Suppose we have 2 images where A is the reference image and B is the crooked image. In an affine transformation, each point (x, y) of image A is mapped to point (u, v) in a new coordinate system. To represent translation, we use homogeneous coordinates for the input vector and get:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{21} & M_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{q} = \mathbf{M}\mathbf{p}$$

where $q \in R^{2\times1}$, $M \in R^{2\times3}$ and $p \in R^{3\times1}$.

By carrying out the matrix multiplication, it can be seen that:

$$\begin{cases} u = M_{11}x + M_{12}y + M_{13} \\ v = M_{21}x + M_{22}y + M_{23} \end{cases}$$

Now we select multiple points in A that match points in B. The goal is to solve for the affine transformation matrix **M** so we oversample the number of points (i.e. more than 2) to minimize the error and produce a more stable transformation. As such, our column vectors **p** and **q** become:

$$P = \begin{bmatrix} x_0 & x_1 & \dots & x_{n-1} \\ y_0 & y_1 & \dots & y_{n-1} \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} u_0 & u_1 & \dots & u_{n-1} \\ v_0 & v_1 & \dots & v_{n-1} \end{bmatrix}$$

and thus, our goal is to solve the linear system of equations

$$\mathbf{Q} = \mathbf{M}\mathbf{P}$$

where $Q \in R^{2\times n}$, $M \in R^{2\times3}$ and $P \in R^{3\times n}$.

The problem is that we are used to solving systems of the form $Ax = b$, where x is the unknown. In our case, we are trying to solve the system $xA = b$. There are two ways to solve this. We could just take the transpose on both sides like so:

$$
\begin{aligned}
MP &= Q \\
(MP)^T &= Q^T && \text{take the transpose on both sides} \\
P^T M^T &= Q^T
\end{aligned}
$$

Now $A = P^T$, $x = M^T$ and $b = Q^T$. Thus, after solving for x, take the transpose to obtain the true solution $M = x^T$.

However, my goal is to use Cholesky factorization to solve the normal equations which requires that A be symmetric positive definite (SPD). This is not the case since we have $A = (P^T)^T P^T = PP^T$ and it is actually $P^T P$ that is SPD.

Thus, a workaround to this is to flatten **M** and **Q** as column vectors, rearrange **P** in a clever manner, and remark the following:

$$
\begin{bmatrix}
x_0 & y_0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_0 & y_0 & 1 \\
x_1 & y_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & y_1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n-1} & y_{n-1} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{n-1} & y_{n-1} & 1
\end{bmatrix}
\begin{bmatrix}
M_{11} \\ M_{12} \\ M_{13} \\ M_{21} \\ M_{22} \\ M_{23}
\end{bmatrix}
=
\begin{bmatrix}
u_0 \\ v_0 \\ u_1 \\ v_1 \\ \vdots \\ u_{n-1} \\ v_{n-1}
\end{bmatrix}
$$

$$\mathbf{P}^* \mathbf{M}^* = \mathbf{Q}^*$$

where $P^* \in R^{2n \times 6}$, $M^* \in R^{6 \times 1}$ and $P^* \in R^{2n \times 1}$.

Since this is in standard form, we obtain the solution as follows:

$$
\begin{aligned}
P^* M^* &= Q^* \\
P^{*T} P^* M^* &= P^{*T} Q^*
\end{aligned}
$$

and we can apply Cholesky factorization to solve for M^* which we can rearrange to get **M**.

Now remember, M is the transformation which goes from reference image to crooked image. Our goal is to align the crooked image, i.e. go from crooked to reference. Thus, we just switch the matrices P and Q, that is, in the matrix P we put the coordinates (u, v) and the matrix Q will be the flattened (x, y).

Finally, after computing the matrix M, we use a grid generator and bilinear interpolation on the crooked image to generate the aligned image.