GRAPH THEORY SMTH312

By

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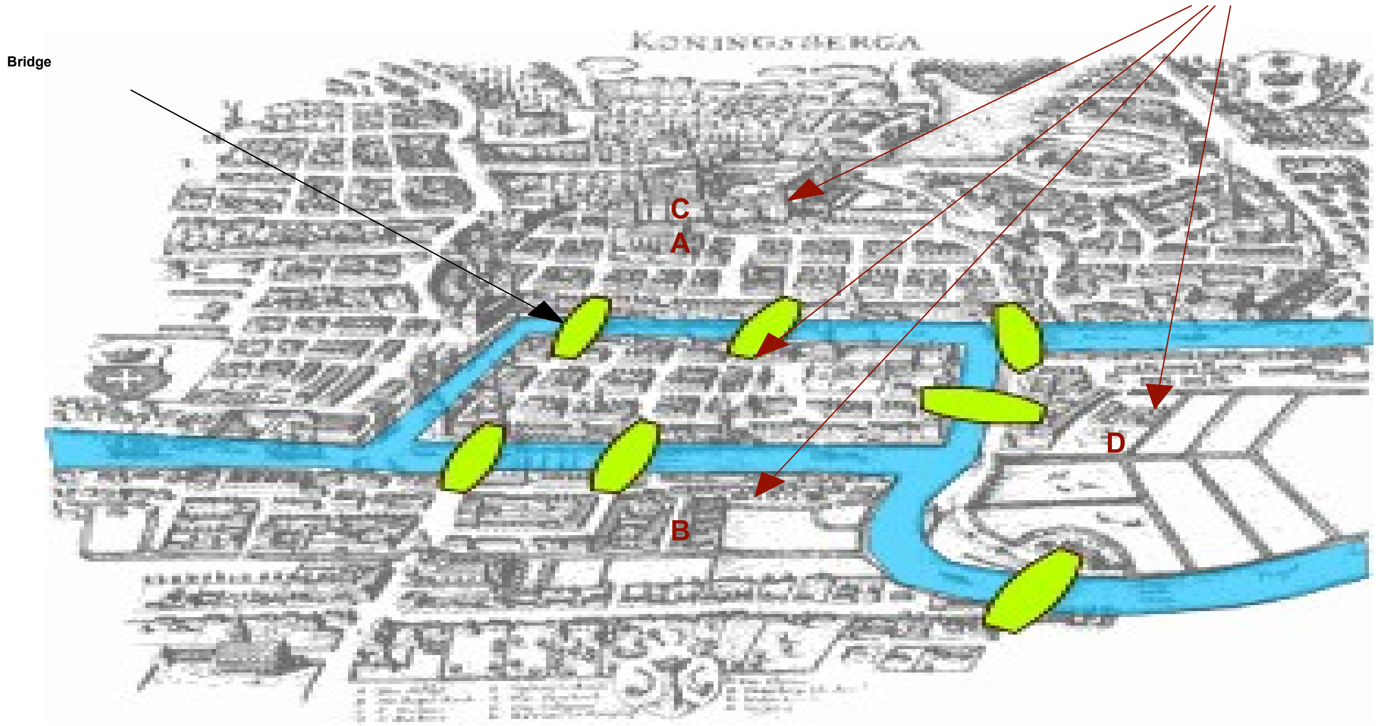
Graph Theory began with **Leonhard Euler** in his study of the Bridges of Königsburg problem. The city of Königsburg exists as a collection of islands connected by bridges as shown in Figure 1.1. The problem Euler wanted to analyze was: Is it possible to go from



An Introduction to Graphs

Some Definitions and Theorems

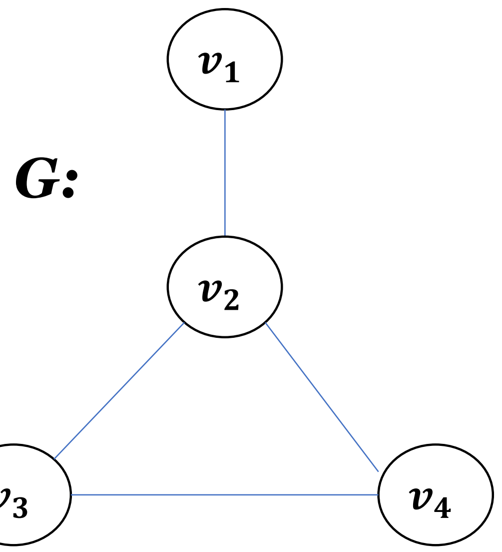
Definition 1.1 (Graph). A graph is a tuple $G = (V, E)$ where V is a (finite) set of vertices and E is a finite collection of edges. The set E contains elements from the union of the one and two element subsets of V. That is, each edge is either a one or two element subset of V. **Vertices is represented by dots and edges by line (curve)**

V(G) is the set of vertices of graph G

E(G) is the set of edges of graph G

$$V(G) = \{v_1, v_2, v_3, v_4\}$$

$$\mathrm{E}(G) = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}$$

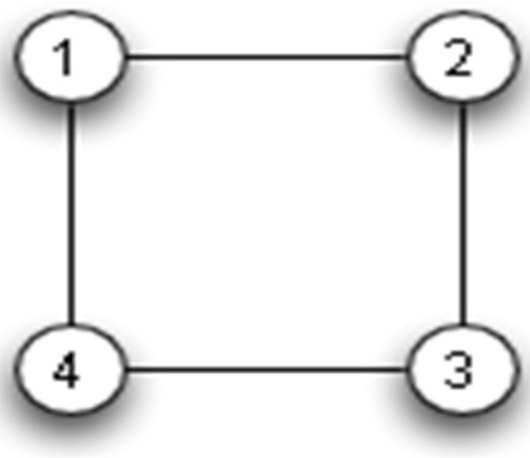


Example 1.7. Consider the set of vertices $V = \{1, 2, 3, 4\}$. The set of edges

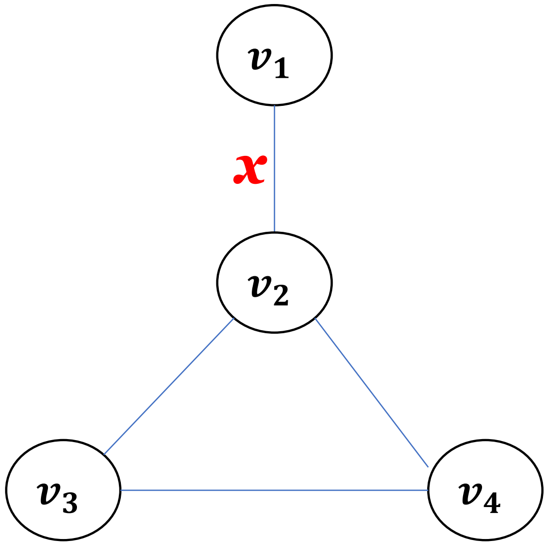
$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$$

Then the graph $G = (V, E)$ has four vertices and four edges. It is usually easier to represent this graphically (as we did for the Bridges of Königsberg Problem). See Figure 2.1 for the visual representation of G. These visualizations are constructed by representing each

G:



G:



Definition 1.2 (**Incident and Adjacent**).

- **Edge** $\boldsymbol{x}$ ***in incident to*** $\boldsymbol{v_1}$ ***and*** $\boldsymbol{v_2}$
- ***Vertex*** $\boldsymbol{v_1}$ ***is adjacent to*** $\boldsymbol{v_2}$
- ***Vertex*** $\boldsymbol{v_3}$ ***is adjacent to*** $\boldsymbol{v_2}$ ***and*** $\boldsymbol{v_4}$

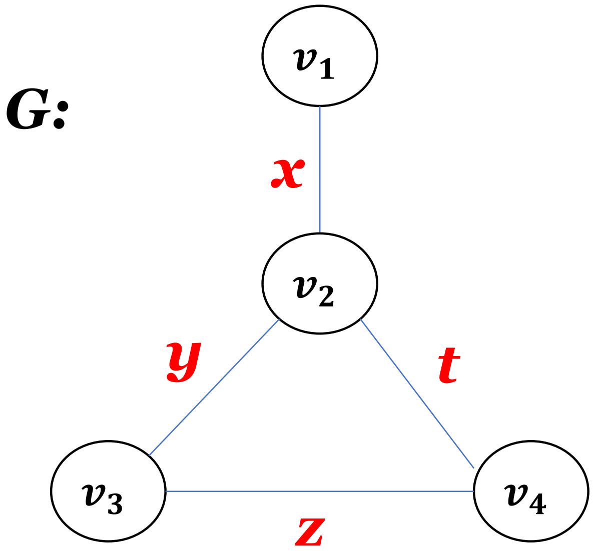
Definition 1.3 (Vertex Adjacency). Let $G = (V, E)$ be a graph. Two vertices v_1 and v_2 are said to be *adjacent* if there exists an edge $e \in E$ so that $e = \{v_1, v_2\}$. A vertex v is self-adjacent if $e = \{v\}$ is an element of E.

Definition 1.4 (Edge Adjacency). Let $G = (V, E)$ be a graph. Two edges e_1 and e_2 are said to be *adjacent* if there exists a vertex v so that v is an element of both v_1 and v_2 (as sets). An edge e is said to be *adjacent* to a vertex v if v is an element of e as a set.

Definition 1.5 (Neighborhood). Let $G = (V, E)$ be a graph and let $v \in V$. The *neighbors* of v are the set of vertices that are adjacent to v.
Formally: (2.1) $N(v) = \{u \in V : \exists e \in E\ (e = \{u, v\} \text{ or } u = v \text{ and } e = \{v\})\}$

$N(v)$ is the set of vertices u in (the set) V such that there exists an edge e in (the set) E so that $e = \{u, v\}$ or $u = v$ and $e = \{v\}$.

Definition 1.8 (**Undirected graph**).

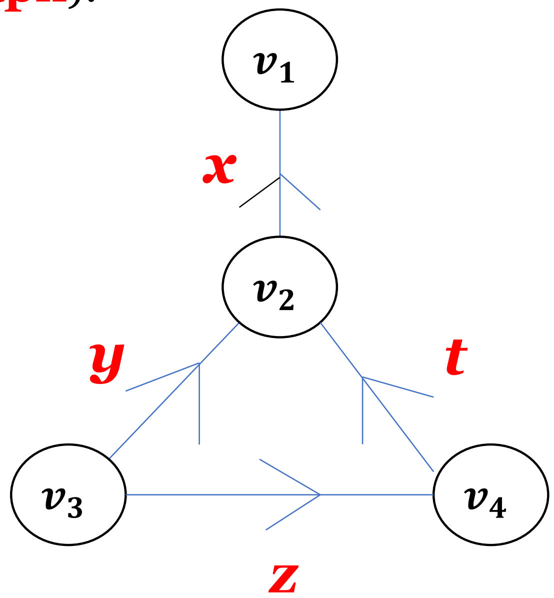


$x = \{v_1, v_2\} = \{v_2, v_1\}$

$OR\ x = v_1 v_2 = v_2 v_1$

Definition 1.8 (**Directed graph**).

_G_:



$x = \{v_1, v_2\} \neq \{v_2, v_1\}$

$OR\ x = v_1 v_2 \neq v_2 v_1$

$n(G)$is the cardinality of the V(G) or **order of a graph G** and *m(G)* is the cardinality of E(G) or **size of a graph G**

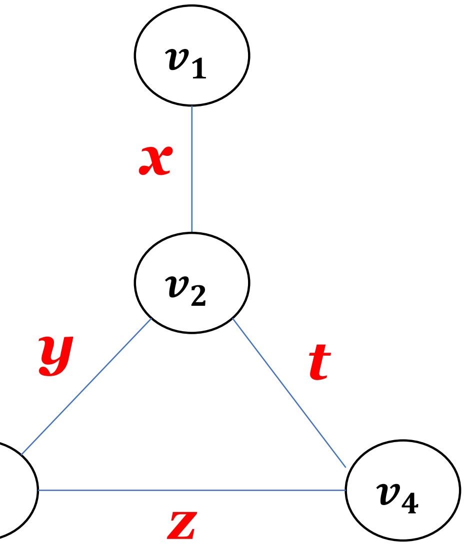
$\boldsymbol{n(G) = 4}$
$\boldsymbol{m(G) = 4}$

Definition 1.8 (**Trivial graph**). A trivial graph is a graph with order $\boldsymbol{n(G) = 1}$

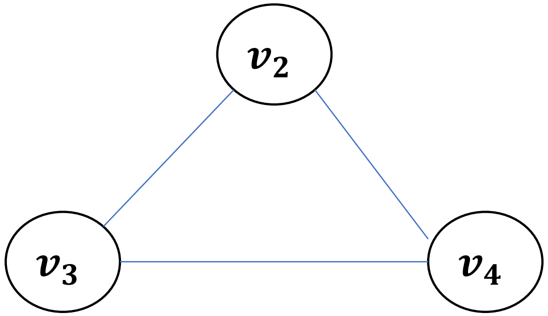
Definition 1.8 (**Subgraph**). A subgraph is a graph that exist with another graph.

- **A subgraph is obtained either by vertex deletion or edge deletion**
- **Doing vertex deletion will result by deleting all edges incident to it**
- **Doing edge deletion does not required to delete any vertex**

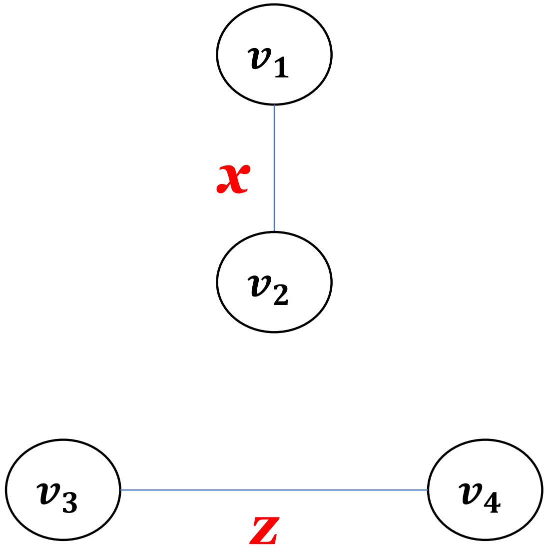
$\boldsymbol{G}$**:**



$\boldsymbol{G \setminus v_1}$ **:**



$G \setminus \{y, t\}$:

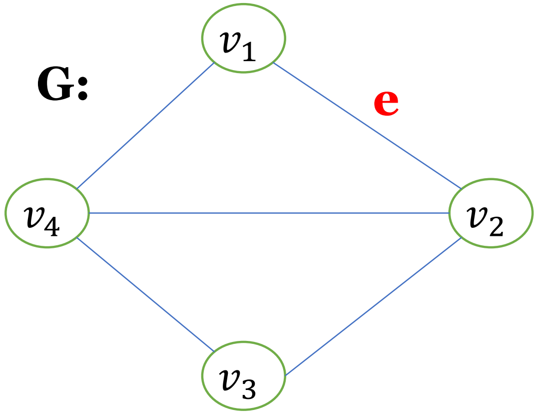


$A\ graph\ F\ is\ a\ subgraph\ of\ G\ (F \sqsubset G)\ if\ \mathbf{V(F)} \sqsubseteq V(G)\ and\ \mathbf{E(F)} \sqsubseteq E(G)$

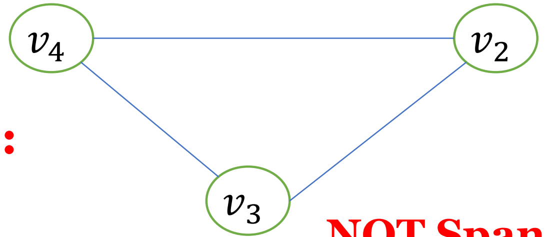
Definition 1.8 (**Spanning subgraph**). A spanning subgraph is a subgraph that contains all the vertices of the original graph. In other words a spanning subgraph is a subgraph obtained only by edge deletion

Example

G:

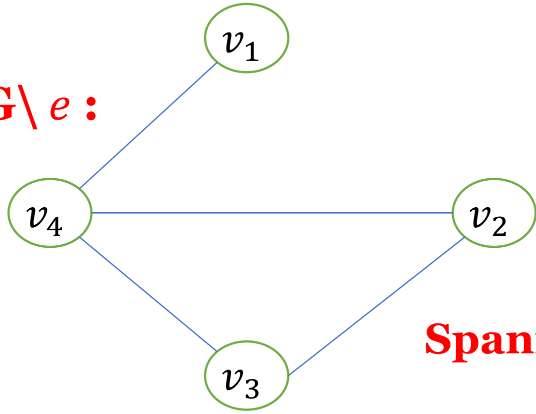


G\ v_1 :



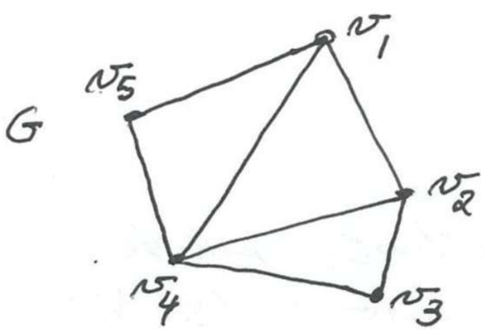
NOT Spanning subgraph

G\ e :



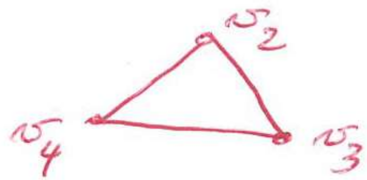
Spanning subgraph

Definition 1.8 (**Induced subgraph**). A subgraph obtained only by vertex deletion is called an induced subgraph



Let $X = \{v_1, v_5\}$

$G - X$:



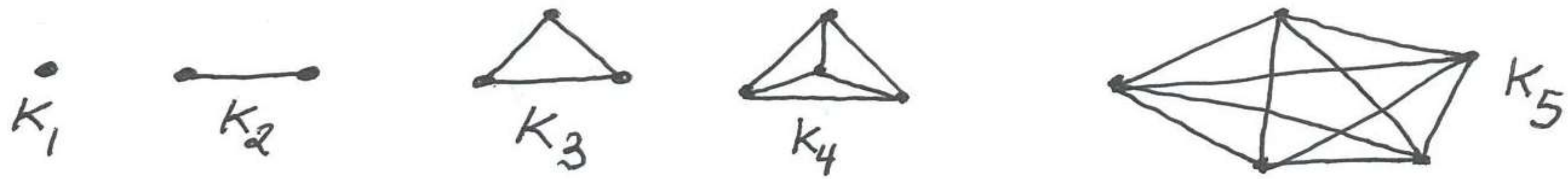
NOTATION

$G[X]$ = The graph induced by the vertices in X.

Examples of Graphs

1) Complete graphs

A complete graph or clique is a graph in which every two distinct vertices are adjacent. The complete graph of order n is denoted by K_n and is called an n-clique.



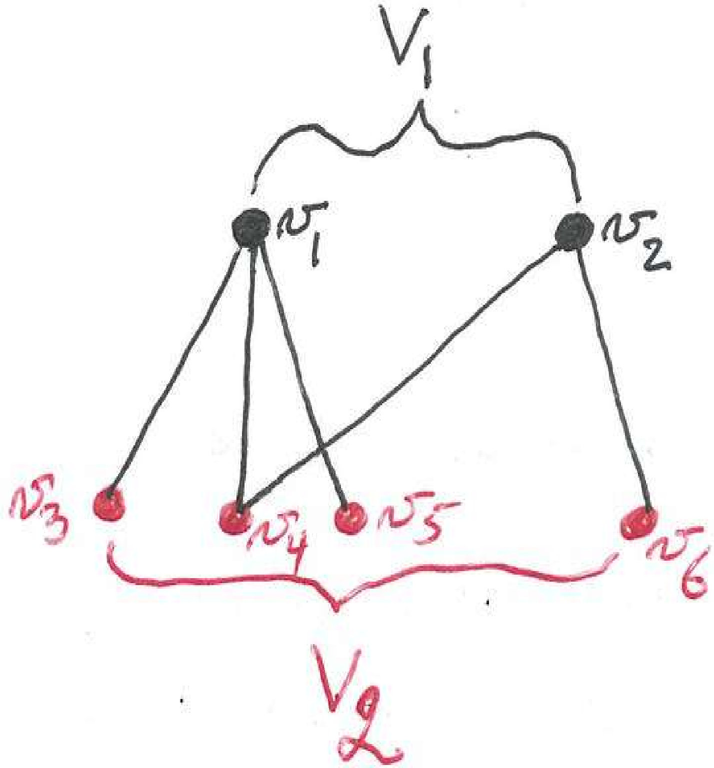
NOTA: The edge of K_n is $e = \frac{n(n-1)}{2}$

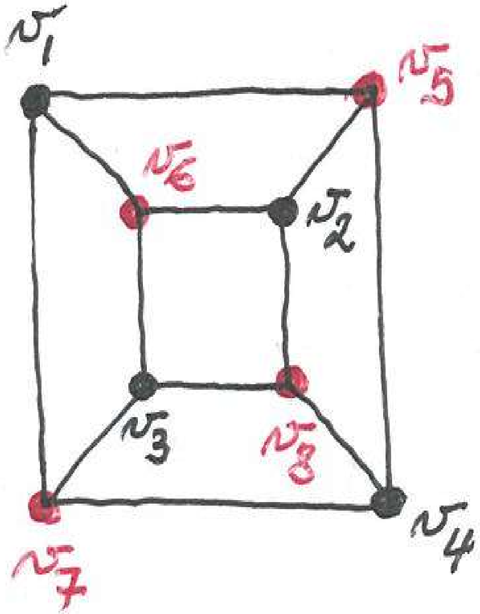
2) Empty graphs

The empty graph is a graph containing no edges.
The empty graph of order n is denoted by $\overline{K}_n$

3) Bipartite graphs

A bipartite graph is a graph whose vertex set can be partitioned into two sets V_1 and V_2 in such a way that each edge of the graph joins a vertex of V_1 to a vertex of V_2.





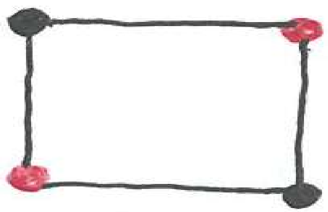
$V_1 = \{ v_1, v_2, v_3, v_4 \}$

$V_2 = \{ v_5, v_6, v_7, v_8 \}$

4) Complete bipartite graphs

A complete bipartite graph is a bipartite graph with partite set V_1 and V_2 having the added property that every vertex V_1 is adjacent to every vertex of V_2. If $|V_1| = r$ and $|V_2| = s$, then this graph is denoted by $K_{r,s}$. A complete bipartite graph of the form $K_{1,s}$ is called a star graph. A complete bipartite graph of the form $K_{n,n}$ is called n-biclique.

$K_{1,4}$



$K_{2,2}$

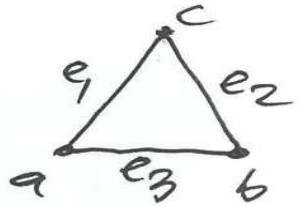
operations on graphs

1) The intersection

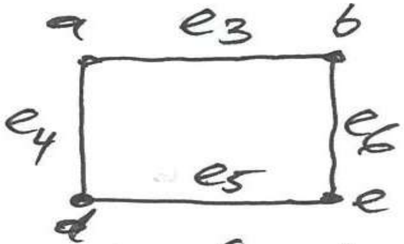
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

Then $G_3 = G_1 \cap G_2 = (V_3, E_3)$ where $V_3 = V_1 \cap V_2$ and $E_3 = E_1 \cap E_2$.

$G_1 =$



G_2:



$V_1 = \{a, b, c\}$

$E_1 = \{e_1, e_2, e_3\}$

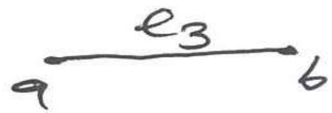
$V_2 = \{a, b, d, e\}$

$E_2 = \{e_3, e_4, e_5, e_6\}$

$V_3 = V_1 \cap V_2 = \{a, b\}$

$E_3 = E_1 \cap E_2 = \{e_3\}$

$G_3 = G_1 \cap G_2$:

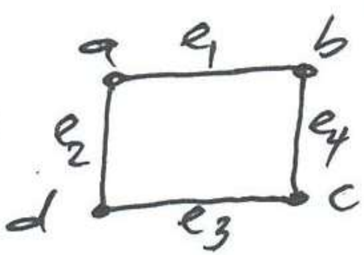


2) Union

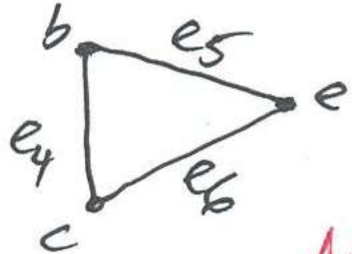
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

Then $G_3 = G_1 \cup G_2 = (V_3, E_3)$ where $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$

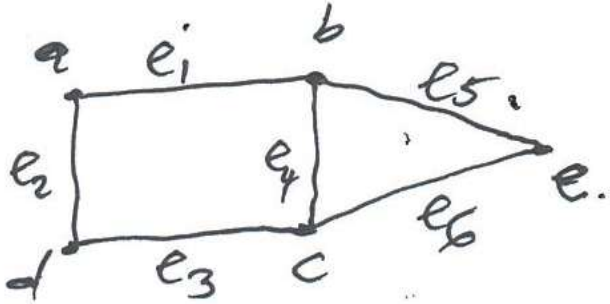
G_1:



G_2:



$G_1 \cup G_2$:



NOTA

If G_1 has edge e_1 and G_2 has edge e_2

Then $G_1 \cup G_2$ has edge $e_1 + e_2$.

3) The join graph

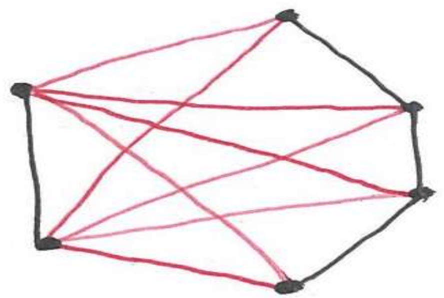
Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$. The join graph $G_3 = G_1+G_2 = (V_3, E_3)$ has $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2 \cup \{uv \mid u \in V_1 \text{ and } v \in V_2\}$.

Example

G_1 :

G_2 :

$G_3 = G_1 + G_2$:



4) The Cartesian Product.

Recall: $A \times B = \{(a,b) : a \in A,\ b \in B\}$

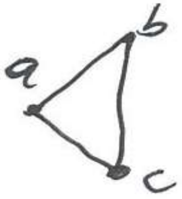
Suppose G_1 and G_2 are two graphs with $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$. Then $G_1 \times G_2$ is the graph with vertex set

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$
$$= \{(u_i, v_j) : u_i \in V(G_1),\ v_j \in V(G_2)\}$$

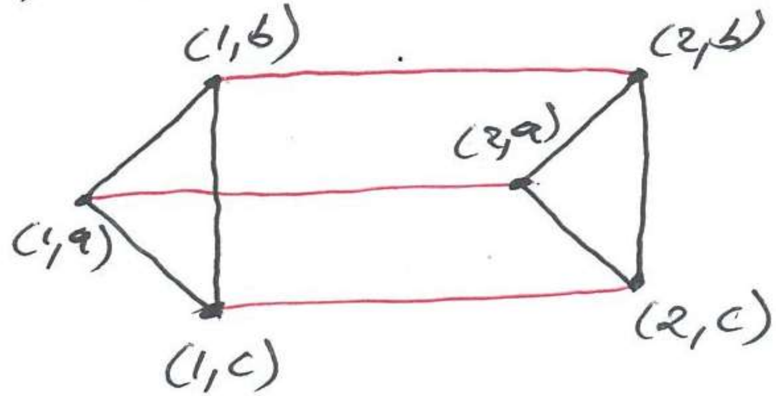
Example

G_1 :

G_2 :



$G_1 \times G_2$:

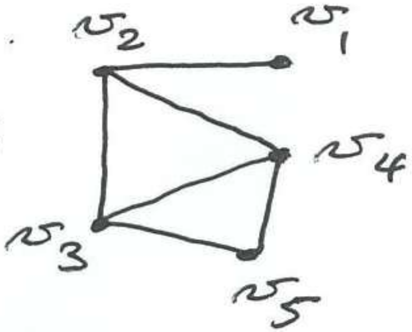


5) Compliment of the graph.

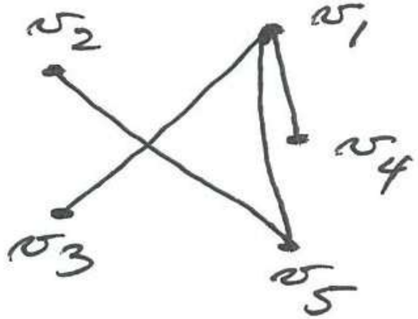
Considering a graph G. The Compliment of G, $\overline{G}$ is a graph with same vertex set of G and with two vertices adjacents only when they are not adjacents in G.

Example

G :



$\overline{G}$:



NOTA:

- If a graph G has v vertices and e edges

Then $\overline{G}$ will have $\frac{v(v-1)}{2} - e$ edges.

G: $v=5$, $e=6$ $\overline{G} = \frac{5(5-1)}{2} - 6 = 10 - 6 = 4$ edges.

- If G is a graph with n vertices

Then $G \cup \overline{G} = K_n$ (complete graph of order n).

Exercise.

Given graph G with 15 edges and $\overline{G}$ with 13 edges. Find the number of vertices in graph G.

Solution

We know that $G \cup \overline{G} = K_n$

The number of edges in K_n is $e = \frac{n(n-1)}{2}$

$$G \cup \overline{G} = \frac{n(n-1)}{2}$$

$$15 + 13 = \frac{n(n-1)}{2}$$

$$n^2 - n - 56 = 0$$

$$(n-8)(n+7) = 0$$

$n = 8$; $n = -7$ (not possible)

$\therefore$ G contains 8 vertices.

The Degree of a Vertex.

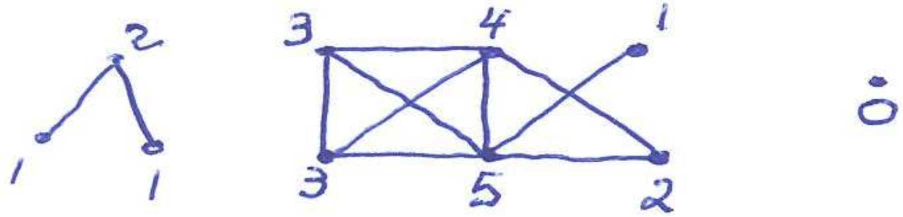
Definition

Let v be a vertex of a graph G. The degree of v is the number of edges of G incident with v. The degree of v is denoted by $\deg_G v$, or simply $\deg v$.

The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$, while the maximum degree of G is the maximum degree among the vertices of G is denoted by $\Delta(G)$.

Example

G:



$\delta(G) = 0$

$\Delta(G) = 5$

Definition

A vertex is called <u>odd</u> or <u>even</u> depending on whether its degree is <u>odd</u> or <u>even</u>.

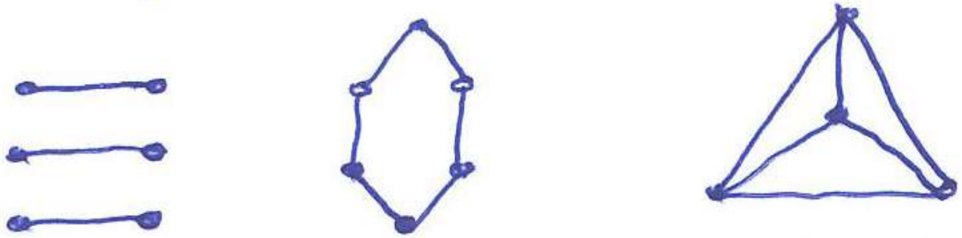
A vertex of degree 0 in G is called an <u>isolated</u> vertex and a vertex of degree 1 is an <u>end-vertex</u> of G.

A vertex adjacent to an end-vertex is called a remote vertex.

Definition (Regular graph)

We say that a graph is regular if all its vertices have the same degree. In particular, if the degree of each vertex is r, then the graph is regular of degree r or is r-regular.

Example



$r=1$ $\quad$ $r=2$ $\quad$ $r=3$

Theorem 1.1

In any graph, the sum of all the vertex degrees is equal to twice the number of edges

proof

Every edge is incident with two vertices; hence, when the degrees of the vertices are summed, each edge is counted twice. $\square$

Corrollary 1.2

In any graph, there is an even number of odd vertices.

proof

Let G be a $(\overset{\text{order}}{\overset{\uparrow}{n}}, \overset{\text{size}}{\overset{\uparrow}{m}})$ graph. If G has no odd vertices, then the result follows immediately. Suppose that G contains $k(\geq 1)$ odd vertices $v_1, v_2, \ldots, v_k$. If G contains even vertices as well, then denote these by $v_{k+1}, \ldots, v_n$.

By Theorem 1.1. we have

$$\sum_{i=\blacksquare}^{k} \deg v_i + \sum_{i=k+1}^{n} \deg v_i = 2m$$

$$\Rightarrow \sum_{i=1}^{k} \deg v_i = \underbrace{2m}_{\text{even}} - \underbrace{\sum_{i=k+1}^{n} \deg v_i}_{\substack{\text{even since the number} \\ \deg v_{k+1}, \ldots, \deg v_n \text{ is even.}}}$$

$$\therefore \sum_{i=1}^{k} \deg v_i \text{ is even}$$

However, each of the number $\deg v_1, \deg v_2, \ldots, \deg v_k$ is odd. Since the sum of an odd number is odd, it follows that k must be even; that is G has an even number of odd vertices.

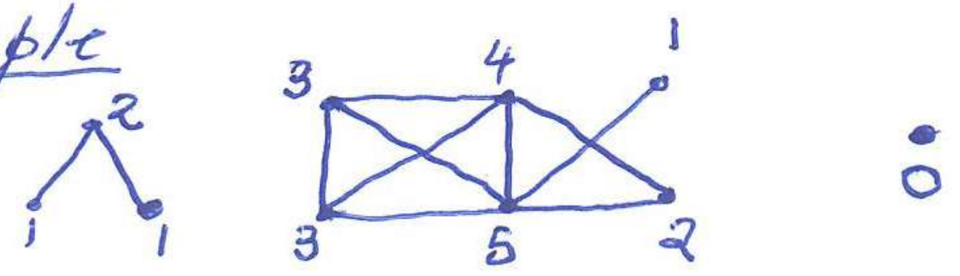
If G has no even vertices, then we have $\deg v_1 + \deg v_2 + \cdots + \deg v_k = 2m$, from which we again conclude that k is even. $\square$

The Degree Sequence.

- It is convenient to list the degrees of vertices in decreasing order by allowing "repeats"
- The result list is called the degree sequence of a graph.

Example

G:



The degree sequence of graph G is

5, 4, 3, 3, 2, 2, 1, 1, 1, 0

Definition

- A sequence is graphical if
  1) $d_i \geq 0$, $\forall i$
  2) $d_i \leq n-1$ with n the order of the graph.
  3) $\sum_{i=1}^{n} d_i$ even.

Example.

The sequence 5, 4, 3, 3, 2, 2, 1, 1, 1, 0 is graphical:

1) $d_i \geq 0$
2) $d_i \leq n-1$, Here $n = 10$ (we have 10 vertices)
3) $\sum_{i=1}^{n} d_i = 5+4+3+3+2+2+1+1+1+0$
   $= 22$ even.

- The above 3 conditions are necessary for a sequence to be graphical, but these conditions are not sufficient.

Example

The sequence 3, 3, 3, 1 is not graphical

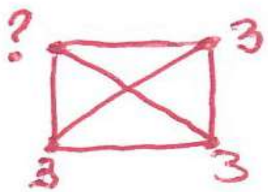
1) $d_i \geq 0 \quad \forall i$ (satisfied)

2) $d_i \leq n-1$

Here $n = 4$

3) $\sum_{i=1}^{n} d_i = 3+3+3+1 = 10$ even (satisfied)

Hence we can't represent the sequence 3, 3, 3, 1 graphically



THE 3 CONDITIONS ARE SATISFIED. HOWEVER THE SEQUENCE <u>IS NOT</u> GRAPHICAL

Theorem 1.3 [Havel - Hakimi]

A sequence $s: d_1, d_2, \cdots, d_n$ of non-negative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ and $n \geq 2$, $d_1 \geq 1$ is graphical if and only if the sequence

$$s_1 = \underbrace{d_2 - 1, \; d_3 - 1, \; \cdots, \; d_{d_1+1} - 1}_{d_1 \text{ terms}}, \; d_{d_1+2}, \cdots, d_n, \quad n \geq 2$$

is graphical.

Example

Ex. 1 $\cancel{3}, 3, 3, 2, 2, 1$ is graphical $\Leftrightarrow 2, 2, 1, 2, 1$ is graphical

$\Leftrightarrow \cancel{2}, 2, 2, 1, 1$ re-order

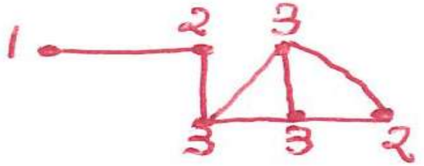
$\Leftrightarrow \cancel{1}, 1, 1, 1$ is graphical

$\Leftrightarrow 0, 1, 1$ is graphical

$\Leftrightarrow \cancel{1}, 1, 0$ re-order

$\Leftrightarrow 0, 0$ is graphical

$\therefore$ The sequence $3, 3, 3, 2, 2, 1$ is graphical.



Ex. 2 $\cancel{3}, 3, 3, 1$ is graphical $\Leftrightarrow \cancel{2}, 2, 0$ is graphical

$\Leftrightarrow 1, -1$ is not graphical

since -1 cannot be a degree of any vertex.

Ex.③ $\cancel{4}, 4, 4, 3, 3, 2$ graphical $\Longleftrightarrow \cancel{3}, 3, 2, 2, 2$ graphical

$\Longleftrightarrow 2, 1, 1, 2$

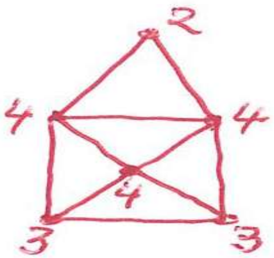
$\Longleftrightarrow \cancel{2}, 2, 1, 1$ re-order

$\Longleftrightarrow 1, 0, 1$

$\Longleftrightarrow \cancel{1}, 1, 0$ re-order

$\Longleftrightarrow 0, 0$ graphical

$\therefore$ 4, 4, 4, 3, 3, 2 is graphical



<u>NOTA</u>

The degre sequences do not always provide enough information to uniquely describe a graph.

isomorphic graphs

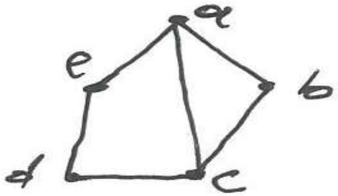
- An isomorphism from a graph G to a graph H is a bijective mapping $\phi: V(G) \longrightarrow V(H)$ Such that $xy \in E(G) \iff \phi(x)\phi(y) \in E(H)$.
  (every edge of G get map to a single edge of H).
- A Homomorphism from a graph G to a graph H is a mapping (not necessarily bijective) $\phi: V(G) \longrightarrow V(H)$ Such that

  $xy \in E(G) \implies \phi(x)\phi(y) \in E(H)$.

  (Every edge of G get map to the edge of H)

Example

Given the graph

G: 

H: 

Consider $\phi : V(G) \longrightarrow V(H)$ given by

$\phi(a) = 1$; $\phi(b) = 2$; $\phi(c) = 3$; $\phi(d) = 2$; $\phi(e) = 3$

Discuss if ϕ is homomorphism.

Check edges: $ab \longrightarrow 12$; $bc \longrightarrow 23$; $cd \longrightarrow 32$;
$de \longrightarrow 23$; $ea \longrightarrow 31$; $ac \longrightarrow 13$.

NOT edges: $eb \longrightarrow 32$; $bd \longrightarrow$ single vertex 2.

Hence, every edge of G get map to the edge of H.

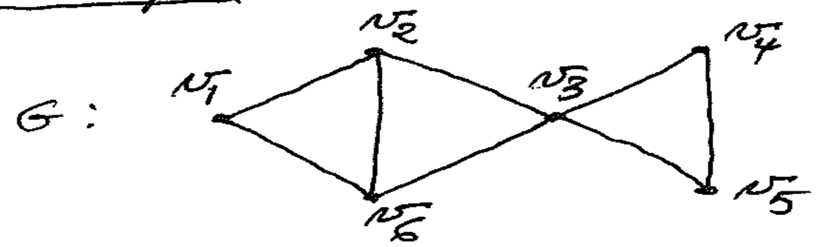
$\therefore$ ϕ is a homorphism from G to H.

Connected graphs

Definition (A walk)

Let $u, v \in V(G)$ (not necessarily distinct)
A $u-v$ walk in G is a finite, alternating sequence of vertices and edges that begin with the vertex u and end with the vertex v.

Example



Find the walk $v_3 - v_4$:

$v_3, v_3v_2, v_2, v_2v_6, v_6, v_6v_3, v_3, v_3v_4, v_4,$
v_4v_5, v_5, v_5v_4, v_4

OR

$v_3, v_2, v_6, v_3, v_4, v_5, v_4$ (This walk has length $\underline{\underline{6}}$)

NOTA

- A closed walk occurs when $u = v$
- A trivial walk is a walk which does not take any edges (Example v_6)

Example

Find closed walk $v_3 - v_3$:

v_3, v_2, v_1, v_6, v_3

This is a closed walk of length 4.

Definition

- A trail is a walk with no repeated edges.

  Example: $v_1 - v_5$ trial:

  $$v_1, v_2, v_6, v_3, v_5$$

- A closed trial is a circuit

  Example: $v_1 - v_1$ Circuit:

  $$v_1, v_2, v_6, v_1$$

- A cycle is a circuit which does not repeat any vertices (except the first and last vertices).

- A path is a walk with no repeated vertices.

  Example: $v_2 - v_5$ PATH:

  $$v_2, v_1, v_6, v_3, v_4, v_5$$

- A closed path is a cycle.

Theorem

Every u-v walk in a graph contains a u-v path.

proof

Let W be a u-v walk in a graph G. If W is closed, the result is easy; we simply use trivial path u. Thus assume W is an open walk, say $W: u = u_0, u_1, u_2, \ldots, u_k = v$.

Note that a vertex may have received more than one label if it occurs more than once in W.

If no vertex is repeated, then W is already a path

otherwise, there are vertices of G that occur
in W twice or more. Let i and $j \in \mathbb{Z}$, $i \neq j$
with $i < j$ such that $u_i = u_j$. That is, the
vertex u_i is repeated as u_j. If we now
delete the vertices $u_i, u_{i+1}, \ldots, u_{j-1}$ from W,
we obtain a u-v walk W_1 which is shorter
than W and has fewer repeated vertices.
If W_1 is a path, we are done; if not, we
continue this process until finally we reach
a stage where no vertices are repeated and
a u-v path is obtained $\square$

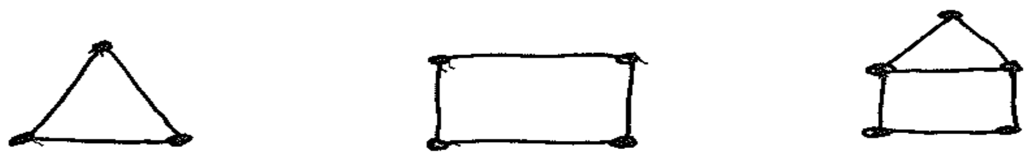
Definition (Path graphs)

A path graph is a graph consisting of a single path. The path graph of order n is denoted by P_n.

P_1 P_2 P_3 P_4

Definition (cycle graphs)

A cycle graph is a graph consisting of a single cycle. The cycle graph of orde n is denoted by C_n $(n \geq 3)$



C_3 C_4 C_5

Definitions:

- A graph G is <u>connected</u> if there exists a path in G between any two of its vertices, and is <u>disconnected</u> otherwise. Every disconnected graph can be split up into a number of connected subgraphs, called component.
- A component of a graph G is a maximal connected subgraph.
- Two vertices u and v in a graph G are connected if $u = v$, or if $u \neq v$ and a u-v path exist in G.
- The number of component of G is denoted by $k(G)$; of course, $k(G) = 1$ if and only if G is connected.

Distance in graphs

Definition

- For a connected graph G, we define the distance $d(u, v)$ between two vertices u and v as the minimum of the lengths of the u-v path of G.

- If G is a disconnected graph, then the distance between two vertices in the same component of G is defined as above.

- If u and v belong to different component of G, then $d(u, v)$ is undefined or $d(u,v)=\infty$

Example

G :

$d(u,u)=0$; $d(x,u)=2$; $d(x,w)=3$; $d(x,v)=5$

NOTA : The distance is the shortest path.

Theorem

Let G be a graph and let $u, v, w \in V(G)$

Then

(i) $d(u,v) \geq 0$ and $d(u,v)=0$ iff $u=v$;

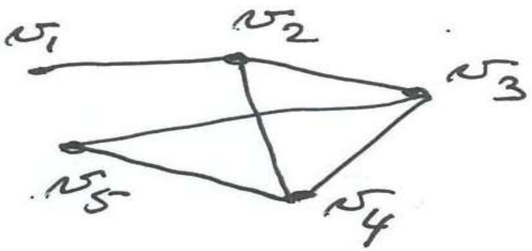
(ii) $d(u,v)=d(v,u)$ (Symmetry property)

(iii) $d(u,v) \leq d(u,w)+d(w,v)$ (Triangle inequality).

<u>Definitions</u>

- The eccentricity, $e(v)$ of a vertex v in a connected graph G is the maximum distance of a vertex from v. $e(v) = \max\{d(u,v) / u, v \in V(G)\}$
- The radius of G, $\operatorname{rad} G$ is the minimum eccentricity among the vertices of G.
- The diameter, $\operatorname{diam} G$ is the maximum eccentricity $\operatorname{diam} G = \max\{e(v) / v \in V(G)\}$
- A vertex v is called a central vertex if $e(v) = \operatorname{rad} G$
- A vertex v is called a peripheral vertex if $e(v) = \operatorname{diam} G$.

Example

G : 

$e(v_1) = 3$; $e(v_2) = 2$; $e(v_3) = 2$

$e(v_4) = 2$

$\operatorname{diam} G = 3$

$\operatorname{rad} G = 2$.

Theorem

For every connected graph

$$\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2 \operatorname{rad}(G)$$

Proof

- $\underline{\operatorname{rad}(G) \leq \operatorname{diam}(G)}$. This is obvious since we know that $\operatorname{rad}(G) = \min\{e(v) \mid v \in V(G)\}$

$\operatorname{diam}(G) = \max\{e(v) \mid v \in V(G)\}$

$\therefore$ clearly, $\operatorname{rad}(G) \leq \operatorname{diam}(G)$

Now, let proof the second inequality,

$$diam(G) \leq 2\, rad(G)$$

We will start by taking two vertices:

Let $u, v \in V(G)$ such that $d(u,v) = diam(G)$

Let w be a central vertex of G:

$$d(u,v) \leq d(u,w) + d(w,v) = e(w) + e(w)$$
$$\leq 2e(w)$$

$d(u,w)$ could be the most eccentricity of w since $e(w)$ represent the furthest path of distance from w to any other vertex. Similarly, $d(w,v)$ is the most eccentricity of w

$\therefore\ d(u,v) \leq 2e(w) = 2\, rad(G)$ since w is the central vertex.

Hence,

$$rad(G) \leq diam(G) \leq 2\, rad(G).$$

compliment of graph G

Theorem

Let G be a disconnected graph. Therefore, $\overline{G}$ is a connected graph and $diam(\overline{G}) \leq 2$.

proof

Let G be a disconnected graph and let G_1 be one of the connected component of G.

Let $V_1 = V(G_1)$ and $V_2 = V(G) \setminus V(G_1)$

Let $u \in V_1$. Then for every $v \in V_2$, $uv \notin E(G)$

$\therefore uv \in E(\overline{G})$, i.e. $d_{\overline{G}}(u,v) = 1$

And for every $v_1, v_2 \in V_2$

$$d_{\bar{G}}(v_1, v_2) \leq 2$$

similarly, $\forall u_1, u_2 \in V_1$:

$$d_{\bar{G}}(u_1, u_2) \leq 2 .$$

$$\therefore \max\{ e_{\bar{G}}(v) \mid v \in V(\bar{G}) \} \leq 2$$

Hence, $\operatorname{diam}(\bar{G}) \leq 2$. $\square$

Cut-Vertices and Bridge

Recap (subgraph).

If $e \in E(G)$, then $G - e$ is the subgraph of G possessing the same vertex set as G and having all the edges of G except e.

If v is a vertex of a graph G containing at least two vertices, then $G - v$ is the subgraph of G whose vertex set consists of all vertices of G except v and whose edge set consists of all edges of G except those incident with v.

Definition

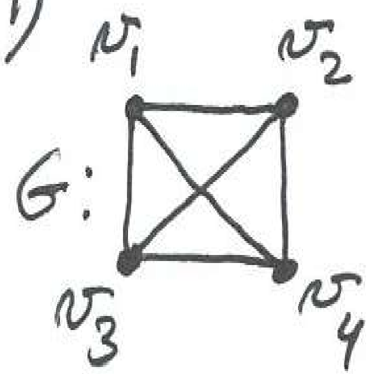
- A vertex v in a graph G is called a cut-vertex of G if $k(G-v) > k(G)$ with $k(G)$ the number of connected component of G.
  If G is a connected graph, then v is a cut-vertex if $G-v$ is disconnected.

- e in a graph G is called a bridge if $k(G-e) > k(G)$. If G is a connected graph, then e is a bridge if $G-e$ is disconnected.

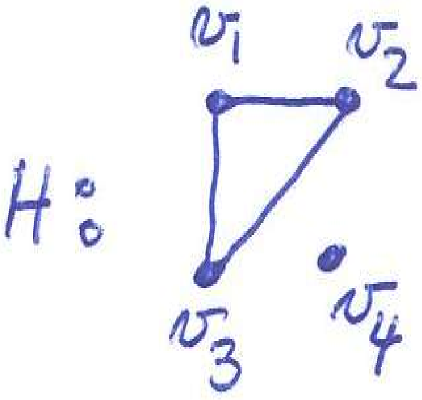
- Recap: A graph G is connected iff $k(G) = 1$

Example

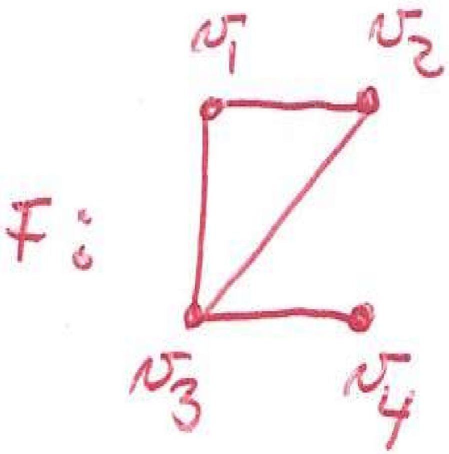
1)



connected graph



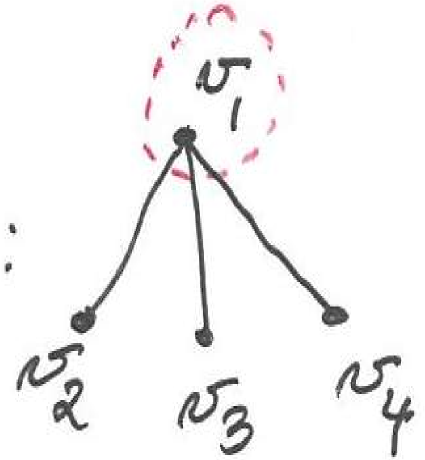
disconnected graph



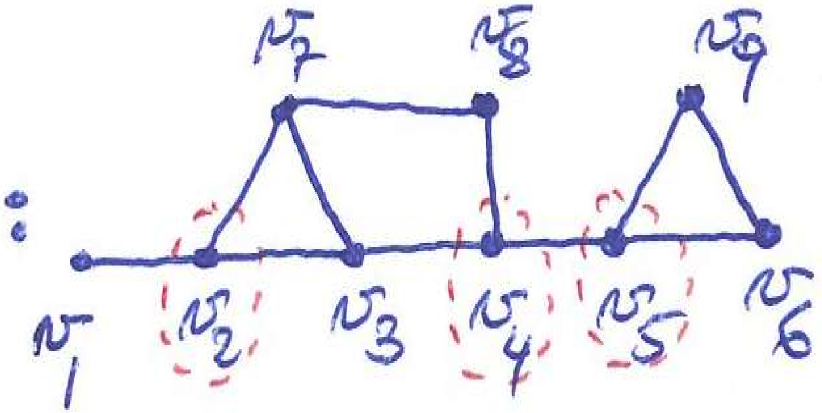
connected graph but easily made disconnected by removing vertex v_3

2) Find the cut-vertex in the graph below.

G:



H:

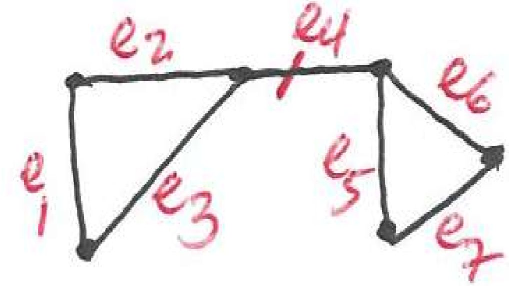


The cut vertex of Graph G is v_1

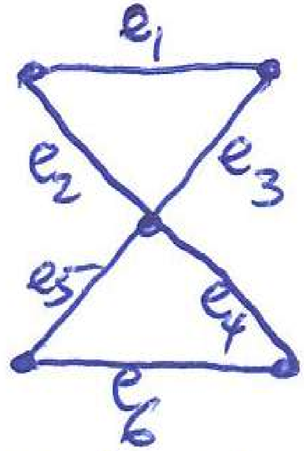
v_2, v_4 and v_5 are cut-vertices of graph H.

3) Find the Bridge in the graph below:

G:



H:



e_4 is the bridge of graph G.

There is No bridge in the graph H.

Theorem

A vertex $v \in V(G)$ is a cut-vertex of G if and only if there exist $u, w \in V(G)$, $u, w \neq v$ such that v is on $u-w$ path of G.

proof

we are going to assume that G is a connected graph.

$\boxed{\Rightarrow}$

Let $v \in V(G)$ be a cut-vertex. Then $G-v$ is a disconnected graph. Let u, w be vertices in different components of $G-v$. Therefore, $u-w$ is not a path in $G-v$. But since G is a connected graph, then there exist $u-w$ path in G. Hence, all such path went through vertex v.

$\boxed{\Leftarrow}$

suppose there exist $u, w \in V(G)$, $u, w \neq v$ such that v lies on every u-w path. then removing vertex v means that there is <u>not</u> any u-w path in $G-v$. Hence, $G-v$ is disconnected, then v is a cut-vertex. $\square$

Theorem

An edge $e \in E(G)$ is a bridge if and only if e is not in any cycle of G.

Recap : - A closed path is a cycle.

- $P \Rightarrow Q$; contrapositive : $\neg Q \Rightarrow \neg P$

proof

we are going to prove by contraposition:

$\boxed{\Rightarrow}$ contrapositive: we have to show that if e is on the cycle then e is not a bridge.

Let $e \in E(G)$ is on the cycle C, then $G-e$, u and v are in the same component ($e = uv$)

Hence, e is not a bridge.

$\boxed{\Leftarrow}$ contrapositive: we have to show that if $e \in E(G)$ is not a bridge then e is on a cycle.

$e = uv$ is not a bridge then there is a path from u to v (Since u and v are in the same component). in the original graph G. Hence, there is a cycle that contains edge e. $\square$

Lemma

Let e be a bridge in a connected graph G. Then $k(G-e) = 2$.

proof

Let $e = uv$ be a bridge, then u and v are in different component of $G-e$. Therefore, $k(G-e) \geq 2$
Now, since the original graph is connected, let $w \in V(G)$ and since G is connected, there exist a path $w-v$ in G. Now, if path $w-v$ does not use edge e, then path $w-v$ is in $G-e$. In a contrary if path $w-v$ uses edge e then there exist path from w to u in $G-e$. Hence, $k(G-e) = 2$.

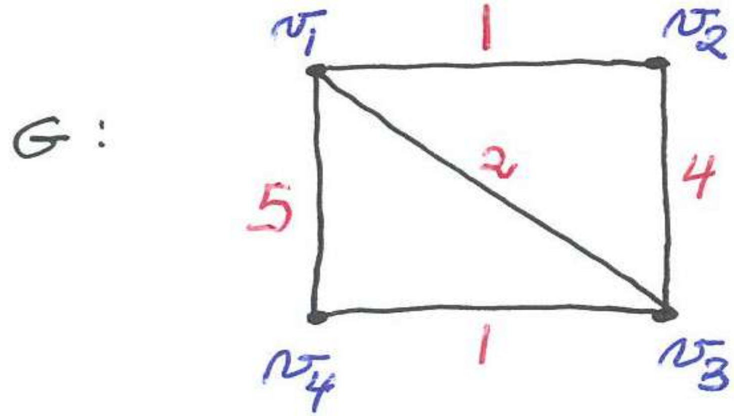
The Shortest Path Algorithm.

Definitions

- A weighted graph is a graph in which each edge e is assigned a positive real number, called the weight of e, and denoted by $w(e)$.
- The length of a path P in a weighted graph G is the sum of the weights of the edges of P.

- For connected vertices u and v of the weighted graph G, the distance $d(u, v)$ between u and v is the minimum of the length of the u-v path of G.
- A u-v path of minimum weight in G is called a shortest u-v path for G.

Example



- The path $P: v_2, v_1, v_3, v_4$ is the shortest path of minimum weight 4.
- The path $Q: v_2, v_3, v_4$ of weight 5 is NOT the shortest path for G, even though path Q contains few edges than path P.

The weight matrix.

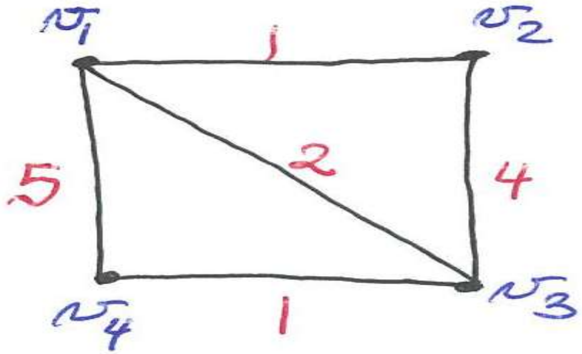
Let G be a weighted graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. It is convenient to represent G by means of a weight matrix $W(G)$ defined as follows:

$$W(G) = \begin{cases} w(v_i v_j) & \text{if } v_i v_j \in E(G) \\ \infty & \text{if } v_i v_j \notin E(G) \end{cases}$$

Example

Given

G :



$$W(G) = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 1 & 0 & 4 & \infty \\ 2 & 4 & 0 & 1 \\ 5 & \infty & 1 & 0 \end{pmatrix}$$

Dijkstra's Algorithm.

Given a connected weighted graph G of order p and a vertex x of G:

1) Set $l(x) = 0$ and for all $v \neq x$, set $l(v) = \infty$ and set $S = V(G)$

2) if $|S| = 1$, then stop; otherwise continue.

3) Among all the vertices in S, let u be one of minimum label $l(u)$;

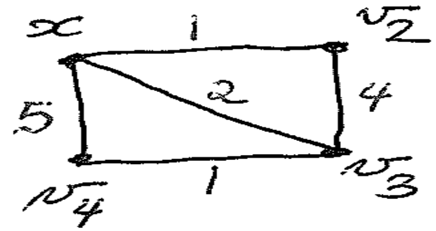
4) For each $v \in S$, if $uv \in E(G)$ and $l(v) > l(u) + w(uv)$, then

- replace $l(v)$ by $l(u) + w(uv)$ and
- Assign to parent v the vertex u.

5) Remove u from S, and return to step 2.

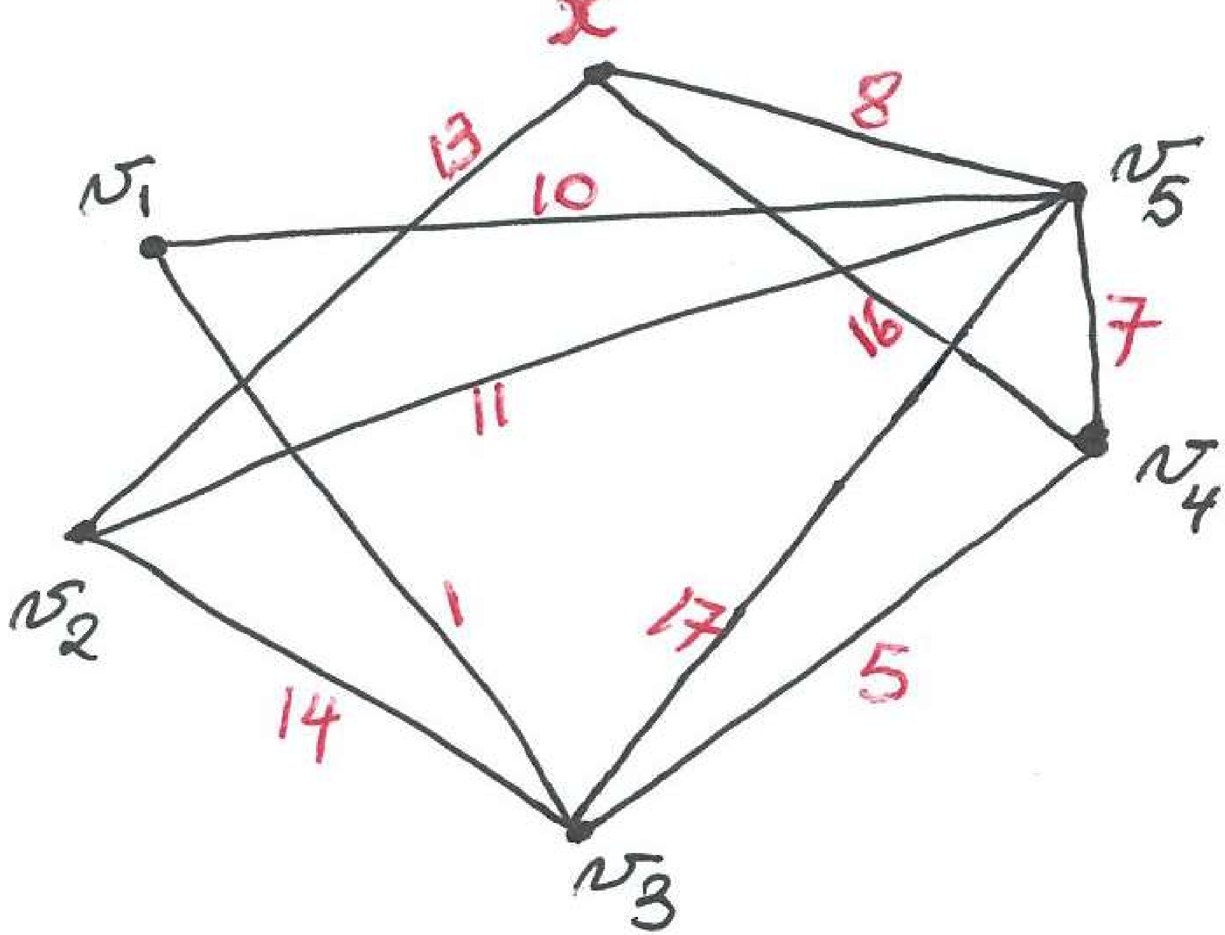
Example

$l(x)$	v_2	v_3	v_4	Removed from S	S
0	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	—	$\{x, v_2, v_3, v_4\}$
	$(1,x)$	$(2,x)$	$(5,x)$	x	$\{v_2, v_3, v_4\}$
		$(2,x)$	$(5,x)$	v_2	$\{v_3, v_4\}$
			$(3,v_3)$	v_3	$\{v_4\}$



shortest path:

v	$d(x,v)$	Q_i
v_2	$l(v_2)=1$	x, v_2
v_3	$l(v_3)=2$	x, v_3
v_4	$l(v_4)=3$	x, v_3, v_4


x
v_1
v_2
v_3
v_4
v_5
13
10
8
7
16
11
1
17
14
5

$l(x)$	v_1	v_2	v_3	v_4	v_5	Removed from S	S
0	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	—	$V(G)$
	$(\infty,-)$	$(13,x)$	$(\infty,-)$	$(16,x)$	$(8,x)$	x	$\{v_1, v_2, v_3, v_4, v_5\}$
	$(18,v_5)$	$(13,x)$	$(25,v_5)$	$(15,v_5)$		v_5	$\{v_1, v_2, v_3, v_4\}$
	$(18,v_5)$		$(25,v_5)$	$(15,v_5)$		v_2	$\{v_1, v_3, v_4\}$
	$(18,v_5)$		$(20,v_4)$			v_4	$\{v_1, v_3\}$
			$(19,v_1)$			v_1	$\{v_3\}$.

The shortest path.

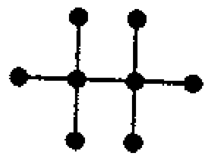
v	$d(x, v)$	P_i
v_1	$\ell(v_1) = 18$	x, v_5, v_1
v_2	$\ell(v_2) = 13$	x, v_2
v_3	$\ell(v_3) = 19$	x, v_5, v_1, v_2
v_4	$\ell(v_4) = 15$	x, v_5, v_4
v_5	$\ell(v_5) = 8$	x, v_5

Trees

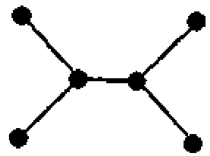
Definition:

- A Graph with NO cycle is acyclic
- A tree is a connected graph which contains no cycle (i.e. A tree is a connected acyclic graph)
- A Forest is a graph that has no cycles (Note that each component of a forest is a tree). A forest is an acyclic graph

T_1 :



T_2 :



T_3 :

Since a tree contains no cycle, it follows that:_

- Every edge of a tree is a bridge

- Any two vertices of a tress are connected by exactly one path

Properties of a Tree

Theorem 4.1:

A graph *T* is a tree if and only if every two distinct vertices of *T* are joined by a unique path

Proof. If T is a tree, then by definition it is connected. Hence, any two vertices are joined by at least one path. Suppose that there are vertices u and v of T that are joined by two or more different paths. Let P and Q be two different u-v paths in T. Then there must be a vertex x (possibly $x = u$) on both paths such that the vertex immediately following x on P is different from the vertex immediately following x on Q. Let y be the first vertex of P following x that also belongs to Q (possible $y = v$). Then the section of P from x to y and the section of Q from x to y produce two x-y paths that have only x and y in common. These two paths produce a cycle in T, which contradicts the fact that T is a tree. Hence every two distinct vertices of T are joined by a unique path.

On the other hand, suppose T is a graph in which any two distinct are joined by a unique path. This implies that T is connected. If T has a cycle containing vertices u and v, the u and v are joined by at least two paths, contradicting our hypothesis. Hence T is acyclic so that T is a tree. $\square$

Theorem 4.2:

A tree *T* of order *p* has size *p-1*

Proof. We proceed by induction on p. If $p = 1$, then $T \cong K_1$ and T has size 0, as required. Let $k \geq 2$ be an integer, and suppose the result is true for all trees of order less than k. Let T be a tree of order $p = k$ and size q, and let $e = uv$ be an edge of T. Since every edge of a tree is a bridge, the graph $T - e$ is disconnected. In fact, $T - e$ is a forest with exactly two components, namely, a tree T_1 containing u, and a tree T_2 containing v. Let T_i $(i = 1, 2)$ be a tree of order p_i and size q_i. Then $p_i < k$ $(i = 1, 2)$. Hence, by the inductive hypothesis, we know that $q_i = p_i - 1$ for $i = 1, 2$. Since $p = p_1 + p_2$ and $q = q_1 + q_2 + 1$,

$$q = (p_1 - 1) + (p_2 - 1) + 1 = p_1 + p_2 - 1 = p - 1.$$

Thus, by induction, the size of a tree is one less than its order. □

Characterizations of a Tree

Theorem 4.3:

Let *T* be a graph of order *p*. Then the following statements are equivalent

i. T is a tree

ii. T is connected and has size p-1

iii. T has no cycles and has size p-1

Please note that you need to prove the following:

$$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$$
$$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$$
$$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$$

Proof. If T is a tree of order p, then, by definition, T is connected and, by Theorem 4.2, T has size $p-1$. So $(i) \Rightarrow (ii)$.

Suppose that T is connected and has size $p-1$. Assume T contains a cycle and let e be an edge of this cycle. Then, by Theorem 2.5, e is not a bridge of T, so $T-e$ is a connected graph of order p and size $p-2$, which is a contradiction (see Exercise 4.3). Therefore T has no cycles. Thus $(ii) \Rightarrow (iii)$.

Suppose that T has no cycles and size $p-1$. In order to prove that T is a tree, we must show that T is connected. Let $T_1, T_2, \ldots, T_k$ be the components of T $(k \geq 1)$, where T_i has order p_i and size q_i for $i = 1, 2, \ldots, k$. Since each component T_i $(i = 1, 2, \ldots, k)$ is a connected graph with no cycles, each T_i is a tree. Thus, by Theorem 4.2, $q_i = p_i - 1$. Hence

$$p-1 = q = \sum_{i=1}^{k} q_i = \sum_{i=1}^{k} (p_i - 1) = \left(\sum_{i=1}^{k} p_i\right) - k = p - k,$$

so $k = 1$ and therefore T is connected. Hence T is a tree. Thus $(iii) \Rightarrow (i)$. □

Theorem 4.3:
Every nontrivial tree contains at least two end-vertices.

Proof. Suppose that T is a tree of order p and size q, and let $d_1, d_2, \ldots, d_p$ denote the degrees of its vertices, ordered so that $d_1 \leq d_2 \leq \cdots \leq d_p$. Since T is connected and nontrivial, $d_i \geq 1$ for each i $(1 \leq i \leq p)$. If T does not contain two end-vertices, then $d_1 \geq 1$ and $d_i \geq 2$ for $2 \leq i \leq p$. So

$$\sum_{i=1}^{p} d_i = d_1 + \sum_{i=2}^{p} d_i \geq 1 + 2(p-1) = 2p - 1.$$

However, by Theorems 1.1 and 4.2,

$$\sum_{i=1}^{p} d_i = 2q = 2(p-1) = 2p - 2,$$

which contradicts the previous inequality. Hence, T contains at least two end-vertices. □