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GRAPH THEORY SMTH312

By

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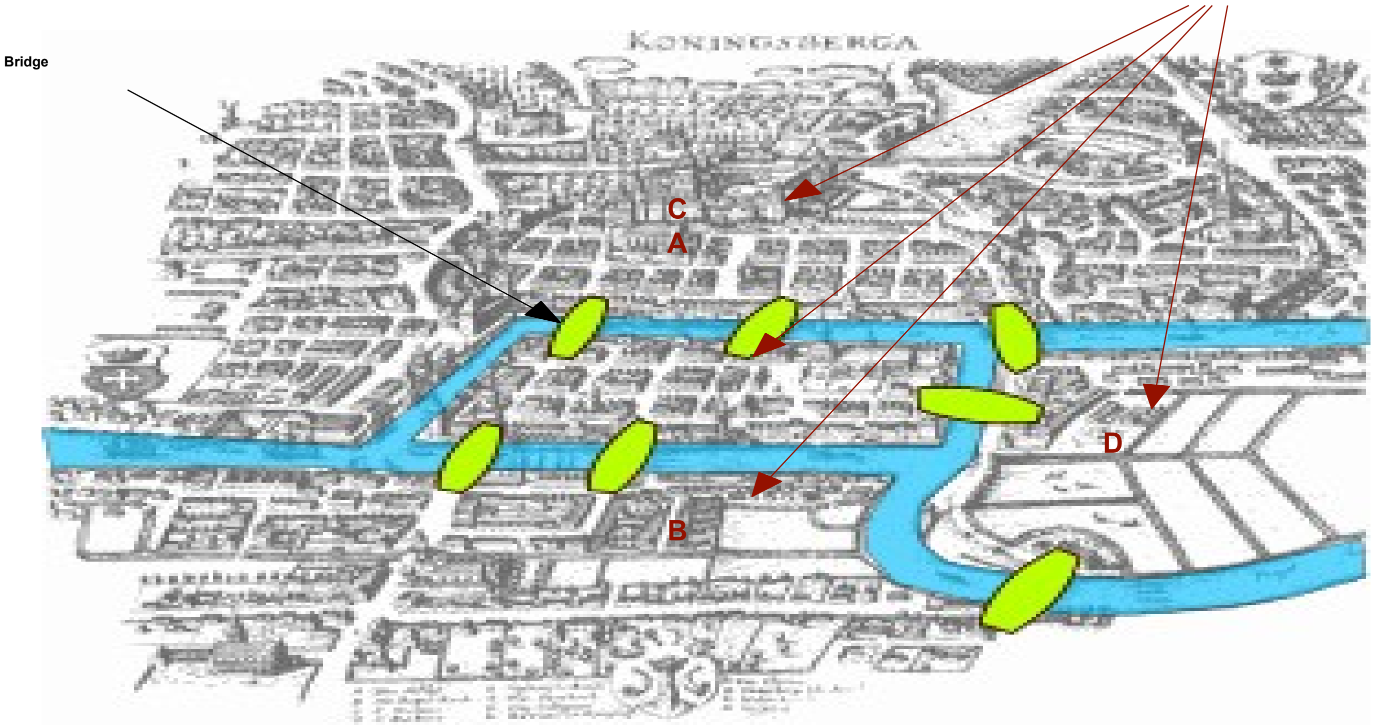
Graph Theory began with **Leonhard Euler** in his study of the Bridges of Königsburg problem. The city of Königsburg exists as a collection of islands connected by bridges as shown in Figure 1.1. The problem Euler wanted to analyze was: Is it possible to go from



An Introduction to Graphs

Some Definitions and Theorems

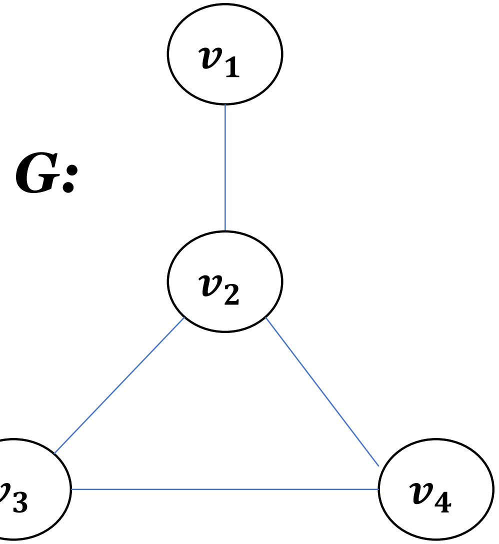
Definition 1.1 (Graph). A graph is a tuple $G = (V, E)$ where V is a (finite) set of vertices and E is a finite collection of edges. The set E contains elements from the union of the one and two element subsets of V. That is, each edge is either a one or two element subset of V. **Vertices is represented by dots and edges by line (curve)**

V(G) is the set of vertices of graph G

E(G) is the set of edges of graph G

$$V(G) = \{v_1, v_2, v_3, v_4\}$$

$$\mathrm{E}(G) = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}$$

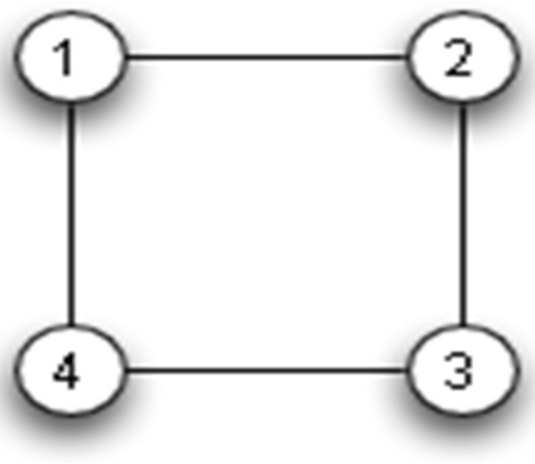


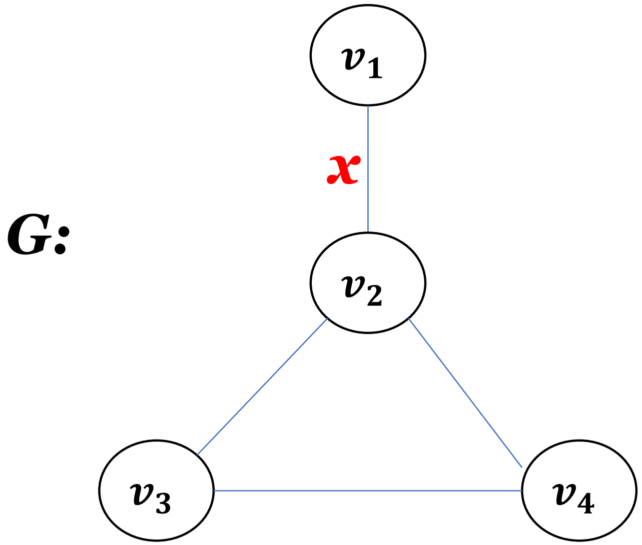
Example 1.7. Consider the set of vertices $V = \{1, 2, 3, 4\}$. The set of edges

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$$

Then the graph $G = (V, E)$ has four vertices and four edges. It is usually easier to represent this graphically (as we did for the Bridges of Königsburg Problem). See Figure 2.1 for the visual representation of G. These visualizations are constructed by representing each

G:





Definition 1.2 (**Incident and Adjacent**).

- **Edge x *in incident to* v_1 *and* v_2**
- ***Vertex* v_1 *is adjacent to* v_2**
- ***Vertex* v_3 *is adjacent to* v_2 *and* v_4**

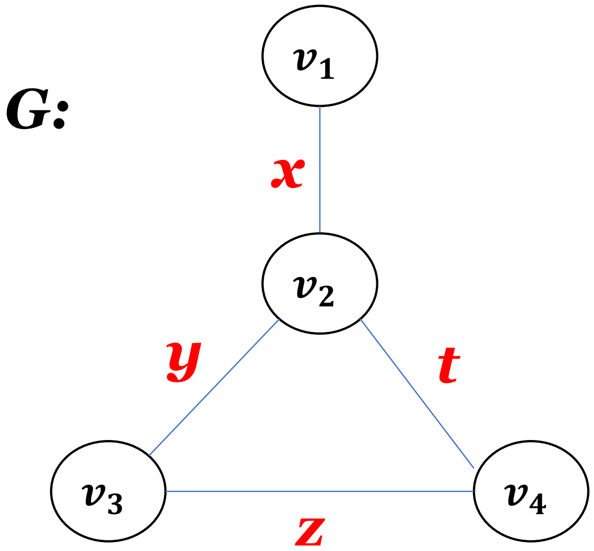
Definition 1.3 (Vertex Adjacency). Let $G = (V, E)$ be a graph. Two vertices v_1 and v_2 are said to be *adjacent* if there exists an edge $e \in E$ so that $e = \{v_1, v_2\}$. A vertex v is self-adjacent if $e = \{v\}$ is an element of E.

Definition 1.4 (Edge Adjacency). Let $G = (V, E)$ be a graph. Two edges e_1 and e_2 are said to be *adjacent* if there exists a vertex v so that v is an element of both v_1 and v_2 (as sets). An edge e is said to be *adjacent* to a vertex v if v is an element of e as a set.

Definition 1.5 (Neighborhood). Let $G = (V, E)$ be a graph and let $v \in V$. The *neighbors* of v are the set of vertices that are adjacent to v.
Formally: (2.1) $N(v) = \{u \in V : \exists e \in E\ (e = \{u, v\} \text{ or } u = v \text{ and } e = \{v\})\}$

$N(v)$ is the set of vertices u in (the set) V such that there exists an edge e in (the set) E so that $e = \{u, v\}$ or $u = v$ and $e = \{v\}$.

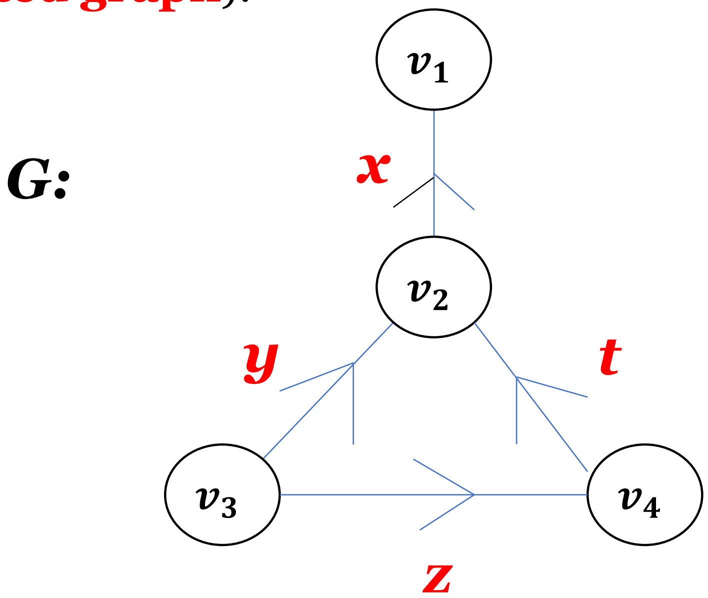
Definition 1.8 (**Undirected graph**).



$x = \{v_1, v_2\} = \{v_2, v_1\}$

$OR\ x = v_1 v_2 = v_2 v_1$

Definition 1.8 (**Directed graph**).



$x = \{v_1, v_2\} \neq \{v_2, v_1\}$

$OR\ x = v_1v_2 \neq v_2v_1$

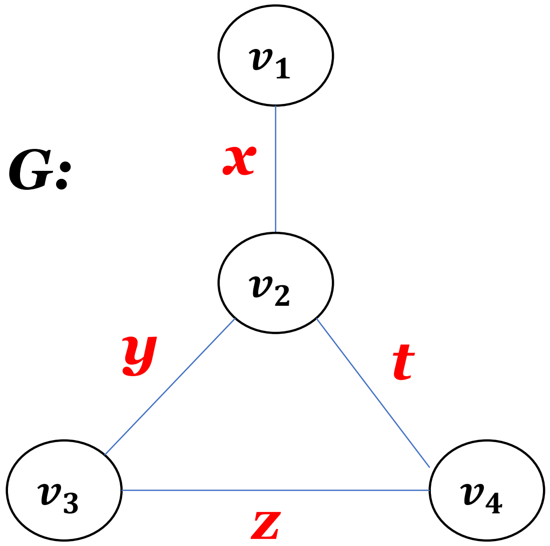
$n(G)$is the cardinality of the V(G) or **order of a graph G** and $m(G)$ is the cardinality of E(G) or **size of a graph G**

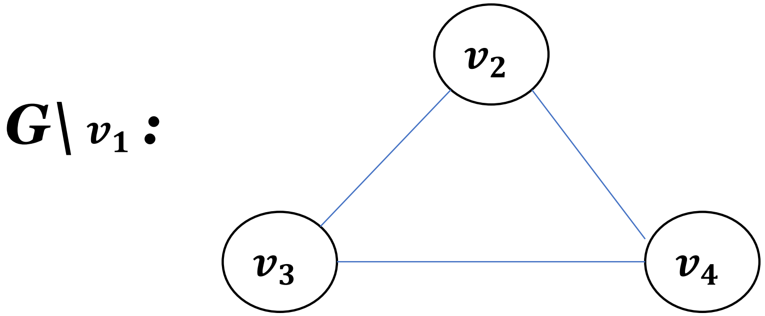
$\boldsymbol{n(G) = 4}$
$\boldsymbol{m(G) = 4}$

Definition 1.8 (**Trivial graph**). A trivial graph is a graph with order $n(G) = 1$

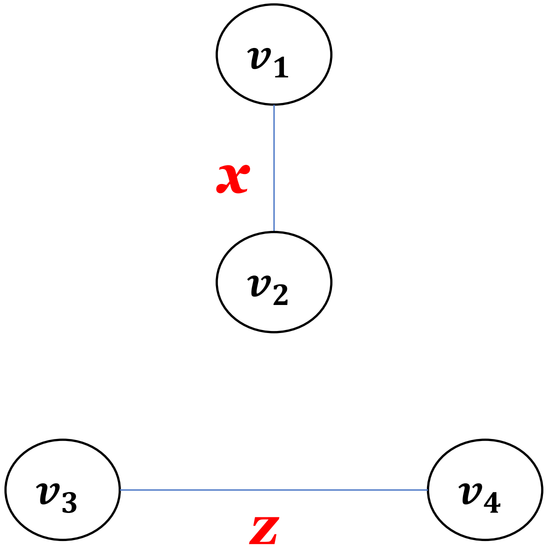
Definition 1.8 (**Subgraph**). A subgraph is a graph that exist with another graph.

- **A subgraph is obtained either by vertex deletion or edge deletion**
- **Doing vertex deletion will result by deleting all edges incident to it**
- **Doing edge deletion does not required to delete any vertex**





$G\setminus\{y,t\}$:

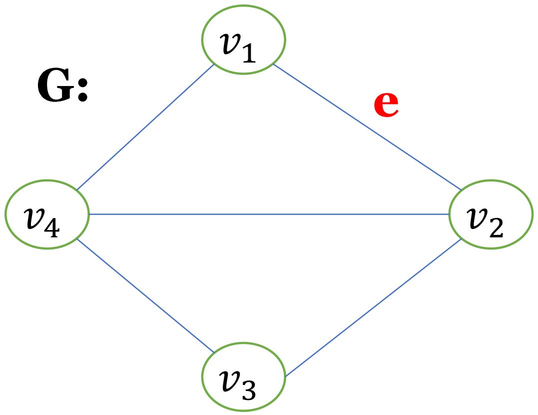


A graph F is a subgraph of G $(F \sqsubset G)$ *if* $\mathbf{V(F)} \sqsubseteq V(G)$ *and* $\mathbf{E(F)} \sqsubseteq E(G)$

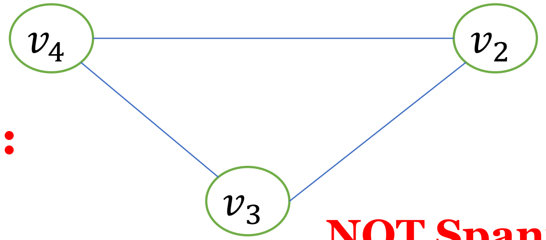
Definition 1.8 (**Spanning subgraph**). A spanning subgraph is a subgraph that contains all the vertices of the original graph. In other words a spanning subgraph is a subgraph obtained only by edge deletion

Example

G:

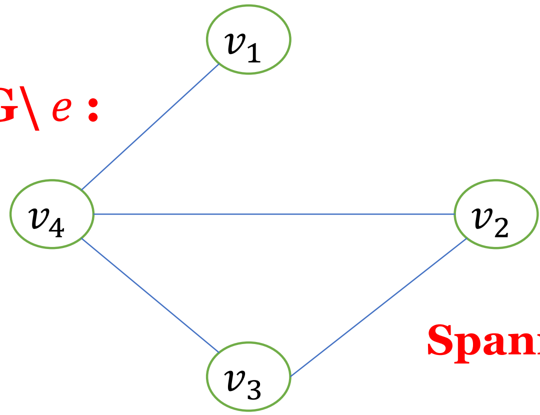


G\ v_1 :



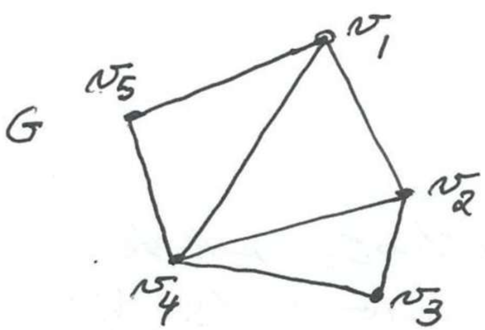
NOT Spanning subgraph

G\ e :



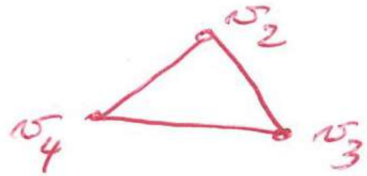
Spanning subgraph

Definition 1.8 (**Induced subgraph**). A subgraph obtained only by vertex deletion is called an induced subgraph



Let $X = \{v_1, v_5\}$

$G - X$:



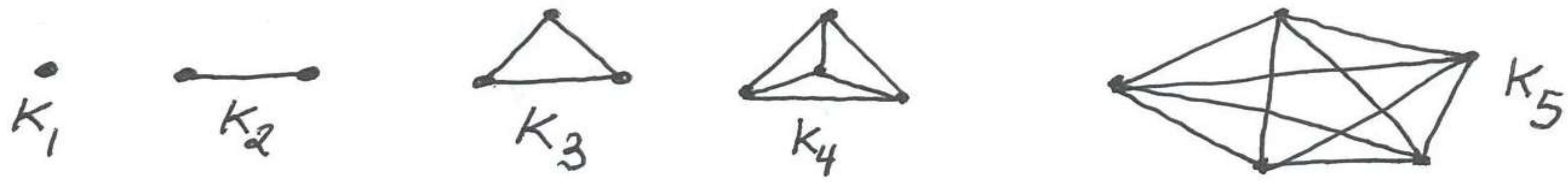
NOTATION

$G[X]$ = The graph induced by the vertices in X.

Examples of Graphs

1) Complete graphs

A complete graph or <u>clique</u> is a graph in which every two distinct vertices are adjacent. The complete graph of order n is denoted by K_n and is called an n-clique.



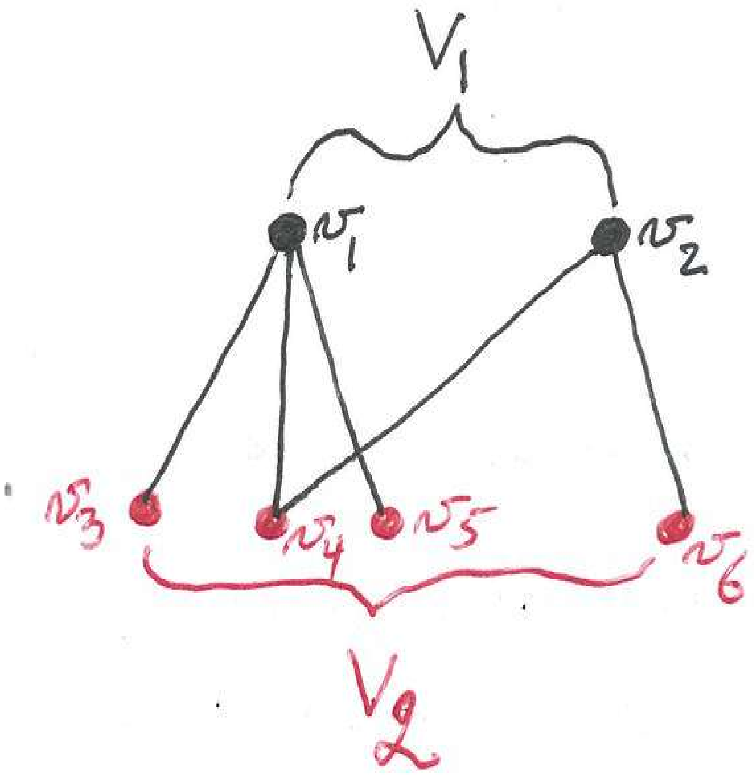
<u>NOTA</u>: The edge of K_n is $e = \frac{n(n-1)}{2}$

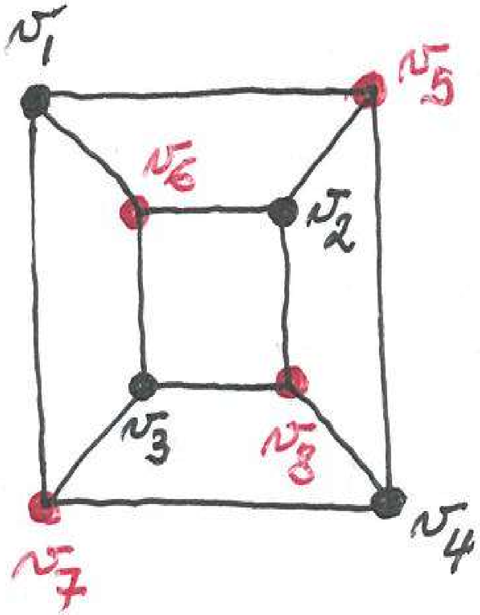
2) Empty graphs

The empty graph is a graph containing no edges. The empty graph of order n is denoted by $\overline{K}_n$

3) Bipartite graphs

A bipartite graph is a graph whose vertex set can be partitioned into two sets V_1 and V_2 in such a way that each edge of the graph joins a vertex of V_1 to a vertex of V_2.





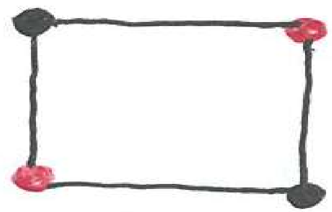
$V_1 = \{v_1, v_2, v_3, v_4\}$

$V_2 = \{v_5, v_6, v_7, v_8\}$

4) Complete bipartite graphs

A complete bipartite graph is a bipartite graph with partite set V_1 and V_2 having the added property that every vertex V_1 is adjacent to every vertex of V_2. If $|V_1| = r$ and $|V_2| = s$, then this graph is denoted by $K_{r,s}$. A complete bipartite graph of the form $K_{1,s}$ is called a star graph. A complete bipartite graph of the form $K_{n,n}$ is called n-biclique.

$K_{1,4}$



$K_{2,2}$

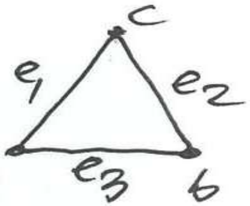
operations on graphs

1) The intersection

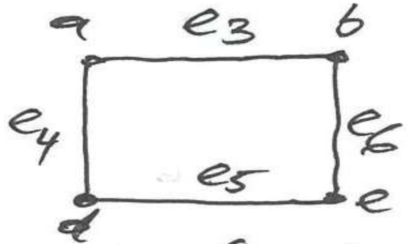
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

Then $G_3 = G_1 \cap G_2 = (V_3, E_3)$ where $V_3 = V_1 \cap V_2$ and $E_3 = E_1 \cap E_2$.

$G_1 =$



G_2:



$V_1 = \{a, b, c\}$

$E_1 = \{e_1, e_2, e_3\}$

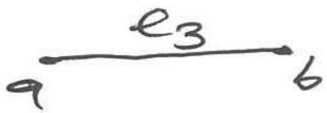
$V_2 = \{a, b, d, e\}$

$E_2 = \{e_3, e_4, e_5, e_6\}$

$V_3 = V_1 \cap V_2 = \{a, b\}$

$E_3 = E_1 \cap E_2 = \{e_3\}$

$G_3 = G_1 \cap G_2)$:

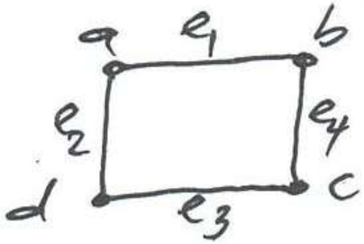


2) Union

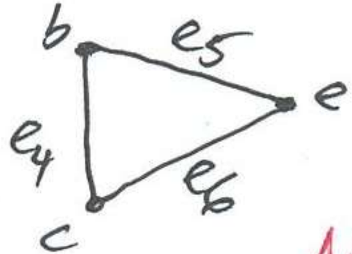
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

Then $G_3 = G_1 \cup G_2 = (V_3, E_3)$ where $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$

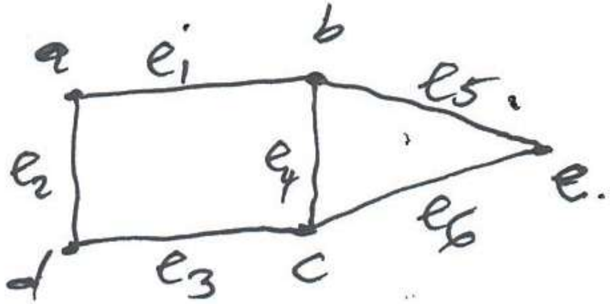
G_1:



G_2:



$G_1 \cup G_2$:



NOTA

if G_1 has edge e_1 and G_2 has edge e_2 Then $G_1 \cup G_2$ has edge $e_1 + e_2$.

3) <u>The join graph</u>

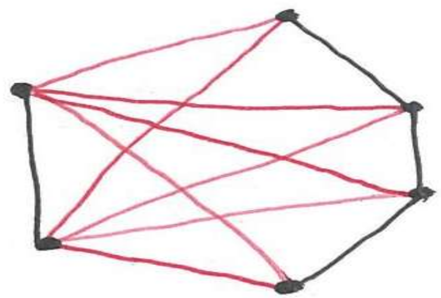
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The join graph $G_3 = G_1 + G_2 = (V_3, E_3)$ has $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2 \cup \{uv \mid u \in V_1 \text{ and } v \in V_2\}$.

<u>Example</u>

G_1 :

G_2 :

$G_3 = G_1 + G_2$:



4) The Cartesian Product.

Recall: $A \times B = \{(a,b) : a \in A,\ b \in B\}$

Suppose G_1 and G_2 are two graphs with $V(G_1) = \{u_1, u_2, \dots, u_n\}$ and $V(G_2) = \{v_1, v_2, \dots, v_n\}$

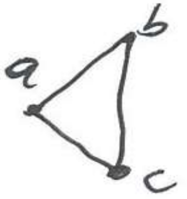
Then $G_1 \times G_2$ is the graph with vertex set

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$
$$= \{(u_i, v_j) : u_i \in V(G_1),\ v_j \in V(G_2)\}$$

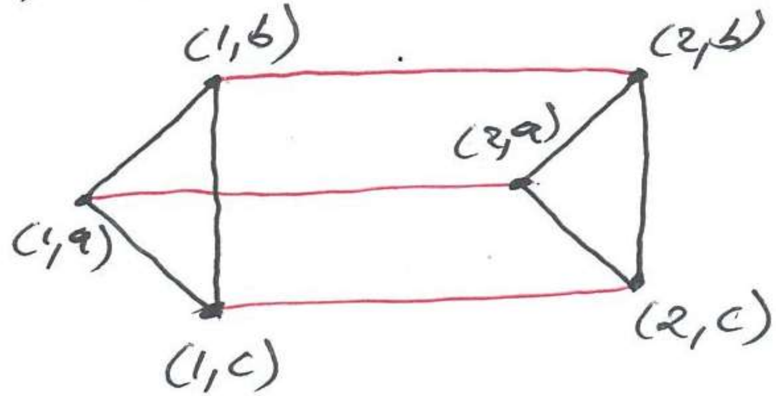
Example

G_1 :

G_2 :



$G_1 \times G_2$:

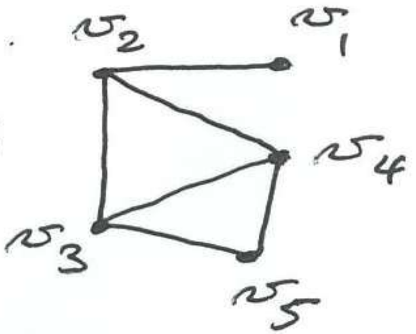


5) Compliment of the graph.

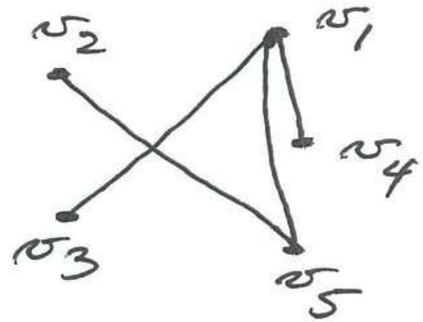
Considering a graph G. The Compliment of G, $\overline{G}$ is a graph with same vertex set of G and with two vertices adjacents only when they are not adjacents in G.

Example

G :



$\overline{G}$:



NOTA:

- If a graph G has v vertices and e edges. Then $\bar{G}$ will have $\frac{v(v-1)}{2} - e$ edges.

G: $v = 5$, $e = 6$ $\quad \bar{G} = \frac{5(5-1)}{2} - 6 = 10 - 6 = 4$ edges.

- If G is a graph with n vertices. Then $G \cup \bar{G} = K_n$ (complete graph of order n).

Exercise.

Given graph G with 15 edges and $\overline{G}$ with 13 edges.
Find the number of vertices in graph G.

Solution

We know that $G \cup \overline{G} = K_n$

The number of edges in K_n is $e = \frac{n(n-1)}{2}$

$$G \cup \overline{G} = \frac{n(n-1)}{2}$$

$$15 + 13 = \frac{n(n-1)}{2}$$

$$n^2 - n - 56 = 0$$

$$(n-8)(n+7) = 0$$

$n = 8$; $n = -7$ (not possible)

$\therefore$ G contains 8 vertices.

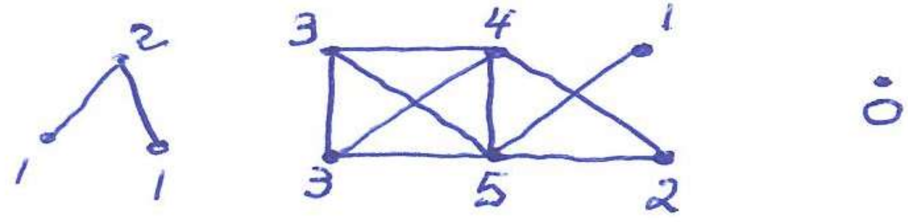
The Degree of a Vertex.

Definition

Let v be a vertex of a graph G. The degree of v is the number of edges of G incident with v. The degree of v is denoted by $\deg_G v$, or simply $\deg v$.

The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$, while the maximum degree of G is the maximum degree among the vertices of G is denoted by $\Delta(G)$.

Example

G: 

$\delta(G) = 0$

$\Delta(G) = 5$

Definition

A vertex is called <u>odd</u> or <u>even</u> depending on whether its degree is <u>odd</u> or <u>even</u>.

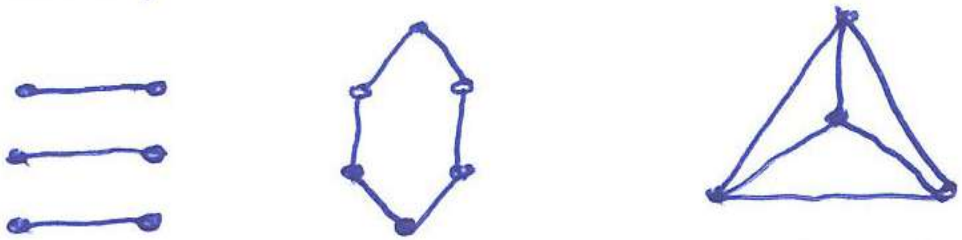
A vertex of degree 0 in G is called an <u>isolated</u> vertex and a vertex of degree 1 is an <u>end-vertex</u> of G.

A vertex adjacent to an end-vertex is called a remote vertex.

Definition (Regular graph)

We say that a graph is <u>regular</u> if all its vertices have the same degree. In particular, if the degree of each vertex is r, then the graph is regular of degree r or is r-regular.

Example



$r=1$ $r=2$ $r=3$

Theorem 1.1

In any graph, the sum of all the vertex degrees is equal to twice the number of edges

proof

Every edge is incident with two vertices; hence, when the degrees of the vertices are summed, each edge is counted twice. □

Corrollary 1.2

In any graph, there is an even number of odd vertices.

proof

Let G be a $(\overset{\substack{\text{order}\\ \uparrow}}{n}, \overset{\substack{\text{size}\\ \uparrow}}{m})$ graph. If G has no odd vertices, then the result follows immediately. Suppose that G contains $k (\geq 1)$ odd vertices $v_1, v_2, \ldots, v_k$. If G contains even vertices as well, then denote these by $v_{k+1}, \ldots, v_n$.

By Theorem 1.1. we have

$$\sum_{i=1}^{k} \deg v_i + \sum_{i=k+1}^{n} \deg v_i = 2m$$

$$\Rightarrow \sum_{i=1}^{k} \deg v_i = \underbrace{2m}_{\text{even}} - \underbrace{\sum_{i=k+1}^{n} \deg v_i}_{\text{even since the number } \deg v_{k+1}, \dots, \deg v_n \text{ is even.}}$$

$\therefore \sum_{i=1}^{k} \deg v_i$ is even

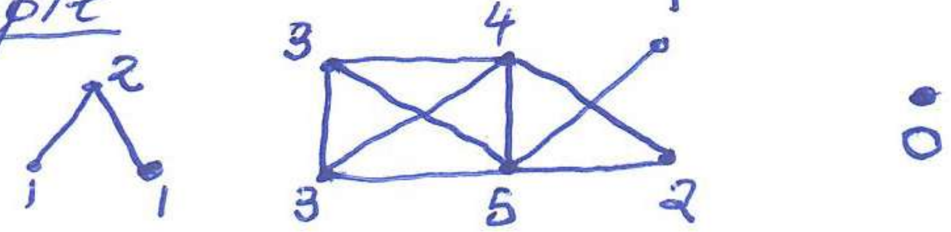
However, each of the number $\deg v_1, \deg v_2, \dots, \deg v_k$ is odd. Since the sum of an odd number is odd, it follows that k must be even; that is G has an even number of odd vertices.

If G has no even vertices, then we have $\deg v_1 + \deg v_2 + \dots + \deg v_k = 2m$, from which we again conclude that k is even. □

The Degree Sequence.

- It is convenient to list the degrees of vertices in decreasing order by allowing "repeats"
- The result list is called the degree sequence of a graph.

Example

G: 

The degree sequence of graph G is

5, 4, 3, 3, 2, 2, 1, 1, 1, 0

Definition

- A sequence is graphical if

  1) $d_i \geq 0$, $\forall i$
  2) $d_i \leq n-1$ with n the order of the graph.
  3) $\sum_{i=1}^{n} d_i$ even.

Example.

The sequence 5, 4, 3, 3, 2, 2, 1, 1, 1, 0 is graphical:

  i) $d_i \geq 0$
  2) $d_i \leq n-1$, Here $n = 10$ (we have 10 vertices)
  3) $\sum_{i=1}^{n} d_i = 5+4+3+3+2+2+1+1+1+0$
  $= 22$ even.

- The above 3 conditions are necessary for a sequence to be graphical, but these conditions are not sufficient.

Example

The sequence 3, 3, 3, 1 is not graphical

1) $d_i \geq 0 \quad \forall i$ (satisfied)

2) $d_i \leq n-1$

Here $n = 4$

3) $\sum_{i=1}^{n} d_i = 3+3+3+1 = 10$ even (satisfied)

Hence we can't represent the sequence 3, 3, 3, 1 graphically

THE 3 CONDITIONS ARE SATISFIED. HOWEVER THE SEQUENCE <u>IS NOT</u> GRAPHICAL

Theorem 1.3 [Havel-Hakimi]

A Sequence $s: d_1, d_2, \ldots, d_n$ of non-negative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ and $n \geq 2$, $d_1 \geq 1$ is graphical if and only if the Sequence

$$s_1 = \underbrace{d_2 - 1,\ d_3 - 1,\ \ldots,\ d_{d_1+1} - 1}_{d_1 \text{ terms}},\ d_{d_1+2}, \ldots, d_n\ ,\ n \geq 2$$

is graphical.

Example

Ex. ① $\cancel{3}, 3, 3, 2, 2, 1$ is graphical $\Leftrightarrow$ $2, 2, 1, 2, 1$ is graphical

$\Leftrightarrow \cancel{2}, 2, 2, 1, 1$ re-order

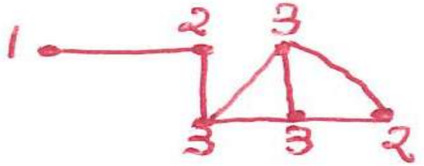
$\Leftrightarrow \cancel{1}, 1, 1, 1$ is graphical

$\Leftrightarrow 0, 1, 1$ is graphical

$\Leftrightarrow \cancel{1}, 1, 0$ re-order

$\Leftrightarrow 0, 0$ is graphical

∴ The sequence 3, 3, 3, 2, 2, 1 is graphical.



Ex. ② $\cancel{3}, 3, 3, 1$ is graphical $\Leftrightarrow \cancel{2}, 2, 0$ is graphical

$\Leftrightarrow 1, -1$ is not graphical

since -1 cannot be a degree of any vertex.

Ex. 3 $\cancel{4}, 4, 4, 3, 3, 2$ graphical $\Leftrightarrow \cancel{3}, 3, 2, 2, 2$ graphical

$\Leftrightarrow 2, 1, 1, 2$

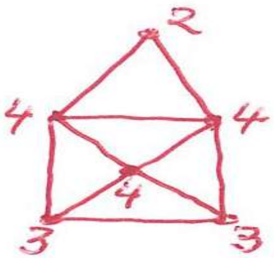
$\Leftrightarrow \cancel{2}, 2, 1, 1$ re-order

$\Leftrightarrow 1, 0, 1$

$\Leftrightarrow \cancel{1}, 1, 0$ re-order

$\Leftrightarrow 0, 0$ graphical

$\therefore$ 4, 4, 4, 3, 3, 2 is graphical



NOTA

The degre sequences do not always provide enough information to uniquely describe a graph.

isomorphic graphs

- An <u>isomorphism</u> from a graph G to a graph H is a bijective mapping $\phi : V(G) \longrightarrow V(H)$ Such that $xy \in E(G) \Longleftrightarrow \phi(x)\phi(y) \in E(H)$.
(every edge of G get map to a single edge of H).
- A <u>Homomorphism</u> from a graph G to a graph H is a mapping (not necessarily bijective) $\phi : V(G) \longrightarrow V(H)$ Such that

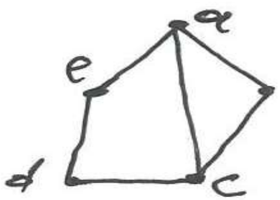
$xy \in E(G) \Longrightarrow \phi(x)\phi(y) \in E(H)$.

(Every edge of G get map to the edge of H)

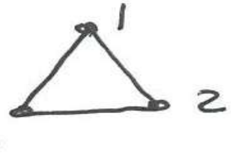
Example

Given the graph

G :



H :



consider $\phi : V(G) \longrightarrow V(H)$ given by

$\phi(a) = 1$; $\phi(b) = 2$; $\phi(c) = 3$; $\phi(d) = 2$; $\phi(e) = 3$

Discuss if ϕ is homomorphism.

Check edges : $ab \longrightarrow 12$; $bc \longrightarrow 23$; $cd \longrightarrow 32$;
$de \longrightarrow 23$; $ea \longrightarrow 31$; $ac \longrightarrow 13$.

NOT edges : $eb \longrightarrow 32$; $bd \longrightarrow$ single vertex 2.

Hence, every edge of G get map to the edge of H.

$\therefore$ ϕ is a homorphism from G to H.

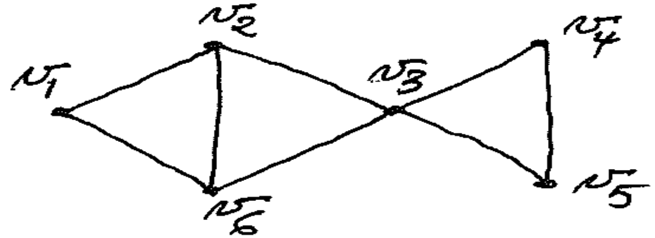
Connected graphs

Definition (A walk)

Let $u, v \in V(G)$ (not necessarily distinct)
A $u-v$ walk in G is a finite, alternating sequence of vertices and edges that begin with the vertex u and end with the vertex v.

Example

G :



Find the walk $v_3 - v_4$:

$v_3, v_3v_2, v_2, v_2v_6, v_6, v_6v_3, v_3, v_3v_4, v_4,$
v_4v_5, v_5, v_5v_4, v_4

OR

$v_3, v_2, v_6, v_3, v_4, v_5, v_4$ (This walk has length $\underline{\underline{6}}$)

NOTA

- A closed walk occurs when $u = v$
- A trivial walk is a walk which does not take any edges (Example v_6)

Example

Find closed walk $v_3 - v_3$:

v_3, v_2, v_1, v_6, v_3

This is a closed walk of length 4.

Definition

- A trail is a walk with no repeated edges.

  Example : $v_1 - v_5$ trial :

  v_1, v_2, v_6, v_3, v_5

- A closed trial is a circuit

  Example : $v_1 - v_1$ Circuit :

  v_1, v_2, v_6, v_1

- A cycle is a circuit which does not repeat any vertices (except the first and last vertices).

- ~~A~~ path is a walk with no repeated vertices.

  Example : $v_2 - v_5$ PATH :

  $v_2, v_1, v_6, v_3, v_4, v_5$

- A closed path is a cycle.

Theorem

Every u-v walk in a graph contains a u-v path.

proof

Let W be a u-v walk in a graph G. If W is closed, the result is easy; we simply use trivial path u. Thus assume W is an open walk, say $W: u = u_0, u_1, u_2, \ldots, u_k = v$.

Note that a vertex may have received more than one label if it occurs more than once in W.

If no Vertex is repeated, then W is already a path otherwise, there are vertices of G that occur in W twice or more. Let i and $j \in \mathbb{Z}$, $i \neq j$ with $i < j$ such that $u_i = u_j$. That is, the vertex u_i is repeated as u_j. If we now delete the vertices $u_i, u_{i+1}, \ldots, u_{j-1}$ from W, we obtain a $u-v$ walk W_1 which is shorter than W and has fewer repeated vertices. If W_1 is a path, we are done; if not, we continue this process until finally we reach a stage where no vertices are repeated and a $u-v$ path is obtained $\square$

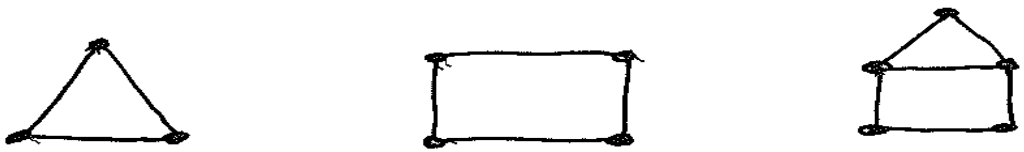
Definition (Path graphs)

A path graph is a graph consisting of a single path. The path graph of order n is denoted by P_n.

P_1 $\quad$ P_2 $\quad$ P_3 $\quad$ P_4

Definition (cycle graphs)

A cycle graph is a graph consisting of a single cycle. The cycle graph of orde n is denoted by C_n $(n \geq 3)$



C_3 $\quad$ C_4 $\quad$ C_5

Definitions:

- A graph G is connected if there exists a path in G between any two of its vertices, and is disconnected otherwise. Every disconnected graph can be split up into a number of connected subgraphs, called component.
- A component of a graph G is a maximal connected subgraph.
- Two vertices u and v in a graph G are connected if $u = v$, or if $u \neq v$ and a u-v path exist in G.
- The number of component of G is denoted by $k(G)$; of course, $k(G) = 1$ if and only if G is connected.

Distance in graphs

Definition

- For a connected graph G, we define the distance $d(u,v)$ between two vertices u and v as the minimum of the lengths of the $u-v$ path of G.

- If G is a disconnected graph, then the distance between two vertices in the same component of G is defined as above.

- If u and v belong to different component of G, then $d(u,v)$ is undefined or $d(u,v)=\infty$

Example

G : 

$d(u,u)=0$; $d(x,u)=2$; $d(x,w)=3$; $d(x,v)=5$

NOTA: The distance is the shortest path.

Theorem

Let G be a graph and let $u, v, w \in V(G)$

Then

(i) $d(u,v) \geq 0$ and $d(u,v)=0$ iff $u=v$;

(ii) $d(u,v)=d(v,u)$ (Symmetry property)

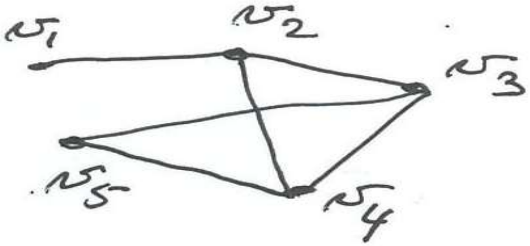
(iii) $d(u,v) \leq d(u,w)+d(w,v)$ (Triangle inequality).

Definitions

- The eccentricity, $e(v)$ of a vertex v in a connected graph G is the maximum distance of a vertex from v. $e(v) = \max\{d(u,v) / u, v \in V(G)\}$
- The radius of G, $\operatorname{rad} G$ is the minimum eccentricity among the vertices of G.
- The diameter, $\operatorname{diam} G$ is the maximum eccentricity $\operatorname{diam} G = \max\{e(v) / v \in V(G)\}$
- A vertex v is called a central vertex if $e(v) = \operatorname{rad} G$
- A vertex v is called a peripheral vertex if $e(v) = \operatorname{diam} G$.

Example

G :



$e(v_1) = 3$; $e(v_2) = 2$; $e(v_3) = 2$

$e(v_4) = 2$

$\operatorname{diam} G = 3$

$\operatorname{rad} G = 2$.

Theorem

For every connected graph

$$\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2\operatorname{rad}(G)$$

Proof

- $\underline{\operatorname{rad}(G) \leq \operatorname{diam}(G)}$. This is obvious since we know that $\operatorname{rad}(G) = \min\{e(v) / v \in V(G)\}$

$\operatorname{diam}(G) = \max\{e(v) / v \in V(G)\}$

$\therefore$ clearly, $\operatorname{rad}(G) \leq \operatorname{diam}(G)$

Now, let proof the second inequality,

$$diam(G) \leq 2\,rad(G)$$

We will start by taking two vertices:

Let $u, v \in V(G)$ such that $d(u,v) = diam(G)$

Let w be a central vertex of G:

$$d(u,v) \leq d(u,w) + d(w,v) = e(w) + e(w)$$
$$\leq 2\,e(w)$$

$d(u,w)$ could be the most eccentricity of w since $e(w)$ represent the furthest path of distance from w to any other vertex. Similarly, $d(w,v)$ is the most eccentricity of w

$\therefore d(u,v) \leq 2\,e(w) = 2\,rad(G)$ since w is the central vertex.

Hence,

$$rad(G) \leq diam(G) \leq 2\,rad(G).$$

compliment of graph G

Theorem

Let G be a disconnected graph. Therefore, $\overline{G}$ is a connected graph and $\text{diam}(\overline{G}) \le 2$.

proof

Let G be a disconnected graph and let G_1 be one of the connected component of G.

Let $V_1 = V(G_1)$ and $V_2 = V(G) \setminus V(G_1)$

Let $u \in V_1$. Then for every $v \in V_2$, $uv \notin E(G)$

$\therefore uv \in E(\overline{G})$, i.e. $d_{\overline{G}}(u,v) = 1$

And for every $v_1, v_2 \in V_2$

$$d_{\bar{G}}(v_1, v_2) \leq 2$$

similarly, $\forall u_1, u_2 \in V_1$:

$$d_{\bar{G}}(u_1, u_2) \leq 2.$$

$\therefore \max\{e_{\bar{G}}(v) \mid v \in V(\bar{G})\} \leq 2$

Hence, $\text{diam}(\bar{G}) \leq 2.$ $\square$

Cut-Vertices and Bridge

Recap (subgraph).

If $e \in E(G)$, then $G-e$ is the subgraph of G possessing the same vertex set as G and having all the edges of G except e.

If v is a vertex of a graph G containing at least two vertices, then $G-v$ is the subgraph of G whose vertex set consists of all vertices of G except v and whose edge set consists of all edges of G except those incident with v.

Definition

- A vertex v in a graph G is called a cut-vertex of G if $k(G-v) > k(G)$ with $k(G)$ the number of connected component of G.
  If G is a connected graph, then v is a cut-vertex if $G-v$ is disconnected.

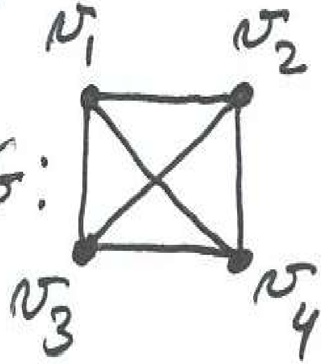
- e in a graph G is called a bridge if $k(G-e) > k(G)$. If G is a connected graph, then e is a bridge if $G-e$ is disconnected.

- Recap: A graph G is connected iff $k(G) = 1$

Example

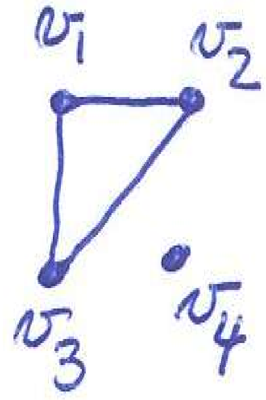
1)

G:



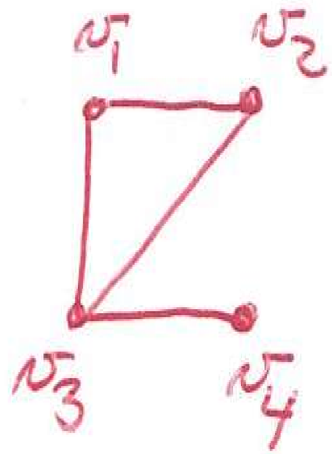
connected graph

H:



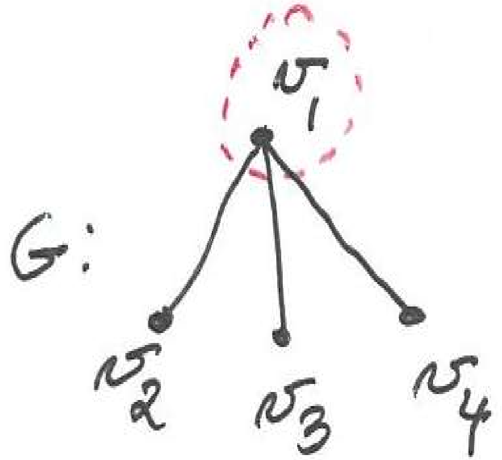
disconnected graph

F:

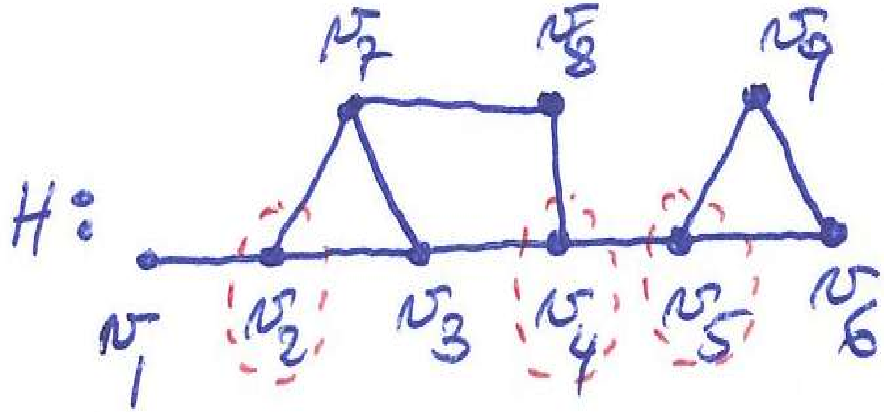


connected graph but easily made disconnected by removing vertex v_3

2) Find the cut-vertex in the graph below.



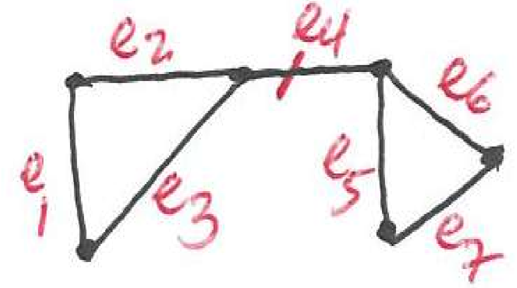
The cut vertex of Graph G is v_1



v_2, v_4 and v_5 are cut-vertices of graph H.

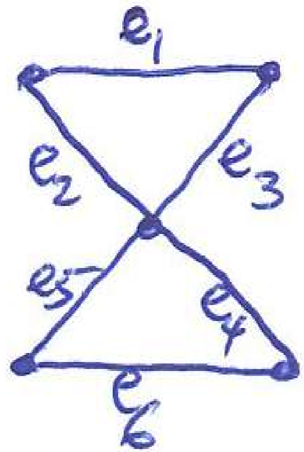
3) Find the Bridge in the graph below:

G:



e_4 is the bridge of graph G.

H:



There is No bridge in the graph H.

Theorem

A vertex $v \in V(G)$ is a cut-vertex of G if and only if there exist $u, w \in V(G)$, $u, w \neq v$ such that v is on $u-w$ path of G.

proof

we are going to assume that G is a connected graph.

$\boxed{\Rightarrow}$

Let $v \in V(G)$ be a cut-vertex. Then $G-v$ is a disconnected graph. Let u, w be vertices in different components of $G-v$. Therefore, $u-w$ is not a path in $G-v$. But since G is a connected graph, then there exist $u-w$ path in G. Hence, all such path went through vertex v.

$\boxed{\Leftarrow}$

suppose there exist $u, w \in V(G)$, $u, w \neq v$ such that v lies on every u-w path. then removing vertex v means that there is <u>not</u> any u-w path in $G-v$. Hence, $G-v$ is disconnected, then v is a cut-vertex. $\square$

Theorem

An edge $e \in E(G)$ is a bridge if and only if e is not in any cycle of G.

Recap: - A closed path is a cycle.

- $P \Rightarrow Q$; contrapositive : $\neg Q \Rightarrow \neg P$

proof

we are going to prove by contraposition:

$\boxed{\Rightarrow}$ contrapositive: we have to show that if e is on the cycle then e is not a bridge.

Let $e \in E(G)$ is on the cycle C, then $G-e$, u and v are in the same component ($e=uv$)

Hence, e is not a bridge.

$\boxed{\Leftarrow}$ contrapositive: we have to show that if $e \in E(G)$ is not a bridge then e is on a cycle.

$e = uv$ is not a bridge then there is a path from u to v (Since u and v are in the same component). in the original graph G. Hence, there is a cycle that contains edge e. $\square$

Lemma

Let e be a bridge in a connected graph G.
Then $k(G-e)=2$.

proof

~~Let~~ $e=uv$ be a bridge, then u and v are in different component of $G-e$. Therefore, $k(G-e)\geq 2$
Now, since the original graph is connected, let $w\in V(G)$ and since G is connected, there exist a path $w-v$ in G. Now, if path $w-v$ does not use edge e, then path $w-v$ is in $G-e$. In a contrary if path $w-v$ uses edge e then there exist path from w to u in $G-e$. Hence, $k(G-e)=2$.

The Shortest Path Algorithm.

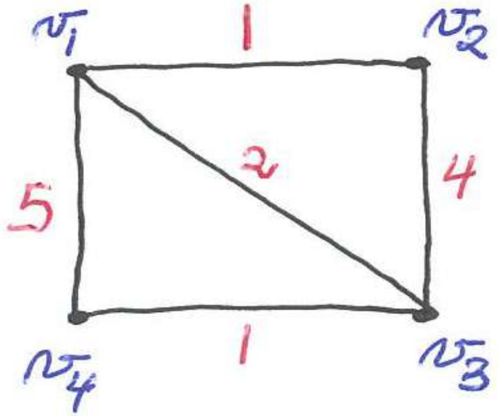
Definitions

- A weighted graph is a graph in which each edge e is assigned a positive real number, called the weight of e, and denoted by $w(e)$.
- The length of a path P in a weighted graph G is the sum of the weights of the edges of P.

- For connected vertices u and v of the weighted graph G, the distance $d(u,v)$ between u and v is the minimum of the length of the $u-v$ path of G.
- A $u-v$ path of minimum weight in G is called a shortest $u-v$ path for G.

Example

G :



- The path $P: v_2, v_1, v_3, v_4$ is the shortest path of minimum weight 4.

- The path $Q: v_2, v_3, v_4$ of weight 5 is NOT the shortest path for G, even though path Q contains few edges than path P.

The weight matrix.

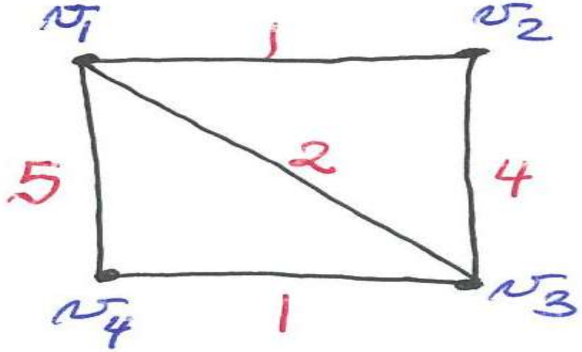
Let G be a weighted graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. It is convenient to represent G by means of a weight matrix $W(G)$ defined as follows:

$$W(G) = \begin{cases} w(v_i v_j) & \text{if } v_i v_j \in E(G) \\ \infty & \text{if } v_i v_j \notin E(G) \end{cases}$$

Example

Given

G :



$$W(G) = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 1 & 0 & 4 & \infty \\ 2 & 4 & 0 & 1 \\ 5 & \infty & 1 & 0 \end{pmatrix}$$

Dijkstra's Algorithm.

Given a connected weighted graph G of order p and a vertex x of G:

1) Set $l(x) = 0$ and for all $v \neq x$, set $l(v) = \infty$ and set $S = V(G)$

2) if $|S| = 1$, then stop; otherwise continue.

3) Among all the vertices in S, let u be one of minimum label $l(u)$;

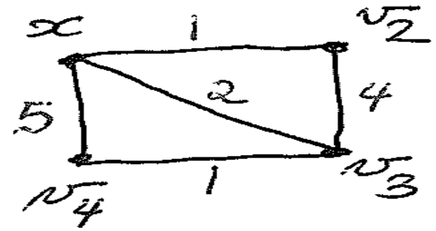
4) For each $v \in S$, if $uv \in E(G)$ and $l(v) > l(u) + w(uv)$, then

- replace $l(v)$ by $l(u) + w(uv)$ and
- Assign to parent v the vertex u.

5) Remove u from S, and return to step 2.

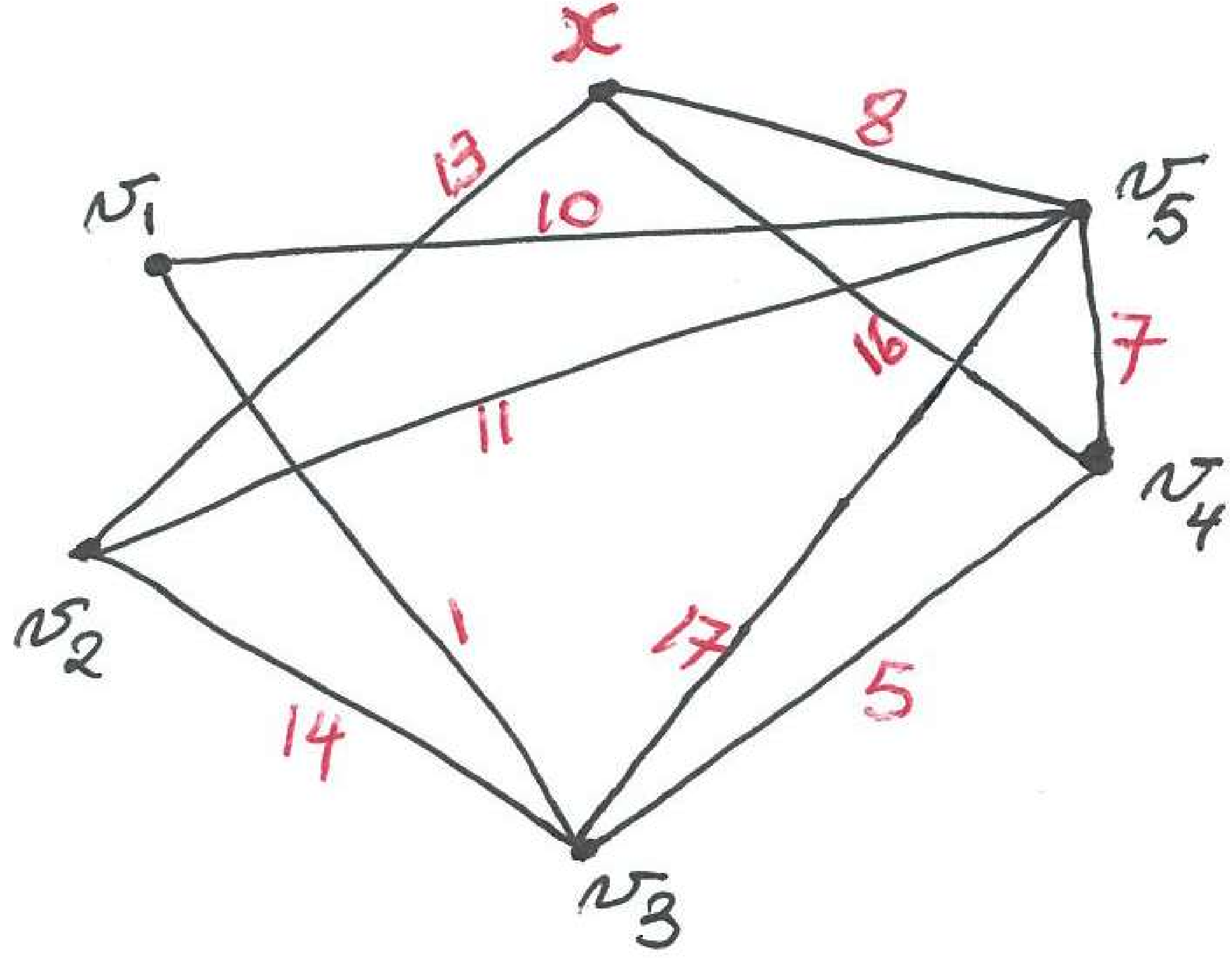
Example

$l(x)$	v_2	v_3	v_4	Removed from S	S
0	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	—	$\{x, v_2, v_3, v_4\}$
	$(1, x)$	$(2, x)$	$(5, x)$	x	$\{v_2, v_3, v_4\}$
		$(2, x)$	$(5, x)$	v_2	$\{v_3, v_4\}$
			$(3, v_3)$	v_3	$\{v_4\}$



shortest path:

v	$d(x, v)$	Q_i
v_2	$l(v_2) = 1$	x, v_2
v_3	$l(v_3) = 2$	x, v_3
v_4	$l(v_4) = 3$	x, v_3, v_4


x
8
13
10
v_1
v_5
7
16
11
v_4
v_2
1
17
5
14
v_3

$l(x)$	v_1	v_2	v_3	v_4	v_5	Removed from S	S
0	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	—	$V(G)$
	$(\infty, -)$	$(13, x)$	$(\infty, -)$	$(16, x)$	$(8, x)$	x	$\{v_1, v_2, v_3, v_4, v_5\}$
	$(18, v_5)$	$(13, x)$	$(25, v_5)$	$(15, v_5)$		v_5	$\{v_1, v_2, v_3, v_4\}$
	$(18, v_5)$		$(25, v_5)$	$(15, v_5)$		v_2	$\{v_1, v_3, v_4\}$
	$(18, v_5)$		$(20, v_4)$			v_4	$\{v_1, v_3\}$
			$(19, v_1)$			v_1	$\{v_3\}$.

The shortest path.

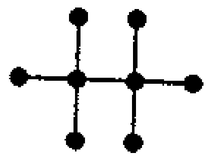
v	$d(x, v)$	P_i
v_1	$l(v_1) = 18$	x, v_5, v_1
v_2	$l(v_2) = 13$	x, v_2
v_3	$l(v_3) = 19$	x, v_5, v_1, v_2
v_4	$l(v_4) = 15$	x, v_5, v_4
v_5	$l(v_5) = 8$	x, v_5

Trees

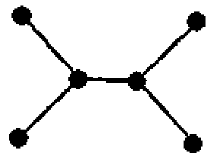
Definition:

- A Graph with NO cycle is acyclic
- A tree is a connected graph which contains no cycle (i.e. A tree is a connected acyclic graph)
- A Forest is a graph that has no cycles (Note that each component of a forest is a tree). A forest is an acyclic graph

T_1 :



T_2 :



T_3 :

Since a tree contains no cycle, it follows that:_

- Every edge of a tree is a bridge

- Any two vertices of a tress are connected by exactly one path

Properties of a Tree

Theorem 4.1:

A graph *T* is a tree if and only if every two distinct vertices of *T* are joined by a unique path

Proof. If T is a tree, then by definition it is connected. Hence, any two vertices are joined by at least one path. Suppose that there are vertices u and v of T that are joined by two or more different paths. Let P and Q be two different u-v paths in T. Then there must be a vertex x (possibly $x = u$) on both paths such that the vertex immediately following x on P is different from the vertex immediately following x on Q. Let y be the first vertex of P following x that also belongs to Q (possible $y = v$). Then the section of P from x to y and the section of Q from x to y produce two x-y paths that have only x and y in common. These two paths produce a cycle in T, which contradicts the fact that T is a tree. Hence every two distinct vertices of T are joined by a unique path.

On the other hand, suppose T is a graph in which any two distinct are joined by a unique path. This implies that T is connected. If T has a cycle containing vertices u and v, the u and v are joined by at least two paths, contradicting our hypothesis. Hence T is acyclic so that T is a tree. □

Theorem 4.2:

A tree *T* of order *p* has size *p-1*

Proof. We proceed by induction on p. If $p = 1$, then $T \cong K_1$ and T has size 0, as required. Let $k \geq 2$ be an integer, and suppose the result is true for all trees of order less than k. Let T be a tree of order $p = k$ and size q, and let $e = uv$ be an edge of T. Since every edge of a tree is a bridge, the graph $T - e$ is disconnected. In fact, $T - e$ is a forest with exactly two components, namely, a tree T_1 containing u, and a tree T_2 containing v. Let T_i $(i = 1, 2)$ be a tree of order p_i and size q_i. Then $p_i < k$ $(i = 1, 2)$. Hence, by the inductive hypothesis, we know that $q_i = p_i - 1$ for $i = 1, 2$. Since $p = p_1 + p_2$ and $q = q_1 + q_2 + 1$,

$$q = (p_1 - 1) + (p_2 - 1) + 1 = p_1 + p_2 - 1 = p - 1.$$

Thus, by induction, the size of a tree is one less than its order. □

Characterizations of a Tree

Theorem 4.3:

Let T be a graph of order p. Then the following statements are equivalent

i. T is a tree

ii. T is connected and has size p-1

iii. T has no cycles and has size p-1

Please note that you need to prove the following:

$$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$$
$$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$$
$$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$$

Proof. If T is a tree of order p, then, by definition, T is connected and, by Theorem 4.2, T has size $p-1$. So $(i) \Rightarrow (ii)$.

Suppose that T is connected and has size $p-1$. Assume T contains a cycle and let e be an edge of this cycle. Then, by Theorem 2.5, e is not a bridge of T, so $T-e$ is a connected graph of order p and size $p-2$, which is a contradiction (see Exercise 4.3). Therefore T has no cycles. Thus $(ii) \Rightarrow (iii)$.

Suppose that T has no cycles and size $p-1$. In order to prove that T is a tree, we must show that T is connected. Let $T_1, T_2, \ldots, T_k$ be the components of T $(k \geq 1)$, where T_i has order p_i and size q_i for $i = 1, 2, \ldots, k$. Since each component T_i $(i = 1, 2, \ldots, k)$ is a connected graph with no cycles, each T_i is a tree. Thus, by Theorem 4.2, $q_i = p_i - 1$. Hence

$$p - 1 = q = \sum_{i=1}^{k} q_i = \sum_{i=1}^{k} (p_i - 1) = \left(\sum_{i=1}^{k} p_i\right) - k = p - k,$$

so $k = 1$ and therefore T is connected. Hence T is a tree. Thus $(iii) \Rightarrow (i)$. $\square$

Theorem 4.3:
Every nontrivial tree contains at least two end-vertices.

Proof. Suppose that T is a tree of order p and size q, and let $d_1, d_2, \ldots, d_p$ denote the degrees of its vertices, ordered so that $d_1 \leq d_2 \leq \cdots \leq d_p$. Since T is connected and nontrivial, $d_i \geq 1$ for each i $(1 \leq i \leq p)$. If T does not contain two end-vertices, then $d_1 \geq 1$ and $d_i \geq 2$ for $2 \leq i \leq p$. So

$$\sum_{i=1}^{p} d_i = d_1 + \sum_{i=2}^{p} d_i \geq 1 + 2(p-1) = 2p - 1.$$

However, by Theorems 1.1 and 4.2,

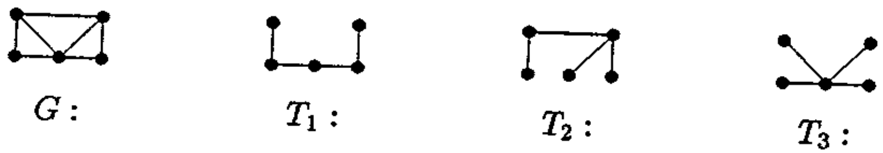
$$\sum_{i=1}^{p} d_i = 2q = 2(p-1) = 2p - 2,$$

which contradicts the previous inequality. Hence, T contains at least two end-vertices. □

Constructing Minimum Spanning Tree

Definitions:

- A Spanning Tree of a connected graph *G* is a tree that is a subgraph of *G* that contains all the vertices of *G.*
- A Spanning forest of a graph *G* is a forest that is a subgraph of *G* that contains all the vertices of *G*.
- A Spanning tree *T* of a connected graph *G* is said to be distance preserving from a vertex $\boldsymbol{v}$ in *G* if $\boldsymbol{d_T(u, v) = d_G(u, v)}\ \forall \boldsymbol{u} \in \boldsymbol{V(G)}$.



Spanning trees of a graph.

Important:

- Every connected graph G contains a spanning tree.
- If G has no cycles, then the graph G itself is a spanning tree.
- If there are cycles in a graph G, then choose any cycles in G and remove one of its edges..

Theorem 4.5 *For every vertex v of a connected graph G, there exists a spanning tree T that is distance-preserving from v.*

Proof. Let n be the maximum distance from v to a vertex of G. For $i = 1, 2, \ldots, n$, let

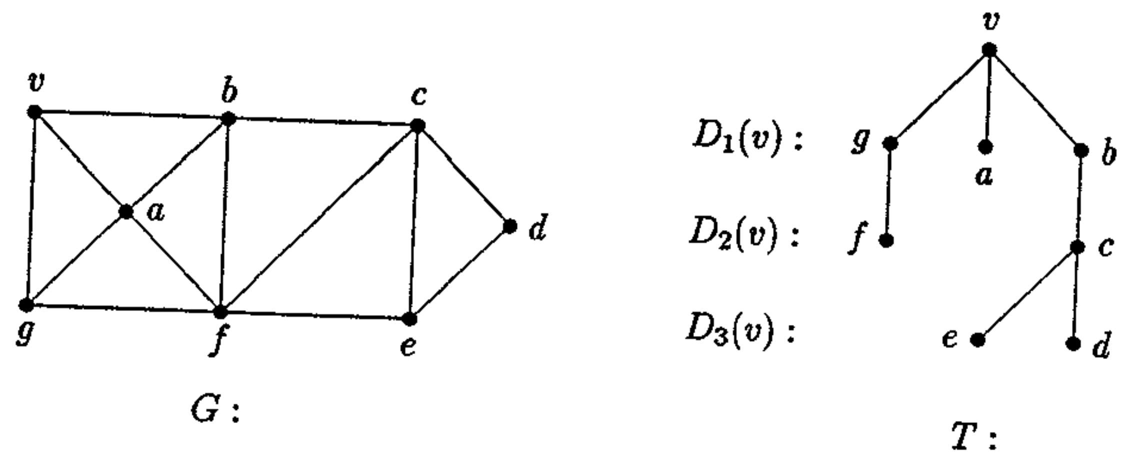
$$D_i(v) = \{u \in V(G) \mid d(u, v) = i\}.$$

Since G is connected, it follows that every vertex $u \neq v$ belongs to $D_i(v)$ for some i $(1 \leq i \leq n)$. Furthermore, such a vertex u is adjacent to at least one vertex of $D_{i-1}(v)$ and possibly to vertices in $D_i(v)$ and $D_{i+1}(v)$ as well. Delete all but one edge that joins u to a vertex of $D_{i-1}(v)$. Also, remove every edge joining u to a vertex of $D_i(v)$. Repeat this process for each $u \neq v$. Let T denote the resulting graph.

From the manner in which T was contructed, it is clear that T is connected since a u-v path exists for each $u \neq v$. It is also clear that T is distance-preserving from v. It remains for us to verify that T is a tree. We need only show that T is acyclic. If this is not the case, then there is a cycle C in T. Let w be a vertex of C whose distance from v is maximum, and let w_1 and w_2 be the vertices adjacent to w on C. Suppose $w \in D_k$. Then $w_i \in D_k$ or $w_i \in D_{k-1}$ for $i = 1, 2$. If $w_1 \in D_k$ or $w_2 \in D_k$, then this contradicts the way in which T was constructed (no edge in T joins two vertices in the same set $D_i(v)$). Thus, both w_1 and w_2 belong to the set D_{k-1}. Once again, this contradicts the way in which T was constructed (exactly one edge in T joins a vertex in D_i to a vertex in D_{i-1}). We deduce, therefore, that T is acyclic and hence is a tree. □

Constructing a spanning Tree from a connected Graph

- Let $\boldsymbol{n}$ be the maximum distance from $\boldsymbol{v}$ to a vertex of $\boldsymbol{G}$ for $\boldsymbol{i = 1, 2, \ldots n}$
- Let $\boldsymbol{D_i(v) = \{u \in V(G)/d(u, v = i)\}}$
- No edge in $\boldsymbol{T}$ joins two vertices in the same set $\boldsymbol{D_i(v)}$
- Exactly one edge in $\boldsymbol{T}$ joins a vertex in $\boldsymbol{D_i}$ to a vertex in $\boldsymbol{D_{i-1}}$



A connected graph G (with vertex v) and a spanning tree T that is distance-preserving from v.

Definitions.

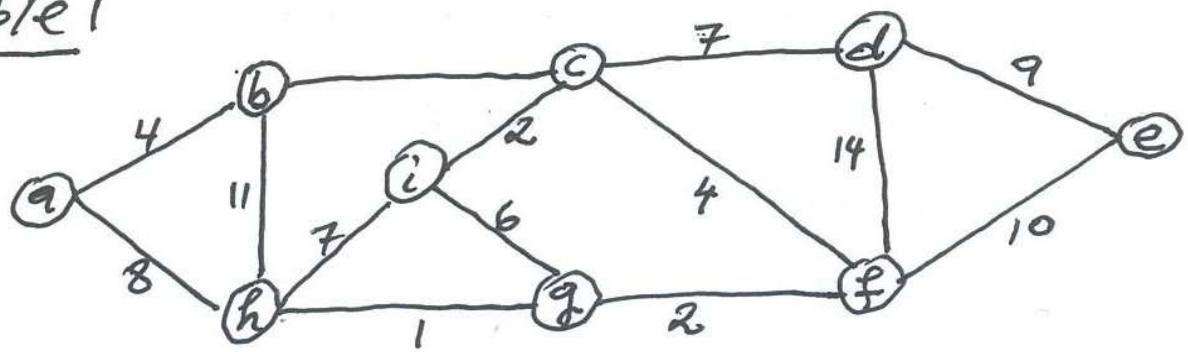
- The weight $w(H)$ of a subgraph H of a weighted graph is the sum of the weights of the edges of H.
- A minimum spanning tree of a connected weighted graph is a spanning tree of minimum weight in the graph.
- The minimum connector Problem:
  Given a connected weighted graph, find a minimum spanning tree in it.

Algorithm (Kruskal)

To construct a minimum spanning tree in a connected weighted graph G, successively choose edges of G of minimum weight in such a way that no cycles are created.

Example 1

G :

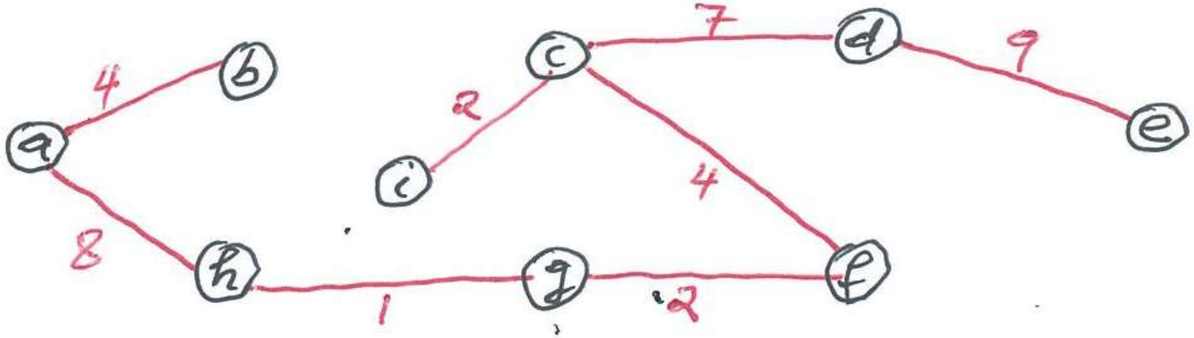


$$w(T) = w(gh) + w(fg) + w(ci) + w(cf) + w(ab) + w(cd) + w(ah) + w(de)$$

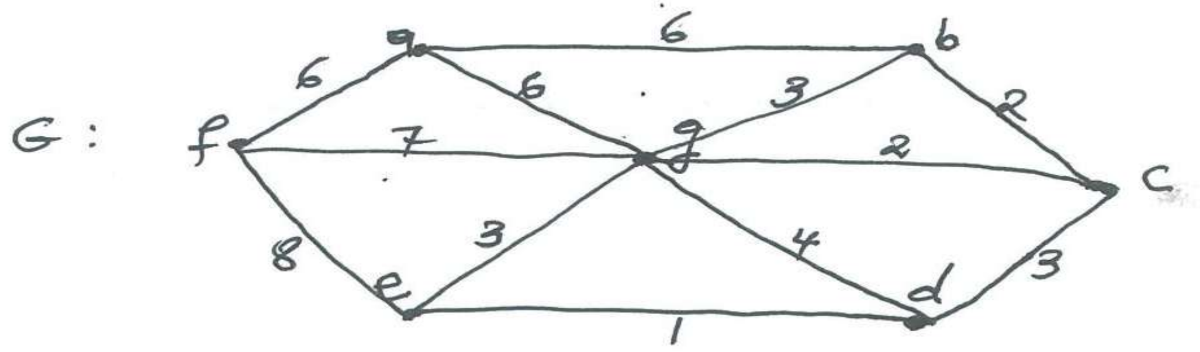
$$w(T) = 1 + 2 + 2 + 4 + 4 + 7 + 8 + 9 = 37$$

Hence, A spanning tree T of G is :

T :

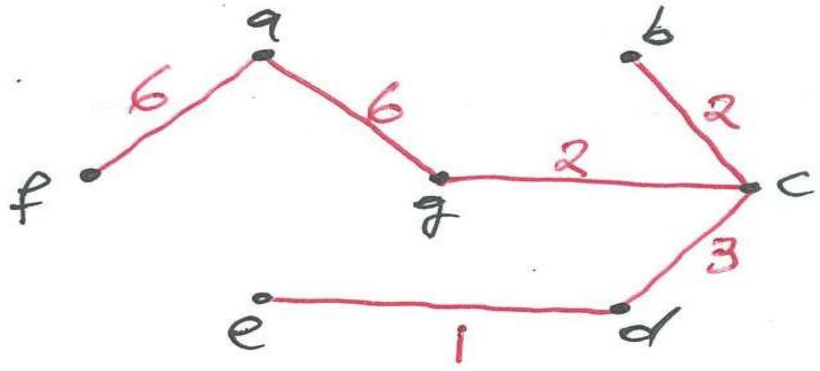


Example 2



$\omega(T) = \omega(de) + \omega(bc) + \omega(cg) + \omega(cd) + \omega(ag) + \omega(af) = 20$

Hence, a spanning tree T of G is :



Eulerian Graph

Recap:

- **Trail:** walk with no repeated edge
- **Path:** walk with no repeated vertex
- **Circuit:** Closed trail
- **Cycle:** closed path

Definitions

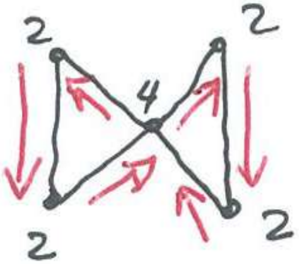
A multigraph is a graph where more than one edge is permitted to connect vertices in pairs.

A graph is said to be **semi-Eulerian** if two vertices have an odd degree.

A graph is said to be **Eulerian** if every vertex has an even degree.

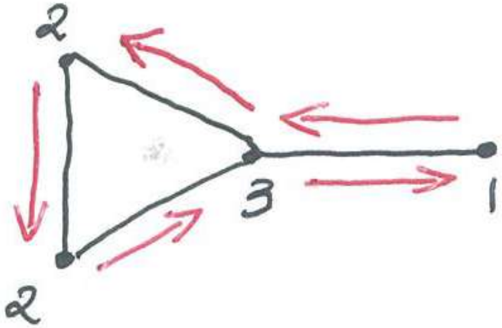
Example

1)



Eulerian graph.

2)



NOT an Eulerian graph.

FLEURY'S ALGORITHM

If G is an eulerian graph, then the following steps can always be carried out, and produce an eulerian circuit in G:

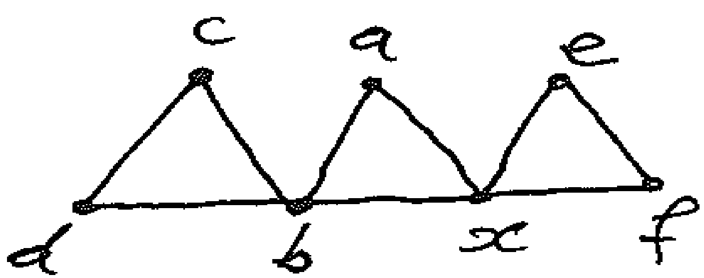
Step 1: choose a starting vertex x

Step 2: At each stage, traverse any available edge, choosing a bridge only if there is no alternative.

step 3: After traversing each edge erase it (erasing any vertices of degree 0 which result) and then choose another available edge.

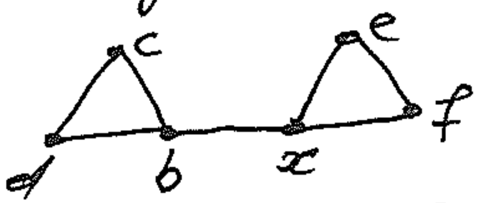
step 4: stop when there are no more edges.

Example

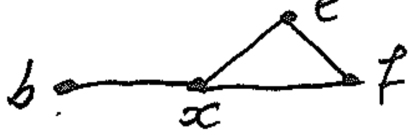


① starting at x, we may choose edge xa

② then edge ab:



③ choose edge bc followed by cd and db

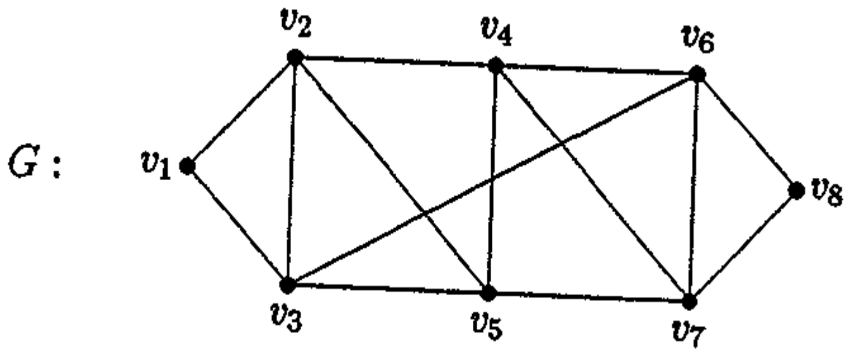


④ Now, there is no alternative, we have to traverse the bridge bx

$\therefore$ The circuit is: $x, a, b, c, d, b, x, e, f, x$.

Exercise

Using Fleury's algorithm, find an eulerian circuit in the accompanying graph G.



The chinese postman Problem.

A postman collects mail at the post office, wishes to deliver the mail along all the street in the area, and then return to the post office. He must of course cover each street in his area at least once. How can the route be planned in order to cover the smallest total distance?

Definition

For a connected weighted graph G.

1) A tour of G is a closed walk that uses each edge at least once.

2) An optimal tour is a tour of minimum weight in G.

Using the above definition, the chinese postman problem becomes:

For a connected weighted graph G, find an optimal tour in G.

- If the map of the postman's area happens to correspond to an eulerian graph, then there is no difficulty with this problem - The postman will simply chooses an eulerian circuit (using Fleury's algorithm), and such a circuit will involve the smallest total distance.
- If the graph under consideration is not an eulerian graph, then the Chinese postman problem can be solved by an algorithm provided by Edmonds and Johnson.

Algorithm.

If G is a connected weighted graph with exactly two vertices u and v of odd degree, then the following steps will produce an optimal tour in G:

Step1: Find a shortest u-v path P in G
(use Dijkstra's Algorithm);

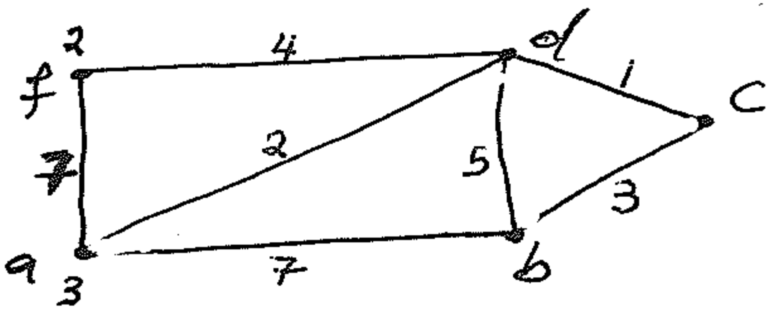
Step 2: Form a new (eulerian) graph G^* by adding to G a duplicate of each edge occuring in path P;

Step 3: Find an eulerian circuit T^* of G^* (use Fleury's algorithm);

Step 4: Interpret T^* as an (optimal) tour of G with some repeated edges.

Example

G :



$l(a)$	b	c	d	f	Removed from S	S
0	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	—	$V(G)=\{a,b,c,d,f\}$
	$(7,a)$	$(\infty,-)$	$(2,a)$	$(7,a)$	a	$\{b,c,d,f\}$
	$(7,a)$	$(3,d)$		$(6,d)$	d	$\{b,c,f\}$
	$(6,c)$			$(6,d)$	c	$\{b,f\}$
				$(6,d)$	b	$\{f\}$

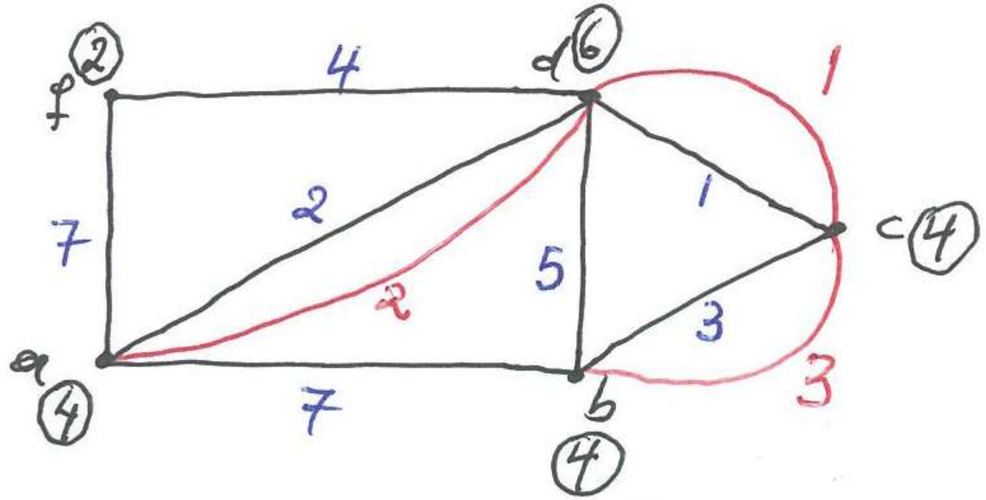
Shortest Path

v	$d(a,v)$	q_i
a	$d(a,a)=0$	
b	$d(a,b)=6$	a, d, c, b *
c	$d(a,c)=3$	a, d, c
d	$d(a,d)=2$	a, d
f	$d(a,f)=6$	a, d, f

* Since a and b have odd degree

- We find that $d(a,b) = 6$ and the shortest a-b path in G is $P: a, d, c, b$.

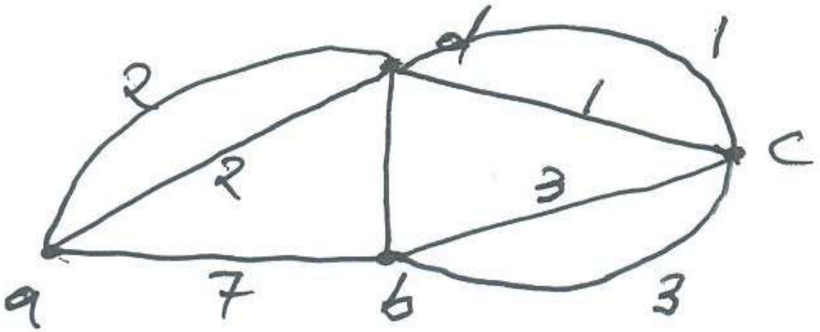
- We now form a multigraph G^* by duplicating the edges ad, dc and cd.



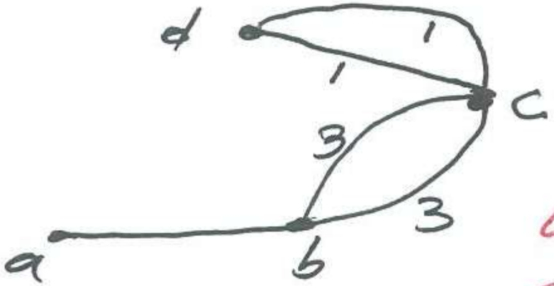
: G^*

- We now apply Fleury's algorithm to the eulerian multigraph G^*

① starting at a, we may choose the edge af followed by the edge fd. We obtain:

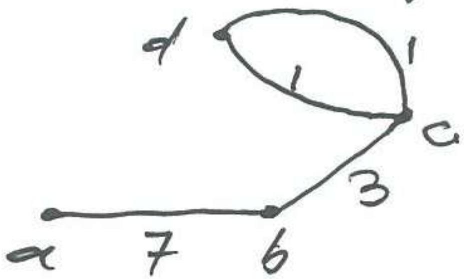


② we choose the edge da, followed by the edge ad and the edge db



we cannot choose edge ba since it is a bridge.

③ we choose edge bc (not choose cb since it is a bridge).



(4) we choose edge cd, followed by dc, cb and ba

Hence, the circuit T^* is

$a, f, d, a, d, b, c, d, c, b, a$

This is the optimal tour of G with weight

$w(T) = 35$

PLANAR GRAPHS

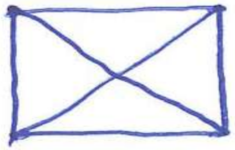
Definitions

- A planar graph is a graph that can be drawn in the plane without any edge crossings.

- A plane graph divides the plane into regions.

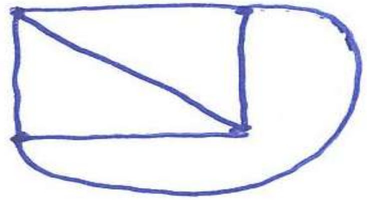
Example

K_4 :



NOT PLANE GRAPH
EDGE CROSSING

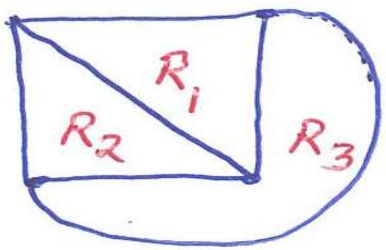
K_4 :



PLANE GRAPH
K_4 can be drawn without edge crossing.

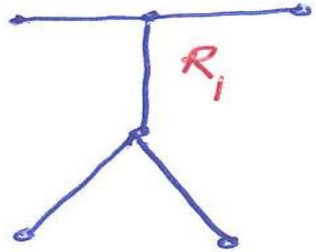
- Every plane graph contains exactly one unbounded region called the exterior region

K_4 :



R_4 = unbounded Region

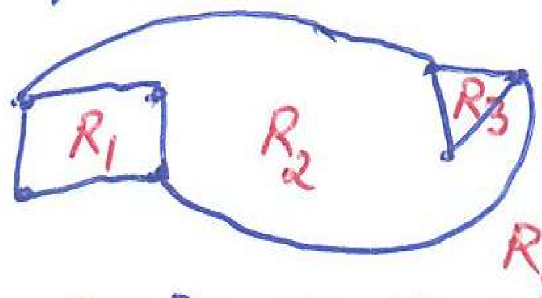
Tree :



- The boundary of a region in a plane graph is the Set of vertices and edges that outline it.

- in every connected plane graph we have $n-m+r=2$

with n the order of graph
m the Size of graph
r the region



$n=7;\ m=9,\ r=4$

$\therefore\ n-m+r=7-9+4=\underline{\underline{2}}$

Theorem (Euler's formula)

If G is a connected plane graph with n vertices, m edges and r regions. Then $n-m+r=2$.

proof

we proceed by induction on m (number of edges)
If $m=0$, then $G \cong K_1$; so, $n=1$, $r=1$ and
$n-m+r=2$. Thus the result is true if $m=0$.
Assume that the result holds for all connected
plane graph with fewer than m edges, where $m \geq 1$
and let G be a connected plane graph with
m edges. Suppose that G has n vertices and
r regions. We will consider two cases:

case 1: If G is a tree, then $r=1$ and $m=n-1$

$$\text{so, } n-m+r = n-(n-1)+1 = 2$$

Case 2: If G is not a tree, then G contains a cycle C. Let e be an edge of C, e is not a bridge. Then $G-e$ is a connected plane graph with n vertices and $m-1$ edges. Furthermore, the two regions incident with e in G produce one region in $G-e$. So, $G-e$ has $r-1$ regions. Applying the inductive hypothesis to $G-e$ we have

$$n-(m-1)+(r-1) = n-m+r$$
$$= 2.$$

$\square$

Theorem

If G is a planar graph with $n \geq 3$ vertices and m edges. Then $m \leq 3n-6$.

proof

Consider a plane graph G, resulting in r regions. Every edge lies on the boundary of either one or two regions. Therefore, if the number of edges on the boundary of a region is summed over all the region, the result is at most $2m$. However, the boundary of every region contains at least three edges, so such a sum is at least $3r$. Hence, $3r \leq 2m$ or $r \leq \frac{2m}{3}$

Applying "Euler's Theorem", we obtain

$$2 = n-m+r \leq n-m+\frac{2m}{3} = n-\frac{m}{3}$$

$\therefore\ m \leq 3n-6.$ $\square$

Theorem

The graph $K_{3,3}$ is nonplanar.

proof

Assume to the contrary, that $K_{3,3}$ is planar.
Consider any plane graph of $K_{3,3}$, resulting
in r regions. Since $K_{3,3}$ is a bipartite graph,
it has no triangles; so the boundary of every
region contains at least four edges.
let x be the number of edges on the boundary
of a region summed over all r_ regions.

Then $x \geq 4r$. However, since $K_{3,3}$ contains no bridges, every edge lies on the boundary of exactly two regions. Thus, the sum x counts each edge twice that is, $x = 2m = 18$. Therefore, $4r \leq x = 18$, equivalently, $r \leq \frac{9}{2}$. Thus, $r \leq 4$. However, by "Euler's Theorem" $r = 2 + m - n = 2 + 9 - 6 = 5$. This produce a contradiction. We deduce, therefore that $K_{3,3}$ is not a planar graph.

Theorem

The graph K_5 is nonplanar.

proof

Assume to the contrary, that K_5 is a planar.
Since K_5 has $n = 5$ vertices and $m = 10$ edges,
we have

$$10 = m > 3n - 6 = 9.$$

~~t~~this is a contradiction since $m \leq 3n - 6$.

Thus, K_3 is nonplanar

$$e = \frac{n(n-1)}{2} = \frac{5(5-1)}{2} = 5 \times 2 = 10 \text{ edges}$$

Theorem

Every planar graph contains a vertex of degree at most 5.

proof

Let G be a planar graph with n vertices and m edges, and $V(G) = \{v_1, v_2, \ldots, v_n\}$. If $n \leq 6$, then the degree of every vertex is at most 5. So we may assume that $n \geq 7$. Since $m \leq 3n-6$.

Therefore,

$$\sum_{i=1}^{n} \deg v_i = 2m \leq 6n - 12.$$

If all the vertices of G have degree 6 or more,

then

$$2m = \sum_{i=1}^{n} \deg v_i \geq 6n$$

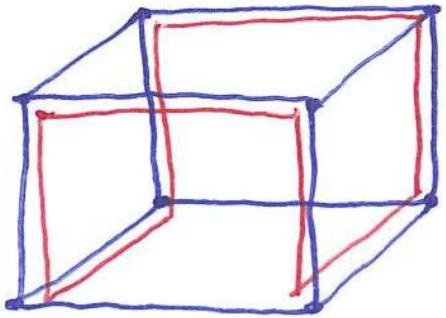
which produced a contradiction. Hence, G must contain a vertex of degree 5 or less. $\square$

HAMILTONIAN GRAPHS

Definitions

- Hamiltonian Path : is a path of graph containing every vertex of graph.
- Hamiltonian cycle : is a cycle of a graph G containing every vertex of G.
- Hamiltonian graph : is a graph possessing a hamiltonian cycle.

Example



Hamiltonian cycle.

Theorem (Dirac's Theorem)

Let G be a graph of order $n \geq 3$.
If $\delta(G) \geq \frac{n}{2}$. Then G is hamiltonian.

[Recall $\delta(G)$ denote the minimum degree in a graph G.]

Corollary

Let G be a graph of order n.
If $\delta(G) \geq \frac{n-1}{2}$, then G contains a hamiltonian path

Proof

If $n=1$, then $G \cong K_1$ and G contains a (trivial) hamiltonian path.
Assume that $n \geq 2$ and let H be a graph obtained

from G by adding a new vertex v and joining v with an edge to every vertex of G.
Then H has order $n+1$, so v has degree n in H.
For $u \in V(G)$:

$$\deg_H u = \deg_G u + 1 \geq \frac{n-1}{2} + 1 = \frac{n+1}{2} = \frac{|V(H)|}{2}$$

Hence, H is a graph of order $n+1$ with $\delta(H) \geq \frac{n+1}{2}$.
By Dirac's theorem, H contains a hamiltonian cycle.
By removing the vertex v from C, we obtain a hamiltonian path in G. $\square$

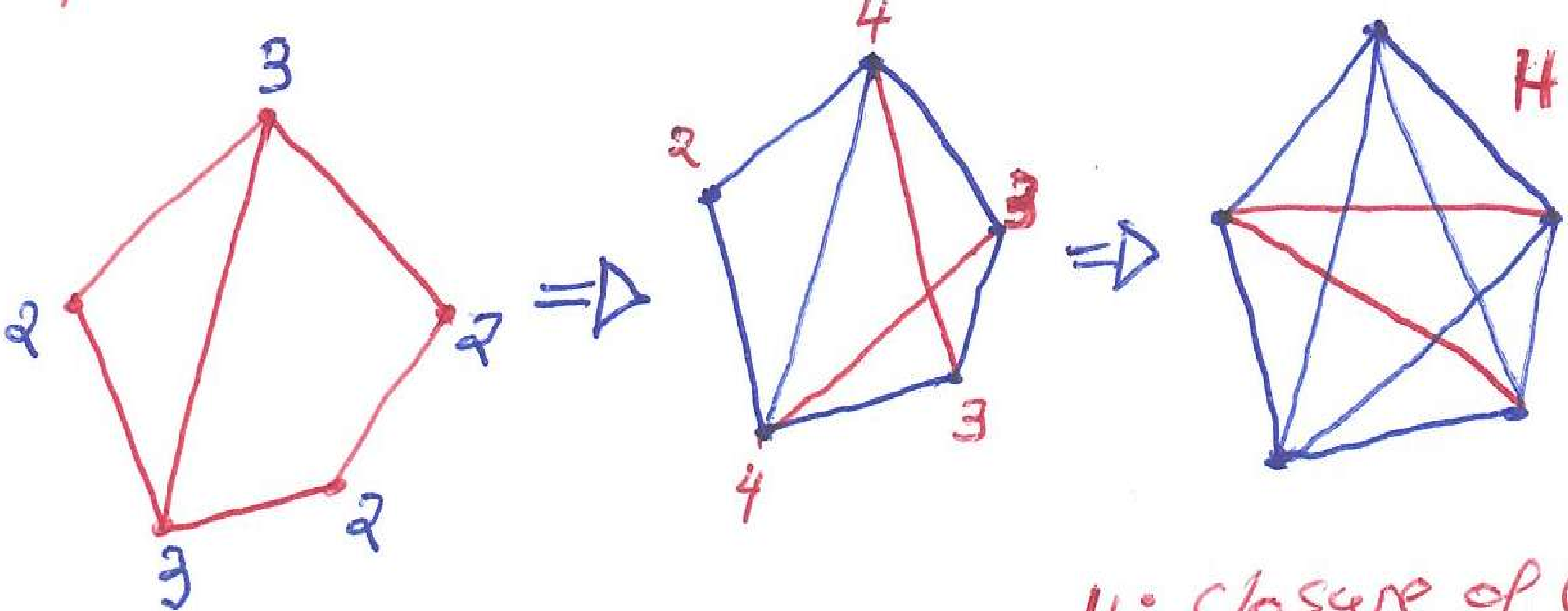
Definition

The closure of a graph G of order n denoted by $C(G)$ is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n. (in the resulting graph at each stage) until no such pair remains.

This mean that: if $d(u) + d(v) \geq n$, then add edge uv

Example

G :



$n=5$

H: Closure of G

Theorem

If G_1 and G_2 are two graph obtained from a graph G of order $n \geq 3$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least n, then $G_1 = G_2$.

Theorem

A graph is hamiltonian if and only if its closure is hamiltonian.

Corollary

Let G be a graph of order $n \geq 3$. If $C(G) \cong K_n$

Then G is hamiltonian.

Corollary

If $\deg u + \deg v \geq n$, $\forall u, v \in V(G)$, nonadjacent

of order $n \geq 3$, then G is hamiltonian.

Corollary

Let G be a graph of order $n \geq 3$ and size m.

If

$$m \geq \binom{n-1}{2} + 2.$$

Then G is hamiltonian.

Proof

If G is complete, then G is hamiltonian.
Assume that G is not complete and that G
satisfies the hypothesis of the corollary.
Let u and v nonadjacent $V(G)$, $u \neq v$ and

define $H = G - \{u, v\}$.
Then, $m(G) = m(H) + \deg(u) + \deg(v)$.
Since $m(H) \leq m(K_{n-2}) = \binom{n-2}{2}$. It follows that

$$\deg u + \deg v = m(G) - m(H)$$

$$\geq \binom{n-1}{2} + 2 - \binom{n-2}{2}$$

$$= \frac{(n-1)!}{2!(n-1-2)!} + 2 - \frac{(n-2)!}{2!(n-2-2)!}$$

$$= \frac{(n-1)!}{2!(n-3)!} + \frac{4}{2} - \frac{(n-2)!}{2!(n-4)!}$$

$$= \frac{(n-1)(n-2)\cancel{(n-3)!}}{2 \times \cancel{(n-3)!}} + \frac{4}{2} - \frac{(n-2)(n-3)\cancel{(n-4)!}}{2 \times \cancel{(n-4)!}}$$

$$= \frac{\cancel{n^2} - 3n + \cancel{2} + \cancel{4} - \cancel{n^2} + 5n - \cancel{6}}{2}$$

$$= \frac{2n}{2}$$

$$= n$$

$\therefore \deg u + \deg v \geq n$

Hence, G is hamiltonian. $\Box$