UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercise solutions: Chapter 5

1. (i) 3 (ii) 7.

2. (i) proper (ii) proper (iii) improper (iv) improper (v) proper (vi) improper.

3. (i) $\dfrac{1}{x+1}$ (ii) $\dfrac{1}{x+2}$ (iii) $\dfrac{x}{x^2+1}$ (iv) $\dfrac{(x+1)^2}{x^2-4x-4}$ (v) $\dfrac{-1}{(x^2+4)(x+2)}$ (vi) $\dfrac{x(x-1)}{x+4}$.

4. Using long division
(i)

$$\begin{array}{rr|rrr} & & & 2x & +9 \\ \hline x & -4 & 2x^2 & +x & +7 \\ & & 2x^2 & -8x & \\ & & & 9x & +7 \\ & & & 9x & -36 \\ & & & & 43 \end{array}$$

So

$$\frac{2x^2+x+7}{x-4} = 2x+9+\frac{43}{x-4}$$

(**quotient** is $2x+9$, **remainder** is 43).

(ii)

$$\begin{array}{rrr|rrrr} & & & & & 3x & -2 \\ \hline x^2 & +2x & +2 & 3x^3 & +4x^2 & +2x & +1 \\ & & & 3x^3 & +6x^2 & +6x & \\ & & & & -2x^2 & -4x & +1 \\ & & & & -2x^2 & -4x & -4 \\ & & & & & & 5 \end{array}$$

So

$$\frac{3x^3+4x^2+2x+1}{x^2+2x+2} = 3x-2+\frac{5}{x^2+2x+2}$$

(**quotient** is $3x-2$, **remainder** is 5).

(iii)

$$\begin{array}{cc|ccc} & & & 4x & 9 \\ \hline x & -5 & 4x^2 & +4x & -2 \\ & & 4x^2 & -20x & \\ & & & 24x & -2 \\ & & & 24x & -120 \\ & & & & 118 \end{array}$$

So

$$\frac{4x^2+4x-2}{x-5} = 4x + 24 + \frac{118}{x-5}$$

(**quotient** is $4x + 24$, **remainder** is 118.)

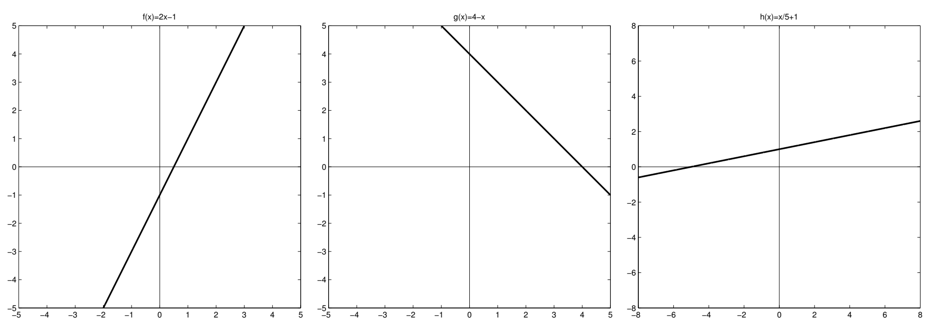
(iv)

$$\begin{array}{ccc|cccc} & & & & & -x & -4 \\ \hline -x^2 & +2x & +1 & x^3 & +2x^2 & -x & -3 \\ & & & x^3 & -2x^2 & -x & \\ & & & & 4x^2 & & -3 \\ & & & & 4x^2 & -8x & -4 \\ & & & & & 8x & +1 \end{array}$$

So

$$\frac{x^3+2x^2-x-3}{-x^2+2x+1} = \frac{8x+1}{-x^2+2x+1}$$

(**quotient** is $-x-4$, **remainder** is $8x+1$).

5. 

6. (i) $x = -9/5$ (ii) $x = 13/3$ (iii) $x = 49/45$.

7. (i) Completing the square gives

$$x^2 + 4x - 7 = (x+2)^2 - 11.$$

Hence, the minimum value taken by the quadratic is -11.
(i) Completing the square gives

$$-2x^2 + 3x + 1 = -2(x - \frac{3}{4})^2 + \frac{17}{8}.$$

Hence, the maximum value taken by the quadratic is $17/8$.

8. (i) $x = 3$ or $x = 5$ (ii) $x = 2$ or $x = -6$ (iii) $x = 3/2$ or $x = -1$
(iv) $x = 3 \pm \sqrt{3}$ (v) $x = \dfrac{2 \pm \sqrt{10}}{3}$ (vi) $x = \dfrac{1 \pm \sqrt{23}i}{4}$

9. The roots can be found by trying factors of the constant term in each case.
(i) $x = 7,\ x = 1,\ x = -2$ (ii) $x = 2,\ x = -2, x = 3, x = 4$.

10. In each case the first root is found by trial and error, then using long division with the related factor and factorising gives the final result.
(a) $x = 2$ is a root so $x - 2$ is a factor. Final factorisation is $(x-2)(x+3)(x+4)$.
(b) $x = 3$ is a root so $x - 3$ is a factor. Final factorisation is $(x-3)(x+1)^2$.
(c) $x = -1$ is a root so $x + 1$ is a factor. Final factorisation is $(x+1)(x^2+1)$.
(d) $x = 0$ is a root so x is a factor. Final factorisation is $x(x+1)(x-1)$.

11. $(x+y)^2 = x^2 + 2xy + y^2$ so $(x+y)^2 = x^2 + y^2 \iff x = 0$ or $y = 0$.
$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x+y)$ so
$(x+y)^3 = x^3 + y^3 \iff x = 0$ or $y = 0$ or $x = -y$.

12. Let P be the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

of positive degree n. If $x - 1$ is a factor of P, then 1 is a root, so

$$P(1) = a_n + a_{n-1} + \ldots + a_1 + a_0 = 0.$$

That is, the sum of the coefficients must be zero. Similarly, if $x + 1$ is a factor, -1 is a root so the coefficients must satisfy

$$P(-1) = (-1)^n a_n + (-1)^{n-1} a_{n-1} + \ldots - a_1 + a_0 = 0.$$

13. (a) $2x + 3 > 1 - 4x \Leftrightarrow 6x > -2 \Leftrightarrow x > -1/3$.

(b) $x^2 - 11x + 24 = (x-8)(x-3)$ so quadratic changes sign at $x = 3$ and $x = 8$. Table of signs:

x values	$x < 3$	$x = 3$	$3 < x < 8$	$x = 8$	$x > 8$
$x - 3$	-	0	+	+	+
$x - 8$	-	-	-	0	+
$(x-3)(x-8)$	+	0	-	0	+

Quadratic is negative for $3 < x < 8$.

(c) $6 + x \le 4x + 3 \Leftrightarrow 3 \le 3x \Leftrightarrow x \ge 1$.

(d) If $0 \le 1 - 2x \le 3$ then $0 \le 1 - 2x$ and $1 - 2x \le 3$. So

$$0 \le 1 - 2x \Leftrightarrow 2x \le 1 \Leftrightarrow x \le \frac{1}{2}$$

and

$$1 - 2x \le 3 \Leftrightarrow -2 \le 2x \Leftrightarrow -1 \le x.$$

That is, $-1 \le x \le 1/2$.

(e)

$$\frac{1}{2x-1} > 1 \Leftrightarrow \frac{1}{2x-1} - 1 > 0 \Leftrightarrow \frac{1-(2x-1)}{2x-1} > 0 \Leftrightarrow \frac{2-2x}{2x-1} > 0 \Leftrightarrow \frac{2(1-x)}{2x-1} > 0.$$

Factors change sign at $x = 1$ and $x = 1/2$. Table of signs:

x values	$x < 1/2$	$x = 1/2$	$1/2 < x < 1$	$x = 1$	$x > 1$
$1 - x$	-	-	-	0	+
$2x - 1$	-	0	+	+	+
$\frac{2(1-x)}{2x-1}$	+	nd	-	0	+

Hence we required $1/2 < x < 1$.

(f)

$$\frac{1}{2x-1} > \frac{1}{3x+2} \Leftrightarrow \frac{1}{2x-1} - \frac{1}{3x+2} > 0 \Leftrightarrow \frac{(3x+2)-(2x-1)}{(2x-1)(3x+2)} > 0$$

$$\Leftrightarrow \frac{x+3}{(2x-1)(3x+2)} > 0.$$

Factors change sign at $x = -3$, $x = -2/3$ and $x = 1/2$. Table of signs:

x values	$x < -3$	$x = -3$	$-3 < x < -\frac{2}{3}$	$x = -\frac{2}{3}$	$-\frac{2}{3} < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$x + 3$	-	0	+	+	+	+	+
$2x - 1$	-	-	-	-	-	0	+
$3x + 2$	-	-	-	0	+	+	+
$\frac{x+3}{(2x-1)(3x+2)}$	-	0	+	nd	-	nd	+

Hence we require $-3 < x < -2/3$ or $x > 1/2$.

14. (a) $|x - 4| < 2$ means

$$x - 4 > -2 \qquad \text{and} \qquad x - 4 < 2$$

so

$$x > 2 \qquad \text{and} \qquad x < 6.$$

That is, $2 < x < 6$.

(b) $|4x + 3| \leq 1$ means

$$4x + 3 \geq -1 \qquad \text{and} \qquad 4x + 3 \leq 1$$

so

$$4x \geq -4 \Leftrightarrow x \geq -1 \qquad \text{and} \qquad 4x \leq -2 \Leftrightarrow x \leq -\tfrac{1}{2}.$$

That is, $-1 \leq x \leq -1/2$.

(c) $|2x + 5| \geq 7$ means

$$2x + 5 \leq -7 \qquad \text{or} \qquad 7 \leq 2x + 5$$

so

$$2x \leq -12 \Leftrightarrow x \leq -6 \qquad \text{or} \qquad 2 \leq 2x \Leftrightarrow 1 \leq x.$$

That is, we require $x \leq -6$ or $x \geq 1$.

(d) $|x-2| < 3$ means

$$x - 2 > -3 \qquad \text{and} \qquad x - 2 < 3$$

so

$$x > -1 \qquad \text{and} \qquad x < 5.$$

That is, $-1 < x < 5$.

(e) From part (b), we have immediately that the required interval is $(-\infty, -1] \cup [-1/2, \infty)$,

(f) $|2 - 3x| \leq 1$ means

$$2 - 3x \geq -1 \qquad \text{and} \qquad 2 - 3x \leq 1$$

so

$$-3x \geq -3 \Leftrightarrow x \leq 1 \qquad \text{and} \qquad -3x \leq -1 \Leftrightarrow x \geq \frac{1}{3}.$$

That is, $1/3 \leq x \leq 1$.

15. (i) Label the equations (1) and (2). Then

$$(1) : 5x + 3y = 1, \qquad 5 \times (2) : 5x - 20y = 70$$

so subtracting (2) from (1) gives

$$23y = -69 \Leftrightarrow y = -3.$$

Substituting this into (1) gives

$$5x - 9 = 1 \Leftrightarrow 5x = 10 \Leftrightarrow x = 2.$$

(ii) Label the equations (1) and (2). Then

$$2 \times (1) : 6x - 14y = -18, \qquad 3 \times (2) : 6x + 9y = 51$$

so subtracting (2) from (1) gives

$$-23y = -69 \Leftrightarrow y = 3.$$

Substituting this into (1) gives

$$6x - 42 = -18 \Leftrightarrow 6x = 24 \Leftrightarrow x = 4.$$

(iii) Label the equations (1) and (2). Then

$$(1) : 4x + 5y = 3, \qquad 2 \times (2) : 6x - 5y = 2$$

so adding (1) and (2) gives

$$10x = 5 \Leftrightarrow x = \frac{1}{2}.$$

Substituting this into (1) gives

$$2 + 5y = 3 \Leftrightarrow 5y = 1 \Leftrightarrow y = \frac{1}{5}.$$

16. (i) Second equation gives $x = 0$ or $y = 1$.
If $x = 0$, first equation is $y - 2 = 0 \Leftrightarrow y = 2$.
If $y = 1$, first equation is $-(x + 1) = 0 \Leftrightarrow x = -1$.
So the solutions are $(0, 2)$ and (-1,1).

(ii) Second equation gives $x = 4 + y^2$. Substitute this into first equation:

$$(4 + y^2)^2 + 2y^2 = 8 \Leftrightarrow 16 + 8y^2 + y^4 + 2y^2 = 8 \Leftrightarrow y^4 + 10y^2 + 8 = 0.$$

This is a quadratic equation for y^2 : solve using the quadratic formula.

$$y^2 = -\frac{10 \pm \sqrt{68}}{2} = -5 \pm \sqrt{17} < 0$$

so there are no real solutions.

(iii) Many methods of solution. Here are two.

- Factorising first equation gives $(2x - y)(2x + y) = 0$, so $y = \pm 2x$.
  Letting $y = 2x$ in the second equation gives

  $$x^2 + 2x - 1 = 0 \Leftrightarrow x = -1 \pm \sqrt{2}.$$

  Since $y = 2x$ we get two solutions $(-1-\sqrt{2}, -2(1+\sqrt{2}))$ and $(-1+\sqrt{2}, -2(1-\sqrt{2}))$.
  Similarly with $y = -2x$ we find $x^2 - 2x + 1 = 0$ leading to two more solutions $(1 - \sqrt{2}, -2(1 - \sqrt{2}))$ and $(1 + \sqrt{2}, -2(1 + \sqrt{2}))$.
- Second equation gives $y = 1 - x^2$. Substitute this into first equation:

  $$4x^2-(1-x^2)^2 = 0 \Leftrightarrow 4x^2-(1-2x^2+x^4) = 0 \Leftrightarrow -x^4+6x^2-1 = 0 \Leftrightarrow x^4-6x^2+1 = 0.$$

  This is a quadratic equation in x^2 : solve using the quadratic formula.

  $$\begin{aligned} x^2 &= \frac{6 \pm \sqrt{36 - 4}}{2} \\ &= 3 \pm \frac{\sqrt{32}}{2} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}. \end{aligned}$$

If $x^2 = 3 + 2\sqrt{2}$ then $x = \pm\sqrt{3 + 2\sqrt{2}} = \pm(1 + \sqrt{2})$.

If $x^2 = 3 - 2\sqrt{2}$ then $x = \pm\sqrt{3 - 2\sqrt{2}} = \pm(-1 + \sqrt{2})$.

So $x = 1 + \sqrt{2}, -1 - \sqrt{2}, -1 + \sqrt{2}, 1 - \sqrt{2}$ are all possible solutions.

Now $y = 1 - x^2$ so when $x^2 = 3 + 2\sqrt{2}, \quad y = -2 - 2\sqrt{2} = -2(1 + \sqrt{2})$ and when $x^2 = 3 - 2\sqrt{2}, \quad y = -2 + 2\sqrt{2} = -2(1 - \sqrt{2})$.

So, all possible solutions (x, y) are:

$(1+\sqrt{2}, -2(1+\sqrt{2})), (-1-\sqrt{2}, -2(1+\sqrt{2})), (-1+\sqrt{2}, -2(1-\sqrt{2})), (1-\sqrt{2}, -2(1-\sqrt{2}))$.

Exercise solutions: Chapter 6

1.

Angle in degrees	$0°$	$30°$	$45°$	$60°$	$90°$
Fraction of circle	0	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$
Angle θ in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

2. (i)

$$\begin{aligned}
\cos 210° &= -\cos 30° = -\frac{\sqrt{3}}{2} \\
\sin 135° &= \sin 45° = \frac{1}{\sqrt{2}} \\
\tan 330° &= -\tan 30° = -\frac{1}{\sqrt{3}} \\
\cot 225° &= \cot 45° = 1 \\
\sec(-60°) &= \sec 60° = 2 \\
\operatorname{cosec}(-240°) &= -\operatorname{cosec} 60° = -\frac{2}{\sqrt{3}}
\end{aligned}$$

(ii)

$$\begin{aligned}
\sin\frac{5\pi}{2} &= \sin\frac{\pi}{2} = 1 \\
\tan\frac{5\pi}{4} &= \tan\frac{\pi}{4} = 1 \\
\cos\frac{13\pi}{4} &= -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \\
\cos\frac{7\pi}{4} &= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\
\cos\left(-\frac{2\pi}{3}\right) &= -\cos\frac{\pi}{3} = -\frac{1}{2}
\end{aligned}$$

3. (i) $$\begin{aligned} \tan\theta &= -2 \text{ so } \theta = \arctan(-2) \text{ giving} \\ \theta &= 1.107 + \pi k, \quad k \in \mathbb{Z} \end{aligned}$$

   (ii) $$\begin{aligned} \sin\theta &= 0.1 \text{ so } \theta = \arcsin(0.1) \text{ giving} \\ \theta &= 0.100 + 2\pi k, \quad \pi - 0.100 + 2\pi k, \quad k \in \mathbb{Z} \end{aligned}$$

   (iii) $$\begin{aligned} \cos\theta &= -0.9 \text{ so } \theta = \arccos(-0.9) \text{ giving} \\ \Rightarrow \theta &= 2.691 + 2\pi k, \ -2.691 + 2\pi k, \quad k \in \mathbb{Z} \end{aligned}$$

4. (i) $\tan x = -\dfrac{1}{\sqrt{3}}$ so $x = \arctan\left(-\dfrac{1}{\sqrt{3}}\right)$ giving $x = -\dfrac{\pi}{6} + k\pi, \quad k \in \mathbb{Z}.$

   (ii) $\cos x = -\dfrac{\sqrt{3}}{2}$ so $x = \arccos\left(-\dfrac{\sqrt{3}}{2}\right)$ giving $x = \pm\dfrac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}.$

5. (i) $\sin\theta = \dfrac{5}{13} \Rightarrow \tan\theta = \dfrac{5}{12}$

   (ii) $\cos\theta = \dfrac{1}{2} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \operatorname{cosec}\theta = \dfrac{2}{\sqrt{3}}$

6. $\tan\alpha = 3 \Rightarrow \dfrac{\sin\alpha}{\cos\alpha} = 3 \Rightarrow \sin\alpha = 3\cos\alpha$ so

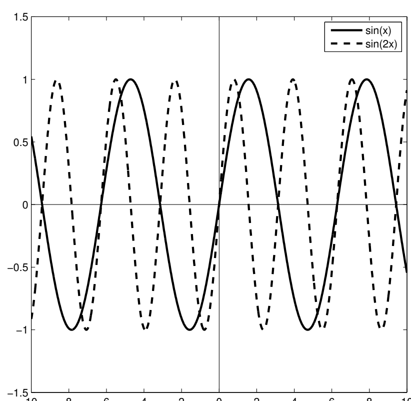
$$\frac{\sin\alpha - \cos\alpha}{\sec\alpha - \operatorname{cosec}\alpha} = \frac{2\cos\alpha}{\dfrac{1}{\cos\alpha} - \dfrac{1}{3\cos\alpha}} = \frac{2\cos\alpha}{\dfrac{3-1}{3\cos\alpha}} = 3\cos^2\alpha.$$

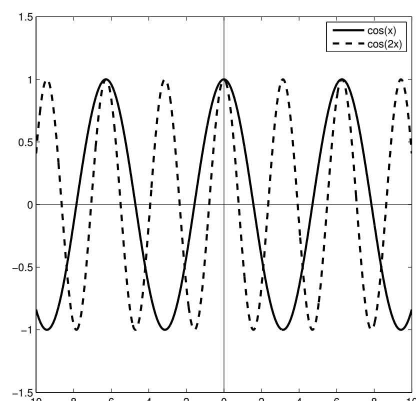
   But $\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow 9\cos^2\alpha + \cos^2\alpha = 1 \Rightarrow \cos^2\alpha = \dfrac{1}{10}$, hence result.

7. (i) $\cos 252^\circ = -\cos 72^\circ$; $\sin 116^\circ = \sin 64^\circ$; $\sin(-10^\circ) = -\sin 10^\circ$; $\tan 187.5^\circ = \tan 7.5^\circ$.

   (ii) $\cos\left(\dfrac{7\pi}{12}\right) = -\sin\left(\dfrac{\pi}{12}\right)$; $\sin\left(\dfrac{9\pi}{8}\right) = -\sin\left(\dfrac{\pi}{8}\right)$; $\tan\left(-\dfrac{11}{12}\pi\right) = \tan\left(\dfrac{\pi}{12}\right)$;

   $\sec\left(\dfrac{7\pi}{5}\right) = -\sec\left(\dfrac{2\pi}{5}\right)$; $\operatorname{cosec}\left(-\dfrac{15\pi}{8}\right) = \cos\left(\dfrac{\pi}{8}\right)$.

8. 



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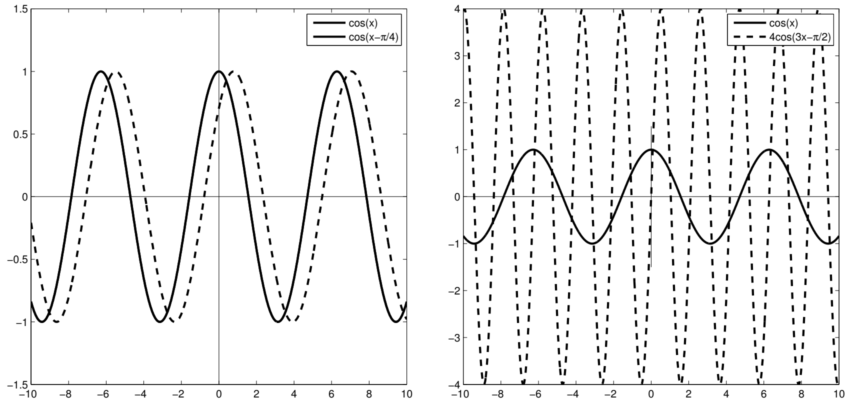
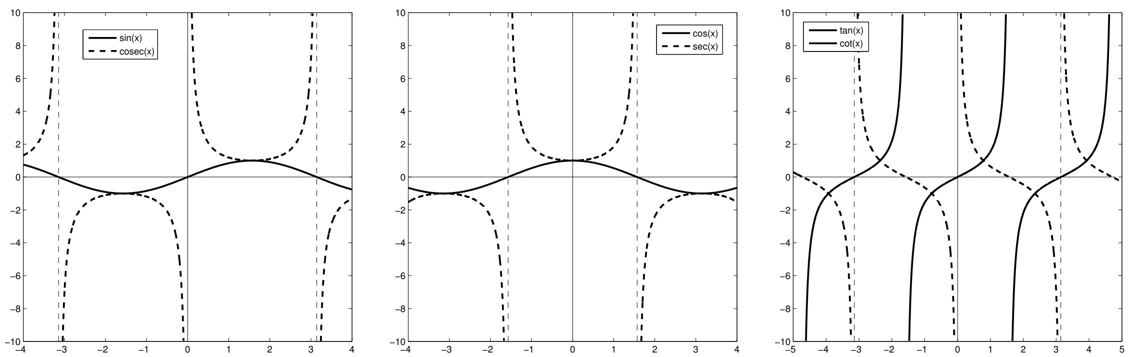




10.

$$f(g(x)) = f(\sin x) = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|.$$

Since $-1 \le \sin x \le 1$, $1-\sin^2 x \ge 0$ for all x and the domain of $f \circ g$ is $\mathbb{R}$.

$$g(f(x)) = g(\sqrt{1-x^2}) = \sin(\sqrt{1-x^2}).$$

The domain of $f(x)$ is $|x| \le 1$, hence the domain of $g \circ f$ is $\{x \in \mathbb{R} : |x| \le 1\}$, or $[-1, 1]$.

11. (a) $(-x)^3 + \sin(-x) = -x^3 - \sin(x)$: odd.

(b) $(-x) + \cos(-x) = -x + \cos(x)$: neither.

(c) $(-x)\sin(-x) + (-x) + 1 = x\sin(x) - x + 1$: neither.

(d) $(-x)^2 \tan(-x) + \sin(-2x) = -x^2 \tan(x) - \sin(2x)$: odd.

(e) $\cos(-\pi x) + \sin^2(-x) + 3 = \cos(\pi x) + [-\sin(x)]^2 + 3 = \cos(\pi x) + \sin^2(x) + 3$: even.

12. $\sin\left(\frac{\pi}{2}-\alpha\right)\cot\left(\frac{\pi}{2}-\alpha\right) = \sin\left(\frac{\pi}{2}-\alpha\right) \cdot \dfrac{\cos\left(\frac{\pi}{2}-\alpha\right)}{\sin\left(\frac{\pi}{2}-\alpha\right)}$

$= \cos\left(\frac{\pi}{2}-\alpha\right) = \cos\frac{\pi}{2}\,\cos\alpha + \sin\frac{\pi}{2}\,\sin\alpha = \sin\alpha.$

13. $\dfrac{\cos\left(\frac{\pi}{2}-\beta\right)}{\sec\left(\frac{\pi}{2}-\beta\right)} = \cos^2\left(\frac{\pi}{2}-\beta\right) = \left(\cos\frac{\pi}{2}\,\cos\beta + \sin\frac{\pi}{2}\,\sin\beta\right)^2 = \sin^2\beta.$

14. (i) $\dfrac{1}{\sin^2\theta} - 1 = \dfrac{1-\sin^2\theta}{\sin^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta.$

(ii) $\sin^3 A + \sin A\cos^2 A = \sin A(\sin^2 A + \cos^2 A) = \sin A.$

(iii) $\dfrac{\sec\alpha - \cos\alpha}{\sin\alpha} = \dfrac{\dfrac{1}{\cos\alpha} - \cos\alpha}{\sin\alpha} = \dfrac{1-\cos^2\alpha}{\cos\alpha}\cdot\dfrac{1}{\sin\alpha} = \dfrac{\sin^2\alpha}{\sin\alpha\cos\alpha} = \tan\alpha.$

(iv) $\dfrac{\cot\theta}{1+\cot^2\theta} = \dfrac{\dfrac{\cos\theta}{\sin\theta}}{1+\dfrac{\cos^2\theta}{\sin^2\theta}} = \dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\sin^2\theta}{\sin^2\theta+\cos^2\theta} = \sin\theta\cos\theta.$

15. (i) $\dfrac{1}{1+\sin x} + \dfrac{1}{1-\sin x} = \dfrac{(1-\sin x)+(1+\sin x)}{1-\sin^2 x} = \dfrac{2}{\cos^2 x} = 2\sec^2 x$

(ii) $\dfrac{\sin^2 A}{\tan A} - \dfrac{\cos^2 A}{\cot A} = \sin^2 A\cdot\dfrac{\cos A}{\sin A} - \cos^2 A\cdot\dfrac{\sin A}{\cos A} = \sin A\cos A - \cos A\sin A = 0.$

16. (i) $\tan A + \cot A = \dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A} = \dfrac{\sin^2 A+\cos^2 A}{\sin A\cos A} = \sec A \operatorname{cosec} A.$

(ii) $\sec^2 A + \operatorname{cosec}^2 A = \dfrac{1}{\cos^2 A} + \dfrac{1}{\sin^2 A} = \dfrac{\sin^2 A+\cos^2 A}{\sin^2 A\cos^2 A} = \sec^2 A\operatorname{cosec}^2 A.$

(iii) $\sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$
$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B = \sin^2 A - \sin^2 B.$

(iv) $\sin^4 A - \cos^4 A = (\sin^2 A - \cos^2 A)(\sin^2 A + \cos^2 A)$
$= \sin^2 A - \cos^2 A = (1-\cos^2 A) - \cos^2 A = 1 - 2\cos^2 A.$

(v) $\dfrac{1+\sin x}{1-\sin x}\cdot\dfrac{1+\sin x}{1+\sin x} = \dfrac{(1+\sin x)^2}{1-\sin^2 x} = \dfrac{(1+\sin x)^2}{\cos^2 x} = \left(\dfrac{1+\sin x}{\cos x}\right)^2$
$= (\sec x + \tan x)^2.$

(vi) $\dfrac{1-\cos x}{1+\cos x}\cdot\dfrac{1-\cos x}{1-\cos x} = \dfrac{(1-\cos x)^2}{1-\cos^2 x} = \dfrac{(1-\cos x)^2}{\sin^2 x} = \left(\dfrac{1-\cos x}{\sin x}\right)^2$
$= (\operatorname{cosec} x - \cot x)^2.$

17. (i) (a) $\cos^2\theta - \sin\theta - \dfrac{1}{4} = 0 \Rightarrow (1-\sin^2\theta) - \sin\theta - \dfrac{1}{4} = 0$

$\Rightarrow \sin^2\theta + \sin\theta - \dfrac{3}{4} = 0 \Rightarrow \left(\sin\theta + \dfrac{3}{2}\right)\left(\sin\theta - \dfrac{1}{2}\right) = 0$

$\Rightarrow \sin\theta = -\dfrac{3}{2}$ or $\sin\theta = \dfrac{1}{2} \Rightarrow \sin\theta = \dfrac{1}{2}$ (as $-1 \le \sin\theta \le 1$) so $\theta = \dfrac{\pi}{6} + 2k\pi$ or $\theta = \dfrac{5\pi}{6} + 2k\pi$ for some $k \in \mathbb{Z}$.

(b) $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

(ii) (a) $2\sin^2\theta + 3\cos\theta = 0 \Rightarrow 2(1-\cos^2\theta) + 3\cos\theta = 0$
$\Rightarrow 2\cos^2\theta - 3\cos\theta - 2 = 0 \Rightarrow (2\cos\theta + 1)(\cos\theta - 2) = 0$
$\Rightarrow \cos\theta = 2$ or $\cos\theta = -\dfrac{1}{2} \Rightarrow \cos\theta = -\dfrac{1}{2}$ (as $-1 \le \cos\theta \le 1$) so $\theta = \dfrac{2\pi}{3} + 2k\pi$ or $\theta = -\dfrac{2\pi}{3} + 2k\pi$ for some $k \in \mathbb{Z}$.

(b) $\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

(iii) (a) $2\sqrt{3}\cos^2\theta = \sin\theta \Rightarrow 2\sqrt{3}(1-\sin^2\theta) - \sin\theta = 0$
$\Rightarrow 2\sqrt{3}\sin^2\theta + \sin\theta - 2\sqrt{3} = 0 \ \Rightarrow (2\sin\theta - \sqrt{3})(\sqrt{3}\sin\theta + 2) = 0$
$\Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$ or $\sin\theta = -\dfrac{2}{\sqrt{3}}$
$\Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$ (as $-1 \le \sin\theta \le 1$) so $\theta = \dfrac{\pi}{3} + 2k\pi$ or $\theta = \dfrac{2\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$.

(b) $\dfrac{\pi}{3}, \dfrac{2\pi}{3}$

(iv) (a) $\tan^2\theta + \cot^2\theta = 2 \Rightarrow \tan^2\theta + \dfrac{1}{\tan^2\theta} - 2 = 0 \Rightarrow \tan^4\theta - 2\tan^2\theta + 1 = 0$
$\Rightarrow (\tan^2\theta - 1)^2 = 0 \Rightarrow \tan^2\theta = 1 \Rightarrow \tan\theta = 1$ or $\tan\theta = -1$ so $\theta = \dfrac{\pi}{4} + k\pi$ or $\theta = -\dfrac{\pi}{4} + k\pi$ for some $k \in \mathbb{Z}$

(b) $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

(v) (a) $\tan^2\theta^\circ - 4\tan\theta^\circ + 1 = 0 \Rightarrow \tan\theta^\circ = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$
$\Rightarrow \theta = 75 + 180k$ or $\theta = 15 + 180k$ or $\theta = 180k - 105$ or $\theta = 180k - 165$ for some $k \in \mathbb{Z}$.

(b) 15, 75, 195, 255.

(vi) (a) $\sin\theta + \dfrac{\sin\theta}{\cos\theta} = 0 \Rightarrow \sin\theta\cos\theta + \sin\theta = 0 \Rightarrow \sin\theta(\cos\theta + 1) = 0$
$\Rightarrow \sin\theta = 0$ or $\cos\theta = -1$ so $\theta = k\pi$ for some $k \in \mathbb{Z}$.

(b) $0, \pi, 2\pi$.

18. We have $\sin\alpha = \dfrac{3}{5}, \cos\alpha = \dfrac{4}{5}, \sin\beta = \dfrac{40}{41}, \cos\beta = \dfrac{9}{41}$ so

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{3}{5}\cdot\frac{9}{41} - \frac{4}{5}\cdot\frac{40}{41} = -\frac{133}{205},$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{4}{5}\cdot\frac{9}{41} - \frac{3}{5}\cdot\frac{40}{41} = -\frac{84}{205}.$$

19. $\sin 105° + \cos 105° = \sin(45° + 60°) + \cos(45° + 60°)$

$= \sin(45°)\cos(60°) + \cos(45°)\sin(60°) + \cos(45°)\cos(60°) - \sin(45°)\sin(60°)$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2} - \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{2}}$

20. $\sin(A+B)\sin(A-B) = (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A - \sin^2 B$ (by exercise 16(iii)).

21. $\cos A\cos(B-A) - \sin A\sin(B-A) = \cos(A + (B-A)) = \cos B$.

22. $\cos[(n+1)A]\cos[(n-1)A] + \sin[(n+1)A]\sin[(n-1)A]$
$= \cos[(n+1)A - (n-1)A] = \cos 2A$.

23. (i) $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{1 + (2\cos^2 A - 1)} = \dfrac{\sin A\cos A}{\cos^2 A} = \tan A$.

(ii) $\dfrac{1-\cos 2A}{1+\cos 2A} = \dfrac{1-(1-2\sin^2 A)}{1+(2\cos^2 A - 1)} = \dfrac{\sin^2 A}{\cos^2 A} = \tan^2 A$.

24. (i) $2\cos(3x)\cos(x) = \cos(3x+x) + \cos(3x-x) = \cos(4x) + \cos(2x)$.

(ii) $2\sin(3x)\cos(5x) = \sin(3x+5x) + \sin(3x-5x) = \sin(8x) - \sin(2x)$.

(iii) $\sin(4x)\sin(x) = \dfrac{1}{2}[\cos(4x-x) - \cos(4x+x)] = \dfrac{1}{2}[\cos(3x) - \cos(5x)]$.

(iv) $\cos(5x)\sin(2x) = \dfrac{1}{2}[\sin(5x+2x) - \sin(5x-2x)] = \dfrac{1}{2}[\sin(7x) - \sin(3x)]$.

25. (i) $\cos(6x) + \cos(4x) = 2\cos\left[\dfrac{1}{2}(6x+4x)\right]\cos\left[\dfrac{1}{2}(6x-4x)\right] = 2\cos(5x)\cos x$.

(ii) $\sin(3x) + \sin(5x) = 2\sin\left[\dfrac{1}{2}(3x+5x)\right]\cos\left[\dfrac{1}{2}(3x-5x)\right] = 2\sin(4x)\cos x$.

(iii) $\sin(x+\alpha) - \sin(x-\alpha) = 2\cos\left[\dfrac{1}{2}(x+\alpha+x-\alpha)\right]\sin\left[\dfrac{1}{2}(x+\alpha-(x-\alpha))\right]$

$= 2\cos x\sin\alpha$.

26. (i) $3\sin x + 4\cos x = R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$
$\Rightarrow R\cos\alpha = 3,\ R\sin\alpha = 4$. So $R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 \Rightarrow R^2 = 5^2 \Leftrightarrow R = 5$. Dividing gives $\tan\alpha = \dfrac{4}{3} \Rightarrow \alpha = \arctan\left(\dfrac{4}{3}\right) = 0.9273$ (in 1st quadrant as $\sin\alpha$ and $\cos\alpha$ are positive).

(ii) $\cos x - 3\sin x = R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$
$\Rightarrow R\cos\alpha = 1, R\sin\alpha = -3$. So $R^2\cos^2\alpha + R^2\sin^2\alpha = 10 \Leftrightarrow R = \sqrt{10}$.
$\tan\alpha = -3 \Rightarrow \alpha = \arctan(-3) + 2\pi = 5.0341$ (in 4th quadrant as $\sin\alpha$ is negative and $\cos\alpha$ is positive).

(iii) $\sin(3x) - \cos(3x) = R\sin(3x + \alpha) = R\sin 3x\cos\alpha + R\cos 3x\sin\alpha$
$\Rightarrow R\cos\alpha = 1, R\sin\alpha = -1$. So $R^2\cos^2\alpha + R^2\sin^2\alpha = 2 \Leftrightarrow R = \sqrt{2}$.
$\tan\alpha = -1 \Rightarrow \alpha = \arctan(-1) + 2\pi = 5.4978$ (in 4th quadrant as $\sin\alpha$ is negative and $\cos\alpha$ is positive).

(iv) $2\sin(\omega x) + 3\cos(\omega x) = R\cos(\omega x + \alpha) = R\cos(\omega x)\cos\alpha - R\sin(\omega x)\sin\alpha$
$\Rightarrow R\cos\alpha = 3, R\sin\alpha = -2$. So $R^2\cos^2\alpha + R^2\sin^2\alpha = 13 \Leftrightarrow R = \sqrt{13}$.
$\tan\alpha = -\dfrac{2}{3} \Rightarrow \alpha = \arctan\left(-\dfrac{2}{3}\right) + 2\pi = 5.6952$ (in 4th quadrant as $\sin\alpha$ is negative and $\cos\alpha$ is positive).

27.
$$f(x) = \cos x - \sin x = A\cos(x + \alpha) = A\cos x\cos\alpha - A\sin x\sin\alpha$$
$$\left.A\cos\alpha = 1, \quad A\sin\alpha = 1\ \right\} \Rightarrow A^2(\cos^2\alpha + \sin^2\alpha) = A^2 = 2 \Rightarrow A = \sqrt{2}.$$
Hence $\cos\alpha = \sin\alpha = 1/\sqrt{2} \Rightarrow \alpha = \pi/4$, so
$$f(x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right).$$
This has minimum value of $-\sqrt{2}$ when $x + \pi/4 = \pi$, i.e. $x = 3\pi/4$.

28.
$$g(t) = \sqrt{3}\cos t - \sin t = A\cos(t + \alpha) = A\cos t\cos\alpha - A\sin t\sin\alpha$$
$$\left.A\cos\alpha = \sqrt{3}, \quad A\sin\alpha = 1\ \right\} \Rightarrow A^2 = 3 + 1 = 4 \Rightarrow A = 2.$$
Hence $\cos\alpha = \sqrt{3}/2, \quad \sin\alpha = 1/2 \Rightarrow \alpha = \pi/6$, so
$$g(t) = 2\cos\left(t + \frac{\pi}{6}\right).$$
This has minimum value of -2 when $t + \pi/6 = \pi$, i.e. $t = 5\pi/6$, and maximum value of 2 when $t + \pi/6 = 0$, i.e. $t = -\pi/6$.

29.
$$g(t) = \sqrt{3}\sin 2t - 3\cos 2t = A\sin(2t + \alpha) = A\sin(2t)\cos\alpha + A\cos(2t)\sin\alpha$$
$$\left.A\cos\alpha = \sqrt{3}, \quad A\sin\alpha = -3\ \right\} \Rightarrow A^2 = 3 + 9 = 12 \Rightarrow A = 2\sqrt{3}.$$
Hence $\cos\alpha = 1/2, \quad \sin\alpha = -\sqrt{3}/2 \Rightarrow \alpha = -\pi/6$ (as $-\pi/2 < \alpha < 0$), so
$$g(t) = 2\sqrt{3}\sin\left(2t - \frac{\pi}{3}\right).$$
This has minimum value of $-2\sqrt{3}$ when $2t - \pi/3 = -\pi/2$, i.e. $t = -\pi/12$, and maximum value of $2\sqrt{3}$ when $2t - \pi/3 = \pi/2$, i.e. $t = 5\pi/12$.

30.

$$\cos(2\theta) - 3\sin\theta + 1 = 1 - 2\sin^2\theta - 3\sin\theta + 1 = -(2\sin\theta - 1)(\sin\theta + 2) = 0.$$

Hence $\sin\theta = 1/2$ or $\sin\theta = -2$ (which cannot be true), so $\theta \in \{\pi/6, 5\pi/6\}$ (as $\theta \in [0, 2\pi]$).

31.

$$6\sin^2 x + \cos x = 5 \Leftrightarrow 6 - 6\cos^2 x + \cos x - 5 = 0 \Leftrightarrow 6\cos^2 x - \cos x - 1 = 0$$
$$\Leftrightarrow (3\cos x + 1)(2\cos x - 1) = 0.$$

For $x \in [0, \pi]$, $\cos x = 1/2 \Rightarrow x = \pi/3 = 1.0472$ and $\cos x = -1/3 \Rightarrow x = 1.9106$, so $x \in \{1.0472, 1.9106\}$.

32.

$$\begin{aligned} \sin(A+B+C) &= \sin(A+B)\cos C + \cos(A+B)\sin C \\ &= [\sin A\cos B + \cos A\sin B]\cos C + [\cos A\cos B - \sin A\sin B]\sin C \\ &= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C. \end{aligned}$$

Similarly,

$$\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C.$$

33. (*a*) $\arcsin\left(\sin\frac{\pi}{8}\right) = \frac{\pi}{8}$.

(*b*) $\arcsin\left(\sin\frac{7\pi}{3}\right) = \arcsin\left(\sin\left[\frac{\pi}{3} + 2\pi\right]\right) = \arcsin\left(\sin\frac{\pi}{3}\cos 2\pi + \cos\frac{\pi}{3}\sin 2\pi\right)$
$= \arcsin\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$.

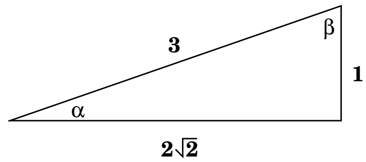
(*c*) The expression $\sin(\arcsin 2)$ makes no sense as the inverse sine is defined only on $[-1, 1]$, i.e. there is no angle whose sine is 2.

34. Let $\arcsin(1/3) = \alpha$ and $\arccos(1/3) = \beta$. From the diagram,

$$\cos\alpha = \frac{2\sqrt{2}}{3}, \qquad \sin\beta = \frac{2\sqrt{2}}{3}$$

so

$$\sin\theta = \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{1}{9} + \frac{8}{9} = 1.$$



35. Let $\arcsin(3/5) = \alpha$ and $\arccos(12/13) = \beta$. From the diagrams,

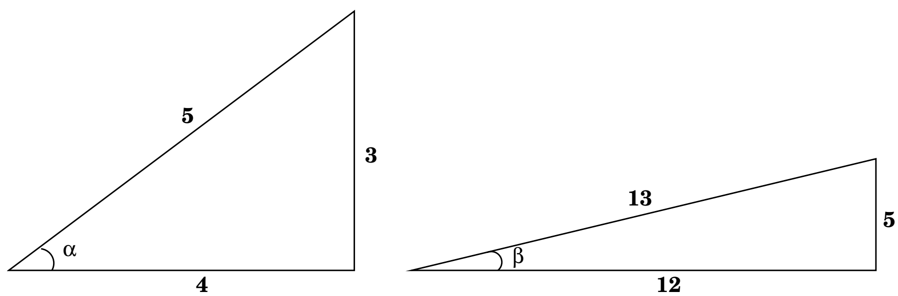
$$\cos\alpha = \frac{4}{5}, \qquad \sin\beta = \frac{5}{13}$$

so

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2.\frac{3}{5}.\frac{4}{5} = \frac{24}{25}, \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{16}{25} - \frac{9}{25} = \frac{7}{25},$$

and

$$\sin\phi = \sin(2\alpha - \beta) = \sin(2\alpha)\cos\beta - \cos(2\alpha)\sin\beta = \frac{24}{25}.\frac{12}{13} - \frac{7}{25}.\frac{5}{13} = \frac{288-35}{325} = \frac{253}{325}.$$



36. If $\arctan(x) = \alpha$ and $\arctan(y) = \beta$ then $x = \tan\alpha$ and $y = \tan\beta$, so

$$\tan(\arctan(x) + \arctan(y)) = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x+y}{1-xy}.$$

Hence

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

provided that $\arctan(x) + \arctan(y) \in Im(\arctan)$, that is, $-\pi/2 < \arctan(x) + \arctan(y) < \pi/2$.

Case 1: $\alpha = \beta = \pi/6 \Rightarrow x = y = 1/\sqrt{3}$. So

$$\arctan\left(\frac{x+y}{1-xy}\right) = \arctan\left(\frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3} = \alpha+\beta.$$

Case 2: $\alpha = \beta = \pi/3 \Rightarrow x = y = \sqrt{3}$. So

$$\arctan\left(\frac{x+y}{1-xy}\right) = \arctan\left(\frac{2\sqrt{3}}{1-3}\right) = \arctan(-\sqrt{3}) = -\frac{\pi}{3} \neq \alpha+\beta.$$

37.

$$\tan\left(\arctan(3x) + \arctan(2x)\right) = \frac{3x + 2x}{1 - 6x^2} = \frac{5x}{1 - 6x^2} = \tan\left(\frac{\pi}{4}\right) = 1.$$

Now

$$\frac{5x}{1 - 6x^2} = 1 \Leftrightarrow 1 - 6x^2 = 5x \Leftrightarrow 6x^2 + 5x - 1 = (6x - 1)(x + 1) = 0 \Leftrightarrow x = -1 \quad \text{or} \quad x = \frac{1}{6}.$$

If $x = -1$ then

$$\arctan(3x) + \arctan(2x) = \arctan(-3) + \arctan(-2) < -\frac{\pi}{2},$$

hence $x = -1$ is NOT a solution. If $x = 1/6$,

$$\arctan(3x) + \arctan(2x) = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) < \frac{\pi}{2},$$

so the only solution is $x = 1/6$.