CIS 185
Practice 5b
Objective:

- Be able to use mathematical induction

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Exercise 1: Let P(n) be the statement that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for the positive integer n.

a) What is the statement P(1)?

$P(1) = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$

b) Show that P(1) is true, completing the basis step of the proof.

$n^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad 1^3 = \left(\frac{2}{2}\right)^2 \qquad 1 = 1$ ✓

c) What is the inductive hypothesis?

For $n \geq 1$ in $P(n)$: $1^3 + 2^3 + \cdots n^3 = \left(\frac{n(n+1)}{2}\right)^2$

d) What do you need to prove in the inductive step?

Prove $P(n) \rightarrow P(n+1)$

e) Complete the inductive step, identifying where you use the inductive hypothesis.

$\left(\frac{(n+1)((n+1)+1)}{2}\right)^2 = 1^3 + 2^3 + \cdots + n^3 + (n+1)^3$

$\left(\frac{(n+1)(n+2)}{2}\right)^2 = \left(\frac{(n+1)n}{2}\right)^2 + (n+1)^3$

$\frac{(n+1)(n+2)}{2} \cdot \frac{(n+1)(n+2)}{2} = \frac{n^2+n}{2} \cdot \frac{n^2+n}{2} + \frac{4(n+1)^3}{4}$

f) Explain why these steps show that this formula is true whenever n is a positive integer.

$\frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} = \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4}$

$= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}$

f) $P(b)$ for constant b

$\forall n \geq b \, (P(n) \rightarrow P(n+1))$

$\therefore \forall n \geq b \; P(n)$ by induction

Exercise 2: Use mathematical induction to prove the inequality

$$n! < n^n$$

for all positive integers $n \geq 2$. $2! < 2^2$ $2 < 4$ ✓

$k \geq n,\ P(k) = k! < k^k$

$P(k+1) = (k+1)! < (k+1)^{k+1}$

$(k+1)! < (k+1)^k \cdot (k+1)$

$\therefore\ n! < n^n$ hold for all positive integers $n \geq 2$

Exercise 3: Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

$n = 1$ $1 \cdot 1! = (1+1)! - 1$

$1 = 2! - 1$

$1 = 1$ ✓

$P(k+1)$

$1 \cdot k! + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)!$

$= (k+1)! + (k+1)(k+1)! - 1$

$= (1 + k + 1)(k+1)! - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1 = ((k+1)+1)! - 1$

$\therefore\ n \cdot n! = n+1! - 1$ hold for postive integer n

Exercise 4: Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

$0^3 - 0 = 0$ ✓

$(k+1)^3 + (k+1) = k^3 + 3k^2 + 3k - k$

$= (k^3 - k) + 3k(k+1)$

$k^3 - k$ is divisble by 6 proven by hypothesis

$3k(k+1)$ is also divisble by 6

$\therefore$ The sum of $k^3 - k$ and $3k(k+1)$ is also divisible by 6