1. Logarithms

Is this equation solvable? x^ln(4)+x^ln(10)=x^ln(25)

Let us look at the equation

$$x^{\ln 4} + x^{\ln 10} = x^{\ln 25}, \tag{1.1}$$

which has the trivial solution $x = 0$. Note that

$$a^{\log_b c} = \left(b^{\log_b a}\right)^{\log_b c} = b^{\log_b a \cdot \log_b c} = \left(b^{\log_b c}\right)^{\log_b a} = c^{\log_b a} \tag{1.2}$$

Therefore, for $x \neq 0$, (1.1) becomes

$$4^{\ln x} + 10^{\ln x} = 25^{\ln x} \tag{1.3}$$

$$1 + \frac{10^{\ln x}}{4^{\ln x}} = \frac{25^{\ln x}}{4^{\ln x}} \tag{1.4}$$

$$1 + \left(\frac{5}{2}\right)^{\ln x} = \left(\left(\frac{5}{2}\right)^2\right)^{\ln x} \tag{1.5}$$

$$\left(\left(\frac{5}{2}\right)^{\ln x}\right)^2 - \left(\frac{5}{2}\right)^{\ln x} - 1 = 0 \tag{1.6}$$

Therefore, using the midnight formula, we have

$$\left(\frac{5}{2}\right)^{\ln x} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 + \sqrt{5}}{2} \equiv \varphi, \tag{1.7}$$

where φ is the golden ratio. So

$$\log_{\frac{5}{2}} \left(\frac{5}{2}\right)^{\ln x} = \log_{\frac{5}{2}} \varphi \tag{1.8}$$

$$\ln x = \log_{\frac{5}{2}} \varphi \tag{1.9}$$

$$e^{\ln x} = e^{\log_{\frac{5}{2}} \varphi} \tag{1.10}$$

$$x = e^{\log_{\frac{5}{2}} \varphi} \approx 1.691 \tag{1.11}$$

2. Continuity

Proving x^2 is continuous but NOT uniformly continuous on (-inf, inf)

We could show that $f(x) = x^2$ is not uniformly continous on $(-\infty, \infty) = \mathbb{R}$ using the following theorem:

Theorem 2.1:
$\lim_{x\to\infty} f'(x) = \infty \implies f$ is not uniformly continous.
We have that $f'(x) = 2x$ is not bounded. But we will not use (2.1) here.

Definition 2.2:
f is continuous on I means

$$\forall a \in I, \forall \epsilon > 0, \exists \delta > 0, \forall x \in I : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{2.1}$$

Definition 2.3:
f is uniformly continous on I means

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in I : |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \epsilon \tag{2.2}$$

Definition 2.4:
f is not uniformly continous on I means

$$\begin{gathered} \exists \epsilon > 0, \forall \delta > 0, \exists x_1, x_2 \in I : |x_1 - x_2| < \delta \\ \text{“but” / “and at the same time”} : |f(x_1) - f(x_2)| \geq \epsilon \end{gathered} \tag{2.3}$$

One can use predicate logic to show that this is the negation of the Definition 2.3.

With these definitions we want to show that x^2 is uniformly continous on $[0,1]$ but not uniformly continous on $\mathbb{R}$

Theorem 2.5:
$f(x) = x^2$ is uniformly continous on $[0,1]$.

Proof: Given $\epsilon > 0$. Lets choose a $\delta > 0$ we don't know yet. So $\delta :\equiv \boxed{}$. Let $x_1, x_2 \in [0,1]$. Suppose $|x_1 - x_2| < \delta$. We can see that

$$|x_1^2 - x_2^2| = |(x_1 - x_2)(x_1 + x_2)| = |x_1 - x_2||x_1 + x_2| < \delta |x_1 + x_2| \leq \delta 2 \tag{2.4}$$

So for a given ϵ we choose $\delta :\equiv \boxed{\epsilon/2}$.
Then we have $|(f(x_1) - f(x_2)| \leq |x_1^2 - x_2^2| \leq 2 \cdot \delta = \epsilon \quad \forall x_1, x_2 \in [0,1]$.
So $f(x) = x^2$ is uniformly continuous on $[0,1]$. □

Theorem 2.6:
$f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof: Looking at the definition 2.4 with it's equation (2.3) we can choose $\epsilon > 0$ as well as $x_1, x_2 \in \mathbb{R}$ freely. So let $\epsilon :\equiv 1$. So given any $\delta > 0$ we have to pick $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_2| < \delta$ and at the same time $|x_1^2 - x_2^2| \geq 1 = \epsilon$. So let again $|x_1^2 - x_2^2| = |x_1^2 - x_2^2||x_1^2 + x_2^2|$ like in (2.4) . If we pick $x_1 :\equiv \frac{\delta}{2}, x_2 :\equiv 0$ we will end up with $|x_1^2 - x_2^2| = \delta^2/4$ which will not be ≥ 1 for any given $\delta > 0$. Similarly, if we pick $x_1 :\equiv \delta + \frac{\delta}{2}, x_2 :\equiv \delta$ we will end up with $|x_1^2 - x_2^2| = (\delta/2)((5\delta)/2) = (5\delta^2)/2$ which will not be ≥ 1 for any given $\delta > 0$. Finally, if we pick $x_1 :\equiv \frac{1}{\delta} + \frac{\delta}{2} \in \mathbb{R}, x_2 :\equiv \frac{1}{\delta} \in \mathbb{R}$ we will end up with $|x_1^2 - x_2^2| = \frac{\delta}{2}(\frac{2}{\delta} + \frac{\delta}{2}) = 1 + \delta^2/4$ which will be $\geq 1 = \epsilon$ for any given $\delta > 0$. So $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ □

Theorem 2.7:
$f(x) = x^2$ is continuous on $\mathbb{R}$

Proof: We must show that $\lim_{x \to a} x^2 = a^2 \quad \forall a \in \mathbb{R}$. Given $\epsilon > 0, a \in \mathbb{R}$.

Choose $\delta = \boxed{}$ we dont know yet. Let $x \in \mathbb{R}$ be arbitrary. Suppose $0 < |x - a| < \delta$. Then we have

$$|x^2 - a^2| = |x - a||x + a| < \delta |x + a|. \tag{2.5}$$

If we would choose δ to be ≤ 1 to begin with, we would have

$$|x + a| = |x - a + 2a| \leq |x - a| + |2a| < 1 + |2a|. \tag{2.6}$$

So we have

$$|x^2 - a^2| < \delta(1 + 2|a|) \tag{2.7}$$

If we choose $\delta = \boxed{\min\left\{1, \frac{\epsilon}{1 + 2|a|}\right\}}$, we have

$$|x^2 - a^2| < \delta(1 + 2|a|) \leq \frac{\epsilon}{1 + 2|a|}(1 + 2|a|) = \epsilon \tag{2.8}$$

□

3. Integration

$$\int_{-1}^{1} \frac{x^2}{1+2^{\sin x}} = \int_0^1 x^2 dx = \frac{1}{3} 1^3 = \frac{1}{3} \tag{3.1}$$

Theorem 3.1:
Let even : $\mathbb{R} \to \mathbb{R}$ be an even function and odd : $\mathbb{R} \to \mathbb{R}$ be an odd function. And let $a, b \in \mathbb{R}$. Then

$$\int_{-a}^{a} \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx = \int_0^a \text{even}(x) dx \tag{3.2}$$

Proof: Using that fact that

$$\begin{aligned}\int_{x=-a}^{x=0} \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx &= \int_{u=a}^{u=0} \frac{\text{even}(-u)}{1+b^{\text{odd}(-u)}} (-1) du \\ &= \int_{u=0}^{u=a} \left(\frac{\text{even}(u)}{1+b^{-\text{odd}(u)}} \right) \frac{b^{\text{odd}(u)}}{b^{\text{odd}(u)}} du \\ &= \int_0^a \frac{b^{\text{odd}(u)} \text{even}(u)}{b^{\text{odd}}(u)+1} du\end{aligned} \tag{3.3}$$

We have that

$$\begin{aligned}\int_{-a}^{a} \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx &= \int_{-a}^{0} \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx + \int_0^a \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx \\ &= \int_0^a \frac{b^{\text{odd}(x)} \text{even}(x)}{b^{\text{odd}(x)}+1} dx + \int_0^a \frac{\text{even}(x)}{1+b^{\text{odd}(x)}} dx \\ &= \int_0^a \frac{\text{even}(x)(b^{\text{odd}(x)}+1)}{b^{\text{odd}(x)}+1} dx \\ &= \int_0^a \text{even}(x) dx\end{aligned} \tag{3.4}$$

□

100 Integralrechnungen (Weltrekord?)

1.

$$\begin{aligned}\int \tan^5 x \sec^3 x dx &= \int \tan^4 x \sec^2 \tan x \sec x dx \\ &= \int (\tan^2 x)^2 \sec^2 \tan x \sec x dx \\ &= \int (\sec^2 x - 1)^2 \sec^2 \tan x \sec x dx \\ &\text{Substitution: } \boxed{\begin{aligned} u &:\equiv \sec x \\ du &:\equiv \sec x \tan x dx \end{aligned}} \\ &= \int (u^2-1)^2 u^2 du \\ &= \int (u^6 - u^4 + u^2) du \\ &= \frac{1}{7} \sec^2 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^2 x + c\end{aligned} \tag{3.5}$$

2.

$$\begin{aligned}\int \frac{\cos(2x)}{\sin x + \cos x} &= \int \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} \\ &= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sin x + \cos x} \\ &= \int (\cos x - \sin x) dx \\ &= \sin x + \cos x + c\end{aligned} \tag{3.6}$$

3.

$$\begin{aligned}\int \frac{x^2+1}{x^4-x^2+1} &= \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}} \\ &= \int \frac{1+(1/x^2)}{x^2-2+(1/x^2)+1} \\ &\text{Substitution: } \boxed{\begin{aligned} u &:\equiv x - \frac{1}{x} \\ du &:\equiv \left(1+\frac{1}{x^2}\right)dx \end{aligned}} \\ &= \int \frac{1}{u^2+1} du \\ &= \tan^{-1}\left(x - \frac{1}{x}\right) + c\end{aligned} \tag{3.7}$$

4.

$$\begin{aligned}\int (x+e^x)^2 dx &= \int (x^2 + 2xe^x + e^{2x}) dx \\ &\text{DI-method for } 2xe^x\text{: } \boxed{\begin{array}{ccc} & D & I \\ + & 2x & e^x \\ - & 2 & e^x \\ + & 0 & e^x \end{array}} \\ &= \frac{1}{3}x^3 + 2xe^x - 2e^x + \frac{1}{2}e^{2x}\end{aligned} \tag{3.8}$$

5.

$$
\begin{aligned}
\int \csc^3 x \sec x \, dx &= \int \frac{1}{\sin^3 x \cos x} \, dx \\
&\quad \text{Pythagorean identity:} \quad \boxed{\sin^2\theta + \cos^2\theta = 1} \\
&= \int \frac{\sin^2 x}{\sin^3 x \cos x} + \frac{\cos^2}{\sin^3 x \cos x} \, dx \\
&= \int \frac{1}{\sin x \cos x} + \frac{\cos x}{\sin^3 x} \, dx \\
&\quad \text{Pythagorean identity again} \\
&= \int \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x} + \frac{\cos x}{\sin^3 x} \, dx \\
&= \int \frac{\sin x}{\cos x} \, dx + \int \frac{\cos x}{\sin x} \, dx + \int \frac{\cos x}{\sin^3 x} \, dx \\
&= \ln|\sec x| + \ln|\sin x| - \frac{1}{2\sin^2 x}
\end{aligned} \tag{3.9}
$$
