Solving Poisson’s equation

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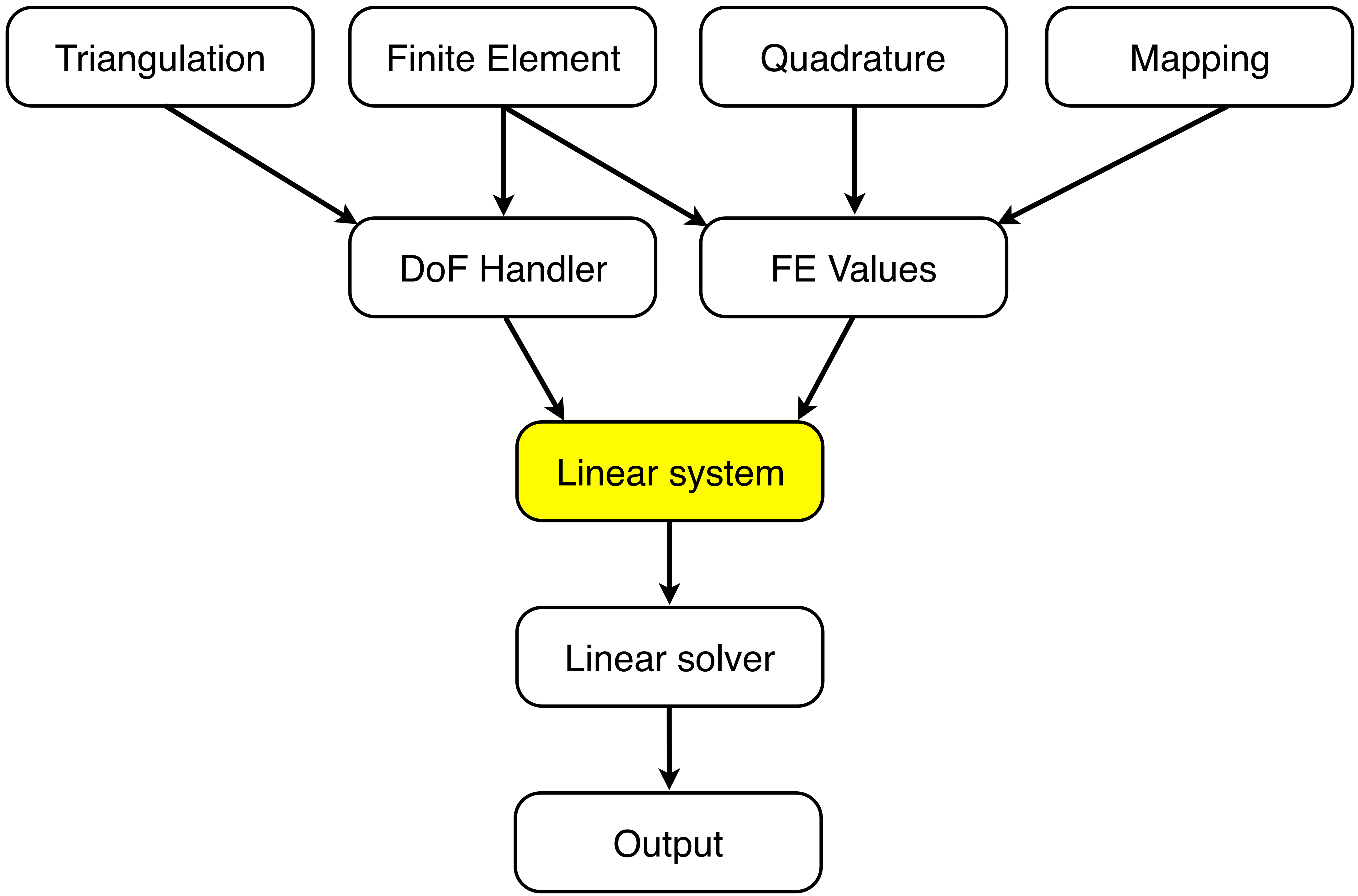
Aims for this module

- First introduction into assembly of sparse linear systems
  - Translation of weak form to assembly loops
  - Applying boundary conditions
- Using linear solvers
- Post-processing and visualisation

Reference material

- Tutorials
  - Step-3 https://dealii.org/8.5.1/doxygen/deal.II/step_3.html
- Documentation
  - https://www.dealii.org/developer/doxygen/deal.II/group__FE__vs__Mapping__vs__FEValues.html
  - https://www.dealii.org/developer/doxygen/deal.II/group__UpdateFlags.html

Structure of a prototypical FE problem



Sparse linear systems

- Minimise data storage
  - Evaluate grid connectivity
- Functions to help set up
  - Connectivity
  - Constraints
- Minimal access times
  - Direct manipulation of (non-zero) entries
  - Matrix-vector operations
    - Skip over zero-entries
- Types
  - Unity (monolithic, contiguous)
  - Block sparse structures
- Sub-organisation (e.g. component-wise)

$$[K]\,\{d\} = \{F\}$$

$$\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} d_1 \\ \hline d_2 \end{array}\right\} = \left\{\begin{array}{c} F_1 \\ \hline F_2 \end{array}\right\}$$

- $$\left(K_{11} - K_{12}K_{22}^{-1}K_{21}\right) d_1 = F_1 - K_{12}K_{22}^{-1}F_2$$
- $$d_2 = K_{22}^{-1}\left(F_2 - K_{21}d_1\right)$$

Constraints on sparse linear systems

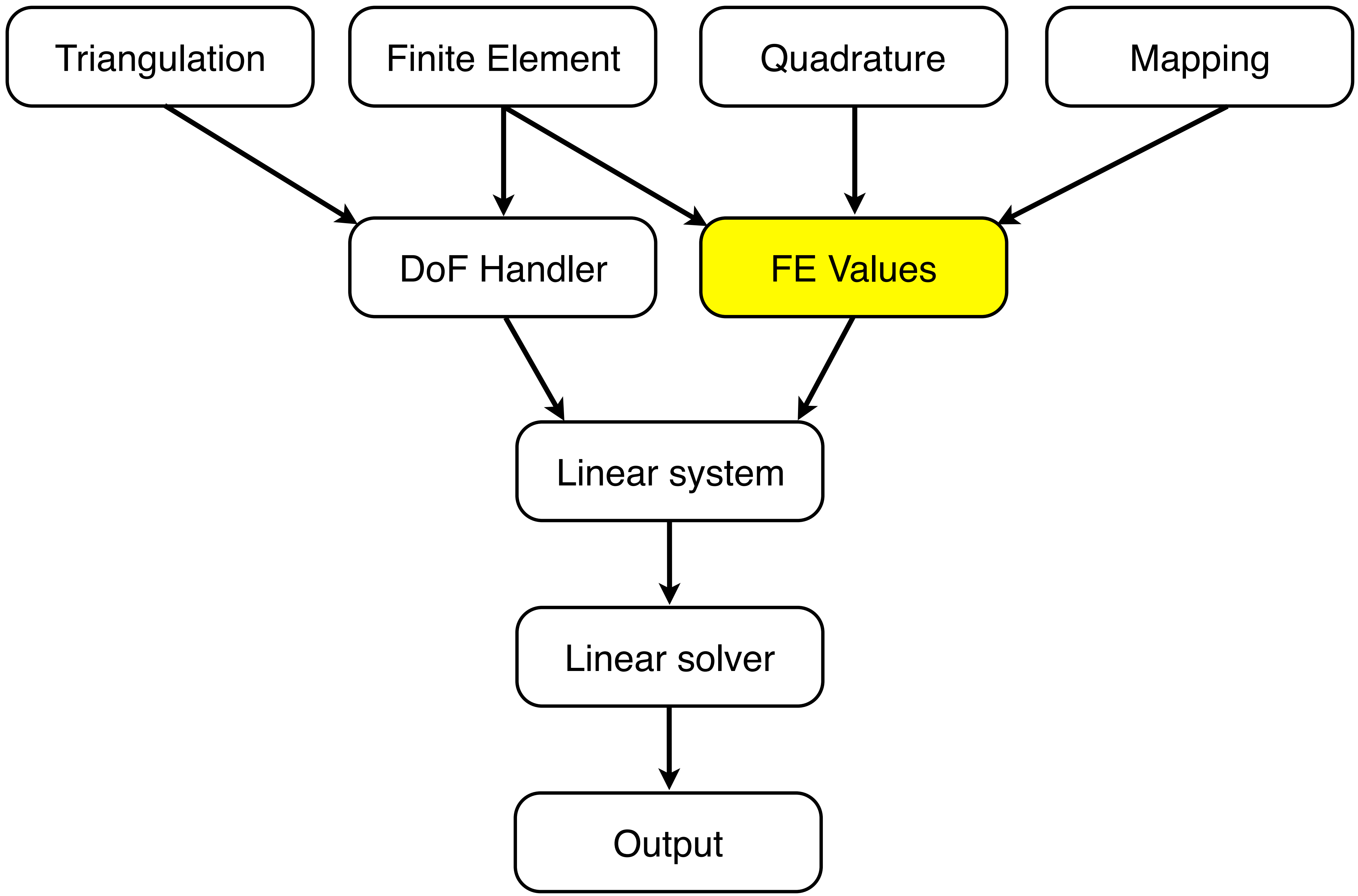
- Strong Dirichlet boundary conditions
  - Apply user-defined spatially-dependent functions to specific boundaries
  - Can restrict to components of a multi-dimensional field
  - Limited to fields with support points on faces
  - Possible to scale system matrix/RHS vector accordingly
    - Better matrix conditioning
- Neumann boundary conditions
  - Implementation dependent
- Other constraints need special consideration
  - Periodic boundary conditions
  - Refinement with hanging nodes
  - Some time-dependent formulations

$$[K]\{d\} = \{F\}$$

$$\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} d_1 \\ \hline d_2 \end{array}\right\} = \left\{\begin{array}{c} F_1 \\ \hline F_2 \end{array}\right\}$$

- $\left(K_{11} - K_{12}K_{22}^{-1}K_{21}\right) d_1 = F_1 - K_{12}K_{22}^{-1}F_2$
- $d_2 = K_{22}^{-1}\left(F_2 - K_{21}d_1\right)$

Structure of a prototypical FE problem



Integration on a cell: the FEValues class

$$K = \int_\Omega \nabla\delta\phi \cdot k\nabla\phi dV$$

$$\approx \delta\phi^I \sum_h \left(\int_{\Omega^h} \nabla N^I \cdot k \nabla N^J dV^h \right) \phi^J$$

$$\approx \underbrace{\delta\phi^I \sum_h \left(\sum_q \nabla N^I \left(\mathbf{x}_q\right) \cdot k_q \nabla N^J \left(\mathbf{x}_q\right)\, w_q \right)}_{K_{IJ}=(\nabla\phi^I, k\nabla\phi^J)} \phi^J$$

$$\approx \delta\phi^I \underbrace{\sum_h \boxed{\left(\sum_q \left[J^h_\square\right]_q^{-1} \nabla_\square N^I \left(\mathbf{x}_q\right) \cdot k_q \left[J^h_\square\right]_q^{-1} \nabla_\square N^J \left(\mathbf{x}_q\right) \left|\det J_q\right| w_q \right)}}_{K_{IJ}} \phi^J$$

$$J^h_\square = \frac{\partial \mathbf{X}^\xi}{\partial \mathbf{X}}$$

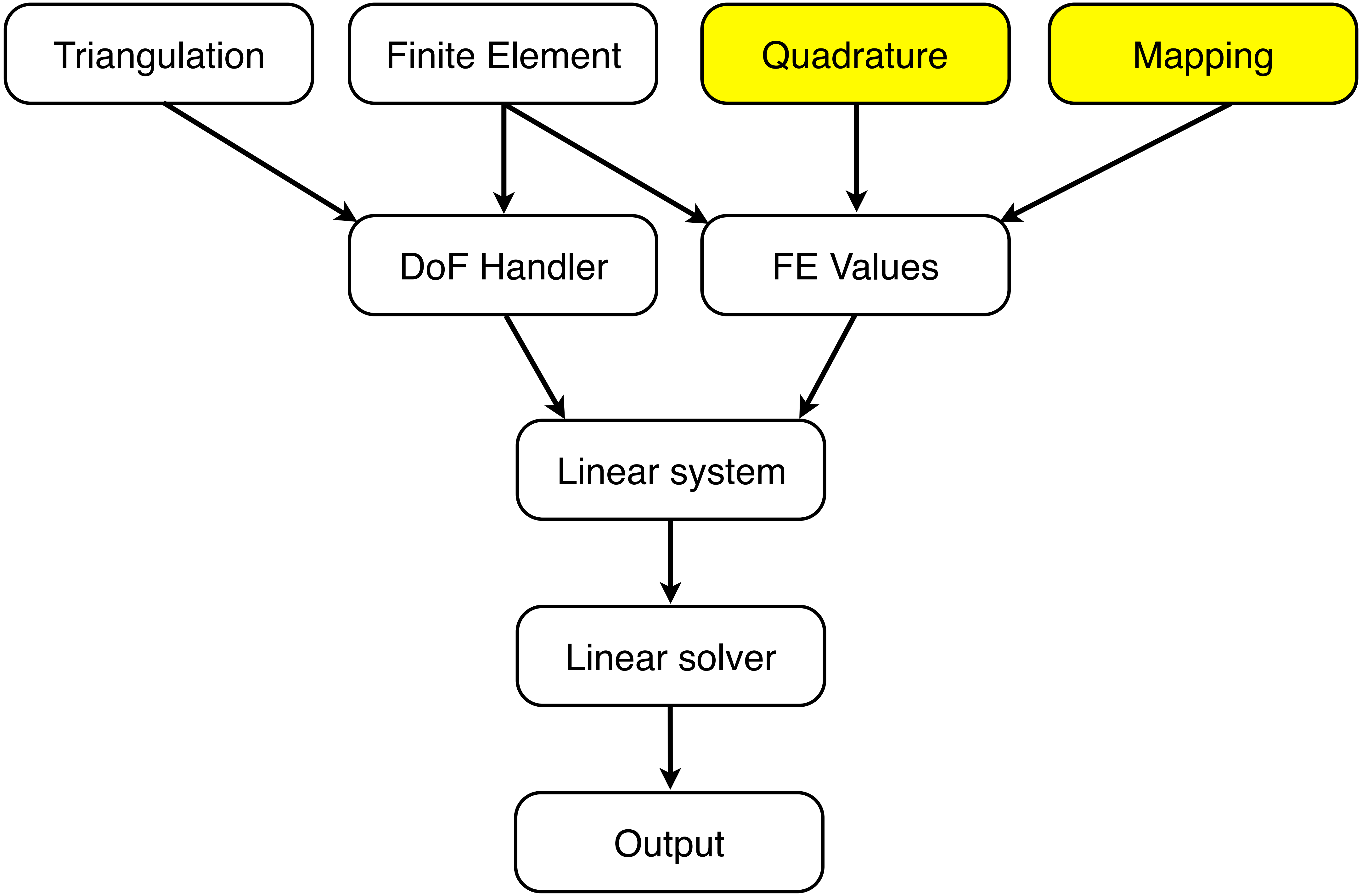
Integration on a cell: the FEValues class

- Object that helps perform integration
- Combines information of:
  - Cell geometry
  - Finite-element system
  - Quadrature rule
  - Mappings
- Can provide:
  - Shape function data
  - Quadrature weights and mapping jacobian at a point
  - Normal on face surface
  - Covariant/contravariant basis vectors
- More ways it can help:
  - Object to extract shape function data for individual fields
  - Natural expressions when coding
- Low level optimisations

$$K_{IJ} = \sum_h \left(\sum_q \boxed{\left[J^h_\square\right]^{-1}_q \nabla_\square N^I\left(\mathbf{x}_q\right) \cdot k_q \left[J^h_\square\right]^{-1}_q \nabla_\square N^J\left(\mathbf{x}_q\right) \left|\det J_q\right| w_q} \right)$$

```
cell_matrix(I,J) += k
      * fe_values.shape_grad (I, q_point)
      * fe_values.shape_grad (J, q_point)
      * fe_values.JxW (q_point);
```

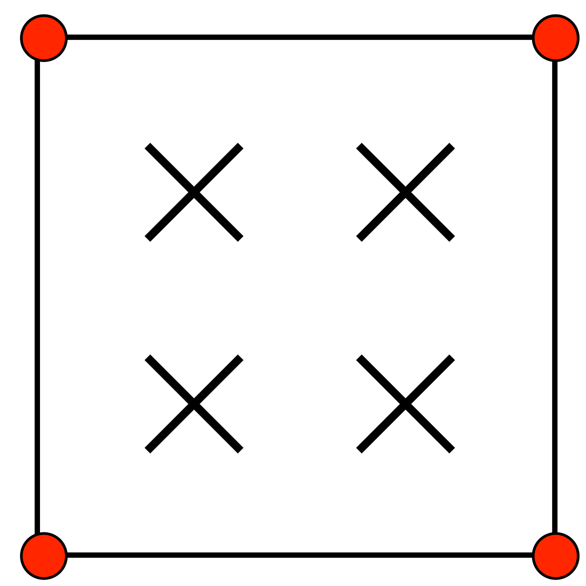
Structure of a prototypical FE problem



Integration on a cell: the Quadrature classes

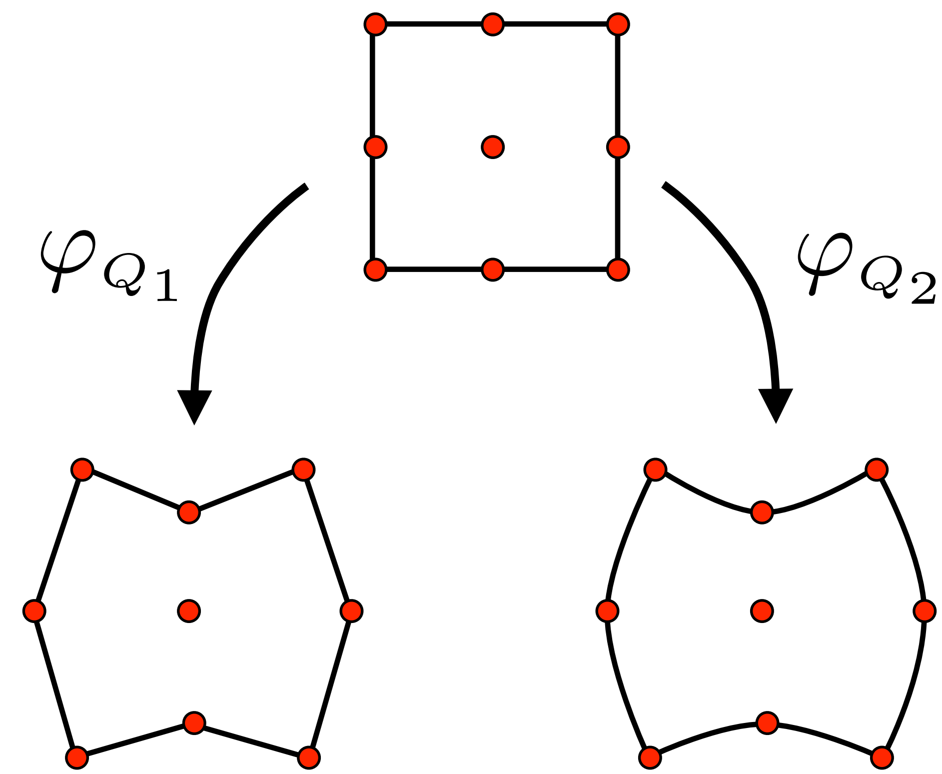
- n-Order Gauss quadrature
- Other rules
    - Gauss Lobatto
    - Simpson
    - Trapezoidal
    - Midpoint
    - A few others
- Anisotropic
    - Tensor product

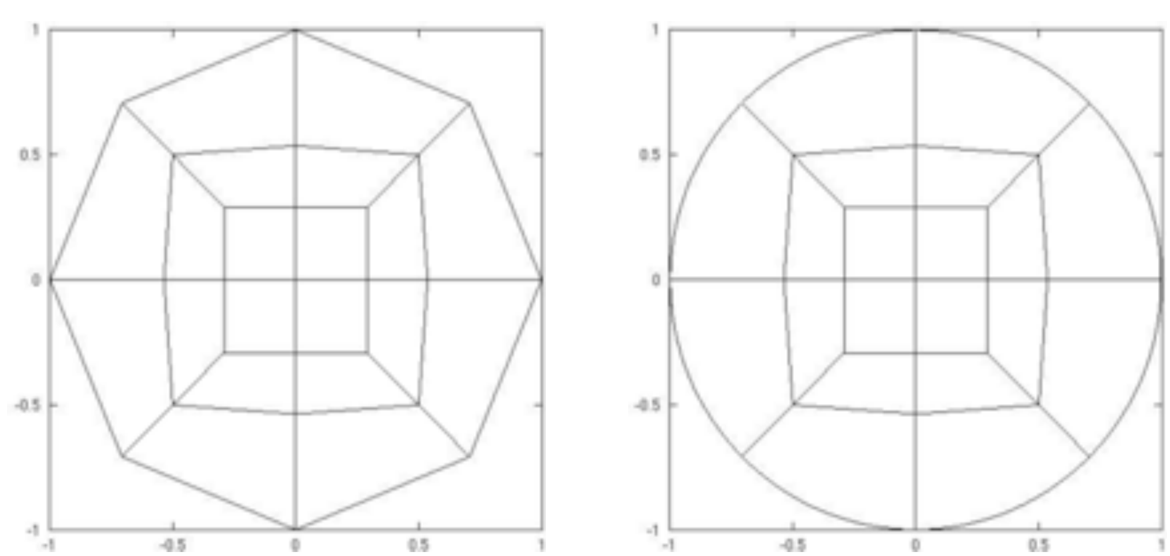
`FE_Q<2>(1)`



Integration on a cell: the Mapping classes

- n-order mappings
  - Increase accuracy of:
    - Integration schemes
    - Surface basis vectors
- Lagrangian / Eulerian
  - Latter useful for fluid and contact problems, data visualisation
- Boundary and interior manifolds





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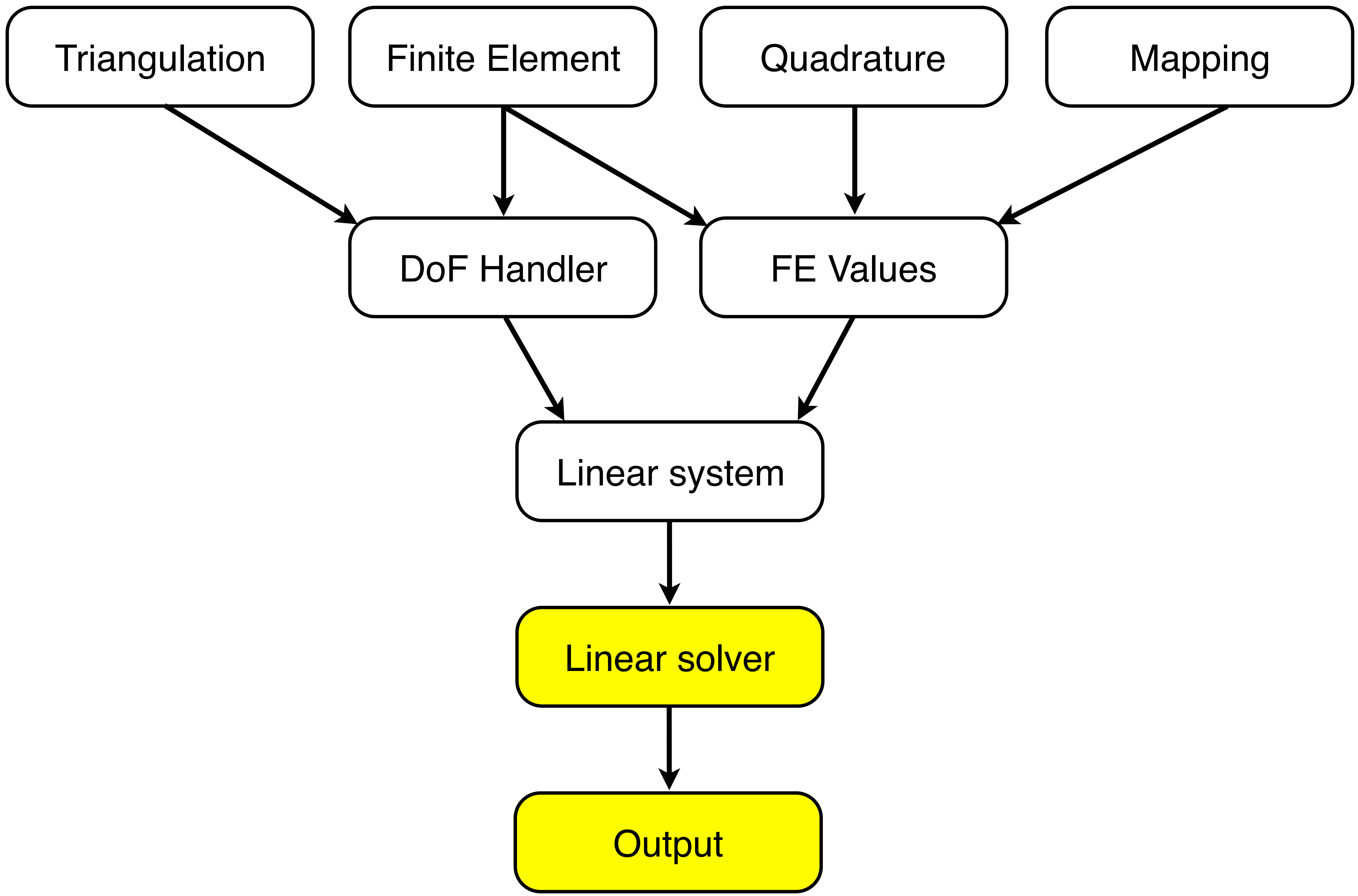

Solving Poisson's equation

- Demonstration: Step-3
  https://www.dealii.org/8.5.1/doxygen/deal.II/step_3.html
  http://www.math.colostate.edu/~bangerth/videos.676.10.html

- Key points
  - Local assembly + quadrature rules
  - Distribution of local contributions to the global linear system
  - Application of boundary conditions
  - Solving a linear system
  - Output for visualisation