1. Vector spaces

1.1 Definition of a Vector Space

Definition 1.20:
A "vector space" is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

- **Commutativity:** $u + v = v + u \quad \forall u, v \in V$
- **Associativity:** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv) \quad \forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}$
- **Additive identity:** $\exists 0 \in V : v + 0 = v \quad \forall v \in V$. ($\exists!$ is not a requirement but a property.)
- **Additive inverse:** $\forall v \in V \quad \exists w \in V : v + w = 0$
- **Multiplicative identity:** $1v = v \quad \forall v \in V$
- **Distributive properties:**
  $a(u + v) = au + av$ and
  $(a + b)v = av + bv \quad \forall a, b \in \mathbb{F}$ and $\forall u, v \in V$

Definition 1.24:
If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$. Let $f, g \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$:

- The "sum" $f + g \in \mathbb{F}^S$ is defined as follows: $(f + g)(x) :\equiv f(x) + g(x) \quad \forall x \in S$
- The "product" $\lambda f \in \mathbb{F}^S$ is defined as follows: $(\lambda f)(x) :\equiv \lambda f(x) \quad \forall x \in S$.

The vector space $\mathbb{F}^n :\equiv \mathbb{F}^{\{1,2,\ldots,n\}}$ is a special case, because each $(x_1, \ldots, x_n) \in \mathbb{F}^n$ can be thought of as a function x from the set $\{1, 2, \ldots, n\}$ to $\mathbb{F}$ by writing $x(k)$ instead of x_k for the k^{th} coordinate of $(x_1, \ldots, x_n)$.

Theorem 1.26:
A vector space has a unique additive identity.

Theorem 1.27:
Every element in a vector space has a unique additive inverse.

Definition 1.28:
For $v, w \in V$, $-v$ denotes the additive inverse of v and $w - v :\equiv w + (-v)$

Definition 1.29:
For the rest of this summary, V denotes a vector space over $\mathbb{F}$.

Theorem 1.30:
$0v = 0 \quad \forall v \in V$

Theorem 1.31:
$a\vec{0} = \vec{0} \quad \forall a \in \mathbb{F}$

Theorem 1.32:
$(-1)v = -v \quad \forall v \in V$

1.2 Subspaces

Definition 1.33:
A subset U of V ($U \subseteq V$) is called a "subspace" of V if U is also a vector space with the same additive identity, addition, and scalar multiplication as on V.

Theorem 1.34:
A subset U of V is a subspace of $V \iff U$ satisfies the following three conditions:

- **Additive identity:** $0 \in U$
- **Closed under addition:** $u, w \in U \implies u + w \in U$
- **Closed under scalar multiplication:** $a \in \mathbb{F}$ and $u \in U \implies au \in U$

1.2.1 Sum of Subspaces

Definition 1.36:
$V_1 + \cdots + V_m :\equiv \{v_1 + \cdots + v_m \mid v_1 \in V_1, \ldots, v_m \in V_m\}$ for V_i's beeing subspaces of V. It is called the "sum from V_1 up to V_m."

Theorem 1.40:
$V_1 + \cdots + V_m$ is the smalles subspace of V containing $V_1, \ldots, V_m$

1.2.2 Direct Sum

Theorem 1.41:
$V_1 + \cdots + V_m$ is called "a direct sum", if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$. In this case:

$$V_1 \oplus \cdots \oplus V_m :\equiv V_1 + \cdots + V_m \tag{1.1}$$

Example 1.42:
example: $\mathbb{F}^3 = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in \mathbb{F} \mid x, y \in \mathbb{F} \right\} \oplus \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \in \mathbb{F} \mid z \in \mathbb{F} \right\}$

Theorem 1.45:
$V_1 + \cdots + V_m$ is a direct sum $\iff$ the only write 0 as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0.

Theorem 1.46:
$U + W$ is a direct sum $\iff U \cap W = \{0\}$

2. Finite Dimensional Vector Spaces

2.1 The definition of span

Definition 2.5:
The "span" of a list of vectors $v_1, \dots, v_m$ is defined as follows:

$$\operatorname{span}(v_1, \dots, v_m) :\equiv \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \dots a_m \in \mathbb{F}\} \tag{2.1}$$

Theorem 2.6:
The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Theorem 2.7:
If $\operatorname{span}(v_1, \dots, v_m) = V$, we say $v_1, \dots, v_m$ "spans" V.

Theorem 2.9:
Such a vector space is called finite-dimensional. $\dim V \neq \infty$

Theorem 2.10:
$\mathcal{P}(\mathbb{F}) = \operatorname{span}(1, z, \dots, z^m)$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m. The degree of a polynomial p is denoted with $\deg p$. We also define $\deg 0 :\equiv -\infty$

2.2 Linear Independence

Definition 2.15:
There are 2 ways to define linear independence. A list of vectors $v_1, \dots, v_m$ is called "linearly independent ", if every combination of these vectors has a unique representation $w = b_1 v_1 + \cdots + b_m v_m \in \operatorname{span}(v_1, \dots, v_m)$. (The list $v_1, \dots, v_m$ is linearly independent , if the choice of b's to yield w is unique.)
Another way to put it: A list of vectors $v_1, \dots, v_m$ is also called linearly independent , if the only way to combine them together and yield zero $\lambda_1 v_1 + \cdots + \lambda_m v_m = 0$, is if we choose every coefficient λ_i to be zero.

$$\begin{aligned} &w = a_1 v_1 + \cdots + a_m v_m \text{ and} \\ &w = c_1 v_1 + \cdots + c_m v_m \\ &\quad\Longleftrightarrow \\ &0 = \underbrace{(a_1 - c_1)}_{=0?} v_1 + \cdots + \underbrace{(a_m - c_m)}_{=0?} v_m \end{aligned} \tag{2.2}$$

if $a_1 = c_1, a_2 = c_2, \dots, a_m = c_m$ is the only solution, the representation of w is unique and the only way to add them up together equaling 0 is $0v_1 + \cdots + 0v_m = 0$. Both of these definitions are equivalent.

Definition 2.16:
Otherwise, $v_1, \dots, v_m$ is called "linearly dependent".

Theorem 2.19:
Linear dependence Lemma:
Suppose $v_1, \dots, v_m \in V$ is a linearly dependent list. $\Longrightarrow \exists k \in \{1, \dots, m\} : v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$.
Furthermore, if k satisfies the conditions above and the k^{th}-term is removed from $v_1, \dots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \dots, v_{k-1})$:

$$\operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_m) = \operatorname{span}(v_1, \dots, v_m) \tag{2.3}$$

Theorem 2.22:

Length of linearly independent list $\leq$ length of spanning list:
In a finite dimensional vector space, the length of every linearly independent list of vectors is less then or equal to the length of every spanning list of vectors. From now on, every vector space we consider is finite dimension if not explicitly mentioned otherwise.

2.3 Bases

Definition 2.26:
A basis of V is a list of vectors $v_1,\dots,v_m$ in V that is linearly independent and spans V.

Definition 2.28:
$v_1,\dots,v_m$ is a basis of $V \overset{\text{def}}{\iff}$ every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \cdots + a_nv_n$.

Theorem 2.30:
Every spanning list of a vector space can be reduced to a basis of the vector space .

Theorem 2.31:
Every finite-dimensional vector space has a basis.

Theorem 2.32:
Every linearly independent list of vectors in a finite-dimensional vector space can be reduced to a basis of the vector space .

Theorem 2.33:
Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

2.4 Dimensions

Theorem 2.34:
Any two bases of a finite-dimensional vector space have the same length (Another wording: basis length does not depend on basis, by 2.22)

Definition 2.35:
$\dim V :\equiv$ length of any basis of V

Theorem 2.37:
If V is finite-dimensional and U is a subspace of V, $\implies \dim U \leq \dim V$ (by 2.22)

Theorem 2.38:
Every linearly independent list of vectors of length $\dim V$ is a basis. (by 2.32)

Theorem 2.39:
If U is a subspace of V and $\dim U = \dim V$, then $U = V$. (by 2.38)

Theorem 2.42:
Every spanning list of vectors in V of length $\dim V$ is a basis of V

Theorem 2.43:
V_1 and V_2 are subspaces of a finite-dimensional vector space

$$\implies \dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) \tag{2.4}$$

3. Linear Maps

3.1 vector space of linear maps

Definition 3.1:
A linear map from V to W is a function $T : V \to W$ with following properties:

- additivity: $T(u+v) = Tu + Tv \qquad \forall u, v \in V$
- homogenity: $T(\lambda v) = \lambda(Tv) \qquad \forall \lambda \in \mathbb{F}\ \forall v \in V$

Definition 3.2:
The set of all linear maps from V to W is denoted by $\mathcal{L}(V, W)$. For The set of all linear maps from V to itself we use $\mathcal{L}(V) :\equiv \mathcal{L}(V, V)$

Theorem 3.4:
Linear map lemma: Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$, where the w's do not have any specific properties. Then there exist a unique linear map $T : V \to W$ or $T \in \mathcal{L}(V, W)$ such that

$$Tv_k = w_k \quad \forall k \in \{1, \ldots, n\}.$$

It is of the form $T(c_1 v_1 + \cdots c_n v_n) \mapsto c_1 w_1 + \cdots + c_n w_n$

Definition 3.5:
The sum and the product of linear maps:
If $S, T \in \mathcal{L}(V, W)$, $v \in V$, $\lambda \in \mathbb{F}$:

- $(S+T)(v) :\equiv Sv + Tv$
- $(\lambda T)(v) :\equiv \lambda(Tv)$

Theorem 3.6:
With these operations above, $\mathcal{L}(V, W)$ is itself a vector space .

Definition 3.7:
Let $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. We define the product $ST \in \mathcal{L}(U, W)$ as follows:

$$(ST)(u) :\equiv S(Tu) \quad \forall u \in U$$

Theorem 3.8:
With these definitions we have

- **Associativity:** $(T_1 T_2) T_3 = T_1 (T_2 T_3)$, whenever T_3 maps into the Domain of T_2 and T_2 maps into the Domain of T_1.
- **Identity:** $TI = IT = T$ for $T \in \mathcal{L}(V, W)$ (The first I is the identity operator on V and the second I the identity operator on W. We could also write $TI_V = I_W T$
- **Distributive properties:** For $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$:
  $(S_1 + S_2)T = S_1 T + S_2 T$ and $S(T_1 + T_2) = ST_1 + ST_2$
- **Non-commutative:** $ST \neq TS$ in general.

Theorem 3.10:
$T \in \mathcal{L}(V, W) \implies T(0) = 0.$
Proof: $T(0) = T(0+0) = T(0) + T(0)$. Subtracting $T(0)$ on both sides ends the proof. $\square$

3.2 Null spaces, ranges and injectivity

Definition 3.11:
$\operatorname{null} T :\equiv \ker T :\equiv \{v \in V \mid Tv = 0\} \subseteq V$ for $T \in \mathcal{L}(V, W)$

Theorem 3.13:
$\operatorname{null} T$ is a subspace of V

Definition 3.14:
A function $T : V \to W$ is called injective if $Tu = Tv \implies u = v$.
Or analogous: $u \neq v \implies Tu \neq Tv$

Definition 3.15:
Let $T \in \mathcal{L}(V, W)$. Then T is injective $\iff \operatorname{null} T = \{0\}$

3.3 Definition of Range

Theorem 3.16:
$\operatorname{range} T = \{Tv \mid v \in V\} \subseteq W$ for $T \in \mathcal{L}(V, W)$

Theorem 3.18:
$T \in \mathcal{L}(V, W) \implies \operatorname{range} T$ is a subspace of W.

Definition 3.19:
If $\operatorname{range} T = W$, T is called "surjektive" or "onto".

Theorem 3.21:
Fundamental theorem of linear maps or **"rank nullity theorem":**
For $T \in \mathcal{L}(V, W)$:

$$\dim V = \underbrace{\dim \operatorname{null} T}_{\text{nullity}} + \underbrace{\dim \operatorname{range} T}_{\text{rank}} \tag{3.1}$$

Theorem 3.22:
If $\dim V > \dim W \implies$ No linear map from V to W is injective.

Theorem 3.24:
If $\dim V < \dim W \implies$ No linear map from V to W is surjective.

Theorem 3.26:
A homogeneous system of linear equations with more variables then equations has nonzero solutions.

Theorem 3.28:
An inhomogeneous system of linear equations with more equations then variables has no solutions for some choice of constant terms.

3.4 Matrices

3.4.1 Representing a Liner Map by a Matrix

Definition 3.31:
Matrix of a linear map, $\mathcal{M}(\mathbf{T})$:
Suppose $T \in \mathcal{L}(L, W)$ and $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ be a basis of W. The "matrix of T" with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m = \sum_{j=1}^{m} A_{j,k}w_j \tag{3.2}$$

If the bases are not clear from the context, one writes $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$. One remembers how to construct $\mathcal{M}(T)$ as follows:

$$\mathcal{M}(T) :\equiv \begin{array}{c} \\ w_1 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \cdots & v_k & \cdots & v_n \end{array} \\ \begin{pmatrix} & & A_{1,k} & & \\ & & \vdots & & \\ & & A_{m,k} & & \end{pmatrix} \end{array} \tag{3.3}$$

3.4.2 Addition and Scalar Multiplication of Matrices

Theorem 3.35:
$\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$ for $S, T \in \mathcal{L}(V, W)$

Theorem 3.38:
$\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$ for $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$

Definition 3.39:
$\mathbb{F}^{m,n}$ denotes the set of all m-by-n matrices with entries in F, where $m, n \in \mathbb{N}$

Theorem 3.40:
$\mathbb{F}^{m,n}$ is a vector space with $\dim \mathbb{F}^{m,n} = mn$.

Definition 3.41:
$(AB)_{j,k} :\equiv \sum_{r=1}^{m} A_{j,r}B_{r,k}$ for $A \in \mathbb{F}^{m,n}$, $B \in \mathbb{F}^{n,p}$, $AB \in \mathbb{F}^{m,p}$

Theorem 3.46:
$(AB)_{j,k} = A_{j,\bullet}B_{\bullet,k}$ if $1 \le j \le m$ and $1 \le k \le p$

Theorem 3.31:
Supp $T \in \mathcal{L}(V, W)$ and let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ be a basis of W. The matrix of T with respect of these bases is the m-by-n Matrix called $\mathcal{M}(T) :\equiv A$ whose entries $A_{j,k}$ are defined by

$$Tv_k :\equiv A_{1,k}w_1 + \cdots + A_{m,k}w_m \tag{3.4}$$

also denoted as $\mathcal{M}(T, (v_1, \dots, v_m), (w_1, \dots, w_m))$.

Theorem 3.50:
Suppose $A \in \mathbb{F}^{m,n}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{F}^{n,1}$. Then $Ab = b_1 A_{\bullet,1} + \cdots + b_n A_{\bullet,n}$

3.5 Invertibility and Isomorphisms

3.5.1 Inverse

Definition 3.59:
$S \in \mathcal{L}(V,W)$ with $ST = I_V \wedge TW = I_W$ is called the "inverse" (map?) of the invertible linear map $T \in \mathcal{L}(V,W)$

Theorem 3.60:
An invertible linear map has a unique inverse.
Proof: Suppose $T \in \mathcal{L}(V,W)$ is invertible and S_1 and S_2 are inverses of T.
$\implies S_1 = S_1 I = S_1(TS_2) = (S_1T)S_2 = IS_2 = S_2 \implies S_1 = S_2$ □

Theorem 3.61:
The inverse is denoted by T^{-1}. $T^{-1}T = I_V$ and $TT^{-1} = I_W$ (?)

Theorem 3.62:
Invertibility of a linear map $\in \mathcal{L}(V,W) \Leftrightarrow$ injectivity and surjectivity.

Theorem 3.65:
If $\dim V = \dim W \neq \infty$, injectivity is equivalent to surjectivity for $T \in \mathcal{L}(V,W)$:
T is invertible $\iff$ T is injective $\iff$ T is surjective.
Proof: The rank-nullity theorem (3.21) or fundamental theorem of linear maps states that

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \tag{3.5}$$

If T is injective, $\dim \operatorname{null} T = 0$ and therefore $\dim \operatorname{range} T = \dim V = \dim W$, which means T is surjective ($\operatorname{range} T = \dim W$). This is because every linearly independent list of vectors of of length $\dim W$ is a basis by **??**.
If T is surjective, we have $\dim \operatorname{range} T = \dim W$ to begin with and therefore 3.5 becomes

$$\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T = \dim V - \dim W = 0 \tag{3.6}$$

which makes T also injective. Since injectivity and surjectivy together imply invertibility, this ends the proof. □

Theorem 3.68:
$ST = I \Leftrightarrow TS = I$

Theorem 3.69:
An isomorphism is an invertible linear map. Two vector spaces are called isomorphic if there is a isomorphism from one vector space to the other. $T : V \to W$

Theorem 3.70:
Two finite-dimensional vector spaces U, V over $\mathbb{F}$ are isomorphic $\iff$ they have the same dimension. $\dim U = \dim V$

Theorem 3.71:
Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$

Theorem 3.72:
Suppose V, W are finite-dimensional . $\implies$ $\mathcal{L}(V,W)$ is finite-dimensional and

$$\dim \mathcal{L}(V,W) = (\dim V) \cdot (dimW) \tag{3.7}$$

3.5.2 Linear maps thought of as matrix multiplication

Definition 3.73:
Let $v \in V$ and $v_1, \ldots, v_m$ be a basis of V such that $v = b_1 v_1 + \cdots + b_m v_m$.
We define $\mathcal{M}(v) :\equiv \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. The vector depends on the basis, but it is not included in the notation.

Recall: $\mathcal{M}(T) :\equiv \begin{array}{c c} & \begin{array}{ccccc} v_1 & \cdots & v_k & \cdots & v_n \end{array} \\ \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array} & \begin{pmatrix} & & A_{1,k} & & \\ & & \vdots & & \\ & & A_{m,k} & & \end{pmatrix} \end{array}$

Theorem 3.75:
$\mathcal{M}_{\bullet,k} = M(Tv_k)$. The k^{th} column of $\mathcal{M}(T)$ equals $(A_{1,k}, \ldots, A_{m,k})^\top$

Theorem 3.76:
Linear maps act like matrix multiplication: Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T) \cdot \mathcal{M}(v) \tag{3.8}$$

Or using different notation:

$$\mathcal{M}(Tv, (v_1, \ldots, v_n), (w_1, \ldots, w_m)) = \mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)) \cdot \mathcal{M}(v) \tag{3.9}$$

Theorem 3.78:
For $T \in \mathcal{L}(V, W)$: $\dim \operatorname{range} T$ = column rank of $\mathcal{M}(T)$

3.5.3 Change of basis

Definition:
$\mathcal{M}(T, (v_1, \ldots, v_n)) :\equiv \mathcal{M}(T, (v_1, \ldots, v_n), (v_1, \ldots, v_n))$

Theorem 3.80:
A square matrix A is called invertible, if there is some square matrix B of the same size such that $AB = BA = I$. We call B the inverse of A denoted by $A^{-1} :\equiv B$. Rules:

- $(A^{-1})^{-1} = A$
- $(AC)^{-1} = C^{-1}A^{-1}$ (Because $(AC)(C^{-1}A^{-1}) = I$ and $(C^{-1}A^{-1})(AC) = I$)

Theorem 3.81:
Matrix of product of linear maps:
Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. If $u_1, \ldots, u_m$ is a basis of U, $v_1, \ldots, v_n$, is a basis of V and $w_1, \ldots, w_p$ is a basis of W. Note that $\dim U = m$, $\dim V = n$, $\dim W = p$. Then we have:

$$\mathcal{M}(ST, (u_1, \ldots, u_m), (w_1, \ldots, w_p)) = \mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_p)) \cdot \mathcal{M}(T, (u_1, \ldots, u_m), (v_1, \ldots, v_n)) \tag{3.10}$$

Or using different notation:

$$\mathcal{M}(ST) = \mathcal{M}(S) \cdot \mathcal{M}(T) \tag{3.11}$$

Theorem 3.84:
Change-of-basis formula:
Suppose $T \in \mathcal{L}(V)$. Let $V = \operatorname{span}(u_1, \ldots, u_m) = \operatorname{span}(v_1, \ldots, v_m)$ such that the u's and v's both form a basis. Let $A = \mathcal{M}(T, (u_1, \ldots, u_m))$ and $B = \mathcal{M}(T, (v_1, \ldots, v_m))$. Let

$C = \mathcal{M}(I, u_1, \ldots, u_m, v_1, \ldots, v_m)$. Then

$$A = C^{-1}BC \tag{3.12}$$

Theorem 3.86:
If $v_1, \ldots, v_m$ is a basis V and $T \in \mathcal{L}$ is invertible, then

$$\begin{gathered}\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1} \text{ or} \\ \mathcal{M}(T^{-1}, (v_1, \ldots, v_m)) = \mathcal{M}(T, (v_1, \ldots, v_m))^{-1}\end{gathered} \tag{3.13}$$

3.6 Products and quotients of vector spaces

3.6.1 Products of vector spaces

Definition 3.87:
The product, addition and scalar multiplication of vector spaces $V_1, \cdots, V_m$ is defined as follows:

$$\begin{aligned} V_1 \times \cdots \times V_m &:\equiv \{(v_1, \dots, v_m) \mid v_1 \in V_1, \dots, v_m \in V_m\} \\ (u_1, \dots, u_m) + (v_1, \dots, v_m) &:\equiv (u_1 + v_1, \dots, u_m + v_m) \\ \lambda(v_1, \dots, v_m) &:\equiv (\lambda v_1, \dots, \lambda v_m) \end{aligned} \tag{3.14}$$

Theorem 3.89:
$V_1 \times \cdots \times V_m$ together with addition and scalar multiplication is a vector space over $\mathbb{F}$.

Theorem 3.92:
$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$

Theorem 3.93:
Let $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ and $\Gamma(v_1, \cdots, v_m) \mapsto v_1 + \cdots + v_m$
Then $v_1 + \cdots + v_m$ is a direct sum $\iff$ Γ is injective.

Theorem 3.94:
$V_1 + \cdots + V_1$ is a direct sum $\iff$ $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$

3.7 Quotient spaces

Definition 3.95:
$v + U :\equiv \{v + U \mid u \in U\}$ for $v \in V$ and $U \subseteq V$ is said to be "a translate" of U.

Definition 3.99:
$V/U :\equiv \{v + U \mid v \in V\}$ is called "quotient space".

Example 3.100:
If $U = \{(x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \implies \mathbb{R}/U$ is the set of all lines with slope 2.

Example 3.101:
If U is a plane in $\mathbb{R}^3 \implies \mathbb{R}^3/U$ is the set of all planes parallel to U.

Theorem 3.101:
$U \subseteq V$ and $v, w \in V$. Then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset$$

That means, two translates of a subspaces are equal or disjoint.

Definition 3.102:
Definition of addition and scalar multiplication on V/U.

$$\begin{aligned} (v + U) + (w + U) &:\equiv (v + w) + U \\ \lambda(v + U) &:\equiv (\lambda v) + U \qquad \forall v, w \in V \text{ and } \forall \lambda \in \mathbb{F} \text{ and } U \subseteq V \end{aligned} \tag{3.15}$$

Theorem 3.103:
V/U is a vector space with additive identity $0 + U$ which is equal to U and the additive inverse $(-v) + U$.
3.104 TODO **3.105** TODO

3.8 Duality

3.8.1 Dual Space and Dual Map

Definition 3.108:
A "linear functional" ϕ is an element of $\mathcal{L}(V,\mathbb{F})$. So $\phi \in \mathcal{L}(V,\mathbb{F})$

Examples:

$$\begin{array}{lll} \phi : \mathbb{R}^3 \to \mathbb{R}, & \phi(x,y,z) & \mapsto 4x - 5y - 2z \\ \phi : \mathbb{F}^n \to \mathbb{F}, & \phi(x_1,\dots,x_n) & \mapsto c_1x_1 + \cdots + c_nx_n \\ \phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}, & \phi(p) & \mapsto 3p''(5) + 7p(4) \\ \phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}, & \phi(p) & \mapsto \int_0^1 p(x)dx \end{array} \tag{3.16}$$

Theorem 3.110:
The dual space of V, denoted by V' or V^*, is the vector space of all linear functionals on V.

$$V^* :\equiv \mathcal{L}(V,\mathbb{F}) \tag{3.17}$$

Theorem 3.111:
$\dim V^* = \dim V$.

Proof: $\dim V^* = \dim \mathcal{L}(V,\mathbb{F}) = (\dim V)\cdot(\dim \mathbb{F}) = \dim V$ □

Definition 3.112:
If $v_1,\dots,v_m$ is a basis of V, then the "dual basis" of $v_1,\dots,v_n$ is the list $\varphi_1,\dots,\varphi_m \in V^*$, where each φ_j is the linear functional such that $\varphi_j(v_k) = \delta_{j,k} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases}$.

Note that this is not the definition of each φ_j.

Theorem 3.114:
Suppose $v_1,\dots,v_m$ is a basis of V. $\varphi_1,\dots,\varphi_m$ is called a dual basis of $v_1,\dots,v_m$ because for every $v \in V$ we have $v = \varphi_1(v)v_1 + \cdots \varphi_m(v)v_m$

Proof: Let $v \in V$, $v = c_1v_1 + \cdots c_mv_m$. If $j \in \{1,\dots,m\}$, then applying φ_j on both sides gives $\varphi_j(v) = c_j$. □

Theorem 3.116:
The dual basis of a basis of V is a basis of the dual space V^*

Definition 3.118:
Suppose $T \in \mathcal{L}(V,W)$. The "dual map" of T is the linear map $T^* \in \mathcal{L}(W^*,V^*)$ defined like this:

$$\forall \phi \in W^* : T^*(\varphi) :\equiv \varphi \circ T$$

- $\varphi \in W^* = \mathcal{L}(W,\mathbb{F})$ and $T^*(\varphi) \in V^* = \mathcal{L}(V,\mathbb{F})$. So T^* is indeed a map from W^* to V^*
- $\varphi,\psi \in W^* \implies T^*(\varphi+\psi) = (\varphi+\psi)\circ T = \varphi\circ T + \psi\circ T = T^*(\varphi) + T^*(\psi)$
- $\lambda \in \mathbb{F}, \varphi \in W \implies T^*(\lambda\varphi) = (\lambda\varphi)\circ T = \lambda(\varphi\circ T) = \lambda T^*(\varphi)$

4. Polynomials

$\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

4.1 Complex numbers

4.1 $z = a + bi$

4.2 Zeros of polynomials

$p : \mathbb{F} \to \mathbb{F}$, $p(z) = a_0 + a_1 z + \cdots + a_n z^m$

Theorem 4.5:
$\lambda \in \mathbb{F}$ is called a zero (or root) of a polynomial if $p(\lambda) = 0$.

Theorem 4.6:
$p \in \mathcal{P}_m(\mathbb{F}), \lambda \in \mathbb{F} : p(\lambda) = 0 \iff \exists q \in \mathcal{P}_{m-1} : p(z) = (z - \lambda)q(z)$

Theorem 4.8:
A polynomial of degree m has at most m zeros.
Proof: For $m = 1$, $a_0 + a_1 z$ has only one zero, which is $-\frac{a_0}{a_1}$
Now for $m > 1$ and assuming the desired result holds for $m - 1$. If $p \in \mathcal{P}_m(\mathbb{F})$ has no zeros, we are done. If p has a zero $\lambda \in \mathbb{F}$, by 4.6, there is a $q \in \mathcal{P}(\mathbb{F})_{m-1}$ such that $p(z) = (z - \lambda)q(z)$. Since q has at most $m - 1$ zeros, p has at most m zeros. $\square$

4.3 Division algorithm for polynomials

Theorem 4.9:
Suppose $p, s \in \mathcal{P}(\mathbb{F}), s \neq 0$. Then $\exists q, r \in \mathcal{P}(\mathbb{F})$ such that $p = sq + r$ and $\deg r < \deg s$.

Theorem 4.12:
Fundamental theorem of algebra, first version: Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Theorem 4.13:
Fundamental theorem of algebra, second version: $p \in \mathcal{P}(\mathbb{C})$, p is non-constant $\implies p$ has a unique factorization $p(z) = c \cdot (z - \lambda_1) \cdots (z - \lambda_m)$ where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

Theorem 4.14:
Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is $\overline{\lambda}$.

Theorem 4.15:
Factorization of a quadratic polynomial: $b, c \in \mathbb{R}$. Then $x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{R} \iff b^2 \geq 4c$.
Proof: $x^2 + bx + c = (x + \frac{b}{2})^2 + (c - \frac{b^2}{4})$. Therefore, if $b^2 < 4c$, the last term is always positive and no factorization is possible because it has no zeros.
If $b^2 \geq 4c$, we define $d^2 :\equiv \frac{b^2}{4} - c$. $\implies x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - d^2 = \left((x + \frac{b}{2}) + d\right)\left((x + \frac{b}{2}) - d\right) = \left(x - (-d - \frac{b}{2})\right)\left(x - (d - \frac{b}{2})\right) = (x - \lambda_1)(x - \lambda_2)$ for $\lambda_1 :\equiv -d - \frac{b}{2}$ and $\lambda_2 :\equiv d - \frac{b}{2}$ $\square$

Theorem 4.16:
Suppose $p \in \mathcal{P}(\mathbb{R})$ is nonconstant. Then p has a unique factorization:

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M) \tag{4.1}$$

where $c, \lambda_1, \cdots, \lambda_m, b_1, \cdots, b_M, c_1, \cdots, c_M \in \mathbb{R}$

5. Eigenvalues and Eigenvectors

5.1 Invariant Subspaces

5.1.1 Eigenvalues

Definition 5.1:
A linear map from a vector space to itself is called an "operator".
(Suppose $T \in \mathcal{L}(V)$, then may be $T|_{V_k}$ is not an operator on a subspace V_k)

Definition 5.2:
Let $T \in \mathcal{L}(V)$. $U \subseteq V$ is called "invariant under T" if $\forall u \in U : Tu \in U$.
Thus U is invariant under T if $T|_U$ is an operator on U.

Example 5.3:
Let $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that $Tp = p'$. Let $U = \mathcal{P}_4(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$. Then U is invariant under T because if $p \in U$, $\deg p = 4$ and $\deg(p') = 3$.

Example 5.4:
Let $T \in \mathcal{L}(V)$. Then $\{0\}, V, \operatorname{null} T, \operatorname{range} T$ are all invariant.
(Sometimes, $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = V$ if T is invertible.)
Invariant subspaces of dimension one:
Take any $v \in V, v \neq 0$ and let $U :\equiv \{\lambda v \mid \lambda \in \mathbb{F}\} = \operatorname{span}(v)$, then U is a one-dimensional subspace of V.
If U is invariant under an operator $T \in \mathcal{L}(V)$, then $Tv \in U$. $\implies \exists \lambda \in \mathbb{F} : Tv = \lambda v$.
Conversely if $Tv = \lambda v$, $\lambda \in \mathbb{F}$, then $\operatorname{span}(v)$ is a one-dimensional subspace of V invariant under T.

Definition 5.5:
Suppose $T \in \mathcal{L}(V)$. $\lambda \in \mathbb{F}$ is called "eigenvalue of T" if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$

Theorem 5.7:
The following are equivalent for $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$:

(a) λ is an eigenvalue of T.
(b) $T - \lambda I$ is not injective.
(c) $T - \lambda I$ is not surjective.
(d) $T - \lambda I$ is not invertible.

Proof: Conditions (a) and (b) are equivalent because the eigenvector v is a solution to $Tv = \lambda v$ which is equivalent to $(T - \lambda I)v = 0$. So there is a non-zero solution to $T - \lambda I$.
(b), (c) and (d) are equivalent by 3.65.

□

Theorem 5.8:
Let $T \in \mathcal{L}(V)$. A vector $v \in V$ is called "an eigenvector" of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$. In other words:
A vector $v \in V, v \neq 0$ is an eigenvector corresponding to $\lambda \iff v \in \operatorname{null}(T - \lambda I_V)$

Theorem 5.11:
Every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.
Proof: Suppose the desired result is false. Then there exists a smallest list of length m of

linearly dependent eigenvectors $v_1,\dots,v_m$ with eigenvalues $\lambda_1,\dots,\lambda_m$ of T. Since an eigenvector is unequal to the zero vector, m must be ≥ 2.
Because of the minimality of m and becaue our list is linearly dependent: $\exists a_1,\dots,a_m \neq 0$ such that $a_1v_1+\dots+a_mv_m=0$. Now we apply $T-\lambda_m I$ on both sides of the equation and get $a_1\lambda_1v_1 - a_1\lambda_mv_1+\dots+a_{m-1}\lambda_{m-1}v_{m-1}-a_{m-1}\lambda_mv_{m-1}+\underbrace{a_m\lambda_mv_m-a_m\lambda_mv_m}_{=0}=0$
From there it follows that: $a_1\underbrace{(\lambda_1-\lambda_m)}_{\neq 0}v_1+\dots+a_{m-1}\underbrace{(\lambda_{m-1}-\lambda_m)}_{\neq 0}v_{m-1}=0$
Which contradicts the minimality of m. Therefore, no such linearly dependent list of eigenvectors can exist. □

Theorem 5.12:
Each operator on V has at most $\dim V$ distinct eigenvalues. content

5.1.2 Polynomials applied to operators

Definition 5.13:
Let $T\in\mathcal{L}(V)$ and $m\in\mathbb{N}^+$

- $T^m :\equiv \underbrace{T\cdots T}_{m \text{ times}}$ or $T^m :\equiv T^{m-1}\cdot T$ such that $T^m\in\mathcal{L}(V)$
- $T^0 :\equiv I_V$
- If T is invertible with inverse T^{-1} then $T^{-m}\in\mathcal{L}(V)$ is defined by $T^{-m} :\equiv (T^{-1})^m$

$\Longrightarrow$ $T^mT^n=T^{m+n}$ and $(T^m)^n=T^{mn}$ when $m,n\in\mathbb{Z}$ when T is invertible. And $m,n\in\mathbb{N}$ if T is not invertible.

Definition 5.14:
For $p\in\mathcal{P}(\mathbb{F})$, $p(z)=a_0+a_1z+a_2z^2+\dots+a_mz^m\ \forall z\in\mathbb{C}$
For $T\in\mathcal{L}(V)$ we define:
$p(T) :\equiv a_0I+a_1T+a_2T^2+\cdots a_mT^m$, $p(T)\in\mathcal{L}(V)$

Theorem 5.17:
Suppose $p,q\in\mathcal{P}(\mathbb{F})$ and $T\in\mathcal{L}(V)$. Then $(pq)(T)=p(T)q(T)=q(T)p(T)$.

Theorem 5.18:
$T\in\mathcal{L}(V)$ and $p\in\mathcal{P}(\mathbb{F})$ $\Longrightarrow$ $\operatorname{null} p(T)$ and $\operatorname{range} p(T)$ are invariant under T.
Proof: Suppose $u\in\operatorname{null}p(T)$ $\Longrightarrow$ $p(T)u=0$. Assoziativiy and distributivity of linear maps imply hat $(p(T))(Tu)=T(p(T)u)=T(0)=0.$ $\Longrightarrow$ $Tu\in\operatorname{null}p(T)$.
Suppose $u\in\operatorname{range}p(T)$ $\Longrightarrow$ $\exists v\in V: u=p(T)v$ $\Longrightarrow$ $Tu=T(p(T)v)=p(T)(Tv)$ $\Longrightarrow$ $Tu\in\operatorname{range}p(T)$. □

5.1.3 The minimal polynomial

Theorem 5.19:
Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.
Proof: Suppose $\dim(V)=n>0$ and $T\in\mathcal{L}(V)$. Choose $v\in V, v\neq 0$. Then $v,Tv,T^2v,\dots,T^nv$ is not linearly independent, because the list has length $n+1$. $\Longrightarrow$ some linear combination of these vectors equals to 0. $\Longrightarrow$ there exists a non-constant polynomial p of smallest degree such that $p(T)v=0$. By the first version of the fundamental theorem of algebra (4.12) $\Longrightarrow$ $\exists\lambda\in\mathbb{C}: p(\lambda)=0$.

$(4.6) \implies \exists q \in \mathcal{P}(\mathbb{C}) : p(z) = (z-\lambda)q(z)\ \forall z \in \mathbb{C}$
$(5.17) \implies 0 = p(T)v = (T-\lambda I)(q(T)v)$. Because q has a smaller degree than p, $q(T)v \neq 0$.
$\implies \lambda$ is an eigenvalue of T with eigenvector $q(T)v$. □

5.1.4 Eigenvalues and the minimal polynomial

Definition 5.20:
A monic polynomial is a polynomial whose highest-degree coefficient equals 1.

Example 5.21:
$p(z) = 2 + 9z^2 + z^7, \deg p = 7$

Theorem 5.22:
Suppose $T \in \mathcal{L}(V)$. Then there exists a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$. Furthermore $\deg p \leq \dim V$

Proof: If $\dim V = 0$, I, is the zero-operator on V and we let $p = 1$ such that $1I\vec{0} = 0$.
Now we use induction on $\dim V$ and we assume $\dim V > 0$. Let $v \in V, v \neq 0$. The list $v, Tv, \ldots, T^{\dim V}$ has length $1 + \dim V$. $\implies$ linear dependence.
By the linear dependence lemma (2.19), there is a smallest positive integer $m \leq \dim V$ such that $c_0 v + c_1 Tv + \cdots + c_{m-1}T^{m-1}v + T^m v = 0$ for some $c_0, c_1, \ldots, c_{m-1} \in \mathbb{F}$.
Let $q(z) :\equiv c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m \in \mathcal{P}(\mathbb{F})$
$\implies q(T)v = 0$. Not that $q(z)$ is a monic polynomial.
If $k \in \mathbb{N}$, then $q(T)(T^k v) = T^k(q(T)v) = T^k(0) = 0$.
By the linear dependence lemma (2.19) $\implies v, Tv, \ldots, T^{m-1}v$ from before are linearly independent $\implies \dim \operatorname{null} q(T) \geq m$
$\implies \dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leq \dim V - m$.
Because $\operatorname{range} q(T)$ is invariant under T (by 5.18), we can apply our induction hypothesis to the operator $T|_{\operatorname{range} q(T)}$.
So there exists monic $s \in \mathcal{P}(\mathbb{F}) : \deg s \leq \dim V - m$ and $s(T|_{\operatorname{range} q(T)}) = 0$
$\implies \forall v \in V : (sq)(T)(v) = s(T)(q(T)v) = 0$, because $q(T)v \in \operatorname{range} q(T)$ and $s(T)|_{\operatorname{range} q(T)} = s\big(T|_{\operatorname{range} q(T)}\big)$. Therefore, sq is a monic polynomial such that $\deg sq \leq \dim V$ and $(sq)(T) = 0$.
Proof of uniqueness: Let $p \in \mathcal{P}(\mathbb{F})$ a monic polynomial of smallest degree such that $p(T) = 0$. Let $r \in \mathcal{P}(\mathbb{F})$ another monic polynomial of same degree such that $p(T) = 0$.
$\implies (p-r)(T) = 0(*)$ and also $\deg(p-r) < \deg p = \deg r$
If $p - r \neq 0$, we could devide $p - r$ by the coefficient of the highest order term in $p - r$ to get a monic polynomial that when applied to T gives the 0-operator(*). This polynomial would have a smaler degree than p or r, which would be a contradiction. Therefore
$p - r = 0 \iff p = r$ □

Definition 5.24:
Let $T \in \mathcal{L}(v)$. The "minimal polynomial of T" is the unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree s.t. $p(T) = 0$

Computation: Find the smallest $m \in \mathbb{N}$ such that:
$c_0 I + c_1 T + \cdots + c_{m-1}T^{m-1} = -T^m$ has a solution $c_0, \ldots, c_{m-1} \in \mathbb{F}$. Solve for $m = 1, 2, \ldots, \dim V$
Even faster (usually), pick $v \in V$ with $v \neq 0$ and consider the equation $c_0 v + c_1 Tv + \cdots + C_{\dim V - 1}T^{\dim V - 1}v = -T^{\dim V}v$. If this equation has a unique solution, as happens most of the time $c_0, c_1, c_2, \ldots, c_{\dim V - 1}, 1$ are the coefficients of the minimal polynomial of T.

Theorem 5.27:
Let $T \in \mathcal{L}(V)$. Then

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.
(b) If V is a complex vector space, the minimal polynomial has the form $(z-\lambda_1)\cdots(z-\lambda_m)$, where $\lambda_1,\dots,\lambda_m$ are the eigenvalues of T, possibly with repetitions.

Proof: Let p be the minimal polynomial of T.

(a) Suppose $\lambda \in \mathbb{F}$ is a zero of p. $\implies p(z) = (z-\lambda)q(z)$, whre q is a monic polynomial with coefficients in $\mathbb{F}$ (see 4.6)
$p(T) = 0 \implies 0 = (T-\lambda I)(q(T)v) \quad \forall v \in V$. Because q is of lesser degree than p, there exists at least one vector $v \in V$ sucht that $q(T)v \neq 0$, which makes $q(T)v$ an eigenvector with eigenvalue λ.
Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of T. Thus there exists $v \in V, v \neq 0$ such that $Tv = \lambda v$. Repeated applications of T on both sides of this equation show that $T^k v = \lambda^k v \quad \forall k \in \mathbb{N}$. $\implies p(T)v = p(\lambda)v$. Because p is the minimal polynomial of T, we have $p(T)v = 0$. $\implies p(\lambda) = 0$. $\implies \lambda$ is a zero of p.
(b) use (a) and the second version of the fundamental theorem of algebra. (4.13)

□

Theorem 5.29:
$T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{F})$: $q(T) = 0 \iff q$ is a multiple of the minimal polynomial of T.
Proof: Let p denote the minimal polynomial of T.
$\Rightarrow$ direction: Suppose $q(T) = 0$. By (4.9) there exists $s, r \in \mathcal{P}(\mathbb{F})$ such that
$$q = ps + r, \quad \deg r < \deg p \tag{5.1}$$
We have
$$0 = q(T) = p(T)s(T) + r(T) = r(T). \tag{5.2}$$
The equation above implies that $r = 0$. Otherwise, dividing r by its highest-degree coefficient would produce a monic polynomial that wehn applies to T gives 0. A contradiction because $\deg r < \deg p$ and p is minimal. Thus 5.2 becomes the equation $q = ps$, as desired
$\Leftarrow$ direction: Suppose $q = ps$ for $q, p, s \in \mathcal{P}(\mathbb{F})$. We have
$$q(T) = p(T)s(T) = 0s(T) = 0, \tag{5.3}$$
as desired. □

Theorem 5.31:
If U is a subspace of V, then the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T_{|U}$
Proof: Suppose p is the miniaml polynomial of T. $\implies p(T)v = 0 \quad \forall v \in V$. In particular, $p(T)u = 0 \quad \forall u \in U$. Thus $p(T_{|U}) = 0$. Now the previous theorem 5.29 ends the proof. □

Theorem 5.32:
$T \in \mathcal{L}(V): T$ is not invertible $\iff$ the constant term of the minimal polynomial of T is 0.
Proof: T is not invertible $\overset{5.7}{\iff}$ 0 is an eigenvalue of T $\overset{5.27}{\iff}$ 0 is a zero of $p \iff$ the constant term of p is 0. (In the first equivalence, we have actually used that 0 is an eigenvalue of T if and only if $T - 0I$ is not invertible, according to 5.7.) □

Theorem 5.39:
If $T \in \mathcal{L}(V)$ and $v_1,\dots,v_n$ is a basis of V. Then the following are equivalent.

(a) The matrix of T with respect to $v_1,\dots,v_n$ is upper triangular.
(b) $\operatorname{span}(v_1,\dots,v_k)$ is invariant under $T \quad \forall k \in \{1,\dots,n\}$

(c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k) \quad \forall k \in \{1, \ldots, n\}$

Theorem 5.40:
Suppose $T \in \mathcal{L}(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. $\implies$

$$(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0 \tag{5.4}$$

Theorem 5.41:
Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

Theorem 5.42:
Let $T \in \text{Ł}(V)$. Then T has an upper-triangular matrix in respect to some basis $V \iff$ the min. polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m), \quad \lambda_1, \ldots, \lambda_m \in \mathbb{F}$

Theorem 5.43:
Let $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

5.2 Diagonalizable Operators

5.2.1 Diagonal Matrices

Definition 5.48:
A "diagonal matrix" is a square matrix that is 0 everywhere except possibly on the diagonal

Definition 5.50:
An operator on V is called "diagonalizable" if the operator has a diagonal matrix with respect to some basis on V

Theorem 5.52:
Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The "eigenspace" of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) :\equiv \operatorname{null}(T - \lambda I) = \{v \in V \mid Tv = \lambda v\} \tag{5.5}$$

Theorem 5.54:
Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T) \tag{5.6}$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\begin{aligned} \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) &= \dim \left(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)\right) \\ &\leq \dim V \end{aligned} \tag{5.7}$$

Theorem 5.46:
Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of $T \in \mathcal{L}(V)$. Then

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$

Theorem 5.58:
$T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues $\implies$ T is diagonalizable.

Theorem 5.62:
Let $T \in \mathcal{L}(V)$. Then T is diagonalizable $\iff$ the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m), \quad \lambda_1,\ldots,\lambda_m \in \mathbb{F}, \quad \lambda_1 \neq \cdots \neq \lambda_m$

Theorem 5.65:
Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T.
$\implies T|_U$ is a diagonalizable operator on U.

Proof: Diagonazability of $T \overset{5.62}{\iff}$ the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for $\lambda_1 \neq \cdots \neq \lambda_m$. By 5.31, the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$. Hence the minimal polynomial of $T|_U$ has the form required by 5.62, which shows that $T|_U$ is diagonalizable. It consits of factors $(z-\lambda_1),(z-\lambda_2),\ldots,(z-\lambda_m)$. □

5.2.2 Commuting Operators

Definition 5.66:
Two operators or matrices A and B "commute" if $ST = TS$