Exercises and solutions for MM101 Tutorial in Week 2

1. Given the sets $S_1 = \{x \in \mathbb{N} : 1 \leq x \leq 5\}$, $S_2 = \{y \in \mathbb{Z} : |y| < 4\}$ and $S_3 = \{z \in \mathbb{Z} : z \text{ is even}\}$, identify the elements in the sets

$$\text{(i) } S_1 \cup S_2, \qquad \text{(ii) } S_2 \cap S_3, \qquad \text{(iii) } S_1 \cap S_2 \cap S_3, \qquad \text{(iv) } S_1 - S_3.$$

As $S_1 = \{1, 2, 3, 4, 5\}$ and $S_2 = \{-3, -2, -1, 0, 1, 2, 3\}$ we have

(i) $S_1 \cup S_2 = \{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$.

(ii) $S_2 \cap S_3 = \{-2, 0, 2\}$.

(iii) $S_1 \cap S_2 \cap S_3 = \{2\}$.

(iv) $S_1 - S_3 = \{1, 3, 5\}$.

2. Identify the members of the following sets:

   (a) $S = \{a \in \mathbb{Z} \mid a = (-1)^i \text{ for some } i \in \mathbb{Z}\}$.

   (b) $T = \{b \in \mathbb{Z} \mid b = 1 + (-1)^j \text{ for some } j \in \mathbb{Z}\}$.

   (c) $U = \{c \in \mathbb{Z} \mid 3 \leq c \leq -3\}$.

   (d) $V = \{d \in \mathbb{Z} \mid d > -2 \text{ or } d < -1\}$.

(a) $\{-1, 1\}$. (b) $\{0, 2\}$. (c) $\emptyset$. (d) $\mathbb{Z}$.

3. Sketch the following sets of real numbers on the real number line and express them using interval notation:

$$\text{(i) } \{x \in \mathbb{R} \;:\; -2 \leq x \leq 2\}, \qquad \text{(ii) } \{x \in \mathbb{R} \;:\; |x| \leq 2\},$$

$$\text{(iii) } \{x \in \mathbb{R} \;:\; x \leq -2 \;\vee\; x \geq 2\}, \qquad \text{(iv) } \{x \in \mathbb{R} \;:\; x \leq -2 \;\wedge\; x \geq 2\}.$$

(i) $\{x \in \mathbb{R} \;:\; -2 \leq x \leq 2\} = [-2, 2]$.

(ii) $\{x \in \mathbb{R} \;:\; |x| \leq 2\} = [-2, 2]$.

(iii) $\{x \in \mathbb{R} \;:\; x \leq -2 \;\vee\; x \geq 2\} = (-\infty, -2] \cup [2, \infty)$.

(iv) There is no standard way of writing $\emptyset$ in interval notation (although e.g. (a, a), $(a, a]$, $[a, a)$ (for any $a \in \mathbb{R}$) and (a, b), $(a, b]$, $[a, b)$, $[a, b]$ when $a > b$ all denote empty intervals!).

4. Evaluate the following.

(i) $\lvert\sqrt{2}\rvert$	(ii)$\lvert -\sqrt{3}\rvert$
(iii)$\lvert\sqrt{3}-\sqrt{2}\rvert$	(iv)$\lvert\sqrt{3}-\sqrt{5}\rvert$
(v)$\lvert\sqrt{2}+\sqrt{5}-\sqrt{3}\rvert$	(vi)$\lvert 1+\sqrt{2}-\sqrt{10}\rvert$
(vii)$\lvert\sqrt{5}-\sqrt{2}-\sqrt{3}-\sqrt{7}\rvert$	(viii)$\lvert\sqrt{5}+\sqrt{7}-\sqrt{3}-\sqrt{2}\rvert$
(ix)$\lvert\sqrt{5}+\sqrt{2}-\sqrt{3}-\sqrt{7}\rvert$	(x)$\lvert\sqrt{6}-\sqrt{3}+\sqrt{5}-\sqrt{8}\rvert$
(xi)$\lvert\sqrt{7}+\sqrt{2}-\sqrt{3}-\sqrt{5}\rvert$	(xii)$\big\lvert\lvert\sqrt{2}-\sqrt{3}\rvert-\lvert\sqrt{5}-\sqrt{7}\rvert\big\rvert$

(i) $|\sqrt{2}| = \sqrt{2}$.

(ii) $|-\sqrt{3}| = \sqrt{3}$.

(iii) $\sqrt{3} > \sqrt{2} \quad \text{so} \quad |\sqrt{3}-\sqrt{2}| = \sqrt{3}-\sqrt{2}$.

(iv) $\sqrt{5} > \sqrt{3} \quad \text{so} \quad |\sqrt{3}-\sqrt{5}| = \sqrt{5}-\sqrt{3}$.

(v) $\sqrt{2}+\sqrt{5} > \sqrt{3} \quad \text{so} \quad |\sqrt{2}+\sqrt{5}-\sqrt{3}| = \sqrt{2}+\sqrt{5}-\sqrt{3}$.

(vi) $\sqrt{10} > \sqrt{9} = 3$ and $\sqrt{2} < 2 \quad$ so $\quad \sqrt{10} > 1+\sqrt{2}$. Thus $|1+\sqrt{2}-\sqrt{10}| = \sqrt{10}-\sqrt{2}-1$.

(vii) $|\sqrt{5}-\sqrt{2}-\sqrt{3}-\sqrt{7}| = |\sqrt{5}-(\sqrt{2}+\sqrt{3}+\sqrt{7})|$ and $\sqrt{5} < \sqrt{2}+\sqrt{3}+\sqrt{7}$ so $|\sqrt{5}-\sqrt{2}-\sqrt{3}-\sqrt{7}| = \sqrt{2}+\sqrt{3}+\sqrt{7}-\sqrt{5}$.

(viii) $|\sqrt{5}+\sqrt{7}-\sqrt{3}-\sqrt{2}| = |\sqrt{5}+\sqrt{7}-(\sqrt{3}+\sqrt{2})|$ and $\sqrt{5}+\sqrt{7} > \sqrt{3}+\sqrt{2}$ so $|\sqrt{5}+\sqrt{7}-\sqrt{3}-\sqrt{2}| = \sqrt{5}+\sqrt{7}-\sqrt{3}-\sqrt{2}$.

(ix) $|\sqrt{5}+\sqrt{2}-\sqrt{3}-\sqrt{7}| = |\sqrt{5}+\sqrt{2}-(\sqrt{3}+\sqrt{7})|$. As $\sqrt{7} > \sqrt{5}$ and $\sqrt{3} > \sqrt{2}$, $\sqrt{5}+\sqrt{2} < \sqrt{3}+\sqrt{7} \quad$ so $\quad |\sqrt{5}+\sqrt{2}-\sqrt{3}-\sqrt{7}| = \sqrt{3}+\sqrt{7}-\sqrt{5}-\sqrt{2}$.

(x) $|\sqrt{6}-\sqrt{3}+\sqrt{5}-\sqrt{8}| = |\sqrt{6}+\sqrt{5}-(\sqrt{3}+\sqrt{8})|$. But

$$(\sqrt{6}+\sqrt{5})^2 = 11+2\sqrt{5}\sqrt{6} = 11+2\sqrt{30} \quad \text{and} \quad (\sqrt{3}+\sqrt{8})^2 = 11+2\sqrt{3}\sqrt{8} = 11+2\sqrt{24}$$

so $\sqrt{6}+\sqrt{5} > \sqrt{3}+\sqrt{8}$ and $|\sqrt{6}-\sqrt{3}+\sqrt{5}-\sqrt{8}| = \sqrt{6}-\sqrt{3}+\sqrt{5}-\sqrt{8}$.

(xi) $|\sqrt{7}+\sqrt{2}-\sqrt{3}-\sqrt{5}| = |\sqrt{7}+\sqrt{2}-(\sqrt{3}+\sqrt{5})|$. We see that $(\sqrt{7}+\sqrt{2})^2 = 9+2\sqrt{14}$ and $(\sqrt{3}+\sqrt{5})^2 = 8+2\sqrt{15}$. But

$$9+2\sqrt{14} > 8+2\sqrt{15} \Leftrightarrow 1+2\sqrt{14} > 2\sqrt{15} \Leftrightarrow 57+4\sqrt{14} > 60$$

which is true (as $\sqrt{14} > 3$), so $\sqrt{7}+\sqrt{2} > \sqrt{3}+\sqrt{5}$ and $|\sqrt{7}+\sqrt{2}-\sqrt{3}-\sqrt{5}| = \sqrt{7}+\sqrt{2}-\sqrt{3}-\sqrt{5}$.

(xii) $\big||\sqrt{2}-\sqrt{3}|-|\sqrt{5}-\sqrt{7}|\big| = \big|\sqrt{3}-\sqrt{2}-(\sqrt{7}-\sqrt{5})\big| = \big|\sqrt{3}+\sqrt{5}-(\sqrt{7}+\sqrt{2})\big| = \big|\sqrt{7}+\sqrt{2}-\sqrt{3}-\sqrt{5}\big| = \sqrt{7}+\sqrt{2}-\sqrt{3}+\sqrt{5}$ from part (xi).

5. Prove that the sum and product of any two rational numbers are rational.

We will denote the two rational numbers by

$$a = \frac{p}{q}, \qquad b = \frac{r}{s}$$

where p, q, r and s are integers with $q \neq 0$ and $s \neq 0$. Then their sum is

$$a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$$

which is rational as $ps + qr$ and qs are integers with $qs \neq 0$. Similarly, their product is

$$ab = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

which is rational as pr and qs are integers with $qs \neq 0$.