14 The Fundamental Theorem of Calculus

14.1 (a) Using Theorem 14.2 we have

$$F(x) = \int_a^x \cos t \, \mathrm{d}t = [\sin t]_a^x = \sin x - \sin a$$

so

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\sin x - \sin a) = \cos x.$$

(b) Using Theorem 14.1 with $f(x) = \cos x$ we have $F'(x) = \cos x$.

14.2 (a) $F'(x) = \cos^3 x$.

(b) $F'(t) = \cos^3 t$, so $F'(x) = \cos^3 x$ as in part (a).

(c) $F'(x) = 0$.

(d) $F'(x) = \displaystyle\int_a^b \cos^3 t \, \mathrm{d}t$.

(e) $F(x) = -\displaystyle\int_b^x \cos^3 t \, \mathrm{d}t$, so $F'(x) = -\cos^3 x$.

(f) $F'(x) = \displaystyle\int_x^3 \cos^2 t \, \mathrm{d}t$.

(g) $F'(x) = \cos\left(\displaystyle\int_a^x \cos^2 t \, \mathrm{d}t\right) \cos^2 x$ (by the Chain Rule).

(h) $F'(x) = \cos x \displaystyle\int_a^x \cos^2 t \, \mathrm{d}t + \sin x \cos^2 x$ (by the Product Rule).

(i) $G(x) = \displaystyle\int_a^x \cos^3 t \, \mathrm{d}t$ so $G'(x) = \cos^3 x$.
$F(x) = G(x^3)$ so (by the Chain Rule) $F'(x) = 3x^2 G'(x^3) = 3x^2 \cos^3(x^3)$.

(j) $F'(x) = \dfrac{\sin^2 x}{1 + \left(\int_6^x \sin^2 t \, \mathrm{d}t\right)^2 + \cos\left(\int_6^x \sin^2 t \, \mathrm{d}t\right)}$.

(k) For a point $a \in [x, 2x]$ we may write

$$F(x) = \int_x^a \cos^2 t \, \mathrm{d}t + \int_a^{2x} \cos^2 t \, \mathrm{d}t = -\int_a^x \cos^2 t \, \mathrm{d}t + \int_a^{2x} \cos^2 t \, \mathrm{d}t$$

so $F'(x) = -\cos^2 x + 2\cos^2(2x)$ (using the Chain Rule as in part (i) to get the second term).

(l) In general,

$$(F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))}.$$

For this particular function, we have $F'(x) = 1/x$ so

$$(F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))} = \frac{1}{\frac{1}{F^{-1}(x)}} = F^{-1}(x),$$

that is, F^{-1} doesn't change when it is differentiated.

(m) $F'(x) = \dfrac{1}{\sqrt{1-x^2}}$ by the First Fundamental Theorem. The derivative of the inverse function is $(F^{-1})'(x) = \dfrac{1}{F'(F^{-1}(x))} = \sqrt{1-(F^{-1}(x))^2}$.
Note that $F = \arcsin$, $F^{-1} = \sin$, and $(F^{-1})' = \cos$.

14.3 For a point $a \in [g(x), h(x)]$ we may write

$$F(x) = \int_{g(x)}^{a} f(t)\,\mathrm{d}t + \int_{a}^{h(x)} f(t)\,\mathrm{d}t. = -\int_{a}^{g(x)} f(t)\,\mathrm{d}t + \int_{a}^{h(x)} f(t)\,\mathrm{d}t.$$

Now use the Chain Rule as in Exercise 14.2 (i) on each term to get

$$F'(x) = -f(g(x)) \cdot g'(x) + f(h(x)) \cdot h'(x)$$

as required.

14.4 (a) $\displaystyle\int_0^1 (x - x^2)\,\mathrm{d}x = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(b) $\displaystyle\int_{\pi/4}^{3\pi/4} (\sin x + \cos x)\,\mathrm{d}x = [-\cos x + \sin x]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \sqrt{2}$

(c) $\displaystyle\int_3^4 (2x^3 - 3x + 1)\,\mathrm{d}x = \left[\frac{1}{2}x^4 - \frac{3}{2}x^2 + x\right]_3^4 = 128 - 24 + 4 - \frac{81}{2} + \frac{27}{2} - 3 = 78$

(d) $\displaystyle\int_1^2 (x - 2x^{1/2} + 3x^{-1/2})\,\mathrm{d}x = \left[\frac{1}{2}x^2 - 4/3x^{3/2} + 6x^{1/2}\right]_1^2$
$\displaystyle = 2 - \frac{8}{3}\sqrt{2} + 6\sqrt{2} - \frac{1}{2} + \frac{4}{3} - 6 = \sqrt{2}\left(6 - \frac{8}{3}\right) - \tfrac{19}{6} = \frac{10}{3}\sqrt{2} - \frac{19}{6}$

(e) $\displaystyle\int_{\pi/3}^{\pi/6} \left(3\csc^2 x - \frac{1}{3}\sec^2 x\right)\,\mathrm{d}x = \left[-3\cot x - \frac{1}{3}\tan x\right]_{\pi/6}^{\pi/3}$
$\displaystyle = -3.\frac{1}{\sqrt{3}} - \frac{1}{3}\sqrt{3} + 3\sqrt{3} + \frac{1}{3}\cdot\frac{1}{\sqrt{3}} = -\sqrt{3} - \frac{1}{3}\sqrt{3} + 3\sqrt{3} + \frac{1}{9}\sqrt{3}$
$\displaystyle = \sqrt{3}\left(-1 - \frac{1}{3} + 3 + \frac{1}{9}\right) = \frac{-9 - 3 + 27 + 1}{9}\sqrt{3} = \frac{16}{9}\sqrt{3}$

14.5 (a) $\displaystyle A = \int_0^{3\pi/2} |\sin x|\,\mathrm{d}x = \int_0^{\pi} \sin x\,\mathrm{d}x + \int_{\pi}^{3\pi/2} (-\sin x)\,\mathrm{d}x = [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{3\pi/2} =$
$\displaystyle -\cos\pi + \cos 0 + \cos\frac{3\pi}{2} - \cos\pi = 1 + 1 + 0 + 1 = 3.$

(b) $\displaystyle A = \int_0^{2\pi} |\sin x|\,\mathrm{d}x = \int_0^{\pi} \sin x\,\mathrm{d}x + \int_{\pi}^{2\pi} (-\sin x)\,\mathrm{d}x = [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} =$
$-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi = 1 + 1 + 1 + 1 = 4.$

(c) $\displaystyle A = \int_0^1 |1 - x^2|\,\mathrm{d}x = \int_0^1 (1 - x^2)\,\mathrm{d}x = \left[x - \frac{x^3}{3}\right]_0^1 = \left(1 - \frac{1}{3}\right) - 0 = \frac{2}{3}.$

(d) $A = \int_0^2 |1-x^2|\,\mathrm{d}x = \int_0^1 (1-x^2)\,\mathrm{d}x + \int_1^2 (x^2-1)\,\mathrm{d}x = \frac{2}{3} + \left[\frac{x^3}{3} - x\right]_1^2 = \frac{2}{3} + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right) = 2.$

(e) $A = \int_0^2 |x-x^2|\,\mathrm{d}x = \int_0^1 (x-x^2)\,\mathrm{d}x + \int_1^2 (x^2-x)\,\mathrm{d}x = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \left(\frac{1}{2} - \frac{1}{3}\right) - 0 + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = 1.$

14.6 (a) $A = \int_0^4 y\,\mathrm{d}x = \int_0^4 x^{\frac{1}{2}}\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{16}{3}.$

(b) $A = \int_{-1}^2 (2-x)(x+1)\,\mathrm{d}x = \int_{-1}^2 (2+x-x^2)\,\mathrm{d}x = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \left(4+2-\frac{8}{3}\right) - \left(-2+\frac{1}{2}+\frac{1}{3}\right) = \frac{9}{2}.$

(c) Curves intersect when $x^2 = x^3$, i.e. $x^2 - x^3 = 0 \implies x^2(x-1) = 0 \implies x = 0$ or $x = 1$.
$A = \int_0^1 (x^2 - x^3)\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$

(d) Curves intersect when $2+x-x^2 = x+1$, i.e. $x^2 - 1 = 0 \implies x = 1$ or $x = -1$.
$A = \int_{-1}^1 \left\{(2+x-x^2) - (1+x)\right\}\,\mathrm{d}x = \int_{-1}^1 (1-x^2)\,\mathrm{d}x = \left[x - \frac{x^3}{3}\right]_{-1}^1 = \frac{4}{3}.$

(e) $A = \int_1^8 y^{\frac{1}{3}}\,\mathrm{d}y = \left[\frac{3}{4}y^{\frac{4}{3}}\right]_1^8 = \frac{3}{4}\left(8^{\frac{4}{3}} - 1^{\frac{4}{3}}\right) = \frac{45}{4}$ OR
$A = \int_0^1 (8-1)\,\mathrm{d}x + \int_1^2 (8-x^3)\,\mathrm{d}x = 7 + \left[8x - \frac{x^4}{4}\right] = 7 + (16-4) - \left(8 - \frac{1}{4}\right) = 19 - \frac{31}{4} = \frac{45}{4}.$

(f) Area of top half of region:
$A = \int_1^4 y\,\mathrm{d}x = \int_1^4 (4-x)^{\frac{1}{2}}\,\mathrm{d}x = \left[-\frac{2}{3}(4-x)^{\frac{3}{2}}\right]_1^4 = \frac{2}{3}\left(0 + 3^{\frac{3}{2}}\right) = 2\sqrt{3}.$
Region is symmetric about the x-axis, so area of the whole region is twice the area of the top half, i.e. $4\sqrt{3}$.

14.7 (a) $\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,\mathrm{d}x = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$

(b) $\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\left[\arctan x\right]_0^b = \lim_{b\to\infty}(\arctan b - \arctan 0) = \frac{\pi}{2}.$

(c) $\int_{-\infty}^0 \frac{\mathrm{d}x}{4+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{\mathrm{d}x}{4+x^2} = \lim_{a\to-\infty}\left[\frac{1}{2}\arctan\frac{x}{2}\right]_a^0 = \lim_{a\to-\infty}\left[-\frac{1}{2}\arctan a\right] = \frac{\pi}{4}.$

(d) $\displaystyle\int_{-\infty}^{0} \cos x \,\mathrm{d}x = \lim_{a\to-\infty} \int_{a}^{0} \cos x \,\mathrm{d}x = \lim_{a\to-\infty} \left[\sin x\right]_a^0 = \lim_{a\to-\infty} (\sin 0 - \sin a)$,
which does not exist: the integral diverges.

(e) $\displaystyle I = \int_3^5 \frac{x}{\sqrt{x^2-9}}\,\mathrm{d}x = \lim_{c\to 3+} \int_c^5 \frac{x}{\sqrt{x^2-9}}\,\mathrm{d}x.$
Substitution: put $u = x^2 - 9$ so $\mathrm{d}u = 2x\mathrm{d}x \implies x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
$\displaystyle I = \lim_{c\searrow 3} \int_{c^2-9}^{16} \frac{1}{2}u^{-\frac{1}{2}}\,\mathrm{d}u = \lim_{c\searrow 3}\left[u^{\frac{1}{2}}\right]_{c^2-9}^{16} = \lim_{c\searrow 3}\left[4 - \sqrt{c^2-9}\right] = 4.$

(f) $\displaystyle\int_1^2 \frac{1}{(x-2)^2}\,\mathrm{d}x = \lim_{c\nearrow 2}\int_1^c \frac{1}{(x-2)^2}\,\mathrm{d}x = \lim_{c\nearrow 2}\left[-(x-2)^{-1}\right]_1^c = \lim_{c\nearrow 2}\left[-\frac{1}{c-2} - 1\right]$,
which does not exist: the integral diverges.