λ

# Abstraction of control and recursion

Programmazione Funzionale

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Università di Trento

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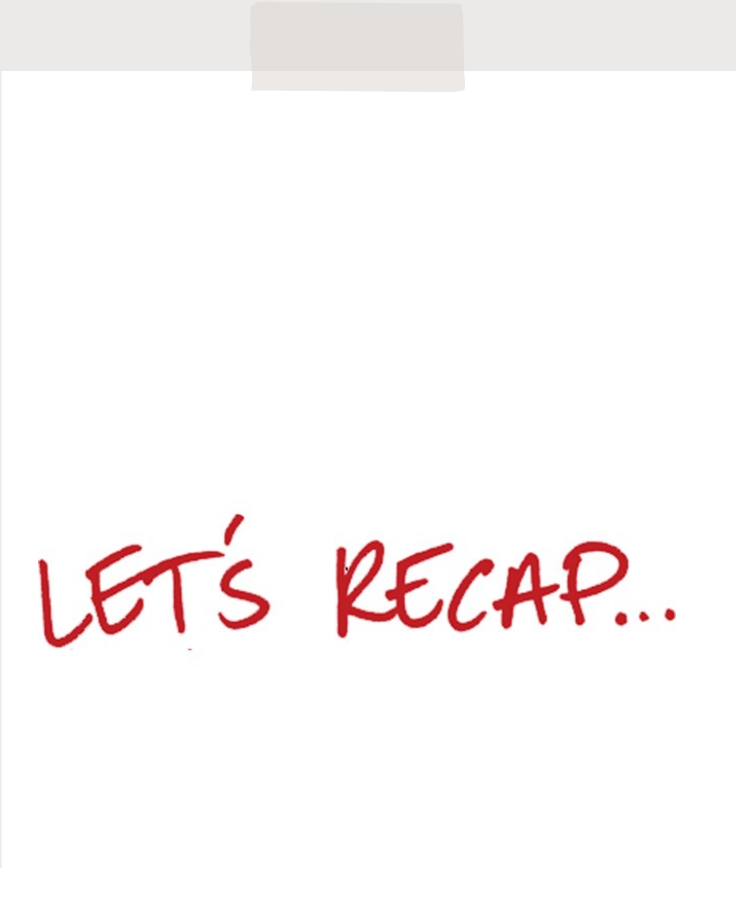
# Tutoring

- Tomorrow 15:30 – 16:30 in pc A201

- For those of you attending the English course who also wants to attend the tutoring hour, please fill in the form in Moodle or drop me an email every time you need it (the timeslot reserved would be Friday 11:30 – 12:30)




# Recap

# What defines the environment?

- Visibility rules (based on the block structure)
- Scope rules
- Rules for the parameter passing
- Binding policy

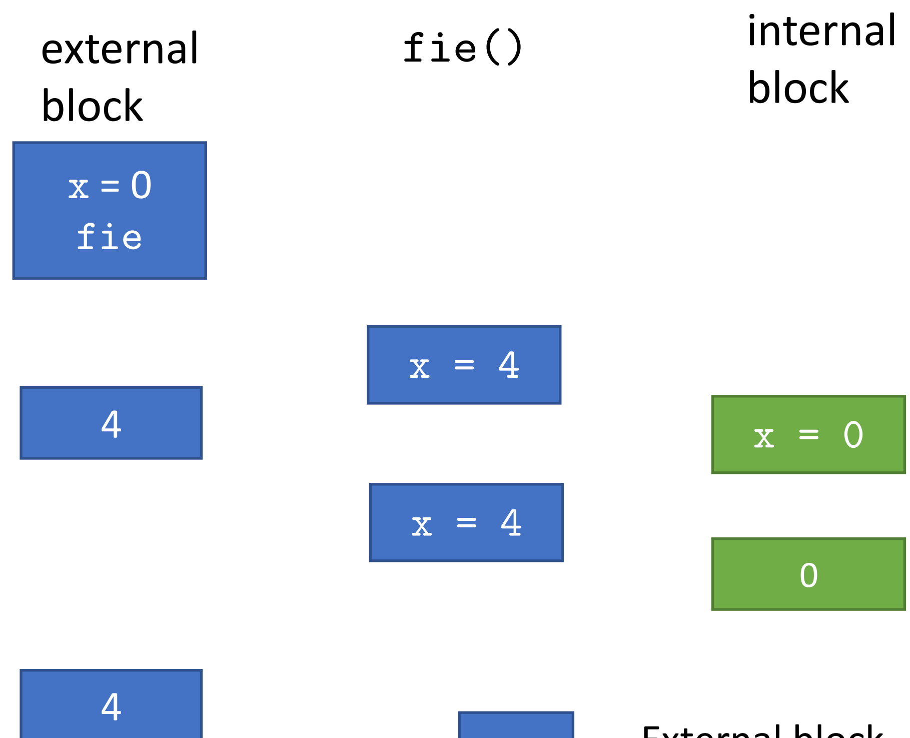
# Static and dynamic scoping

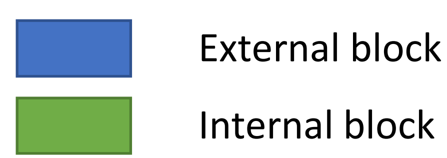
- Rules of visibility. A local declaration in a block is visible in this block, and in all nested blocks, as long as these do not contain another declaration of the same name, that hides or masks the previous declaration
- Static scoping: a block A is nested in block B if block B textually includes block A
- Dynamic scoping: a block A is nested in block B if block B has been most recently activated and has not yet been deactivated

# Static scoping

- In static scoping a non-local name is resolved in the block that textually includes it

```
{int x=0;
 void fie (int n){
     x = n+1;
 }
 fie(3);
 write (x);
     {int x =0;
      fie(3);
      write (x);
      }
  write (x);
}
```



# Dynamic scoping

- In dynamic scoping a non-local name is resolved in the block that has been most recently activated and has not yet been deactivated

```
{int x=0;
 void fie (int n){
     x = n+1;
 }
 fie(3);
 write (x);
     {int x=0;
      fie(3);
      write (x);
      }
  write (x);
}
```

external block

x = 0
fie

4

4

fie()

x = 4

x = 4

internal block

x = 0

4

External block

Internal block

# Exercise 4.1

- Consider the following program fragment written in a pseudo-language which uses static scope and where the primitive `read(Y)` allows the reading of the variable `Y` from standard input. State what the printed value(s) is (are).

```
...
int X = 0;
int Y;
void fie(){
    X++;
}
void foo(){
    X++;
    fie();
}
read(Y);
if Y > 0{
    int X = 5;
    foo();}
else
    foo();
write(X);
```

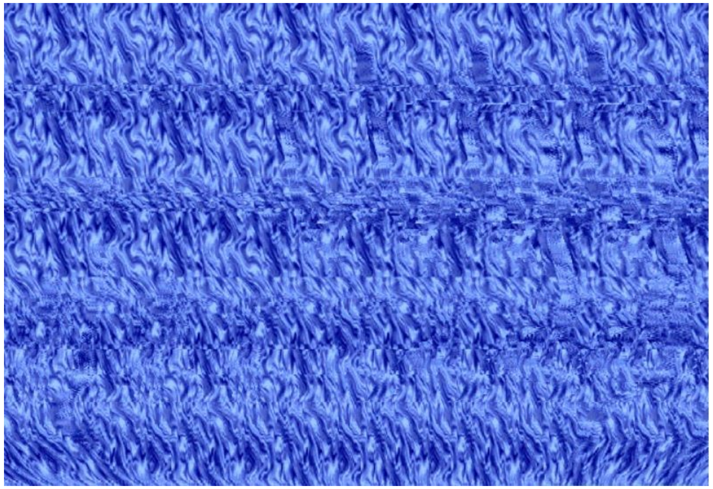
# Exercise 4.2

- Consider the following program fragment written in a pseudo-language that uses dynamic scoping. State what the printed value(s) is (are).

```
...
int X;
X = 1;
int Y;
void fie() {
    foo();
    X = 0; }
void foo(){
    int X;
    X = 5; }
read(Y);
if Y > 0 {
    int X;
    X = 4;
    fie();
}
else fie();
write(X);
```

# What defines the environment?

- Visibility rules (based on the block structure)
- Scope rules
- Rules for the parameter passing
- Binding policy



# Abstraction of control

# Abstraction of control

- Main mechanism: subprogram/procedure/function
- Subprogram: piece of code identified by its name, with a local environment and exchanging information with the rest of the code using parameters
- Two main constructs

definition

```
int foo (int n, int a) {
     int tmp=a;
     if (tmp==0) return n;
     else return n+1;
}
...
```

use

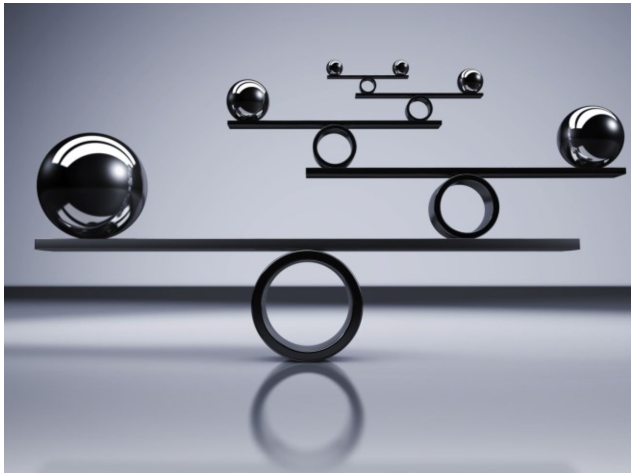
```
int x;
x = foo(3,0);
x = foo(x+1,1);
```

# Mechanisms for exchanging information with external code

- Parameters
- Return value
- Nonlocal environment

# Methods for passing parameters

- Three classes of parameters:
  - Input parameters
  - Output parameters
  - Input/output parameters



# Call by value

# Call by value

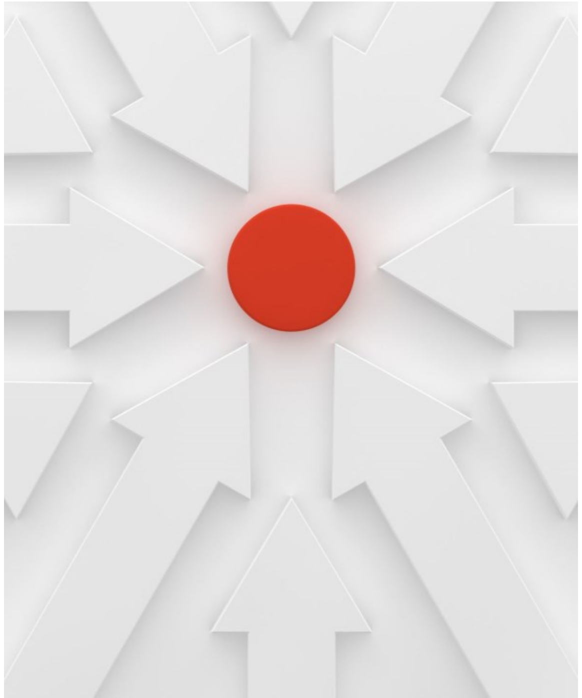
- The value is the actual one (r-value) assigned to the formal parameter, that is treated like a local variable
- Transmission from `main` to `proc` ⇨
- Modifications to the formal parameter do not affect the actual one
- On procedure termination, the formal parameter is destroyed (together with the local environment)
- No way to be used to transfer information from the callee to the caller!

# An example

```
int y = 1;
void foo (int x) {
    x = x+1;
}
...
y = 1;
foo(y+1);
```

- x assumes the initial value 2
- x is incremented to 3
- x is destroyed
- y+1 is evaluated, and its value assigned to x
- y is still 1

- The formal parameter `x` is a local variable
- There is no link between `x` in the body of `foo` and `y` (`y` never changes its value)
- On exit from `foo`, `x` is destroyed
- It is not possible to transmit data from `foo` via the parameter

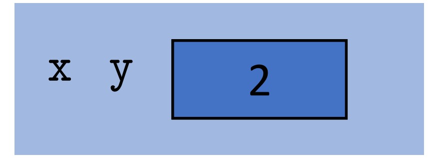


# Call by reference

# Call by reference (or variable)

- A reference (address) to the actual parameter (an expression with l-value) is passed to the function
- **The actual parameter must be an expression with l-value**
- References to the formal parameter are references to the actual one (aliasing)
- Transmission from and to `main` and `proc` ⇔
- Modifications to the formal parameter are transferred to the actual one
- On procedure termination the link between formal and actual is destroyed

# An example




```
int y = 1;
void foo (reference int x) {
    x = x+1;
}
...
y = 1;
foo(y);
```

x is another name for y

x is incremented to 2

x and its link with y are destroyed

a reference is passed

y is 2

- A reference (address, pointer) is passed
- `x` is an alias of `y`
- The actual value is an l-value
- On exit from `foo`, the link between `x` and the address of `y` is destroyed
- Transmission: Two-way between `foo` and the caller

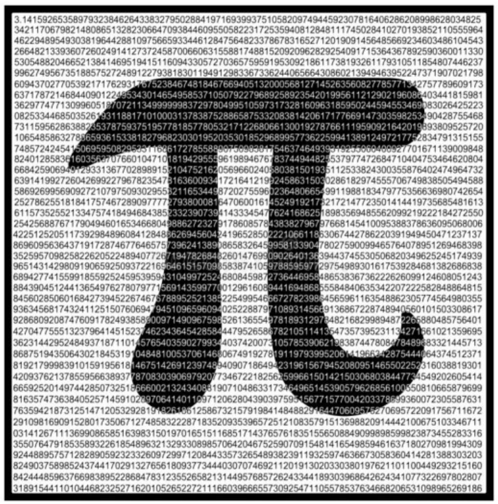
# Call by value vs call by reference

- Simple semantics
- Call could be expensive due to copy  operations
- Need for other mechanisms to communicate with the called procedure

**Call by value**

- Complicated semantics: aliasing
- Call is efficient
- Reference to formal parameter slightly more expensive

**Call by reference**



# Call by constant

# Call by constant/read -only

- Read-only parameter method: ⇨
- Procedures are not allowed to change the value of the formal parameter (could be statically controlled by the compiler)
- Implementation could be at the discretion of the compiler ("large" parameters passed by reference, "small" by value)
- In Java: `final`

```
void foo (final int x){ //x cannot be modified
```

- In C/C++: `const`


# Call by result

# Call by result

- The actual parameter is an expression that evaluates to an l-value
- No link between the formal and the actual parameter in the body
- The local environment is extended with an association between the formal parameter and a new variable
- When the procedure terminates, the value of the formal parameter is assigned to the location corresponding to l-value of the actual parameter
- Output-only communication: no way to communicate from `main` to `proc` ⇦

# An example

```
void foo (result int x) {
     x = 8;
}
...
y = 1;
foo(y);
```

x is a local variable

x is 8

the value of x is assigned to the current l-value of y

x is destroyed

y is 8

- Dual of the call by value
- No link between `x` and `y` in the body of `foo`
- When `foo` ends, the value of `x` is assigned to the location obtained with the l-value of `y`
- It is important when the l-value of `y` is determined (when the function is called or when it terminates)

# Call by value result

# Call by value-result

- Bidirectional communication using the formal parameter as a local variable ⇔
- The actual parameter must be an expression that can yield an l-value
- At the call, the actual parameter is evaluated and the r-value assigned to the formal parameter.
- At the end of the procedure, the value of the formal parameter is assigned to the location corresponding to the actual parameter

# An example

```
void foo (value-result int x) {
     x = x+1;
}
...
y = 8;
foo(y);
```

x is a local variable

the value of y assigned to x

x = 9

the value of x assigned to y

x is destroyed

y is 9

- No link between `x` and `y` in the body of `foo`
- When `foo` ends, the value of `x` is assigned to the location obtained with the l-value of `y`

# Call by name

# Call by name

- Aim: give a precise semantics to parameter passing
- Copy-rule mechanism of the actual parameter to the formal one
- A call to P is the same as executing the body of P after substituting the actual parameters for the formal one
- “Macro expansion", implemented in a semantically correct way: every time the formal parameter appears we re-evaluate the actual one
- Input and output parameters ⇔
- Appears to be simple but … it is not that simple: it has to deal with variables with the same name
- No longer used by any imperative language

# Actual parameter evaluation

```
int y;
void fie (int x){
     int y;
     x = x + 1; y = 0;
}
...
z = 1;
fie(z);
```

x is z

x is 2

y is 0

z is 2

# An example

```
int x=0;
int foo (name int y){
    int x = 2;
    return x + y;
}
...
int a = foo(x+1);
```

- Blindly applying the copy rule would lead us to a result of x+x+1=5
- Incorrect result as it would depend on the name of the local variable
- With a body `{int z = 2; return z + y;}` the result would have been z+x+1=3

- When the body contains the same name of the actual parameter, we say that it is captured by the local declaration
- In order to avoid substitutions in which the actual parameter is captured by the local declaration, we impose that the formal parameter – even after the substitution – is evaluated in the environment of the caller and not of the callee
- Substitute the actual parameter together with its evaluation environment – fixed at the time of the call

# Actual parameter evaluation

- A pair <`exp`,`env`> is passed (closure), where
  - `exp` is the actual parameter, not evaluated
  - `env` is the evaluation environment
- Every time the formula is used, exp is evaluated in env

```
int y;
void fie (int x){
     int y;
     x = x + 1; y = 0;
}
...
y = 1;
fie(y);
```

<x is y (external), {y/1}>

x is 2 and {y/2}

y (local) is 0

y is 2

# Call by name vs call by value-result

```
void fiefoo (valueresult int x,
valueresult int y) {
      x = x+1;
      y = 1;
}
...
int i = 1;
int[] A = new int[5];
A[1]=4;
fiefoo(i,A[i]);
```

x is 1, y is A[1]

x is 2

y is 1

i is 2, A[1] is 1

call- by value-result

```
void fiefoo (name int x,
name int y) {
      x = x+1;
      y = 1;
}
...
int i = 1;
int[] A = new int[5];
A[1]=4;
fiefoo(i,A[i]);
```

x is i, y is A[i]

x is 2

y is 1

i is 2, A[1] is 4, A[2]=1

call- by name

# Summing up

| Call type | Direction | Link between formal and actual parameters | | | Actual parameter l-value? | Implementation |
|---|---|---|---|---|---|---|
| | | Before | During | After | | |
| Value | ⇨ | * | | | NO | Copy |
| Reference | ⬄ | * | * | * | YES | Reference |
| Constant | ⇨ | * | | | NO | Copy and/or reference |
| Result | ⇦ | | | * | YES | Copy |
| Value-result | ⬄ | * | | * | YES | Copy |
| Name | ⬄ | | Every time it appears | | Can be | Closure |

# Exercise 4.3

- Say what will be printed by the following code fragment written in a pseudo-language which uses dynamic scope; the parameters are passed by reference.

```
{int x = 2;
 int fie(reference int
y){
    x = x + y;
 }
 {int x = 5;
 fie(x);
 write(x);
 }
 write(x);
}
```

# Exercise 4.4

- State what will be printed by the following fragment of code written in a pseudo-language which uses static scope and call by name.

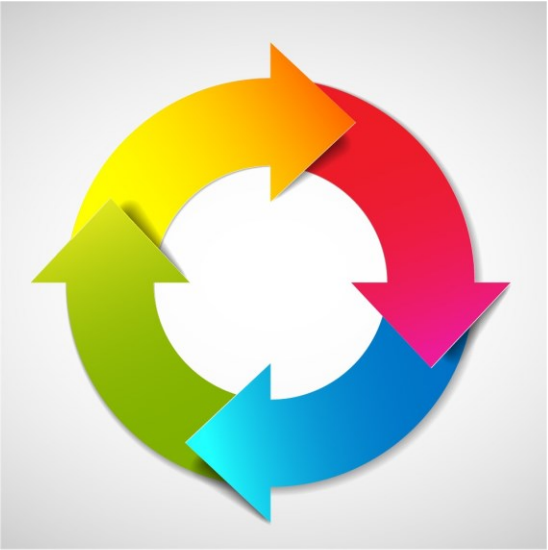
```
{int x = 2;
 void fie(name int y){
    x = x + y;
 }
 {int x = 5;
    {int x = 7}
    fie(x++);
    write(x);
 }
 write(x);
}
```

Remember that x++ means that x is increased by 1 but the value of the expression is returned before the increment

# Exercise 4.5

- State what will be printed by the following code fragment written in a pseudo-language which allows value-result parameters.

```
int X = 2;
void foo (value-result int Y){
    Y++;
    write(X);
    Y++;
 }
foo(X);
write(X);
```



# Recursion in ML

# Few suggestions for thinking in a recursive way

1. Carefully read the problem to solve
2. Think about the most trivial case for our function and for which we have an immediate result (e.g., the smallest set, 0, the empty list, the empty string) – without caring about recursive call
3. Think about the recursive case:
    1. Think you are able to solve the problem in the case immediately easier
    2. Build the more complex case

# Reversing a list

- Example: `reverse([1,2,3])` is `[3,2,1]`
  - Base case: empty list to empty list
  - Induction: reverse the tail of the list (recursively) and then append the head

```
> fun reverse L =
    if L = nil then nil
    else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3,4];
val it = [4, 3, 2, 1]: int list
> reverse["ab","bc","cd"];
val it = ["cd", "bc", "ab"]: string list
```

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list
```

| | |
|---|---|
| | |
| | |
| | |
| | |
| `reverse` | Definition of `reverse` |
| | |

Environment Before the call

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3];
```

| | |
|---|---|
| | |
| | |
| | |
| `L` | `[1,2,3]` |
| | |
| `reverse` | Definition of `reverse` |
| | |

Added in call to `reverse ([1,2,3])`

Environment Before the call

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3];
```

| | |
|---|---|
| | |
| | |
| `L` | `[2,3]` |
| `L` | `[1,2,3]` |
| | |
| `reverse` | Definition of `reverse` |
| | |

Added in call to `reverse ([2,3])`

Added in call to `reverse ([1,2,3])`

Environment Before the call

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3];
```

| | | |
|---|---|---|
| | | |
| L | [3] | Added in call to `reverse ([3])` |
| L | [2,3] | Added in call to `reverse ([2,3])` |
| L | [1,2,3] | Added in call to `reverse ([1,2,3])` |
| | | Environment Before the call |
| `reverse` | Definition of `reverse` | |
| | | |

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3];
```

| | |
|---|---|
| L | nil |
| L | [3] |
| L | [2,3] |
| L | [1,2,3] |
| | |
| reverse | Definition of `reverse` |
| | |

Added in call to `reverse (nil)`

Added in call to `reverse ([3])`

Added in call to `reverse ([2,3])`

Added in call to `reverse ([1,2,3])`

Environment Before the call

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3];
```

| | | |
|---|---|---|
| | | |
| L | [3] | Added in call to `reverse ([3])` |
| L | [2,3] | Added in call to `reverse ([2,3])` |
| L | [1,2,3] | Added in call to `reverse ([1,2,3])` |
| | | Environment Before the call |
| `reverse` | Definition of `reverse` | |
| | | |

`nil@[3] =[3]`

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3];
```

| | |
|---|---|
| | |
| | |
| `L` | `[2,3]` |
| `L` | `[1,2,3]` |
| | |
| `reverse` | Definition of `reverse` |
| | |

Added in call to `reverse ([2,3])`

Added in call to `reverse ([1,2,3])`

Environment Before the call

`[3]@[2] =[3,2]`

# How does the function execution works?

- The arguments are evaluated
- An addition to the environment: call-by-value

```
> fun reverse L =
     if L = nil then nil
     else reverse (tl L) @ [hd L];
val reverse = fn: ''a list -> ''a list

> reverse [1,2,3];
val it = [3, 2, 1]: int list
```

| | | |
|---|---|---|
| | | |
| | | |
| | | Added in call to `reverse ([2,3])` |
| L | [1,2,3] | Added in call to `reverse ([1,2,3])` |
| | | Environment Before the call |
| `reverse` | Definition of `reverse` | |
| | | |

[3,2]@[1] =[3,2,1]

# Different ways for writing functions

- Syntax `fn` (corresponds with λ in the λ-calculus, that we will see later)

```
fn <param> => <expression>;
```

- We can also directly apply the function to the parameter (anonymous function)

```
(fn n => n+1) 5;
```

- We can associate the functions to a name, just like values

```
> val increment = fn n => n+1;
val increment = fn: int -> int
```

- In case the function is recursive use rec

```
> val rec fact n = fn 0 => 1
                 |n => n*fact(n-1);
```

- Syntactic sugar notation for functions with names (no need to specify rec)

```
> fun increment n = n+1;
val increment = fn: int -> int
```

# Nonlinear recursion

- A function can call itself recursively multiple times
- Example: Number of combinations of *k* things out of *n*
  - Written $\binom{n}{k}$
  - Can be shown to be equal to $\frac{n!}{(n-k)!k!}$
  - We can also use the following recursive definition

# Combinations

- Base case. If $k = 0$ the number of ways to pick 0 out of $n$ is 1. Similarly, if $k = n$, there is exactly one way to pick $n$ out of $n$.

$$\binom{n}{0} = \binom{n}{n} = 1$$

- Induction. If $0 < k < n$ to select $k$ out of $n$ we can
  - reject the first thing and select $k$ out of the remaining $n - 1$
  - pick the first thing, and pick $k - 1$ out of the remaining $n - 1$
  - formally

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- We assume that 0 <=k<=n
- We use this to write a ML program

# Combinations

```
> fun comb(n,k) = (* assumes 0 <= k <= n *)
    if k=0 orelse k=n then 1
    else comb(n-1,k) + comb(n-1,k-1);
val comb = fn: int * int -> int
```

- Without using `orelse`:

```
> fun comb (n,k) =
    if k=0 then 1
    else
        if k=n then 1 else comb(n-1,k)+comb(n-1,k-1);
val comb = fn: int * int -> int

> comb (5,0);
val it = 1: int
> comb (5,5);
val it = 1: int
> comb (5,2);
val it = 10: int
```

# Why the type is fn: int*int -> int?

```
> fun comb(n,m) = (* assumes 0 <= m <= n *)
    if m=0 orelse m=n then 1
    else comb(n-1,m) + comb(n-1,m-1);
val comb = fn: int * int -> int
```

- Result of `if` clause is an integer
- Therefore `comb(n-1,m)+comb(n-1,m-1)` is an integer
- Therefore, so are `comb(n-1,m)` and `comb(n-1,m-1)`
- Therefore `comb` returns an integer
- The expression `n-1` and `m-1` must be integers (because of -1)
- `comb` maps pairs of integers to an integer

# Mutual recursion

- How can we do mutual recursion (functions, types) in languages where a name must be declared before use?
  1. Relax this rule for functions and/or types
     - Java via methods

```
{void f(){
    g();
}
{void g(){
    f();
 }
}
```

     - Pascal by pointer types

```
type list = ^elem;
type elem = record
     info: integer;
     next: list;
end
```

^T denotes the type of pointers to objects of type T

# Mutual recursion

2. Incomplete definitions
   - Ada

```
type elem;
type list is access elem;
type elem is record
     info: integer;
     next: list;
end
```

   - C

```
struct elem;
struct elem {
     int info;
     elem *next }
end
```

   - Pascal

```
procedure fie(A:integer); forward;
procedure foo(B: integer);
     begin ... fie(3); ... end
procedure fie;
     begin ... foo(4); ... end
```

# Mutual recursion

- Two functions can call one another recursively
- Example. A function that takes a list `L` and produces a list with the first, third, fifth etc. elements of `L`
- Two functions:
  - `take(L)` takes the first element of `L` and then alternates
  - `skip(L)` skips the first element and then calls `take`

# First attempt

```
> fun take(L) =
    if L = nil then nil
    else hd(L) :: skip(tl(L));
> fun skip(L) =
    if L = nil then nil
    else take(tl(L));

> fun take(L) =
    if L = nil then nil
    else hd(L) :: skip(tl(L));
poly: : error: Value or constructor (skip) has not been
declared
Found near if L = nil then nil else hd (L) :: skip (tl (L))
```

Does not work!

# Solution: keyword `and`

```
fun
          <definition of first function>
and
          <definition of second function>
and ...

fun
take(L) =
          if L = nil then nil
          else hd(L) :: skip(tl(L))
and
skip(L) =
          if L = nil then nil
          else take(tl(L));
val skip = fn: ''a list -> ''a list
val take = fn: ''a list -> ''a list

> take ([1,2,3,4,5]);
val it = [1, 3, 5]: int list
```

# Summary

- Scoping mechanisms
- Abstraction of control and methods for parameter passing
- Recursion in ML

# Readings

- Chapter 4 of the reference book
  - Maurizio Gabbrielli and Simone Martini "Linguaggi di Programmazione -  Principi e Paradigmi", McGraw-Hill