Connectivity

Graph Theory

Maximum and minimum degree of a graph

Let us recall the following definitions:

Definition 1.

- The maximum degree of a graph G, denoted by $\triangle(G)$, is defined to be $\triangle(G) = max\{d(v) : V(G)\}$.
- Similarly, the minimum degree of a graph G, denoted by $\delta(G)$, is defined to be $\delta(G) = mind\{(v) : V(G)\}$. Note that for any vertex v in G, we have $\delta(G) \leq d(v) \leq \triangle(G)$.

Theorem 2.

If a graph G with minimum degree $\delta(G) \geq 2$, then G contains a cycle.

Proof.

Suppose G is a graph G with minimum degree $\delta(G) \geq 2$. Let $P = (v_0, ..., v_i, ..., v_{n-1}, v_n)$ be the longest path in G. Consider the degree of the last vertex v_n. From the hypothesis we know that the degree v_n is is at least 2 $(d(v_n) \leq 2)$. We know that one the v_n's adjacent vertex is v_{n-1} because consecutive vertices in a path are adjacent. Since v_n has at least 2 adjacent vertices, it must be that it has another adjacent vertex besides v_{n-1}, say w. If w does not lie on P, then we can extend P by going from v_n to w. □

Proof Continuation

Proof.

This would mean that this path is longer than P and thus contradicts P being the longest path. So w cannot be the other adjacent vertex. This means that the other adjacent vertex of v_n must be in the path P. Among the other neighbours of v_n that are on the path, let's pick one of them and call it v_i. Then we can go from v_n to v_i and proceed along the path and back to v_n. Then $(v_n, v_i, v_{i+1}, ..., v_{n-1}, v_n)$ is a cycle. $\square$

Recall some definitions

Definition 3.

The **complement** or inverse of a graph G, denoted by $\overline{G}$ is a graph with $V(G) = V(\overline{G})$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

Note that for a graph G and its complement $\overline{G}$, we have:

- $G \cup \overline{G} = K_n$;
- $V(G) = V(\overline{G})$;
- $E(G) \cup E(\overline{G}) = E(K_n)$;

Theorem 4.

If a graph is bipartite with minimum degree $\delta \leq 1$, then G contains a path of order $2\delta(G)$

Theorem 5.

A graph G and its complement cannot both be disconnected.

Proof.

Let G be a graph and its complement $\overline{G}$.
Case 1: Let G be a disconnected graph ($V(G) = V(\overline{G})$). Choose distinct vertices $u, v \in V(G)$. Since G is disconnected it has at least 2 components.
Suppose u and v are in different components.

□

Proof Continuation

Proof.

Since u and v are in different components it must be that u and v are not adjacent to each other, $uv \notin E(G)$. This means that $uv \in E(\overline{G})$ by definition of complement. Therefore, u and v are connect in $\overline{G}$.
Now suppose that u and v are in the same component of G.
There exists a vertex $w \in V(G)$ such that $uw \notin E(G)$ and $vw \notin E(G)$. So $uw \in E(\overline{G})$ and $vw \in E(\overline{G})$. Thus uwv is a uv path in $\overline{G}$. So by definition of connectedness , $\overline{G}$ is connected.
Case 2: Similarly if $\overline{G}$ is disconnected then G is connected. **i.e** both G and $\overline{G}$ cannot be disconnected.

□