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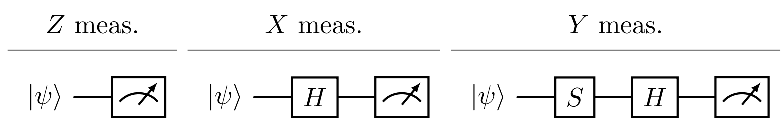
Quantum Circuits

1. Estimate α and β using quantum tomography on a single qubit pure state.

   (a) Generate a quantum state $|\psi\rangle = 0.8\,|0\rangle + 0.6e^{j\pi/3}\,|1\rangle$.

   (b) Perform each of the following measurements N times, storing the number of times the measurement "0" has occurred.

   (c) Using the measurement results estimate $|\psi\rangle_N$ for $N = 10, 100, 1000$. Fill the following table and describe how the results are obtained.

N	$\|\psi\rangle_N$
10	
100	
1000	

Hint 1: Rewrite the state as $|\psi\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{j\varphi}\,|1\rangle$ *with* $0 \leq \theta \leq \pi$ *and* $0 \leq \varphi < 2\pi$. *Then,* θ *and* φ *are the two real numbers to be estimated through quantum state tomography.*

Hint 2: In the Z *measurement setup, we have that* $|\alpha|^2$ *is the probability of the measurement outcome "0". Using the frequentist approach of probability we can approximate this probability with* $n_{\mathrm{z}}^{(0)}/N$ *where* $n_{\mathrm{z}}^{(0)}$ *are the number of times the measurement "0" has occurred using the measurement setup* Z. *In this way, we find that* $\cos^2(\theta/2) = n_{\mathrm{z}}^{(0)}/N$. *Finally, we obtain*

$$\theta = 2\arccos\left(\sqrt{\frac{n_{\mathrm{z}}^{(0)}}{N}}\right) + 2k\pi = 2\arccos\left(\sqrt{\frac{n_{\mathrm{z}}^{(0)}}{N}}\right) .$$

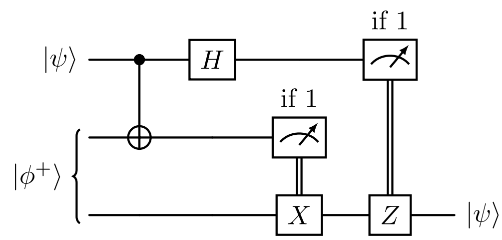
with $k \in \mathbb{Z}$.

Hint 3: In the X *measurement setup, the quantum state before the measuring gate is*

$$\alpha\,|+\rangle + \beta\,|-\rangle = \frac{\alpha+\beta}{\sqrt{2}}\,|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}\,|1\rangle \ .$$

Then, we have that $\left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2$ *is the probability that the outcome of the measurement is "0".*

2. Consider the following circuit executing the quantum teleportation algorithm



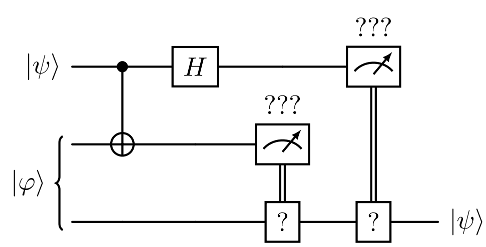
where $|\phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

(a) Generate a quantum state $|\psi\rangle = 0.8\,|0\rangle + 0.6\,e^{j\pi/3}\,|1\rangle$ and verify via tomography the output.

(b) What happens if $|\psi^+\rangle$ is used instead of $|\phi^+\rangle$ as entangled pair?

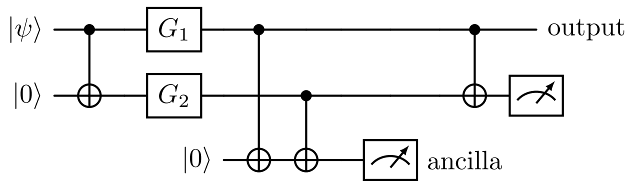
(c) In what ways can the teleportation circuit be adapted based on the specific entangled pair employed? Develop quantum circuits, one for each entangled pair

$$\left|\phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad \left|\psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad \left|\psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$



where $|\varphi\rangle = |\phi^-\rangle\,, |\psi^+\rangle\,, |\psi^-\rangle$

3. Consider the following circuit of an quantum error detection scheme



(a) Generate a quantum state $|\psi\rangle = 0.8\,|0\rangle + 0.6\,e^{j\pi/3}\,|1\rangle$ and verify via tomography the output when $G_1 = G_2 = I$.

(b) Repeat the experiment for all the combinations of $G_1 \in \mathcal{P}$ and $G_2 \in \mathcal{P}$, where $\mathcal{P} = \{I, X, Y, Z\}$, and fill the following table. For which combinations the output is equal to $|\psi\rangle$? Considering that the output state is correct whenever the ancilla is "0", in which cases we have an undetected error?

G_1	G_2	Output Error [Yes/No]	Ancilla Meas. [0/1]
I	I		
I	X		
I	Z		
I	Y		
X	I		
X	X		
X	Z		
X	Y		
Z	I		
Z	X		
Z	Z		
Z	Y		
Y	I		
Y	X		
Y	Z		
Y	Y		