7 First-order ordinary differential equations

The general form for a first-order ordinary differential equation can be written as

$$F\left(x,\, y,\, \frac{dy}{dx}\right) = 0,$$

where F is a particular function.

For all the cases we consider, we will be able to solve this equation for the first derivative, so we can write

$$\frac{dy}{dx} = f(x,\, y), \tag{7.1}$$

and it is equations of this form that we will solve in this Chapter.

We first look at specific forms of the function $f(x,\, y)$ where solutions can more easily be obtained.

7.1 Directly integrable equations

The simplest type of first-order differential equations are those for which the right hand side (RHS) of equation (7.1) is only a function of x, so that

$$\frac{dy}{dx} = f(x). \tag{7.2}$$

For this type of equations we can (hopefully) integrate the function $f(x)$ by x to obtain the general solution,

$$y(x) = \int f(x)\, dx + C,$$

where C is a constant. If we know the value of y at a particular value of x then we are able to determine the value of C. For instance if $F(x)$ is the anti-derivative of $f(x)$, so that $F'(x) = f(x)$, and we know that $y = y_0$ at $x = x_0$, then we see that

$$\begin{aligned} y(x) &= F(x) + C, \\ \Rightarrow \qquad y_0 &= F(x_0) + C, \\ \Rightarrow \qquad C &= y_0 - F(x_0), \end{aligned}$$

which finally gives the solution,

$$y(x) = y_0 + F(x) - F(x_0).$$

Examples 7.1

(a) Find y as a function of x if $\dfrac{dy}{dx} = x^3$.

(b) Solve $\dfrac{dy}{dx} = e^{2x}$ to find the solution that satisfies the condition $y = 1$ at $x = 0$.

✓ Watch Video: Example 7.1

✓ Solve Exercise: Tutorial question 5 (a).

7.2 Separable equations

The examples in the previous section were in fact simple versions of **separable equations**.

Definition 7.1 *The general form of a* ***separable*** *first-order ODE is*

$$\frac{dy}{dx} = g(x)h(y), \tag{7.3}$$

i.e. the right hand side factorises as a product of a function of the independent variable and a function of the dependent variable.

We can solve this type of equation by separating the functions of x and y and then integrating as follows:

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} &= g(x) \qquad \text{assuming that } h(y) \neq 0 \\ \Rightarrow \quad \int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\ \Rightarrow \quad \int \frac{1}{h(y)}\,dy &= \int g(x)\,dx. \end{aligned}$$

Now, provided we can do these two integrals, we have found a general solution. Note that although there are two separate integrations involved there will only be one independent arbitrary constant. This is because the two constants of integration can be combined into a single constant.

If $h(y) = 0$ has a solution $y = a$, where a is a constant, then looking at equation (7.3) it is clear that $y(x) = a$ is actually a solution of the first-order differential equation. It is often the case that a constant solution like this will not be obtainable from the general solution found above. These peculiar solutions are called ***singular solutions*** of the ODE. However, unless the constant a is equal to the initial value of y (i.e. the value stated in the condition $y(x_0) = y_0$) then this will not be a particular solution. In most cases the singular solution will not be of any practical significance in a real (modelled) problem.

Method Separable equations: $\dfrac{dy}{dx} = g(x)h(y)$

- Write ODE in form $\dfrac{1}{h(y)}\dfrac{dy}{dx} = g(x)$.
- Integrate $\displaystyle\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$ to find a general solution.
- Check for singular solutions by finding the constant value solutions of $h(y) = 0$.
- Apply the initial condition $y(x_0) = y_0$, if it is given, to find the integration constant.

Examples 7.2

(a) Solve: $\dfrac{dy}{dx} = x^2 y$ (for $y \geq 0$).

(b) Solve: $x(1+x)\dfrac{dy}{dx} - y = 1$ subject to the condition $y(1) = 1$.

✓ Watch Video: Example 7.2

✓ Solve Exercise: Tutorial question 5 (b)-(d).