University of Strathclyde
Department of Mathematics and Statistics

MM102: Applications of Calculus

Lecture Notes for Week 3

Contents

2 Volumes of Revolution, Arc Length and Surface Area of Revolution

2.1 Volumes of Revolution

First we recall the definition of the Riemann integral. Let $[a, b]$ be a closed, finite interval ($a, b \in \mathbb{R}$, $a < b$) and f a continuous function defined on $[a, b]$.

A partition $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ of the interval $[a, b]$ is a finite collection of points $x_j \in [a, b]$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Denote by $\Delta x_j := x_j - x_{j-1}$ the length of the subinterval $[x_{j-1}, x_j]$. The lower and upper sums of f for P are defined by

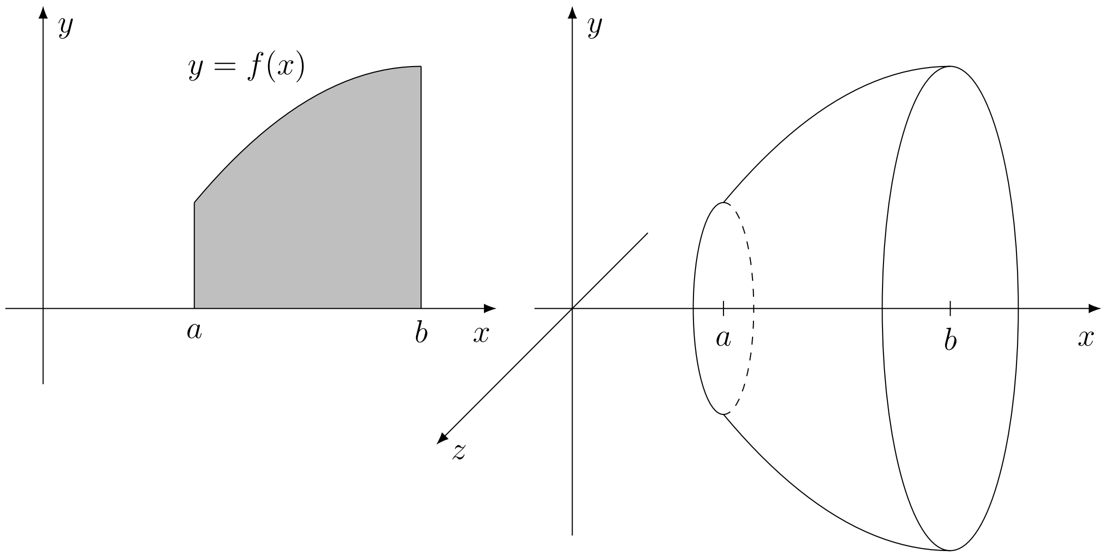
$$L(f, P) = \sum_{j=1}^{n} m_j \Delta x_j, \qquad m_j = \min_{t \in [x_{j-1}, x_j]} f(t) \qquad \text{(lower sum)},$$

$$U(f, P) = \sum_{j=1}^{n} M_j \Delta x_j, \qquad M_j = \max_{t \in [x_{j-1}, x_j]} f(t) \qquad \text{(upper sum)},$$

respectively. The integral $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ is the unique number I for which

$$L(f, P) \le I \le U(f, P) \qquad \text{for all partitions } P.$$

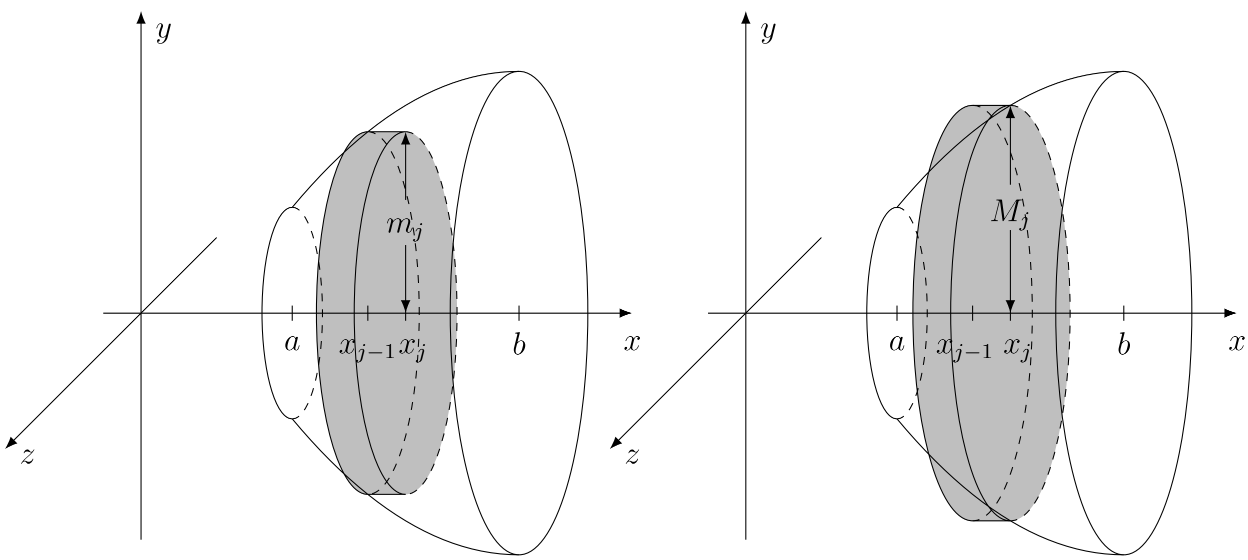
Now let us consider a solid that is obtained from rotation of the region between a curve $y = f(x)$ and the x-axis through $360°$ about the x-axis, where f is a continuous function defined on the interval $[a, b]$ with $f(x) \ge 0$ for $x \in [a, b]$.



Let us now consider a partition P of the interval $[a, b]$. If m_j denotes again the minimum of f on the interval $[x_{j-1}, x_j]$ and $\Delta x_j = x_j - x_{j-1}$, then the disc with radius m_j and sides at x_{j-1} and x_j lies inside the solid. The volume of this disc is

$$\pi m_j^2 \Delta x_j.$$

We can also consider the disc with radius M_j where M_j is the maximum of f on $[x_{j-1}, x_j]$. This disc contains the slice of the solid between $x = x_{j-1}$ and $x = x_j$ and has volume $\pi M_j^2 \Delta x_j$.



If we add the volumes of all these discs, we can enclose the volume V of the solid as follows:

$$\sum_{j=1}^{n} \pi m_j^2 \Delta x_j \leq V \leq \sum_{j=1}^{n} \pi M_j^2 \Delta x_j. \tag{2.1}$$

Consider now the function $g(x) := \pi(f(x))^2$, $x \in [a,b]$ and denote the minima and maxima of g on $[x_{j-1}, x_j]$ by $m_{g,j}$ and $M_{g,j}$, respectively, i.e.

$$m_{g,j} = \min_{t\in[x_{j-1},x_j]} g(t), \qquad M_{g,j} = \max_{t\in[x_{j-1},x_j]} g(t).$$

Since $f(x) \geq 0$ for all $x \in [a,b]$, we have

$$m_{g,j} = \min_{t\in[x_{j-1},x_j]} \pi(f(t))^2 = \pi m_j^2,$$

$$M_{g,j} = \max_{t\in[x_{j-1},x_j]} \pi(f(t))^2 = \pi M_j^2.$$

This shows that the left-hand and right-hand sides in (2.1) are the lower and upper sums for the function g, respectively, and hence

$$L(g,P) \leq V \leq U(g,P).$$

Since this is true for all partitions P, it follows that V is equal to the integral over g, i.e.

$$\boxed{V = \pi \int_a^b \big(f(x)\big)^2 \mathrm{d}x.} \tag{2.2}$$

Roughly speaking, the volume is approximated by the following sum

$$\sum_{j=1}^{n} \pi\big(f(x_j)\big)^2 \Delta x_j,$$

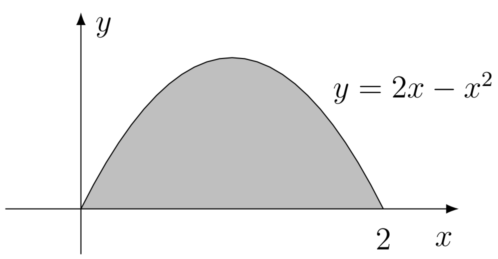
which can be used as an aid to memorise the formula.

Examples 2.1.

(i) Sketch the finite region bounded by $y = 2x - x^2$ and the x-axis. Hence find the volume generated when the region is rotated through $360°$ about the x-axis.

The curve cuts the x-axis when

$$2x - x^2 = 0 \quad\Longleftrightarrow\quad x(2-x) = 0 \quad\Longleftrightarrow\quad x = 0 \quad \text{or} \quad x = 2.$$



With the function $f(x) = 2x - x^2$ defined on the interval $[0, 2]$ we obtain

$$\begin{aligned} V &= \pi \int_0^2 \big(f(x)\big)^2 \mathrm{d}x = \pi \int_0^2 \big(2x - x^2\big)^2 \mathrm{d}x \\ &= \pi \int_0^2 \big(4x^2 - 4x^3 + x^4\big) \mathrm{d}x = \pi \left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right]_0^2 \\ &= \pi\left(\frac{4}{3} \times 8 - 16 + \frac{1}{5} \times 32\right) = \left(10 + \frac{2}{3} - 16 + 6 + \frac{2}{5}\right)\pi \\ &= \frac{10+6}{15}\pi = \frac{16\pi}{15}\,. \end{aligned}$$

(ii) Find the volume of a circular cone of radius r and height h.

Solution in video

(iii) Find the volume of the ellipsoid generated by rotating the ellipse

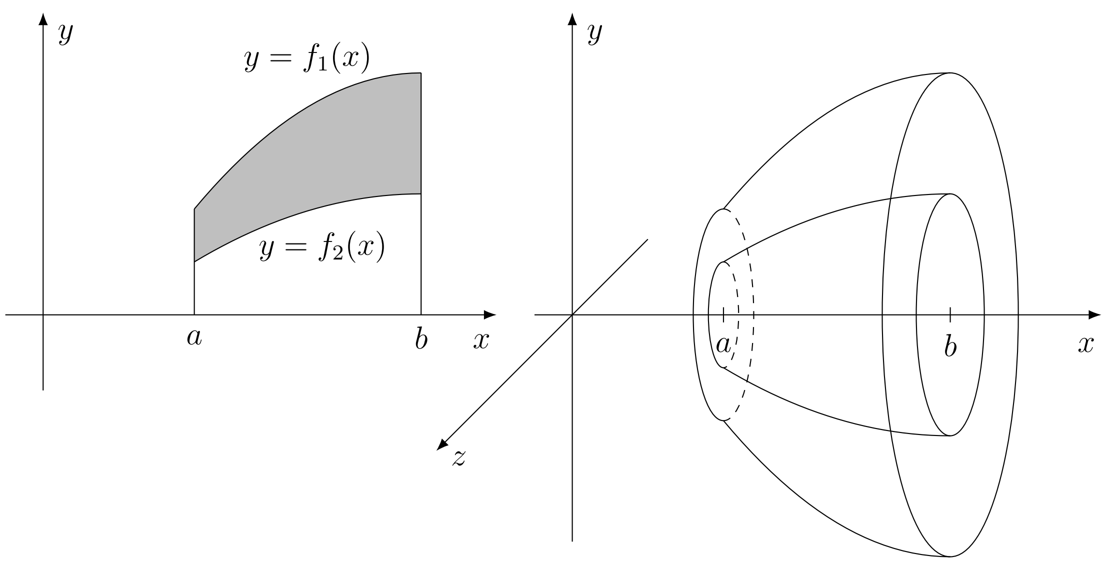
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

through $360°$ about the x-axis, where a, b are positive constants.

Solution in video

Volumes of bodies obtained from rotating regions bounded by two graphs

Let f_1 and f_2 be two functions defined on the interval $[a, b]$ and consider the region bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ and the lines $x = a$ and $x = b$. What is the volume of the solid generated by rotating this region about the x-axis through $360°$?



Let us assume that $f_1(x) \geq f_2(x)$ for $x \in [a,b]$. The volume of the solid generated by the region between the curves $y = f_1(x)$ and $y = f_2(x)$ is the difference of the volumes of the solids generated by the regions below the two curves, i.e.

$$V = \pi \int_a^b \big(f_1(x)\big)^2 \mathrm{d}x - \pi \int_a^b \big(f_2(x)\big)^2 \mathrm{d}x.$$

Hence

$$\boxed{V = \pi \int_a^b \Big(\big(f_1(x)\big)^2 - \big(f_2(x)\big)^2\Big) \mathrm{d}x.} \tag{2.3}$$

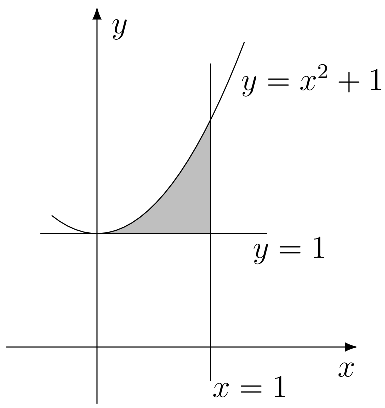
Examples 2.2.

(i) Sketch the finite region between the curve $y = x^2 + 1$ and the lines $x = 1$ and $y = 1$. Hence find the volume of the solid generated when this region is rotated through $360°$ about the x-axis.

The curves $y = x^2 + 1$ and $y = 1$ intersect when $x^2 + 1 = 1$, i.e. when $x = 0$. Hence the finite region is between $x = 0$ and $x = 1$.

Set

$$f_1(x) = x^2 + 1, \qquad f_2(x) = 1.$$



Then $f_1(x) \geq f_2(x)$ for $x \in [0,1]$ and the volume of the solid is equal to

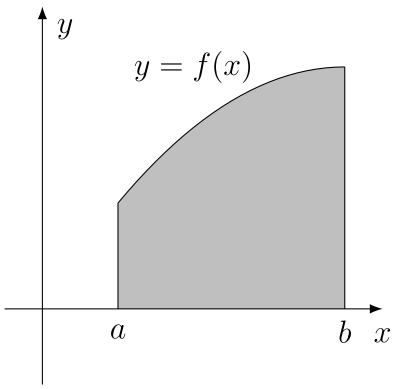
$$\begin{aligned} V &= \pi \int_0^1 \Big(\big(x^2+1\big)^2 - 1^2\Big) \mathrm{d}x \\ &= \pi \int_0^1 \Big(x^4 + 2x^2\Big) \mathrm{d}x \\ &= \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3\right]_0^1 = \pi \left(\frac{1}{5} + \frac{2}{3}\right) = \frac{13\pi}{15}\,. \end{aligned}$$

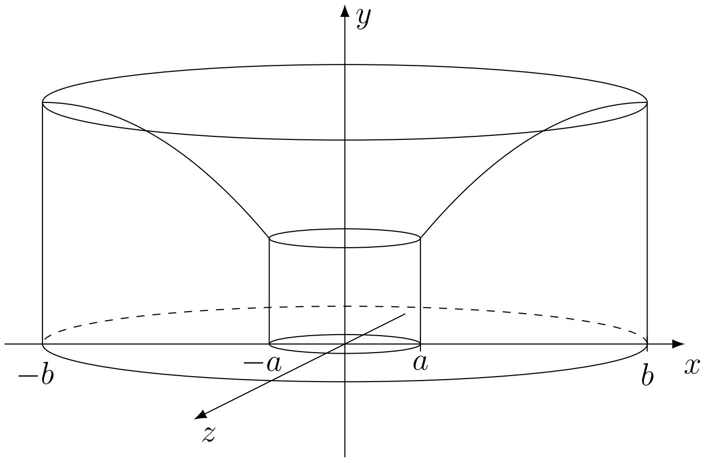
(ii) Sketch the finite region between the curves $y = x^2 + 2$ and $y = 5x - x^2$. Hence find the volume of the solid generated when this region is rotated through $360°$.

Solution in video

Rotation about the y-axis

Next we consider a region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. We are interested in the volume of the solid that is obtained from rotating this region about the y-axis through $360°$ and not about the x-axis as before.





For a partition $P = \{x_0, x_1, \ldots, x_n\}$ we consider the 'shells' obtained by rotating the rectangle with base $[x_{j-1}, x_j]$ and height $f(x_j)$. The volume of such a shell is

$$\begin{aligned}\pi x_j^2 f(x_j) - \pi x_{j-1}^2 f(x_j) &= \pi f(x_j)(x_j + x_{j-1})(x_j - x_{j-1}) \\ &= 2\pi f(x_j)\frac{x_j + x_{j-1}}{2}\,\Delta x_j \approx 2\pi f(x_j)x_j\Delta x_j.\end{aligned}$$

If we take the sum of these volumes we obtain an approximation to the volume we want to calculate:

$$V \approx \sum_{j=1}^{n} 2\pi f(x_j)x_j\Delta x_j.$$

One can prove that when the partition becomes finer, this sum converges to the integral

$$\int_a^b 2\pi x f(x)\mathrm{d}x.$$

Hence the volume of the solid that is obtained from rotation about the y-axis is equal to

$$\boxed{V = 2\pi \int_a^b x f(x)\mathrm{d}x.}$$

Example 2.3.
Find the volume of the body that is obtained when the region bounded by $y = x^2$, $y = 0$ and $x = 1$ is rotated through $360°$ about the y-axis.

We set

$$f(x) = x^2, \qquad x \in [0, 1].$$

Then

$$V = 2\pi \int_0^1 x f(x) \mathrm{d}x = 2\pi \int_0^1 x^3 \, \mathrm{d}x = 2\pi \left[\frac{1}{4}x^4\right]_0^1 = \frac{\pi}{2}.$$

As before we can also consider a region that is bounded by two curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$. If $f_1(x) \geq f_2(x)$ for $x \in [a, b]$, then the volume that is obtained when this region is rotated about the y-axis through $360°$ is equal to

$$\boxed{V = 2\pi \int_a^b x\big(f_1(x) - f_2(x)\big)\mathrm{d}x.}$$

Example 2.4.
Find the volume of the torus generated by rotating the circle

$$(x - 3)^2 + y^2 = 1$$

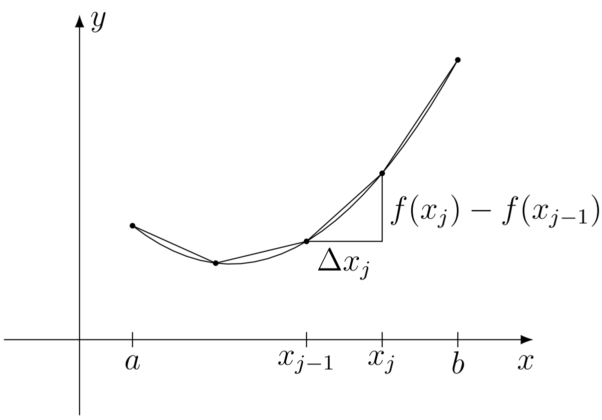
through $360°$ about the y-axis.

Solution in video

2.2 Arc Length of a Curve

Let us consider the graph of a continuously differentiable function f defined on the interval $[a, b]$. In order to calculate the arc length, s, of this curve, we choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$ and approximate the curve by a sequence of straight lines from

$$\big(x_{j-1}, f(x_{j-1})\big) \qquad \text{to} \qquad \big(x_j, f(x_j)\big).$$



The length of such a piece is

$$\begin{aligned}\ell_j &= \sqrt{\big(x_j - x_{j-1}\big)^2 + \big(f(x_j) - f(x_{j-1})\big)^2} \\ &= \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}}\right)^2}\,\Delta x_j \qquad \text{(where } \Delta x_j = x_j - x_{j-1}\text{).}\end{aligned}$$

It follows from the Mean Value Theorem (see Chapter 6) that

$$\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} = f'(\xi_j) \qquad \text{for some } \xi_j \in (x_{j-1}, x_j).$$

Taking the sum over all intervals we obtain an approximation, $s(f,P)$, of the arc length s:

$$s(f,P) := \sum_{j=1}^{n} l_j = \sum_{j=1}^{n} \sqrt{1 + \big(f'(\xi_j)\big)^2}\,\Delta x_j.$$

Define the function $g(x) := \sqrt{1 + (f'(x))^2}$, $x \in [a,b]$, and let $m_{g,j}$ and $M_{g,j}$ be the minima and maxima of g on $[x_{j-1}, x_j]$, respectively. Since

$$m_{g,j} \le \sqrt{1 + \big(f'(\xi_j)\big)^2} \le M_{g,j},$$

we obtain

$$L(g,P) \le s(f,P) \le U(g,P).$$

If the partition gets finer so that $\max\{\Delta x_1, \ldots, \Delta x_n\}$ tends to 0, then $L(g,P)$ and $U(g,P)$ both converge to the integral over g, and hence also $s(f,P)$ converges to the integral over g. This means that the arc length s, which is approximated by $s(f,P)$, is equal to the integral over g, i.e.

$$\boxed{s = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\,\mathrm{d}x.}$$

Examples 2.5.

(i) Find the arc length of the curve

$$y = x^{3/2}, \qquad x \in [0,1].$$

Let $f(x) = x^{3/2}$; its first derivative is

$$f'(x) = \frac{3}{2}x^{1/2}.$$

Hence the arc length is equal to

$$s = \int_0^1 \sqrt{1 + \big(f'(x)\big)^2}\,\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x$$

$$= \left[\frac{4}{9}\cdot\frac{2}{3}\Big(1 + \frac{9}{4}x\Big)^{3/2}\right]_0^1$$

$$= \frac{8}{27}\left(\Big(\frac{13}{4}\Big)^{3/2} - 1\right) = \frac{1}{27}\Big(13\sqrt{13} - 8\Big).$$

(ii) The curve of a hanging chain or cable ('catenary' from Latin: catena = chain) is given by

$$y = \cosh x, \qquad x \in [-1, 1].$$

Find the length of the curve.

Solution in video

2.3 Surface Area of Revolution

In this subsection we consider the area of the surface that is obtained from the graph of a function f that is rotated about the x-axis through $360°$. For a partition $P = \{x_0, x_1, \ldots, x_n\}$ we can consider the area of the surface that is obtained from rotating the straight line from

$$\big(x_{j-1}, f(x_{j-1})\big) \qquad \text{to} \qquad \big(x_j, f(x_j)\big)$$

about the x-axis. Its area equals

$$2\pi\frac{f(x_j) + f(x_{j-1})}{2}\sqrt{\big(x_j - x_{j-1}\big)^2 + \big(f(x_j) - f(x_{j-1}\big)^2}$$

$$= 2\pi\frac{f(x_j) + f(x_{j-1})}{2}\sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}}\right)^2}\,\Delta x_j.$$

If we add all these areas and make the partition finer, we obtain an integral. The area of the surface that is obtained from the graph of f by rotating about the x-axis is equal to

$$\boxed{S = 2\pi\int_a^b f(x)\sqrt{1 + \big(f'(x)\big)^2}\,\mathrm{d}x.}$$

Example 2.6.

Find the area of the surface that is generated by rotating the following curve about the x-axis:

$$y = x^3, \qquad x \in [0, 1].$$

Solution in video