University of Strathclyde
Department of Mathematics and Statistics
MM101 Introduction to Calculus

Exercise solutions for chapters 9 to 16

These are *outline* solutions only, so they are written as concisely as possible. You may have to fill in some intermediate steps.

When you write your own solutions to these exercises you should aim to make them easy to follow, rather than concise!

9 Limits

9.1 (a) The limit is $l = 7$. As $f(x) = 7$ for all x, $|f(x) - 7| = 0 < \epsilon$ for all $\epsilon > 0$. Hence any $\delta > 0$ will do.

(b) The limit is $l = 0$. As $|x - 0| = |x| < \epsilon$ exactly when $|x| < \epsilon$, $\delta = \epsilon$ works.

(c) The limit is $l = 3$. As in part (b), $|x - 3| < \epsilon$ exactly when $|x - 3| < \epsilon$, so again $\delta = \epsilon$ works.

(d) The limit is $l = \frac{8}{3}$. We require that $|4x - \frac{8}{3}| < \epsilon$. Dividing by 4 yields $|x - \frac{2}{3}| < \frac{\epsilon}{4}$. So the choice $\delta = \frac{\epsilon}{4}$ works.

(e) The limit is $l = 9$. We need $|x^2 - 9| < \epsilon$. We have that $|x^2 - 9| = |(x+3)\cdot(x-3)| = |x+3|\cdot|x-3|$. As x needs to be close to 3, we choose $|x - 3| < 1$, so $-1 < x - 3 < 1$. Adding 6 shows that $5 < x + 3 < 7$, so $|x + 3| < 7$. Hence we choose $\delta = \min(\dfrac{\epsilon}{7}, 1)$. Then if $|x - 3| < \delta$, it follows that $|x^2 - 9| = |x + 3| \cdot |x - 3| < 7 \cdot \delta \leq 7 \cdot \dfrac{\epsilon}{7} = \epsilon$.

(f) The limit is $l = 16$. We need $|7x - 5 - 16| = |7x - 21| < \epsilon$. Dividing by seven gives $|x - 3| < \frac{\epsilon}{7}$, so $\delta = \frac{\epsilon}{7}$ works.

(g) The limit is $l = 25$. To start we split $|x^2 + 7x - 5 - 25| = |x^2 - 9 + 7x - 21| \leq |x^2 - 9| + |7x - 21|$ and then use the results obtained in parts (e) and (f). With $\delta = \frac{\epsilon}{14}$ we have that both $|x^2 - 9| < \frac{\epsilon}{2}$ and $|7x - 21| < \frac{\epsilon}{2}$ as long as $|x - 3| < \delta$. We find that $|f(x) - 25| =\leq |x^2 - 9| + |7x - 21| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ as required.

9.2 (a) $\displaystyle \lim_{y\to4} \frac{y^2 - 16}{y - 4} = \lim_{y\to4} \frac{(y-4)(y+4)}{y-4} = \lim_{y\to4}(y+4) = 8$

(b) $\displaystyle \lim_{f\to4} \frac{f^2 - 2f - 8}{f - 4} = \lim_{f\to4} \frac{(f-4)(f+2)}{f-4} = \lim_{f\to4}(f+2) = 6$

(c) $\displaystyle \lim_{x\to4} \frac{(x-4)(x+2)}{x+3} = 0$

9.2 (d) $\lim_{x\to1}\dfrac{x^2+6x-7}{x^2+4x-5}=\lim_{x\to1}\dfrac{(x-1)(x+7)}{(x-1)(x+5)}=\lim_{x\to1}\dfrac{x+7}{x+5}=\dfrac{8}{6}=\dfrac{4}{3}$

(e) $\lim_{x\to3/2}\dfrac{2x^2+5x-12}{4x^2+4x-15}=\lim_{x\to3/2}\dfrac{(2x-3)(x+4)}{(2x-3)(2x+5)}=\lim_{x\to3/2}\dfrac{x+4}{2x+5}=\dfrac{11/2}{8}=\dfrac{11}{16}$

(f) $\lim_{x\to c}\dfrac{x^2-c^2}{x-c}=\lim_{x\to c}\dfrac{(x-c)(x+c)}{x-c}=\lim_{x\to c}(x+c)=2c$

(g) $\lim_{x\to c}\dfrac{x^3-c^3}{x-c}=\lim_{x\to c}\dfrac{(x-c)(x^2+xc+c^2)}{x-c}=\lim_{x\to c}(x^2+xc+c^2)=3c^2$

(h) $\lim_{x\to-2}\dfrac{x^3+8}{x+2}=\lim_{x\to-2}\dfrac{(x+2)(x^2-2x+4)}{x+2}=\lim_{x\to-2}(x^2-2x+4)=4+4+4=12$

(i) $\lim_{x\to25}\dfrac{x-25}{\sqrt{x}-5}=\lim_{x\to25}\dfrac{(\sqrt{x}-5)(\sqrt{x}+5)}{\sqrt{x}-5}=\lim_{x\to25}(\sqrt{x}+5)=10$

(j) $\lim_{x\to7}\left(\dfrac{1}{x}-\dfrac{1}{7}\right)\dfrac{1}{x-7}=\lim_{x\to7}\dfrac{7-x}{7x}\cdot\dfrac{1}{x-7}=\lim_{x\to7}\left(-\dfrac{1}{7x}\right)=-\dfrac{1}{49}$

(k) $\lim_{x\to2}\dfrac{x^2+5x-14}{x^2+5x+6}=0$

(l) $\lim_{x\to2}\dfrac{x^2-x-6}{x^2+3x-10}=\lim_{x\to2}\dfrac{(x+2)(x-3)}{(x+5)(x-2)}$: limit does not exist.

9.3 (a) $\lim_{h\to0}\dfrac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to0}\dfrac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})}$

$=\lim_{h\to0}\dfrac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to0}\dfrac{1}{\sqrt{a+h}+\sqrt{a}}=\dfrac{1}{2\sqrt{a}}.$

(b) $\lim_{x\to1}\dfrac{1-\sqrt{x}}{1-x}=\lim_{x\to1}\dfrac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})}=\lim_{x\to1}\dfrac{1-x}{(1-x)(1+\sqrt{x})}=\lim_{x\to1}\dfrac{1}{(1+\sqrt{x})}=\dfrac{1}{2}.$

(c) $\lim_{x\to0}\dfrac{1-\sqrt{1-x^2}}{x}=\lim_{x\to0}\dfrac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x(1+\sqrt{1-x^2})}=\lim_{x\to0}\dfrac{x}{1+\sqrt{1-x^2}}=0.$

(d) $\lim_{x\to0}\dfrac{1-\sqrt{1-x^2}}{x^2}=\lim_{x\to0}\dfrac{1}{1+\sqrt{1-x^2}}=\dfrac{1}{2}.$

9.4 We use the results: $\lim_{x\to0}\dfrac{\sin(ax)}{bx}=\dfrac{a}{b},\quad \lim_{x\to0}\dfrac{ax}{\sin(bx)}=\dfrac{a}{b}\ (b\neq0).$

(a) 3 (b) $\dfrac{1}{3}$ (c) $\dfrac{5}{7}$ (d) $\dfrac{12}{25}$

(e) $\lim_{x\to0}\dfrac{\sin(3x)\tan(5x)}{4x^2}=\lim_{x\to0}\dfrac{\sin(3x)}{4x}\dfrac{\sin(5x)}{x}\dfrac{1}{\cos(5x)}=\dfrac{3}{4}\cdot5\cdot\dfrac{1}{1}=\dfrac{15}{4}$

(f) $\lim_{x\to0}\dfrac{1-\cos^2x}{x^2}=\lim_{x\to0}\dfrac{\sin^2x}{x^2}=1.$

9.4 (g) We use $\cos x = \sin(\pi/2 - x)$:

$$\lim_{x\to\pi/2} \frac{\cos x}{\pi - 2x} = \lim_{x\to\pi/2} \frac{\sin(\pi/2 - x)}{2(\pi/2 - x)} = \frac{1}{2} \lim_{y\to 0} \frac{\sin y}{y} = \frac{1}{2}$$

with $y = \pi/2 - x$.

9.5 If $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} f(x) = 0$ as $f(x)$ is always smaller in size than $g(x)$. If $\lim_{x\to a} g(x) = 5$, then we can only conclude that $f(x)$ does not tend to a limit which has modulus greater than 5: we do not have enough information to say anything else, in particular, we do not know whether $\lim_{x\to a} f(x)$ exists or not.

9.6 (a) For any $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then both $|f(x) - l| < \epsilon$ and $|h(x) - l| < \epsilon$ (Choose the minimum of the two δs in the limit definitions for f and g). Then for $0 < |x - a| < \delta$

$$l - \epsilon < f(x) \le g(x) \le h(x) < l + \epsilon,$$

and so $|g(x) - l| < \epsilon$ and hence $\lim_{x\to a} g(x) = l$.

(b) The result remains the same; the main line in the proof of part (a) becomes

$$l - \epsilon < f(x) < g(x) < h(x) < l + \epsilon,$$

which leads to the same conclusion. (However, note that at least one of f and h must now be discontinuous at a.)

9.7 Suppose that $\lim_{x\to a} f(x) = l$. That means that for every $\epsilon > 0$ there is a $\delta > 0$ such that when $0 < |x - a| < \delta$ then $|f(x) - l| < \epsilon$. The same δ then works to show that $\lim_{h\to 0} g(h) = l$: when $0 < |h| < \delta$ then $0 < |a + h - a| < \delta$ and so $|f(a + h) - l| < \epsilon$, that is $|g(h) - l| < \epsilon$. But $\lim_{h\to 0} g(h) = l$ is just another way of writing $\lim_{h\to 0} f(a + h) = l$. The same type of argument can be used to show that if $\lim_{h\to 0} f(a + h) = m$, then $\lim_{x\to a} f(x) = m$. So when either of the two limits exists, they both exist and are the same.

9.8 We need to look at the distance between $|f(x)|$ and $|l|$, that is $||f(x)| - |l||$. Remember that

$$||a| - |b|| \le |a - b| \qquad \text{for all } a, b \in \mathbb{R}.$$

The proof is now straightforward. Because $\lim_{x\to a} f(x) = l$ there is, for every $\epsilon > 0$, a $\delta > 0$ such that when $0 < |x - a| < \delta$ then $|f(x) - l| < \epsilon$. For the same δ we then have

$$||f(x)| - |l|| \le |f(x) - l| < \epsilon$$

and so $\lim_{x\to a} |f(x)| = |l|$.

9.9 To see this, given an ϵ, choose $N = 1/\epsilon$. Then $x > N = 1/\epsilon \iff 1/x < \epsilon$ and so

$$\left|\frac{\sin x}{x} - 0\right| = \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right| = \frac{1}{x} < \epsilon$$

as required.