Exercises and outline solutions for MM101 tutorial in week 7

1. Prove, using the limit definition, that if $f(x) = 1/x$ then $f'(a) = -1/a^2$ for all $a \neq 0$.

$$f'(a) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{a+h} - \frac{1}{a}\right) = \lim_{h\to 0} \frac{a-(a+h)}{h(a+h)a} = \lim_{h\to 0} \frac{-1}{(a+h)a} = -\frac{1}{a^2}.$$

2. Find the derivatives of $f(x) = \dfrac{1}{\sin x}$, $g(x) = \dfrac{x^2+1}{x^3-1}$, and $h(x) = \dfrac{\sin^7(x)}{\cos(x)}$.

$$\begin{aligned}
f'(x) &= \frac{-\cos x}{\sin^2 x} = -\csc x \cot x.\\
g'(x) &= \frac{2x(x^3-1) - (x^2+1)3x^2}{(x^3-1)^2} = -\frac{x(x^3+3x+2)}{(x^3-1)^2}.\\
h'(x) &= \frac{7\sin^6(x)\sin'(x) - \sin^7(x)\cos'(x)}{\cos^2(x)} = \frac{7\sin^6(x)\cos^2(x) + \sin^8(x)}{\cos^2(x)}.
\end{aligned}$$

3. Find $f'(x)$ for the functions defined by the following expressions.

   (a) $f(x) = \sin(x + x^2)$.

   (b) $f(x) = \sin^2 x + \sin(x^2)$.

   (c) $f(x) = \sin^2(\sqrt{x}) + \sin(\sqrt{x}^2)$.

   (d) $f(x) = \cos x \, \sin x$.

   (e) $f(x) = \cos(\sin x)$.

   (a) $f'(x) = (1+2x)\cos(x+x^2)$.

   (b) $f'(x) = 2\sin x \cos x + 2x \cos x^2$.

   (c) $f'(x) = \dfrac{\sin\sqrt{x}\,\cos\sqrt{x}}{\sqrt{x}} + \cos x$.

   (d) $f'(x) = -\sin^2 x + \cos^2 x = 2\cos^2 x - 1$.

   (e) $f'(x) = -\sin(\sin x)\cdot\cos x$.

4. Find $f'(x)$ for the functions defined by the following expressions.

   (a) $f(x) = \arctan(\arctan x)$.

   (b) $f(x) = \arccos(1+x^2)$.

   (c) $f(x) = \arcsin\left(\dfrac{1}{\sqrt{1+x^2}}\right)$ for $x \geq 0$.

   (a) $f'(x) = \dfrac{1}{1+\arctan^2 x}\cdot\dfrac{1}{1+x^2}$.

   (b) $f'(x) = \dfrac{-2x}{\sqrt{1-(1+x^2)^2}} = \dfrac{-2x}{\sqrt{-2x^2-x^4}}$. This is not defined for any $x \in \mathbb{R}$. The domain of arccos is $[-1, 1]$.

(c) $f'(x) = \dfrac{1}{\sqrt{1-(\frac{1}{\sqrt{1+x^2}})^2}} \cdot \dfrac{-x}{(1+x^2)^{3/2}} = \dfrac{1}{\sqrt{\frac{x^2}{1+x^2}}} \cdot \dfrac{-x}{(1+x^2)^{3/2}} = \dfrac{-1}{1+x^2}$.

(Note that $f(x) = \arctan\dfrac{1}{x} = \dfrac{\pi}{2} - \arctan x$. This can be seen from a right-angled triangle with sides 1, x, and $\sqrt{1+x^2}$.)