SOME SOLVE PROBLEMS

(1) Consider the complex series

$$\sum_{k=1}^{\infty} \frac{\sin kz}{k^2},$$

show that it is absolutely convergent when z is real but it becomes divergent when z is non-real.

Solution

(a) When z is real, we have

$$\left|\frac{\sin kz}{k^2}\right| \leq \frac{1}{k^2} \quad \forall k>0,\ k\in \mathbb{Z}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is known to be convergent, then $\sum_{k=1}^{\infty} \frac{\sin kz}{k^2}$ is absolutely convergent $\forall z$ by virtue of the comparison test.

(b) When z is non-real, we let $z = x+iy$, $y\neq 0$

Hence, $\frac{\sin kz}{k^2} = \frac{e^{-ky}e^{ikx} - e^{ky}e^{-ikx}}{2k^2 i}$,

we deduce that

$$\left|\frac{\sin kz}{k^2}\right| \geqslant \frac{e^{k|y|} - e^{-k|y|}}{2k^2} \longrightarrow \infty \text{ as } k\to\infty$$

Since $\left|\frac{\sin kz}{k^2}\right|$ is unbounded as $k\to\infty$, the series is divergent.

② Find the circle of convergence for each of the following power series:

(a) $\sum_{k=1}^{\infty} \frac{1}{k}(z-i)^k$

Solution

By the ratio test, we have

$$R = \lim_{k\to\infty} \frac{1/k}{1/(k+1)} = 1$$

$\therefore$ the circle of conv. is $|z-i| = 1$

(b) $\sum_{k=1}^{\infty} k^{\ln k}(z-2)^k$

using the root test, we have

$$\frac{1}{R} = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{k^{\ln k}}$$

To evaluate the limit, we consider the logarithm,

$$\ln\left(\lim_{k\to\infty} \sqrt[k]{k^{\ln k}}\right) = \lim_{k\to\infty} \ln\left(\sqrt[k]{k^{\ln k}}\right) = \lim_{k\to\infty} \frac{(\ln k)^2}{k} = 0$$

So, $\frac{1}{R} = \lim_{k\to\infty} \sqrt[k]{k^{\ln k}} = e^0 = 1$

$\therefore$ The circle of conv. is $|z-2| = 1$.

(c) $\sum_{k=1}^{\infty} \left(\frac{z}{k}\right)^k$

Solution

By the ratio Test, we have

$$R = \lim_{k\to\infty} \frac{\left(1/k\right)^k}{\left(1/k+1\right)^{k+1}} = \lim_{k\to\infty} (k+1)\left(1+\frac{1}{k}\right)^k = \infty$$

So the circle of Conv. is the whole complex plane.

(d) $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k^2} \cdot z^k$

Solution

By the root test, we have

$$\frac{1}{R} = \lim_{k\to\infty} \sqrt[k]{\left(1+\frac{1}{k}\right)^{k^2}} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e$$

$\therefore$ the circle of Conv. is $|z| = \frac{1}{e}$.

(3) Find the Laurent expansion of

$$f(z) = \sin\left(z - \frac{1}{z}\right)$$

The function has singularity at $z=0$ so that the annulus of Conv. is $|z|>0$. The Laurent Coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \oint_C \frac{\sin\left(z-\frac{1}{z}\right)}{z^{n+1}}\, dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

where C is chosen to be the unit circle $|z|=1$

Take $z=e^{i\theta}$ so that $dz=ie^{i\theta}d\theta$, then

$$C_n=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\sin(2i\sin\theta)}{e^{i(n+1)\theta}}\,d\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} i\sinh(2\sin\theta)(\cos n\theta-i\sin n\theta)\,d\theta.$$

Note that $\sinh(2\sin\theta)$ is an odd function in θ, hence,

$$\int_{-\pi}^{\pi}\sinh(2\sin\theta)\cos n\theta\,d\theta=0.$$

Finally,

$$C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh(2\sin\theta)\sin n\theta\,d\theta,\quad n=1,\pm1,\pm2,\ldots$$

(4) Find the Laurent Expansion of

$$f(z)=\frac{1}{z-k}$$

That is valid the domain $|z|>k$, where k is real and $|k|<1$. Using the Laurent expansion, deduce that

$$\sum_{k=1}^{\infty}k^n\cos n\theta=\frac{k\cos\theta-k^2}{1-2k\cos\theta+k^2},$$

$$\sum_{k=1}^{\infty}k^n\sin n\theta=\frac{k\sin\theta}{1-2k\cos\theta+k^2}$$

Solution

For $|z| > k$, where $|k| < 1$, we have

$$\frac{1}{z-k} = \frac{1}{z\left(1-\frac{k}{z}\right)}$$

$$= \frac{1}{z}\left(1 + \frac{k}{z} + \frac{k^2}{z^2} + \cdots\right) \text{ for } \left|\frac{k}{z}\right| < 1$$

Take the point $z = e^{i\theta}$, which lies inside the region of conv. of the above Laurent series, Substituting into infinite series

$$\frac{1}{e^{i\theta}-k} = \frac{1}{e^{i\theta}}\left(1 + ke^{-i\theta} + k^2e^{-2i\theta} + \cdots + k^n e^{-in\theta} + \cdots\right)$$

Rearranging the terms

$$\frac{e^{i\theta}(e^{-i\theta}-k)}{(e^{i\theta}-k)(e^{-i\theta}-k)} = \frac{1-k(\cos\theta + i\sin\theta) - (1-2k\cos\theta + k^2)}{1-2k\cos\theta+k^2}$$

$$= \frac{k\cos\theta - k^2 - ik\sin\theta}{1-2k\cos\theta+k^2}$$

$$= \sum_{n=1}^{\infty} k^n \cos n\theta - i\sum_{n=1}^{\infty} k^n \sin n\theta$$

Equating the real and imaginary parts, we obtain the desired result.

(5) By evaluating the contour integral

$$\oint_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z},$$

show that

$$\int_0^{2\pi} \cos^{2n}(\theta)\, d\theta = 2\pi \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$$

Solution

on $|z|=1$, $z=e^{i\theta}$; $\left(z+\frac{1}{z}\right)^{2n} = 2^{2n}\cos^{2n}$; $\frac{dz}{z} = i\,d\theta$

$$I = \oint_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z} = 2^{2n} i \int_0^{2\pi} \cos^{2n}(\theta)\, d\theta$$

on the other hand, the integrand can be expanded as

$$\frac{1}{z}\left(z^{2n} + 2nC_1 z^{2n-2} + \cdots + 2nC_n + \cdots + 2nC_{2n}\frac{1}{z^{2n}}\right), \quad 0<|z|<\infty$$

Since the integrand is analytic everywhere except at $z=0$, the above expansion is the Laurent Series of the integrand valid in the annulus: $|z|>0$.

$$\oint_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z} = 2\pi i \left(\text{Coefficient of } \frac{1}{z} \text{ in the Laurent Series}\right)$$

$$= 2\pi i\, {}_{2n}C_n.$$

Hence,

$$\int_0^{2\pi} \cos^{2n}(\theta)\, d\theta = 2\pi \frac{{}_{2n}C_n}{2^{2n}}$$

$$= 2\pi \frac{(2n)!}{(n!)^2 2^{2n}}$$

$$= 2\pi \frac{1\cdot 3\cdot 5, \ldots (2n-1)}{2\cdot 4\cdot 6 \ldots (2n)}$$

(6) For each of the following Complex functions, do the following:

a) find all its singularities in $\mathbb{C}$

b) write the principal part of the function at each singularity;

c) for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity

d) compute the residue of the function at each singularity.

6.1) $f(z) = \dfrac{1}{(\cos z)^2}$

Solution

$f(z)$ is singular at $\cos z = 0$, i.e., $z = n\pi + \frac{\pi}{2}$

Let $w = z - n\pi - \frac{\pi}{2}$

Then

$$\frac{1}{(\cos z)^2} = \frac{1}{\left(\cos\left(w + n\pi + \frac{\pi}{2}\right)\right)^2} = \frac{1}{(\sin w)^2}$$

Since $\sin w$ has a zero of multiplicity one at $w = 0$, $f(z)$ has a pole of order 2 at $z = n\pi + \frac{\pi}{2}$. So

$$\frac{1}{(\sin w)^2} = \frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + \sum_{n\geq 0} a_n w^n.$$

Since

$$(\sin w)^2 = \left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!}\right) = w^2 + \sum_{n=4}^{\infty} b_n w^n$$

we have

$$1 = \left(\frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + \sum_{n\geq 0} a_n w^n\right)\left(w^2 + \sum_{n=4}^{\infty} b_n w^n\right).$$

Comparing the coefficients of 1 and w on both sides, we obtain

$a_{-2} = 1$

$a_{-1} = 0$

So the principal part of $f(z)$ at $z = n\pi + \frac{\pi}{2}$ is

$$\frac{1}{\left(z - n\pi - \frac{\pi}{2}\right)^2}$$

with residue 0.

Given $f(z) = \dfrac{\cos z}{z^2 - z^3}$

a) Find the principal part of $f(z)$

b) Find the Residue of $f(z)$

Solution

The function has singularities at $\{z^2 - z^3 = 0\}$
$\Rightarrow \{z^2(1-z) = 0\} \Rightarrow z = 0$ (pole of order 2)
$z = 1$ (Single pole)

Suppose that the Laurent series of $f(z)$ at $z = 0$ is given by

$$\frac{\cos z}{z^2 - z^3} = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n$$

$$\cos z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-2}}{(2n-2)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - z^2/2! + z^4/4! - \cdots$$

Hence,

$$(z^2 - z^3)\left(\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n\right) = \cos z = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

comparing the coefficient of 1 and z both sides, we obtain that:

$a_{-2} = 1$ and $a_{-1} - a_{-2} = 0$. Hence $a_{-1} = a_{-2}$

$\therefore$ the principal part of $f(z)$ at $z = 0$ is

$$\frac{1}{z^2} + \frac{1}{z}.$$

(b) $\underset{z=0}{\operatorname{Res}} f(z) = 1$

$$\underset{z=0}{\operatorname{Res}} f(z) = \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} \left\{ z^2 \frac{\cos z}{z^2 - z^3} \right\}$$

$$= \lim_{z \to 0} \frac{d}{dz} \left\{ \frac{\cos z}{1-z} \right\}$$

$$= \lim_{z \to 0} \frac{-\sin z(1-z) + \cos z(1-z)}{(1-z)^2} = 1$$

(*) $z = 1$ (Single pole)

$$a_{-1} = \underset{z=1}{\operatorname{Res}} f(z) = \lim_{z \to 1} \frac{(z-1)\cos z}{z^2(1-z)} = -\lim_{z \to 1} \frac{\cos z}{z^2} = -\cos($$

$$= -\cos(1)$$

$$\frac{\cos z}{z^2 - z^3} = \frac{a_{-1}}{(z-1)} + \sum_{n \geq 0} a_n z^n$$

Hence, the principal part of $f(z)$ at $z=1$ is $-\dfrac{\cos(1)}{z-1}$.