16 Integration

Integration is closely related to differentiation. Every derivative yields information about integrals. For example, if

$$F(x) = e^{x^2}, \qquad \text{then} \qquad F'(x) = 2xe^{x^2},$$

and by the Second Fundamental Theorem

$$\int_a^b 2xe^{x^2}\mathrm{d}x = F(b) - F(a) = e^{b^2} - e^{a^2}.$$

Every evaluation of a definite integral in this way involves on the right hand side an expression of the form $F(b) - F(a)$, which we recall may be written as $[F(x)]_a^b$. This allows us to write simply

$$\int_a^b 2xe^{x^2}\mathrm{d}x = \left[e^{x^2}\right]_a^b = e^{b^2} - e^{a^2}$$

without the need to introduce names for the functions involved.

Evaluating definite integrals $\displaystyle\int_a^b f$ in this fashion relies on the knowledge of a function F such that $F' = f$. Any such function F is called a **primitive** or **antiderivative** of f. These terms mean exactly the same. "Primitive" expresses the fact that f stems from F via $f = F'$. "Antiderivative" expresses the fact that F is related to f by the reverse process of differentiation. However, you must not be misled by this terminology. There is no way to 'undo' differentiation, simply because if $f = F'$ then also $f = (F + C)'$ where C is any constant function. Consequently, F is *a* primitive or antiderivative, not *the* primitive or antiderivative.

For continuous functions f the First Fundamental Theorem guarantees the existence of a primitive, namely

$$F(x) = \int_a^x f \tag{16.1}$$

for any choice of the lower limit of integration a. A different lower limit of integration usually yields a different primitive.

$$G(x) = \int_b^x f = \int_b^a f + \int_a^x f = F(x) + C,$$

where C is the constant $C = \displaystyle\int_b^a f$.

Unfortunately, there is no way of knowing in advance whether F in (16.1) will turn out to be a familiar **elementary function**, that is, a function that can be written in terms of rational functions, power functions, trigonometric functions and their inverses, logarithms, and exponentials. For example,

$$F(x) = \int_0^x e^{-t^2}\mathrm{d}t$$

is a perfectly well behaved function, and it is very important in statistics ($\frac{2}{\sqrt{\pi}}F$ is the **error function**). Nevertheless, it is impossible to express F in terms of elementary functions.

While differentiation is a rather algorithmic (and sometimes boring) process that involves a number of well-defined rules and always leads to a result, integration is a lot more difficult. For example, of the two very similar looking functions $f(x) = e^{-x^2}$ and $f(x) = xe^{-x^2}$ only the latter has a primitive (antiderivative) that can be written in a simple form.

If differentiation is a technique, integration is more of an art. But every good artist also needs solid technique, and so most of the remainder of this chapter is devoted to methods of integration. These methods allow us to transform difficult integrals into easier ones. The more 'easy' integrals we know, the higher our chances of success.

16.1 The indefinite integral

Definition 16.1 *The statement*

$$F = \int f \qquad \text{or} \qquad F(x) = \int f(x)\,\mathrm{d}x$$

means that $F' = f$; in other words, F is a primitive of f. The symbol $\int f$ or $\int f(x)\,\mathrm{d}x$ is called the ***indefinite integral*** *of f.*

Note that Definition 16.1 is *not* a complete definition of the symbols $\int f$ and $\int f(x)\,\mathrm{d}x$. In fact, the notation here, although it is perfectly standard, is rather problematic.

The problem arises because the primitive of a function f is not uniquely defined: as we noted above, if F is a primitive of f, then so is $F + C$, where C is any constant function. This non-uniqueness can cause trouble. For example, both the following statements are true:

$$\int x\,\mathrm{d}x = \frac{1}{2}x^2 \quad (*) \qquad \text{and} \qquad \int x\,\mathrm{d}x = \frac{1}{2}x^2 + 1. \quad (**)$$

However, by subtracting equation $(*)$ from equation $(**)$, we find

$$\begin{aligned} \int x\,\mathrm{d}x - \int x\,\mathrm{d}x &= \frac{1}{2}x^2 + 1 - \frac{1}{2}x^2 \\ &\Longrightarrow\ 0 = 1. \end{aligned}$$

The flaw in the logic occurs on the left-hand side. Because the symbol $\int x\,\mathrm{d}x$ is ambiguous, we can't assume that it means the same thing each time it occurs. (An everyday analogy: "a litre of water minus a litre of water" means "no water", but "a heap of sand minus a heap of sand" doesn't mean "no sand", because "a heap" isn't a fixed quantity of sand.)

To deal with this problem, we need to distinguish between *particular* primitives or antiderivatives such as $F(x) = \frac{1}{2}x^2$ for $f(x) = x$, and the *general* primitive or antiderivative, which in this case is given by $G(x) = \frac{1}{2}x^2 + C$ where C is an arbitrary constant[8]. Unfortunately, some textbooks use $\int f$ to mean "any particular primitive of f", while some use it to mean "the general primitive of f", so we have to make sure we know which one is meant in any given situation.

As a good rule of thumb, if you are doing any calculations that involve combining or manipulating indefinite integrals as though they were functions or algebraic expressions, you are best to include the constant of integration.[9] In books with extensive tables of integrals the constant is usually omitted for brevity. However, only if you're very sure of yourself, or if you're working in a situation where "any particular primitive of f" will do, can you safely omit the constant of integration.

16.2 Integrals of simple functions

Because of our experience with derivatives, we already know quite a few useful indefinite integrals, which are summarised in Table 1.

Theorem 16.1 *If f and g are continuous and $c \in \mathbb{R}$ is constant, then*

$$\int cf = c \cdot \int f$$

and

$$\int (f + g) = \int f + \int g.$$

Proof: For any primitives $F = \int f$ and $G = \int g$ we have $F' = f$ and $G' = g$. Then

$$(cF)' = cF' = cf$$

and so cF is a primitive for cf, that is, $c \int f = \int cf$. Also,

$$(F + G)' = F' + G' = f + g,$$

and so $F + G$ is a primitive for $f + g$, that is, $\int f + \int g = \int (f + g)$. □

[8]As in Chapter 14, we are a bit sloppy about notation and use the symbol C to denote both the constant $C \in \mathbb{R}$ and the constant function $C(x) = C$.

[9]This is particularly important when you use integration to solve **differential equations**, which you will encounter in MM102 and throughout your maths degree!

$\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$	$\displaystyle\int x^n \,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C, \text{ if } n \neq -1$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$	$\displaystyle\int \cos x \,\mathrm{d}x = \sin x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$	$\displaystyle\int \sin x \,\mathrm{d}x = -\cos x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$	$\displaystyle\int \sec^2 x \,\mathrm{d}x = \tan x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$	$\displaystyle\int \sec x \tan x \,\mathrm{d}x = \sec x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$	$\displaystyle\int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = \arcsin x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\arctan x = \dfrac{1}{1+x^2}$	$\displaystyle\int \frac{1}{1+x^2} \,\mathrm{d}x = \arctan x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\ln\lvert x\rvert = \dfrac{1}{x}$	$\displaystyle\int \frac{1}{x} \,\mathrm{d}x = \ln\lvert x\rvert + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$	$\displaystyle\int e^x \,\mathrm{d}x = e^x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$	$\displaystyle\int \cosh x \,\mathrm{d}x = \sinh x + C$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$	$\displaystyle\int \sinh x \,\mathrm{d}x = \cosh x + C$

Table 1: Standard derivatives and integrals.

Corollary 16.1 *If f and g are continuous, then*

$$\int_a^b cf = c \cdot \int_a^b f$$

and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof: These identities are, of course, nothing else but our Theorems 13.2 and 13.3. They follow from Theorem 16.1 since each definite integral may (by the Second Fundamental Theorem) be written as the difference of the values at a and b of the corresponding indefinite integral. $\square$

Example 16.1 Find $\int (3x^3 + 6x^{1/2})\, \mathrm{d}x$.

$$\int (3x^3+6x^{1/2})\,\mathrm{d}x = 3\int x^3\,\mathrm{d}x + 6\int x^{1/2}\,\mathrm{d}x = 3\left(\frac{1}{4}x^4\right) + 6\left(\frac{2}{3}x^{3/2}\right) + C = \frac{3}{4}x^4 + 4x^{3/2} + C.$$

Note that although we've split the integral up we only end up with a single constant of integration C. This is because the sum of any number of arbitrary constants is just an arbitrary constant.

Example 16.2 Find $\int (4\sin x + \frac{1}{2}\sec^2 x)\, \mathrm{d}x$.

$$\int (4\sin x + \frac{1}{2}\sec^2 x)\,\mathrm{d}x = -4\cos x + \frac{1}{2}\tan x + C.$$

Example 16.3 Find $\int_1^4 (7t^{1/2} - 4t^{-1/2})\, \mathrm{d}t$.

$$\begin{aligned}\int_1^4 (7t^{1/2} - 4t^{-1/2})\,\mathrm{d}t &= \left[7\left(\frac{2}{3}\right)t^{3/2} - 4(2t^{1/2})\right]_1^4 = \left[\frac{14}{3}t^{3/2} - 8t^{1/2}\right]_1^4 \\ &= \left(\frac{14}{3}(8) - 8(2)\right) - \left(\frac{14}{3} - 8\right) = \frac{74}{3}.\end{aligned}$$

Example 16.4 What is wrong with the following?

$$\int_{-1}^{2} \frac{1}{x}\,\mathrm{d}x = \Big[\ln|x|\Big]_{-1}^{2} = \ln 2 - \ln|-1| = \ln 2 - \ln 1 = \ln 2.$$

This illustrates a nasty trap when using tables of integration casually — especially when ln is involved! The problem is that the integral on the left-hand side isn't defined. Recall (Definition 13.2) that we have defined the definite integral for a *continuous* integrand. However, at $x = 0$ the integrand $1/x$ is not continuous, or even defined, and as $x \to 0$ the indefinite integral doesn't tend to a limit. It's easiest to see this by considering $\displaystyle\int_0^2 \frac{1}{x}\,\mathrm{d}x$ and $\displaystyle\int_{-1}^0 \frac{1}{x}\,\mathrm{d}x$, which are clearly both undefined because $\ln 0$ is undefined. By naïvely applying the result from our table of integration, we are essentially cancelling out terms of the form $\ln 0$, which we aren't entitled to do any more than we're entitled to write "$\infty - \infty = 0$".

16.3 Integration by parts

The product rule for differentiation gives rise to a very useful theorem on integration.

Theorem 16.2 ***Integration by Parts.***
If f' and g' are continuous, then

$$\int fg' = fg - \int f'g,$$

that is,

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x.$$

Furthermore,

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x.$$

Proof: The product rule, $(fg)' = f'g + fg'$, can be written as $fg' = (fg)' - f'g$. Integrating this gives

$$\int fg' = \int (fg)' - \int f'g.$$

But fg is certainly a primitive of $(fg)'$, and so $\displaystyle\int (fg)' = fg$, which proves the first two lines of the theorem. The third line follows immediately from either of the first two by the Second Fundamental Theorem. $\square$

The following example shows that in many cases there are two possible approaches, a successful and an unsuccessful one.

Example 16.5 Find

$$I(x) = \int x \cos x \, \mathrm{d}x.$$

(i) An unsuccessful approach. Set $f(x) = \cos x$ and $g'(x) = x$, so that $f'(x) = -\sin x$ and $g(x) = x^2/2$. So

$$I(x) = \int x \cos x \, \mathrm{d}x = \frac{1}{2}x^2 \cos x - \int \frac{1}{2}x^2(-\sin x) \, \mathrm{d}x = \frac{1}{2}x^2 \cos x \, \mathrm{d}x + \int \frac{1}{2}x^2 \sin x \, \mathrm{d}x.$$

This is of no use: if anything, the new integral on the right hand side is harder than the one we started with. We have increased the power of x, when decreasing it would have been more helpful.

(ii) A successful approach. Set $f(x) = x$ and $g'(x) = \cos x$, so that $f'(x) = 1$ and $g(x) = \sin x$. This gives

$$I(x) = x \sin x - \int 1 \cdot \sin x \, \mathrm{d}x = x \sin x + \cos x + C.$$

Example 16.6 Find $\int xe^x \mathrm{d}x$.

$$I(x) = \int xe^x \mathrm{d}x.$$

Set $f(x) = x$ and $g'(x) = e^x$ so that $f'(x) = 1$ and $g(x) = e^x$. Then

$$I(x) = xe^x - \int e^x \mathrm{d}x = xe^x - e^x + C = (x - 1)e^x + C.$$

Example 16.7 Evaluate

$$\int_0^{\pi/2} x \sin x \, \mathrm{d}x.$$

Set $f(x) = x$ and $g'(x) = \sin x$ so that $f'(x) = 1$ and $g(x) = -\cos x$. Then

$$\int_0^{\pi/2} x \sin x \, \mathrm{d}x = [-x \cos x]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, \mathrm{d}x = 0 + [\sin x]_0^{\pi/2} = 1 - 0 = 1.$$

Often, integration by parts has to be used more than once.

Example 16.8 Evaluate

$$P = \int_0^1 (x+1)^2 e^x \, \mathrm{d}x.$$

$$\begin{aligned}
P &= \int_0^1 (x+1)^2 e^x \, \mathrm{d}x \\
&= \left[(x+1)^2 e^x\right]_0^1 - \int_0^1 2(x+1)e^x \, \mathrm{d}x \\
&= \left[(x+1)^2 e^x\right]_0^1 - \left[2(x+1)e^x\right]_0^1 + \int_0^1 2e^x \, \mathrm{d}x \\
&= \left[(x+1)^2 e^x - 2(x+1)e^x + 2e^x\right]_0^1 \\
&= \left((1+1)^2 e^1 - 2(1+1)e^1 + 2e^1\right) - \left(e^0 - 2e^0 + 2e^0\right) \\
&= 2e - 1.
\end{aligned}$$

A handy trick is to use $g' = 1$ when the integrand is a single function f.

Example 16.9 Find $\displaystyle\int \ln x \, \mathrm{d}x$.

With $f(x) = \ln x$ we have $f'(x) = \frac{1}{x}$, and with $g'(x) = 1$ we have $g(x) = x$. Hence

$$\int \ln x \, \mathrm{d}x = x \ln x - \int \frac{1}{x} \cdot x \, \mathrm{d}x = x \ln x - x + C.$$

There are many more applications of this technique. Another example is $\displaystyle\int \arctan x \, \mathrm{d}x$. This looks hard to start with, but after integration by parts with $g' = 1$ we find

$$\int \arctan x \, \mathrm{d}x = x \arctan x - \int \frac{x}{1+x^2} \, \mathrm{d}x,$$

which appears a lot more promising already. We will learn how to evaluate the remaining integral $\displaystyle\int \frac{x}{1+x^2} \, \mathrm{d}x$ in Section 16.4.

In some cases, the original integral resurfaces on the right hand side. Care may then be required!

Example 16.10 Find $\displaystyle\int \sin x \, \cos x \, \mathrm{d}x$.

We use $f(x) = \sin x$ and $g'(x) = \cos x$ so that $f'(x) = \cos x$ and $g(x) = \sin x$. Then

$$\int \sin x \, \cos x \, \mathrm{d}x = \sin^2 x - \int \sin x \, \cos x \, \mathrm{d}x.$$

But this means that

$$2 \int \sin x \, \cos x \, \mathrm{d}x = \sin^2 x + C_2 \implies \int \sin x \, \cos x \, \mathrm{d}x = \frac{1}{2} \sin^2 x + C_1.$$

Note that we can represent the constant of integration in terms of C_1 or C_2, where $C_2 = 2C_1$.

It is interesting to check what happens if we make a different choice of f and g.

We use $f(x) = \cos x$ and $g'(x) = \sin x$ so that $f'(x) = -\sin x$ and $g(x) = -\cos x$. Then

$$\int \sin x \, \cos x \, \mathrm{d}x = -\cos^2 x - \int \sin x \, \cos x \, \mathrm{d}x.$$

But this means that

$$2 \int \sin x \, \cos x \, \mathrm{d}x = -\cos^2 x + D_2 \implies \int \sin x \, \cos x \, \mathrm{d}x = -\frac{1}{2} \cos^2 x + D_1,$$

where D_1 or D_2 represents the constant of integration.

How can it be that we get two different results? Remember that the indefinite integral is defined up to a constant. The expression $\int \sin x \, \cos x \, \mathrm{d}x$ stands for any function F with $F'(x) = \sin x \, \cos x$. It is easily checked that both $F(x) = \frac{1}{2} \sin^2 x$ and $G(x) = -\frac{1}{2} \cos^2 x$ have derivative $\sin x \, \cos x$. In fact, F and G differ by a constant:

$$F(x) - G(x) = \frac{1}{2} \sin^2 x - \left(-\frac{1}{2} \cos^2 x \right) = \frac{1}{2} \left(\sin^2 x + \cos^2 x \right) = \frac{1}{2}.$$

In terms of the constants C_1 and D_1, the results obtained above are made consistent by setting $D_1 - C_1 = \frac{1}{2}$.

Example 16.11 Try to find $\displaystyle\int \frac{1}{x} \, \mathrm{d}x$ using integration by parts.

We set $f(x) = \dfrac{1}{x}$ and $g'(x) = 1$, so $f'(x) = -\dfrac{1}{x^2}$ and $g(x) = x$, to find

$$\int \frac{1}{x} \, \mathrm{d}x = \frac{1}{x} \cdot x - \int (-\frac{1}{x^2}) \cdot x \, \mathrm{d}x = 1 + \int \frac{1}{x} \, \mathrm{d}x,$$

that is,

$$\int \frac{1}{x} \, \mathrm{d}x = 1 + \int \frac{1}{x} \, \mathrm{d}x. \tag{16.2}$$

But this looks impossible: if we subtract $\displaystyle\int \frac{1}{x} \, \mathrm{d}x$ from both sides of the equation we seem to arrive at $0 = 1$. What's gone wrong?

Again, the explanation lies with the constant of integration. Equation (16.2) means that any function F with $F'(x) = \dfrac{1}{x}$ can be written as $F(x) = 1 + G(x)$ where $G(x)$ is another function with $G'(x) = \dfrac{1}{x}$. Of course, $G = F + C$ for some constant C, but it is not necessarily true that $F = G$. In fact, here $F = 1 + G$.

Sometimes all tricks have to be used in a single integration. In the next example, we use $g' = 1$, integrate by parts twice, and then find our original integral on the right hand side.

Example 16.12 Find $\displaystyle\int \sin(\ln x)\,\mathrm{d}x$.

We start with $f(x) = \sin(\ln x)$ and $g'(x) = 1$. Then $g(x) = x$ and $f'(x) = \frac{\cos(\ln x)}{x}$.

$$\begin{aligned}
\int \sin(\ln x)\,\mathrm{d}x &= x\sin(\ln x) - \int x\frac{\cos(\ln x)}{x}\,\mathrm{d}x \\
&= x\sin(\ln x) - \int \cos(\ln x)\,\mathrm{d}x \\
&= x\sin(\ln x) - \left[x\cos(\ln x) + \int x\frac{\sin(\ln x)}{x}\,\mathrm{d}x\right] \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,\mathrm{d}x.
\end{aligned}$$

For the second integration by parts we have used $u(x) = \cos(\ln x)$ and $v'(x) = 1$. The final result is now

$$\int \sin(\ln x)\,\mathrm{d}x = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C.$$

After a lengthy computation like this it is prudent to check the result. Compute the derivative of $\dfrac{x}{2}(\sin(\ln x) - \cos(\ln x))$ and check that it is indeed $\sin(\ln x)$.

16.4 Integration by substitution

The chain rule for differentiation gives rise to the most important method of integration.

Theorem 16.3 ***Substitution Rule.***
If f and g' are continuous and F is any primitive of f, then

$$\int (f \circ g) \cdot g' = F \circ g,$$

that is,

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = F(g(x)).$$

Furthermore,

$$\int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = \left[F(g(x))\right]_a^b.$$

Proof: The chain rule applied to $F \circ g$ reads

$$\begin{aligned}(F \circ g)' &= (F' \circ g) \cdot g' = (f \circ g) \cdot g', \qquad \text{that is,} \\ (F \circ g)'(x) &= F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).\end{aligned}$$

This means that $F \circ g$ is a primitive of $(f \circ g) \cdot g'$, which proves the first two lines of the theorem. The third line follows immediately from either of the first two by the Second Fundamental Theorem. □

Example 16.13 Find $\displaystyle\int af(ax+b)\,\mathrm{d}x$ in terms of a primitive $F = \displaystyle\int f$ of f.

If $F' = f$, then

$$\int af(ax+b)\,\mathrm{d}x = F(ax+b).$$

In this example, $g(x) = ax + b$ and so $g'(x) = a$. The integrand can hence be written as $af(ax+b) = f(g(x)) \cdot g'(x)$. We can easily check the result by differentiating the right-hand side: $\dfrac{\mathrm{d}}{\mathrm{d}x}F(ax+b) = aF'(ax+b) = af(ax+b)$.

In this type of problem, where $g(x) = ax + b$, we usually have to make adjustments so that the constant a appears as a factor in the integrand.

Example 16.14 Find $\displaystyle\int \cos(4x+17)\,\mathrm{d}x$.

$$\int \cos(4x+17)\,\mathrm{d}x = \frac{1}{4}\int 4\cos(4x+17)\,\mathrm{d}x = \frac{1}{4}\sin(4x+17) + C.$$

In the examples that we have seen so far, application of the Substitution Rule immediately led to a primitive. In practice, matters will rarely be so simple. Instead, the substitution rule is normally used to transform an integral into an easier one. When we have successfully written the integrand in the form $f(g(x)) \cdot g'(x)$, we still have to find a primitive F for f. This task of finding $F = \displaystyle\int f$ might be rather involved and require, for example, integration by parts or further substitutions.

To illustrate how the Substitution Rule is used to simplify integrals, we start by looking at definite integrals. The last line of the Substitution Rule, Theorem 16.3, tells us that

$$\int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = [F(g(x))]_a^b\,.$$

Writing

$$[F(g(x))]_a^b = F(g(b)) - F(g(a)) = [F(u)]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u,$$

we find that

$$\int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

Example 16.15 Find $\displaystyle\int_a^b \tan x\,\mathrm{d}x$.

It can sometimes be a bit tricky to spot the functions f and g. The key is to look out for a function and its derivative. Writing

$$\int_a^b \tan x\,\mathrm{d}x = -\int_a^b \frac{-\sin x}{\cos x}\,\mathrm{d}x$$

suggests using $g'(x) = -\sin x$ and $g(x) = \cos x$. The factor $\dfrac{1}{\cos x}$ can then be written as $f(\cos x)$ where $f(u) = \dfrac{1}{u}$. Thus,

$$\begin{aligned}
\int_a^b \tan x\,\mathrm{d}x &= -\int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = -\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u \\
&= -\int_{\cos a}^{\cos b} \frac{1}{u}\,\mathrm{d}u = -[\ln|u|]_{\cos a}^{\cos b} = \ln|\cos a| - \ln|\cos b|.
\end{aligned}$$

Example 16.16 Find $\displaystyle\int_a^b \cot x\,\mathrm{d}x$.

This is similar to the integration of tan. In this case, $\cot x = \dfrac{\cos x}{\sin x}$, and this suggests that we try the sine or cosine for g. Since in the Substitution Rule g' appears in the numerator, we pick $g(x) = \sin x$ and $g'(x) = \cos x$. Then, with $f(x) = \dfrac{1}{x}$ we have

$$\int_a^b \cot x\,\mathrm{d}x = \int_a^b \frac{\cos x}{\sin x}\,\mathrm{d}x = [\ln|\sin x|]_a^b = \ln|\sin b| - \ln|\sin a|.$$

Example 16.17 Find $\displaystyle\int_a^b \frac{1}{x\ln x}\,\mathrm{d}x$.

We choose $g'(x) = \dfrac{1}{x}$ and $g(x) = \ln x$. Then $\frac{1}{\ln x}$ is $f(\ln x)$ where $f(u) = \dfrac{1}{u}$. Thus,

$$\begin{aligned}\int_a^b \frac{1}{x \ln x}\,\mathrm{d}x &= \int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u \\ &= \int_{\ln a}^{\ln b} \frac{1}{u}\,\mathrm{d}u = [\ln|u|]_{\ln a}^{\ln b} = \ln|\ln b| - \ln|\ln a| \\ &= \ln\frac{|\ln b|}{|\ln a|} = \ln\left|\frac{\ln b}{\ln a}\right| = \ln|\log_a b|\,.\end{aligned}$$

The procedure can be summarised as follows.

1. Pick appropriate functions f and g.
2. Substitute:
   (a) u for $g(x)$;
   (b) $\mathrm{d}u$ for $g'(x)\,\mathrm{d}x$;
   (c) $g(a)$ for a and $g(b)$ for b in the limits of integration.
3. Evaluate $\displaystyle\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$.

The rule that $\mathrm{d}u$ is substituted for $g'(x)\,\mathrm{d}x$ is most easily remembered by noting that

$$g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}$$

and formally 'multiplying' both sides of this equation by $\mathrm{d}x$.

Example 16.18 Find $\displaystyle\int_a^b \sin^n x \cos x\,\mathrm{d}x$.

We choose $f(u) = u^n$ and $g(x) = \sin x$. Then $\mathrm{d}u = \cos x\,\mathrm{d}x$ (as $g'(x) = \cos x$) and we get

$$\int_a^b \sin^n x \cos x\,\mathrm{d}x = \int_{\sin a}^{\sin b} u^n\,\mathrm{d}u = \frac{1}{n+1}\left(\sin^{n+1} b - \sin^{n+1} a\right).$$

In the last three examples, we have computed definite integrals. But by inspection of the results, we can also easily read off indefinite integrals:

$$\int \tan x\,\mathrm{d}x = -\ln|\cos x|,$$

$$\int \frac{1}{x \ln x} \mathrm{d}x = \ln|\ln x|,$$

and

$$\int \sin^n x \cos x \, \mathrm{d}x = \frac{1}{n+1} \sin^{n+1} x.$$

To find indefinite integrals directly without first computing definite integrals, we use the following procedure.

1. Pick appropriate functions f and g.
2. Substitute:
   (a) u for $g(x)$;
   (b) $\mathrm{d}u$ for $g'(x)\,\mathrm{d}x$.
3. Find a primitive $F(u) = \int f(u)\,\mathrm{d}u$.
4. Replace u by $g(x)$ to obtain $F(g(x))$.

With some practice, the entire procedure can be carried out without even introducing names for the functions f and g.

Example 16.19 Find $\displaystyle\int \frac{x}{1+x^2}\,\mathrm{d}x$.

We choose $u = 1 + x^2$ and then have $\mathrm{d}u = 2x\,\mathrm{d}x$ (because $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$). Then

$$\int \frac{x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+x^2| + C = \frac{1}{2}\ln(1+x^2) + C.$$

So far, all our examples were rather straightforward in that the factor $g'(x)$ was readily spotted in the integrand. Only in Examples 16.14 and 16.19 did we have to make minor adjustments so that $g'(x)$ appeared properly. However, the Substitution Rule is far more general, and some of the most interesting applications occur when $g'(x)$ does *not* appear in the integrand.

One important method is to substitute u for a 'troublesome' expression in x, and then to replace also all other occurences of x in terms of u. It is crucial that after the substitution the original variable of integration does not appear at all in the integral.

Example 16.20 Find $\displaystyle\int_0^1 \frac{x^2}{\sqrt{x+1}}\,\mathrm{d}x$.

We set $u = x + 1$ so that $\mathrm{d}u = \mathrm{d}x$. Then $x^2 = (u-1)^2 = u^2 - 2u + 1$ and

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x+1}}\,\mathrm{d}x &= \int \frac{u^2 - 2u + 1}{\sqrt{u}}\,\mathrm{d}u \\
&= \int (u^{3/2} - 2u^{1/2} + u^{-1/2})\,\mathrm{d}u \\
&= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + C,
\end{aligned}$$

and thus

$$\int_0^1 \frac{x^2}{\sqrt{x+1}}\,\mathrm{d}x = \frac{2}{5}2^{5/2} - \frac{4}{3}2^{3/2} + 2\cdot 2^{1/2} - \frac{2}{5} + \frac{4}{3} - 2 = \frac{14}{15}\sqrt{2} - \frac{16}{15}.$$

Example 16.21 Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x \sin x}{(1+\cos x)^{3/2}}\,\mathrm{d}x$.

We use $u = 1 + \cos x$, and so $\mathrm{d}u = -\sin x\,\mathrm{d}x$. Also, if $x = 0$, $u = 2$ and if $x = \pi/2$, $u = 1$, so

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos x \sin x}{(1+\cos x)^{3/2}}\,\mathrm{d}x &= \int_2^1 \frac{u-1}{u^{3/2}}\,(-\mathrm{d}u) = \int_1^2 (u^{-1/2} - u^{-3/2})\mathrm{d}u \\
&= [2u^{1/2} + 2u^{-1/2}]_1^2 = \left[2\left(\frac{u+1}{\sqrt{u}}\right)\right]_1^2 \\
&= 2\left[\frac{3}{\sqrt{2}} - 2\right] = 3\sqrt{2} - 4.
\end{aligned}$$

Although we usually carry out a substitution by defining u as a function of x, sometimes it is more convenient to write x as a function of u. This technique is sometimes (for example in ADAMS) called *inverse substitution*. However, it is really just another application of Theorem 16.3. Instead of $\displaystyle\int f(x)\,\mathrm{d}x$, we write $\displaystyle\int f(h(u))\cdot h'(u)\,\mathrm{d}u$, where now $x = h(u)$. Whichever way we look at it, whether we specify $u = g(x)$ or $x = h(u)$, the fact remains that we substitute u for x as the variable of integration.

If we specify x in terms of u, instead of specifying u in terms of x, the procedure is the following.

1. Substitute:
   (a) $h(u)$ for x;
   (b) $h'(u)\,\mathrm{d}u$ for $\mathrm{d}x$.

2. Find a primitive $F(u)$.

3. Replace u by $h^{-1}(x)$ to obtain $F(h^{-1}(x))$.

The rule that $h'(u)\,\mathrm{d}u$ is substituted for $\mathrm{d}x$ is most easily remembered by noting that

$$h'(u) = \frac{\mathrm{d}h(u)}{\mathrm{d}u} = \frac{\mathrm{d}x}{\mathrm{d}u}$$

and formally multiplying both sides of this equation by $\mathrm{d}u$.

Example 16.22 Find $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x$.

We use $x = a\tan u$; then $\mathrm{d}x = a(1+\tan^2 u)\,\mathrm{d}u$, and so

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \int \frac{a(1+\tan^2 u)}{a^2+a^2\tan^2 u}\,\mathrm{d}u = \frac{1}{a}\int \mathrm{d}u = \frac{u}{a} + C = \frac{1}{a}\arctan\frac{x}{a} + C.$$

In this particular case, the integral can also be found by substitution and using the fact that $\arctan'(x) = \dfrac{1}{1+x^2}$.

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a^2}\int \frac{1}{1+(x/a)^2}\,\mathrm{d}x = \frac{1}{a^2}\int \frac{a}{1+v^2}\,\mathrm{d}v = \frac{1}{a}\arctan v + C = \frac{1}{a}\arctan\frac{x}{a} + C,$$

where we have used $v = \dfrac{x}{a}$ and substituted $\mathrm{d}v$ for $\dfrac{1}{a}\,\mathrm{d}x$.

Example 16.23 Find $\displaystyle\int \sqrt{1-x^2}\,\mathrm{d}x$.

We use $x = \sin u$ and $\mathrm{d}x = \cos u\,\mathrm{d}u$. Then for $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have $x \in [-1, 1]$ and $\cos u \geq 0$. The integral becomes

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \int \sqrt{1-\sin^2 u}\,\cos u\,\mathrm{d}u = \int \cos^2 u\,\mathrm{d}u.$$

The integral in u can be evaluated in various ways. Using integration by parts, we get

$$\begin{aligned}\int \cos^2 u\,\mathrm{d}u &= \sin u\,\cos u + \int \sin^2 u\,\mathrm{d}u \\ &= \sin u\,\cos u + \int (1-\cos^2 u)\,\mathrm{d}u \\ &= u + \sin u\,\cos u - \int \cos^2 u\,\mathrm{d}u,\end{aligned}$$

and so

$$\int \cos^2 u\,\mathrm{d}u = \frac{u + \sin u\,\cos u}{2} + C.$$

The original integral is therefore

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{\arcsin x + \sin(\arcsin x)\cdot\cos(\arcsin x)}{2} + C = \frac{\arcsin x + x\sqrt{1-x^2}}{2} + C.$$

NB: from above we have $\displaystyle\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\arcsin(1) = \frac{\pi}{4}$. Can you see why?