Exercises and outline solutions for MM101 tutorial in week 6

1. Evaluate the following limits.

$$\text{(a)} \lim_{x\to -a} \frac{x^2-a^2}{x+a}; \qquad \text{(b)} \lim_{a\to -x} \frac{x^2-a^2}{x+a}.$$

(a) $\displaystyle\lim_{x\to -a} \frac{x^2-a^2}{x+a} = \lim_{x\to -a} \frac{(x+a)(x-a)}{x+a} = \lim_{x\to -a}(x-a) = -2a.$

(b) $\displaystyle\lim_{a\to -x} \frac{x^2-a^2}{x+a} = \lim_{a\to -x}(x-a) = 2x.$

2. Evaluate the following limits.

$$\text{(a)} \lim_{x\to 0} \frac{\sin 7x}{x/6}; \qquad \text{(b)} \lim_{x\to 0} \frac{\tan x}{x}.$$

(a) $\displaystyle\lim_{x\to 0} \frac{\sin 7x}{x/6} = 42 \lim_{x\to 0} \frac{\sin 7x}{7x} = 42.$

(b) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right) = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1.$

3. The function $f(x)$ satisfies $0 \le |f(x)| < |2x|$ for all $x \neq 0$. Prove that $\lim_{x\to 0} f(x) = 0$ using the ϵ–δ definition of the limit.

Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{2}$. Then we have $0 < |x-0| < \delta \implies 0 < |2x| < \epsilon$. But since $0 \le |f(x)| < |2x|$, we see that $0 < |x-0| < \delta \implies 0 \le |f(x)| < \epsilon$, and thus that $\lim_{x\to 0} f(x) = 0$.

4. For each of the following:

   (i) find the limit $l = \lim_{x\to a} f(x)$ for the given value of a; and

   (ii) prove that it is the limit by finding, for an arbitrary $\epsilon > 0$, a suitable $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever x satisfies $0 < |x-a| < \delta$.

$$\text{(a)}\ f(x) = 4x,\ a = \frac{2}{3}; \qquad \text{(b)}\ f(x) = 2x^2,\ a = 2.$$

(i) The limit is $l = \frac{8}{3}$. We require that $|4x - \frac{8}{3}| < \epsilon$. Dividing by 4 yields $|x - \frac{2}{3}| < \frac{\epsilon}{4}$. So the choice $\delta = \frac{\epsilon}{4}$ works.

(ii) The limit is $l = 8$. We require that $|2x^2 - 8| < \epsilon$, which is equivalent to $|x+2| \cdot |x-2| < \epsilon/2$. If we require $|x-2| < 1$ then $-1 < x-2 < 1$, so $-1+2 < x < 1+2$, so $3 < x+2 < 5$, and so $|x+2| < 5$. Thus, when $|x-2| < 1$ we know that $|x^2-4| < 5|x-2|$. Therefore, if we choose $\delta = \min\left(1, \frac{\epsilon}{10}\right)$ then $|x-2| < \delta \implies |x^2-4| < \epsilon/2 \implies |2x^2-8| < \epsilon$, and the proof is done.