favorite	Uniform	125
1005	1	3	6	9	422	44	31	1066	102	44	Uniform	Uniform	335.05
1006	1	2	3	4	378	16	16	228	12	16	Favorite	Favorite	150
1007	1	3	8	12	369	25	15	75	2	25	Favorite	Favorite	127.5
1008	1	2	6	8	404	13	12	1488	118	13	Favorite	Uniform	122.5
1009	1	4	6	9	420	20	16	2743	163	16	Uniform	Uniform	164.96
1010	2	3	5	6	333	15	15	1967	56	15	Favorite	Uniform	112.62
1011	1	1	5	5	323	9	9	218	1	3	Favorite	Favorite	211.92
1012	1	3	4	9	167	17	17	1961	124	17	Uniform	Uniform	125.04
1013	1	1	4	4	381	15	15	267	51	15	Favorite	Uniform	167.5
1014	1	4	5	7	399	36	24	6411	355	36	Favorite	Uniform	202.5
1015	1	2	4	7	444	14	14	2804	133	14	Uniform	Uniform	105.5
1016	1	2	3	5	371	6	6	1746	61	6	Favorite	Uniform	117.5
1017	1	1	6	6	355	11	11	1367	49	11	Favorite	Uniform	175
1018	1	1	7	7	346	18	16	1029	106	14	Uniform	Uniform	181.71
1019	1	2	6	7	211	7	7	428	26	0	Favorite	Uniform	144.93
1020	1	2	4	7	278	12	12	1080	49	8	Uniform	Uniform	182.36
1021	1	3	4	10	448	6	6	1016	56	6	Favorite	Uniform	100
1022	1	2	3	6	447	14	14	1617	58	11	Favorite	Uniform	199.93
1023	1	1	8	8	471	18	18	2131	67	18	Favorite	Uniform	172.5
1024	1	1	4	4	189	22	18	267	33	12	Favorite	Uniform	348.09
1025	1	2	5	7	427	16	16	3781	288	16	Favorite	Uniform	185.02
1026	1	2	5	8	351	5	5	182	26	3	Favorite	Uniform	139.99
1027	1	2	6	6	453	12	10	4036	146	12	Uniform	Uniform	135
1028	1	4	3	7	418	51	41	2947	241	51	Uniform	Uniform	388.01
1029	1	3	3	13	424	17	17	1606	132	17	Favorite	Uniform	152.51
1030	1	5	2	7	444	26	26	7766	583	26	Favorite	Uniform	205.01
1031	1	1	7	7	126	8	8	515	21	8	Favorite	Uniform	100
1032	1	1	7	7	290	9	9	1233	82	8	Favorite	Uniform	144.99
1033	1	4	3	7	448	20	10	4788	370	20	Uniform	Uniform	117.5
1034	1	2	4	6	181	14	14	221	19	14	Favorite	Uniform	125

Ridge: Coefficients

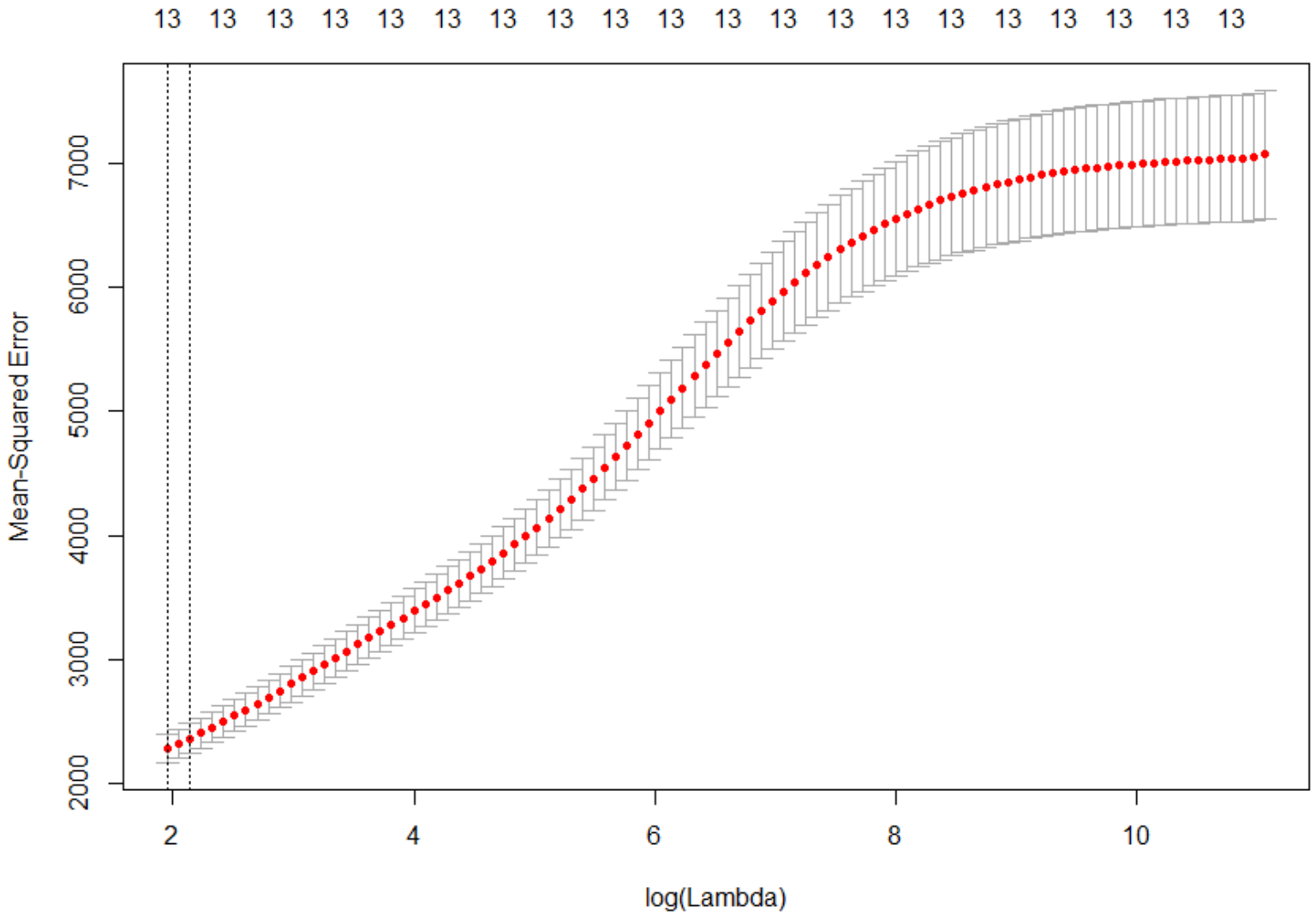


$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\sum_{i=1}^{d}\beta_i{}^2\right)$$

How to choose λ

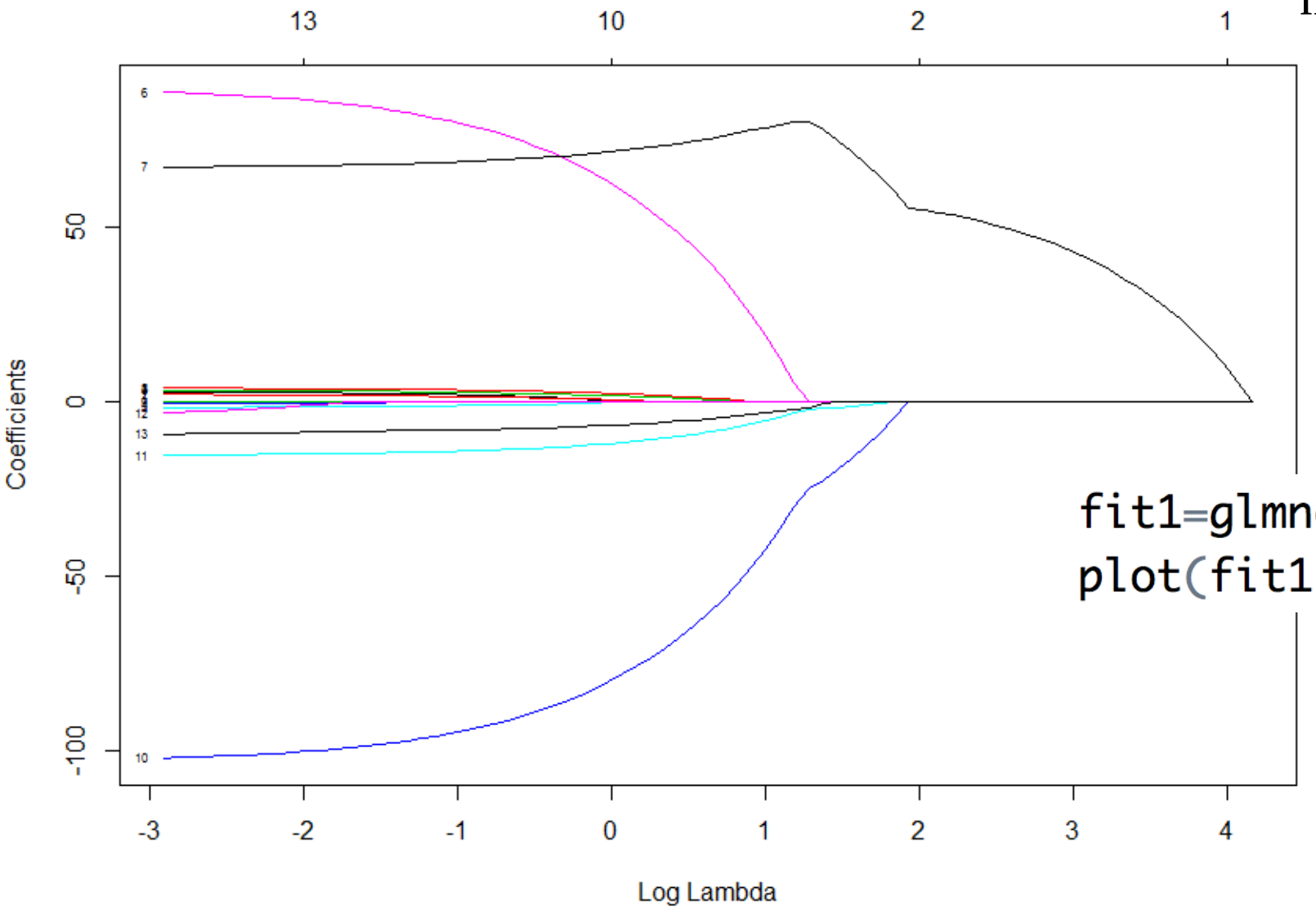
- Choose a value for λ.
- Run *k*-fold cross validation and calculate mean "squared" error on the **test dataset**.
- Repeat the above 2 steps for various λ values.
- Pick λ that gives least error. R recommends the range from min λ to 1 stdev away.
- Pick λ by looking at the coefficients corresponding to these λ values, in addition to the minimum error.

Ridge: MSE



LASSO: Coefficients

$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\sum_{i=1}^{d}|\beta_i|\right)$$




```
fit1=glmnet(train,y,alpha=1)   #LASSO
plot(fit1,xvar="lambda",label=TRUE)
```

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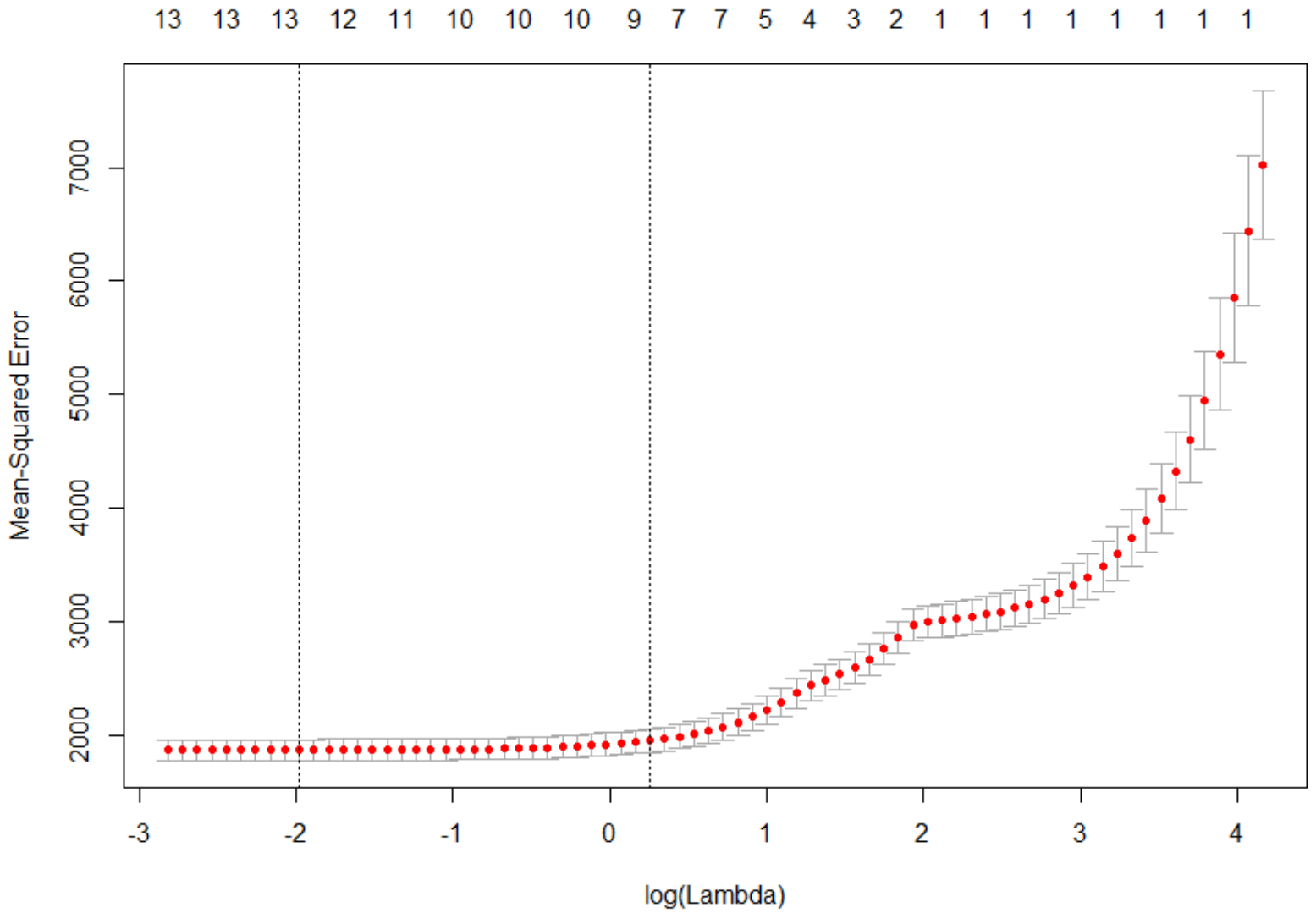
LASSO: MSE



Ridge vs Lasso Regression

Ridge	LASSO
• Reduces the magnitude of coefficients (does not make them zero) • Solution is unique – however, doesn't do feature selection since it keeps all the terms • Stable even if $n<<p$ • Very stable even in multicollinear case	• Values of irrelevant coefficients set to zero • Good for feature selection • Doesn't do well when $n<<p$ • Out of sample performance can be impacted for multicollinear case

Elastic-net

- A mix-between Ridge and LASSO regression

$$\min_{\beta}\left(\sum_{j=1}^{N}(y_j-\sum_{i=0}^{d}\beta_i\cdot x_i)^2+\lambda\left(\sum_{i=1}^{d}\left(\alpha|\beta_i|+(1-\alpha)\beta_i^{\,2}\right)\right)\right)$$

- α=0 is Ridge
- α=1 is LASSO
- α=0.5 is an equal mix of the two

- Keep in mind that Ridge regression can't zero out coefficients; thus, you either end up including all the coefficients in the model, or none of them.

- In contrast, the LASSO does both parameter shrinkage and variable selection automatically.

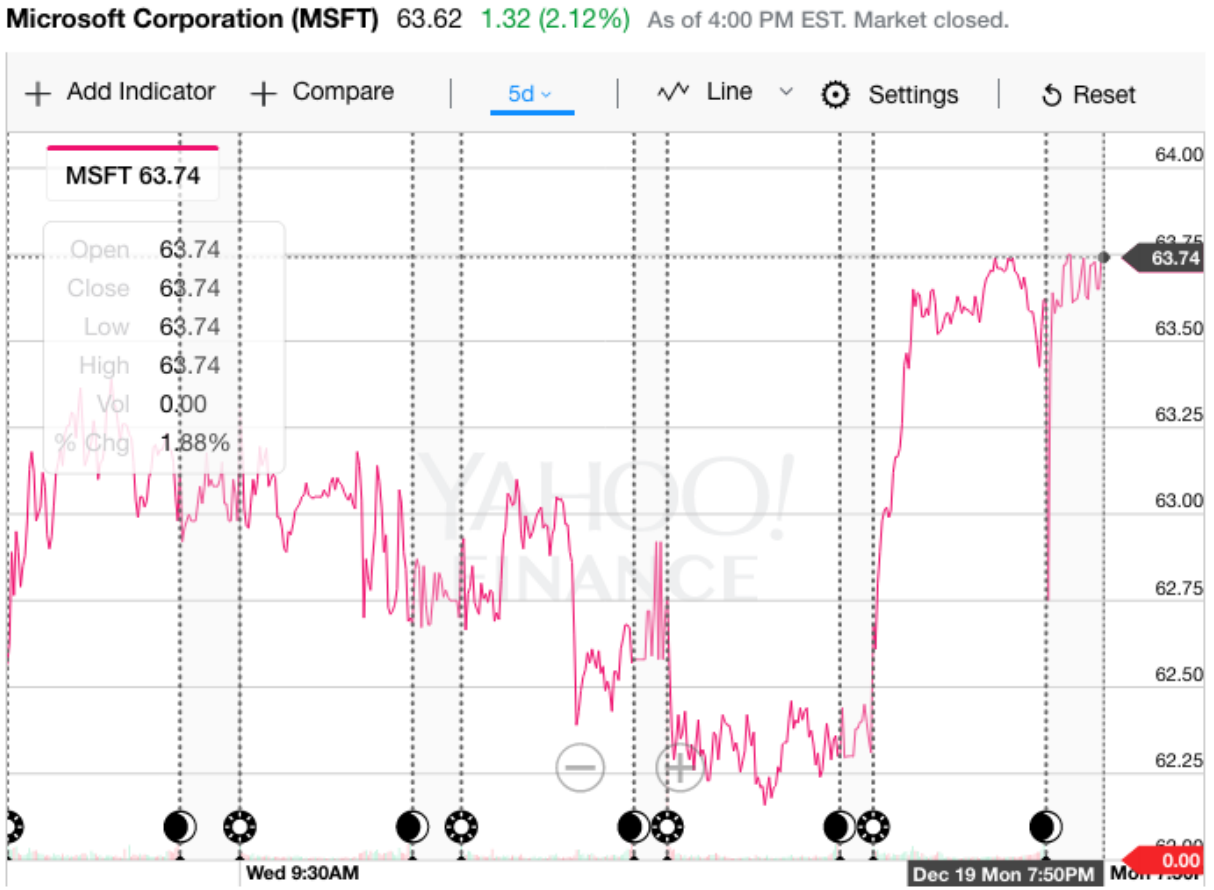
- If some of your covariates are highly correlated, you may want to look at the Elastic Net instead of the LASSO.

TIME SERIES FORECASTING

Why Time Series

Causal independent variables are

- Unknown to us
- Not available
- Might not fit the data well
- Difficult to forecast



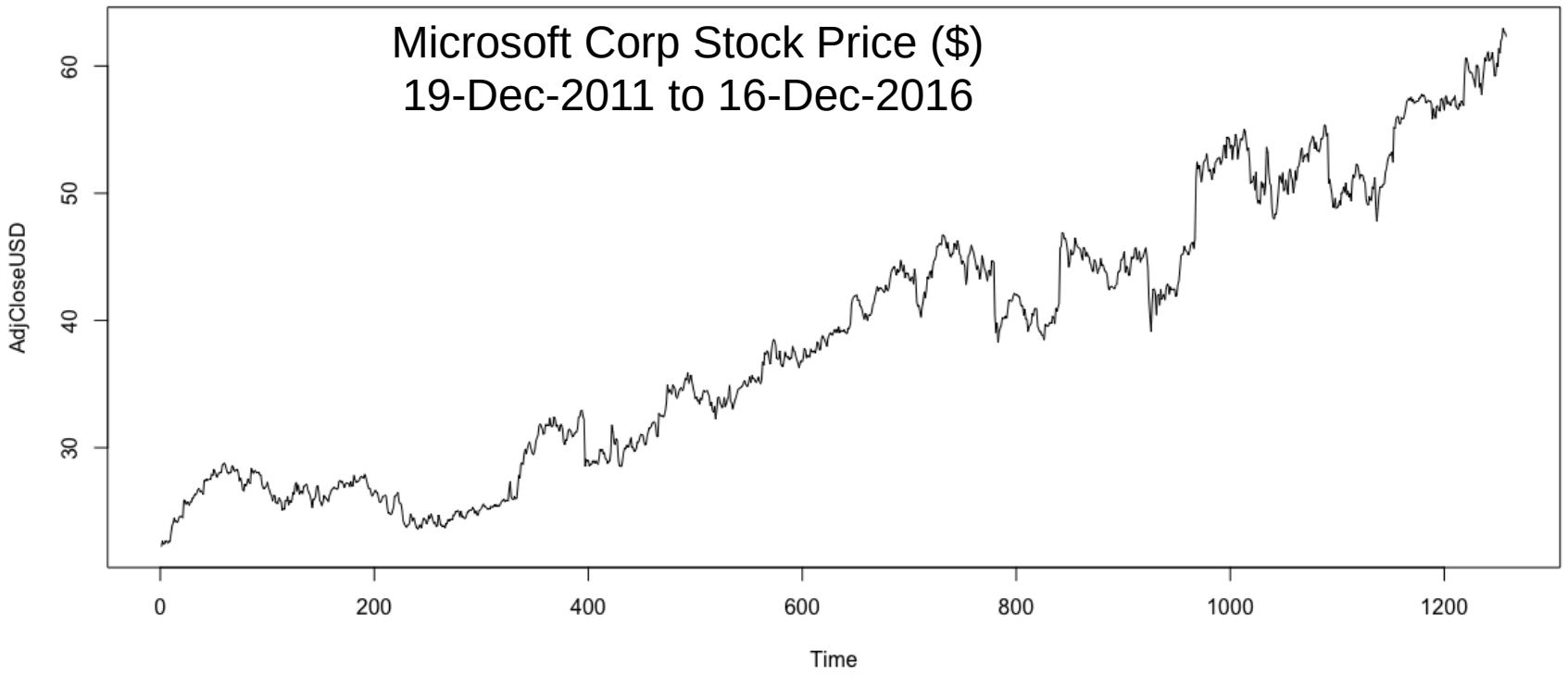
Typical Time Series

$$\hat{y}_{t+1} = f(y_t, y_{t-1}, y_{t-2} \ldots) + f(x_1, x_2, x_3 \ldots)$$

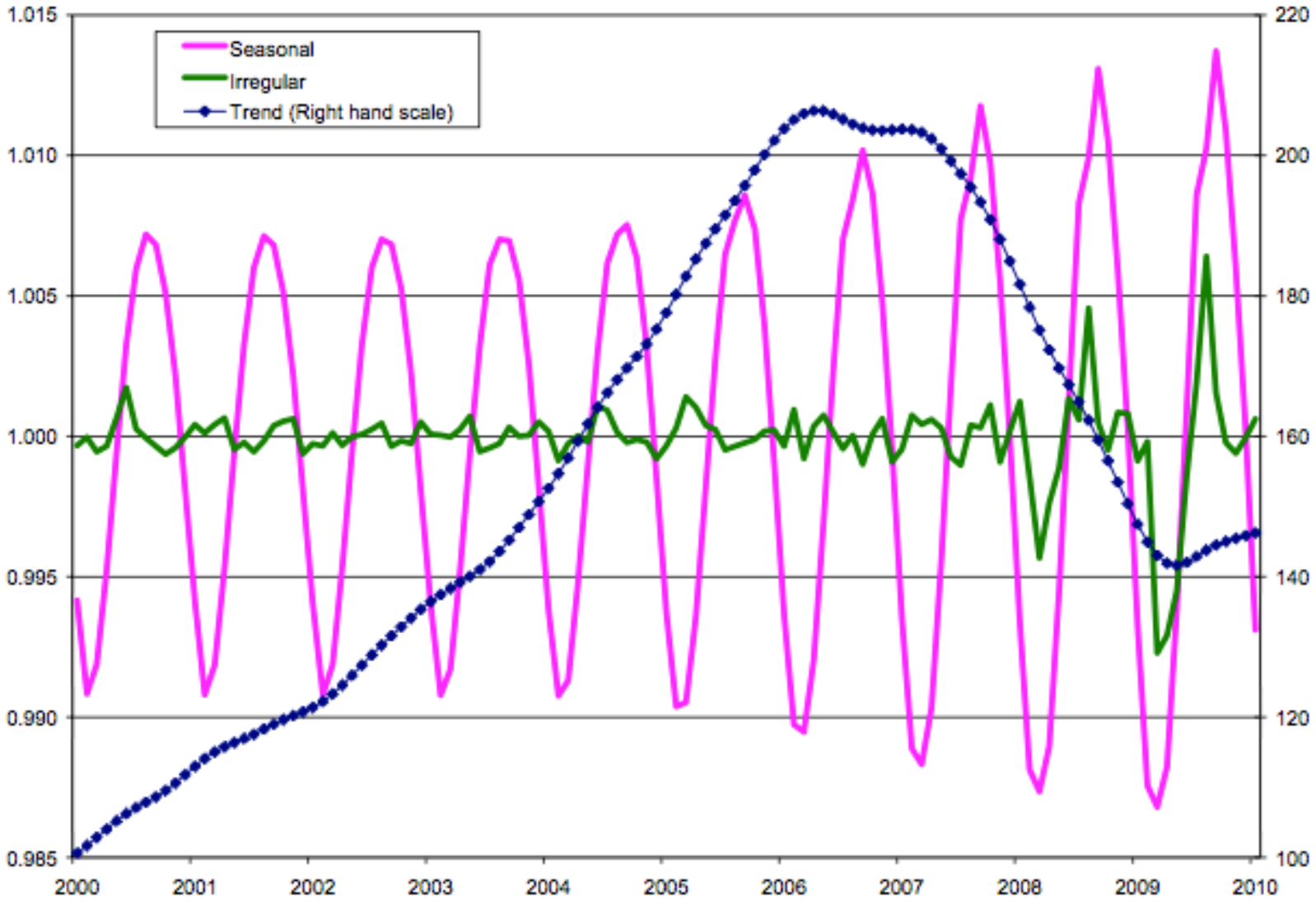
f can be linear or nonlinear

Components of Time Series

- Trend
- Seasonality
- Random component



Trend, Seasonality and Randomness



US Air Carrier Traffic – Revenue Passenger Miles ('000) *RPM*

Month	U.S. Air Carrier Traffic Statistics - Revenue Passenger Miles
1996-01	41972194
1996-02	42054796
1996-03	50443045
1996-04	47112397
1996-05	49118248
1996-06	52880510
1996-07	55664750
1996-08	57723208
1996-09	47035464
1996-10	49263120
1996-11	43937074
1996-12	48539606
1997-01	45850623
1997-02	42838949
1997-03	53620994
1997-04	49282817
1997-05	51191842
1997-06	54707221

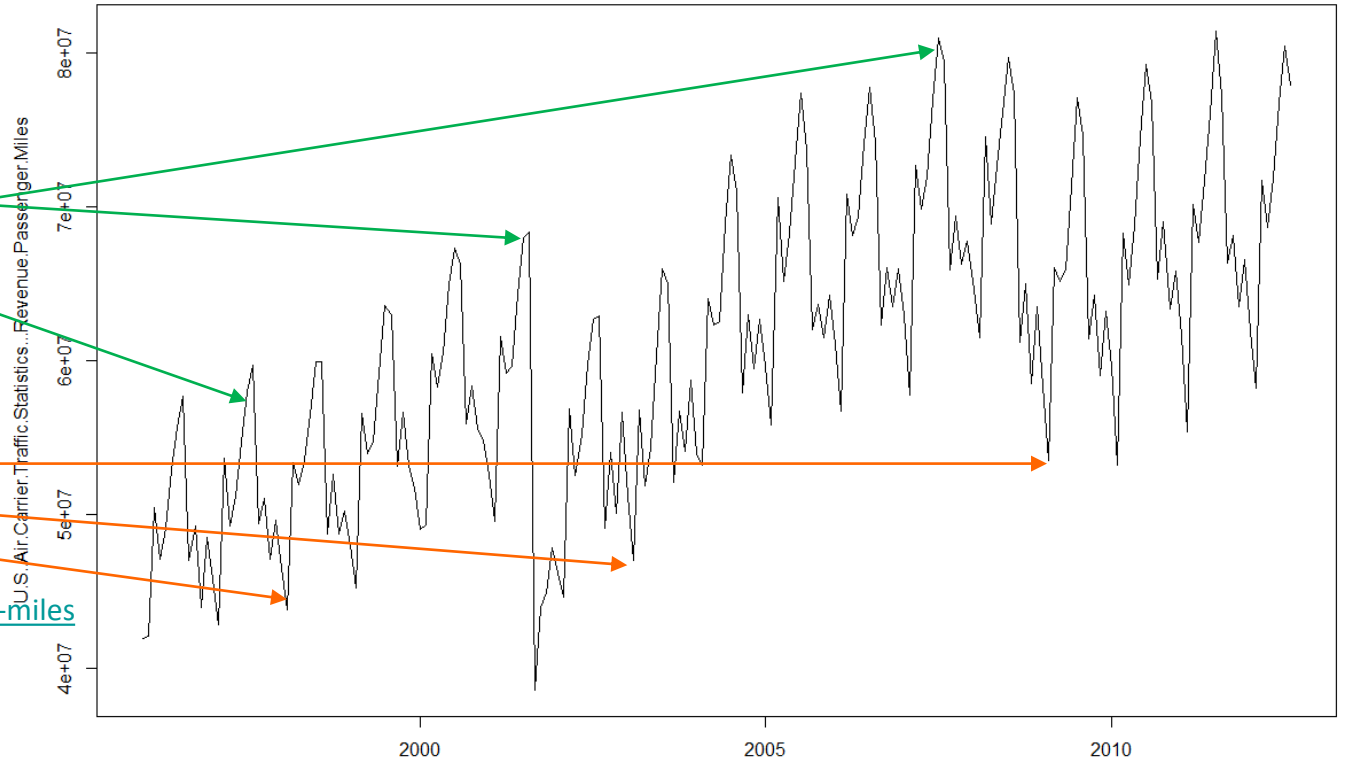
20 years of data from Jan 1996

Data sources: http://www.bts.gov/xml/air_traffic/src/index.xml and https://datamarket.com/data/set/281x/us-air-carrier-traffic-statistics-revenue-passenger-miles

US Air Carrier Traffic – Revenue Passenger Miles ('000) *RPM*

R code: milestimeseries <- ts(miles, frequency = 12, start = c(1996,1))

```
> milestimeseries <- ts(miles, frequency = 12, start = c(1996,1))
> milestimeseries
          Jan      Feb      Mar      Apr      May      Jun      Jul
1996 41972194 42054796 50443045 47112397 49118248 52880510 55664750
1997 45850623 42838949 53620994 49282817 51191842 54707221 57995025
1998 46514139 43769273 53361926 51968480 53515798 56460422 59939170
1999 47988560 45241211 56555731 53920855 54674958 59213000 63572248
2000 49045412 49306303 60443541 58286680 60533783 64903295 67346377
2001 52634354 49532578 61575055 59151645 59662416 64353323 67965298
2002 46224031 44615129 56897729 52542164 55116060 59745343 62664511
2003 51197175 47040806 56766580 51857453 54335598 60272900 65962215
2004 53979786 53179693 64035864 62340117 62530704 68866398 73335888
2005 59629608 55795165 70595861 65145552 68268899 72952959 77432998
2006 61035027 56729212 70799794 68120559 69352606 74099239 77798621
2007 63016013 57793832 72700241 69836156 71933109 76926452 80988340
2008 64667106 61504426 74575531 68906882 72725750 76162105 79707545
2009 58373786 53506580 66027341 65166300 65868254 71350227 77136799
2010 59651061 53240066 68307090 64953250 68850904 74474550 79304441
2011 61630362 55391206 70158268 67683558 71711448 76057910 81423215
2012 61940180 58243763 71696039 68669228 71887523 76760759 80499353
          Aug      Sep      Oct      Nov      Dec
1996 57723208 47035464 49263120 43937074 48539606
1997 59715433 49418190 51058879 47056048 49654209
```



Data sources: http://www.bts.gov/xml/air_traffic/src/index.xml and https://datamarket.com/data/set/281x/us-air-carrier-traffic-statistics-revenue-passenger-miles
Last accessed: 31-Mar-2016

	Aug	Sep	Oct	Nov	Dec
1996	57723208	47035464	49263120	43937074	48539606
1997	59715433	49418190	51058879	47056048	49654209
1998	59927214	48751280	52578217	48734375	50208641
1999	63003663	53131972	56653901	53215500	51746821
2000	66256804	55900504	58373996	55590325	54822970
2001	68377080	38601868	43964788	44915764	47836501
2002	62944816	49096035	54019748	50106814	56656594
2003	64989766	52121480	56724551	54128776	58739845
2004	70961522	57881042	63021142	59453943	62680310

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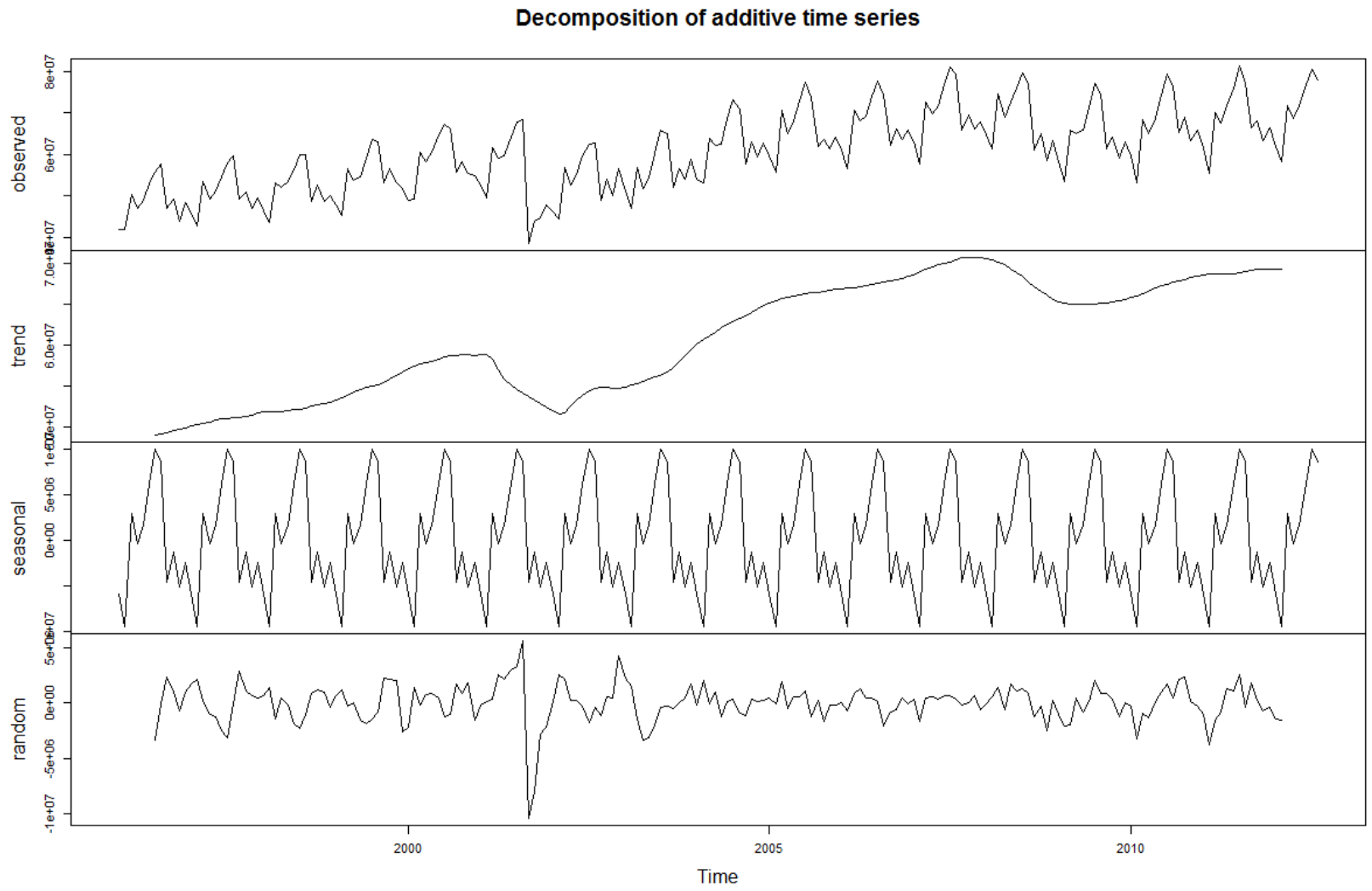
Decomposing Time Series into the 3 Components – Revenue Passenger Miles (RPM)

R code: decompose(milestimeseries, type = "additive")



Time Series

CURVE FITTING / REGRESSION ON TIME METHODS

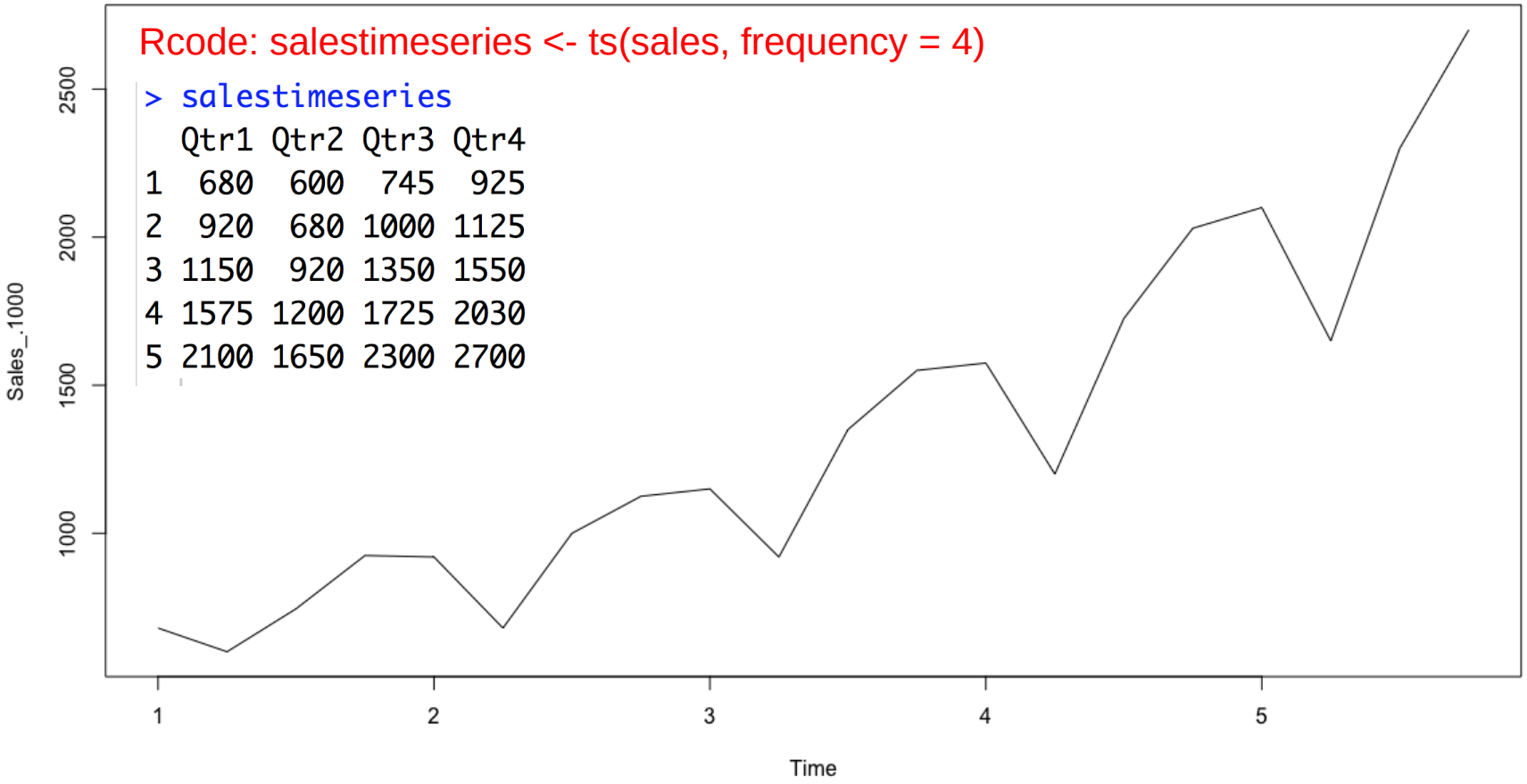
Regression on Time

Use when trend is the most pronounced

Quarter	Sales_$1000
1	680
2	600
3	745
4	925
5	920
6	680
7	1000
8	1125
9	1150
10	920
11	1350
12	1550
13	1575
14	1200
15	1725
16	2030
17	2100
18	1650
19	2300
20	2700

Rcode: salestimeseries <- ts(sales, frequency = 4)

```
> salestimeseries
  Qtr1 Qtr2 Qtr3 Qtr4
1  680  600  745  925
2  920  680 1000 1125
3 1150  920 1350 1550
4 1575 1200 1725 2030
5 2100 1650 2300 2700
```



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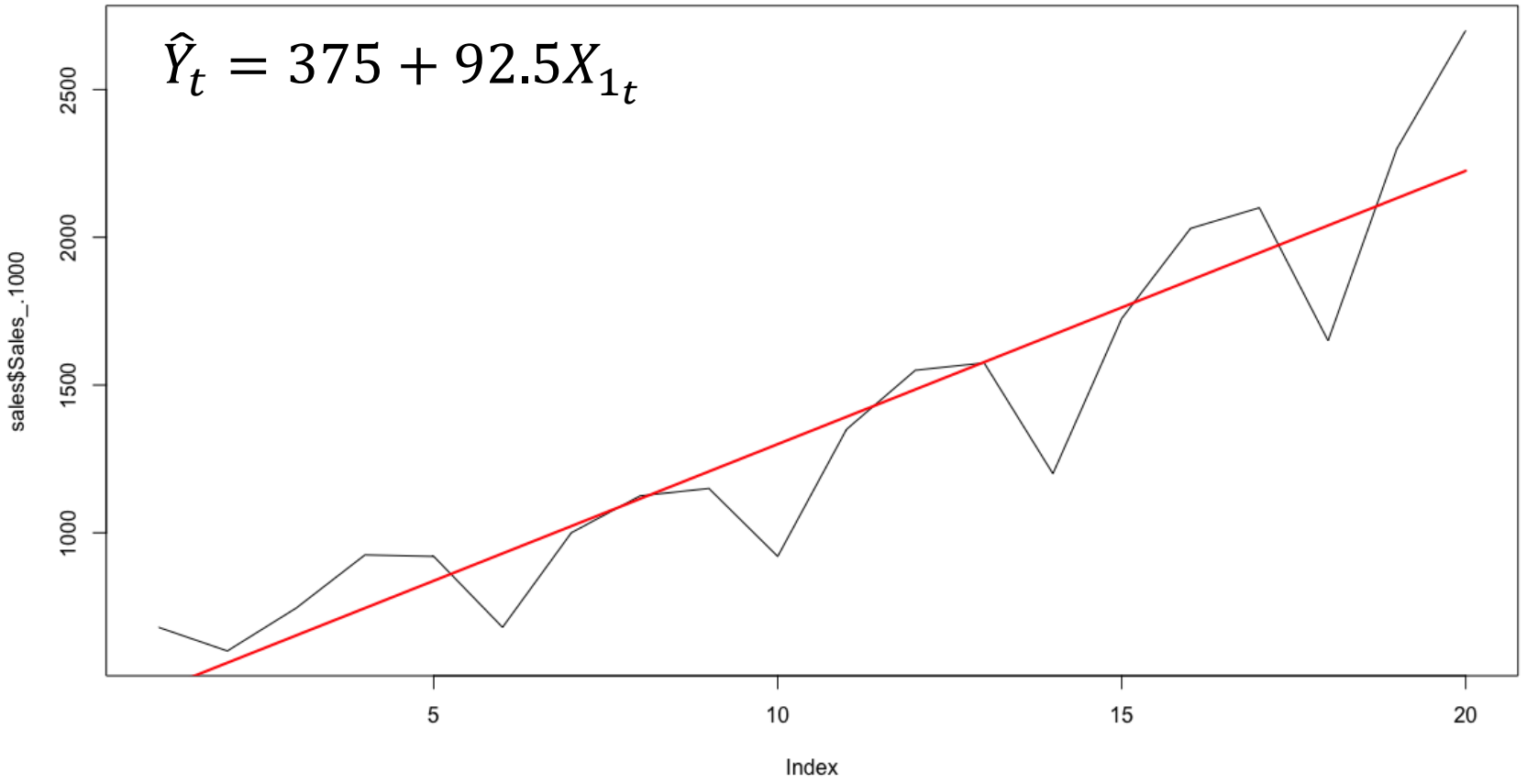
Regression Analysis

Rcode: sales <- data.frame(sales)
sales$time <- seq(1:20)
saleslm1 <- lm(sales$Sales_.1000 ~ sales$time)

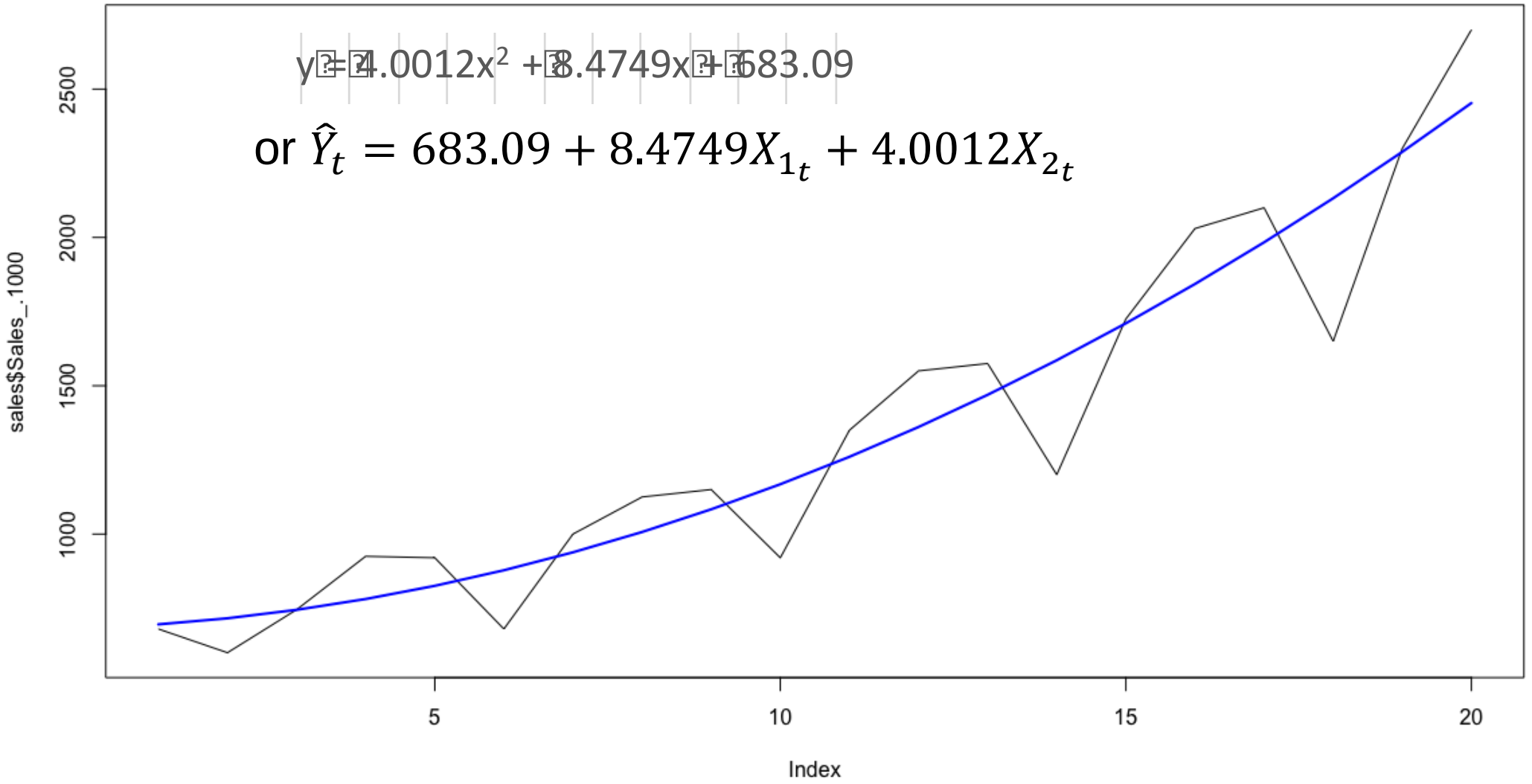
$$\hat{Y}_t = 375 + 92.5X_{1_t}$$



	Sales_.1000	time
1	680	1
2	600	2
3	745	3
4	925	4
5	920	5
6	680	6
7	1000	7
8	1125	8
9	1150	9
10	920	10
11	1350	11
12	1550	12
13	1575	13
14	1200	14
15	1725	15
16	2030	16
17	2100	17
18	1650	18
19	2300	19
20	2700	20

Quadratic Trend

Rcode: saleslm2 <- lm(sales$Sales_.1000 ~ poly(sales$time, 2, raw=TRUE))



Incorporating Seasonality – Dummy Variable Approach

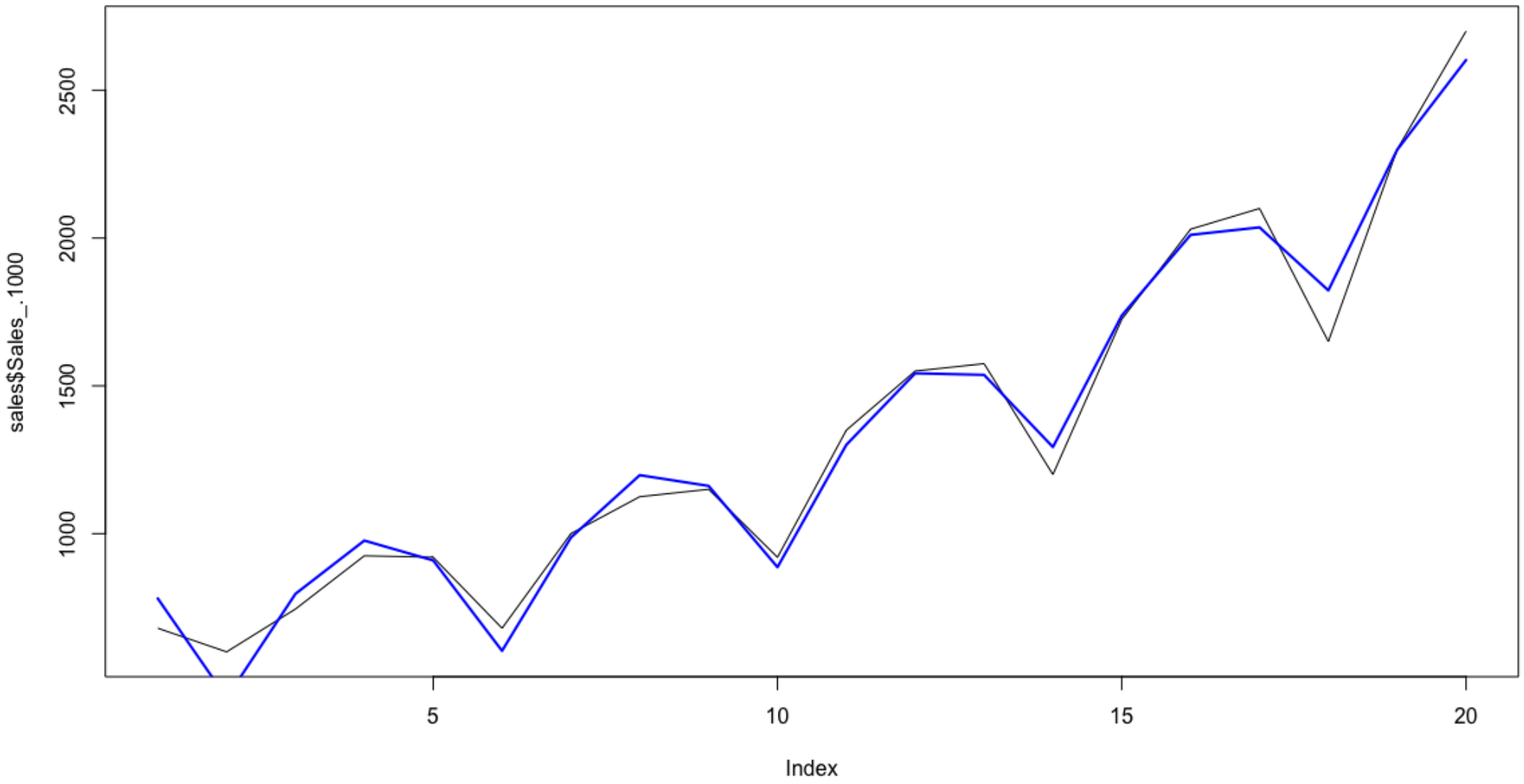
Quarter	Value of X_{3_t}	X_{4_t}	X_{5_t}
1	1	0	0
2	0	1	0
3	0	0	1
4	0	0	0

$$\hat{Y}_t = \beta_0 + \beta_1 X_{1_t} + \beta_2 X_{2_t} + \beta_3 X_{3_t} + \beta_4 X_{4_t} + \beta_5 X_{5_t} + \varepsilon_t$$

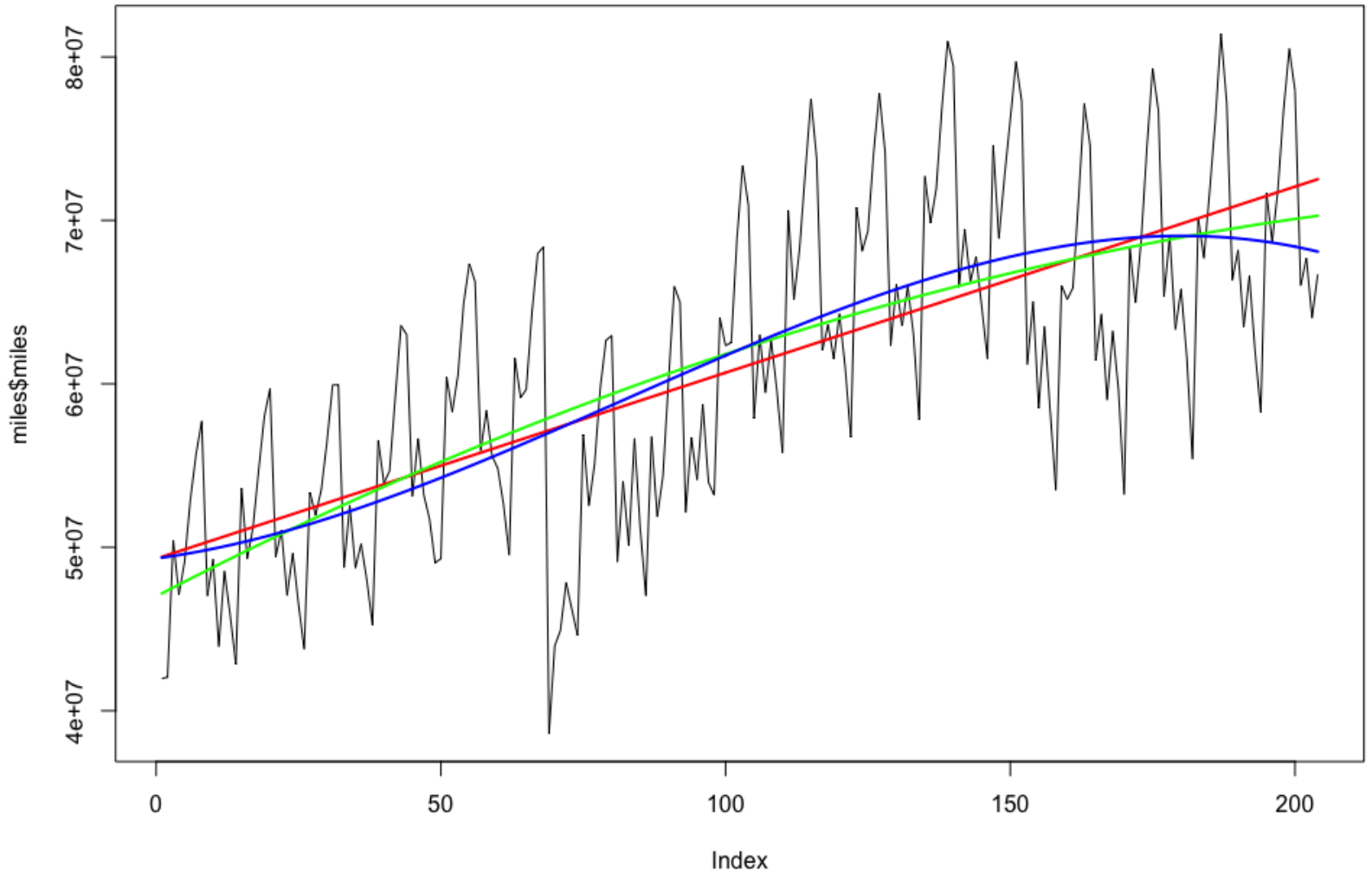
Rcode: sales$seasonal <- as.factor(rep(c(1:4),5))
saleslm2s <- lm(sales$Sales_.1000 ~ poly(time, 2, raw=TRUE)+seasonal, data=sales)

```
> sales
   Sales_.1000 time seasonal
1          680    1        1
2          600    2        2
3          745    3        3
4          925    4        4
5          920    5        1
6          680    6        2
7         1000    7        3
8         1125    8        4
9         1150    9        1
10         920   10        2
11        1350   11        3
12        1550   12        4
13        1575   13        1
14        1200   14        2
15        1725   15        3
16        2030   16        4
17        2100   17        1
18        1650   18        2
19        2300   19        3
20        2700   20        4
```

Seasonal Regression Models

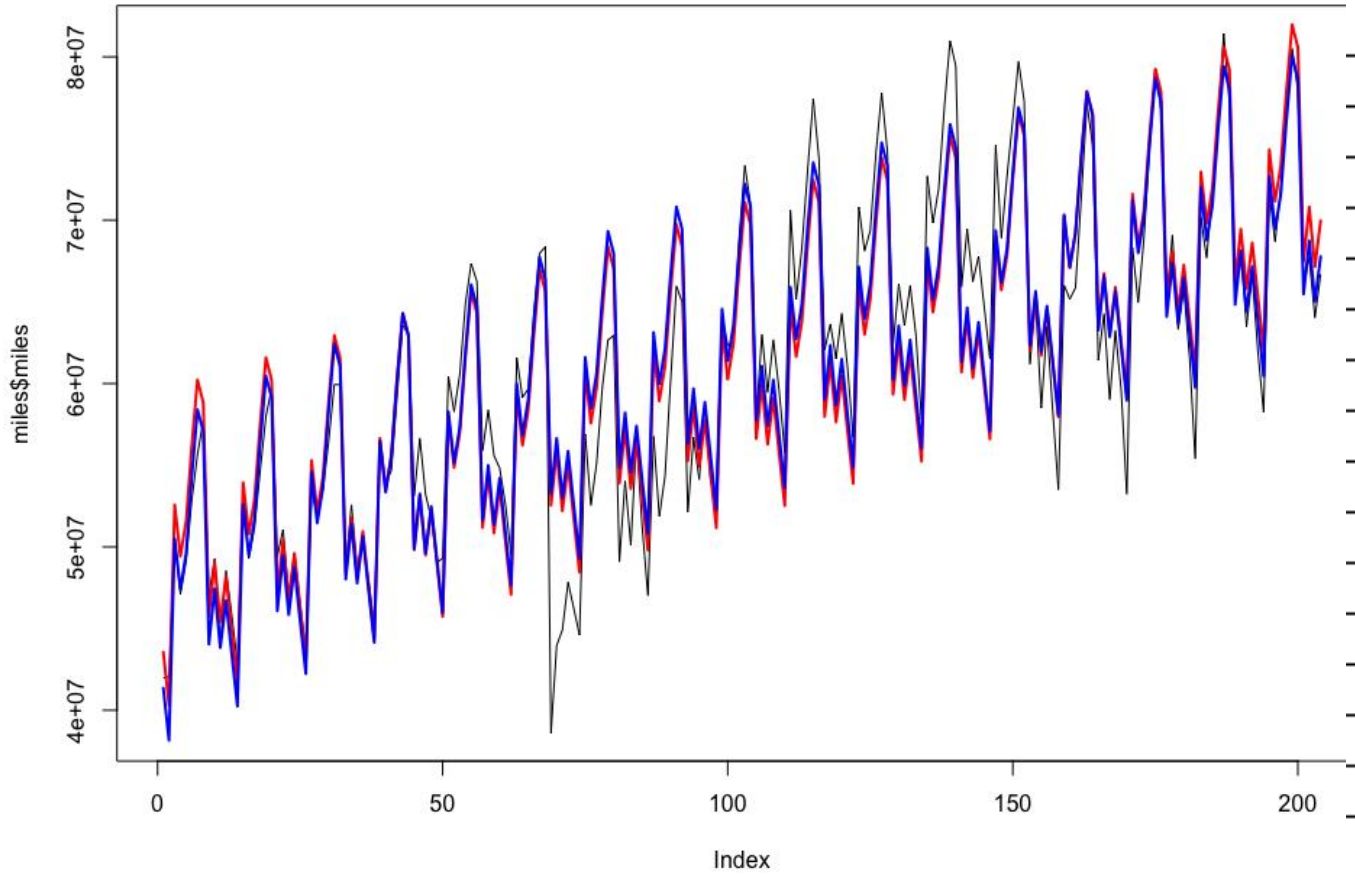


Seasonal Regression Models - RPM



	miles	time	var3
1	41972194	1	
2	42054796	2	
3	50443045	3	
4	47112397	4	
5	49118248	5	
6	52880510	6	
7	55664750	7	
8	57723208	8	
9	47035464	9	
10	49263120	10	
11	43937074	11	
12	48539606	12	
13	45850623	13	
14	42838949	14	
15	53620994	15	

Seasonal Regression Models - RPM



	miles	time	seasonal
1	41972194	1	1
2	42054796	2	2
3	50443045	3	3
4	47112397	4	4
5	49118248	5	5
6	52880510	6	6
7	55664750	7	7
8	57723208	8	8
9	47035464	9	9
10	49263120	10	10
11	43937074	11	11
12	48539606	12	12
13	45850623	13	1
14	42838949	14	2
15	53620994	15	3

Example

Year	Quarter	Time variable (this is created)	Revenues (in $M)
2016	I	1	10.2
	II	2	12.4
	III	3	14.8
	IV	4	15
2017	I	5	11.2
	II	6	14.3
	III	7	18.4
	IV	8	18

```
Call:
lm(formula = y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max
-3.5595 -0.9384  0.4405  1.3265  1.9286

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  10.0393     1.5531   6.464  0.00065 ***
x             0.9440     0.3076   3.069  0.02196 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.993 on 6 degrees of freedom
Multiple R-squared: 0.6109,     Adjusted R-squared: 0.5461
F-statistic: 9.422 on 1 and 6 DF,  p-value: 0.02196
```

What is the Regression equation?

$$y = 10.0393 + 0.9440x$$

Incorporating Seasonality – Another Approach

- Take the trend prediction and actual value.
- Depending on additive or multiplicative model, compute the deviation and map it as seasonality effect for each prediction.
- Take averages of the seasonality value. Use this to make future predictions.

Seasonality: Multiplicative

Time	Observed values TSI* (assuming no impact of cyclicality)	Predicted values (per the regression) T*	SI* = TSI/T
1	10.2	10.983	0.929
2	12.4	11.927	1.040
3	14.8	12.871	1.150
4	15.0	13.815	1.086
5	11.2	14.759	0.759
6	14.3	15.703	0.911
7	18.4	16.647	1.105
8	18.0	17.591	1.023

* T: Trend; S: Seasonal; I: Irregular

Quarterly Seasonality

Time	Observed values TSI* (assuming no impact of cyclicality)	Predicted values (per the regression) T*	SI* = TSI/T
1	10.2	10.983	0.929
2	12.4	11.927	1.040
3	14.8	12.871	1.150
4	15.0	13.815	1.086
5	11.2	14.759	0.759
6	14.3	15.703	0.911
7	18.4	16.647	1.105
8	18.0	17.591	1.023

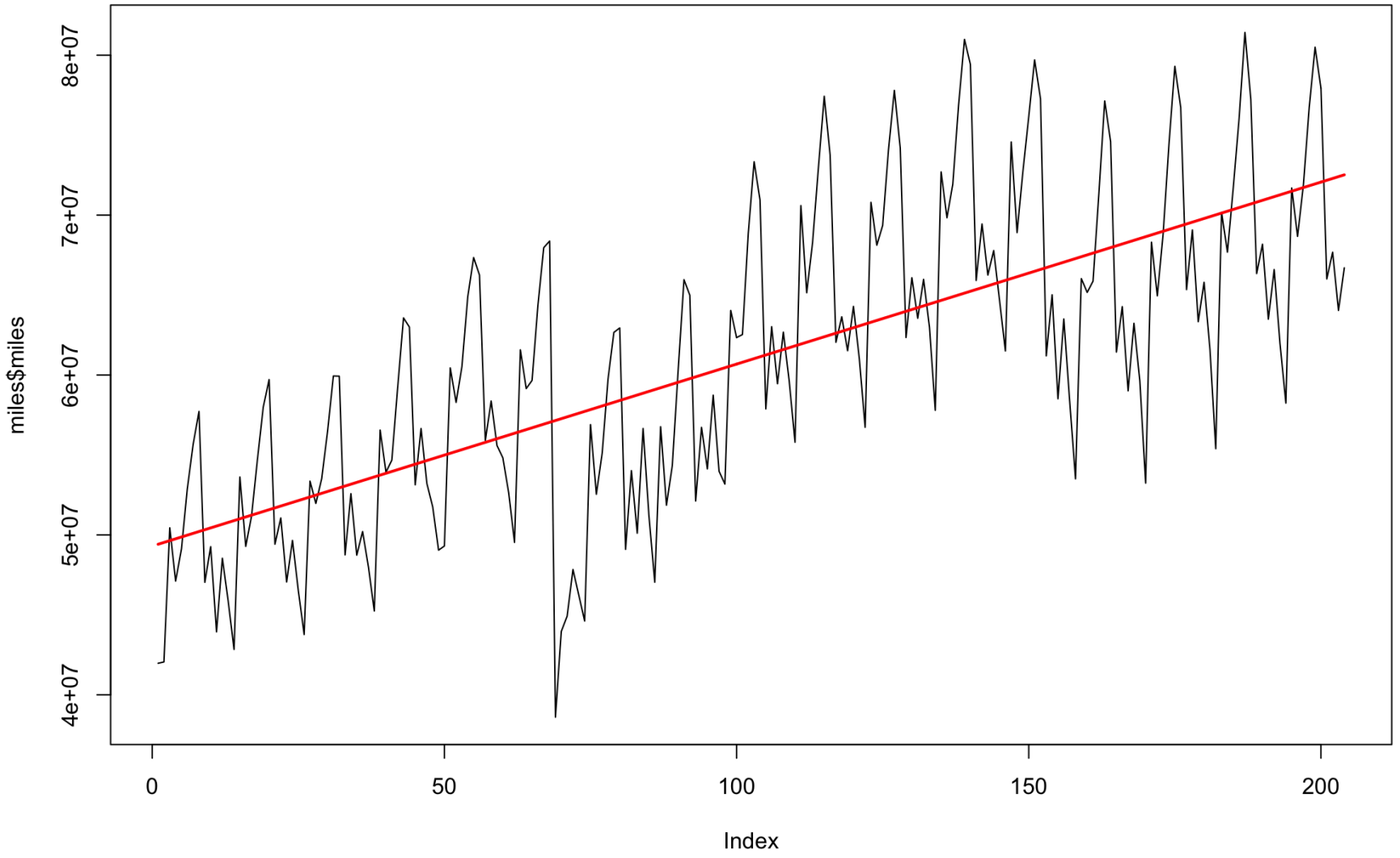
Time	Average seasonality factor
Q1	$0.844 \left(= \frac{0.929+0.759}{2}\right)$
Q2	0.975
Q3	1.127
Q4	1.054

Computations

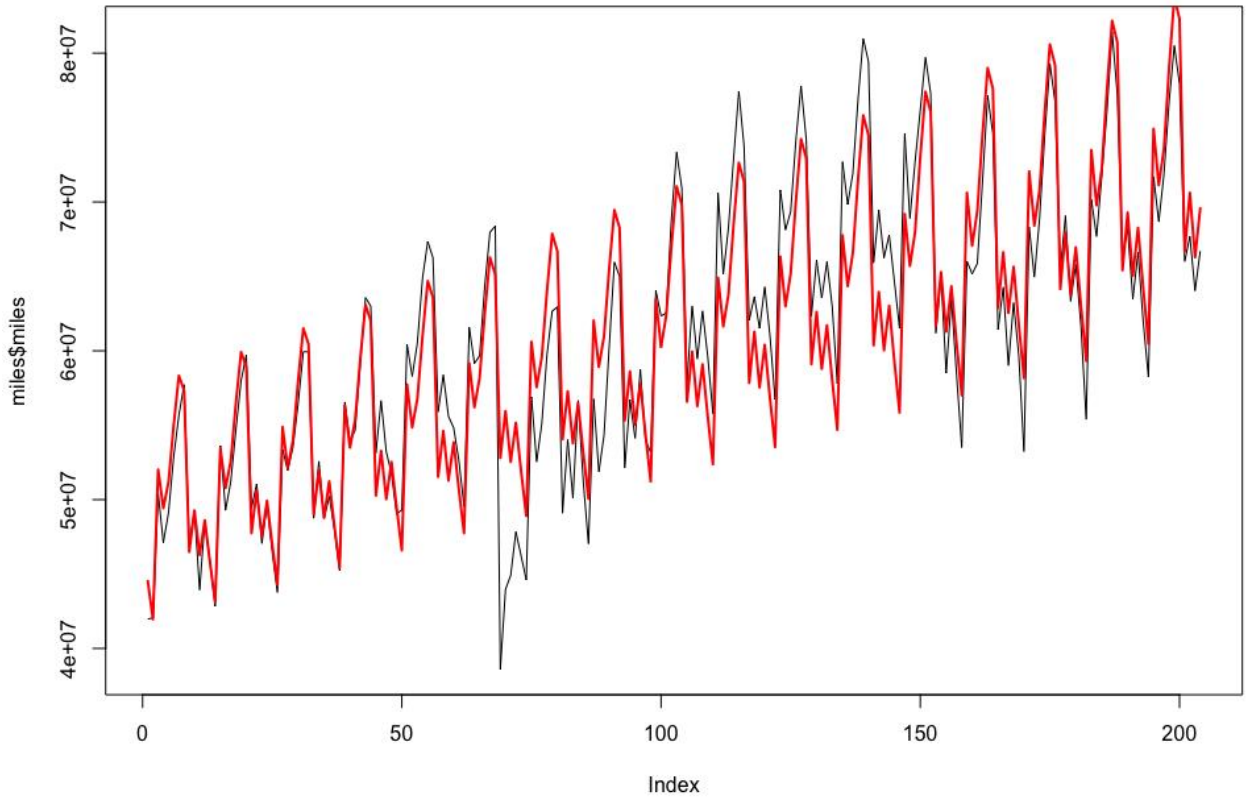
$$y = 10.0393 + 0.9440x$$

- Trend Y_9 =10.039 + 0.944(9) = 18.535
- Corrected for seasonality and randomness: 18.535 * 0.844 = 15.643

Regression on Time - RPM

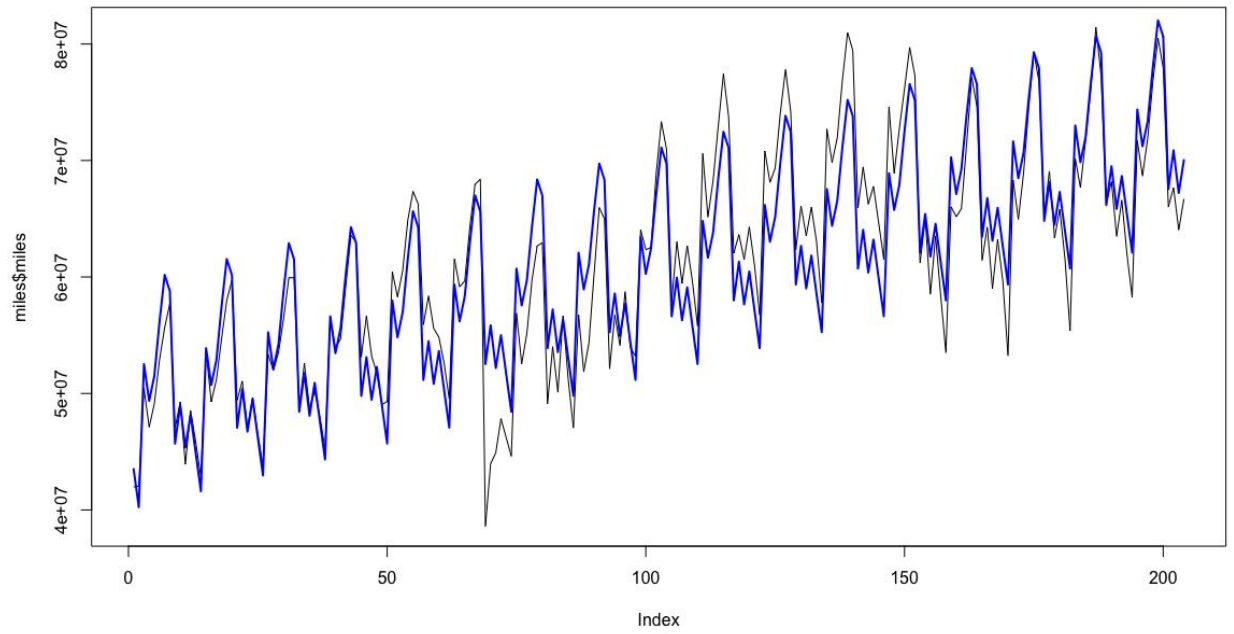


Seasonality: Multiplicative



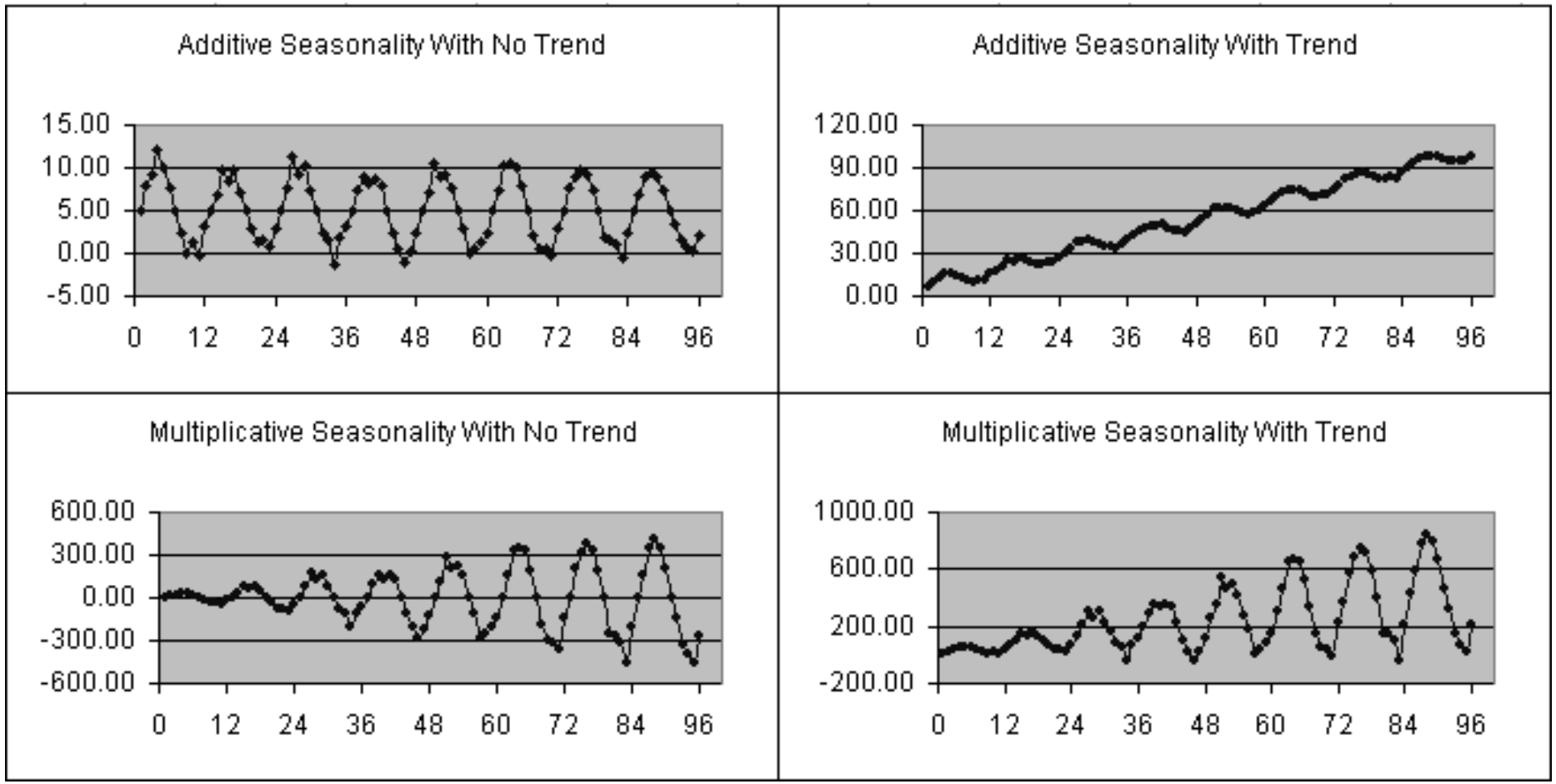
	miles	time	seasonal	mae
1	41972194	1	1	0.849386
2	42054796	2	2	0.8491019
3	50443045	3	3	1.016129
4	47112397	4	4	0.946865
5	49118248	5	5	0.9849257
6	52880510	6	6	1.057953
7	55664750	7	7	1.111125
8	57723208	8	8	1.149602
9	47035464	9	9	0.9346292
10	49263120	10	10	0.9766855
11	43937074	11	11	0.8691307
12	48539606	12	12	0.9580177
13	45850623	13	1	0.9029174
14	42838949	14	2	0.8417232
15	53620994	15	3	1.051224

Seasonality: Additive



	miles	time	seasonal	mae
1	41972194	1	1	-7442550
2	42054796	2	2	-7473763
3	50443045	3	3	800670.5
4	47112397	4	4	-2643793
5	49118248	5	5	-751757.1
6	52880510	6	6	2896690
7	55664750	7	7	5567114
8	57723208	8	8	7511757
9	47035464	9	9	-3289802
10	49263120	10	10	-1175962
11	43937074	11	11	-6615823
12	48539606	12	12	-2127106
13	45850623	13	1	-4929905
14	42838949	14	2	-8055394
15	53620994	15	3	2612836

Additive or Multiplicative



Issues with Regressing on Time

If there is no trend or if seasonality and fluctuations are more important than trend, then the coefficients behave weirdly

Advanced Time Series Methods

HOLT-WINTERS

Holt-Winters – Important Concepts

SMOOTHING

Simple Moving Averages

	Number of products sold	SMA (K=2)	SMA (K=3)
1			
2	15		
3	20	17.5	
4	16	18	17
5	13	14.5	16.333333
6	18	15.5	15.666667
7	14	16	15
8	15	14.5	15.666667
9	17	16	15.333333
10	20	18.5	17.333333
11	20	20	19

Weighted Moving Averages

	Number of products sold	WMA (K=2)	WMA (K=3)
1			
2	15		
3	20	=(A3*2+A2*1)/3	
4	16	=(A4*2+A3*1)/3	=(A4*3+A3*2+A2*1)/6
5	13	=(A5*2+A4*1)/3	=(A5*3+A4*2+A3*1)/6
6	18	=(A6*2+A5*1)/3	=(A6*3+A5*2+A4*1)/6
7	14	=(A7*2+A6*1)/3	=(A7*3+A6*2+A5*1)/6
8	15	=(A8*2+A7*1)/3	=(A8*3+A7*2+A6*1)/6
9	17	=(A9*2+A8*1)/3	=(A9*3+A8*2+A7*1)/6
10	20	=(A10*2+A9*1)/3	=(A10*3+A9*2+A8*1)/6
11	20	=(A11*2+A10*1)/3	=(A11*3+A10*2+A9*1)/6
12	18	=(A12*2+A11*1)/3	=(A12*3+A11*2+A10*1)/6
13	20	=(A13*2+A12*1)/3	=(A13*3+A12*2+A11*1)/6

1	Number of products sold	WMA (K=2)	WMA (K=3)
2	15		
3	20	18.3333333	
4	16	17.3333333	17.1666667
5	13	14	15.1666667
6	18	16.3333333	16
7	14	15.3333333	15.1666667
8	15	14.6666667	15.1666667
9	17	16.3333333	15.8333333
10	20	19	18.1666667
11	20	20	19.5
12	18	18.6666667	19
13	20	19.3333333	19.3333333

Exponential ~~Weighted Moving Averages~~ or Exponential Smoothing

Averaging over long periods dampens fluctuations, removing not only the noise but also trend and seasonality.

Moving averages over short recent periods maintain trend and seasonality but determining an optimum number for periods is tricky, even when using metrics like MAE. If averaged over too few periods, irregularities continue to remain and if averaged over long periods, dampening again becomes a problem.

Exponential smoothing **retains all older periods** while giving a greater weight to more recent periods (hence not a MOVING average).

Caution: It doesn't make any one method superior for all situations.

Stationary Model: Exponential ~~Weighted Moving Averages~~ or Exponential Smoothing

$$\hat{Y}_{t+1} = \hat{Y}_t + \alpha\left(Y_t - \hat{Y}_t\right)$$

Above equation indicates that the predicted value for time period t+1 $(\hat{Y}_{t+1})$ is equal to the predicted value for the previous period $(\hat{Y}_t)$ plus an adjustment for the error made in predicting the previous period's value $(\alpha\left(Y_t - \hat{Y}_t\right))$.

The parameter α can assume any value between 0 and 1 ($0 \leq \alpha \leq 1$).

Exponential Smoothing in Other Ways

$\hat{Y}_{t+1} = \hat{Y}_t + \alpha(Y_t - \hat{Y}_t)$ can be rewritten variously as

$$= \alpha Y_t + (1-\alpha)\hat{Y}_t$$

$$= Y_t - (1-\alpha)\left(Y_t - \hat{Y}_t\right)$$

$$= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots + \alpha(1-\alpha)^n Y_{t-n} + \cdots$$

Various Ways of Understanding Exponential Smoothing

- Forecast
  - Interpolation between previous *forecast* and previous *observation* $= \alpha Y_t + (1-\alpha)\hat{Y}_t$
  - Previous *forecast* plus fraction of previous error $= \hat{Y}_t + \alpha(Y_t - \hat{Y}_t)$
  - Previous *observation* minus fraction 1- of previous error $= Y_t - (1-\alpha)\left(Y_t - \hat{Y}_t\right)$
  - *Exponentially weighted (i.e., discounted) moving average*

$$= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots + \alpha(1-\alpha)^n Y_{t-n} + \cdots$$

Holt-Winters

MODEL BUILDING

Holt-Winters Method

- 3 components – Trend, Seasonality and Level.

Additive Seasonality

$$\hat{y}_t = l_{t-1} + b_{t-1} + s_{t-m}$$

$$\hat{y}_{t+h|t} = l_t + hb_t + s_{t+h-m}$$

Multiplicative Seasonality

$$\hat{y}_t = (l_{t-1} + b_{t-1})s_{t-m}$$

$$\hat{y}_{t+h|t} = (l_t + hb_t)s_{t+h-m}$$

Holt-Winters Method

- 3 weights – smoothing parameters – are used to update components at each period.

The 3 smoothing equations are:

Additive Seasonality

$$l_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(l_{t-1} + b_{t-1})$$

$$b_t = \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1}$$

$$s_t = \gamma(y_t - (l_{t-1} + b_{t-1})) + (1-\gamma)s_{t-m}$$

Multiplicative Seasonality

$$l_t = \alpha\frac{y_t}{s_{t-m}} + (1-\alpha)(l_{t-1} + b_{t-1})$$

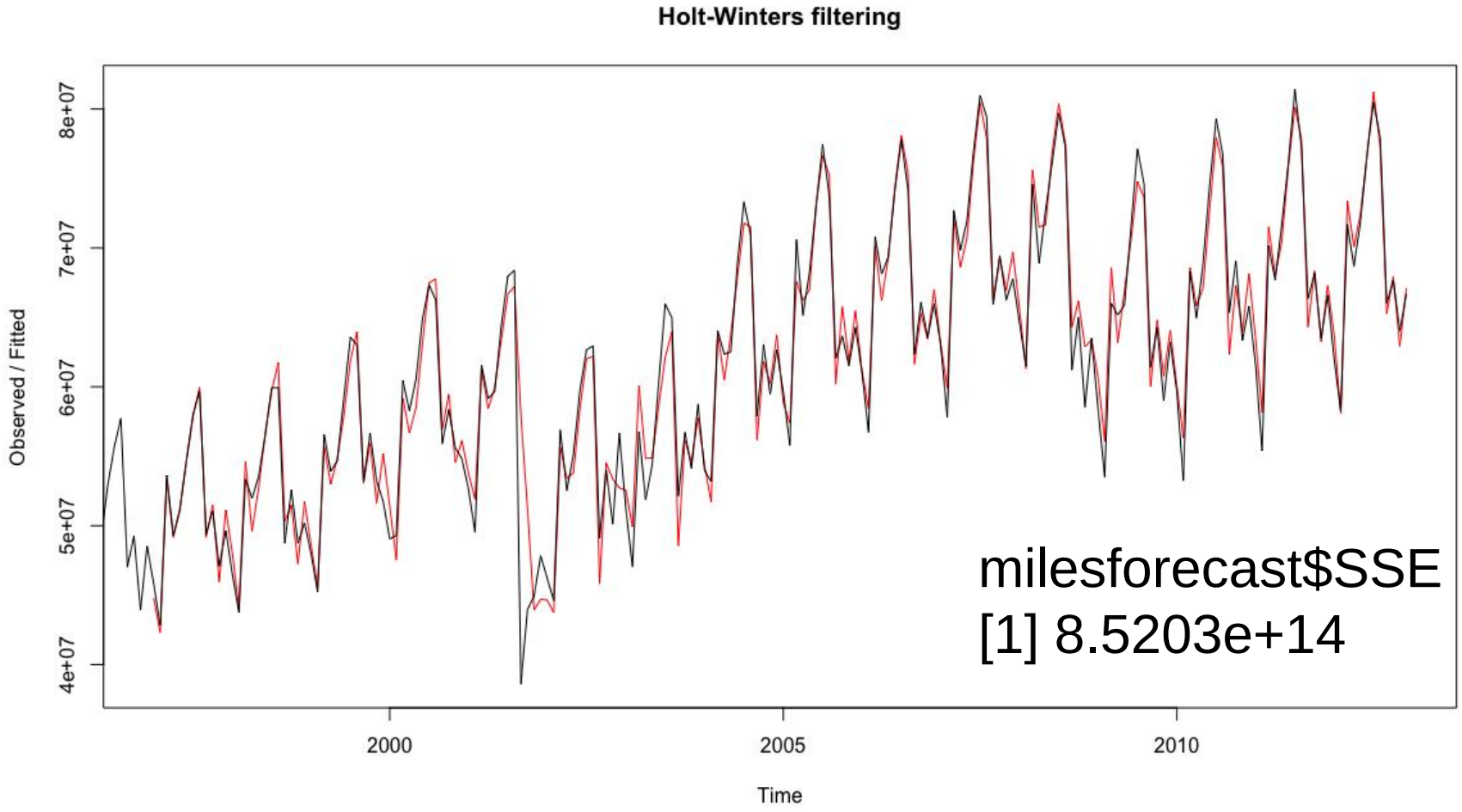
$$b_t = \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1}$$

$$s_t = \gamma\frac{y_t}{l_{t-1} + b_{t-1}} + (1-\gamma)s_{t-m}$$

Recall Exponential Smoothing form: $\hat{Y}_t = \alpha Y_{t-1} + (1-\alpha)\hat{Y}_{t-1}$

Holt-Winters Method: *RPM*

R code: HoltWinters(milestimeseries)



CSE 7302c

Holt-Winters Method: *RPM*

```
> milesforecast
Holt-Winters exponential smoothing with trend and additive seasonal component.

Call:
HoltWinters(x = milestimeseries)

Smoothing parameters:
 alpha: 0.5024519
 beta : 0
 gamma: 0.6698853

Coefficients:
           [,1]
a   71153143.7
b     181508.7
s1   -8052713.5
s2  -12314754.8
s3    2106026.2
s4    -447843.5
s5    2758176.7
s6    7232862.9
s7   11309888.8
s8    8022549.6
s9   -4490660.8
s10  -2715650.2
s11  -7315682.9
s12  -4384446.7
> milesforecast$fitted
             xhat    level    trend       season
Jan 1997 44780388 48804078 181508.7  -4205198.60
Feb 1997 42307860 49523328 181508.7  -7396976.76
```

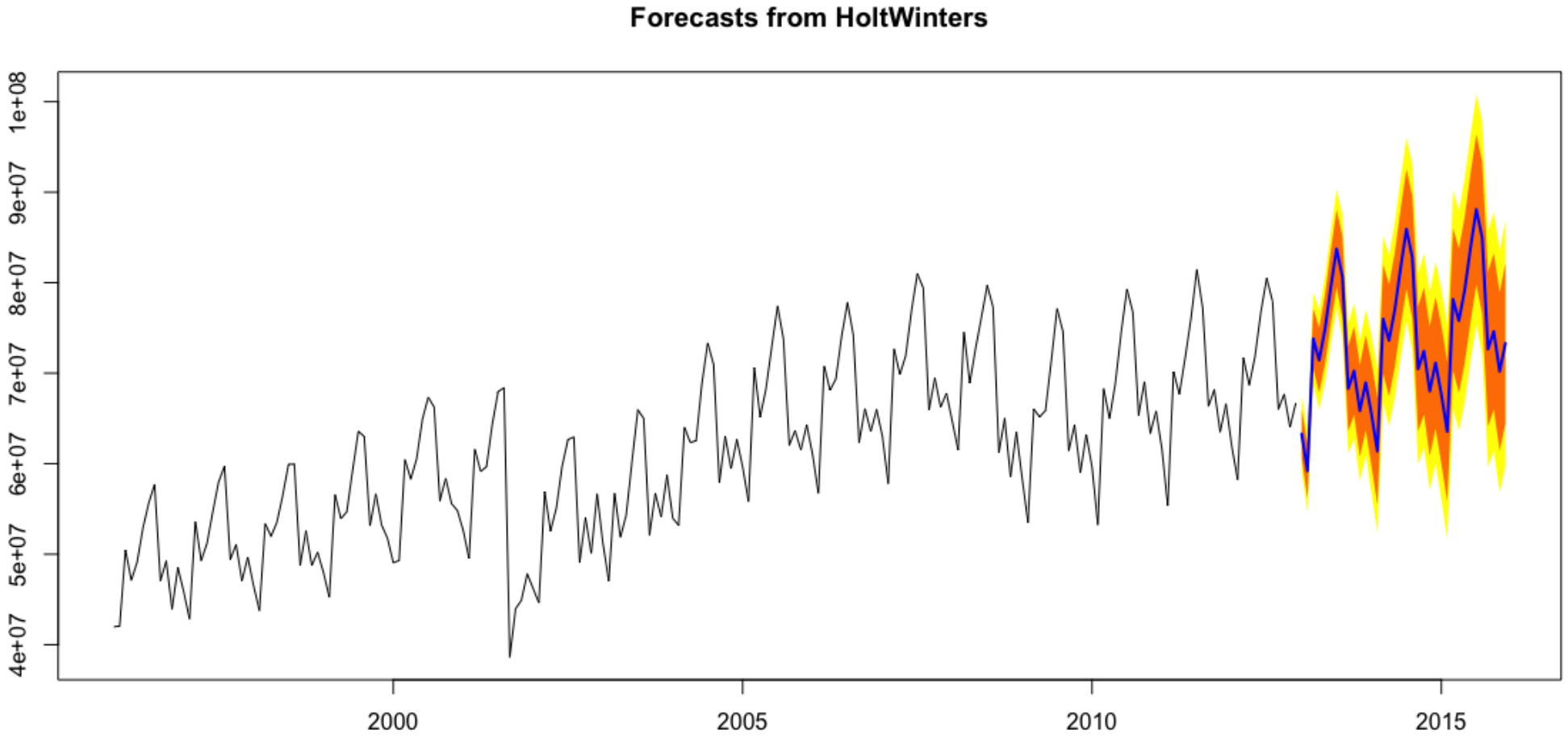
$$l_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(l_{t-1} + b_{t-1})$$
$$b_t = \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1}$$
$$s_t = \gamma(y_t - (l_{t-1} + b_{t-1})) + (1-\gamma)s_{t-m}$$

$$\hat{y}_{t+h|t} = l_t + hb_t + s_{t+h-m}$$

R refers to *l* as *a*

Holt-Winters Method: Forecasting

R code: forecast.HoltWinters(milesforecast, h=36) or forecast(milesforecast, h=36) depending on the version of R being used.



Advanced Time Series Methods

ARIMA

ARIMA – Important Concepts

ACF-PACF AND STATIONARITY

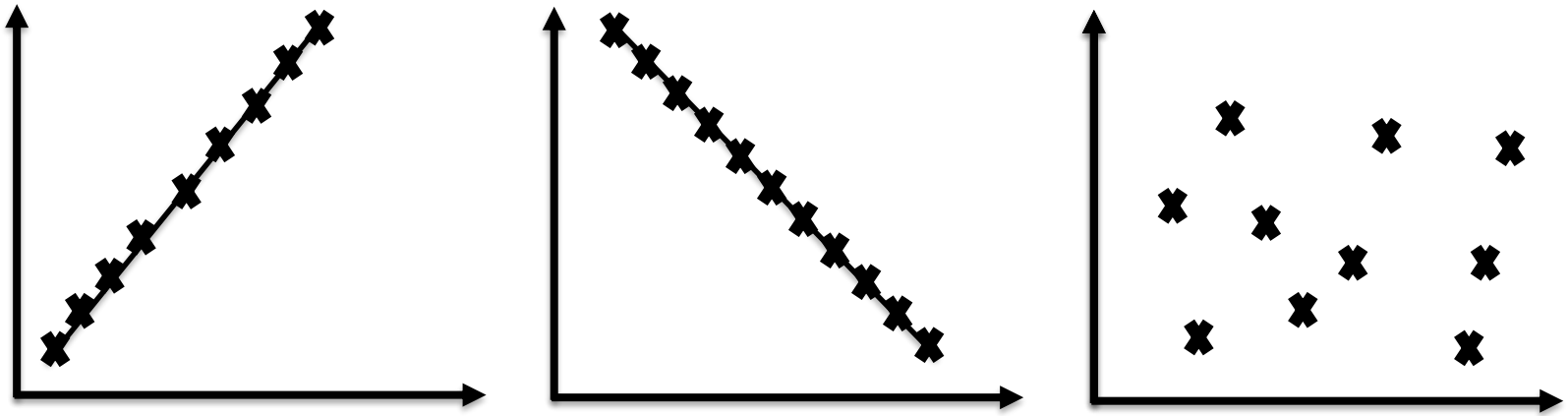
Time Series Descriptive Statistics

- In descriptive statistics covered earlier (central tendencies, measures of variability, distributions, correlations, etc.), the order of observations in the data was of no consequence.

- In time series descriptive statistics, order of observations is of primary importance and so **autocorrelations** play a vital role in identifying the models and their characteristics.

Correlation Coefficient

Correlation coefficient, *r*, is a number between -1 and 1 and tells us how well a regression line fits the data.



r = 1 r = -1 r =0

It gives the strength and direction of the relationship between two variables.

Autocorrelation (ACF) and Partial ACF (PACF)

- ACF: n^{th} lag of ACF is the correlation between a day and n days before that.

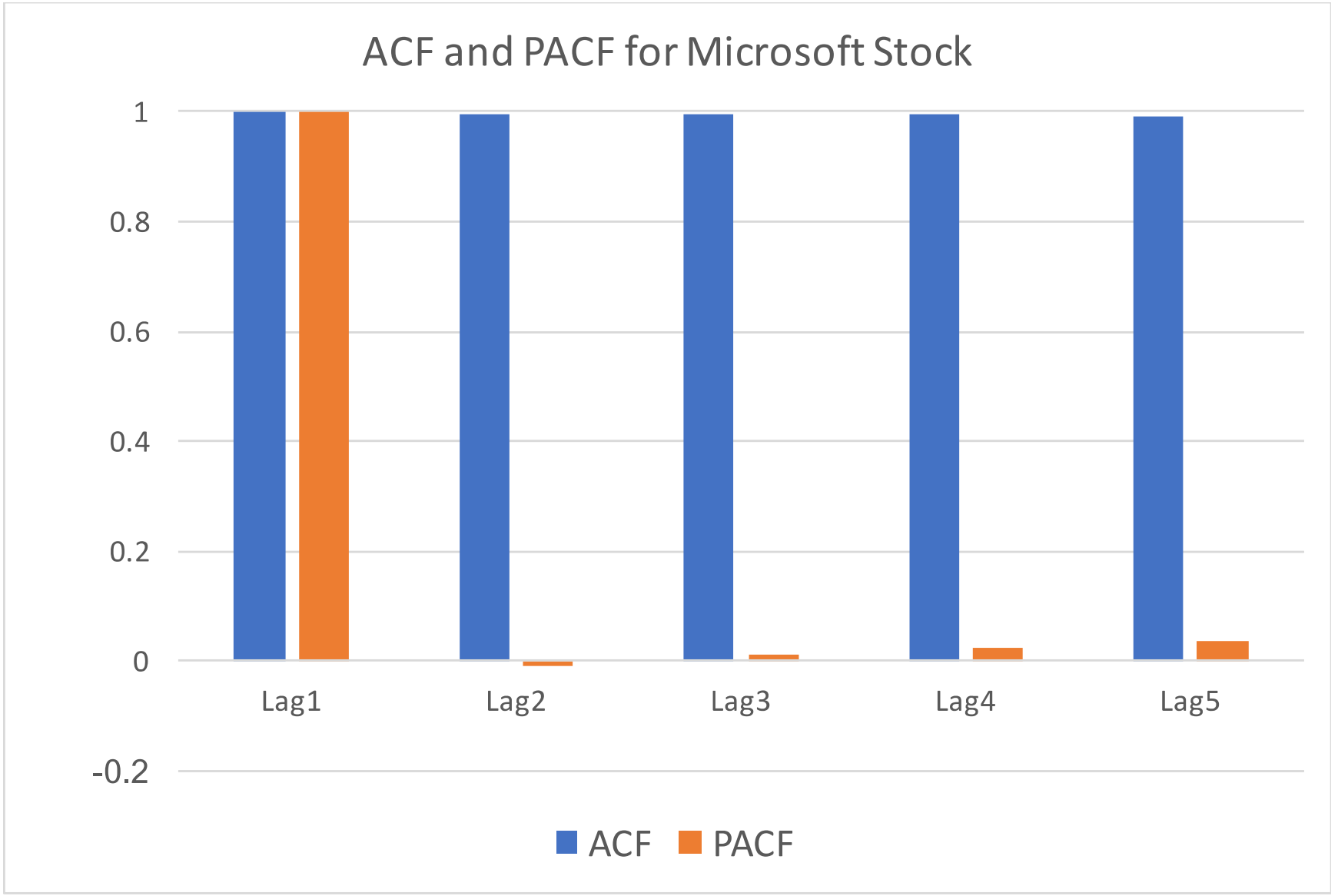
- PACF: The same as ACF with all intermediate correlations removed. It is the k_{th} coefficient of the ordinary least squares regression.

$$[y_t] = \beta_0 + \sum_{i=1}^{k} \beta_i [y_{t-i}] \text{ where}$$

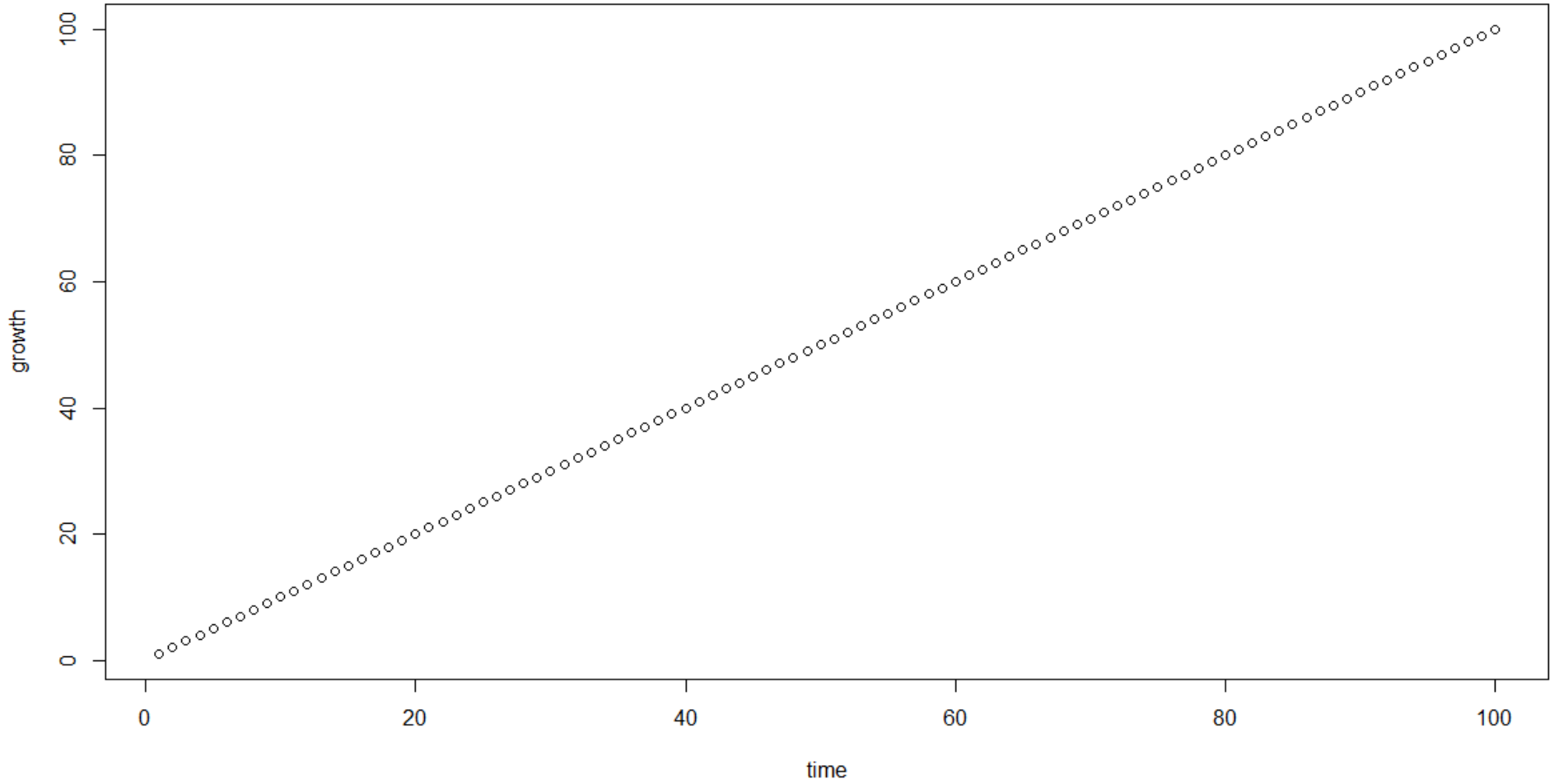
$[y_t]$ is the input time series, k is the lag order and β_i is the i_{th} coefficient of the linear multiple regression.

EXCEL ACTIVITY

Autocorrelation (ACF) and Partial ACF (PACF)

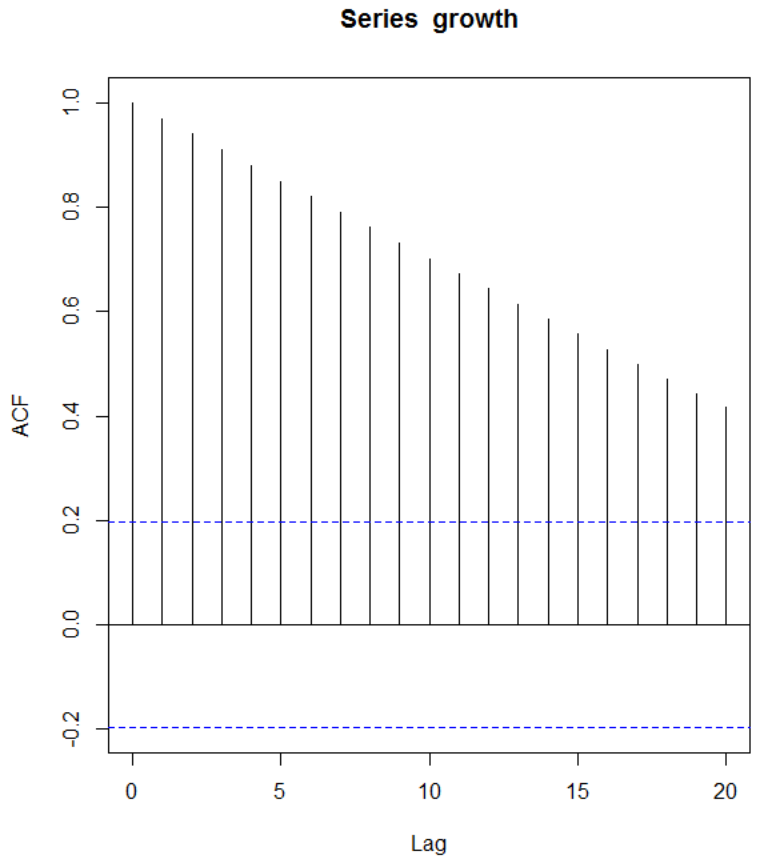


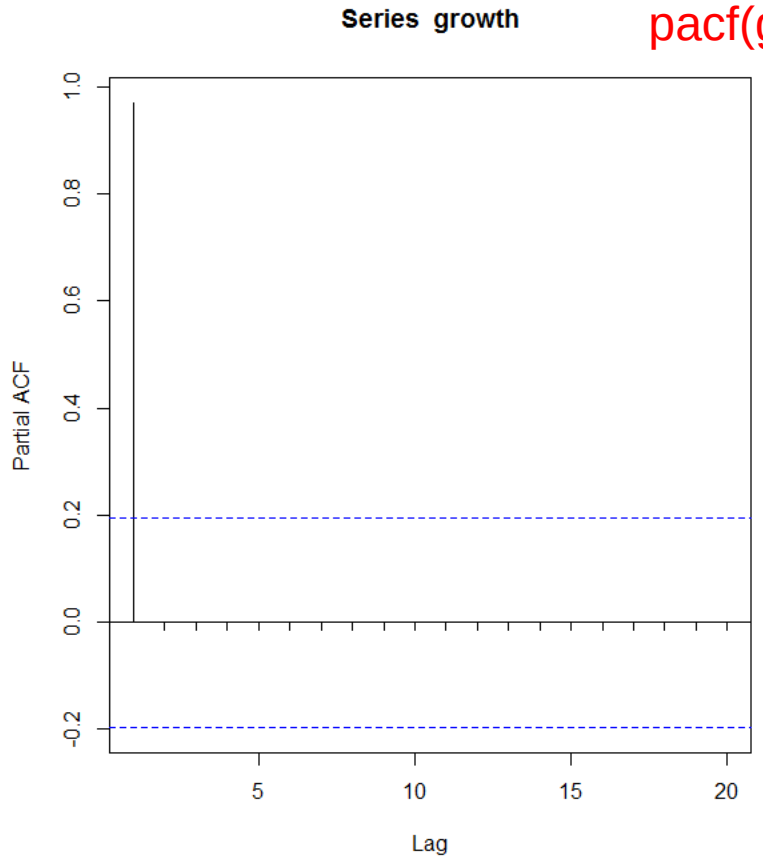
ACF and PACF – Idealized Trend



ACF and PACF – Idealized Trend

R code: growthT <- ts(growthT)
acf(growthT)
pacf(growthT)

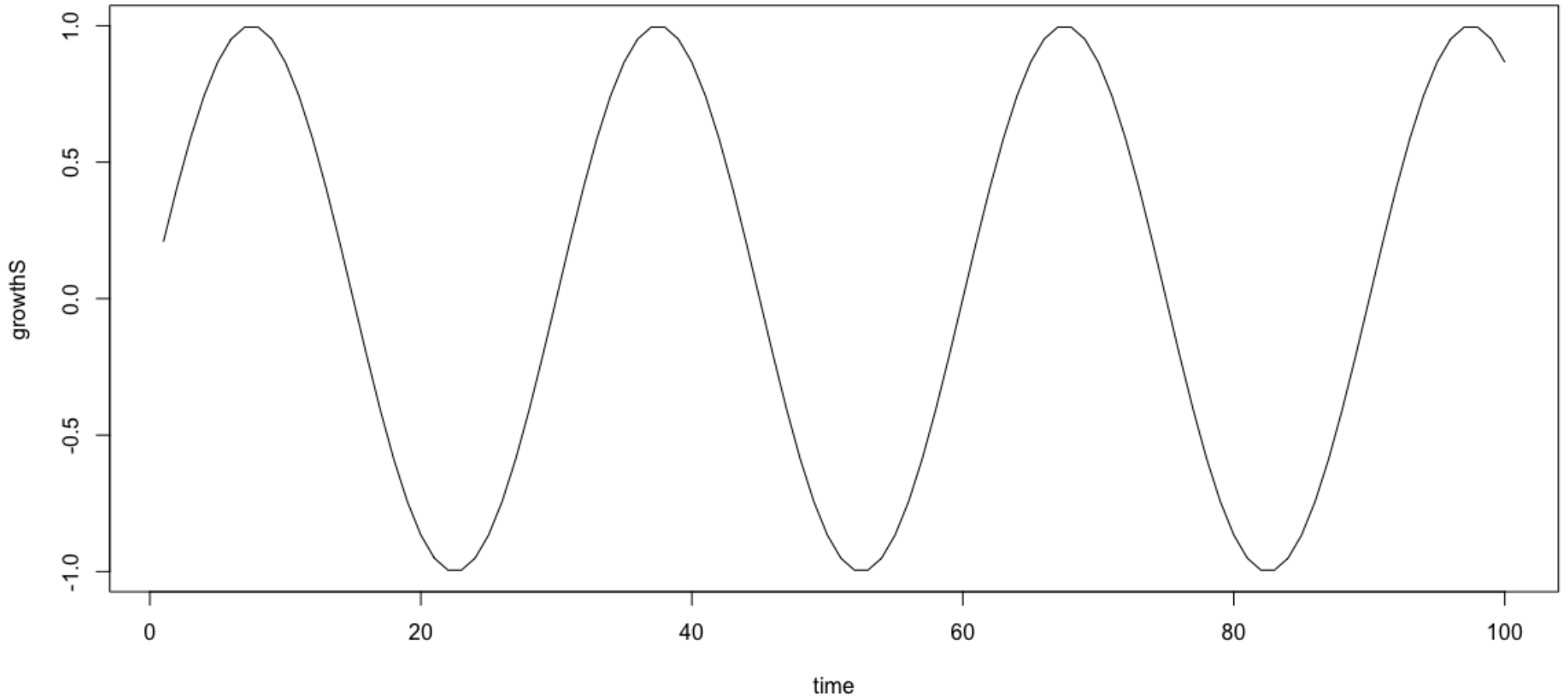




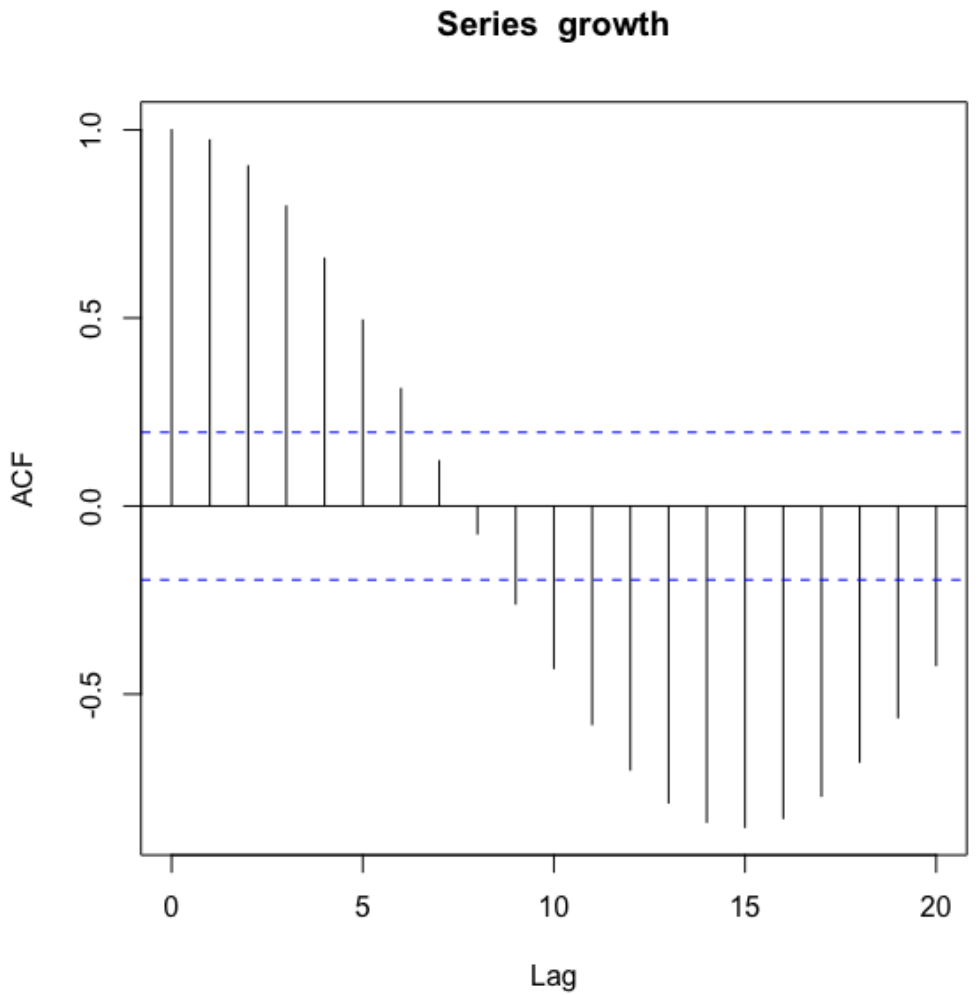
95% CI: $0 \pm \frac{1.96}{\sqrt{n}}$

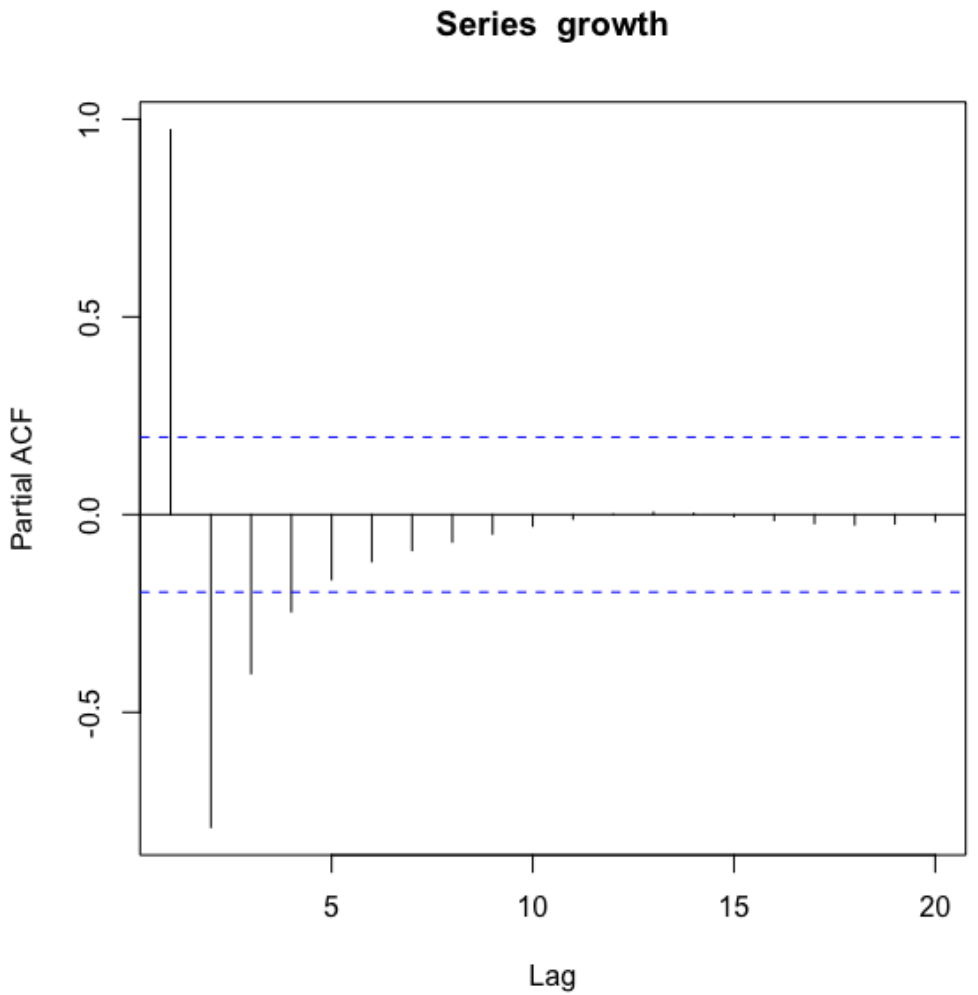
- ACF is a bar chart of correlation coefficients of the time series and its lags.
- PACF is a plot of the partial correlation coefficients of the time series and its lags.

ACF and PACF – Idealized Seasonality

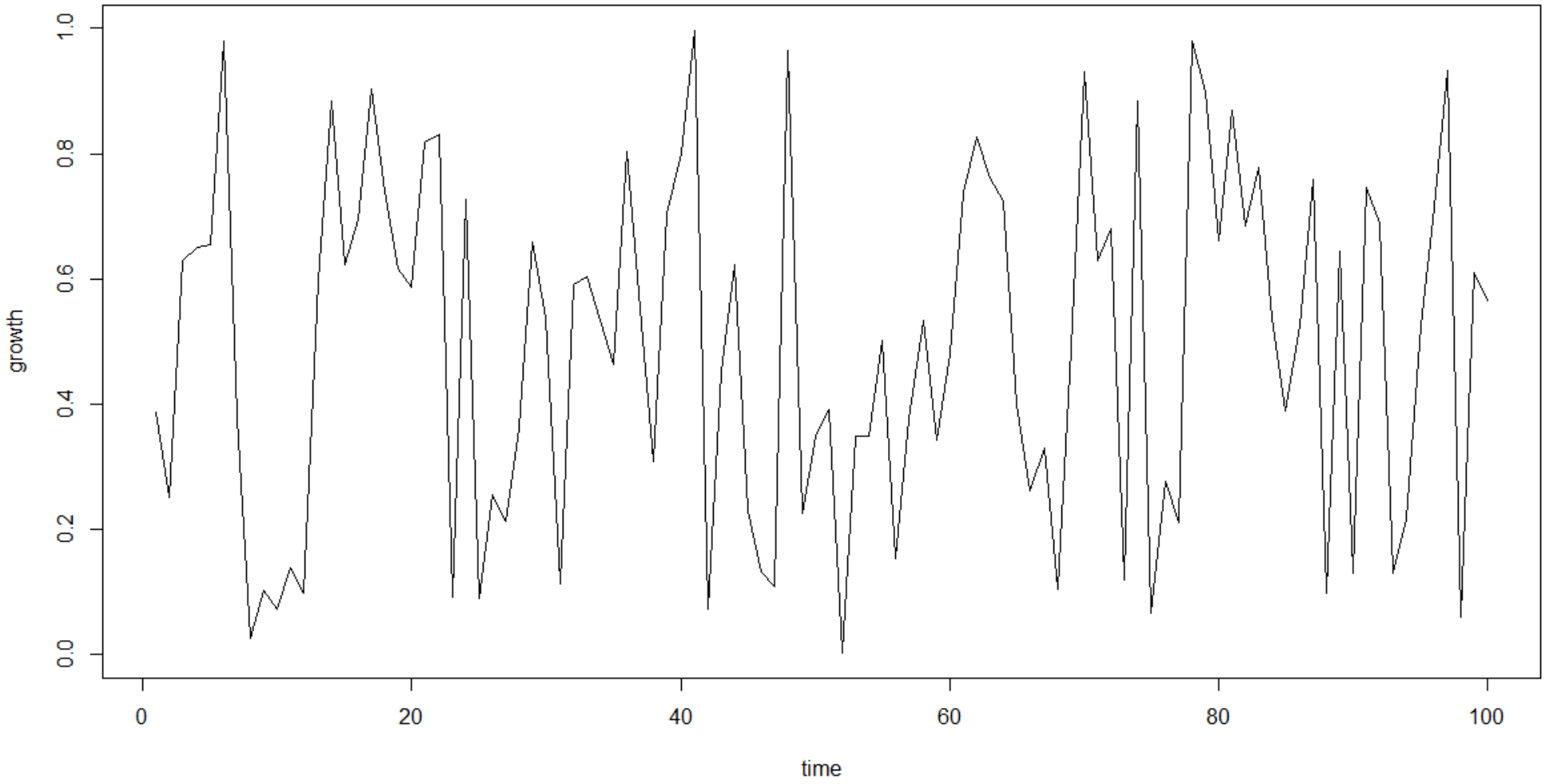


ACF and PACF – Idealized Seasonality

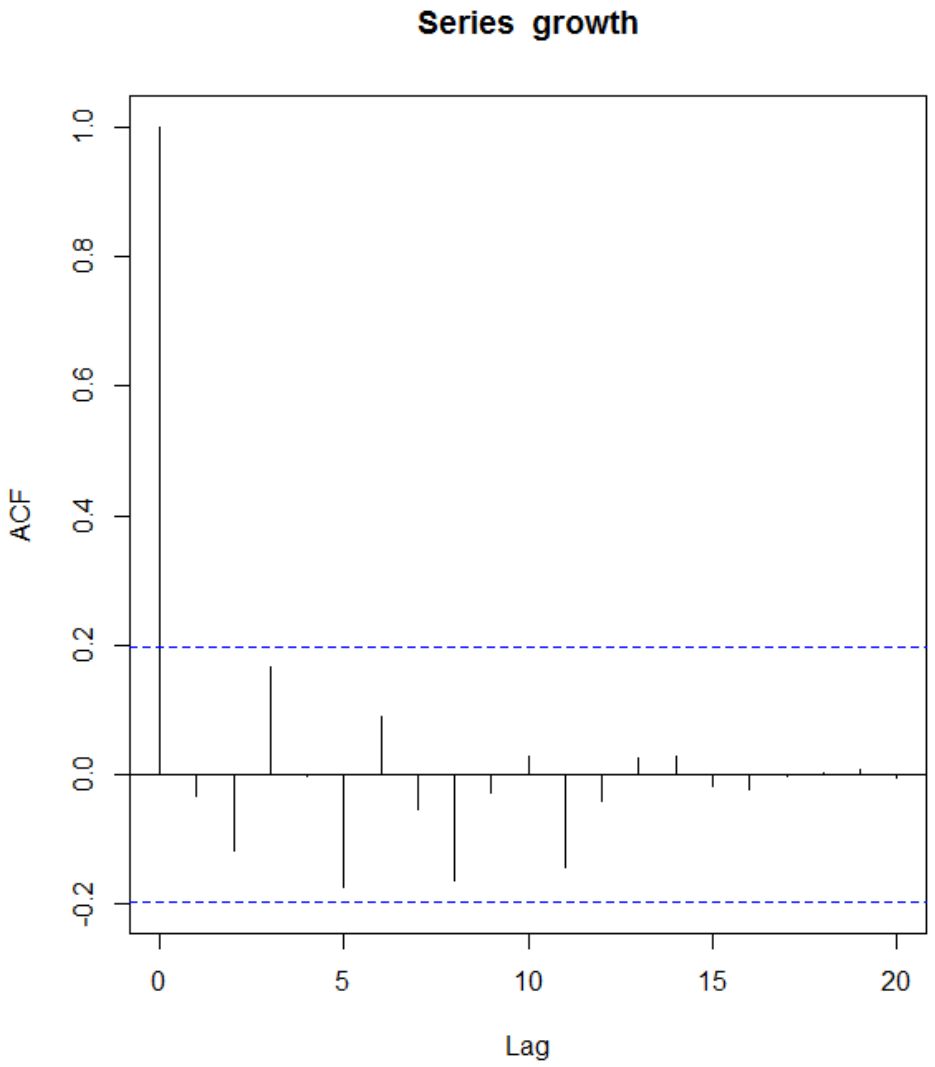


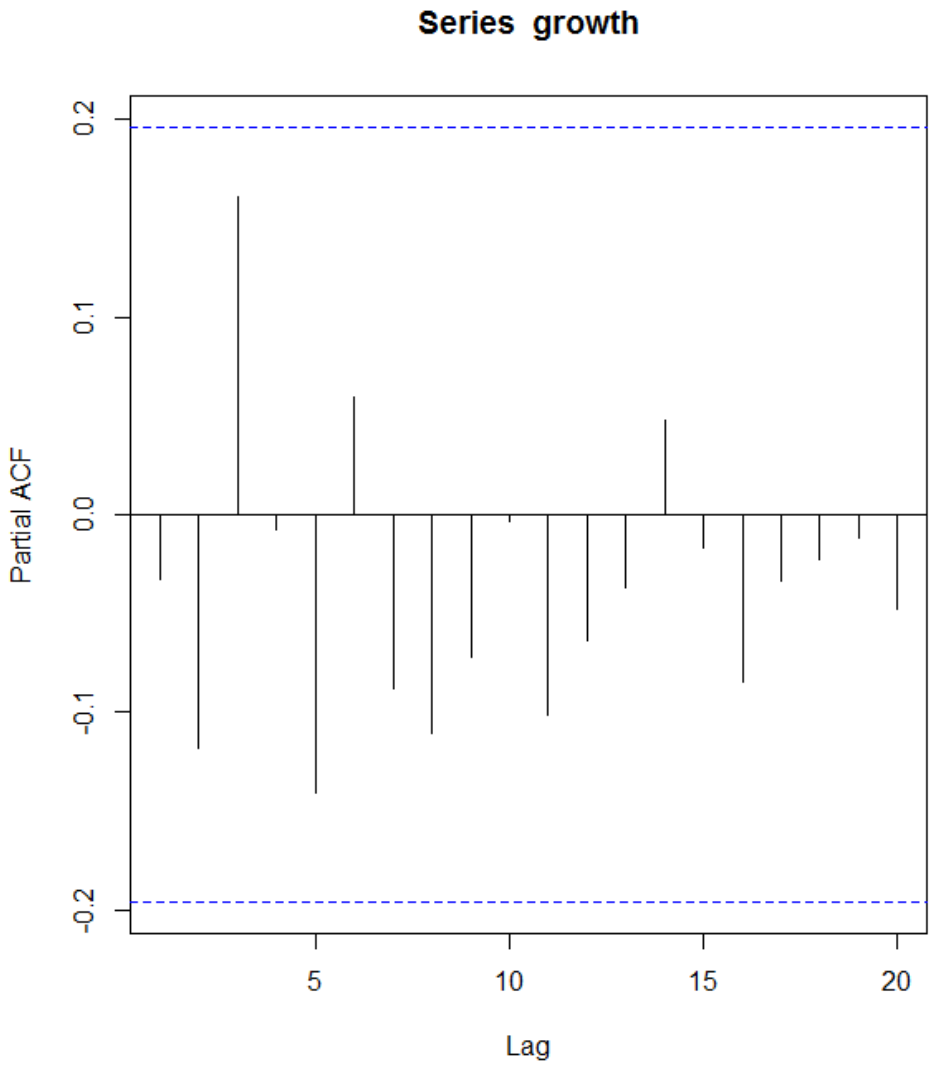


ACF and PACF – Idealized Randomness



ACF and PACF – Idealized Randomness

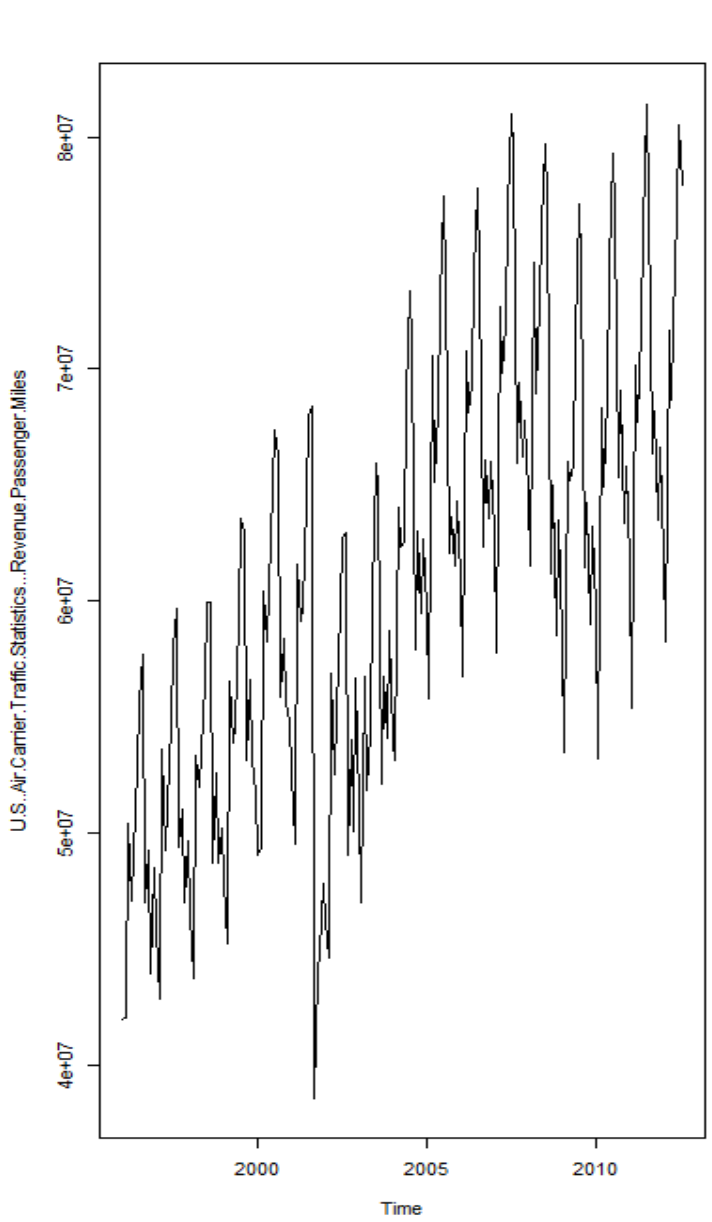


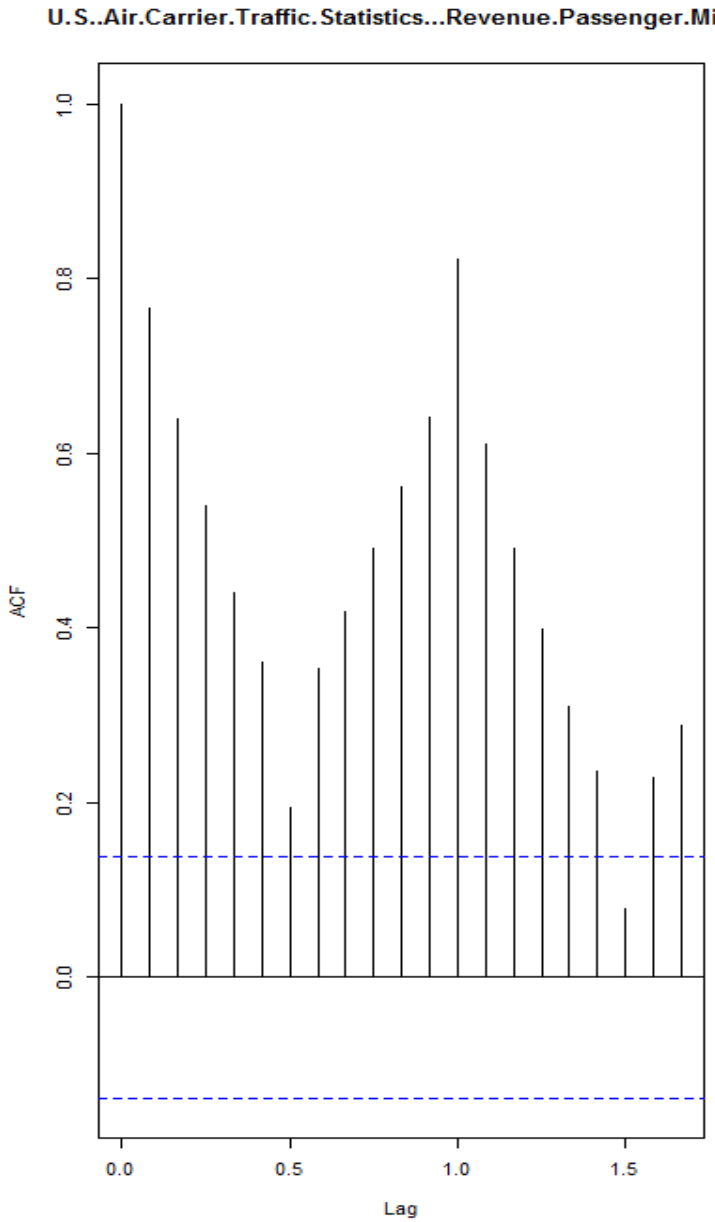


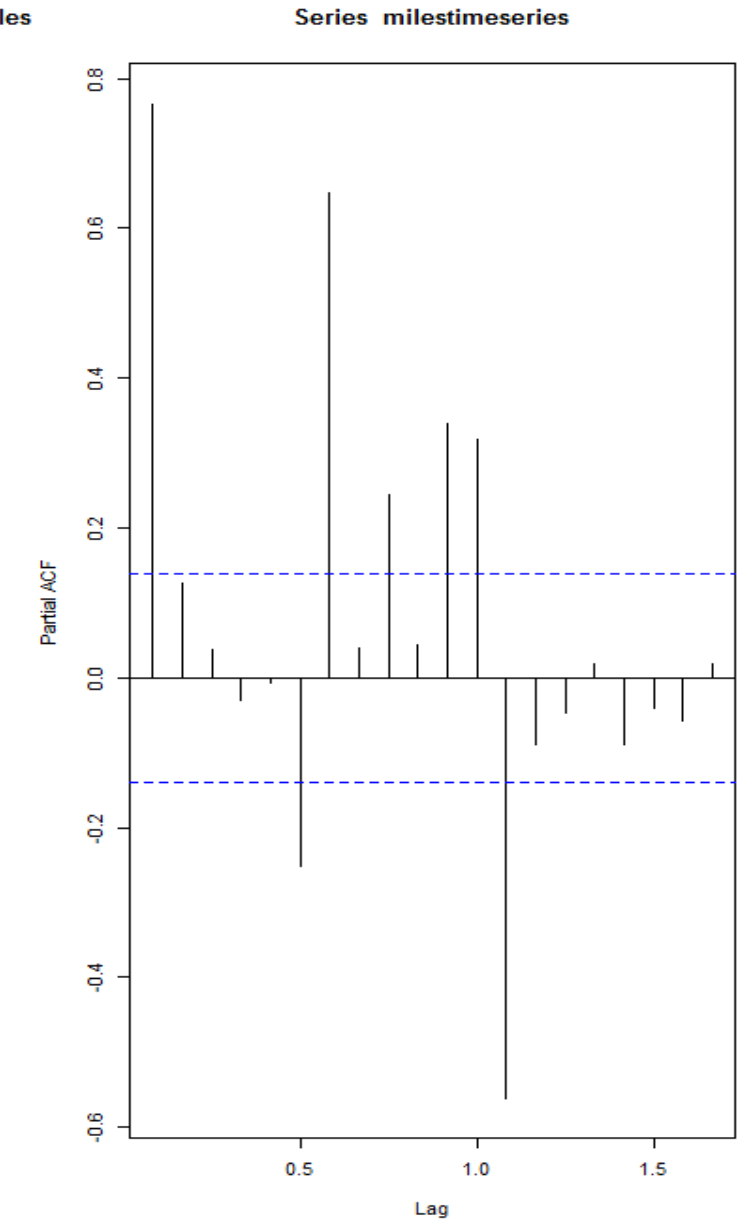
ACF and PACF – Idealized Trend, Seasonality and Randomness

- Ideal Trend: **Decay**ing ACF and **Cutoff** after 1 or 2 lags of PACF
- Ideal Seasonality: **Decay**ing Cyclicality in ACF and **Cutoff** after a few lags of PACF with some positive and some negative
- Ideal Random: A spike may or may not be present; even if present, magnitude will be small

ACF and PACF (Real-world): Decomposing Time Series into the 3 Components – RPM







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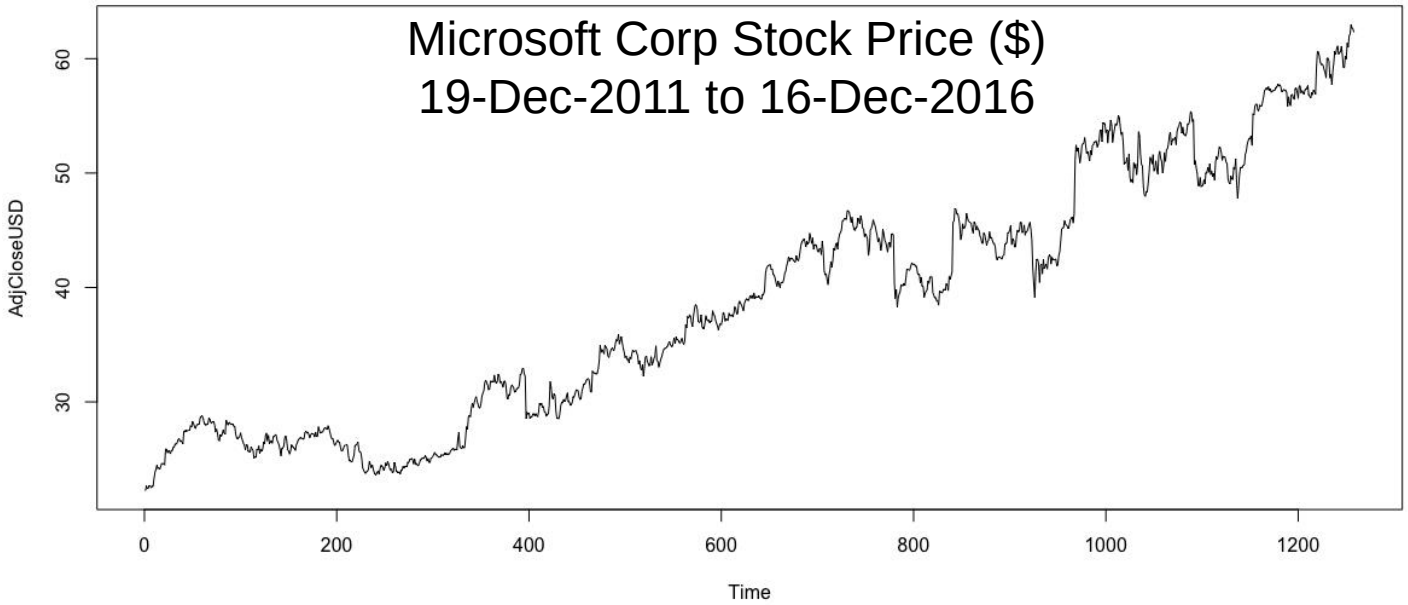
Stationary and Non-Stationary

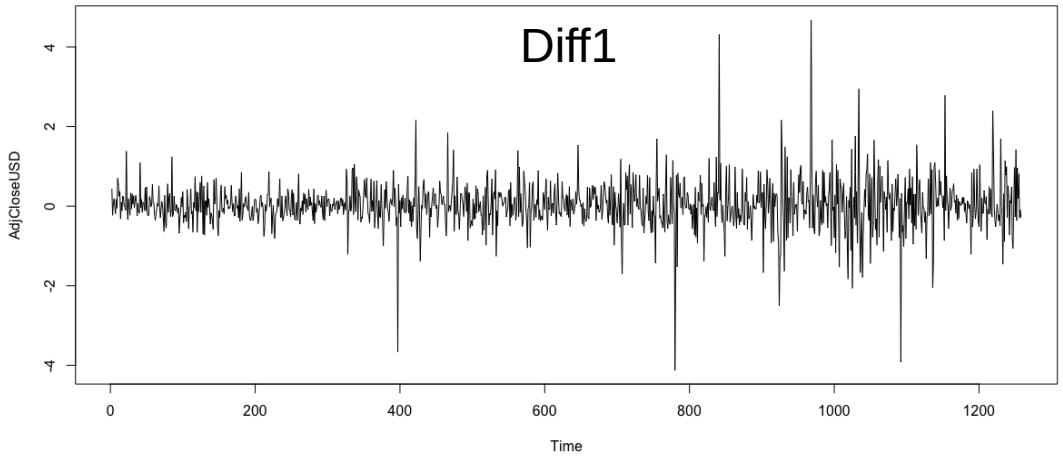
- Stationary data has constant statistical properties – mean, variance, autocorrelation, etc. – over time
- If the data is stationary, forecasting is easier!
- Differencing to convert non-stationary to stationary

EXCEL ACTIVITY

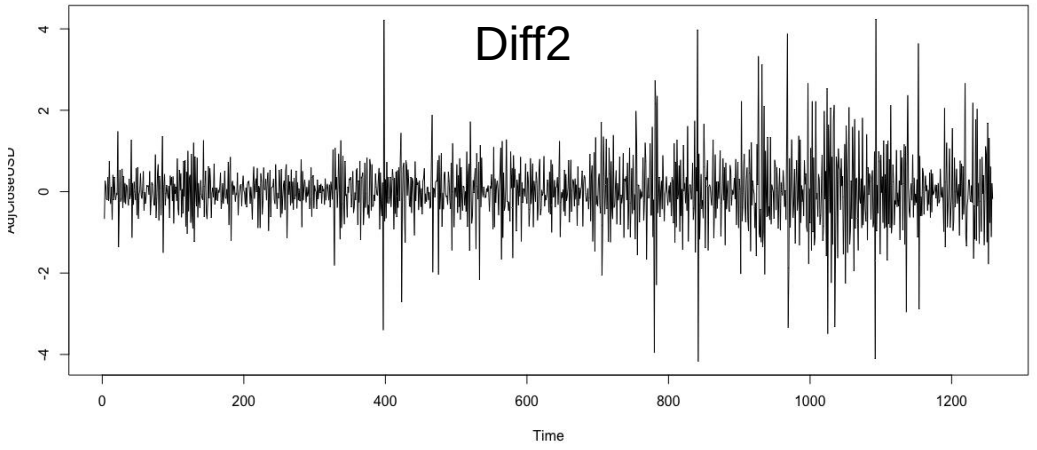
Removing Trend from Data



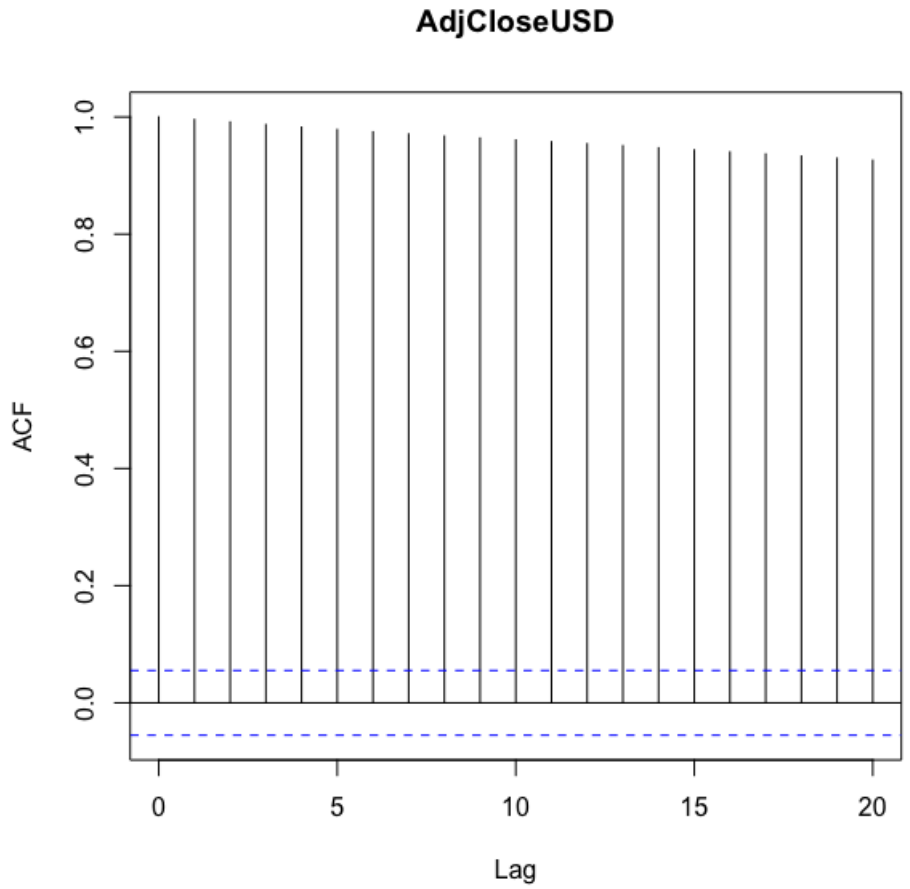
Rcode: MStimeseriesdiff1 <- diff(MStimeseries, differences=1)

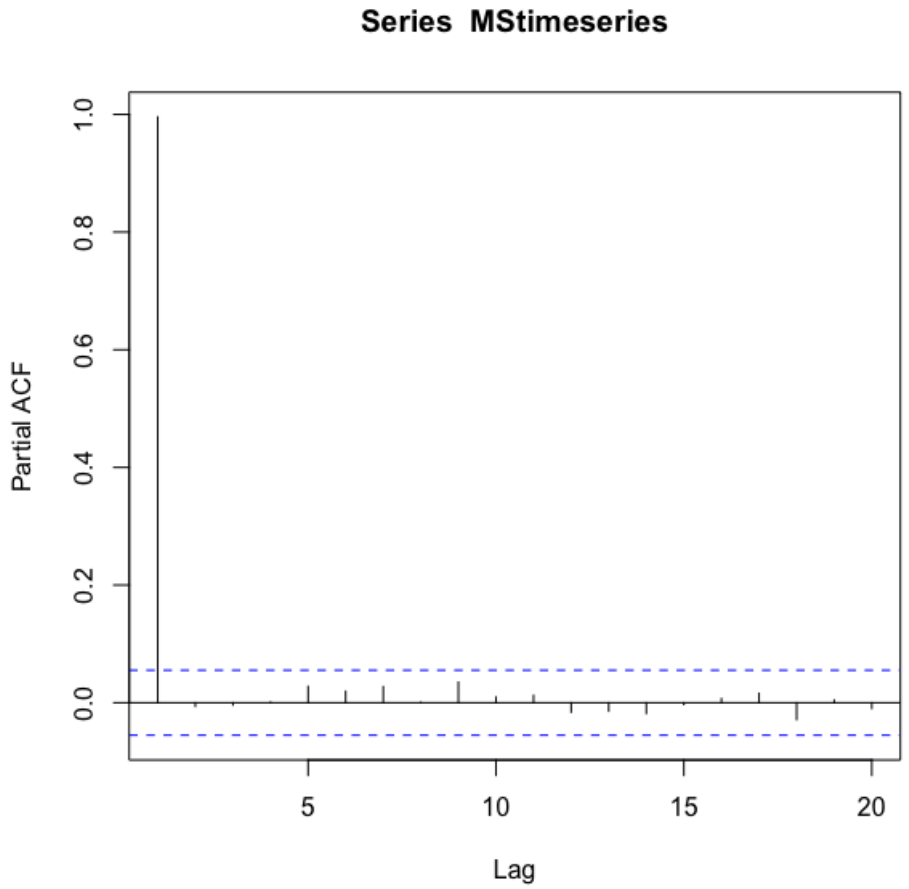


Rcode: MStimeseriesdiff2 <- diff(MStimeseries, differences=2)



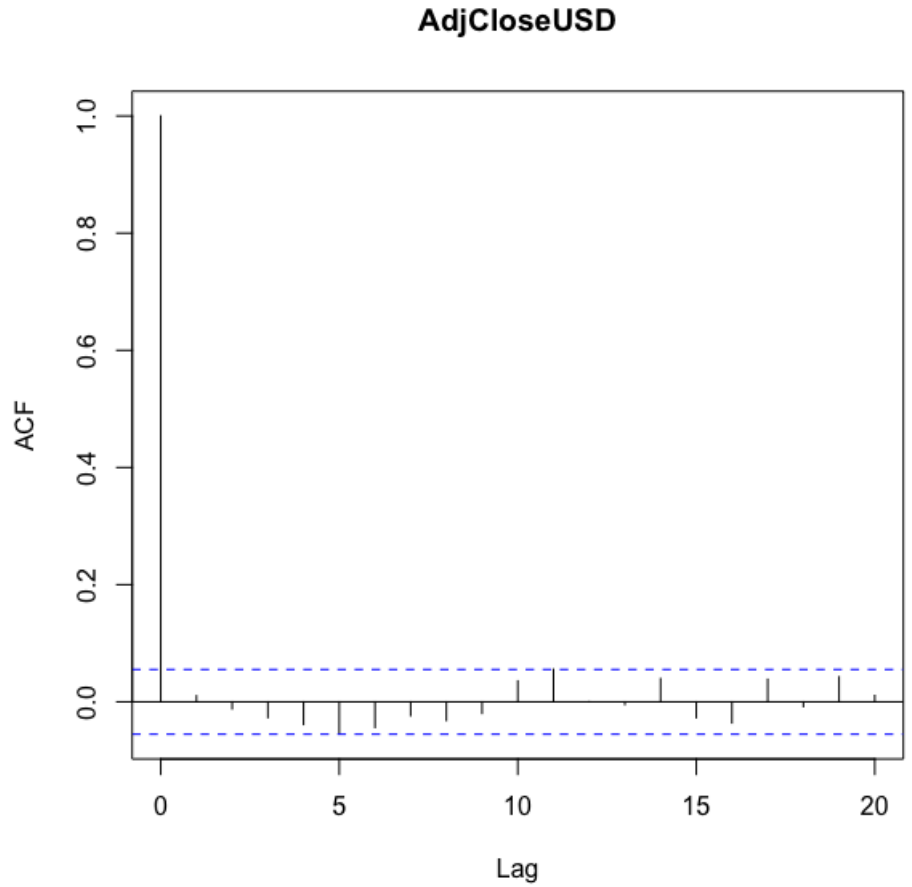
ACF and PACF of Non-Stationary Data

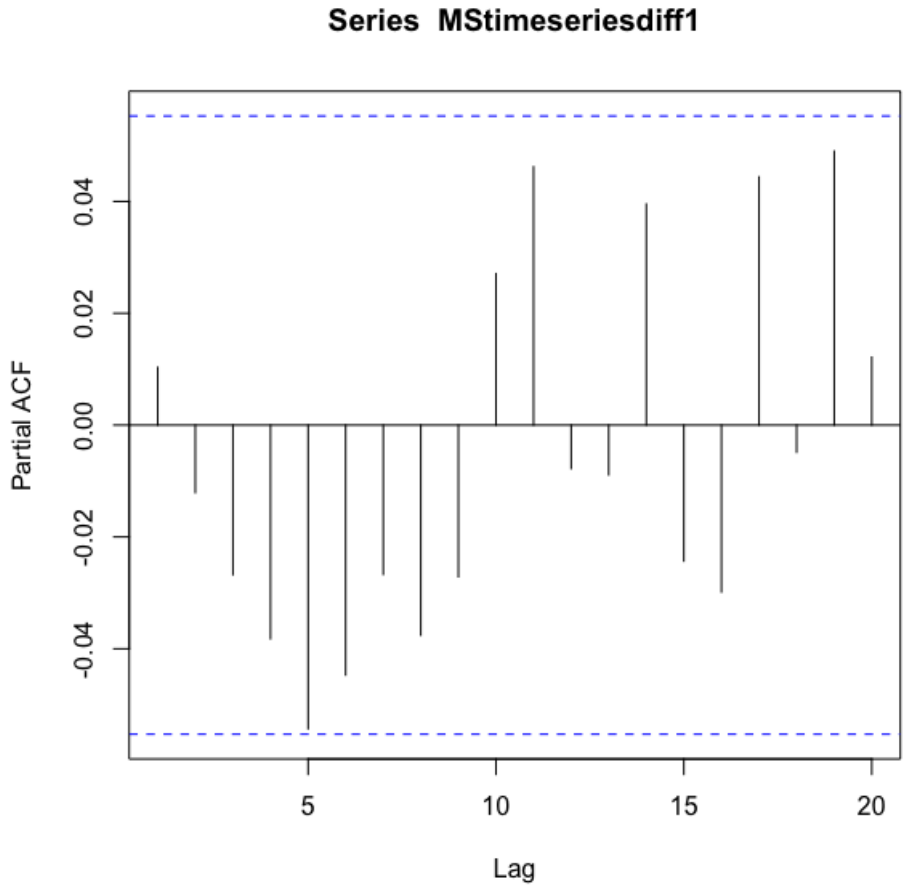




Price of Microsoft stock is highly correlated with the previous day's value.

ACF and PACF of Stationary Data





Daily changes in Microsoft stock price are essentially random.

ACF and PACF of Stationary and Non-Stationary

- Non-stationary series have an ACF that remains significant for half a dozen or more lags, rather than quickly declining to zero.

- You must difference such a series until it is stationary before you can identify the process. *Tip: In practice, never perform more than 2 non-seasonal differences, or one seasonal and one non-seasonal difference; also, never perform 2 seasonal differences.*

ARIMA

MODEL BUILDING

Box–Jenkins Methodology

- Model identification and model selection
  - Make sure variables are stationary. Difference as necessary to get a constant mean and transformations to get constant variance.
  - Check for seasonality: Decays and spikes at regular intervals in ACF plots.
- Parameter estimation
  - Compute coefficients that best fit the selected model.
- Model checking
  - Check if residuals are independent of each other and constant in mean and variance over time (white noise).

http://www.ncss.com/wp-content/themes/ncss/pdf/Procedures/NCSS/The_Box-Jenkins_Method.pdf

AutoRegressive Integrated Moving Average - ARIMA(p,d,q) Model

- *p* is the number of autoregressive [AR(p)] terms (a linear regression of the current value of the series against one or more prior values of the series)
  - Maximum lag beyond which PACF is 0 when PACF shows a **sharp cutoff** and/or lag1 autocorrelation is **positive**, i.e., series is slightly ***underdifferenced***.

AutoRegressive Integrated Moving Average - ARIMA(p,d,q) Model

- *d* is the number of non-seasonal differences (order of the differencing) used to make the time series stationary [I(d)]

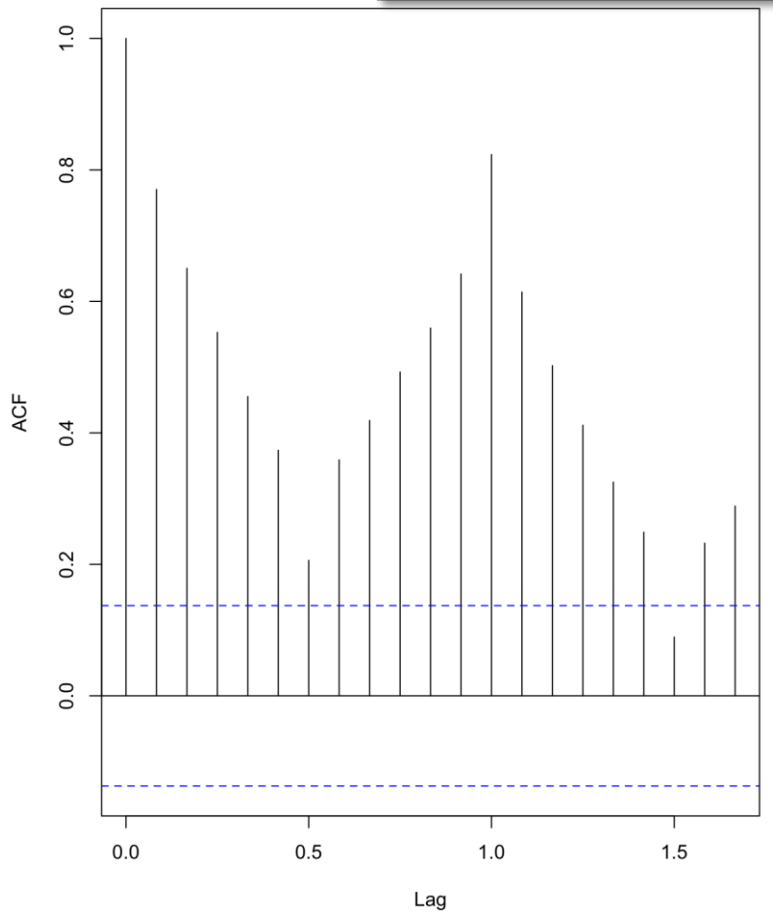
AutoRegressive Integrated Moving Average - ARIMA(p,d,q) Model

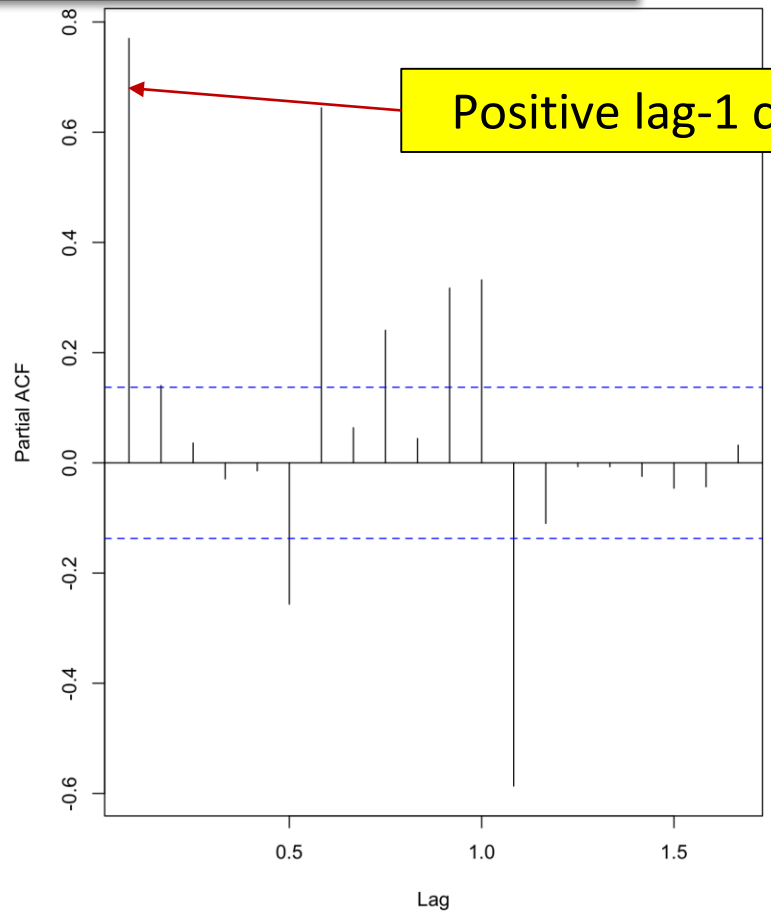
- *q* is the order of the moving average [MA(q)] model
  - Maximum lag beyond which the ACF is 0 when ACF shows a **sharp cutoff** and/or lag1 autocorrelation is **negative**, i.e., the series is slightly ***overdifferenced***.

For a comprehensive list of rules for identification of ARIMA models, see https://people.duke.edu/~rnau/411arim3.htm and https://people.duke.edu/~rnau/arimrule.htm.

AutoRegressive Integrated Moving Average - ARIMA(p,d,q) Model – Effect of Differencing

```
acf(milestimeseries, lag.max=20)
pacf(milestimeseries, lag.max=20)
```

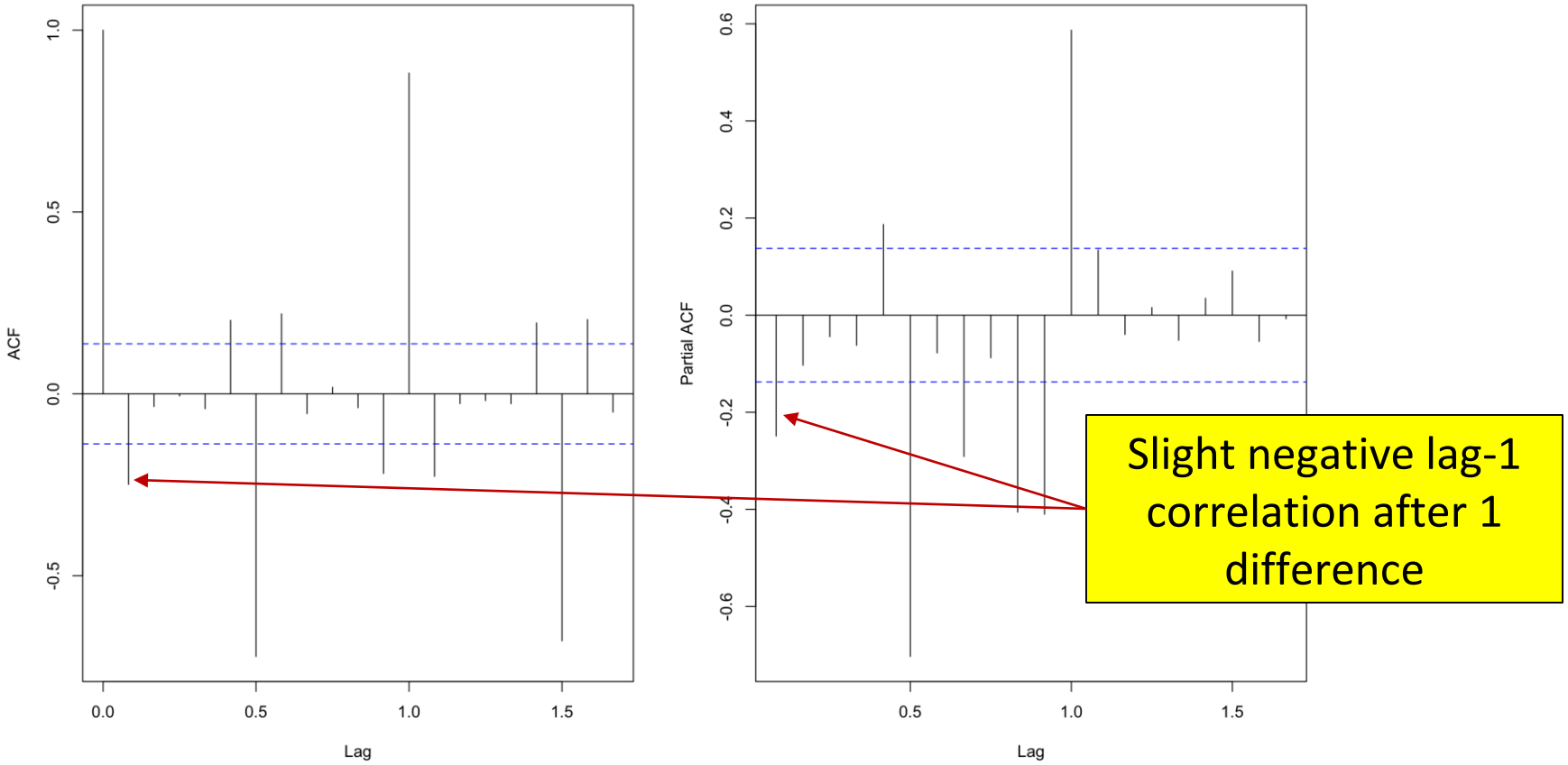




Positive lag-1 correlation

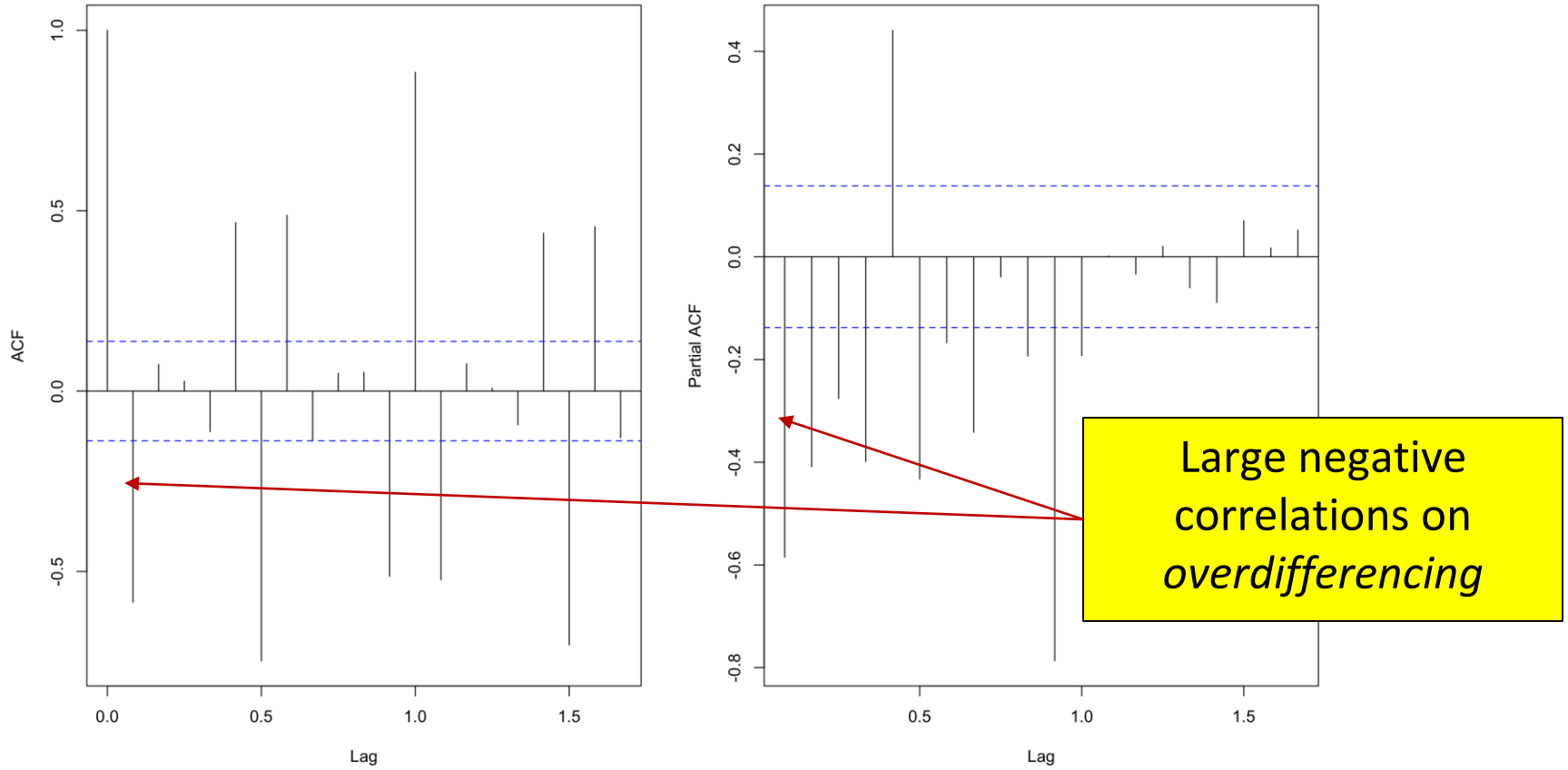
AutoRegressive Integrated Moving Average - ARIMA(p,d,q) Model – Effect of Differencing

```
milestimeseriesdiff1 <- diff(milestimeseries, differences=1)
milestimeseriesdiff1
acf(milestimeseriesdiff1, lag.max=20)
pacf(milestimeseriesdiff1, lag.max=20)
```



AutoRegressive Integrated Moving Average - ARIMA(p,d,q) Model – Effect of Differencing

```
milestimeseriesdiff2 <- diff(milestimeseries, differences=2)
milestimeseriesdiff2
acf(milestimeseriesdiff2, lag.max=20)
pacf(milestimeseriesdiff2, lag.max=20)
```



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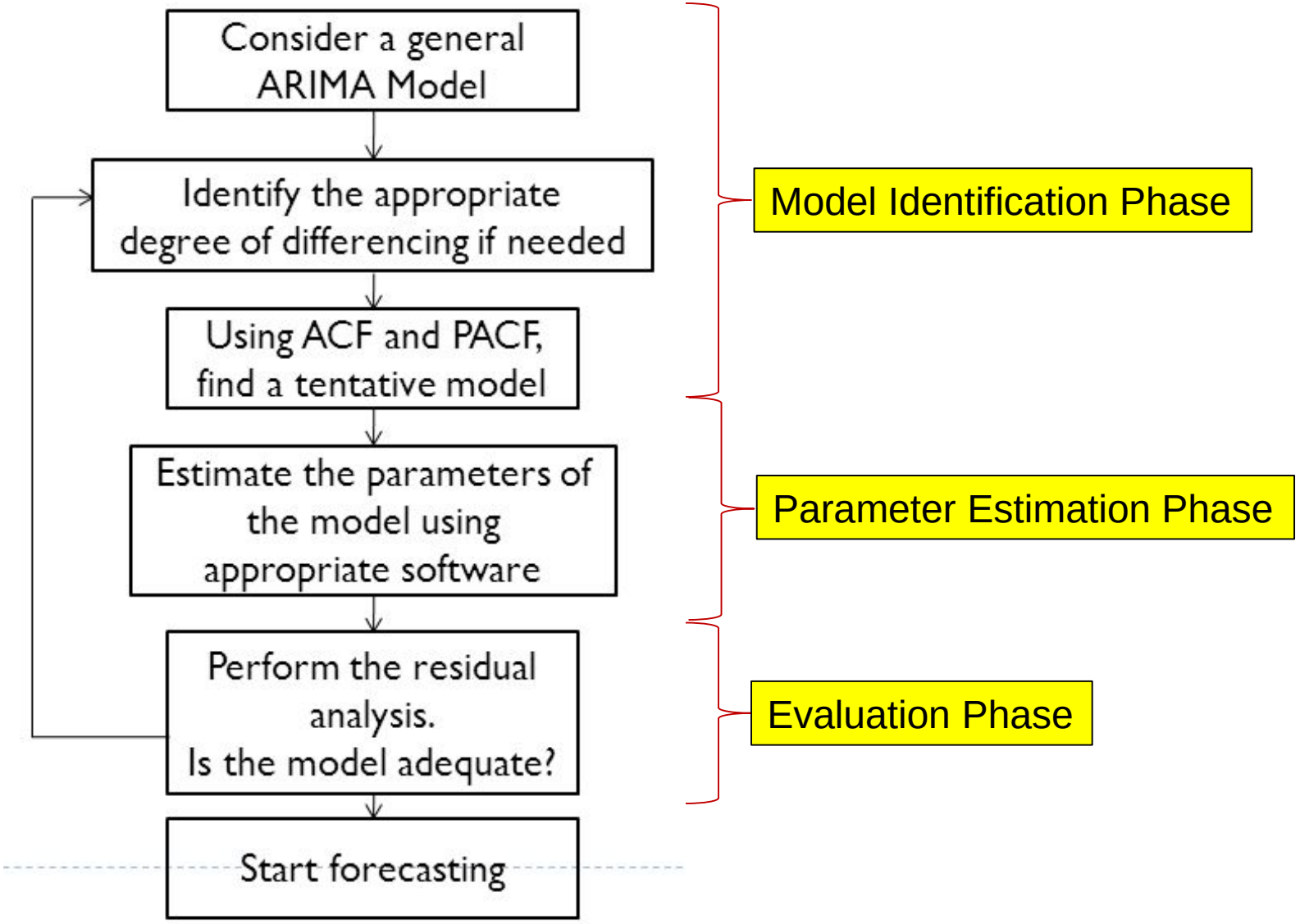
- Non-seasonal ARIMA models are denoted ARIMA(p,d,q)
- Seasonal ARIMA (SARIMA) models are denoted ARIMA(p,d,q)$(P,D,Q)_m$, where m refers to the number of periods in each season and (P,D,Q) refer to the autoregressive, differencing and moving average terms of the seasonal part of the ARIMA model.

Time Series Model Building Using ARIMA

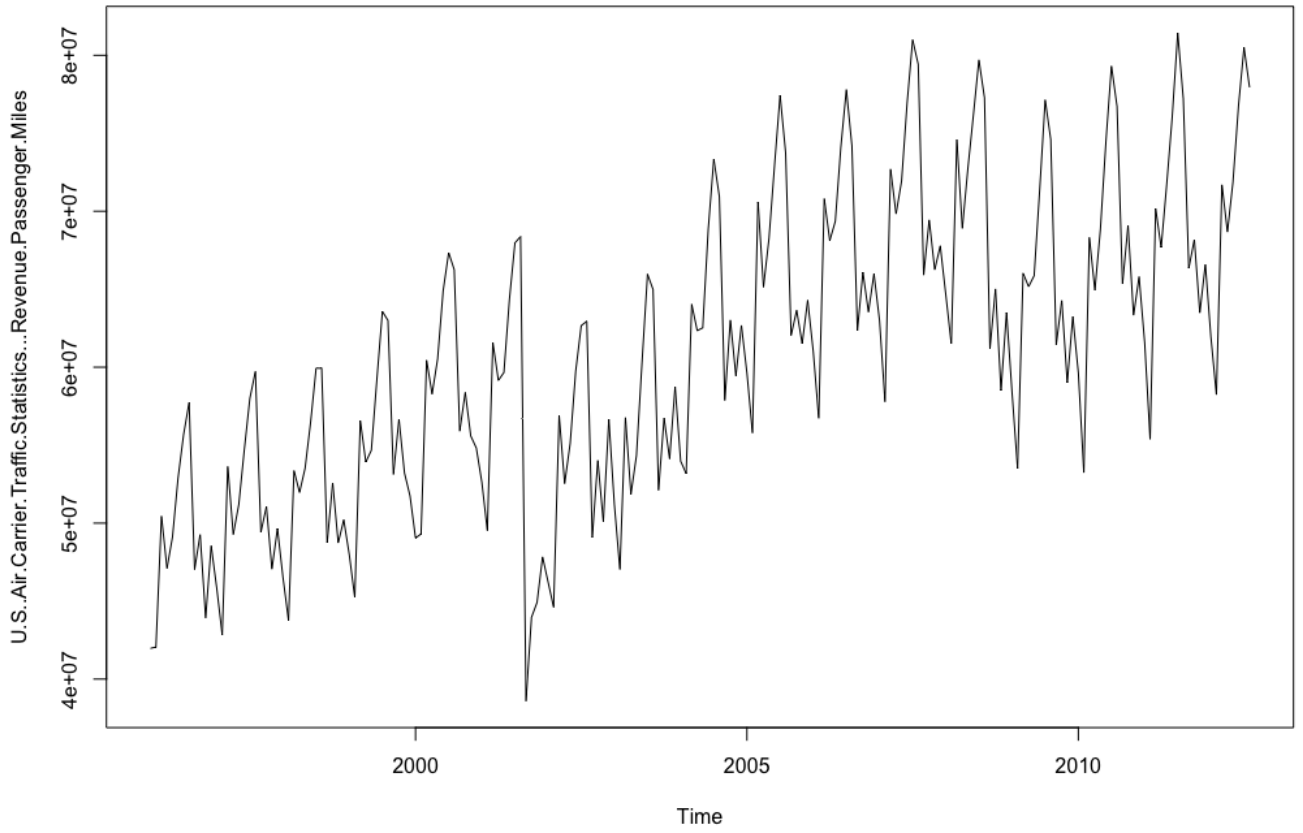


Time Series Model Building Using ARIMA

Identification Phase

Step 1: Plot the data (transform data to stabilize variance, if required)

R code: plot(milestimeseries)

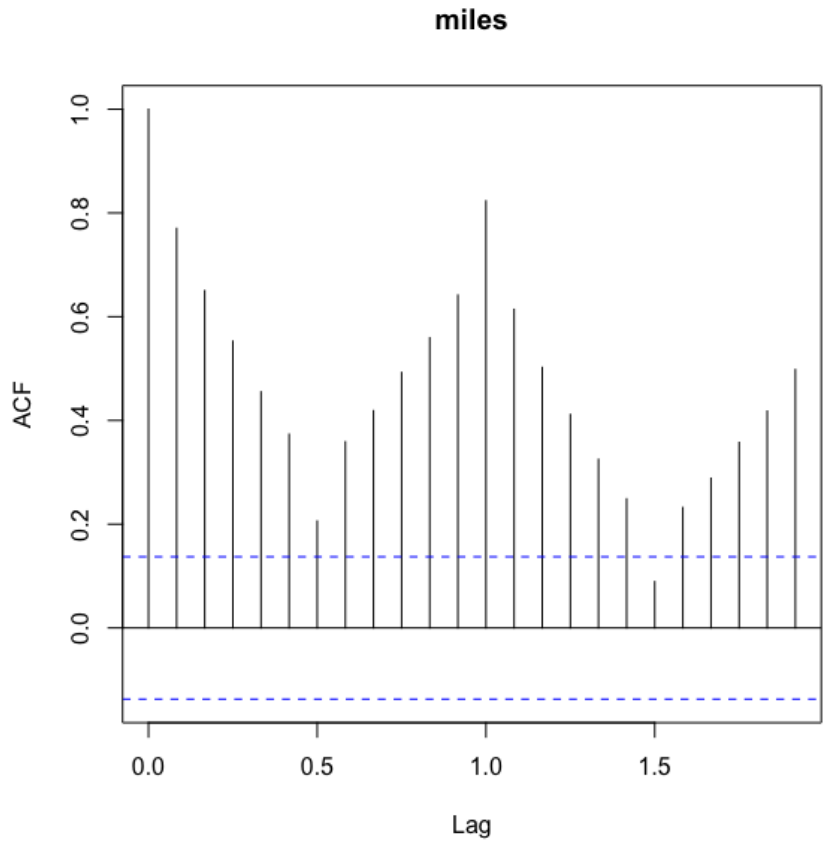


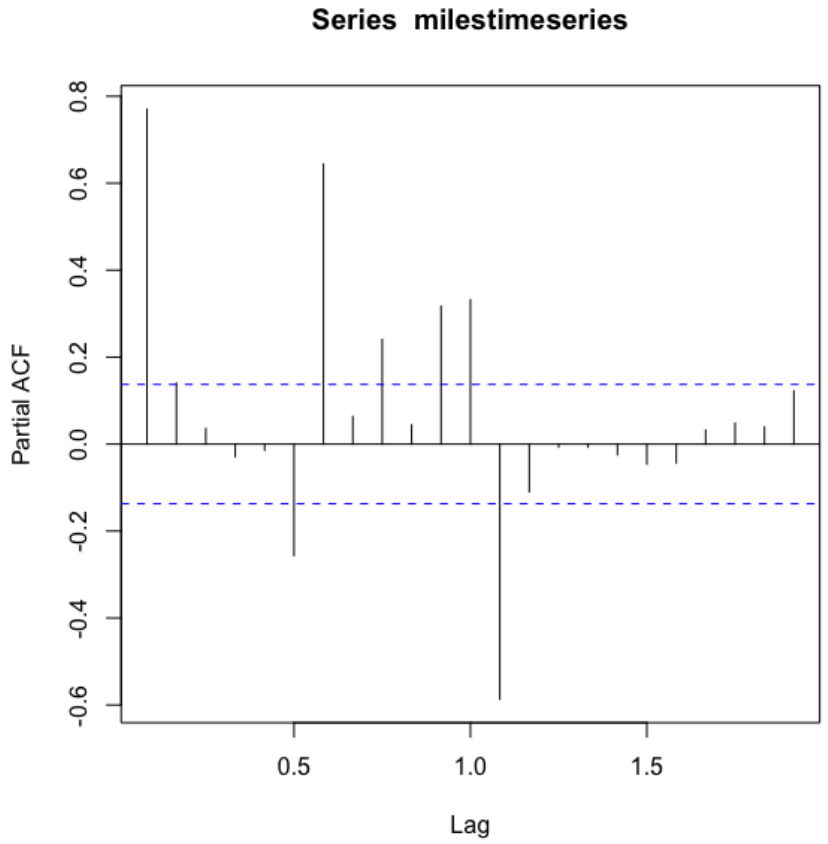
Time Series Model Building Using ARIMA

Identification Phase

Step 2: Plot ACF and PACF to get preliminary understanding of the processes involved.

R code: acf(milestimeseries) pacf(milestimeseries)

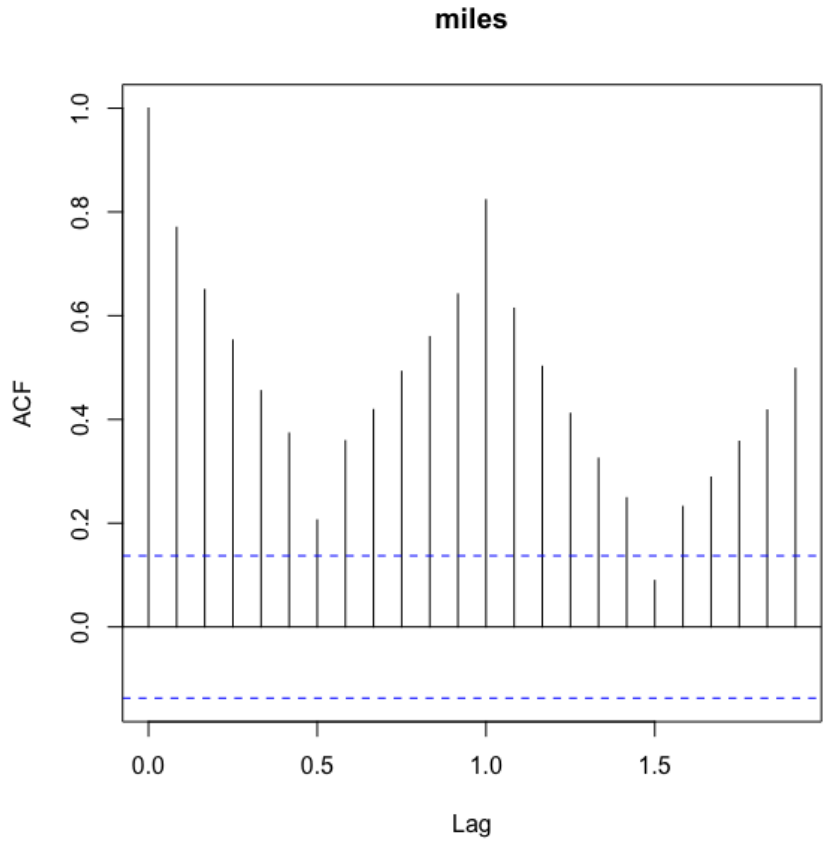




Time Series Model Building Using ARIMA

Identification Phase

Step 2: Plot ACF and PACF to get preliminary understanding of the processes involved.

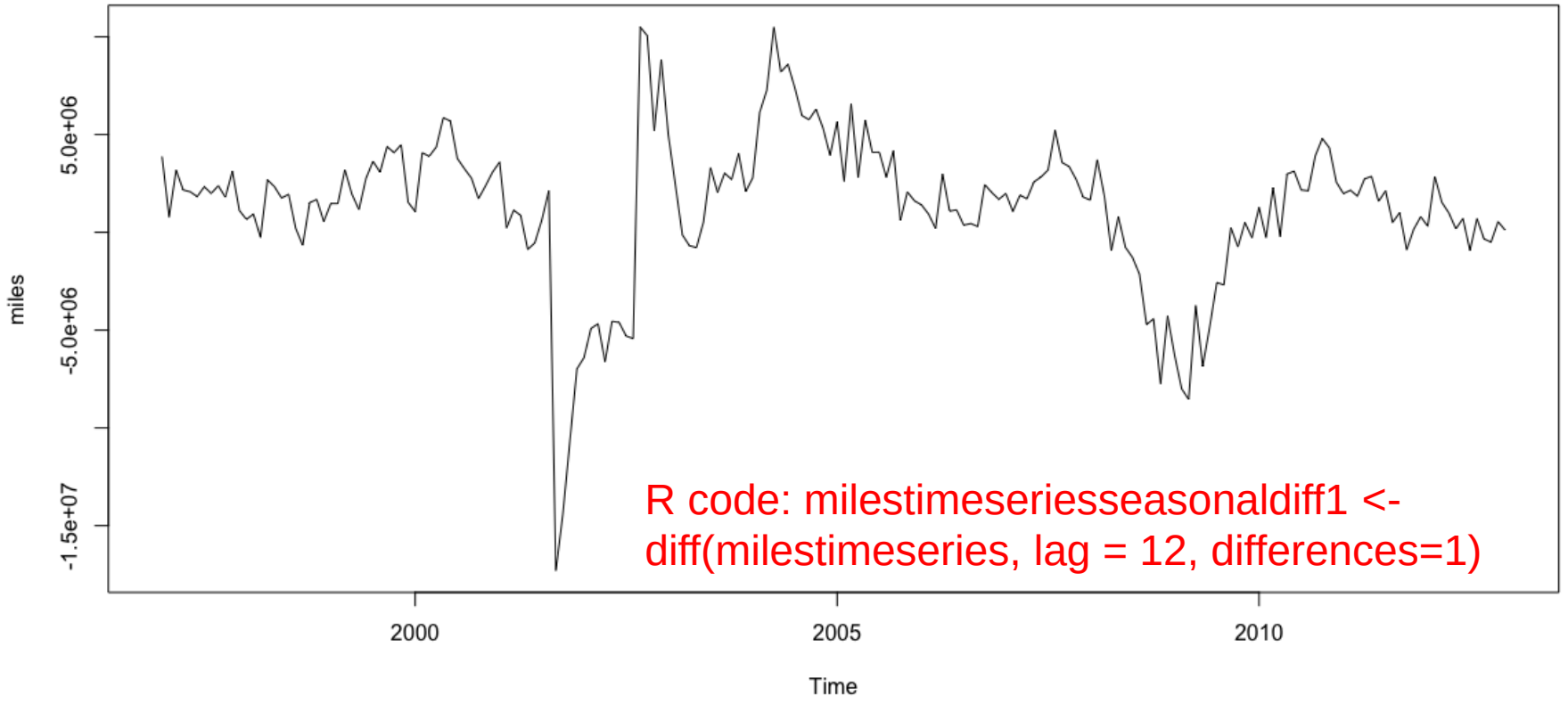


The suspension bridge pattern in ACF (also, positive and negative spikes in PACF) suggests non-stationarity and strong seasonality.

Time Series Model Building Using ARIMA

Identification Phase

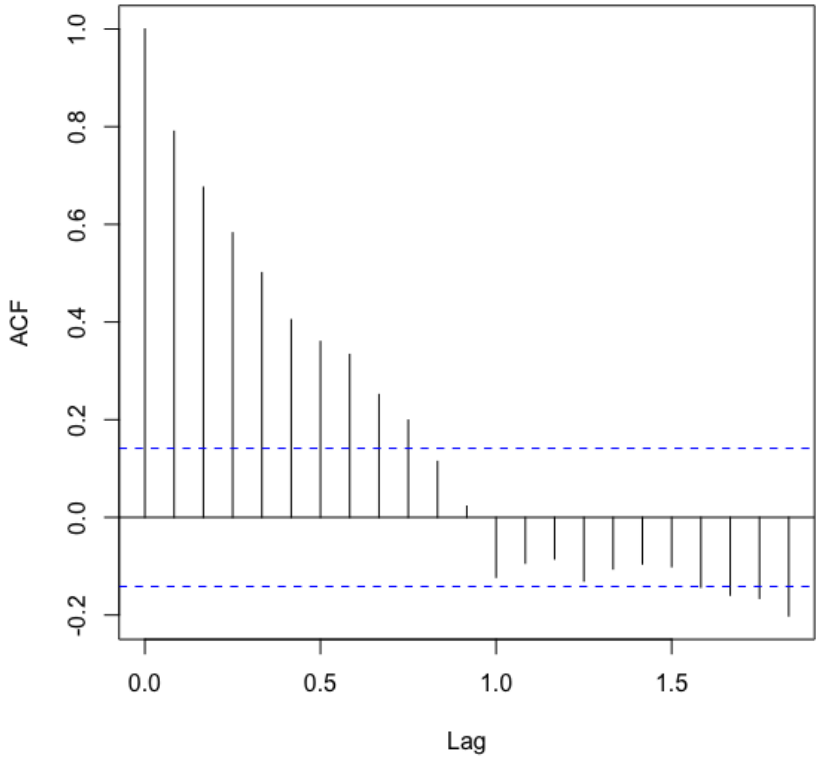
Step 3: Perform seasonal differencing (t_0-t_{12}, t_1-t_{13}, etc.) on the **original** time series to get seasonal stationarity. This is the same as an ARIMA(0,0,0)(0,1,0)$_{12}$ model.

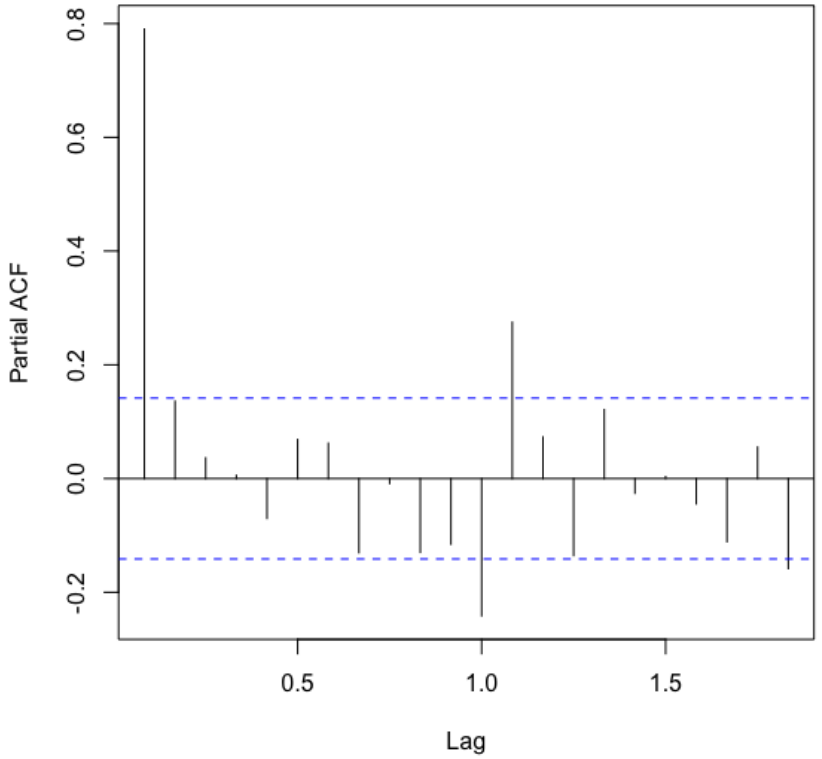


Time Series Model Building Using ARIMA

Identification Phase

Step 4: Check ACF and PACF of seasonally differenced data to explore remaining dependencies and identify model(s).





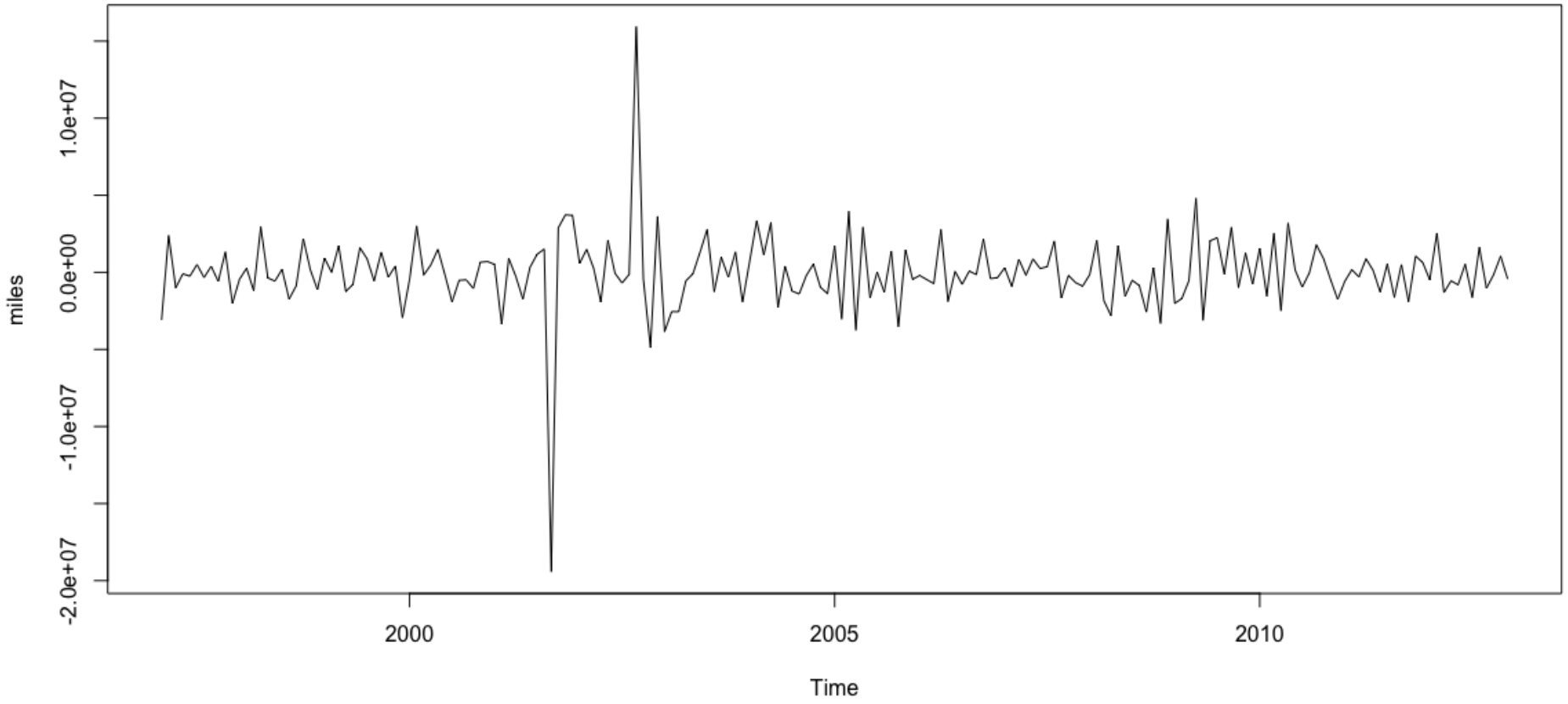
Positive autocorrelation indicates *underdifferencing* requiring either an AR term or a non-seasonal differencing to fix it.

Time Series Model Building Using ARIMA

Identification Phase

Step 5: Perform a non-seasonal differencing on seasonally differenced data. This is like an ARIMA(0,1,0)(0,1,0)$_{12}$ model.

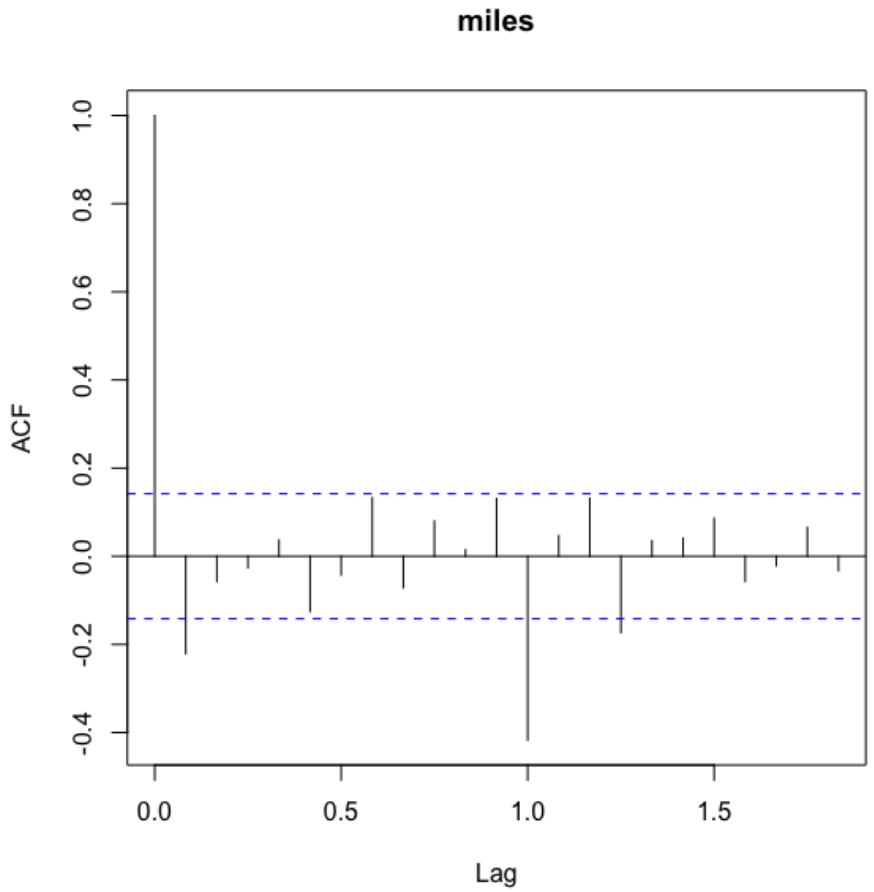
Rcode: milestimeseriesSeasNoSeasdiff1 <- diff(milestimeseriesseasonaldiff1, differences=1)

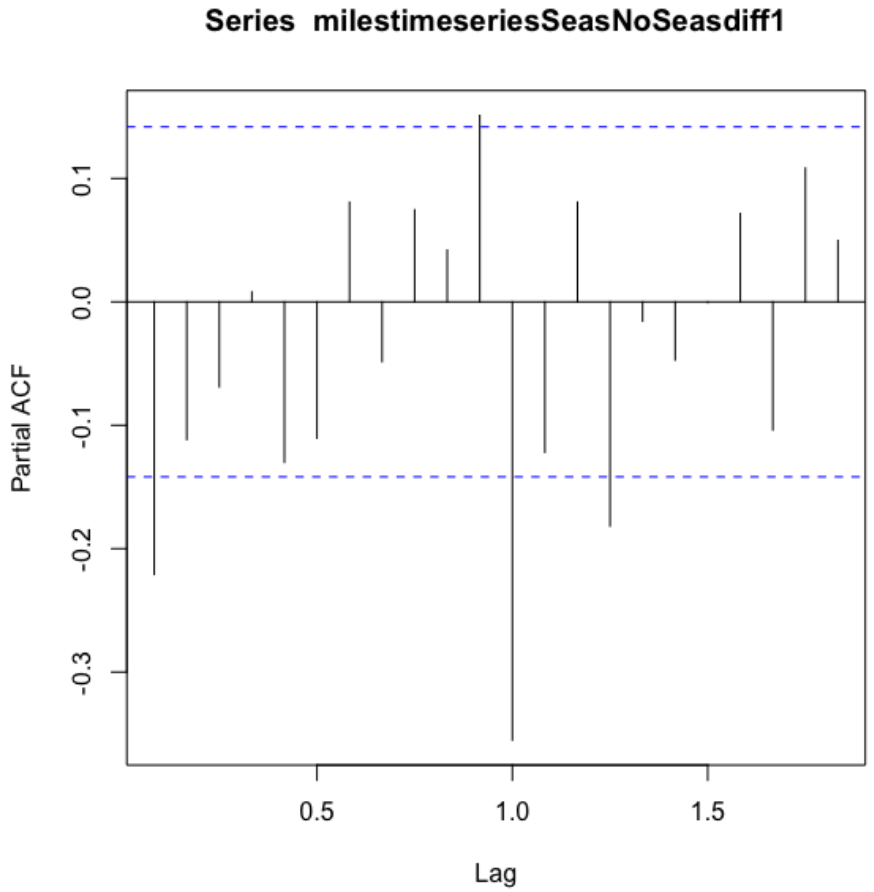


Time Series Model Building Using ARIMA

Identification Phase

Step 6: Check ACF and PACF to explore remaining dependencies.





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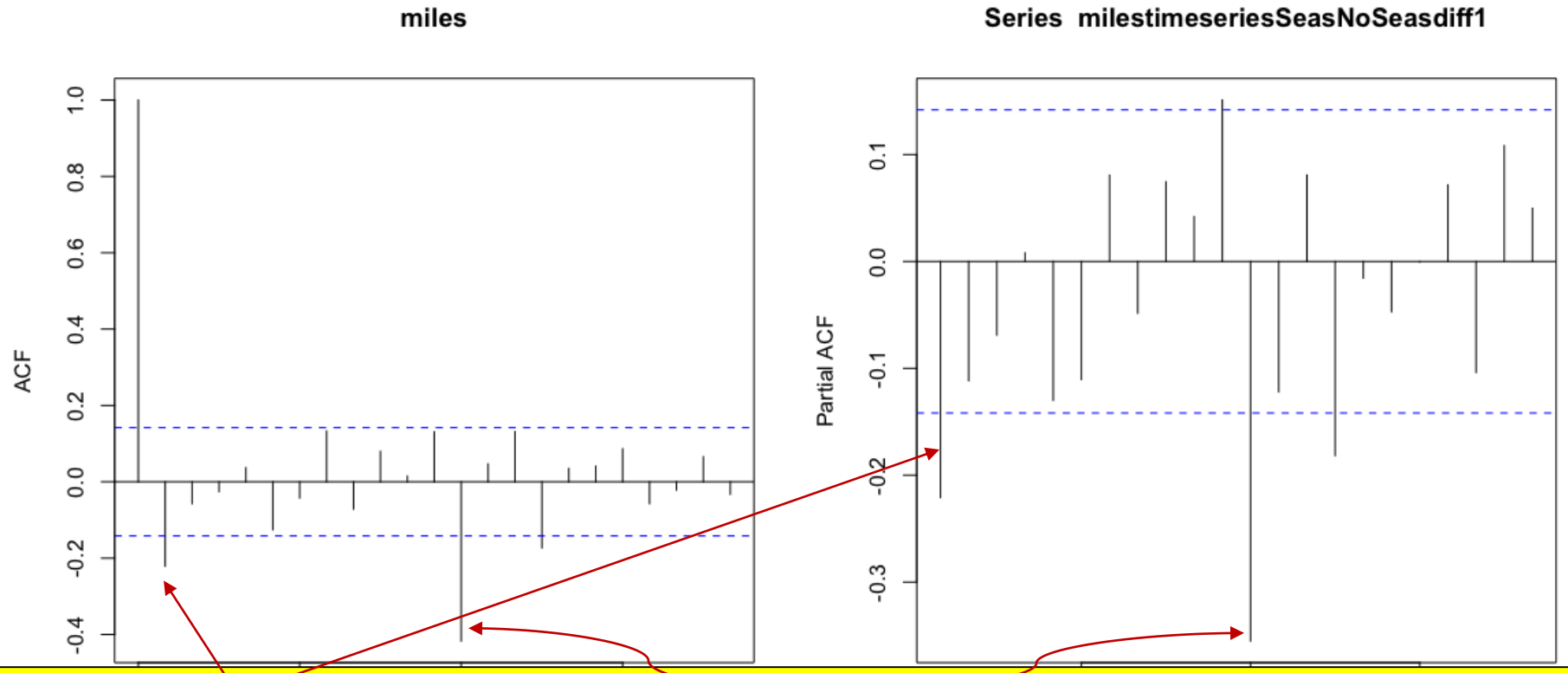
Time Series Model Building Using ARIMA

Identification Phase

Step 7: This indicates an ARIMA(0,1,1)(0,1,1)$_{12}$ model.

R code: Arima(milestimeseries, order = c(0,1,1), seasonal = c(0,1,1), include.drift = TRUE)



As lag1 and the significant lag at seasonal period are **negative**, include a MA(1) and SMA(1) terms. Negative autocorrelation indicates *overdifferencing* requiring MA term to fix it.

Time Series Model Building Using ARIMA

Parameter Estimation Phase

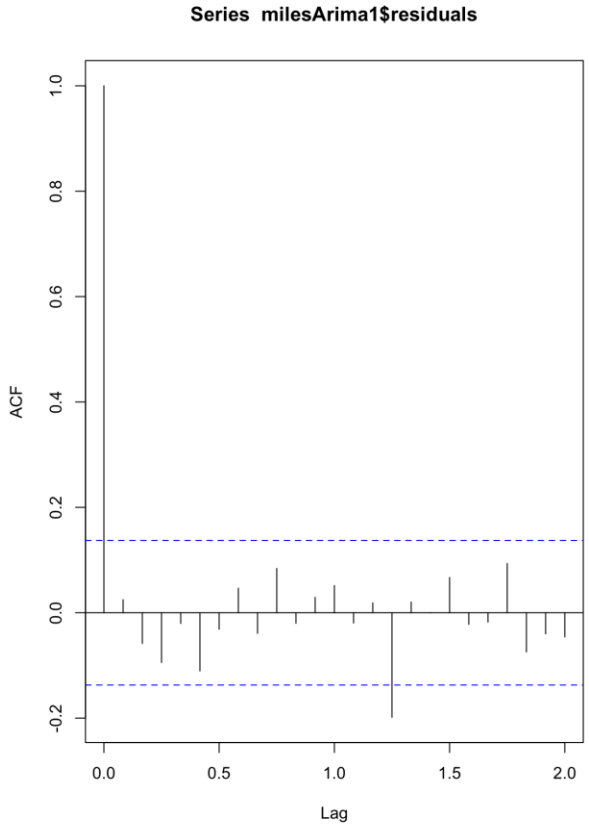
Step 7: Calculate parameters using the identified model(s). Use AIC to pick the best model.

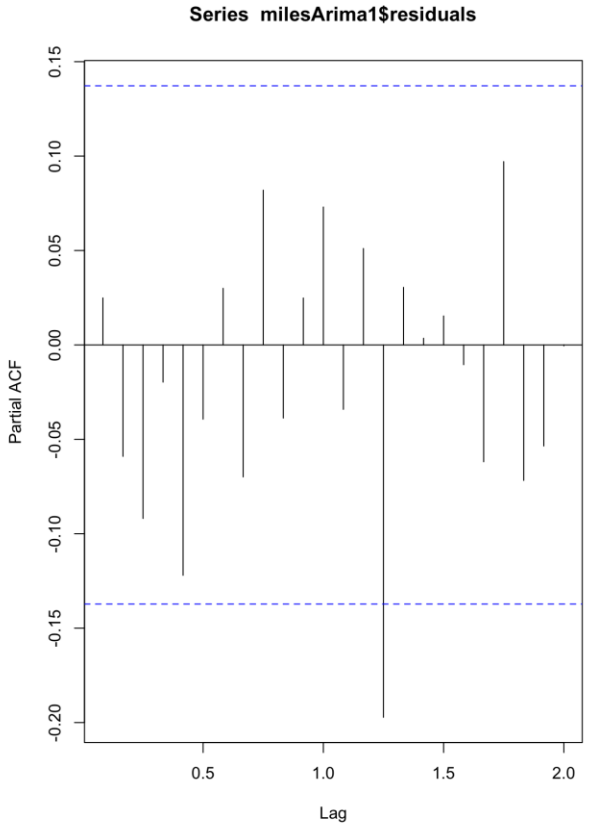
```
Series: milestimeseries
ARIMA(0,1,1)(0,1,1)[12]

Coefficients:
          ma1     sma1
      -0.2757  -0.7277
s.e.   0.0791   0.0653

sigma^2 estimated as 3.974e+12:  log likelihood=-3045.1
AIC=6096.21   AICc=6096.34   BIC=6105.97
```

Time Series Model Building Using ARIMA

Evaluation Phase

Step 8: Check ACF and PACF of the residuals to evaluate model. The residuals indicate white noise. Indicates a good model that can be used for forecasting.





Rcode: acf(milesArima1$residuals, lag.max = 24)
pacf(milesArima1$residuals, lag.max = 24)

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Time Series Model Building Using ARIMA

Evaluation Phase

Step 8: The residuals indicate white noise. Can be checked using Ljung-Box test.

R code: Box.test(milesArima1$residuals, lag=24, type="Ljung-Box")

$$Q^* = n(n+2)\sum_{k=1}^{h}\frac{{r_k}^2}{n-k}$$

h is the maximum lag being considered
n is the # of observations (length of the time series)
r_k is the autocorrelation

* For non-seasonal time series, use *h = min(10, n/5)*
For seasonal time series, use *h = min(2m, n/5)*, where m is the seasonal period

```
        Box-Ljung test

data:  milesArima1$residuals
X-squared = 23.114, df = 24, p-value = 0.5131
```

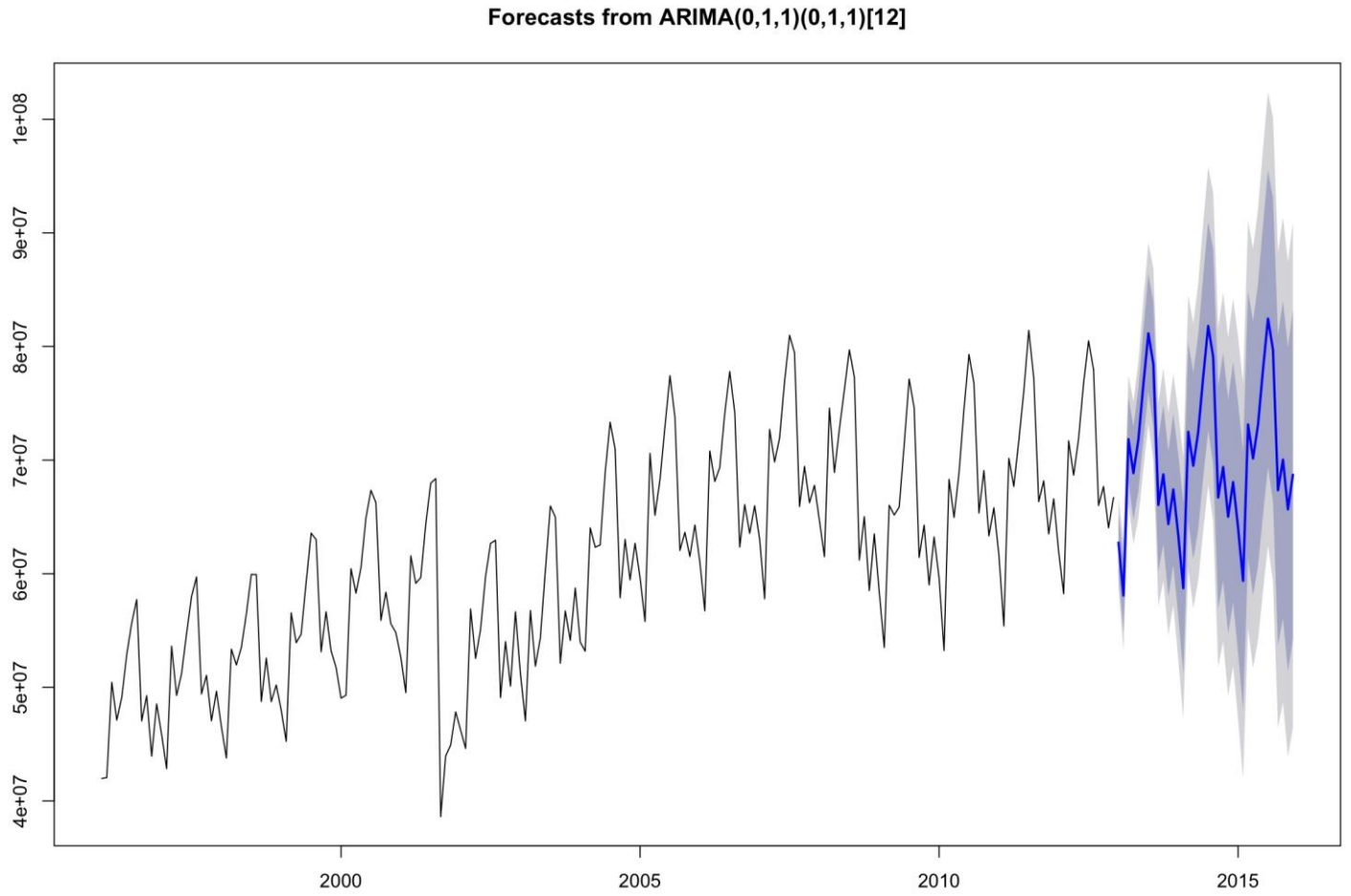
If residuals are white noise (purely random), then *Q* has a χ^2 distribution with *h-p* degrees of freedom, where *p* is the number of parameters estimated in the model

* http://robjhyndman.com/hyndsight/ljung-box-test/

Time Series Model Building Using ARIMA

Forecasting Phase

Step 9: Start forecasting. R code: forecast.Arima(milesArima1, h=36) or forecast(milesArima1, h=36)



Time Series Model Building Using ARIMA

Alternate Steps 5-9 from previous model

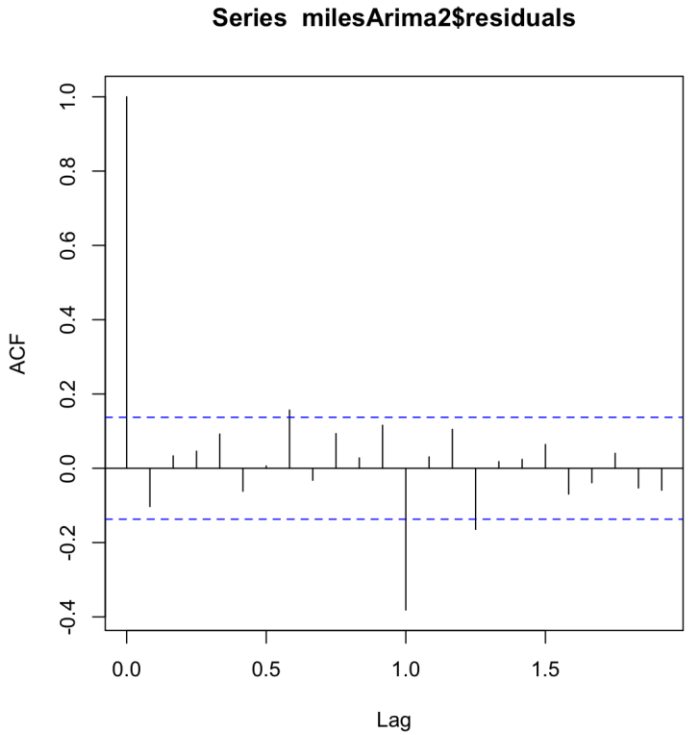
Step 5: Add an AR term to seasonally differenced data. This is like an ARIMA(1,0,0)(0,1,0)$_{12}$ model.

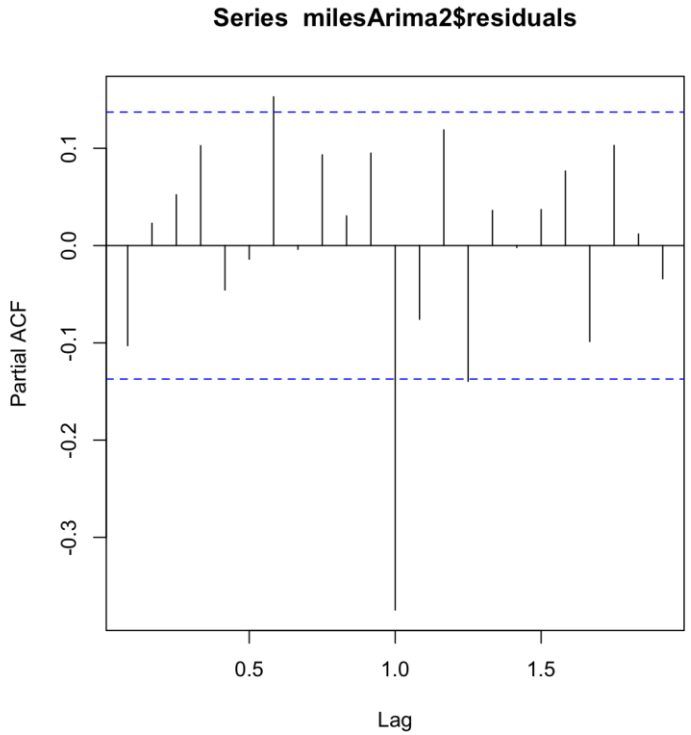
```
Series: milestimeseries
ARIMA(1,0,0)(0,1,0)[12] with drift

Coefficients:
          ar1      drift
      0.7886  109198.14
s.e.  0.0436   65385.94

sigma^2 estimated as 5.541e+12:  log likelihood=-3088.87
AIC=6183.73   AICc=6183.86   BIC=6193.5
```

Time Series Model Building Using ARIMA

Alternate Steps 5-9 from previous model

Step 6: Check ACF and PACF to identify remaining dependencies. Strong negative autocorrelation at the seasonal period indicates need for a seasonal MA term for an ARIMA(1,0,0)(0,1,1)$_{12}$ model.






```
        Box-Ljung test

data:  milesArima2$residuals
X-squared = 61.725, df = 24, p-value = 3.632e-05
```

Time Series Model Building Using ARIMA

Alternate Steps 5-9 from previous model

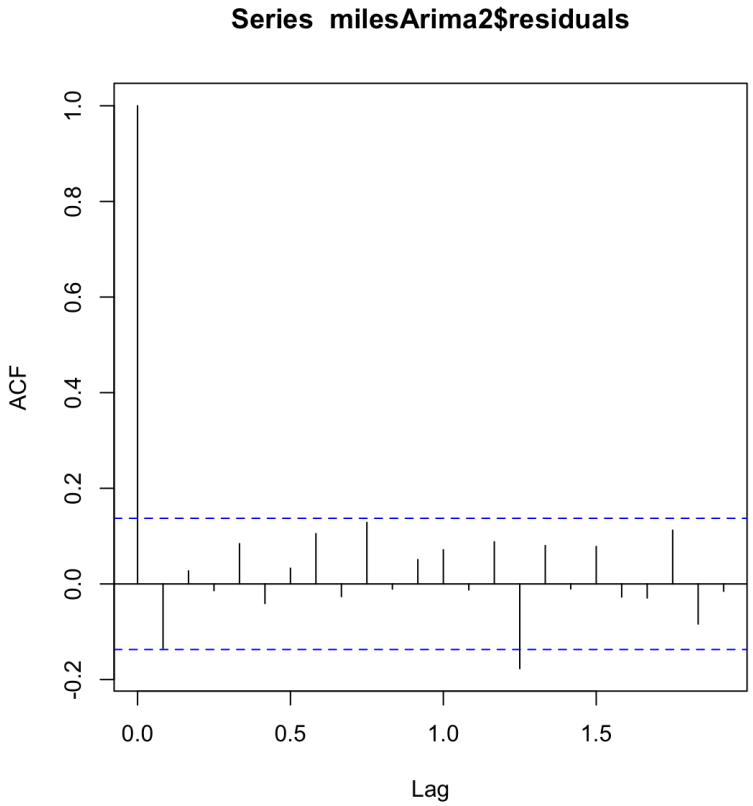
Step 7: ARIMA(1,0,0)(0,1,1)$_{12}$ model.

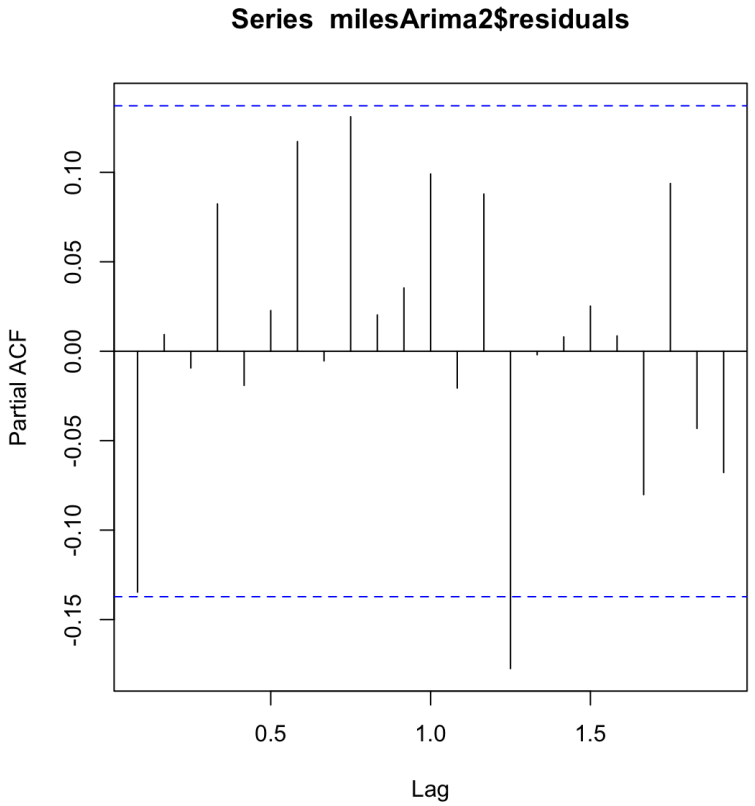
```
Series: milestimeseries
ARIMA(1,0,0)(0,1,1)[12] with drift

Coefficients:
          ar1     sma1      drift
      0.8527  -0.7254  109302.05
s.e.  0.0378   0.0678   25762.84

sigma^2 estimated as 3.923e+12:  log likelihood=-3059.73
AIC=6127.46   AICc=6127.67   BIC=6140.49
```

Time Series Model Building Using ARIMA

Alternate Steps 5-9 from previous model

Step 8: The residuals indicate white noise. Indicates a good model that can be used for forecasting.



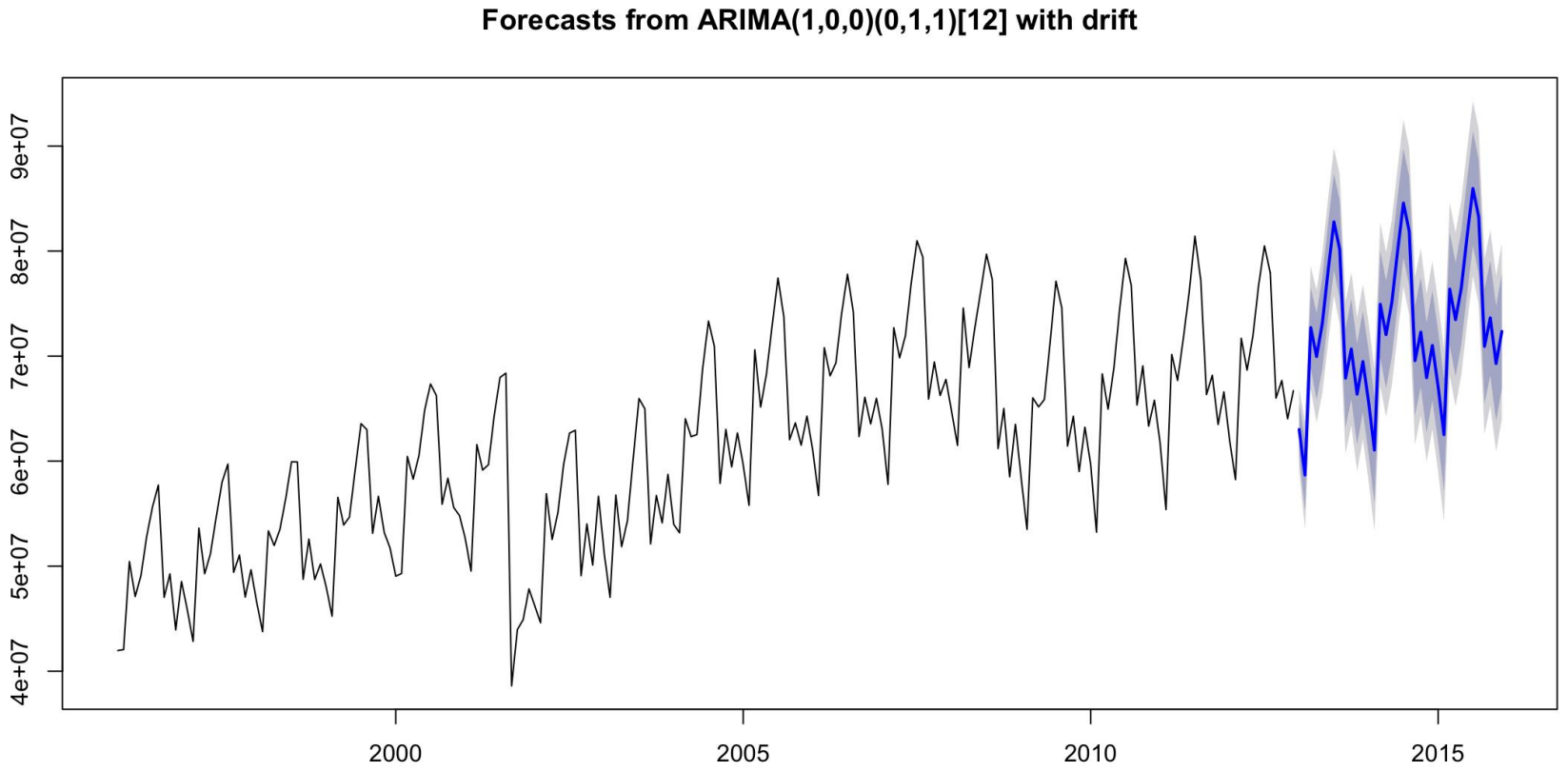



```
        Box-Ljung test

data:  milesArima2$residuals
X-squared = 30.776, df = 24, p-value = 0.1603
```

Time Series Model Building Using ARIMA

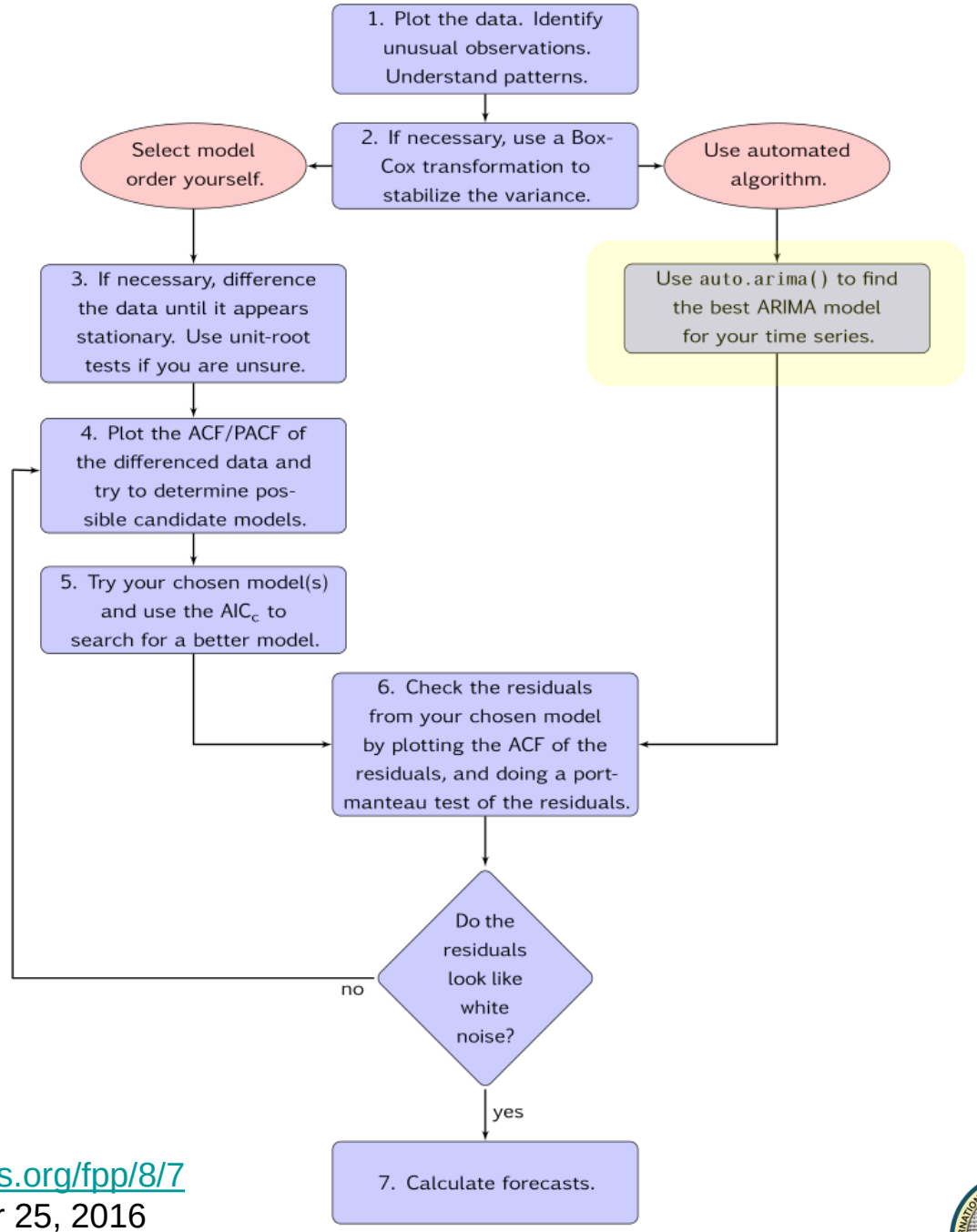
Alternate Steps 5-9 from previous model

Step 9: Start forecasting.



Model Selection in Practice

There are techniques that automate model selection



Source: https://www.otexts.org/fpp/8/7
Last accessed: November 25, 2016

Model Selection in Practice

auto.arima() in R using Hyndman-Khandakar algorithm.

1. Find *d* for stationarity.
2. *p* and *q* are selected using a stepwise search to minimize AICc on the differenced data.
   a. The best model among the following is selected as the current model: ARIMA(2,*d*,2), ARIMA(0,*d*,0), ARIMA(1,*d*,0), ARIMA(0,*d*,1). If *d*=0, constant *c* is included; if *d*≥1, *c* is set to 0.
   b. *p* and *q* are varied by ±1.
   c. *c* is included/excluded.
   d. 2(b) and (c) are repeated till AICc doesn't reduce further.

Source: https://www.otexts.org/fpp/8/7
Last accessed: November 25, 2016

Forecast using Auto ARIMA

```
Series: milestimeseries
ARIMA(1,0,1)(0,1,1)[12] with drift

Coefficients:
         ar1      ma1     sma1      drift
      0.9092  -0.2128  -0.7257  108456.69
s.e.  0.0358   0.0873   0.0673   31546.77

sigma^2 estimated as 3.834e+12:  log likelihood=-3056.9
AIC=6123.81   AICc=6124.13   BIC=6140.09
```

R code: auto.arima(milestimeseries,ic='aic')

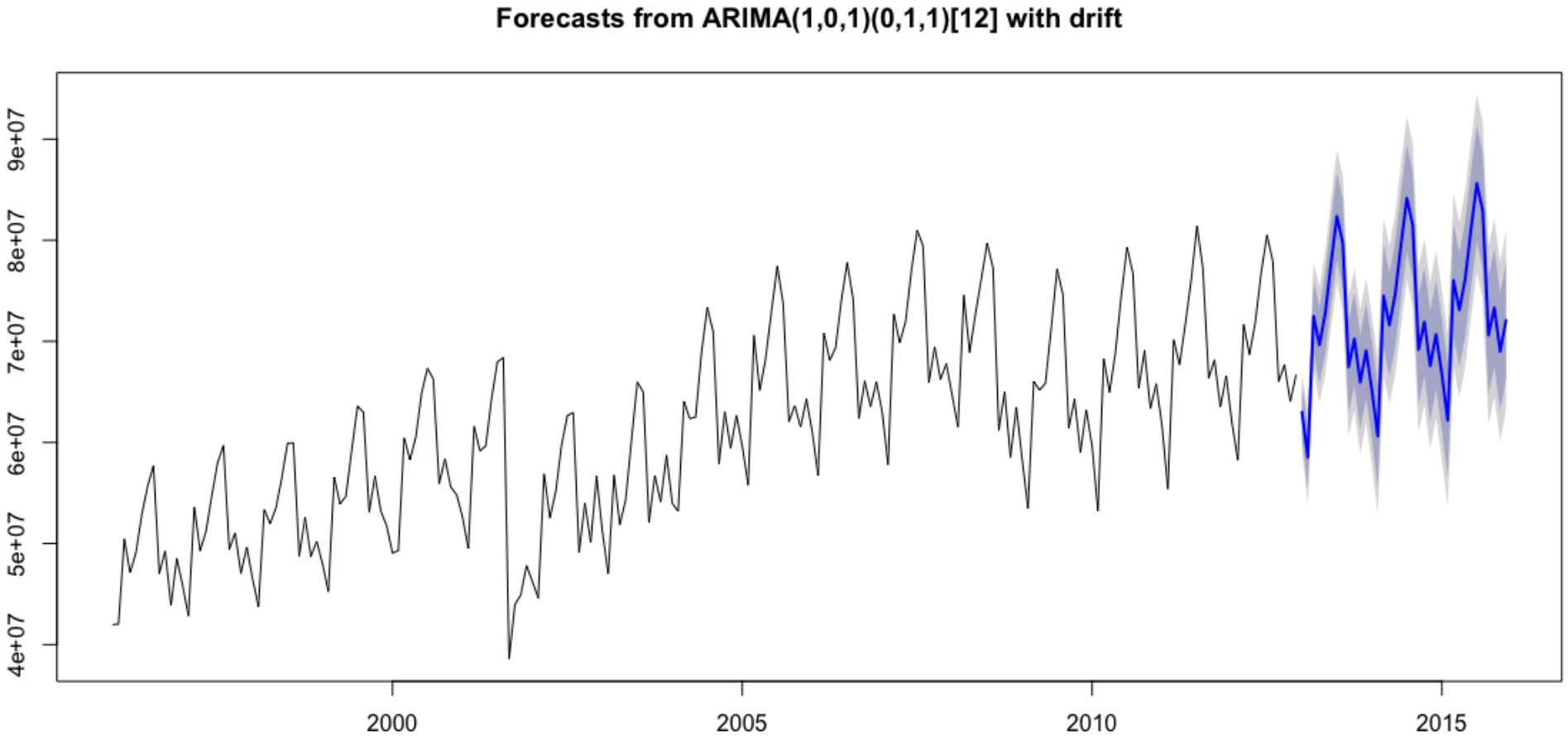
Auto ARIMA

```
Series: milestimeseries
ARIMA(1,0,0)(0,1,1)[12] with drift

Coefficients:
         ar1     sma1      drift
      0.8527  -0.7254  109302.05
s.e.  0.0378   0.0678   25762.84

sigma^2 estimated as 3.923e+12:  log likelihood=-3059.73
AIC=6127.46   AICc=6127.67   BIC=6140.49
```

Manual ARIMA2

Forecast using Auto-ARIMA - RPM

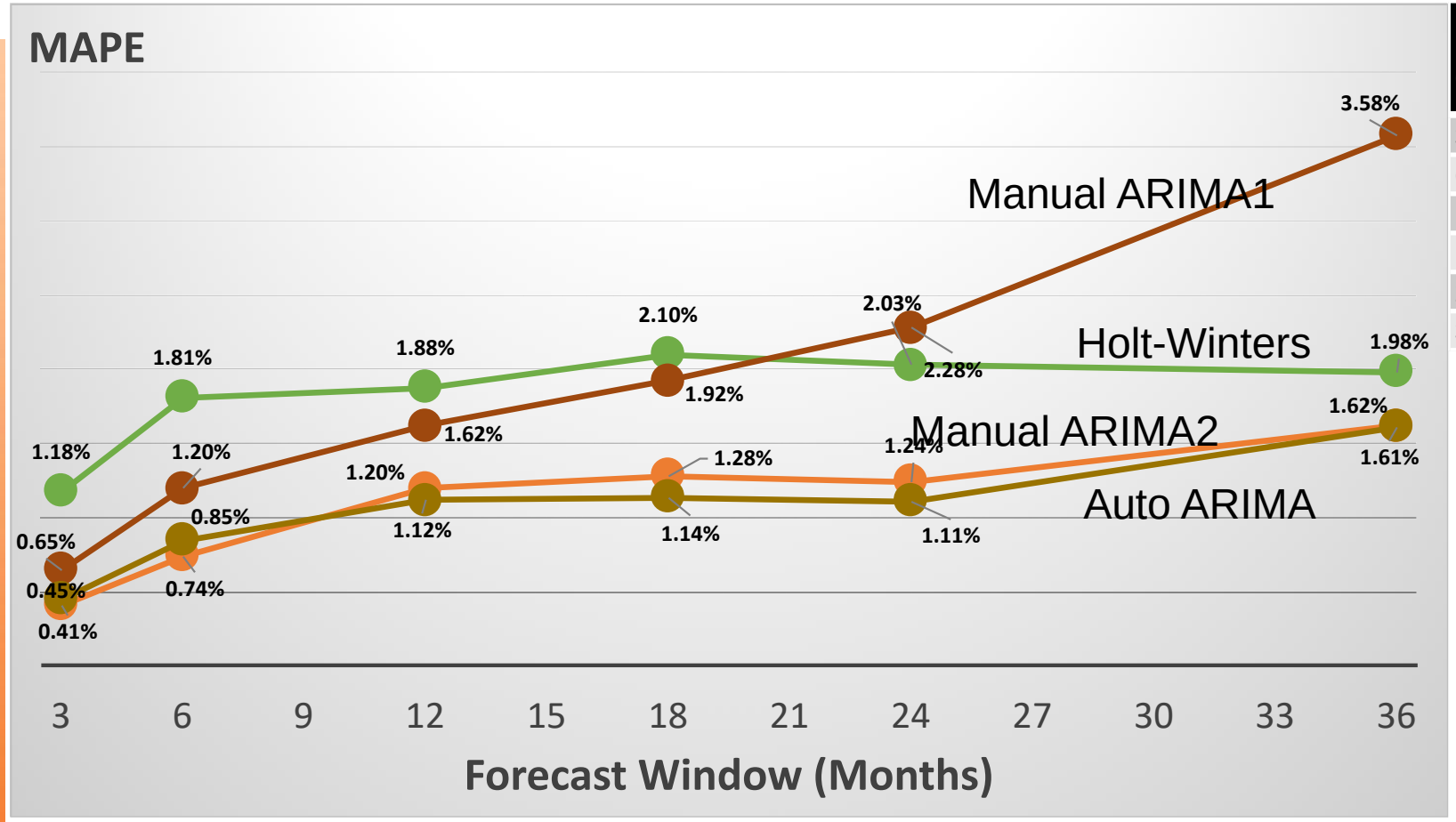
R code: forecast.Arima(milesAutoArima, h=36)



Goodness of Fit

- MAE (Mean absolute error) $\frac{\sum|y_i-\hat{y}_i|}{n}$
- MSE (Mean square error) $\frac{\sum(y_i-\hat{y}_i)^2}{n}$
- RMSE (Root mean square error) $\sqrt{\frac{\sum(y_i-\hat{y}_i)^2}{n}}$
- MAPE (Mean absolute percent error) $\frac{1}{n}\left(\frac{\sum|y_i-\hat{y}_i|}{y_i}\right)*100$

Comparing Models - MAPE



Forecast window (months)	Holt-Winters	Manual ARIMA1	Manual ARIMA2	Auto ARIMA
3	1.18%	0.65%	0.41%	0.45%
6	1.81%	1.20%	0.74%	0.85%
12	1.88%	1.62%	1.20%	1.12%
18	2.10%	1.92%	1.28%	1.14%
24	2.03%	2.28%	1.24%	1.11%
36	1.98%	3.58%	1.62%	1.61%

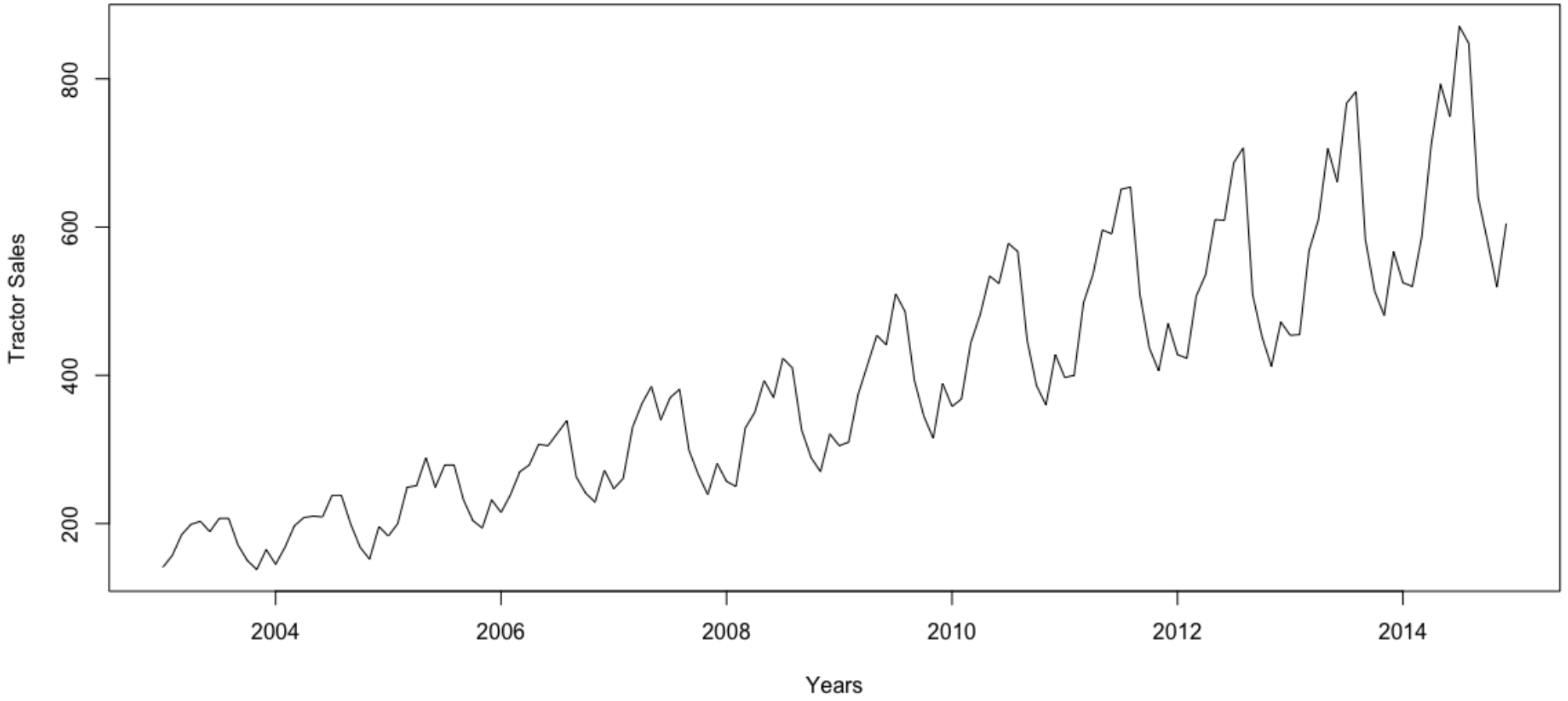
Manufacturing Case Study

FORECASTING TRACTOR SALES

From: http://ucanalytics.com/blogs/step-by-step-graphic-guide-to-forecasting-through-arima-modeling-in-r-manufacturing-case-study-example/

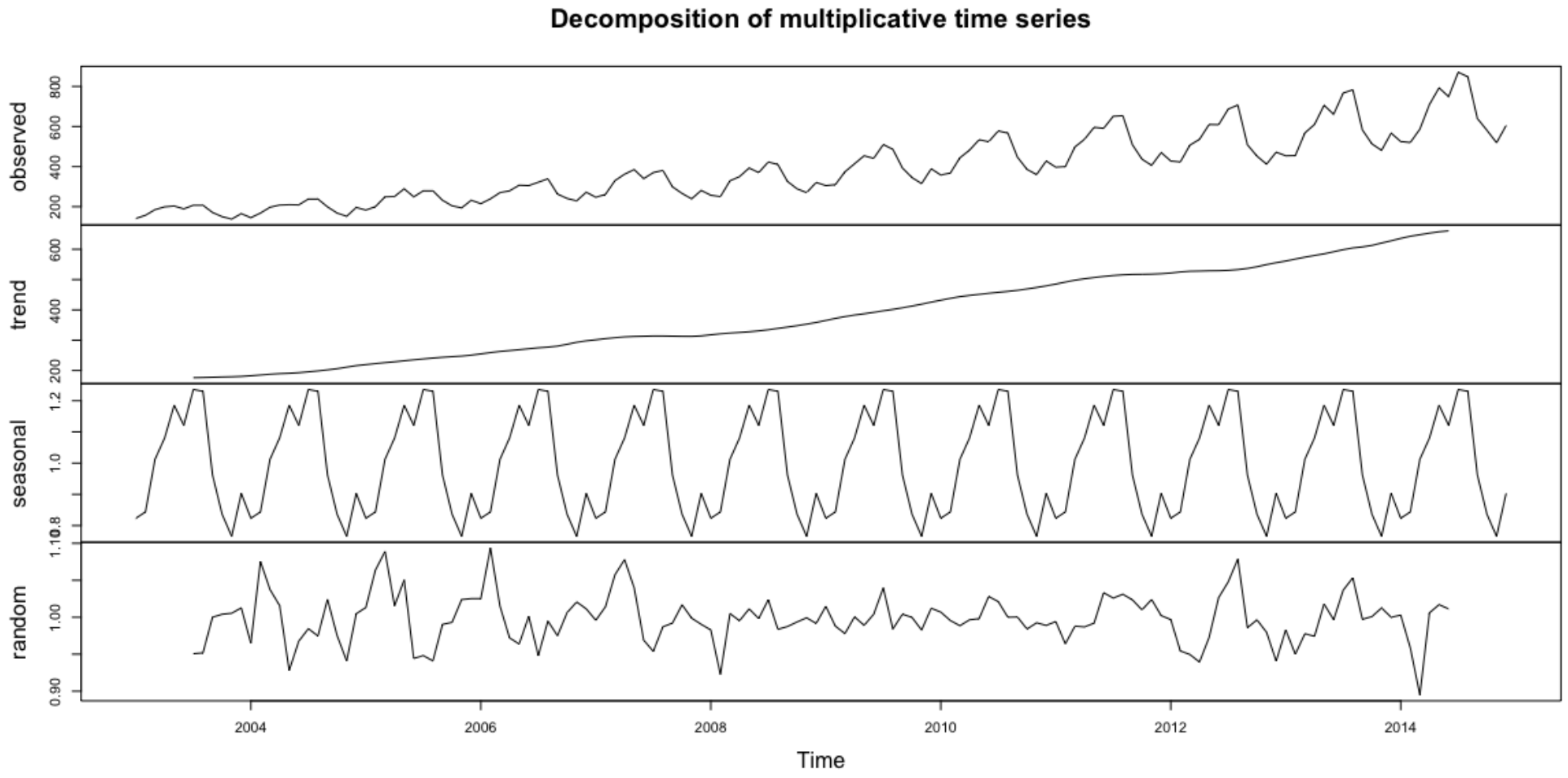
Tractor Sales Case Study

Step 1: Convert data into time series, and plot and decompose into components to understand it.



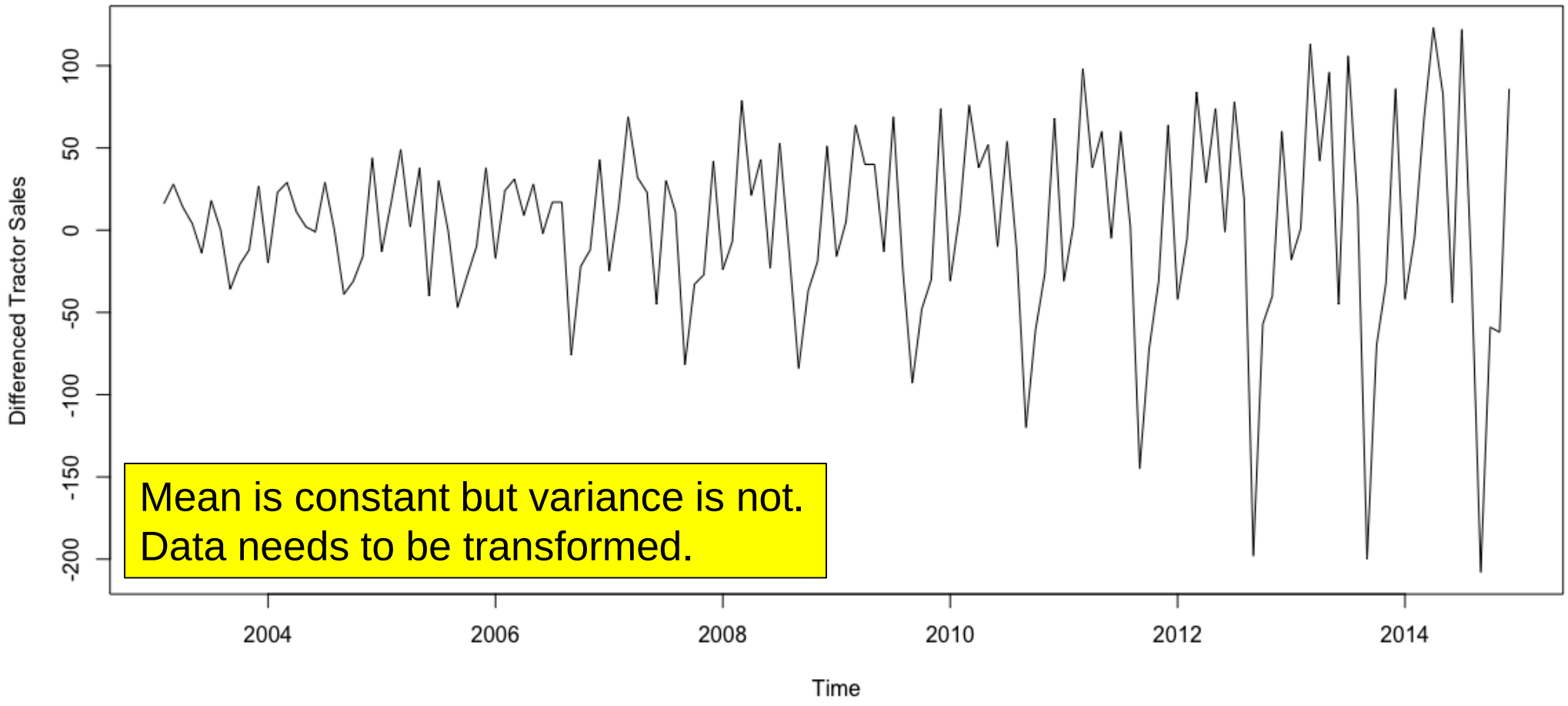
Tractor Sales Case Study

Step 1: Convert data into time series, and plot and decompose into components for exploratory data analysis.



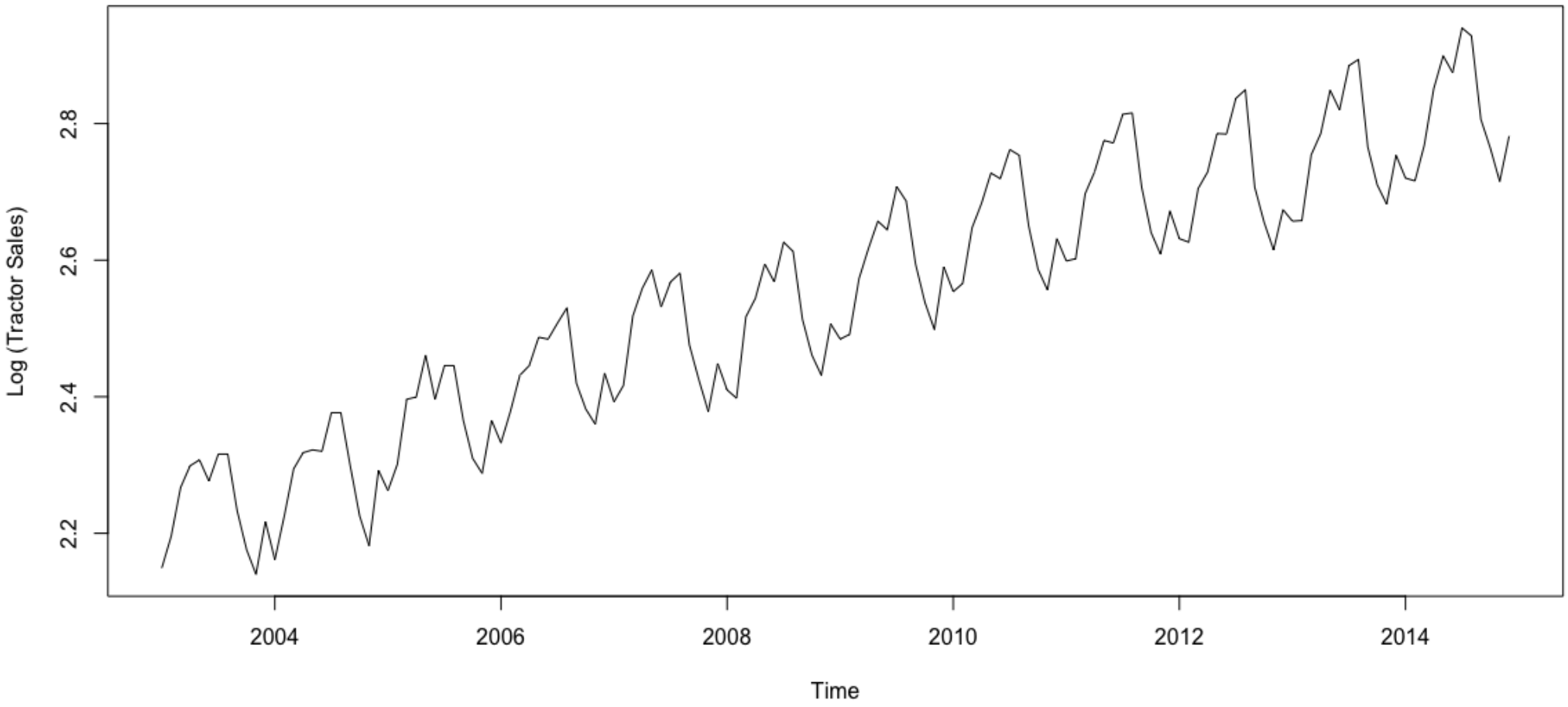
Tractor Sales Case Study

Step 2: Difference the data to make it stationary.



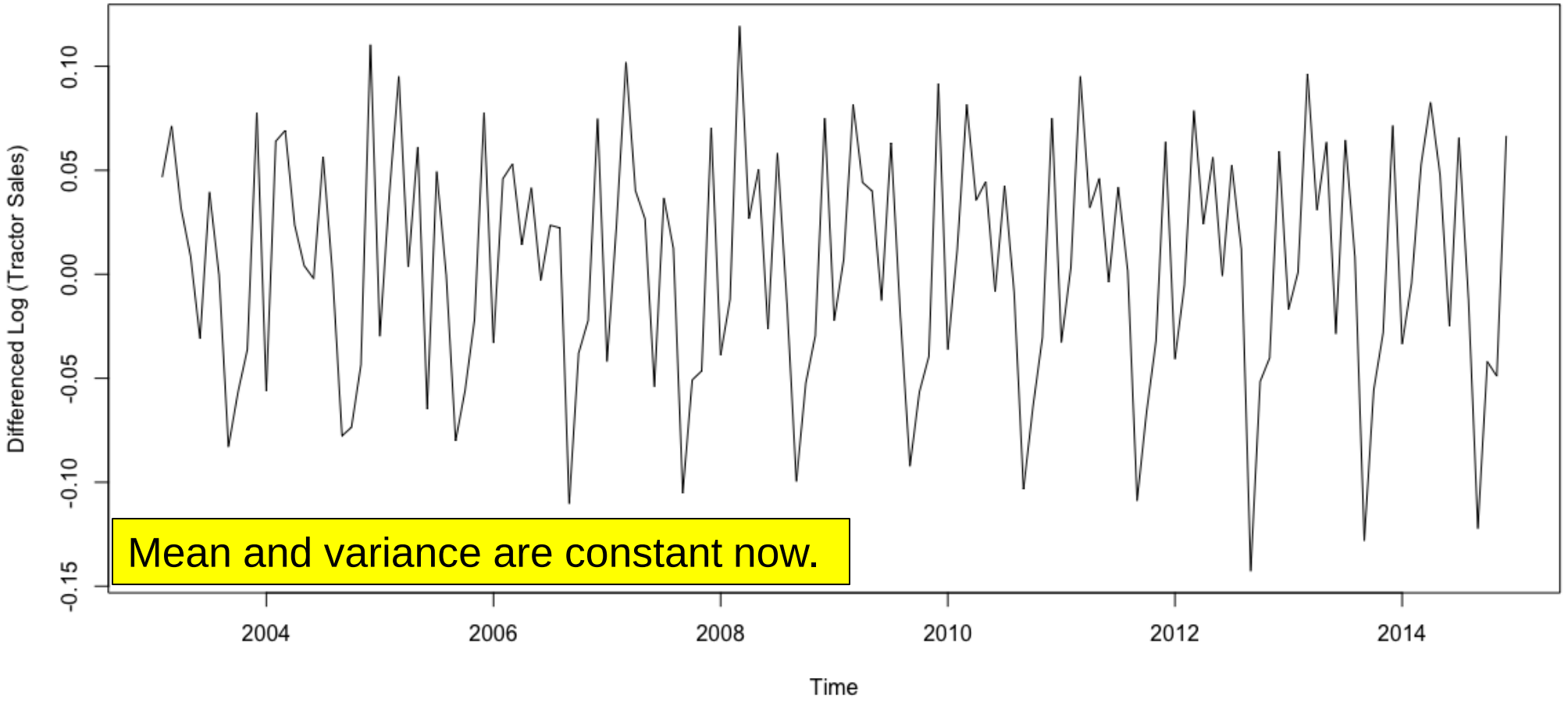
Tractor Sales Case Study

Step 3: Log transform the data.



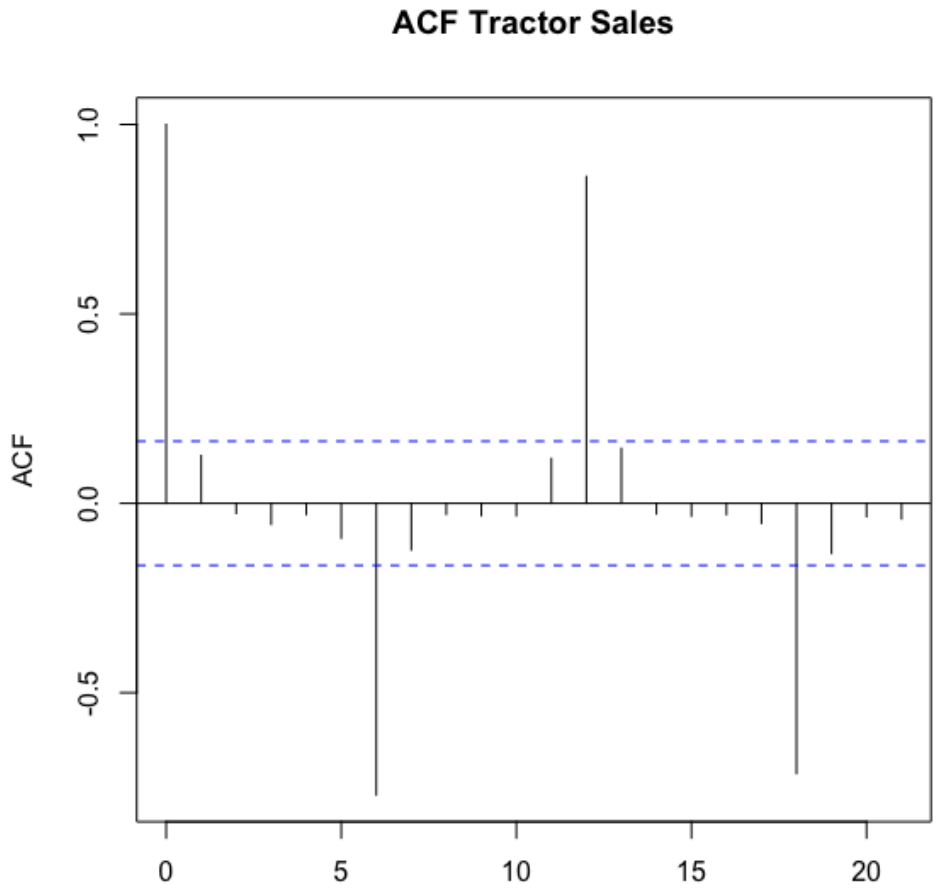
Tractor Sales Case Study

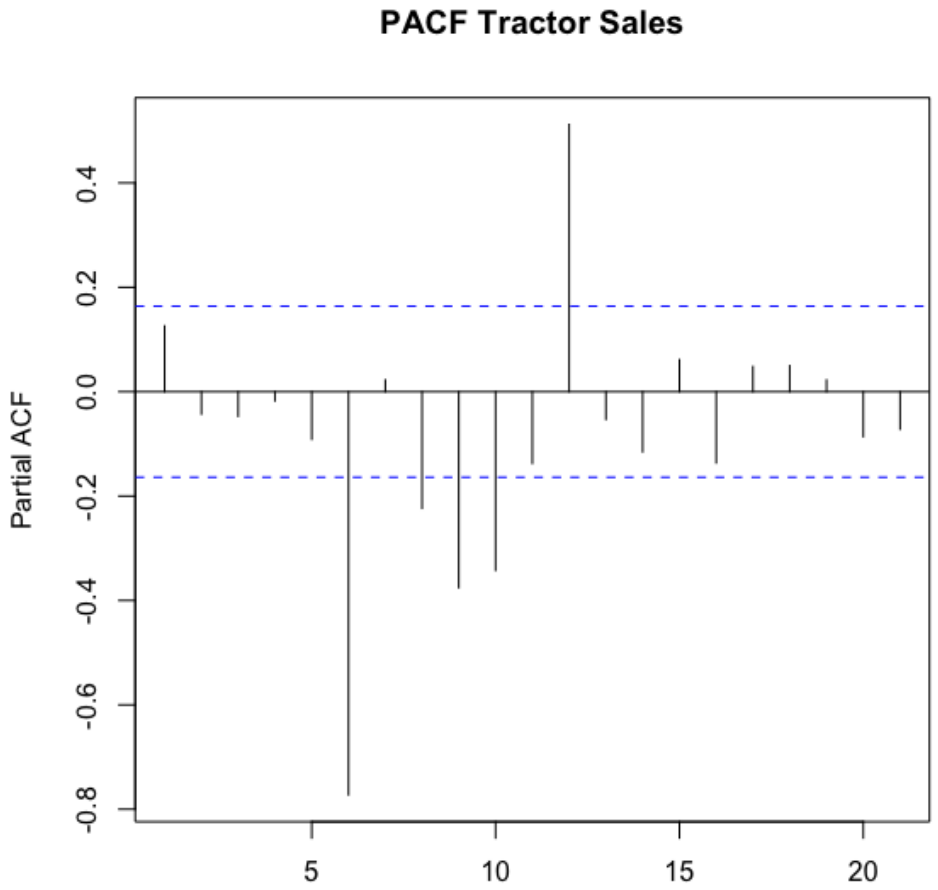
Step 4: Difference the log transformed data to check for stationarity.



Tractor Sales Case Study

Step 5: Check ACF and PACF to explore remaining dependencies.





There is a strong seasonality.

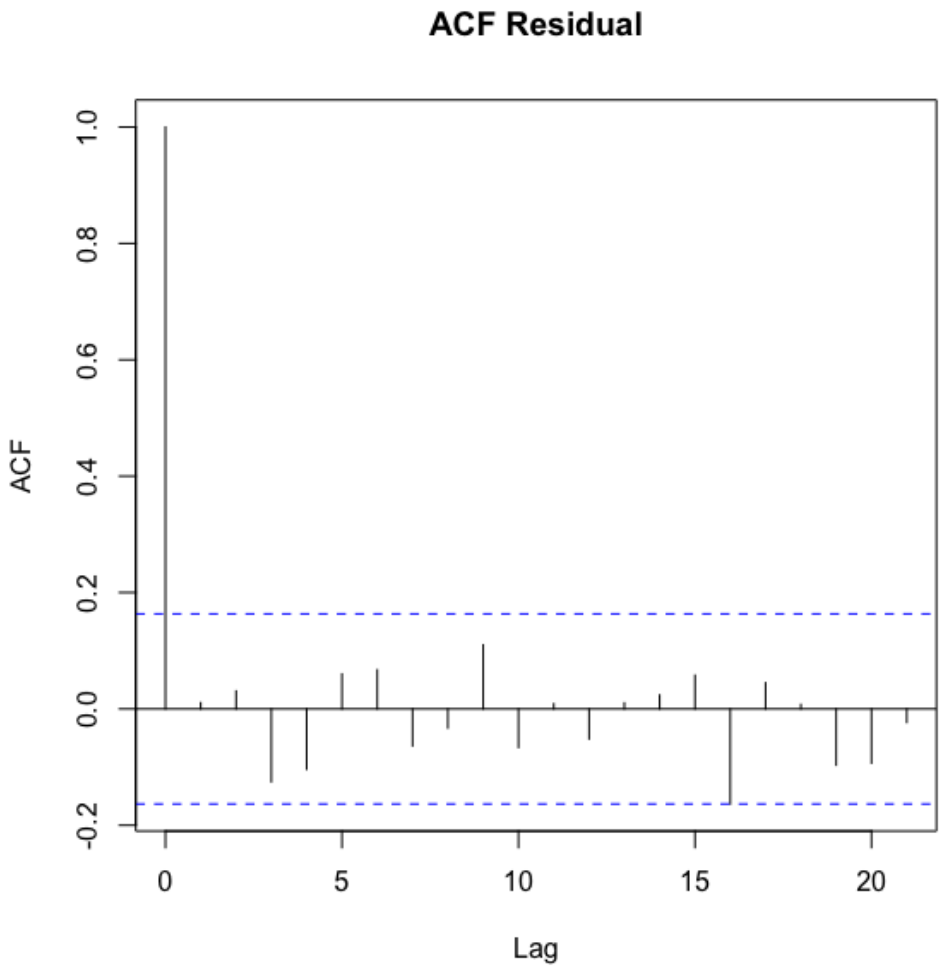
Tractor Sales Case Study

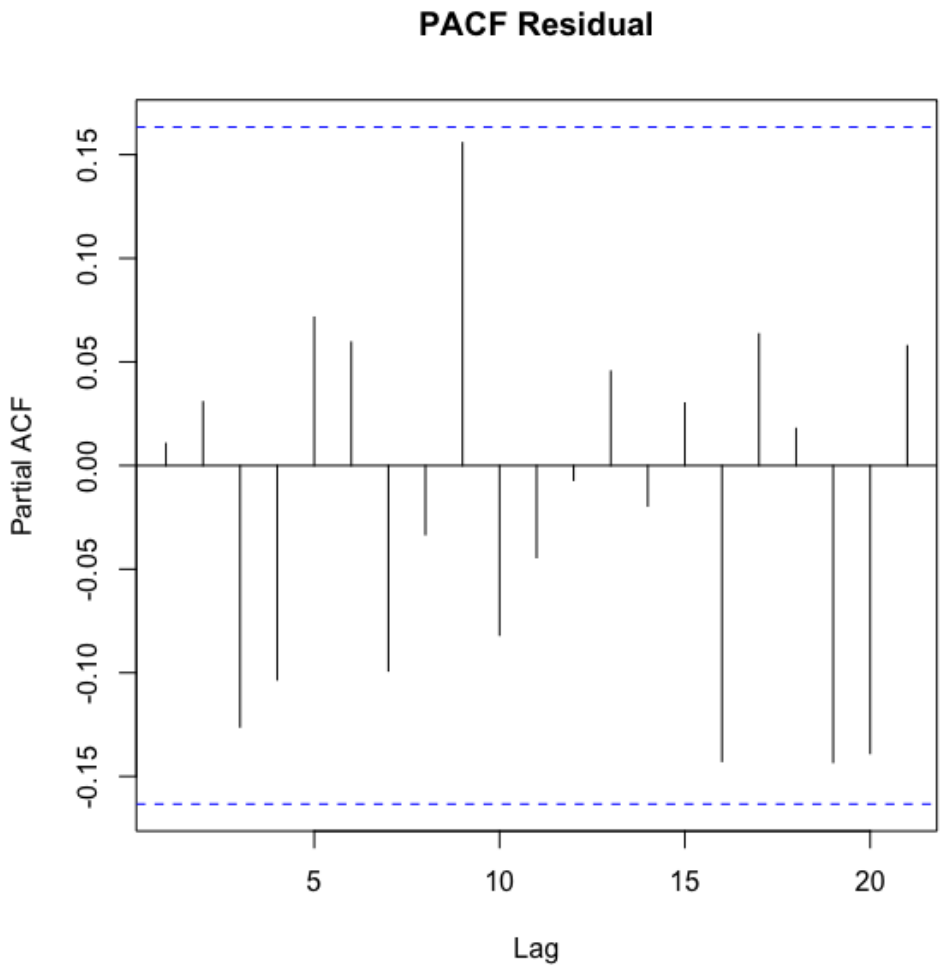
Step 6: Run Auto ARIMA.

```
Series: log10(TractorSalesTS)
ARIMA(0,1,1)(0,1,1)[12]

Coefficients:
          ma1     sma1
      -0.4047  -0.5529
s.e.   0.0885   0.0734

sigma^2 estimated as 0.0002571:  log likelihood=354.4
AIC=-702.79   AICc=-702.6   BIC=-694.17
```

Tractor Sales Case Study

Step 7: Check ACF and PACF of residuals to ensure they are white noise.





CSE 73026

Tractor Sales Case Study

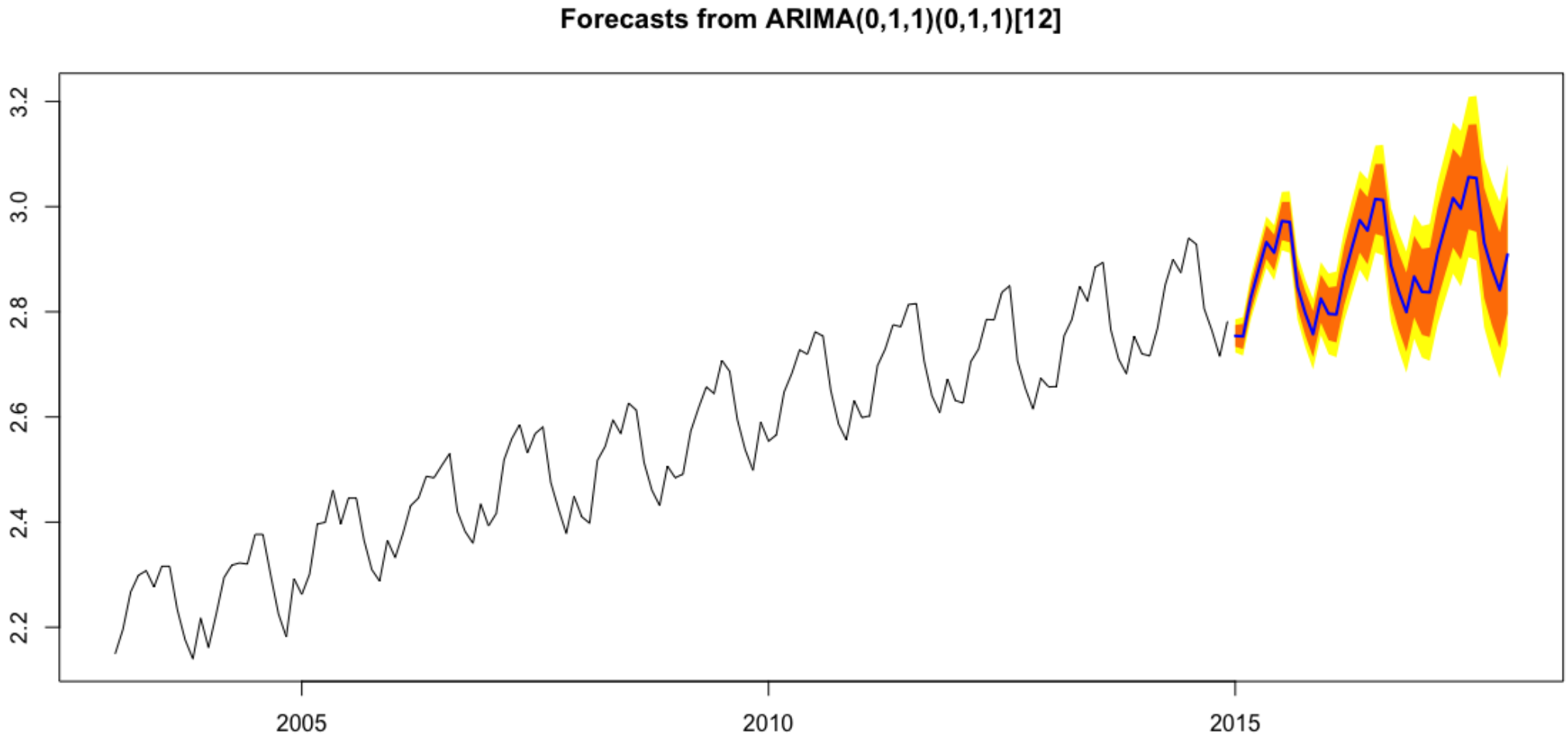
Step 7: Use Box-Ljung test to verify residuals are white noise.

It is good enough to do the verification visually on ACF and PACF.

```
        Box-Ljung test

data:  LogTractorSalesARIMA$residuals
X-squared = 26.219, df = 24, p-value = 0.3422
```

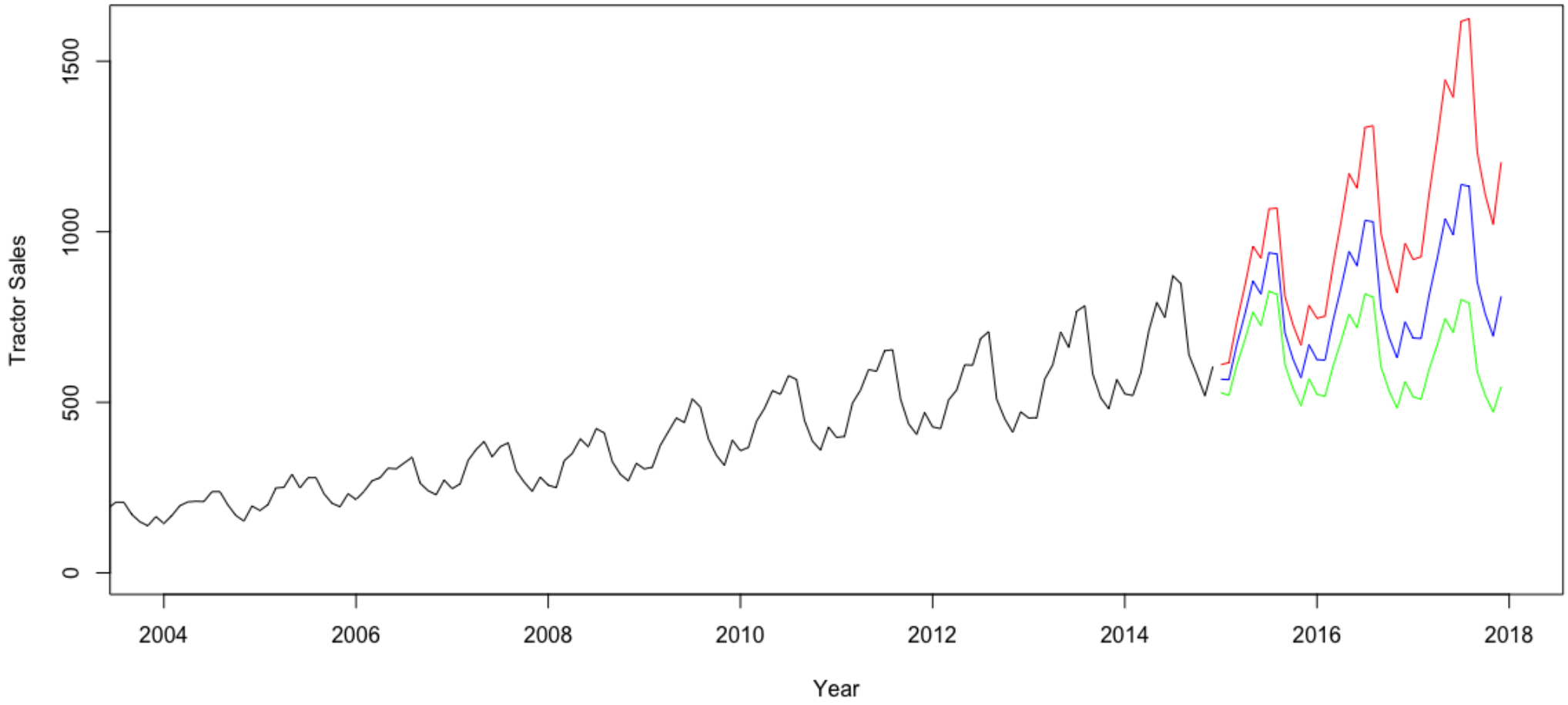
Tractor Sales Case Study

Step 8: Forecast.



Tractor Sales Case Study

Step 8: Forecast on the original scale and not the logged data.



Classification

NAÏVE BAYES ALGORITHM

Classification Problems with Multiple Classes

- Given an article, predict which section of the newspaper (Current News, International, Arts, Sports, Fashion, etc.) it is supposed to go to
- Given a photo of a car number plate, identify which state it belongs to
- Given an audio clip of a song, identify the genre
- Given an email, predict whether it is spam or not spam (a 2-class problem)

Classification Problems

- All classification problems are essentially equivalent to evaluating conditional probability
- $P(Y_i \mid X)$, *i.e.,* given certain evidence X, what is the probability that this is from class Y_i
- Logistic Regression solves this problem by modelling the probabilistic relationship between X and Y (sigmoid function, linear in X, etc.) **directly**
- Such models are called Discriminative Models

Naïve Bayes Algorithm

- Naïve Bayes makes predictions - $P(Y_i \mid X)$ - using Bayes theorem after modelling the joint probability of X and Y

$$Recall\ Conditional\ Probability = \frac{Joint\ Probability}{Marginal\ Probability}$$

- These type of methods are called Generative Learning Models

- A simple classifier that performs surprisingly well on a large class of problems

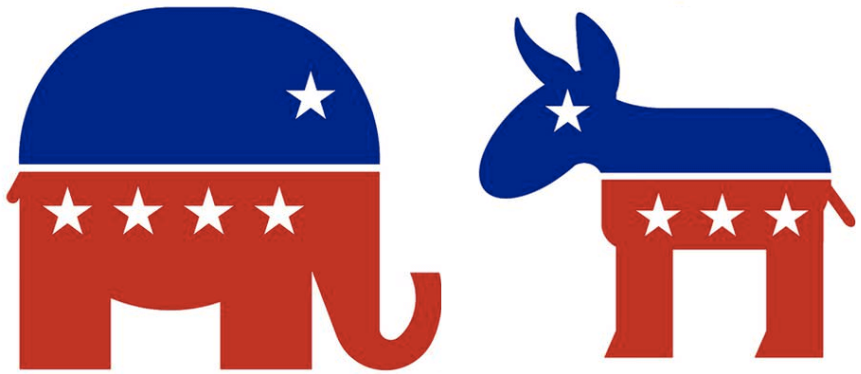
US House of Congress Voting Patterns

Class	V1	V2	V3	V4	V5	V6	V7
republican	n	y	n	y	y	y	n
republican	n	y	n	y	y	y	n
democrat	NA	y	y	NA	y	y	n
democrat	n	y	y	n	NA	y	n
democrat	y	y	y	n	y	y	n
democrat	n	y	y	n	y	y	n
democrat	n	y	n	y	y	y	n
republican	n	y	n	y	y	y	n
republican	n	y	n	y	y	y	n
democrat	y	y	y	n	n	n	y
republican	n	y	n	y	y	n	n
republican	n	y	n	y	y	y	n
democrat	n	y	y	n	n	n	y
democrat	y	y	y	n	n	y	y
republican	n	y	n	y	y	y	n
republican	n	y	n	y	y	y	n
democrat	y	n	y	n	n	y	n
democrat	y	NA	y	n	n	n	y
republican	n	y	n	y	y	y	n
democrat	y	y	y	n	n	n	y

1	**Class Name: 2 (democrat, republican)**
2	handicapped-infants: 2 (y,n)
3	water-project-cost-sharing: 2 (y,n)
4	adoption-of-the-budget-resolution: 2 (y,n)
5	physician-fee-freeze: 2 (y,n)
6	el-salvador-aid: 2 (y,n)
7	religious-groups-in-schools: 2 (y,n)
8	anti-satellite-test-ban: 2 (y,n)
9	aid-to-nicaraguan-contras: 2 (y,n)
10	mx-missile: 2 (y,n)
11	immigration: 2 (y,n)
12	synfuels-corporation-cutback: 2 (y,n)
13	education-spending: 2 (y,n)
14	superfund-right-to-sue: 2 (y,n)
15	crime: 2 (y,n)
16	duty-free-exports: 2 (y,n)
17	export-administration-act-south-africa: 2 (y,n)

House Votes 1984 Dataset: Voting patterns of Members of Congress.
A data frame with 435 observations on 17 variables. 168 Republicans, 267 Democrats

Republican or Democrat?



Republican – R – Red

Democrat – D - Donkey

Given a Congressman's voting pattern (v1 = *y*, v2 = *n*), what is the probability that this person is a Democrat?

P(D | v1 = *y*, v2 = *n*) = ?

Prior Belief - Simplest Solution

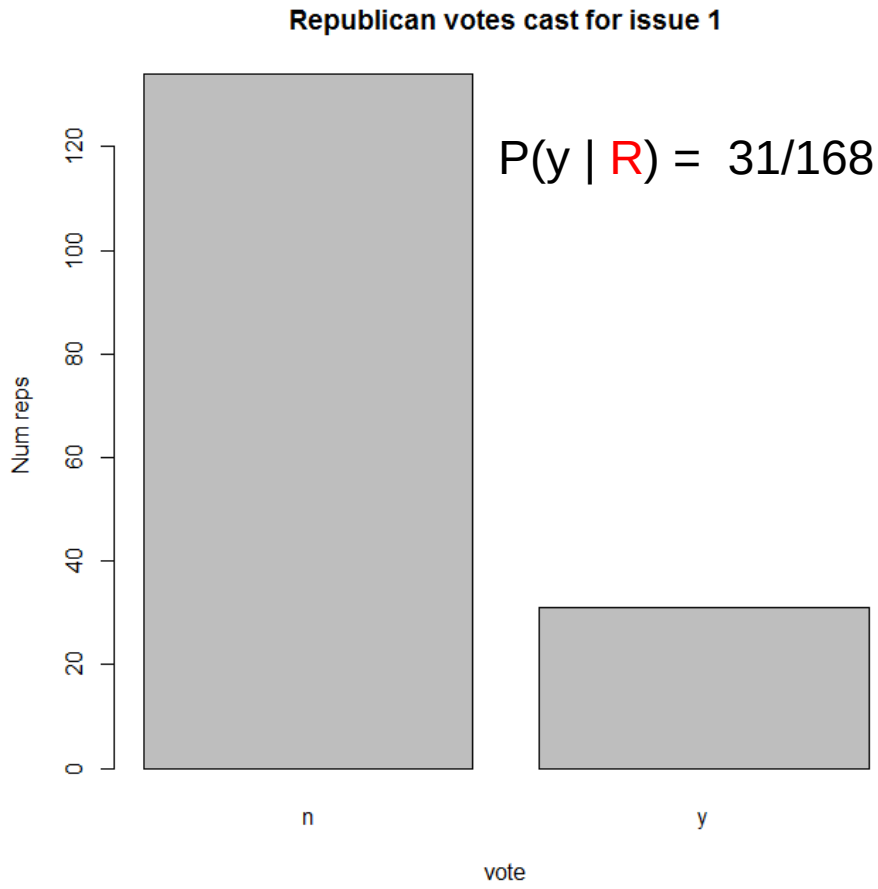
- The house has a majority of Democrats
  - 168 Republicans, 267 Democrats
- Probability of a random person being Democrat is
  - P(D) = 267/435 = 0.61
- Can we do better by incorporating the evidence of their voting patterns?

Voting Patterns for V1

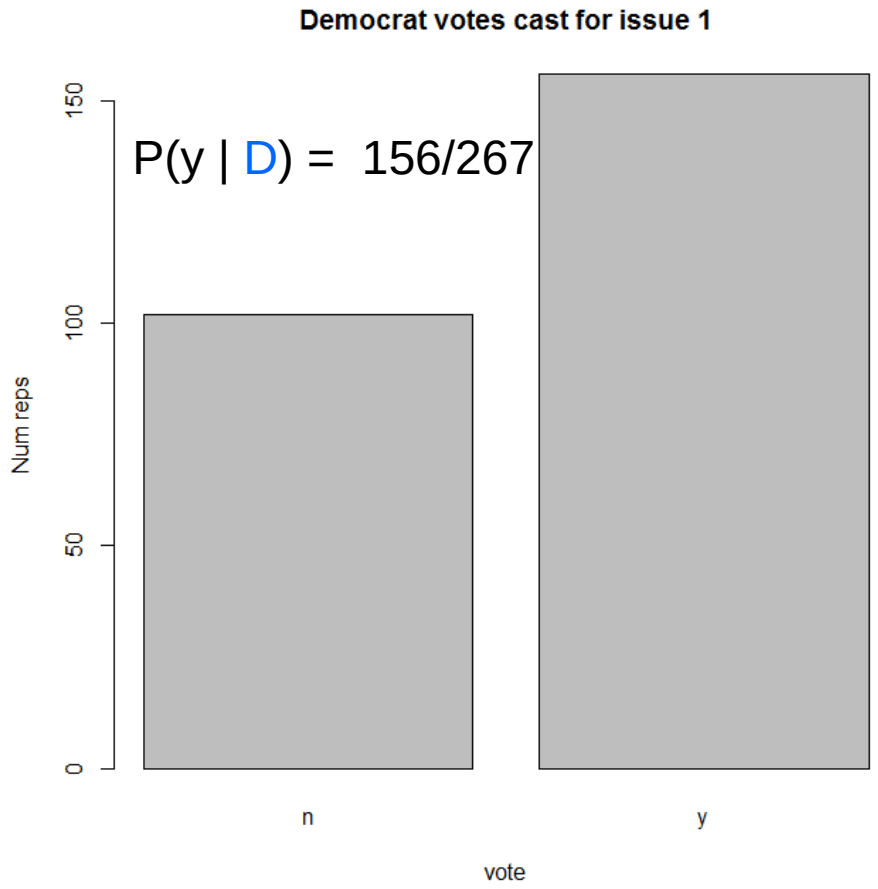
Handicapped Infants. The vote failed to pass: 236 to 187

We are interested in P(D | **v1 = *y***, v2 = *n*) = ?

```
> Repub <-  HouseVotes84$Class=="republican"
> Democrat <- HouseVotes84$Class=="democrat"
> plot(as.factor(HouseVotes84[Repub,2]))
> title(main="Republican votes cast for issue 1",
xlab="vote", ylab="Num reps")
> plot(as.factor(HouseVotes84[Democrat,2]))
> title(main="Democrat votes cast for issue 1", x
lab="vote", ylab="Num reps")
```



P(y | R) = 31/168

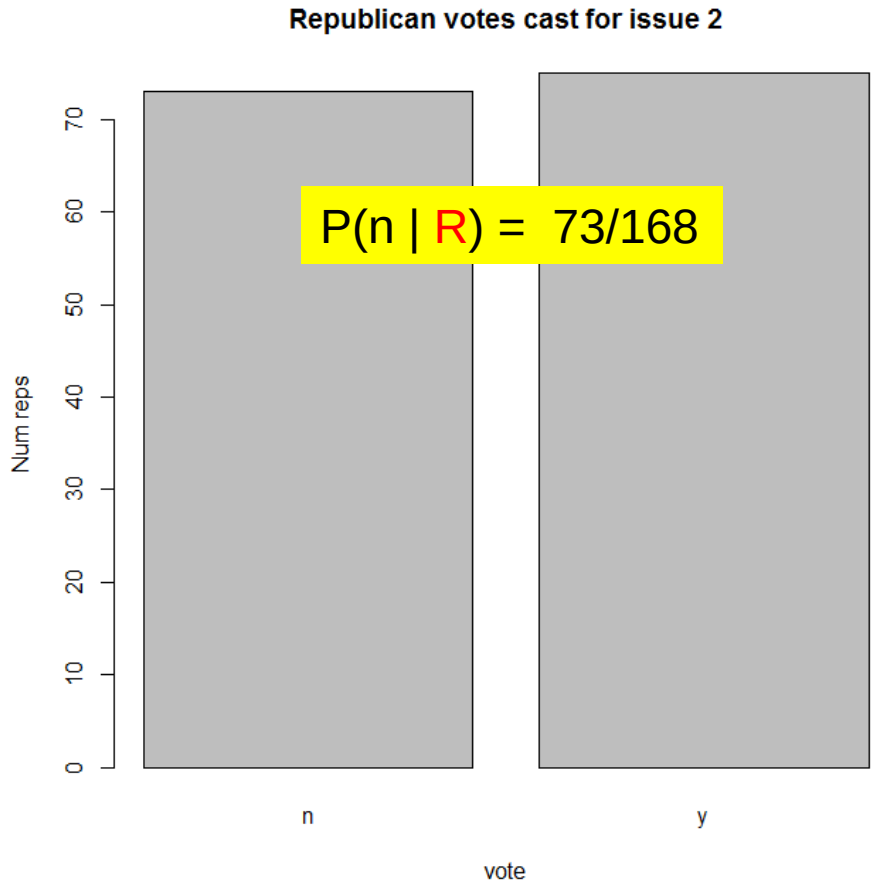


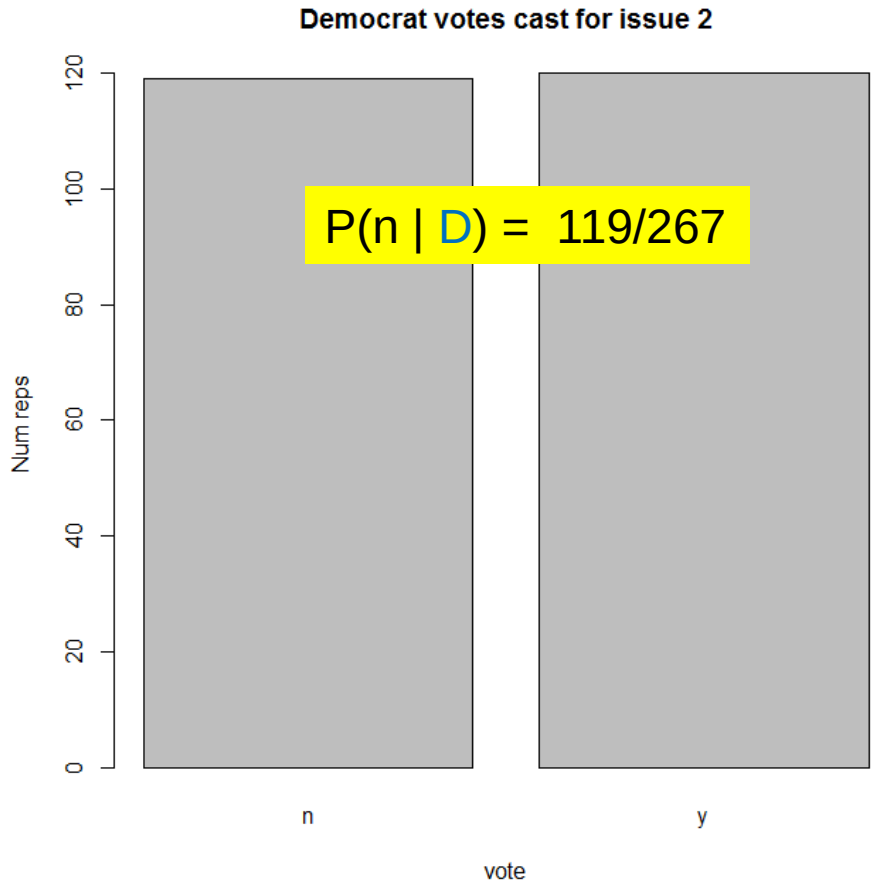
P(y | D) = 156/267

Voting Patterns for V2

Water-project-cost-sharing. The vote passed: 195 to 192

We are interested in
P(D | v1 = *y*, **v2 = *n***) = ?





Bayes Theorem

$$P(A|B) = ?$$

$$P(A|B) = \frac{P(A) * P(B|A)}{P(B)}$$

$$P(D|\text{v1} = y, \text{v2} = n) = ?$$

$$P(D|\text{v1} = y, \text{v2} = n) = \frac{P(D) * P(\text{v1} = y, \text{v2} = n|D)}{P(\text{v1} = y, \text{v2} = n)}$$

Naïve Bayes

Naïve Assumption: *Conditional probability of each feature given the class, is independent of all other features*

$$P(\text{v1} = y, \text{v2}=n \mid D) = P(\text{v1} = y \mid D) * P(\text{v2} = n \mid D)$$

$$P(D \mid \text{v1} = y, \text{v2}=n) = \frac{P(D) * P(\text{v1} = y \mid D) * P(\text{v2} = n \mid D)}{P(\text{v1} = y, \text{v2}=n)}$$

Naïve Bayes

We are trying to decide, given the voting pattern, if that person is a Democrat or a Republican.

$$P(D|\text{v1} = y, \text{v2}=n) = \frac{P(D) * P(\text{v1} = y|D) * P(\text{v2} = n|D)}{P(\text{v1} = y, \text{v2}=n)}$$

$$P(R|\text{v1} = y, \text{v2}=n) = \frac{P(R) * P(\text{v1} = y|R) * P(\text{v2} = n|R)}{P(\text{v1} = y, \text{v2}=n)}$$

Whichever probability is higher, we would classify the person into that party.

Note that the denominator is the same for both. So we need to focus only on numerator.

Naïve Bayes

$$P(D|\text{v1} = y, \text{v2}=n) \propto P(D) * P(\text{v1} = y|D) * P(\text{v2} = n|D)$$

P(D) = 267/435 (267 Democrats among 435 Congressmen)

$$P(D|\text{v1} = y, \text{v2}=n) \propto \frac{267}{435} * \frac{156}{267} * \frac{119}{267} = 0.15$$

From voting pattern slide

$$P(R|\text{v1} = y, \text{v2}=n) \propto \frac{168}{435} * \frac{31}{168} * \frac{73}{168} = 0.03$$

Since the conditional probability for being Democrat is higher, he is likely to be Democrat.

Naïve Bayes: Voting patterns

```
library(e1071)
nb_model <- naiveBayes(Class~.,data = trainHouseVotes84)
```

```
Naive Bayes Classifier for Discrete Predictors

Call:
naiveBayes.default(x = X, y = Y, laplace = laplace)

A-priori probabilities:
Y
  democrat republican
 0.6111111  0.3888889

Conditional probabilities:
            V1
Y                    n         y
  democrat   0.4066986 0.5933014
  republican 0.8195489 0.1804511

            V2
Y                    n         y
  democrat   0.5119617 0.4880383
  republican 0.4586466 0.5413534
```

Naïve Bayes Assumption

- The key assumption of independence of features, is almost never true
- Still Naïve Bayes does surprisingly well in a lot of situations
- It works best when all the predictor variables are categorical variables
- Very frequently used in text mining, character image analysis problems

Resources

Linear Regression

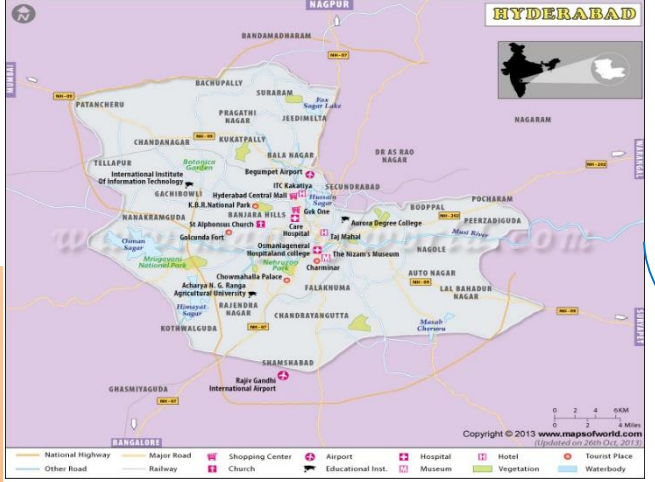
- http://people.duke.edu/~rnau/regintro.htm
- https://onlinecourses.science.psu.edu/stat501/node/250/

Logistic Regression

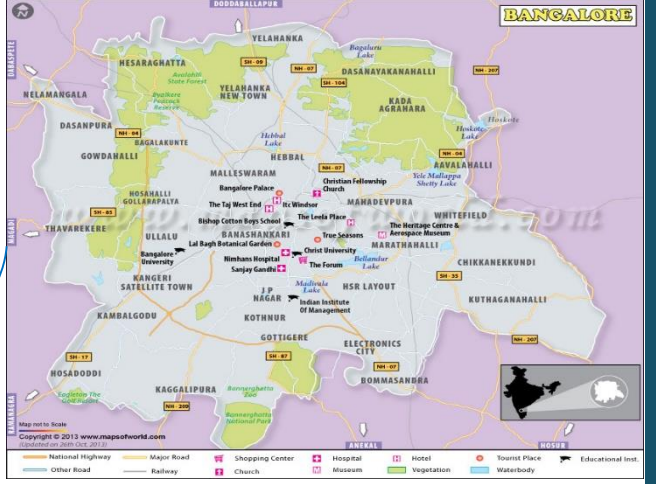
- https://onlinecourses.science.psu.edu/stat504/node/149/

Time Series Forecasting

- https://www.otexts.org/fpp2/ Forecasting: Principles and Practice
- http://people.duke.edu/~rnau/411home.htm
- https://onlinecourses.science.psu.edu/stat510/node/41/



INSOFE
Inspire...Educate...Transform.



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Facebook:	https://www.facebook.com/insofe
Twitter:	https://twitter.com/Insofeedu
YouTube:	http://www.youtube.com/InsofeVideos
SlideShare:	http://www.slideshare.net/INSOFE
LinkedIn:	http://www.linkedin.com/company/international-school-of-engineering

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