8 Second-order and higher-order equations

In this Chapter we will consider second-order ordinary differential equations (with a brief mention of higher-order ODEs), in which case the general form of the ODE can be written as

$$F\left(x,\, y,\, \frac{dy}{dx},\, \frac{d^2y}{dx^2}\right) = 0,$$

where F is a particular function.

As with the first-order equations, for all the cases we consider we will be able to solve this equation for the highest derivative, so we can write

$$\frac{d^2y}{dx^2} = f\left(x,\, y,\, \frac{dy}{dx}\right). \tag{8.1}$$

with appropriate initial or boundary conditions.

With first-order ODEs we saw that a solution was found by performing a single integration, which introduced a single integration constant. Because this constant was unknown there was a family of solutions, such as $y(x) = C\,e^{2x}$. For any constant C we had a different solution but, as long as the ODE satisfied some smoothness constraints, all solutions could be written in this general solution form. A particular solution was then found using an initial condition, which determined the constant C. We also saw that the ODE and the initial condition had a unique solution.

When we consider second-order ODEs we might imagine that two integrations are necessary. For instance, solving

$$\frac{d^2y}{dx^2} = y \qquad\qquad \Rightarrow \qquad\qquad y(x) = c_1\,e^{x} + c_2\,e^{-x}$$

leads to two constants c_1 and c_2. To find these two constants we need two initial conditions or two boundary conditions. The set of solutions to this differential equation is then equal to all linear combinations of e^x and e^{-x}, i.e. $\mathrm{span}(\{e^x,\, e^{-x}\})$. For more difficult equations we will need to prove that **any** solution is a linear combination of two different functions and also that two initial/boundary conditions are all we need to find the two constants of integration.

We first look at three special cases of second-order ODEs which can in fact be easily reduced to first-order equations. They are really first-order ODEs in disguise. Then the techniques in the previous Chapter can be used. In the following three sections we consider special forms of the function $F\left(x,\, y,\, \frac{dy}{dx},\, \frac{d^2y}{dx^2}\right)$: where this function does not explicitly depend either on y; or on x; or on x and $\frac{dy}{dx}$.

8.1 Second-order ODEs: y not appearing explicitly

We consider an ODE of the form

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0. \tag{8.2}$$

In this case, we use a substitution and define a new dependent variable $p(x)$ as,

$$p(x) = \frac{dy}{dx}, \tag{8.3}$$

which means that

$$\frac{dp}{dx} = \frac{d^2y}{dx^2},$$

and therefore equation (8.2) takes the form,

$$F\left(x, p, \frac{dp}{dx}\right) = 0. \tag{8.4}$$

This ODE is now a first-order ODE in the dependent variable $p(x)$ and the independent variable x. We can therefore (hopefully) solve this ODE using the methods described in the previous Chapter to find the solution $p(x)$. Equation (8.3) must then be integrated to find the solution y as a function of x. Finding $p(x)$ by solving the first-order ODE in equation (8.4) will lead to an integration constant, let's call it c_1, so the solution for $p(x)$ will involve this integration constant. When we then find $y(x)$ by integrating equation (8.3) another integration constant will appear, c_2. To determine these two integration constants we will need two initial or boundary conditions.

Method Second-order ODEs of the form $F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$ [no explicit y dependence]

- Set $\frac{dy}{dx} = p(x)$ so that $\frac{d^2y}{dx^2} = \frac{dp}{dx}$
- Write the ODE as $F\left(x,\, p,\, \frac{dp}{dx}\right) = 0$ – a first-order ODE for p and x
- Solve this for p as a function of x (remember the integration constant c_1)
- Write down the relationship between p and y, $\frac{dy}{dx} = p(x)$
- Integrate to obtain the solution $y(x) = \int p(x)\, dx + c_2$.
- Find the integration constants c_1 and c_2 from any conditions on the solution that are given.

Examples 8.1

(a) Solve: $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = x.$

(b) Solve: $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 2\dfrac{dy}{dx} - 1.$

✓ Watch Video: Example 8.1

✓ Solve Exercise: Tutorial question 1.

8.2 Second-order ODEs: x not appearing explicitly

We now consider an ODE of the form

$$F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0. \tag{8.5}$$

In this case, we use a substitution and define a new dependent variable $p(y)$ which is a function of y (so y is now acting as the independent variable). We set

$$p(y) = \frac{dy}{dx}\,, \tag{8.6}$$

which means that

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\,\frac{dp}{dy}$$

using the chain rule.

Therefore equation (8.5) now takes the form,

$$F\left(y, p, p\frac{dp}{dy}\right) = 0. \tag{8.7}$$

This ODE is now a first-order ODE in the dependent variable $p(y)$ with the independent variable y. If we can solve this ODE, using the methods described in the previous Chapter to find the solution $p(y)$, we then use equation (8.6), which is directly integrable, to find the solution,

$$\int \frac{1}{p(y)}\,dy = x + c_2,$$

remembering that the solution for $p(y)$ will also contain an integration constant c_1. To determine the two integration constants, c_1 and c_2, we will need two initial or boundary conditions.

Method Second-order ODEs of the form $F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$ [no explicit x dependence]

- Substitute $\dfrac{dy}{dx} = p(y)$ so that $\dfrac{d^2y}{dx^2} = p\,\dfrac{dp}{dy}$
- Write the ODE in the form $F\left(y, p, p\,\dfrac{dp}{dy}\right) = 0$ – a first-order ODE for p and y
- Solve this ODE for p as a function of y, (the expression for p contains one arbitrary constant c_1)
- Integrate $\displaystyle\int \frac{1}{p(y)}\,dy = x + c_2$.
- Find the integration constants c_1 and c_2 from any conditions on the solution that are given.

Examples 8.2

(a) Solve: $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0.$

(b) Solve: $\dfrac{d^2y}{dx^2} = 2y\dfrac{dy}{dx}$, subject to $y = 0$ and $\dfrac{dy}{dx} = 4$ at $x = 1$.

✓ Watch Video: Example 8.2

✓ Solve Exercise: Tutorial question 2.

8.3 Second-order ODEs: x and dy/dx not appearing explicitly

We consider an ODE of the form

$$F\left(y, \frac{d^2y}{dx^2}\right) = 0. \tag{8.8}$$

In this case, we rearrange the equation and write it in the form,

$$\frac{d^2y}{dx^2} = f(y).$$

If we now multiply the whole equation by $2\dfrac{dy}{dx}$ we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}. \tag{8.9}$$

But the left hand side of this equation can be written as

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right).$$

Using this fact, and integrating equation (8.9) with respect to x we get

$$\begin{aligned}\int \frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right) dx &= \int 2f(y)\,\frac{dy}{dx}\,dx, \\ \Rightarrow \qquad \left(\frac{dy}{dx}\right)^2 &= 2\int f(y)\,dy + c_1.\end{aligned}$$

If we can perform the integration on the right hand side then we can take the square root of both sides to give an expression for $\dfrac{dy}{dx}$ as a function of y, i.e. $\dfrac{dy}{dx} = \pm\sqrt{2F(y)+c_1}$, where F is the anti-derivative of f.

[Note that the $\pm$ means there are two possible solutions. When we apply the initial conditions you should be able to decide which is the correct sign.]

We can hopefully now solve this first-order differential equation because it is a directly integrable equation,

$$\pm\int \frac{1}{\sqrt{2F(y)+c_1}}\,dy = x + c_2.$$

To determine the two integration constants, c_1 and c_2, and whether the $+$ or $-$ sign should be used, we will need two initial or boundary conditions.

Method Second-order ODEs of the form $F\left(y, \dfrac{d^2y}{dx^2}\right) = 0$

[no explicit x or dy/dx dependence]

- Write the ODE in the form $\dfrac{d^2y}{dx^2} = f(y)$
- Multiply through by $2\dfrac{dy}{dx}$ to get $2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = 2\,f(y)\,\dfrac{dy}{dx}$
- Integrate with respect to x giving $\left(\dfrac{dy}{dx}\right)^2 = \displaystyle\int 2\,f(y)\,dy \ + c_1$
- Integrate the RHS and take the square root to get, $\dfrac{dy}{dx} = \pm\sqrt{2F(y) + c_1}$
- Integrate this ODE to get $\displaystyle\int \frac{\pm 1}{\sqrt{2F(y) + c_1}}\, dy = x + c_2$
- Use any initial or boundary conditions to obtain integration constants and decide which sign to use.

Examples 8.3

(a) Solve: $\dfrac{d^2y}{dx^2} = 2y(1 + y^2)$, with $y = 0$ and $\dfrac{dy}{dx} = 1$ at $x = 0$.

✓ Watch Video: Example 8.3

✓ Solve Exercise: Tutorial question 3.

8.4 Second-order linear ODEs

The previous sections in this Chapter considered the solutions of second-order differential equations of a simplified type, i.e. ones where y, or x, or x and dy/dx did not explicitly appear in the ODE.In general though, a second-order ODE will explicitly contain x, y, dy/dx and (of course, since it is a second-order ODE) d^2y/dx^2. Solving these equations is often very difficult.

8.4.1 Introduction: terminology

In this Chapter we will consider another specific type of equation: one in which y, and its derivatives, only appear linearly. In this case the equation must be of the form

$$a_2(x)\,\frac{d^2y}{dx^2} + a_1(x)\,\frac{dy}{dx} + a_0(x)\,y = f(x), \tag{8.10}$$

where the functions $a_2(x)$, $a_1(x)$, $a_0(x)$, $f(x)$ may be nonlinear functions of x.

As well as the ODE above we may be given two initial conditions

$$y(x_0) = y_0, \tag{8.11}$$

$$\frac{dy}{dx}(x_0) = y_0', \tag{8.12}$$

where y_0 and y_0' are given values of the variable and its derivative at $x = x_0$. This is an Initial Value Problem.

Alternatively, we may be given two boundary conditions

$$y(x_0) = y_0, \tag{8.13}$$

$$y(x_1) = y_1, \tag{8.14}$$

where y_0 and y_1 are given values of the variable at two different values of x, $x = x_0$ and $x = x_1$. This is a Boundary Value Problem.

We start with two definitions:

Definition 8.1 *Equation (8.10) is a* ***homogeneous*** *linear ODE if* $f(x) \equiv 0$.

Definition 8.2 *Equation (8.10) is an* ***inhomogeneous*** *or* ***nonhomogeneous*** *linear ODE if* $f(x) \neq 0$ *(for some x in the range of x values we are interested in).*

So

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + \sin(x)\, y = 0, \tag{8.15}$$

is a homogeneous second-order linear ODE, and

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + \sin(x)\, y = \cos(x), \tag{8.16}$$

is a nonhomogeneous second-order linear ODE.

[Note: the meaning of homogeneous here is very different to the meaning for first-order equations in the last Chapter.]

Equation (8.10) can also be written as a linear mapping (a second-order linear differential operator) L_2 using the definition:

$$L_2(y) \;:= \left[a_2(x)\, D^2 + a_1(x)\, D + a_0(x)\right](y), \tag{8.17}$$

so that the homogeneous equation is

$$L_2(y) = 0, \tag{8.18}$$

and the nonhomogeneous equation is

$$L_2(y) = f(x). \tag{8.19}$$

We first consider various results concerning **homogeneous** second-order linear equations, and in the first instance we will consider the Initial Value Problem, i.e. equation (8.10) with initial conditions (8.11) and (8.12). We will also assume that the functions $a_2(x)$, $a_1(x)$, $a_0(x)$ and $f(x)$ are all continuous on some interval I of the real line, $\mathbb{R}$.

We first state one of the main theorems of this section, which is actually true for homogeneous or nonhomogeneous equations:

Theorem 8.3 *If the functions $a_2(x)$, $a_1(x)$, $a_0(x)$, and $f(x)$ are continuous in an open interval $I \subset \mathbb{R}$ then there exists a unique function $y(x)$ satisfying the ODE,*

$$\begin{aligned} a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\, y &= f(x), \\ \textit{with} \quad y(x_0) &= y_0, \\ \frac{dy}{dx}(x_0) &= y_0', \end{aligned}$$

for all $x \in I$ and some $x_0 \in I$.

Essentially this means that we will be able to find the general solution $y(x; c_1, c_2)$ which involves two unknown constants c_1 and c_2 which then will be found from the two initial conditions in order to obtain the unique particular solution for the problem. To prove this theorem the idea is to turn the second-order ODE into a system of two first-order ODEs and proceed similarly to the case of first-order ODEs. We will study two-dimensional systems of ODEs later, so we skip the proof for now.

8.4.2 Homogeneous second-order ODEs: introductory results

We will now look at how to construct the general solution for the homogeneous second-order ODE. First we prove a relatively simple but crucial result:

Theorem 8.4 Principle of superposition for homogeneous equations
Let y_1 and y_2 be two solutions of the homogeneous second-order linear equation

$$L_2(y) = 0. \tag{8.20}$$

Then, for any constants c_1 and c_2, their linear combination $y = c_1 y_1 + c_2 y_2$ also solves (8.20).

Proof 8.4 *We have previously proved that D is a linear mapping and so L_2 given by (8.17) is also a linear mapping (remember D^2 is just D applied twice, it is not a nonlinear term). From the properties of linear mappings we know that*

$$L_2(c_1 y_1 + c_2 y_2) = c_1 L_2(y_1) + c_2 L_2(y_2).$$

However, y_1 and y_2 satisfy the homogeneous equation (8.20) so $L_2(y_1) = 0$ and $L_2(y_2) = 0$. Therefore $L_2(c_1 y_1 + c_2 y_2) = 0$ and $c_1 y_1 + c_2 y_2$ also satisfies the homogeneous equation. ∎

If we are able to find two solutions to the homogeneous equation then we have a general solution of the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$. Using the initial conditions $y(x_0) = y_0$ and $\frac{dy}{dx}(x_0) = y_0'$ we should therefore be able to determine the two unknown constants c_1 and c_2 and hence find the particular solution.

A number of questions now arise:

- If we have found two solutions, and therefore a general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$, can we always find the constants c_1 and c_2 from the initial conditions?
- Is every solution of the ODE of the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$? In other words, is this **the** general solution?

✓ Quiz 1: Second-order linear equations

8.4.3 Linear independence of functions and the Wronskian

These questions turn out to be related to showing that $y_1(x)$ and $y_2(x)$ are linearly independent, something we looked at in the first half of this class but recap here:

Definition 8.5 *A set of functions $S = \{y_1, y_2, \ldots, y_n\}$ is **linearly independent** on an interval $I \subset \mathbb{R}$ if when*

$$c_1\, y_1(x) + c_2\, y_2(x) + \ldots + c_n\, y_n(x) = 0 \quad \forall\, x \in I \quad (c_i \text{ constants}),$$

then $c_i = 0 \quad \forall\, i = 1, 2, \ \ldots\ , n.$

In other words, the only linear combination of the functions that is identically zero (i.e. $= 0$ for all $x \in I$) is the linear combination where the constants c_i are all zero.

For a set of n functions which have continuous derivatives up to order $(n-1)$ we can test for linear independence as follows:

Suppose $y_1, y_2, \ldots, y_n$ are functions whose first $n-1$ derivatives exist at all points of the interval $I \in \mathbb{R}$. Consider

$$c_1\, y_1(x) + c_2\, y_2(x) + \ldots + c_n\, y_n(x) = 0, \qquad c_i \in \mathbb{R}, \quad \forall\, x \in I. \tag{8.21}$$

Differentiate this equation repeatedly $n-1$ times to obtain the set of n linear equations for $c_1, c_2, \ldots, c_n$

$$\begin{array}{ccccccccc} c_1\ y_1 & + & c_2\ y_2 & + & \ldots & + & c_n\ y_n & = & 0 \\ c_1\ y_1{}' & + & c_2\ y_2{}' & + & \ldots & + & c_n\ y_n{}' & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ c_1\ y_1{}^{(n-1)} & + & c_2\ y_2{}^{(n-1)} & + & \ldots & + & c_n\ y_n{}^{(n-1)} & = & 0\ . \end{array}$$

This set of linear equations is equivalent to,

$$\begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1{}' & y_2{}' & \ldots & y_n{}' \\ \vdots & \vdots & \vdots & \vdots \\ y_1{}^{(n-1)} & y_2{}^{(n-1)} & \ldots & y_n{}^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From the results in the linear algebra part of the class, if the determinant of this system is zero there may be an infinite number of solutions and the set of functions must be linearly dependent.

On the other hand, the set of functions is linearly independent if the determinant is non-zero. In other words, this system will have only a trivial solution (i.e. $c_i = 0$ for all i) if

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ \vdots & \vdots & \vdots & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix} \neq 0 \ .$$

Because of the importance of the determinant of the matrix above it is given a particular name:

Definition 8.6 *The* ***Wronskian*** *of the functions $y_1, \, y_2, \ldots, \, y_n$, is defined as*

$$W[y_1, \, y_2, \ldots, \, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ \vdots & \vdots & \vdots & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix} \ .$$

Hence we have proved,

Theorem 8.7 *Suppose that $y_1, \, y_2, \ldots, \, y_n$ are functions whose first $n-1$ derivatives exist in the interval I. If the Wronskian $W[y_1, \, y_2, \ldots, \, y_n]$ is not identically equal to zero in this interval, then $y_1, \, y_2, \ldots, \, y_n$ are linearly independent in I.*

[Note: The converse of Theorem 8.7 is false. If $W[y_1, \, y_2, \ldots, \, y_n] \equiv 0$, then the set $\{y_1, \, y_2, \ldots, \, y_n\}$ may be linearly independent or linearly dependent.]

Method Linear independence of a set of functions $\{y_1, \, y_2, \ldots, \, y_n\}$ in an interval I

- Calculate the Wronskian

  $$W[y_1, \, y_2, \ldots, \, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ \vdots & \vdots & \vdots & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix} \ .$$

  For instance, for a set of two functions the Wronskian is

  $$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

- If there is at least one point $x_0 \in I$ for which the Wronskian is not zero then the set of functions is linearly independent.

Examples 8.4

Use the Wronskian to show that the following sets of functions are linearly independent on any interval $I \in \mathbb{R}$:

(a) $\{\sin(x), \cos(x)\}$

(b) $\{1,\ x,\ x^2,\ x^3\}$

Show that the following sets of functions are linearly dependent:

(c) $\{e^x,\ e^{-x},\ \cosh(x)\}$

(d) $\{e^x, \cosh(x),\ \sinh(x)\}$

✓ Watch Video: Example 8.4

✓ Solve Exercise: Tutorial question 4.

8.4.4 The general solution of linear second-order homogeneous ODEs: Fundamental Set

We now go back to our second-order linear homogeneous ODE (8.20) with a general solution $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$. First we make the following definition:

Definition 8.8 *A linearly independent set $\{y_1, y_2\}$ of solutions of the homogeneous equation (8.20) is called a* ***fundamental set of solutions.***

We now prove that there is **always** a fundamental set of solutions, and we will use the fact that there is a unique solution to the Initial Value Problem (Theorem 8.3). We will then eventually be able to prove that if $y_1(x)$ and $y_2(x)$ are a fundamental set of solutions (i.e. they are linearly independent) then all solutions of the ODE must be of the form $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$, so that this is **the** general solution of the homogeneous ODE.

Theorem 8.9 *There exists a Fundamental Set of solutions of (8.20) on I.*

Proof 8.9 *For some $x_0 \in I$, let y_1 be the unique solution of (8.20) satisfying*

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0,$$

and let y_2 be the unique solution of (8.20) satisfying

$$y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then, for the Wronskian $W[y_1, y_2]$,

$$W[y_1, y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

so the Wronskian is non-zero at at least one point ($x = x_0$) and therefore $\{y_1, y_2\}$ is linearly independent on I. ∎

Theorem 8.10 *A set of solutions $\{y_1, y_2\}$ of (8.20) on $I \subset \mathbb{R}$ is a Fundamental Set of solutions (i.e. is linearly independent on I) if and only if*

$$\exists\ x_0 \in I \ \text{ such that } \ W[y_1, y_2](x_0) \neq 0.$$

Proof 8.10 *From the derivation of the Wronskian and its relationship to linear independence we have already shown that,*

$$\text{If } \exists\ x_0 \in I \text{ such that } W[y_1, y_2](x_0) \neq 0 \text{ then } \{y_1, y_2\} \text{ is linearly independent.}$$

As we are given that y_1 and y_2 are solutions of (8.20) they form a Fundamental Set of solutions.

To prove the converse, we use a proof by contradiction.
Suppose there exists $x_0 \in I$ such that

$$W[y_1, y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0.$$

Then, there exists a non-zero vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ *satisfying*

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8.22}$$

Using this non-zero vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, *consider* $y = c_1\, y_1 + c_2\, y_2$, *which satisfies (8.20) by Theorem 8.4. From (8.22),*

$$y(x_0) = y'(x_0) = 0. \tag{8.23}$$

However, $y \equiv 0$ *satisfies (8.20) and the initial conditions (8.23), and so by the uniqueness of the solution of this IVP, we deduce*

$$y = c_1\, y_1 + c_2\, y_2 \equiv 0\ ;$$

which shows that y_1 *and* y_2 *are linearly dependent, since c_1 and c_2 are not both zero. A contradiction of* $\{y_1, y_2\}$ *being a Fundamental Set of solutions of (8.20)* .∎

Theorem 8.11 *Let* $\{y_1, y_2\}$ *be a Fundamental Set of solutions of (8.20) on the interval $I \subset \mathbb{R}$. Then, any solution y of (8.20) on I can be written uniquely in the form*

$$y = c_1\, y_1 + c_2\, y_2$$

where c_1 and c_2 are constants.

Proof 8.11 *Let y be any solution of (8.20) on I. At some point $x_0 \in I$, suppose*

$$y(x_0) = y_0, \quad y'(x_0) = y_0' \qquad (y_0,\ y_0' \text{ are constants}). \tag{8.24}$$

Consider $z = c_1\, y_1 + c_2\, y_2$ *where c_1 and c_2 are constants. By Theorem 8.4, z is a solution of (8.20), and for z to satisfy the same initial conditions as y at $x = x_0$, we require*

$$\begin{aligned} c_1\, y_1(x_0) + c_2\, y_2(x_0) &= y_0 \\ c_1\, y_1'(x_0) + c_2\, y_2'(x_0) &= y_0', \end{aligned}$$

which, in matrix form, gives

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} . \tag{8.25}$$

Since $\{y_1, y_2\}$ *is a Fundamental Set, it follows from Theorem 8.10 that*

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

so that (8.25) has a unique solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$*, and for these values of* c_1 *and* c_2*,*

$$z = c_1\, y_1 + c_2\, y_2$$

satisfies (8.20) on I *and the same initial conditions as* y*. Hence, by the uniqueness of the solution of this IVP, we conclude that*

$$y = z = c_1\, y_1 + c_2\, y_2.$$

Since y *was any solution of (8.20) on* I*, the Theorem is proved.* ∎

Thus, if $\{y_1, y_2\}$ is a Fundamental Set of solutions (i.e. a linearly independent set of solutions) of

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0 \quad \text{on } I \subset \mathbb{R}$$

where $a_j(x)\,,\ j = 0, 1, 2$ are continuous on I with $a_2(x) > 0$ for all $x \in I$, then

$$y = c_1\, y_1 + c_2\, y_2$$

is a general solution, and **every** solution has this form. In particular this means that there are no singular solutions. Thus, in this linear theory, this is **the** general solution rather than **a** general solution.

Definition 8.12 *Points in* I *where the coefficients* $a_j(x)\,,\ j = 0, 1, 2$ *in the ODE*

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0 \quad \textit{on } I \subset \mathbb{R}$$

satisfy the conditions $a_j(x)\,,\ j = 0, 1, 2$ *are continuous on* I *with* $a_2(x) > 0$ *for all* $x \in I$ *are called* ***ordinary points****. Points in* I *where the coefficients do not satisfy all of these conditions are called* ***singular points****.*

If the interval I on which we seek to solve the differential equation contains singular points, then the situation is not as clear cut. We illustrate this using an example.

Consider the differential equation

$$xy'' - 2y' = 0.$$

There is a singular point at $x = 0$ (we had insisted that $a_2 > 0$ in I). By noting that y does not appear explicitly, we can solve this equation, obtaining a general solution

$$y = c_1 x^3 + c_2. \tag{8.26}$$

On any interval $I \subset \mathbb{R}$, $y_1(x) = x^3$ and $y_2(x) = 1$ are linearly independent (you can check this using the Wronskian). Therefore, on any interval I not including the singular point $x = 0$, we can conclude that any solution is of the form (8.26). However, on any open interval that does contain $x = 0$, it is easy to check that

$$y = \begin{cases} x^3, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

satisfies the differential equation, but this solution is not of the form (8.26) - it is a singular solution.
Note that

$$y' = \begin{cases} 3x^2, & x > 0 \\ 0, & x \leq 0 \end{cases} \qquad \text{and} \qquad y'' = \begin{cases} 6x, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

Derivatives at $x = 0$ being obtained from the definition of a derivative using left/right limits.

Examples 8.5

For each of the following, show that the given set is a Fundamental Set of solutions for the given equation:

(a) $x^2y'' + xy' - y = 0 \ \ (x > 0)$; $\qquad \{x,\ x^{-1}\}$.

(b) $x^2y'' - xy' + y = 0 \ \ (x > 0)$; $\qquad \{x,\ x\ln x\}$.

✓ Watch Video: Example 8.5

✓ Solve Exercise: Tutorial question 5.