λ

Logic Programming – Part I

Programmazione Funzionale
2024/2025
Università di Trento
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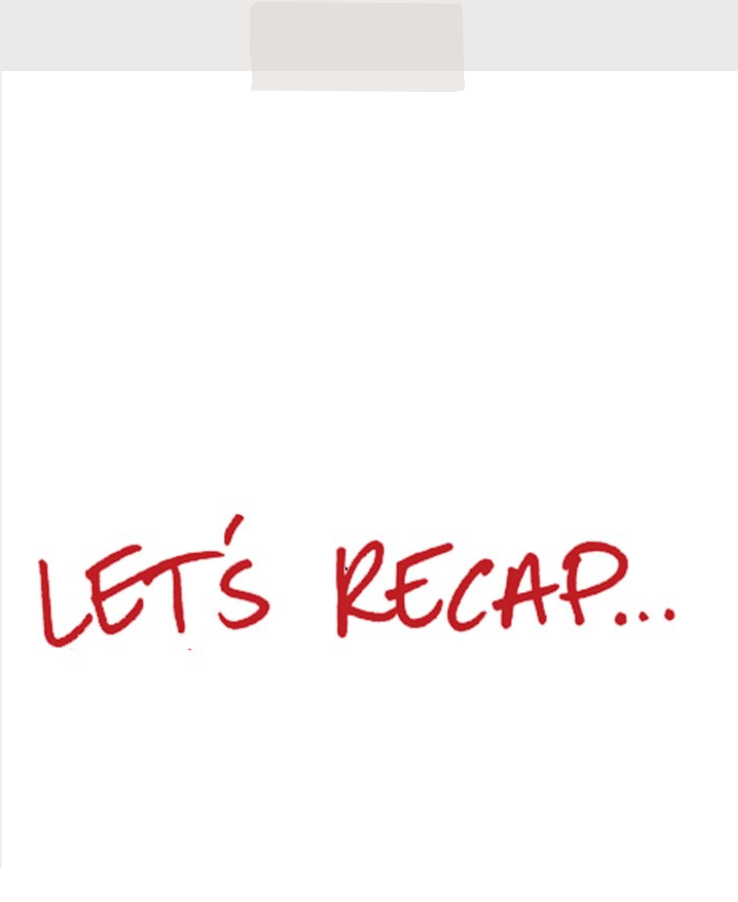
Next lectures

- Tuesday May 27: short seminar
- Thursday May 29: exam simulation
- Last lecture: June 3

Today

- Recursion in lambda calculus
- Logic Programming
- Prolog
  - Syntax
  - Resolution and unification
  - Arithmetic
  - Functions
  - Lists



Recap

The λ-calculus

- We have seen so far a version of λ-calculus including constants (0,1,2) and functions (+,*)
- The pure λ-calculus, however, is a very limited language
  - Expressions: Only variables, application and abstraction
  - For example, $\lambda x.x + 2$ should be invalid, since 2 is not a variable

Booleans

- $true = \lambda x. \lambda y. x$
- $false = \lambda x. \lambda y. y$
- If a then b else c = a b c
- Boolean operations
  - not = $\lambda x. x\ false\ true$
    - not x = if x then false else true
    - not true → $(\lambda x. x\ false\ true) true \rightarrow (true\ false\ true) \rightarrow false$
  - and = $\lambda x. \lambda y. x\ y\ false$
    - and x y = if x then y else false
  - or = $\lambda x. \lambda y. x\ true\ y$
    - or x y = if x then true else y

Pairs

- Encoding of a pair (a,b)
  - (a,b) = $\lambda x. if\ x\ then\ a\ else\ b$
  - fst $=\ \lambda f. f\ true$
  - snd = $\lambda f. f\ false$

Natural numbers

- n is represented by the higher-order function that maps any function f to its n-fold composition
- In other words, the “value" of the numeral n is equivalent to the number of times the function is applied to its argument.
- More formally

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n\ times}$$

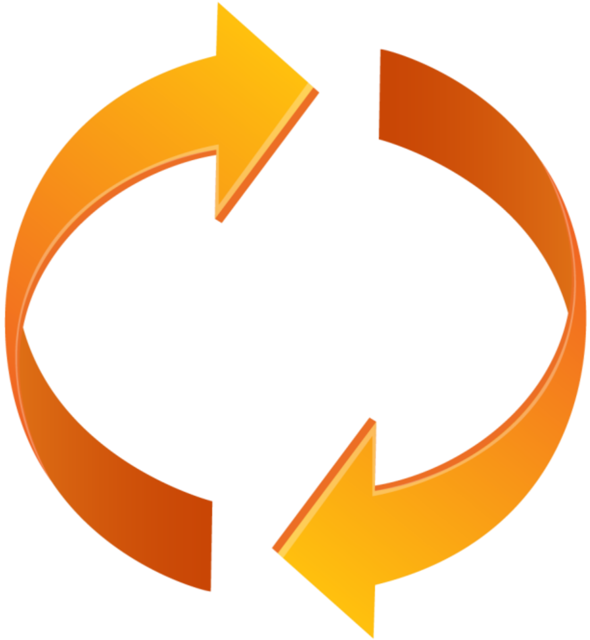
- That is $n = \lambda f.\lambda x.$<apply f n times to x>

Natural numbers: function definition

Number	Function definition	Lambda-expression
0	$0\ f\ x = \ x$	$\lambda f.\lambda x.x$
1	$1\ f\ x = \ f\ x$	$\lambda f.\lambda x.f\ x$
2	$2\ f\ x = \ f(f\ x)$	$\lambda f.\lambda x.f(f\ x)$
3	$3\ f\ x = \ f(f(f\ x))$	$\lambda f.\lambda x.f(f(f\ x))$
…	…	
n	$n\ f\ x = \ f^n\ x$	$\lambda f.\lambda x.f^n\ x$

Natural number operations: addition

- $n + m$ means: "apply f n times to the result of applying f m times to x"
- $\lambda n.\lambda m.\lambda f.\lambda x.nf(mfx)$



Recursion

Recursion in λ-calculus

- We claimed that Lambda-calculus is powerful
- We saw how to define expressions:
  - Booleans and their operations
  - Pairs
  - Numbers and their operations

Recursion

- How to implement recursion in the λ-calculus?
  - Functional paradigm: using recursion
  - But how do we implement recursion?
- We cannot give a name to λx … ., but have to implement recursion using only abstraction and application
- Trivial example

```
fun f n = if n=0 then 1 else n*f(n-1);
```

- What is this function?

Implementing recursion

- Suppose we want to write the factorial function which takes a number n and computes n!

```
λn.if (n=0) then 1 else (n *(f (n-1)))
```

- This does not work. Because what is the unbound variable f?
- It would work if we could somehow make f be the function above

Eliminating recursion

- To give access to the function f, what about passing f as another parameter?
- Making f a parameter, we get
$$\lambda f.\lambda n.if\ n = 0\ then\ 1\ else\ n * f(n-1)$$
- We have then eliminated the recursion

Recursion

- We can write the function as

  $$G = \lambda f.\lambda n.\texttt{ if n=0 then 1 else n * f(n-1)}$$

- In other words, we look for $f = G(f)$ where G is a higher-order function which takes a function as argument, and returns a function
- “Solving" this equation gives us f
- G is a function that if we give it a function f able to compute the next step, then it returns the factorial function, that is G is a description of the factorial function but we need the application
- In ML, this is equivalent to define

```
fun g f n = if n=0 then 1 else n*f(n-1);
```

- But how do we solve this problem?

Y

The Y-combinator

The general problem

- Given a function G, find f such that $f \ =_\beta \ Gf$
- This means to find a fixpoint of the operator G
- The Y combinator is one way to compute such a fixpoint

$$Y \ = \ \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

- The Y combinator is the solution to our problem: it is a function that applied to G returns the function f we were looking for, that is Y is a function that allows us to call again G

The general problem

- We started from a function `fact`:

  $\lambda n.\texttt{if}\ n = 0\ \texttt{then 1 else n*f(n-1)}$
- We wrote a function `ps_fact` G, which is no longer recursive

  $G = \lambda f.\lambda n.\texttt{ if n=0 then 1 else n * f(n-1)}$
- We need a function that allows us to compute the fixpoint
- This is what Y does!
- By applying the Y combinator to the pseudo-recursive function, we obtain our factorial function `fact`:

  `Y ps_fact = fact`
- `ps_fact` describes what the recursion does (given the next step), while `Y ps_fact` is the application of the recursive function, that is the factorial function

The Y combinator

$Y\ e =$

$(\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))e \mapsto$

$(\lambda x.e(xx))(\lambda x.e(xx)) \mapsto$

$e(\lambda x.e(xx))(\lambda x.e(xx)) =_\beta e(Y\ e)$

- Therefore, $Y\ e\ =\ e(Y\ e)$ and so $YG = G(YG)$, i.e., YG is a fixpoint for G
  - We can use Y to achieve recursion for G

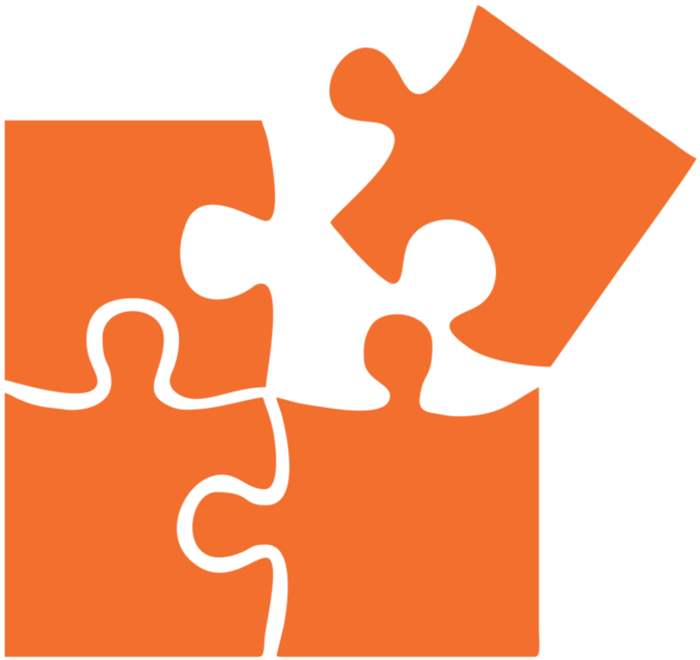
Example

- `ps_fact` =

  $$\lambda f.\lambda n. if\ n\ =\ 0\ then\ 1\ else\ n\ *\ (f\ (n-1))$$

- The second argument of `ps_fact` is the integer
- The first argument is the function to call in the body
  - We'll use Y to make this function recursively call fact

$$(Y\ ps\_fact)1\ =\ \big(ps\_fact\ (Y\ ps\_fact)\big)1\ \rightarrow$$
$$if\ 1\ =\ 0\ then\ 1\ else\ 1\ *\ ((Y\ ps\_fact)\ 0)\ \rightarrow$$
$$1\ *\ ((Y\ ps\_fact)\ 0)\ =$$
$$1\ *\ (ps\_fact\ (Y\ ps\_fact)\ 0)\ \rightarrow$$
$$1\ *\ (if\ 0\ =\ 0\ then\ 1\ else\ 0\ *\ ((Y\ ps\_fact)\ (-1))\ \rightarrow$$
$$1\ *\ 1\ \rightarrow\ 1$$



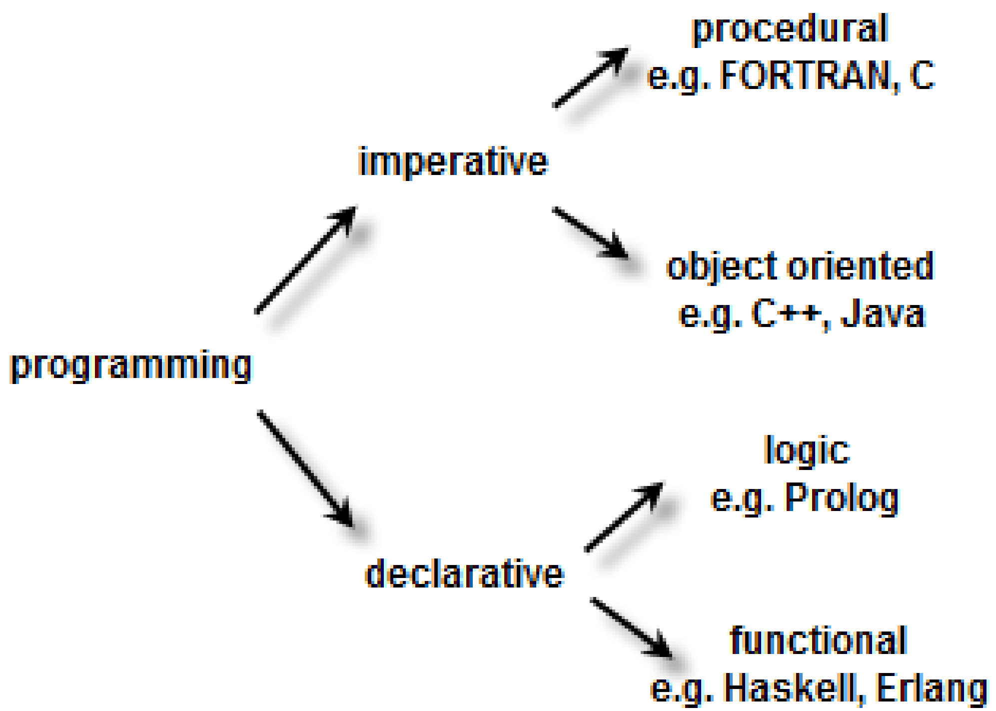
Logic Programming

Different characteristics

- Imperative (how to do?)
  - Specify a sequence of operations that modify a state (statements)

- Declarative (what to do?)
  - What needs to be solved to get the result

… different languages



… different languages

- Imperative (how to do?)
  - Classical: Fortran, Pascal, C
  - Object-oriented: Smalltalk, C++, Java
  - Scripting: Perl, Python, Javascript

- Declarative (what to do?)

```
int main(){
    printf("Hello World");
    return 0;
}
```

```
public class HelloWorld{
  public static void
main(String[] args) {
    System.out.println("He
llo World"); }}
```

```
print ‘‘Hello, world!\n’’
```

… different languages

- Imperative (how to do?)
  - Classical: Fortran, Pascal, C
  - Object-oriented: Smalltalk, C++, Java
  - Scripting: Perl, Python, Javascript

- Declarative (what to do?)
  - Functional: ML, Ocaml
  - Logic: Prolog

```
output = program (input)
```

```
program (input, output)
```

Logic Programming Concepts

- The programmer states a collection of axioms from which theorems can be proven
- The programmer states a goal, and the language implementation attempts to find a collection of axioms and inference steps (including choices of values for variables) that together imply the goal
- Prolog is the most widely used such language

Logic Programming Concepts

- In most logic languages, axioms are written in a standard form known as a Horn clause
- A Horn clause consists of a head, or consequent term `H`, and a body consisting of terms `Bi`
- We write

```
H:- B1, B2 , ... , Bn
```

- The semantics of this statement are that when the `Bi` are all true, we can deduce that `H` is true as well
- We can read this as "`H`, if `B1, B2, ...`, and `Bn`"
- Horn clauses can be used to capture most, but not all, logical statements

Horn clauses

- In order to derive new statements, a logic programming system combines existing statements, through a process known as resolution
  - If we know that A and B imply C, and that C implies D, we can deduce that A and B imply D
  - Terms such as A, B, C, and D may consist not only of constants, but also predicates applied to atoms or to variables
  - During resolution, free variables may acquire values through unification with expressions in matching terms

```
C :- A, B;
D :- C;
----------
D:- A,B
```

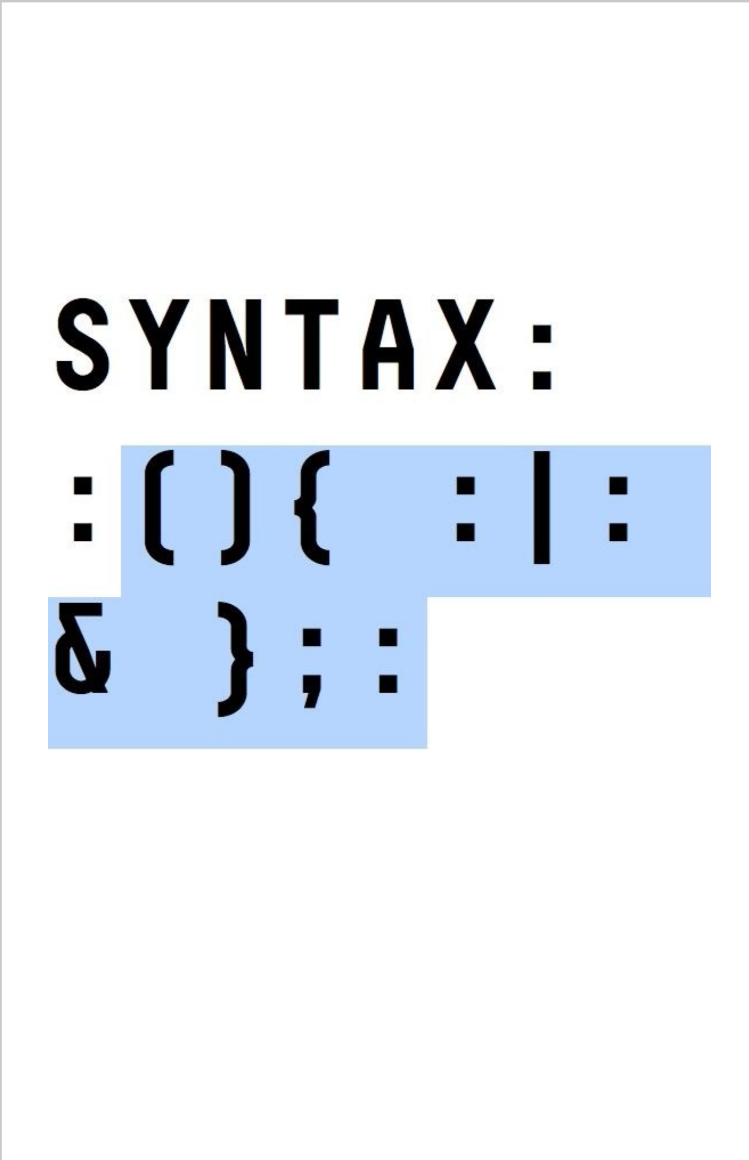
```
p(X) :- q(X);
q(1);
------------------
p(1)
```

Prolog

```
rainy(rochester).
rainy(seattle).
cold(seattle).
```

```
snowy(X):-rainy(X),cold(X).
```

```
?- son(X,Y).
X = charlie, Y = bob;
```



Syntax

Prolog

- A Prolog interpreter runs in the context of a database of clauses (Horn clauses) that are assumed to be true
- Each clause is composed of terms
- A term may be:
  - a constant
  - a variable
  - a structure

Prolog

- A term may be:
  - a constant may be an atom or a number
    - An atom: looks like an identifier beginning with a lowercase letter, a sequence of “punctuation” characters, or a quoted character string
    - A number
  - a variable: like an identifier beginning with an uppercase letter
    - Variables can be instantiated to (i.e., can take on) arbitrary values at run time as a result of unification
  - structure:
    - a logical predicate or
    - a data structure

```
bob horse2
'horse' mario
...
```

```
123 -234
3.14
```

```
X AA List ...
```

```
sum(2,3)
bigger(horse,duck)
...
```

Structures

- Structures consist of an atom, called the functor, and a list of arguments

```
teaches(scott,cs254)
bin_tree(foo,bin_tree(bar,glarch))
```

functor arguments

Functors in Prolog are a completely different concept from functors in ML

- Prolog requires the opening parenthesis to come immediately after the functor, with no intervening space
- Arguments can be arbitrary terms: constants, variables, or (nested) structures
- Conceptually, the programmer may think of certain structures as logical predicates
- We use the term predicate to refer to the combination of a functor and an "arity" (number of arguments)

Clauses: facts and rules

- The clauses in a Prolog database can be classified as
  - facts or
  - rules
- Both end with a period
- A fact is a Horn clause without a right-hand side
- Thus it looks like this (the implication symbol is implicit)

```
rainy(rochester).
```

- A fact can be expressed as `p(t1,...,tn)` where `p` is the name of the fact and `t1, ..., tn` are terms

Facts and rules

- A rule has a right-hand side:

  `snowy(X):-rainy(X),cold(X).`
- The token :- is the implication symbol, and the comma indicates “and”
- `X` is snowy if `X` is rainy and `X` is cold
- A program is a sequence of clauses

An example of Prolog program

lowercase for atoms

uppercase for variables

Clauses:
Fact and rules

```
% facts:
rainy(rochester).
rainy(seattle).
cold(rochester).

% rules for "X is snowy"
snowy(X):-rainy(X),cold(X).
```

Query (or goal)

- Goal: the predicate we wish to prove to be true.
- Clause with an empty left-hand side
  - Queries do not appear in Prolog programs
  - Rather, one builds a database of facts and rules and then initiates execution by giving the Prolog interpreter (or the compiled Prolog program) a query to be answered (i.e., a goal to be proven)
  - In most implementations of Prolog, queries are entered with a special ?− version of the implication symbol

Asking for a query (goal)

- Typing the following:

```
rainy(seattle).
rainy(rochester).
?- rainy(C).
```

  the Prolog interpreter would respond with `C = seattle.`

- Of course, `C = rochester` would also be a valid answer, but Prolog will find `seattle` first, because it comes first in the database
  - One of the differences between Prolog and pure logic

More solutions for a query (goal)

- To find all possible solutions, we can ask the interpreter to continue by typing a semicolon:

```
C = seattle;
C = rochester
```

- With another semicolon, the interpreter will indicate that no further solutions are possible:

```
C = seattle;
C = rochester;
false.
```

- Given

```
rainy(seattle).
rainy(rochester).
cold(rochester).
snowy(X):-rainy(X),cold(X).
```

  the query `?-` `snowy(C)` yields only one solution:

```
C=Rochester.
```

Let’s try

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Declarative and procedural interpretation

- A clause can be interpreted
  - in a declarative way
    - H :- $B_1, \ldots, B_n$ If $B_1, \ldots, B_n$ are true, then also H is true
    - A query is a formula for which we want to prove that is a logical consequence of the program
  - In a procedural way
    - H :- $B_1, \ldots, B_n$ To prove/compute H it is necessary first to prove/compute $B_1, \ldots, B_n$
    - A predicate is the name of a procedure, whose defining clauses constitute the body
    - The goal is a sort of main

If you are curious to try …

- You can try with SWI-Prolog

  https://www.swi-prolog.org/
- It is also installed on laboratory machines

- `?- [‘namefile.pl’].`
- Where namefile.pl contains facts and rules, while the goal is inserted via prompt. Alternatively:
- `?- consult(namefile).`

- If you want to edit and then reload:

```
?- edit(filename).
?- make.
```

- To exit:

```
?- halt.
```

Resolution and unification

Resolution

- The resolution principle (Robinson) says that if `C1` and `C2` are Horn clauses and the head of `C1` matches one of the terms in the body of `C2`, then we can replace the term in `C2` with the body of `C1`

Resolution

- Consider the following

```
takes(alice, his201).
takes(alice, cs254).
takes(bob, art302).
takes(bob, cs254).
classmates(X, Y) :- takes(X, Z), takes(Y, Z).
```

- If we let `X` be `alice` and `Z` be `cs254`, we can replace the first term on the right-hand side of the last clause with the (empty) body of the second clause, yielding the new rule

```
classmates(alice, Y) :- takes(Y, cs254).
```

- In other words, `Y` is a classmate of `alice` if `Y` takes `cs254`.

Unification

- The pattern-matching process used to associate `X` with `alice` and `Z` with `cs254` is known as unification
- Variables that are given values as a result of unification are said to be instantiated

Unification in Prolog

- Unification is a key feature in Prolog
- Two terms unify
  - if they are identical

    `?- takes(alice, cs254).` -> unifies directly with the fact
  - they can be made identical by substituting variables

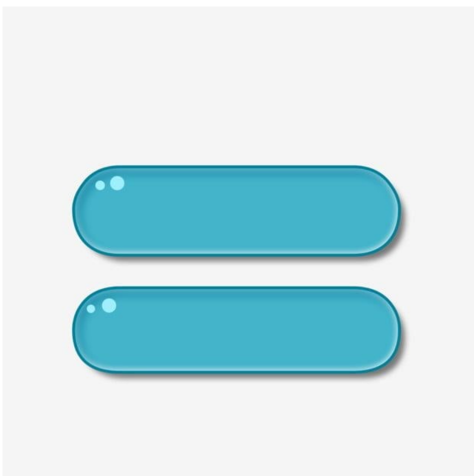
    `?- takes(alice, X).` -> variable `X` is instantiated with `cs254`

```
takes(alice, cs254)
takes(bob, cs254)
```

- The idea is unifying the goal with the head of a rule
  - If succeeds, clauses in body become subgoals
  - Continue until all subgoals are satisfied
    - If search fails, backtrack and try untried subgoals

Unification in Prolog

- The unification rules for Prolog are as follows:
  - A constant unifies only with itself
  - Two structures unify if and only if they have the same functor and the same number of arguments, and the corresponding arguments unify recursively
  - A variable unifies with anything
    - If the other thing has a value, then the variable is instantiated
    - If the other thing is an uninstantiated variable, then the two variables are associated in such a way that if either is given a value later, that value will be shared by both.



Equality

Equality in Prolog

- Equality in Prolog is defined in terms of "unifiability"
- The goal `?-A=B.` succeeds if and only if `A` and `B` can be unified

Example

```
?- a = a.
true. % constant unifies with itself

?- a = b.
false. % but not with another constant

?- foo(a, b) = foo(a, b).
true.

?- X = a.
X = a. % only one possibility

?- foo(a, b) = foo(X, b).
X = a. % arguments must unify only one possibility
```

Equality

- Two variables can be unified without instantiating them
- If we type

```
?- A = B.
```

the interpreter will respond

```
A = B.
```

Arithmetic

Arithmetic

- The usual arithmetic operators are available in Prolog, but they play the role of predicates, not of functions
- `+(2, 3)`, which may also be written `2 + 3`, is a two-argument structure, not a function call
- This means that it will not unify with 5

```
?- (2 + 3) = 5.
false.
```

Arithmetic

- To handle arithmetic, Prolog provides a built-in predicate, `is`, that unifies its first argument with the arithmetic value of its second argument

```
?- is(X, 1+2).
X=3.


?- X is 1+2.
X = 3.

?- 1+2 is 4-1.
false.

?- X is Y.
<error> Arguments are not sufficiently instantiated

?- Y is 1+2, X is Y.
Y=X, Y = 3.
```

Let’s try

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Event code

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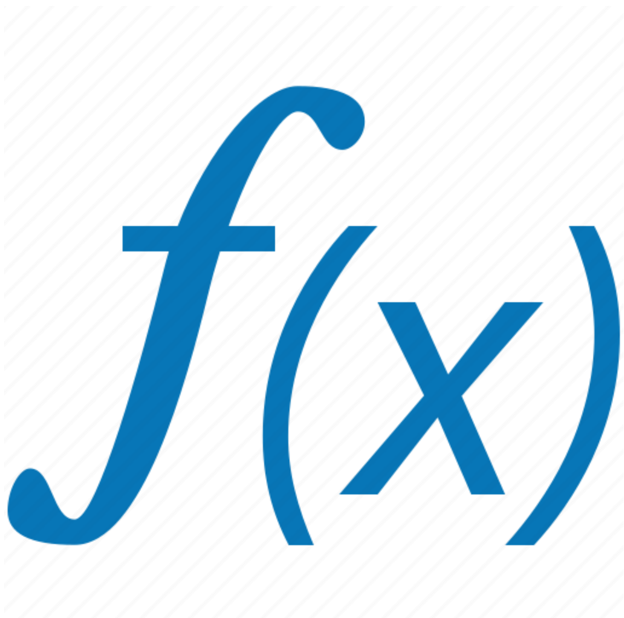
No mutable variables

- `=` and `is` operators do not perform assignment
  - Variables take on exactly one value ("unified")
- Example
  - `foo(...,X) :- ... X = 1,... % true only if X = 1`
  - `foo(...,X) :- ... X = 1, ..., X = 2, ... % always fails`
  - `foo(...,X) :- ... X is 1,... % true only if X = 1`
  - `foo(...,X) :- ... X is 1, ..., X is 2, ... % always fails: X can't be unified with 1 & 2 at the same time`



Functions

Function parameters and return value

- `increment(X,Y) :- Y is X+1.`
- `?-increment(1,Z).`
  `Z=2.`
- `?-increment (1,2).`
  `true.`
- `?-increment(Z,2).`
  `Arguments are not sufficiently instantiated.`

X+1 cannot be evaluated since X has not yet been instantiated.

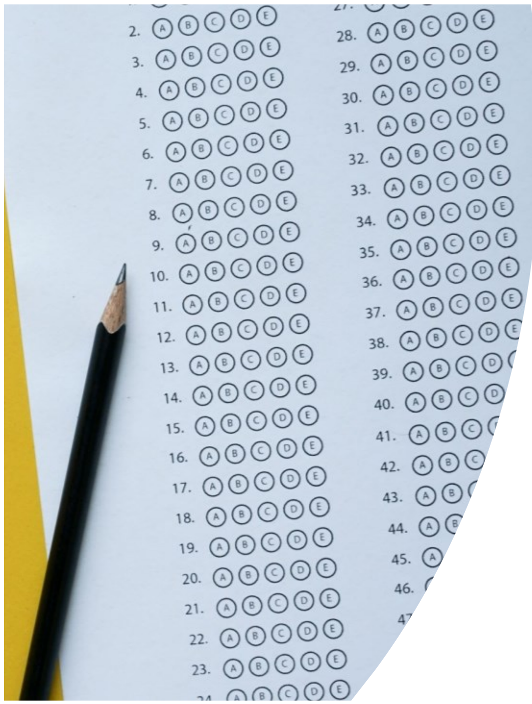
- `addN(X,N,Y):- Y is X+N.`
- `?- addN(1,2,Z)`
  `Z = 3.`

Recursion

- `addN(X,0,X).`
- ```
  addN(X,N,Y):- X1 is X+1,
                N1 is N-1,
                addN(X1,N1,Y).
  ```
- `?-addN(1,2,Z).`

  `Z=3.`

addN is defined as recursively adding 1 to X N times
  ```



Lists

Lists

- A list is a finite sequence of elements
- List elements in Prolog are enclosed in square brackets
- Example: `[a,c,2,'hi',[W,3]]`
- The length of a list is the number of elements it has
- Differently from ML, all sorts of Prolog terms can be elements of a list
- There is a special list: the empty list [ ]

Head and Tail

- A non-empty list can be thought of as consisting of two parts
  - The head
  - The tail
- As in ML, the head is the first item in the list
- The tail is everything else
  - The tail is the list that remains when we take the first element away
  - The tail of a list is always a list

Lists

- The construct `[a, b, c]` is shorthand for the compound structure `.(a, .(b, .(c, [])))`, where `[]` is an atom (the empty list) and . is a built-in cons-like predicate.
- How does it work matching?

```
?-[X,1,Z]=[a,_,17]
X=a,
Z=17
```

- `_` is the anonymous variable: used when we need a variable but we are not interested in how it is instantiated. Each occurrence is independent, i.e., it can be bound to something different

Vertical bar notation

- Prolog adds an extra convenience: an optional vertical bar that delimits the tail of the list
- Using this notation, `[a, b, c]` can be expressed as `[a | [b, c]]`, `[a, b | [c]]`, or `[a, b, c | []]`
- `[H|T]` is syntactically similar to ML `h::t`

```
?-[Head|Tail] = [a,b,c].
 Head = a.
 Tail = [b,c].
```

- The vertical bar notation is particularly useful when the tail of the list is a variable

Examples

- ?-[X,Y,Z]=[1,2,3].

  X=1.

  Y=2.

  Z=3.
- ?-[1,2,3,4]=[_,X|_].

  X = 2.
- ?-[1,2|X]=[1,2,3,4,5].

  X= [3,4,5].

Defining more complex predicates: `member` and `sorted`

```
member(X, [X|T]).
member(X, [H|T]) :- member(X, T).

sorted([]). % empty list is sorted
sorted([X]). % singleton is sorted
sorted([A, B | T]) :- A =< B, sorted([B | T]).
% compound list is sorted if first two elements
are in order and the remainder of the list
(after first element) is sorted
```

- Here `=<` is a built-in predicate that operates on numbers

append (or concatenate)

- append(L1,L2,L3) succeeds when L3 unifies with L2 appended at the end of L1, that is L3 is the concatenation of L1 and L2.
- Given this definition:

```
  append([], L2, L2). /*if L1 is empty,
then L3 = L2 */
  append([H | L1], L2, [H | L3]) :-
append(L1, L2, L3) /*prepending a new
element to L1, means prepending it to L3
as well*/
```

Examples

- `?- append([a, b, c], [d, e], L).`
  `L = [a, b, c, d, e]`
- `?- append(X, [d, e], [a, b, c, d, e]).`
  `X = [a, b, c]`
- `?- append([a, b, c], Y, [a, b, c, d, e]).`
  `Y = [d, e]`
- `?- append (X,Y,[a,b,c])`
  `X=[], Y=[a,b,c];`
  `X=[a], Y=[b,c]; ...`

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Readings

- Chapter 12 of the reference book
  - Maurizio Gabbrielli and Simone Martini "Linguaggi di Programmazione - Principi e Paradigmi", McGraw-Hill
- Few slides from the University of Maryland

Summary

- Recursion in lambda calculus
- Logic Programming
- Prolog
  - Syntax
  - Resolution and unification
  - Arithmetic
  - Functions
  - Lists

Next time

- Logic Programming (second part)