Station 1 - Answer Key
Triangle Inequality Theorem

1) **Easy:** Which of the following sets of three numbers could be the side lengths of a triangle? ***Answer: 4, 5, 6***
   a) 4, 5, 6
   b) 7, 20, 9
   c) ½, ⅙, ⅓
   d) 3.4, 11.3, 9.8
   e) $\sqrt{5}$, $\sqrt{14}$, $\sqrt{19}$

2) **Easy:** The lengths of two sides of a triangle are 7 cm and 3 cm. If the number of centimeters in the perimeter is a whole number, what is the number of centimeters in the positive difference between the greatest and least possible perimeters.
Answer: Minimum perimeter > 14 (7+3+4) Maximum Perimeter < 20 (7+3+10) Difference = 6

3) **Medium:** What are the constraints on x if the side lengths of a triangle are $2x + 3$, $3x + 8$, and $6x + 7$?
Answer:

$2x + 3 + 3x + 8 > 6x + 7$	$2x + 3 < 3x + 8 + 6x + 7$	$3x + 8 < 2x + 3 + 6x + 7$
$5x + 4 > 6x$	$2x + 3 < 9x + 15$	$3x + 8 < 8x + 10$
$x < 4$	$7x > -12$	$-2 < 5x$
	$x > -12/7$	$x > -2/5$

$-\frac{2}{5} < x < 4$

4) **Challenge:** In △ABC, AB = 5, BC = 11. What are the constraints on the perimeter if is △ABC and obtuse triangle? (Hint: Use the Pythagorean Theorem)

Maximum Perimeter < 32 $(11 + 5 + 16)$
Minimum Perimeter: > $16+\sqrt{146}$ $(5 + 11 + \sqrt{5^2+11^2})$

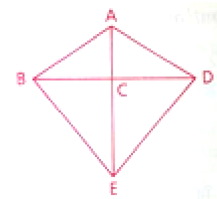
<u>Station 2</u>

Equidistance Theorem and Parallel Bisector Characterization Theorem

1) **Easy**: Given: $\overline{AB} \cong \overline{AD}$

$\overline{BC} \cong \overline{CD}$

Prove: $\overline{BE} \cong \overline{ED}$



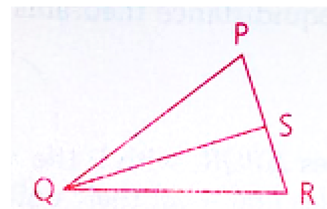
A and C are equidistant to B and D, $\therefore$ A and C form $\perp$ bisector of BD. E lies on AC, $\therefore$ E isequidistant to B and D

2) **Medium:** Given: $\overline{PS} \cong \overline{SR}$

$\overline{PQ} \cong \overline{QR}$

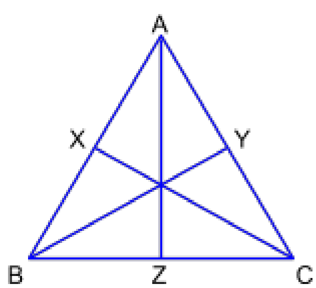
Prove: $\overline{QS}$ is an altitude

(Do not use triangle congruency postulates)



Since $\overline{PQ} \cong \overline{QR}$, QS is equidistant to the endpoints of PR. Since PS $\cong$ PR, S is equidistant to the endpoints of PR and QS bisects PR. So, QS is $\perp$ bisector of PR which originates at the vertex opposite side PR and forms a right angle at PR. $\therefore$ QS is the altitude to base PR.

3) **Challenge:** Without using any triangle congruency postulates, prove that if each of the altitudes of a triangle bisects the side to which it is drawn, then the triangle is equilateral.



AZ, BY and CX are altitudes and, because, BZ $\cong$ CZ, BX $\cong$ AX, AY $\cong$ CY, they are also medians. It follows that $\angle$AXC, $\angle$CXB, $\angle$AYB, $\angle$CYB, $\angle$AZC, and $\angle$AZB are all right angles and are all congruent given the definition of an altitude. And AZ $\cong$ AZ by the reflexive property, therefore $\triangle$ABZ $\cong$ $\triangle$ACZ by SAS.

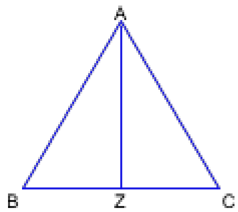
$\angle$XBC $\cong$ $\angle$YCB by CPCTC, XB $\cong$ YC and BC $\cong$ BC by the reflexive property, so $\triangle$XCB $\cong$ $\triangle$YCB by SAS. By CPCTC BX $\cong$ YC, and by the transitive property AY $\cong$ AX. AB $\cong$ AC by addition (AX + BX = AY + CY).

$\triangle$CXA $\cong$ $\triangle$CXB by SAS, so BC $\cong$ AC by CPCTC. By the transitive property, AB $\cong$ BC $\cong$ AC and $\triangle$ ABC is equilateral.

Station 3

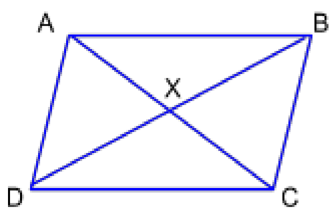
Case of the Missing Diagram

1) **Easy:** If the altitude of a triangle also bisects an angle of a triangle, then the triangle is isosceles.



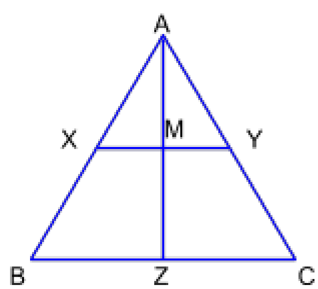
Given AZ is an altitude, AZ bisects ∠A, we can conclude that ∠AZB and ∠AZC are congruent right angles and ∠BAZ≅∠CAZ. AZ ≅ AZ by the reflexive property, so △AZB ≅ △AZC by ASA. AB ≅ AC by CPCTC

2) **Medium:** If each pair of opposite sides of a four-sided figure are congruent, the segments joining opposite vertices bisect each other.



Since AD ≅ BC and AB ≅ DC, and DB ≅ DB by the reflexive property, △BDA ≅ △BDC by SSS. Likewise, since AC ≅ AC by the reflexive property, △ACB ≅ △ACD by SSS. ∠DAX≅∠BCX by CPCTC and ∠ADX ≅ ∠CBX by CPCTC, ∴ △XAD ≅ △XCB by ASA. So, by CPCTC, AX ≅ CX and DX ≅ BX

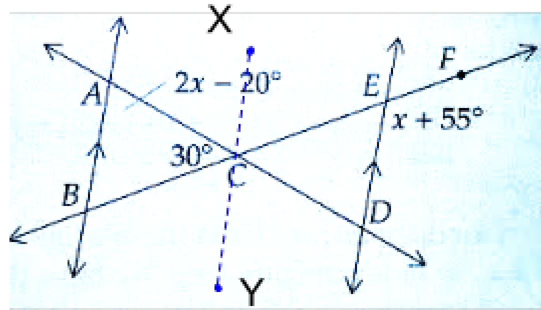
3) **Challenge:** The midpoint of the altitude to the base of an isosceles triangle is equidistant the midpoints of the legs of that triangle.



△ABC is isosceles with base BC, so AB ≅ AC. AZ is the altitude to side BC and, by definition, forms two congruent right angles, ∠AZB and ∠AZC. △AZB ≅ △AZC by HL Postulate. AB≅AC and ∠XAM ≅∠YAM by CPCTC. AX ≅ AY by division and AM ≅ AM by the reflexive property. △AMX ≅ △AMY by SAS, so XM ≅ YM by CPCTC

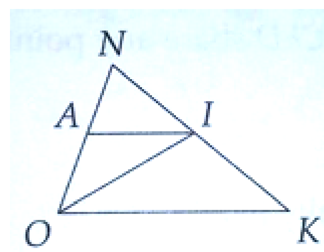
Station 4
Parallel Lines and Related Angles

1) **Easy**[1]: Find $m\angle CED$



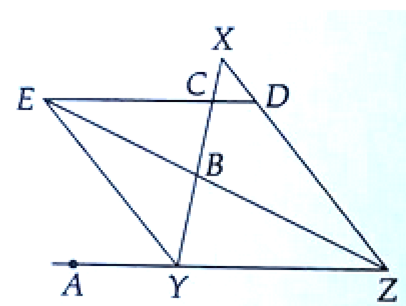
Draw $XY \| AB$ (Parallel Postulate)
Since $AB \| ED$, $XY \| ED$ (Transitive Property)
$\angle ACX = 2x - 20$ *(|| lines $\Rightarrow$ AIC)*
$\angle BCY \$ \angle FED$ *(|| lines $\Rightarrow$ Same Side Ext $\cong$)*
$\angle BCY \cong ECX$ *(vertical angles)*
$\angle YCX = \angle BCY + \angle ACB + \angle ACX = 180$
$180 = x + 55 + 30 + 2x + 20$
$x = 25$
$m\angle CED = 180 - (25 + 75)$
$m\angle CED = 80^o$

2) **Medium**[2]: Given: $\overline{KO} = \overline{KN}$, $\overline{IA} = \overline{IN}$
Prove: $\overline{IA} \| \overline{KO}$



1)	KO ≅ KN and IA ≅ IN	Given
2)	$\angle KNO \cong \angle KON$	If sides $\cong$ $\Rightarrow$ opp $\angle$'s $\cong$
3)	$\angle INA \cong \angle IAN$	Same as 2
4)	$\angle IAN \cong \angle KON$	Transitive Property
5)	$\overline{IA} \| \overline{KO}$	$\cong$ Corresp. $\angle$'s $\Rightarrow$ || lines

3) ***Challenge***[3]: *Given:* $\angle DZE \cong \angle EZY$
$\angle CYE \cong \angle AYE$
$\overline{ED} \| \overline{YZ}$
Prove: $\overline{CD} = \overline{DZ} - \overline{CY}$



1)	$\angle DZE \cong \angle EZY$	*Given*
2)	$\angle CYE \cong \angle AYE$	*Given*
3)	$\overline{ED} \| \overline{YZ}$	*Given*
4)	$\angle CEZ \cong \angle DZE$	*|| lines $\Rightarrow$ AIC*
5)	$ED \cong DZ$	*Sides opp. $\cong \angle$'s are $\cong$*
6)	$\angle CEY \cong \angle AYE$	*|| lines $\Rightarrow$ AIC*
7)	$\angle CEY \cong \angle CYE$	*Transitive Property*
8)	$EC \cong CY$	*Sides opp. $\cong \angle$'s are $\cong$*
9)	$EC + CD = ED$	*Assumed from diagram*
10)	$CY + CD = ED$	*Substitution Property*
11)	$CY + CD = DZ$	*Substitution Property*
12)	$CD = DZ - CY$	*Subtraction Property*

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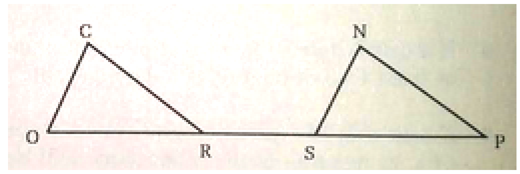
[1] AoPS Problem #2.22 pg 36
[2] AoPs #3.30(b)
[3] AoPS #3.47

Station 5

Arithmetic Properties, Bisectors and Trisectors

1) Easy:

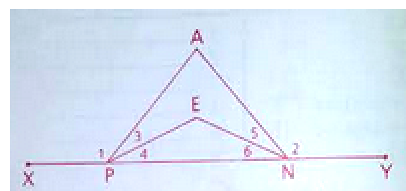
Given: $\overline{CO} \cong \overline{NS}$
$\overline{OS} \cong \overline{RP}$
$\overline{CO} \parallel \overline{NS}$
Prove: $\overline{CR} \parallel \overline{NP}$



1) $\overline{CO} \cong \overline{NS}$ — *Given*
2) $\overline{OS} \cong \overline{RP}$ — *Given*
3) $\overline{CO} \parallel \overline{NS}$ — *Given*
4) $RS \cong RS$ — *Reflexive Property*
5) $OR \cong SP$ — *Subtraction Property*
6) $\angle COR \cong \angle NSP$ — *|| lines ⇒ Corresponding ∠'s ≅*
7) $\triangle COR \cong \triangle NSP$ — *SAS (1,6,5)*
8) $\angle CRO \cong \angle NPS$ — *CPCTC*
9) $\overline{CR} \parallel \overline{NP}$ — *Corresponding ∠'s ≅ ⇒ || lines*

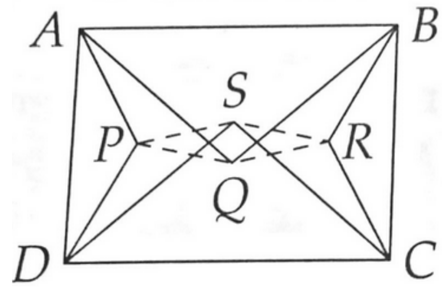
2) Medium:

Given: $\angle 1 \cong \angle 2$
$\overline{PE}$ bisects ∠APN
$\overline{NE}$ bisects ∠ANP
Prove: ∠XPE ≅ ∠ENY



1) $\angle 1 \cong \angle 2$ — *Given*
2) $\angle 1$ \$ $\angle APN$ — *If 2 ∠ form straight ∠ ⇒ \$*
3) $\overline{PE}$ *bisects ∠APN* — *Given*
4) $\angle 3 \cong \angle 4$ — *If ∠ bisected by segment, then divided into 2 ≅ segments*
5) $\angle 2$ \$ $\angle ANP$ — *Same as 2*
6) $\overline{NE}$ *bisects ∠ANP* — *Given*
7) $\angle 5 \cong \angle 6$ — *Same as 4*
8) $\angle APN \cong \angle ANP$ — *Substitution*
9) $\angle 3 \cong \angle 5$ — *Substitution*
10) $\angle XPE \cong \angle ENY$ — *Addition*

3) Challenge: Each angle in rectangle ABCD is trisected by a pair of segments. The angle trisectors meet at P, Q, R, and S, as shown in the diagram to the right. Prove that PQRS is a rhombus.



1)	$\angle BAD$, $\angle ABC$, $\angle BCD$, and $\angle CDA$ are right	Def. of rectangle
2)	$AD \cong BC$ and $AB \cong DC$	Def of rectangle
3)	$\angle BAD$ trisected by AP and AQ	Given
4)	$\angle ABC$ trisected by BR and BQ	Given
5)	$\angle ADC$ trisected by DP and DS	Given
6)	$\angle BCD$ trisected by CR and CS	Given
7)	$\angle DAP \cong \angle CBR$	Division Property
8)	$\angle ADP \cong \angle BCR$	Division Property
9)	$\triangle DAP \cong \triangle CBR$	ASA (7,2,8)
10)	$AP \cong BR$ and $DS \cong CS$	CPCTC
11)	$\angle PAQ \cong \angle RBQ$	Division Property
12)	$\angle PDS \cong \angle RCS$	Division Property
13)	$\angle QAB \cong \angle QBA$	Division Property
14)	$AQ \cong BQ$	If base $\angle$'s $\cong$ $\Rightarrow$ opp. sides $\cong$
15)	$\angle SDC \cong \angle SCD$	Division Property
16)	$DS \cong CS$	Same as 14
17)	$\triangle PAQ \cong \triangle RBQ$	SAS (10, 11, 14)
18)	$\triangle PDQ \cong \triangle RCQ$	SAS (10, 12 ,16)
19)	$PS \cong RS$	CPCTC
20)	$PQ \cong RQ$	CPCTC
21)	$\angle QAB \cong \angle SDC$	Division Property
22)	$\angle QBA \cong \angle SCD$	Division Property
23)	$\triangle AQB \cong \triangle DQC$	ASA (21, 2, 22)
24)	$AQ \cong DS$	CPCTC
25)	$\angle PAS \cong \angle PDQ$	Division Property
26)	$AP \cong PD$	If base $\angle$'s $\cong$ $\Rightarrow$ opp. sides $\cong$
27)	$\triangle PAS \cong \triangle PDQ$	SAS (24, 25, 26)
28)	$PS \cong PQ$	CPCTC
29)	$PS \cong RQ$	Transitive Property
30)	$PR \cong RQ$	CPCTC
31)	PQRS is a rhombus	If a quadrilateral has 4 congruent sides, then it is a rhombus.

Station 6

Equations of Lines (including Perpendicular and Parallel)

1) **Easy:** A = (-6,2), B = (8,4), and C = (2, 9). If $\overline{CM}$ is a median, find the coordinates of M. Then show that $\overline{CM}$ is not perpendicular to $\overline{AB}$

Midpoint M of AB = (1, 3)
Slope of AB = (4-2)÷(8+6) = 2/14
Slope of CM = (9-3)÷(2-1) = 6
CM not ⊥ AB because the slopes are not opposite reciprocals

2) **Medium:** If A = (6,10), B = (1, 5), and C = (7,0), determine by means of slopes what type of triangle △ABC.

Slope of AB = $\frac{10-5}{6-1} = \frac{5}{5}$
Slope of BC = $\frac{5-0}{1-7} = -\frac{5}{6}$
Slope of AC = -10
Triangle is scalene

3) **Challenge:** The three altitudes of a triangle intersect at a common point called the "orthocenter". Given a triangle with vertices A=(2,4), B=(-4,0), and C=(4,0), find the coordinates of the orthocenter.

Segment	**AB**	**BC**	**AC**
Slope	$\frac{0-4}{-4-2} = \frac{2}{3}$	0	-2
Slope of Altitude	$-\frac{3}{2}$	Undefined	½
Equation of Altitude	*Vertex: C=(4,0)* ⇒ $y-0 = -\frac{3}{2}(x-4)$ $y = -\frac{3}{2}(x-4)$ $y = -\frac{3}{2}x+6$	*Vertex: A=(2,4)* ⇒ $y-4 = \frac{1}{0}(x-2)$ $0(y-4) = 1(x-2)$ $0 = x-2$ $x = 2$	Vertex B: (-4,0) $y-0 = \frac{1}{2}(x+4)$ $y = \frac{1}{2}x+2$
Substitution	$y = -\frac{3}{2}(2)+6 = 3$	$x = 2$	$y = \frac{1}{2}(2)+2 = 3$

Coordinates of the orthocenter are (2, 3)

Station 7
Complementary and Supplementary Angles

1) **Easy:** The supplement of an angle is five times its complement. What is the measure of the angle?

$180 - x = 5(90 - x)$
$180 - x = 450 - 5x$
$4x = 270$
$x = 67.5$

2) **Medium:** If four times the supplement of an angle is added to eight times the angles complement, the sum is equivalent to three straight angles. Find the measure of the angle that is supplementary to the complement.

$4(180 - x) + 8(90 - x) = 3(180)$
$720 - 4x + 720 - 8x = 540$
$12x = 900$
$x = 75$
Complement of 75 = 15
Supplement of 15 is 165

3) **Challenge:** Given: $\angle A$ is complementary to $\angle B$
$\angle C$ is complementary to $\angle B$
$m\angle A = (3x + y)^{\circ}$
$m\angle B = (x + 4y + 2)^{\circ}$
$m\angle C = (3y - 3)^{\circ}$
Find: $m\angle B$

$\angle A$ compl $\angle B \Rightarrow m\angle A + m\angle B = 90$
$3x + y + x + 4y + 2 = 90$
$4x + 5y = 88$

$\angle C$ compl $\angle B \Rightarrow m\angle C + m\angle B = 90$
$3y - 3 + x + 4y + 2 = 90$
$7y + x = 91$

Substitution: $x = 91 - 7y$
$4(91 - 7y) + 5y = 88$
$364 - 28y + 5y = 88$
$23y = 276$
$y = 12$

$m\angle C = (3y - 3) = 33$
$m\angle B = 90 - 33 = 57^{\circ}$

Station 8

Logic and Chains of Reasoning

1) **Easy:** Write the converse, inverse, and contrapositive of the following conditional statement:

Jody can talk on the phone only if she has finished her homework.

Converse: Only if she has finished her homework, Jody can talk on the phone.
Inverse: Jody cannot talk on the phone only if she has not finished her homework.
Contrapositive: Only if she has not finished her homework, Jody cannot talk on the phone.

2) **Medium:** What conclusion can be drawn from the following?

$\sim e \Rightarrow \sim d \quad b \Rightarrow c \quad \sim b \Rightarrow \sim a \quad c \Rightarrow d$

$d \Rightarrow e \qquad a \Rightarrow b$

$a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e$

$a \Rightarrow e$

3) **Challenge:**

What conclusion can be drawn from the following Lewis Carroll Puzzle?

No kitten, that loves fish, is unteachable.
No kitten without a tail will play with a gorilla.
Kittens with whiskers always love fish.
No teachable kitten has green eyes.
No kittens have tails unless they have whiskers.

Statement	Contrapositive
loves fish ⇒ teachable	~teachable ⇒ ~loves fish
~tail ⇒ ~play w gorilla	plays w gorilla ⇒ tail
whiskers ⇒ loves fish	~loves fish ⇒ ~ whiskers
teachable ⇒ ~green eyes	green eyes ⇒ teachable
tail ⇒ whiskers	~whiskers ⇒ tails

plays w gorilla ⇒ tail ⇒ whiskers ⇒ loves fish ⇒ teachable ⇒ ~ green eyes.

If a kitten plays with a gorilla then it does not have green eyes.

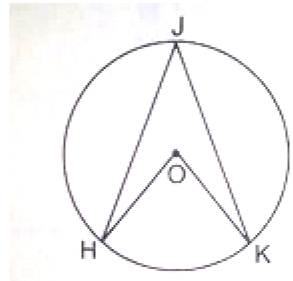
Station 9

Congruent Triangles (CPCTC and Detour Proofs)

1) Easy: Given: ⊙O

$\overline{HJ} = \overline{JK}$

Prove: $\angle H \cong \angle K$

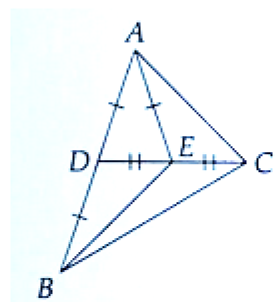


1)	⊙O	Given
2)	HJ ≅ JK	Given
3)	HO ≅ KO	Radii of ⊙ ≅
4)	Draw JO	2 pts determine a line
5)	JO ≅ JO	Reflexive Property
6)	△JOH ≅ △JOK	SSS (2,3,5)
7)	∠H ≅ ∠K	CPCTC

2) Medium[4]: Given: $\overline{AD} = \overline{BD} = \overline{AE}$

$\overline{DE} = \overline{EC}$

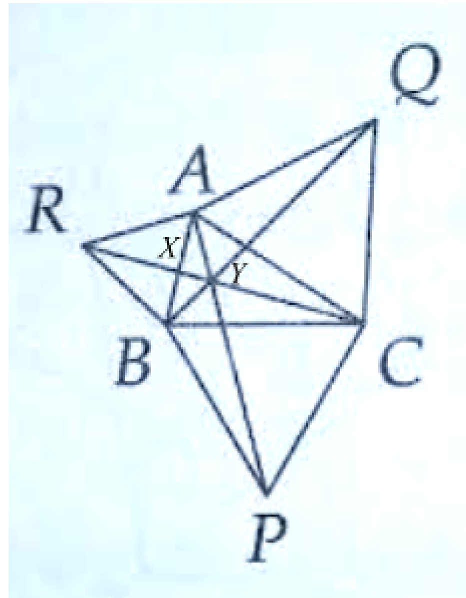
Prove: $\overline{AC} = \overline{BE}$



1)	AD ≅ BD ≅ AE	Given
2)	DE ≅ EC	Given
3)	∠ADE ≅ ∠AED	If base ∠'s ≅ ⇒ opp. sides ≅
4)	∠ADE $ ∠EDB	If 2 ∠'s form str. ∠ ⇒ $
5)	∠AED $ ∠AEC	Same as 4
6)	△BDE ≅ △AEC	SAS
7)	AC ≅ BE	CPCTC

[4] Source: AoPS #3.32

3) Challenge[5]: Given: △BCP, △CAQ, and △ABR are equilateral
Prove: $\overline{AP} = \overline{BQ} = \overline{CR}$



1)	△BCP, △CAQ, △ABR equilateral	Given
2)	BR ≅ AR ≅ AB	Equilateral △ ⇒ ≅ sides
3)	∠RBA ≅ ∠RAB	If sides ≅ ⇒ opp <<'s ≅
4)	RX ≅ RX	Reflexive Property
5)	△RAX ≅ △RBX	SAS (2,3,4)
6)	AX ≅ BX	CPCTC
7)	RC ⊥ bisects AB	Equidistance Theorem
8)	BC ≅ AC	PBC
9)	AC ≅ AQ ≅ CQ	Def. of Equilateral △
10)	BC ≅ BP ≅ CP	Def. of Equilateral △
11)	AQ ≅ BP	Transitive Property
12)	CQ ≅ CP	Transitive Property
13)	△AQC ≅ △BPC	SSS (8,11,12)
14)	∠CAQ ≅ ∠CBP	CPCTC
15)	AC ≅ BC	CPCTC
16)	∠CAB ≅ ∠CBA	If sides of △ ≅ ⇒ opp. ∠'s ≅
17)	∠QAB ≅ ∠PBA	Addition Property
18)	AB ≅ AB	Reflexive Property
19)	△APB ≅ △BQA	SAS(11, 17, 18)
20)	RC ≅ RC	Reflexive Property
21)	△RBC ≅ △RAC	SSS (8, 20, 2)
22)	∠RAC ≅ ∠RBC	CPCTC
23)	∠PAB ≅ ∠QBA	CPCTC
24)	∠CAP ≅ ∠CBQ	Subtraction Property
25)	∠QAC ≅ ∠RAB	Equilateral △'s are equiangular
26)	∠RAC ≅ ∠QAB	Substitution
27)	△RAC ≅ △QAB	SAS (9, 26 ,18)
28)	△APB = △RCB	Transitive Property
29)	AP ≅ BQ ≅ CR	CPCTC

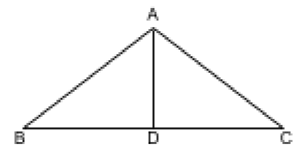
[5] Source: AoPS #3.50

Station 10
Altitudes and Medians

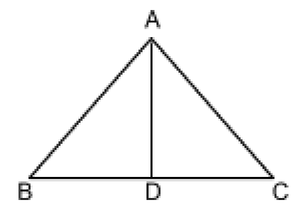
1) Easy: Given: $\overline{AD}$ is the median to $\overline{BC}$

$\overline{AD} = 3x + 2$, $\overline{BD} = 5x - 3$, $\overline{DC} = 2x + 6$

Find the length of $\overline{AD}$



Since AD is median to BC, BD = DC and $5x + 3 = 2x + 6$ $x = 1$ and AD = 8



2) Medium: If a median of a triangle is also an altitude, prove that the median also bisects its vertex angle.

Given: AD is median to BC, AD is altitude to BC

Prove: AD bisects ∠A

1)	AD is median to BC	Given
2)	BD ≅ CD	Medians bisects side of △
3)	AD is altitude to BC	Given
4)	∠ADB and ∠ADC are right	Altitude intersects side to form right ∠'s
5)	AD ≅ AD	Reflexive Property
6)	△ADB ≅ △ADC	SAS (2,4,5)
7)	∠DAB ≅ ∠DAC	CPCTC
8)	AD bisects ∠A	If a segment divides ∠ into 2 ≅ ∠'s, then it bisects the ∠

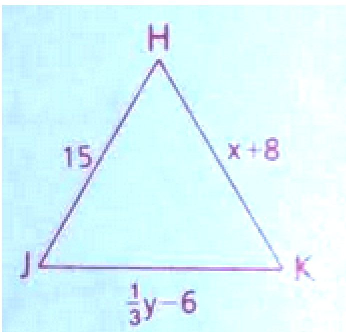
3) Challenge: The medians of a triangle intersect at a common point called the "centroid". Given a triangle with vertices A=(2,4), B=(-4,0), and C=(4,0), find the coordinates of the centroid. **(0.75, 1.5)**

Segment	**AB**	**BC**	**AC**
Midpoint	(-1, 2)	(0,0)	(3,2)
Vertex	C=(4,0)	A=(2,4)	B=(-4,0)
Slope of Median	$-\frac{2}{5}$	2	$\frac{2}{7}$
Equation of Median	$y - 0 = -\frac{2}{5}(x-4)$ $y = -\frac{2}{5}(x-4)$ $y = -\frac{2}{5}x + \frac{8}{5}$	$y - 4 = 2(x-2)$ $y - 4 = 2x - 4$ $y = 2x$	$y - 0 = \frac{2}{7}(x+4)$ $y = \frac{2}{7}x + \frac{8}{7}$
Standard Form	$2x + 5y = 8$	$2x - y = 0$	$2x - 7y = -8$
Substitution	$2x + 5(2x) = 8$ $12x = 8$ $x = \frac{3}{4}$	$y = 1.5$	$2x - 7(2x) = -8$ $-12x = -8$ $x = 3/4$

Station 11
Types of Triangles

1) **Easy:** Determine whether the △ABC with A=(2,4), B=(-4,0), and C=(4,0), is equilateral, isosceles, or scalene.
Scalene

2) **Medium:** If △HJK is equilateral, what are the values of *x* and *y*?



Since △HJK is equilateral, HJ ≅ HK ≅ JK
HJ ≅ HK
$x + 8 = 15$
$x = 7$

HJ ≅ JK
$\frac{1}{2}y - 6 = 15$
$y = 18$

3) **Challenge:** How many different isosceles triangles can you find that have sides that are whole-number lengths and have a perimeter of 18? Identify each of the ones you find by stating the side lengths. **7**

	AB	BC	AC
1)	2	8	8
2)	4	7	7
3)	6	6	6
4)	8	8	2
5)	7	7	4
6)	8	2	8
7)	7	4	7

<u>Station 12</u>
Extra Algebra Practice

1) The lengths of the base and a leg of an isosceles triangle are in the ratio of 3:2. The perimeter of the triangle is 105. Find the length of the base.

   3x + 2x + 2x = 105
   2x = 105
   x = 15
   Base = 3(15) = 45

2) Which of the following points lie on line $8x - 7y = -15$?

   A. (-1, 1)
   B. ($\frac{1}{2}$, $\frac{19}{7}$)
   C. ($\frac{13}{8}$, 4)
   D. (1, -1)
   E. (2, -$\frac{1}{7}$)

   Substitute the coordinates (-1,1) for the *x* and *y,* respectively, in the equation:
   $8(-1) - 7(1) = -15$
   $-8 - 7 = -15$