

1. Basics

1.1 Polynomials

the cover-up method & why it works! (for partial fractions decomposition)

https://www.youtube.com/watch?v=fgPviiiv_oZs

$$\frac{2x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \quad (1.1)$$

We have

$$\begin{aligned} A &= \frac{2(1)}{(1-2)(1-3)} = \frac{1}{2} \\ B &= \frac{2(2)-1}{(4-1)(2-3)} = -3 \\ C &= \frac{2(3)-1}{(3-1)(3-2)} = \frac{5}{2} \end{aligned} \quad (1.2)$$

And therefore

$$\frac{2x-1}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} + \frac{-3}{x-2} + \frac{\frac{5}{2}}{x-3} \quad (1.3)$$

It works because

$$\frac{2x-1}{(x-2)(x-3)} = A + \frac{B}{x-2}(x-1) + \frac{C}{x-3}(x-1) \quad (1.4)$$

2. Logarithms

Is this equation solvable? $x^{\ln(4)} + x^{\ln(10)} = x^{\ln(25)}$

Let us look at the equation

$$x^{\ln 4} + x^{\ln 10} = x^{\ln 25}, \quad (2.1)$$

which has the trivial solution $x = 0$. Note that

$$a^{\log_b c} = \left(b^{\log_b a}\right)^{\log_b c} = b^{\log_b a \cdot \log_b c} = \left(b^{\log_b c}\right)^{\log_b a} = c^{\log_b a} \quad (2.2)$$

Therefore, for $x \neq 0$, (2.1) becomes

$$4^{\ln x} + 10^{\ln x} = 25^{\ln x} \quad (2.3)$$

$$1 + \frac{10^{\ln x}}{4^{\ln x}} = \frac{25^{\ln x}}{4^{\ln x}} \quad (2.4)$$

$$1 + \left(\frac{5}{2}\right)^{\ln x} = \left(\left(\frac{5}{2}\right)^2\right)^{\ln x} \quad (2.5)$$

$$\left(\left(\frac{5}{2}\right)^{\ln x}\right)^2 - \left(\frac{5}{2}\right)^{\ln x} - 1 = 0 \quad (2.6)$$

Therefore, using the midnight formula, we have

$$\left(\frac{5}{2}\right)^{\ln x} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 + \sqrt{5}}{2} \equiv \varphi, \quad (2.7)$$

where φ is the golden ratio. So

$$\log_{\frac{5}{2}} \left(\frac{5}{2}\right)^{\ln x} = \log_{\frac{5}{2}} \varphi \quad (2.8)$$

$$\ln x = \log_{\frac{5}{2}} \varphi \quad (2.9)$$

$$e^{\ln x} = e^{\log_{\frac{5}{2}} \varphi} \quad (2.10)$$

$$x = e^{\log_{\frac{5}{2}} \varphi} \approx 1.691 \quad (2.11)$$

3. Continuity

Proving x^2 is continuous but NOT uniformly continuous on $(-\infty, \infty)$

We could show that $f(x) = x^2$ is not uniformly continuous on $(-\infty, \infty) = \mathbb{R}$ using the following theorem:

Theorem 3.1:

$\lim_{x \rightarrow \infty} f'(x) = \infty \implies f$ is not uniformly continuous.

We have that $f'(x) = 2x$ is not bounded. But we will not use (3.1) here.

Definition 3.2:

f is continuous on I means

$$\forall a \in I, \forall \epsilon > 0, \exists \delta > 0, \forall x \in I : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \quad (3.1)$$

Definition 3.3:

f is uniformly continuous on I means

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in I : |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \epsilon \quad (3.2)$$

Definition 3.4:

f is not uniformly continuous on I means

$$\exists \epsilon > 0, \forall \delta > 0, \exists x_1, x_2 \in I : |x_1 - x_2| < \delta$$

(3.3)

“but” / “and at the same time” : $|f(x_1) - f(x_2)| \geq \epsilon$

One can use predicate logic to show that this is the negation of the Definition 3.3.

With these definitions we want to show that x^2 is uniformly continuous on $[0, 1]$ but not uniformly continuous on \mathbb{R}

Theorem 3.5:

$f(x) = x^2$ is uniformly continuous on $[0, 1]$.

Proof: Given $\epsilon > 0$. Lets choose a $\delta > 0$ we don't know yet. So $\delta := \boxed{}$. Let $x_1, x_2 \in [0, 1]$.

Suppose $|x_1 - x_2| < \delta$. We can see that

$$|x_1^2 - x_2^2| = |(x_1 - x_2)(x_1 + x_2)| = |x_1 - x_2||x_1 + x_2| < \delta |x_1 + x_2| \leq \delta 2 \quad (3.4)$$

So for a given ϵ we choose $\delta := \boxed{\epsilon/2}$.

Then we have $|f(x_1) - f(x_2)| \leq |x_1^2 - x_2^2| \leq 2 \cdot \delta = \epsilon \quad \forall x_1, x_2 \in [0, 1]$.

So $f(x) = x^2$ is uniformly continuous on $[0, 1]$. □

Theorem 3.6:

$f(x) = x^2$ is not uniformly continuous on \mathbb{R} .

Proof: Looking at the definition 3.4 with it's equation (3.3) we can choose $\epsilon > 0$ as well as $x_1, x_2 \in \mathbb{R}$ freely. So let $\epsilon := 1$. So given any $\delta > 0$ we have to pick $x_1, x_2 \in \mathbb{R}$ such that

$|x_1 - x_2| < \delta$ and at the same time $|x_1^2 - x_2^2| \geq 1 = \epsilon$. So let again $|x_1^2 - x_2^2| = |x_1^2 - x_2^2||x_1^2 + x_2^2|$ like in (3.4). If we pick $x_1 := \frac{\delta}{2}, x_2 := 0$ we will end up with $|x_1^2 - x_2^2| = \delta^2/4$ which will not be ≥ 1 for any given $\delta > 0$. Similarly, if we pick $x_1 := \delta + \frac{\delta}{2}, x_2 := \delta$ we will end up with

$|x_1^2 - x_2^2| = (\delta/2)((5\delta)/2) = (5\delta^2)/2$ which will not be ≥ 1 for any given $\delta > 0$. Finally, if we pick

$x_1 := \frac{1}{\delta} + \frac{\delta}{2} \in \mathbb{R}, x_2 := \frac{1}{\delta} \in \mathbb{R}$ we will end up with $|x_1^2 - x_2^2| = \frac{\delta}{2}(\frac{2}{\delta} + \frac{\delta}{2}) = 1 + \delta^2/4$ which will be

$\geq 1 = \epsilon$ for any given $\delta > 0$. So $f(x) = x^2$ is not uniformly continuous on \mathbb{R} □

Theorem 3.7:

$f(x) = x^2$ is continuous on \mathbb{R}

Proof: We must show that $\lim_{x \rightarrow a} x^2 = a^2 \quad \forall a \in \mathbb{R}$. Given $\epsilon > 0, a \in \mathbb{R}$.

Choose $\delta = \boxed{}$ we don't know yet. Let $x \in \mathbb{R}$ be arbitrary. Suppose $0 < |x - a| < \delta$. Then we have

$$|x^2 - a^2| = |x - a||x + a| < \delta|x + a|. \quad (3.5)$$

If we would choose δ to be ≤ 1 to begin with, we would have

$$|x + a| = |x - a + 2a| \leq |x - a| + |2a| < 1 + |2a|. \quad (3.6)$$

So we have

$$|x^2 - a^2| < \delta(1 + 2|a|) \quad (3.7)$$

If we choose $\delta = \boxed{\min\left\{1, \frac{\epsilon}{1 + 2|a|}\right\}}$, we have

$$|x^2 - a^2| < \delta(1 + 2|a|) \leq \frac{\epsilon}{1 + 2|a|}(1 + 2|a|) = \epsilon \quad (3.8)$$

□

4. Integration

$$\int_{-1}^1 \frac{x^2}{1+2^{\sin x}} dx = \int_0^1 x^2 dx = \frac{1}{3} 1^3 = \frac{1}{3} \quad (4.1)$$

Theorem 4.1:

Let $\text{even} : \mathbb{R} \rightarrow \mathbb{R}$ be an even function and $\text{odd} : \mathbb{R} \rightarrow \mathbb{R}$ be an odd function. And let $a, b \in \mathbb{R}$. Then

$$\int_{-a}^a \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx = \int_0^a \text{even}(x) dx \quad (4.2)$$

Proof: Using that fact that

$$\begin{aligned} \int_{x=-a}^{x=0} \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx &= \int_{u=a}^{u=0} \frac{\text{even}(-u)}{1+b^{\text{odd}(-u)}} (-1) du \\ &= \int_{u=0}^{u=a} \left(\frac{\text{even}(u)}{1+b^{-\text{odd}(u)}} \right) \frac{b^{\text{odd}(u)}}{b^{\text{odd}(u)}} du \\ &= \int_0^a \frac{b^{\text{odd}(u)} \text{even}(u)}{b^{\text{odd}(u)} + 1} du \end{aligned} \quad (4.3)$$

We have that

$$\begin{aligned} \int_{-a}^a \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx &= \int_{-a}^0 \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx + \int_0^a \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx \\ &= \int_0^a \frac{b^{\text{odd}(x)} \text{even}(x)}{b^{\text{odd}(x)} + 1} dx + \int_0^a \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx \\ &= \int_0^a \frac{\text{even}(x)(b^{\text{odd}(x)} + 1)}{b^{\text{odd}(x)} + 1} dx \\ &= \int_0^a \text{even}(x) dx \end{aligned} \quad (4.4)$$

□

100 Integralrechnungen (Weltrekord?)

1.

$$\begin{aligned} \int \tan^5 x \sec^3 x dx &= \int \tan^4 x \sec^2 x \tan x \sec x dx \\ &= \int (\tan^2 x)^2 \sec^2 x \tan x \sec x dx \\ &= \int (\sec^2 x - 1)^2 \sec^2 x \tan x \sec x dx \\ \text{Substitution: } &\boxed{\begin{array}{l} u \equiv \sec x \\ du \equiv \sec x \tan x dx \end{array}} \quad (4.5) \\ &= \int (u^2 - 1)^2 u^2 du \\ &= \int (u^6 - u^4 + u^2) du \\ &= \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + c \end{aligned}$$

2.

$$\begin{aligned}
 \int \frac{\cos(2x)}{\sin x + \cos x} &= \int \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} \\
 &= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sin x + \cos x} \\
 &= \int (\cos x - \sin x) dx \\
 &= \sin x + \cos x + c
 \end{aligned} \tag{4.6}$$

3.

$$\begin{aligned}
 \int \frac{x^2 + 1}{x^4 - x^2 + 1} &= \int \frac{1 + \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}} \\
 &= \int \frac{1 + (1/x^2)}{x^2 - 2 + (1/x^2) + 1}
 \end{aligned}$$

Substitution:

$$\begin{aligned}
 u &\equiv x - \frac{1}{x} \\
 du &\equiv \left(1 + \frac{1}{x^2}\right) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{1}{u^2 + 1} du \\
 &= \tan^{-1}\left(x - \frac{1}{x}\right) + c
 \end{aligned} \tag{4.7}$$

4.

$$\int (x + e^x)^2 dx = \int (x^2 + 2xe^x + e^{2x}) dx$$

DI-method for $2xe^x$:

	D	I
+	$2x$	e^x
-	2	e^x
+	0	e^x

$$= \frac{1}{3}x^3 + 2xe^x - 2e^x + \frac{1}{2}e^{2x}$$

5.

$$\begin{aligned}
\int \csc^3 x \sec x \, dx &= \int \frac{1}{\sin^3 x \cos x} \, dx \\
&\quad \text{Pythagorean identity: } \boxed{\sin^2 \theta + \cos^2 \theta = 1} \\
&= \int \frac{\sin^2 x}{\sin^3 x \cos x} + \frac{\cos^2}{\sin^3 x \cos x} \, dx \\
&= \int \frac{1}{\sin x \cos x} + \frac{\cos x}{\sin^3 x} \, dx \\
&\quad \text{Pythagorean identity again} \\
&= \int \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x} + \frac{\cos x}{\sin^3 x} \, dx \\
&= \int \frac{\sin x}{\cos x} \, dx + \int \frac{\cos x}{\sin x} \, dx + \int \frac{\cos x}{\sin^3 x} \, dx \\
&= \ln|\sec x| + \ln|\sin x| - \frac{1}{2\sin^2 x} + c
\end{aligned} \tag{4.9}$$

6.

$$\begin{aligned}
\int \frac{\cos x}{\sin^2 x - 5\sin x - 6} \, dx &\quad \text{Substitution: } \boxed{\begin{array}{l} u \equiv \sin x \\ du \equiv \cos x \, dx \end{array}} \\
&= \int \frac{1}{u^2 - 5u - 6} \, du \\
&= \int \frac{1/7}{u-6} + \frac{-1/7}{u+1} \, du \\
&= \frac{1}{7} \ln|u-6| - \frac{1}{7} \ln|u+1| \\
&= \frac{1}{7} \ln \left| \frac{\sin x - 6}{\sin x + 1} \right| + c
\end{aligned} \tag{4.10}$$

We have been using the fact that $\frac{1}{u^2-5u-6} = \frac{1}{(u-6)(u+1)} = \frac{a_1}{u-6} + \frac{a_2}{u+1} \iff$

$1 = a_1(u+1) + a_2(u-6) = u(a_1+a_2) + (a_1-6a_2)$. Comparing the coefficients leads to the following system of equations:

$$\begin{aligned}
a_1 + a_2 &= 0 \\
a_1 - 6a_2 &= 1
\end{aligned} \tag{4.11}$$

Which implies that $a_1 = \frac{1}{7}$ and $a_2 = -\frac{1}{7}$ and therefore $\frac{1}{u^2-5u-6} = \frac{1}{(u-6)(u+1)} = \frac{1/7}{u-6} + \frac{-1/7}{u+1}$

7.

$$\int \frac{1}{\sqrt{e^x}} \, dx = \int \frac{1}{e^{\frac{x}{2}}} \, dx = \int e^{-\frac{x}{2}} \, dx = -2e^{-\frac{x}{2}} = \frac{-2}{\sqrt{e^x}} + c \tag{4.12}$$