11 Derivatives

11.1 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+3x+2) = \lim\limits_{h\to 0}\dfrac{1}{h}\left\{(x+h)^2+3(x+h)+2-x^2-3x-2\right\}$

$$= \lim_{h\to 0}\frac{1}{h}\left\{2xh+h^2+3h\right\} = \lim_{h\to 0}\left\{2x+h+3\right\} = 2x+3$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2x-1}\right) = \lim\limits_{h\to 0}\dfrac{1}{h}\left\{\dfrac{1}{2(x+h)-1}-\dfrac{1}{2x-1}\right\}$

$$= \lim_{h\to 0}\frac{1}{h}\left\{\frac{2x-1-2x-2h+1}{[2(x+h)-1][2x-1]}\right\} = \lim_{h\to 0}\frac{-2}{[2(x+h)-1][2x-1]} = -\frac{2}{(2x-1)^2}.$$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x^2+3}\right) = \lim\limits_{h\to 0}\dfrac{1}{h}\left\{\dfrac{1}{(x+h)^2+3}-\dfrac{1}{x^2+3}\right\}$

$$= \lim_{h\to 0}\frac{1}{h}\left\{\frac{x^2+3-x^2-2xh-h^2-3}{[(x+h)^2+3][x^2+3]}\right\}$$

$$= \lim_{h\to 0}\frac{-2x-h}{[(x+h)^2+3][x^2+3]} = -\frac{2x}{(x^2+3)^2}.$$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{1/3}\right) = \lim\limits_{h\to 0}\dfrac{1}{h}\left\{(x+h)^{1/3}-x^{1/3}\right\} = \lim\limits_{h\to 0}\dfrac{1}{h}\dfrac{(x+h)-x}{(x+h)^{2/3}+x^{1/3}(x+h)^{1/3}+x^{2/3}}$

$$= \lim_{h\to 0}\frac{1}{(x+h)^{2/3}+x^{1/3}(x+h)^{1/3}+x^{2/3}} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

11.2 $f'(x)$ is not defined if $x\in\mathbb{Z}$ and $f'(x)=0$ for all other x.

11.3

$$\begin{aligned}
\frac{\mathrm{d}x^n}{\mathrm{d}x} &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}\\
&= \lim_{h\to 0}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}x^{n-k}h^k-x^n}{h}\\
&= \lim_{h\to 0}\frac{\sum\limits_{k=1}^{n}\binom{n}{k}x^{n-k}h^k}{h}\\
&= \lim_{h\to 0}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1}\\
&= \binom{n}{1}x^{n-1}h^{1-1} = nx^{n-1}
\end{aligned}$$

as $\lim\limits_{h\to 0}h^{k-1}=0$ for $k>1$.

11.4 (a)

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{[f(x+h)+c]-[f(x)+c]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x). \end{aligned}$$

(b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x). \end{aligned}$$

(c)

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{f(c[x+h])-f(cx)}{h} \\ &= \lim_{h\to 0}\frac{c[f(cx+ch)-f(cx)]}{ch} = c\cdot\lim_{ch\to 0}\frac{f(cx+ch)-f(cx)}{ch} \\ &= cf'(cx). \end{aligned}$$

11.5 In all of the following, $f'(x) = 3x^2$.

(a)$f'(3) = 3\cdot 3^2 = 27$; $f'(9) = 3\cdot 9^2 = 243$; $f'(25) = 3\cdot 25^2 = 1875$.

(b)$f'((\sqrt{3})^2) = f'(3) = 3\cdot 3^2 = 27$; $f'(3^2) = f'(9) = 3\cdot 9^2 = 243$; $f'(5^2) = f'(25) = 3\cdot 25^2 = 1875$.

(c)$f'(a^2) = 3(a^2)^2 = 3a^4$; $f'(x^2) = 3(x^2)^2 = 3x^4$.

11.6 As in the preceding question, with $f(x) = x^3$ we have $f'(x) = 3x^2$ and so $f'(x^2) = 3x^4$. By contrast, $g(x) = f(x^2) = (x^2)^3 = x^6$ and so $g'(x) = 6x^5$.

11.7 With $g(x) = f(-x)$ we have (use $c = -1$ in Exercise 11.4) $g'(x) = -f'(-x)$. But because f is even $g(x) = f(-x) = f(x)$ and so $g'(x) = f'(x)$. It follows that $f'(x) = -f'(-x)$, which means that f' is odd.

11.8 (a) An odd function f satisfies $f(-x) = -f(x)$ for all x. We have

$$\begin{aligned} f'(-a) &= \lim_{h\to 0}\frac{f(-a+h)-f(-a)}{h} = \lim_{h\to 0}\frac{-f(a-h)+f(a)}{h} = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} \\ &= \lim_{k\to 0}\frac{f(a+k)-f(a)}{k} = f'(a), \end{aligned}$$

where we have set $k = -h$ and used the fact that

$$h\to 0 \iff -h\to 0.$$

So $f'(-a) = f'(a)$ for all a, which shows that f' is even.

(b) With $g(x) = f(-x)$ we have $g'(x) = -f'(-x)$. But because f is odd $g(x) = f(-x) = -f(x)$ and so $g'(x) = -f'(x)$ by Exercise 11.4 (b). It follows that $f'(x) = f'(-x)$, which means that f' is even.

11.9 $g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \dfrac{f(x+c+h) - f(x+c)}{h} = f'(x+c).$

11.10 $f'(x) = h'(x+t)$ using the previous question, and, likewise, $g'(t) = h'(x+t)$. So $f'(x) = h'(x+t)$ while $g'(x) = h'(2x)$.

11.11 $f'(x+p) = \lim_{h \to 0} \dfrac{f(x+p+h) - f(x+p)}{h} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = f'(x).$