Exercises and outline solutions for MM101 tutorial in week 8

1. Suppose that the functions f and g are continuous for all $x \in [0,6]$. Determine the value of the integral

$$I = \int_1^6 (f-g)$$

given that

$$\int_0^6 f = 10, \qquad \int_1^6 g = 4, \qquad \text{and} \qquad \int_1^0 f = 2.$$

(You should carefully note which definitions or theorems you are using at each stage of your argument.)

$$\begin{aligned} I = \int_1^6 (f-g) &= \int_1^6 f + \int_1^6 (-g) && \text{by Theorem 13.3} \\ &= \int_1^6 f - \int_1^6 g && \text{by Theorem 13.2} \\ &= \int_0^6 f - \int_0^1 f - \int_1^6 g && \text{by Theorem 13.1} \\ &= \int_0^6 f + \int_1^0 f - \int_1^6 g && \text{by Definition 13.4} \\ &= 10 + 2 - 4 = 8. \end{aligned}$$

2. Without evaluating any integrals, find $f'(x)$ for the functions defined by the following expressions.

(a) $f(x) = \int_a^x t^2 \, \mathrm{d}t$; (b) $f(x) = \int_x^a t^2 \, \mathrm{d}t$;

(c) $f(x) = \int_a^b t^2 \, \mathrm{d}t$; (d) $f(x) = \sin\left(\int_a^x \cos^2 t \, \mathrm{d}t\right)$;

(e) $f(x) = \int_a^b x t^2 \, \mathrm{d}t$; (f) $f(x) = \int_a^x x t^2 \, \mathrm{d}t$.

(a) $f'(x) = x^2$.

(b) $f(x) = -\int_a^x t^2 \, \mathrm{d}t$, and so $f'(x) = -x^2$.

(c) $f'(x) = 0$.

(d) $f'(x) = \cos\left(\int_a^x \cos^2 t \, \mathrm{d}t\right) \cdot \cos^2 x$.

(e) $f(x) = x \int_a^b t^2 \, \mathrm{d}t$, and so $f'(x) = \int_a^b t^2 \, \mathrm{d}t$.

(f) $f'(x) = \int_a^x t^2 \, \mathrm{d}t + x^3$.

3. For what values of the real number n do the following integrals exist?

(a) $\int_0^1 x^n \, \mathrm{d}x$; (b) $\int_1^\infty x^n \, \mathrm{d}x$; (c) $\int_0^\infty x^n \, \mathrm{d}x$.

(a) For $n > 0$, x^n is bounded and continuous on $[0,1]$ so the integral exists. For $n < 0$ and $n \neq -1$, x^n is only bounded on $(0,1]$ so we have the improper integral

$$\int_0^1 x^n \,\mathrm{d}x = \lim_{a\to 0^+} \int_a^1 x^n \,\mathrm{d}x = \lim_{a\to 0^+} \frac{1-a^{n+1}}{n+1},$$

so the integral is defined only when $n > -1$. For $n = -1$, the improper integral is

$$\int_0^1 x^{-1} \,\mathrm{d}x = \lim_{a\to 0^+} \int_a^1 x^{-1} \,\mathrm{d}x = \lim_{a\to 0^+} (-\ln a),$$

and is also undefined.

(b) The improper integral is defined by $\displaystyle\int_1^\infty x^n \,\mathrm{d}x = \lim_{b\to\infty} \int_1^b x^n \,\mathrm{d}x$. For $n \neq -1$, we therefore have

$$\int_1^\infty x^n \,\mathrm{d}x = \lim_{b\to\infty} \frac{b^{n+1}-1}{n+1},$$

while for $n = -1$, we have

$$\int_1^\infty x^{-1} \,\mathrm{d}x = \lim_{b\to\infty} \ln b,$$

Thus the integral is defined for $n < -1$ only.

(c) If the integral is defined then we can write $\displaystyle\int_0^\infty x^n \,\mathrm{d}x = \int_0^1 x^n \,\mathrm{d}x + \int_1^\infty x^n \,\mathrm{d}x$. By (a), the first integral is defined if and only if $n > -1$. By (b), the second integral is defined if and only if $n < -1$. We conclude that $\displaystyle\int_0^\infty x^n \,\mathrm{d}x$ is undefined for every real number n.