15 Logarithms and Exponentials

15.1 Express the following in terms of $\ln 2$ and $\ln 3$.

(a) $\ln 16$ (b) $\ln 12$ (c) $\ln 36$ (d) $\ln 2\sqrt{2}$ (e) $\ln \frac{9}{8}$ (f) $\ln \sqrt{13.5}$.

15.2 Find the derivatives of the following functions.

(a) $f(x) = \ln(x^2 + 2x)$ (b) $f(x) = -\ln(\cos x)$ (c) $f(x) = x\,[\sin(\ln x) + \cos(\ln x)]$

(d) $f(x) = x \ln x$ (e) $f(x) = \ln x - \ln\sqrt{1+x^2} - \dfrac{1}{x}\arctan x$ (f) $f(x) = \ln(\tan x)$.

15.3 Use the properties of logarithms to expand each of the following expressions, and hence determine their derivatives with respect to x.

(a) $\ln[(5x+1)^3]$ (b) $\ln[(3x^3+1)^{1/3}]$ (c) $\ln[(x-1)(x-2)(x-3)]$

(d) $\ln\left(\dfrac{x+4}{x^2-7}\right)$ (e) $\ln\left(\dfrac{(2x+1)^{1/3}}{(3x-2)^{1/4}}\right)$ (f) $\ln\left(\dfrac{x\sqrt{2x^2+3}}{(x-1)\sqrt{1-3x^2}}\right)$.

15.4 Use logarithmic differentiation to find the derivatives of the following functions.

(a) $f(x) = \dfrac{(x-1)^3(x+2)^2}{x+1}$ (b) $g(x) = \dfrac{(3x-2)^{\frac{1}{2}}(x+1)^3}{1+2x}$ (c) $h(x) = \dfrac{(x+2)^{\frac{1}{2}}}{(3x^2+1)^{\frac{3}{2}}}$.

15.5 Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^x$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{x^x}$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^x)^x$ (d) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}x^x$.

15.6 Simplify the following expressions.

(a) $\ln(e^{3x})$ (b) $\ln\left(\dfrac{1}{e^x}\right)$ (c) $e^{\ln x + \ln y}$ (d) $\ln(x^2 e^{2x})$ (e) $\ln(e^{\ln(e^x)})$.

15.7 Find the derivatives of the following functions.

(a) $f(x) = x^2 e^x$ (b) $g(x) = e^{\sin x}$ (c) $h(x) = e^{1+\tan(2x)}$ (d) $i(x) = \arctan(e^x)$

(e) $j(x) = e^{x \sin x}$ (f) $k(x) = e^{e^x}$ (g) $l(x) = e^{e^{e^x}}$.

15.8 Find the derivatives of the following functions.

(a) $f(x) = x^2 a^x$ (b) $g(x) = 10^{\cos x}$ (c) $h(x) = 2^{1+\ln(2x)}$ (d) $i(x) = \arctan(a^x)$

(e) $j(x) = 3^{x\cos x}$ (f) $k(x) = a^{a^x}$ (g) $l(a) = a^{a^a}$ (h) $m(x) = (\ln x)^{\ln x}$.

15.9 Compute $e^{x^2}\dfrac{\mathrm{d}}{\mathrm{d}x}(e^{-x^2})$, $e^{x^2}\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{-x^2})$, and $e^{x^2}\dfrac{\mathrm{d}^3}{\mathrm{d}x^3}(e^{-x^2})$.

15.10 Find all continuous functions f that satisfy

(a) $\int_0^x f = e^x$,

(b) $\int_0^{x^2} f = 1 - e^{2x^2}$.

15.11 Let $F_{A,B}(x) = Ae^x \cos x + Be^x \sin x$. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}F_{A,B}(x) = F_{A+B,B-A}(x)$ and use this result to find $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}F_{A,B}(x)$.

15.12 Find the derivatives of the following functions.

(a) $f(x) = \log_{10}(\sin x)$ (b) $g(x) = \log_2(\exp(x))$ (c) $h(x) = \sin(\log_a x)$

(d) $i(x) = \log_{(e^x)} \sin x$ (e) $j(x) = \log_x a$ (f) $k(a) = \log_x a$.

15.13 Find $\lim_{x\to\infty} a^x$ for $0 < a < 1$.

15.14 Prove the following.

(a) $\log_3 2$ is irrational. [Hint: assume that $\log_3 2 = \frac{r}{s}$ with integers r and s, and use the definition of the logarithm.]

(b) If p and q are two distinct primes, then $\log_p q$ is irrational.

15.15 Verify the following identities by using the definitions of sinh and cosh.

(a) $\sinh x \cosh y + \cosh x \sinh y = \sinh(x + y)$;

(b) $\cosh x \cosh y + \sinh x \sinh y = \cosh(x + y)$.

Deduce similar results for $\sinh(x - y)$, $\cosh(x - y)$, $\sinh(2x)$ and $\cosh(2x)$.

15.16 Differentiate the following expressions with respect to x.

(a) $\cosh\sqrt{1 - x^2}$ (b) $x^2 \sinh(3x^5)$ (c) $\ln(\tanh x)$ (d) $\ln[\sinh(x^3 + 3x)]$.

15.17 (a) What are the natural domain and range of the function cosh? Sketch its graph.

(b) Explain why the function cosh with its natural domain does not have an inverse, but the function $z = \cosh x$ defined on the restricted domain $0 \le x < \infty$ does.

(c) Express $\cosh x$ in terms of e^x and show that $\cosh^{-1}(z) = \ln\left[z + \sqrt{z^2 - 1}\right]$.

Find the derivative of $\cosh^{-1}$.

15.18 (a) What are the natural domain and range of the function tanh? Sketch its graph.

(b) Express $\tanh x$ in terms of e^x and show that $\tanh^{-1}(y) = \dfrac{1}{2}\ln\left(\dfrac{1+y}{1-y}\right)$

for $-1 < y < 1$. Find the derivative of $\tanh^{-1}$.

15.19 Differentiate the following expressions with respect to x.

(a) $\cosh^{-1}(2x)$ (b) $\tanh^{-1}(\sin x)$ (c) $\sinh^{-1}\sqrt{x}$ (d) $\tanh^{-1}(e^{5x^2})$

15.20 Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\tanh^{-1}\left[\tan\left(\dfrac{x}{2}\right)\right]\right\} = \dfrac{1}{2}\sec x$.

15.21 If $f(x) = \left(\sinh^{-1} x\right)^2$, verify that $(1+x^2)f''(x) + xf'(x) = 2$.