University of Strathclyde, Department of Mathematics and Statistics

MM102 Applications of Calculus
Exercises for Week 2
Solutions

Q1. 1(a) $\displaystyle\int \cos^2 x \cdot \sin^7 x \,\mathrm{d}x$

Solution:
We have an odd power of $\sin x$. Therefore we can substitute

$$u = \cos x, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \quad\Longrightarrow\quad \mathrm{d}u = -\sin x \,\mathrm{d}x.$$

Hence

$$\int \cos^2 x \cdot \sin^7 x \,\mathrm{d}x = \int \cos^2 x \cdot (\sin^2 x)^3 \sin x \,\mathrm{d}x$$

$$= \int \cos^2 x \cdot (1-\cos^2 x)^3 \sin x \,\mathrm{d}x$$

$$= \int u^2(1-u^2)^3(-\mathrm{d}u) = -\int u^2(1-3u^2+3u^4-u^6)\,\mathrm{d}u$$

$$= \int (u^8-3u^6+3u^4-u^2)\,\mathrm{d}u$$

$$= \frac{1}{9}u^9 - 3\times\frac{1}{7}u^7 + 3\times\frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \boxed{\frac{1}{9}\cos^9 x - \frac{3}{7}\cos^7 x + \frac{3}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$$

1(b) $\displaystyle\int \cos^7 x \,\mathrm{d}x$

Solution:
We have an odd power of $\cos x$. Therefore we can substitute

$$u = \sin x, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \quad\Longrightarrow\quad \mathrm{d}u = \cos x \,\mathrm{d}x.$$

Hence

$$\int \cos^7 x \,\mathrm{d}x = \int (\cos^2 x)^3 \cos x \,\mathrm{d}x$$

$$= \int (1-\sin^2 x)^3 \cos x \,\mathrm{d}x$$

$$= \int (1-u^2)^3 \mathrm{d}u = \int (1-3u^2+3u^4-u^6)\,\mathrm{d}u$$

$$= u - u^3 + 3 \times \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \boxed{\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

1(c) $\displaystyle\int \sin^4 x \cdot \cos^4 x \,\mathrm{d}x$

Solution:
We have an even power of $\cos x$ and use the double angle formula three times:

$$\int \sin^4 x \cdot \cos^4 x \,\mathrm{d}x = \int \left(\sin^2 x\right)^2 \cdot \left(\cos^2 x\right)^2 \mathrm{d}x$$

$$= \int \left(\frac{1}{2}\Big(1 - \cos(2x)\Big)\right)^2 \cdot \left(\frac{1}{2}\Big(1 + \cos(2x)\Big)\right)^2 \mathrm{d}x$$

$$= \int \frac{1}{4}\Big(1 - \cos(2x)\Big)^2 \cdot \frac{1}{4}\Big(1 + \cos(2x)\Big)^2 \mathrm{d}x$$

$$= \frac{1}{16}\int \Big(\Big(1 - \cos(2x)\Big)\Big(1 + \cos(2x)\Big)\Big)^2 \mathrm{d}x$$

$$= \frac{1}{16}\int \Big(1 - \cos^2(2x)\Big)^2 \mathrm{d}x = \frac{1}{16}\int \left(\sin^2(2x)\right)^2 \mathrm{d}x = \frac{1}{16}\int \left(\frac{1}{2}\Big(1 - \cos(4x)\Big)\right)^2 \mathrm{d}x$$

$$= \frac{1}{16}\int \frac{1}{4}\Big(1 - \cos(4x)\Big)^2 \mathrm{d}x = \frac{1}{64}\int \Big(1 - 2\cos(4x) + \cos^2(4x)\Big)\mathrm{d}x$$

$$= \frac{1}{64}\int \left(1 - 2\cos(4x) + \frac{1}{2}\Big(1 + \cos(8x)\Big)\right)\mathrm{d}x = \frac{1}{64}\int \left(\frac{3}{2} - 2\cos(4x) + \frac{1}{2}\cos(8x)\right)\mathrm{d}x$$

$$= \frac{1}{64}\left(\frac{3}{2}x - 2 \times \frac{1}{4}\sin(4x) + \frac{1}{2} \times \frac{1}{8}\sin(8x)\right) + C$$

$$= \boxed{\frac{3}{128}x - \frac{1}{128}\sin(4x) + \frac{1}{1024}\sin(8x) + C}$$

1(d) $\displaystyle\int \cos^4 x \,\mathrm{d}x$

Solution:
We have an even power of $\cos x$ and use the double angle formula twice:

$$\int \cos^4 x \,\mathrm{d}x = \int \left(\cos^2 x\right)^2 \mathrm{d}x = \int \left(\frac{1}{2}\Big(1 + \cos(2x)\Big)\right)^2 \mathrm{d}x$$

$$= \frac{1}{4}\int \Big(1 + 2\cos(2x) + \cos^2(2x)\Big)\mathrm{d}x = \frac{1}{4}\int \left[1 + 2\cos(2x) + \frac{1}{2}\Big(1 + \cos(4x)\Big)\right]\mathrm{d}x$$

$$= \frac{1}{4}\int\left[\frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\right]\mathrm{d}x = \frac{1}{4}\left[\frac{3}{2}x + 2\times\frac{1}{2}\sin(2x) + \frac{1}{2}\times\frac{1}{4}\sin(4x)\right] + C$$

$$= \boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$$

1(e) $\int \cos^3 x \cdot \sin^5 x \,\mathrm{d}x$

Solution:

We have odd powers of $\sin x$ and $\cos x$ and can substitute either $u = \sin x$ or $u = \cos x$.

In the first case we obtain

$$u = \sin x, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \quad \Longrightarrow \quad \mathrm{d}u = \cos x \,\mathrm{d}x.$$

Hence

$$\int \cos^3 x \cdot \sin^5 x \,\mathrm{d}x = \int \cos^2 x \cdot \sin^5 x \cdot \cos x \,\mathrm{d}x$$

$$= \int \left(1 - \sin^2 x\right) \cdot \sin^5 x \cdot \cos x \,\mathrm{d}x = \int (1 - u^2)u^5 \,\mathrm{d}u = \int (u^5 - u^7)\,\mathrm{d}u$$

$$= \frac{1}{6}u^6 - \frac{1}{8}u^8 + C = \boxed{\frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C}$$

Alternative solution: we can use $u = \cos x$:

$$u = \cos x, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \quad \Longrightarrow \quad \mathrm{d}u = -\sin x \,\mathrm{d}x.$$

Hence

$$\int \cos^3 x \cdot \sin^5 x \,\mathrm{d}x = \int \cos^3 x \cdot (\sin^2 x)^2 \cdot \sin x \,\mathrm{d}x$$

$$= \int \cos^3 x \cdot \left(1 - \cos^2 x\right)^2 \cdot \sin x \,\mathrm{d}x$$

$$= \int u^3(1 - u^2)^2(-\mathrm{d}u) = -\int u^3(1 - 2u^2 + u^4)\,\mathrm{d}u$$

$$= -\int \left(u^3 - 2u^5 + u^7\right)\mathrm{d}u = -\left(\frac{1}{4}u^4 - 2\times\frac{1}{6}u^6 + \frac{1}{8}u^8\right) + C$$

$$= \boxed{-\frac{1}{4}\cos^4 x + \frac{1}{3}\cos^6 x - \frac{1}{8}\cos^8 x + C}$$

which differs from the first solution only by a constant.

1(f) $\displaystyle\int_0^{\pi/2} \sin^2 x \cdot \cos^5 x \,\mathrm{d}x$

Solution:

We have an odd power of $\cos x$; hence we can make the following substitution:

$$\begin{aligned} u &= \sin x, && \mathrm{d}u = \cos x \,\mathrm{d}x, \\ x &= 0 && \Longrightarrow \quad u = \sin 0 = 0, \\ x &= \frac{\pi}{2} && \Longrightarrow \quad u = \sin\frac{\pi}{2} = 1, \end{aligned}$$

which yields

$$\begin{aligned} &\int_0^{\pi/2} \sin^2 x \cdot \cos^5 x \,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x \cdot \left(1 - \sin^2 x\right)^2 \cos x \,\mathrm{d}x \\ &= \int_0^1 u^2\left(1-u^2\right)^2 \mathrm{d}u = \int_0^1 \left(u^2 - 2u^4 + u^6\right)\mathrm{d}u \\ &= \left[\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right]_0^1 = \frac{1}{3} - \frac{2}{5} + \frac{1}{7} - 0 = \boxed{\frac{8}{105}} \end{aligned}$$

1(g) $\displaystyle\int \sin^4 x \,\mathrm{d}x$

Solution:

$$\begin{aligned} \int \sin^4 x \,\mathrm{d}x &= \int \left(\sin^2 x\right)^2 \mathrm{d}x = \int \left(\frac{1}{2}\Big(1 - \cos(2x)\Big)\right)^2 \mathrm{d}x \\ &= \frac{1}{4}\int \Big(1 - 2\cos(2x) + \cos^2(2x)\Big)\mathrm{d}x \\ &= \frac{1}{4}\int \Big(1 - 2\cos(2x) + \frac{1}{2}\big(1 + \cos(4x)\big)\Big)\mathrm{d}x \\ &= \frac{1}{4}\int \Big(\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\Big)\mathrm{d}x \\ &= \frac{1}{4}\left(\frac{3}{2}x - 2 \times \frac{1}{2}\sin(2x) + \frac{1}{2} \times \frac{1}{4}\sin(4x)\right) + C \\ &= \boxed{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C} \end{aligned}$$

1(h) $\displaystyle\int_0^1 \sin^2(\pi x)\,\mathrm{d}x$.

Solution:

We use the double angle formula:

$$\begin{aligned}\int_0^1 \sin^2(\pi x)\,\mathrm{d}x &= \int_0^1 \frac{1}{2}\Big(1-\cos(2\pi x)\Big)\mathrm{d}x = \frac{1}{2}\Big[x - \frac{1}{2\pi}\sin(2\pi x)\Big]_0^1 \\ &= \frac{1}{2}\Big(1 - \frac{1}{2\pi}\sin(2\pi) - \Big(0 - \frac{1}{2\pi}\sin 0\Big)\Big) = \boxed{\frac{1}{2}}\end{aligned}$$

1(i) $\displaystyle\int_0^1 \sin^3\Big(\frac{\pi x}{2}\Big)\mathrm{d}x$

Solution:

We have an odd power of $\sin\Big(\dfrac{\pi x}{2}\Big)$. Therefore we can substitute

$$\begin{aligned}&u = \cos\Big(\frac{\pi x}{2}\Big), \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{\pi}{2}\sin\Big(\frac{\pi x}{2}\Big), \quad \Longrightarrow \quad \mathrm{d}u = -\frac{\pi}{2}\sin\Big(\frac{\pi x}{2}\Big)\mathrm{d}x, \\ &x = 0 \quad \Longrightarrow \quad u = \cos 0 = 1, \\ &x = 1 \quad \Longrightarrow \quad u = \cos\Big(\frac{\pi}{2}\Big) = 0.\end{aligned}$$

Hence

$$\begin{aligned}&\int_0^1 \sin^3\Big(\frac{\pi x}{2}\Big)\mathrm{d}x = \int_0^1 \sin^2\Big(\frac{\pi x}{2}\Big)\sin\Big(\frac{\pi x}{2}\Big)\mathrm{d}x \\ &= \int_0^1 \Big(1 - \cos^2\Big(\frac{\pi x}{2}\Big)\Big)\sin\Big(\frac{\pi x}{2}\Big)\mathrm{d}x \\ &= \int_1^0 (1-u^2)\Big(-\frac{2}{\pi}\mathrm{d}u\Big) = \frac{2}{\pi}\int_1^0 (u^2-1)\mathrm{d}u \\ &= \frac{2}{\pi}\Big[\frac{1}{3}u^3 - u\Big]_1^0 = \frac{2}{\pi}\Big(0 - \Big(\frac{1}{3} - 1\Big)\Big) = \boxed{\frac{4}{3\pi}}\end{aligned}$$

1(j) $\displaystyle\int \sin^2 x \cdot \cos^2 x\,\mathrm{d}x$

Solution:

We have an even power of $\sin x$ and $\cos x$ and use the double angle formula twice:

$$\begin{aligned}&\int \sin^2 x \cdot \cos^2 x\,\mathrm{d}x = \int \frac{1}{2}\big(1-\cos(2x)\big)\cdot\frac{1}{2}\big(1+\cos(2x)\big)\mathrm{d}x \\ &= \frac{1}{4}\int \big(1-\cos^2(2x)\big)\mathrm{d}x = \frac{1}{4}\int \sin^2(2x)\mathrm{d}x \\ &= \frac{1}{8}\int \big(1-\cos(4x)\big)\mathrm{d}x = \frac{1}{8}\Big(x - \frac{1}{4}\sin(4x)\Big) + C = \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}\end{aligned}$$

Q2. 2(a) $\displaystyle\int \cos(4x)\cdot\cos x\,\mathrm{d}x$

Solution:
We use the formula

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}\Big[\cos(\alpha-\beta)+\cos(\alpha+\beta)\Big]$$

to obtain

$$\int \cos(4x)\cdot\cos x\,\mathrm{d}x = \int \frac{1}{2}\Big[\cos(3x)+\cos(5x)\Big]\mathrm{d}x$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)\right)+C = \boxed{\frac{1}{6}\sin(3x)+\frac{1}{10}\sin(5x)+C}$$

2(b) $\displaystyle\int \sin(7x)\cdot\cos(2x)\,\mathrm{d}x$

Solution:
We use the formula

$$\sin\alpha\cdot\cos\beta = \frac{1}{2}\Big[\sin(\alpha-\beta)+\sin(\alpha+\beta)\Big]$$

to obtain

$$\int \sin(7x)\cdot\cos(2x)\,\mathrm{d}x = \int \frac{1}{2}\Big[\sin(5x)+\sin(9x)\Big]\mathrm{d}x$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos(5x)-\frac{1}{9}\cos(9x)\right)+C = \boxed{-\frac{1}{10}\cos(5x)-\frac{1}{18}\cos(9x)+C}$$

2(c) $\displaystyle\int_0^{\pi/2} \sin(3x)\cdot\sin(2x)\,\mathrm{d}x$

Solution: We use the formula

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}\Big[\cos(\alpha-\beta)-\cos(\alpha+\beta)\Big]$$

to obtain

$$\int_0^{\pi/2} \sin(3x)\sin(2x)\,\mathrm{d}x = \int_0^{\pi/2}\frac{1}{2}\Big(\cos(3x-2x)-\cos(3x+2x)\Big)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\pi/2}\big(\cos x-\cos(5x)\big)\mathrm{d}x = \frac{1}{2}\Big[\sin x-\frac{1}{5}\sin(5x)\Big]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\sin\frac{\pi}{2}-\frac{1}{5}\sin\frac{5\pi}{2}-\Big(\sin 0-\frac{1}{5}\sin 0\Big)\right) = \frac{1}{2}\Big(1-\frac{1}{5}\Big) = \boxed{\frac{2}{5}}$$

Q3. 3(a) $\displaystyle\int \frac{1}{1+\sin x}\,\mathrm{d}x$

Solution:
With the substitution

$$t = \tan\frac{x}{2}, \qquad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t, \qquad \sin x = \frac{2t}{1+t^2}$$

we obtain

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x = \int \frac{1}{1+\dfrac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{\dfrac{1+t^2+2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{2}{1+2t+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{(1+t)^2}\,\mathrm{d}t = -\frac{2}{1+t} + C = \boxed{-\frac{2}{1+\tan\frac{x}{2}} + C}$$

3(b) $\displaystyle\int \frac{1}{1-3\cos x}\,\mathrm{d}x$

Solution:
With the substitution

$$t = \tan\frac{x}{2}, \qquad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t, \qquad \cos x = \frac{1-t^2}{1+t^2}$$

we obtain

$$\int \frac{1}{1-3\cos x}\,\mathrm{d}x = \int \frac{1}{1-3\dfrac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{1+t^2-3(1-t^2)}\,\mathrm{d}t = \int \frac{2}{4t^2-2}\,\mathrm{d}t$$

$$= \frac{1}{2}\int \frac{1}{t^2-\frac{1}{2}}\,\mathrm{d}t = \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{\sqrt{2}}\right)\left(t-\frac{1}{\sqrt{2}}\right)}\,\mathrm{d}t$$

We need a partial fraction decomposition of the integrand:

$$\frac{1}{\left(t+\frac{1}{\sqrt{2}}\right)\left(t-\frac{1}{\sqrt{2}}\right)} = \frac{A}{t+\frac{1}{\sqrt{2}}} + \frac{B}{t-\frac{1}{\sqrt{2}}}.$$

Multiplying both sides by the common denominator yields

$$1 = A\left(t-\frac{1}{\sqrt{2}}\right) + B\left(t+\frac{1}{\sqrt{2}}\right).$$

Setting $t = -\frac{1}{\sqrt{2}}$ we obtain $1 = -\frac{2}{\sqrt{2}}A$ and hence $A = -\frac{1}{\sqrt{2}}$.

Setting $t = \frac{1}{\sqrt{2}}$ we obtain $1 = \frac{2}{\sqrt{2}} B$ and hence $B = \frac{1}{\sqrt{2}}$.
Hence

$$\int \frac{1}{1-3\cos x}\,\mathrm{d}x = \frac{1}{2}\int \left(-\frac{1}{\sqrt{2}} \cdot \frac{1}{t+\frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{t-\frac{1}{\sqrt{2}}}\right)\mathrm{d}t$$

$$= \frac{1}{2}\left(-\frac{1}{\sqrt{2}} \ln\left|t+\frac{1}{\sqrt{2}}\right| + \frac{1}{\sqrt{2}} \ln\left|t-\frac{1}{\sqrt{2}}\right|\right) + C$$

$$= \boxed{-\frac{1}{2\sqrt{2}} \ln\left|\tan\frac{x}{2} + \frac{1}{\sqrt{2}}\right| + \frac{1}{2\sqrt{2}} \ln\left|\tan\frac{x}{2} - \frac{1}{\sqrt{2}}\right| + C}$$

3(c) $\displaystyle\int \frac{1}{\sin^2 x \cdot (1+\cos x)}\,\mathrm{d}x$

Solution:
With the substitution

$$t = \tan\frac{x}{2}, \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$$

we obtain

$$\int \frac{1}{\sin^2 x \cdot (1+\cos x)}\,\mathrm{d}x = \int \frac{1}{\left(\frac{2t}{1+t^2}\right)^2 \cdot \left(1+\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{\frac{4t^2}{\left(1+t^2\right)^2} \cdot \frac{2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{\left(1+t^2\right)^2}{4t^2}\,\mathrm{d}t = \frac{1}{4}\int \frac{1+2t^2+t^4}{t^2}\,\mathrm{d}t = \frac{1}{4}\int \left(t^{-2}+2+t^2\right)\mathrm{d}t$$

$$= \frac{1}{4}\left(-t^{-1}+2t+\frac{1}{3}t^3\right) + C = \boxed{-\frac{1}{4\tan\frac{x}{2}} + \frac{1}{2}\tan\frac{x}{2} + \frac{1}{12}\tan^3\frac{x}{2} + C}$$

Q4. 4(a) $\displaystyle\int e^{2x}\sin(2x)\,\mathrm{d}x$

Solution:
We denote the integral by I and integrate by parts. Then

$$I = \int e^{2x}\sin(2x)\,\mathrm{d}x$$

$$\begin{bmatrix} u = e^{2x}, & v' = \sin(2x) \\ u' = 2e^{2x}, & v = -\frac{1}{2}\cos(2x) \end{bmatrix}$$

$$= e^{2x}\Big(-\frac{1}{2}\cos(2x)\Big) - \int 2e^{2x}\Big(-\frac{1}{2}\cos(2x)\Big)\mathrm{d}x$$

$$= -\frac{1}{2}e^{2x}\cos(2x) + \int e^{2x}\cos(2x)\,\mathrm{d}x$$

$$\begin{bmatrix} f = e^{2x}, & g' = \cos(2x) \\ f' = 2e^{2x}, & g = \frac{1}{2}\sin(2x) \end{bmatrix}$$

$$= -\frac{1}{2}e^{2x}\cos(2x) + e^{2x}\frac{1}{2}\sin(2x) - \int 2e^{2x}\cdot\frac{1}{2}\sin(2x)\,\mathrm{d}x$$

$$= -\frac{1}{2}e^{2x}\cos(2x) + \frac{1}{2}e^{2x}\sin(2x) - \underbrace{\int e^{2x}\sin(2x)\,\mathrm{d}x}_{=I+C}$$

$$= -\frac{1}{2}e^{2x}\cos(2x) + \frac{1}{2}e^{2x}\sin(2x) - I - C.$$

Solving for I we obtain

$$2I = -\frac{1}{2}e^{2x}\cos(2x) + \frac{1}{2}e^{2x}\sin(2x) - C$$

$$\Longrightarrow \quad I = -\frac{1}{4}e^{2x}\cos(2x) + \frac{1}{4}e^{2x}\sin(2x) - \frac{C}{2}$$

and hence

$$\int e^{2x}\sin(2x)\,\mathrm{d}x = \boxed{\frac{1}{4}e^{2x}\Big(-\cos(2x) + \sin(2x)\Big) + C'}$$

with a new constant C'.

4(b) $\int e^{3x}\sin x\,\mathrm{d}x$

Solution:

We denote the integral by I and integrate by parts. Then

$$I = \int e^{3x}\sin x\,\mathrm{d}x$$

$$\begin{bmatrix} u = e^{3x} & v' = \sin x, \\ u' = 3e^{3x}, & v = -\cos x \end{bmatrix}$$

$$= e^{3x}(-\cos x) - \int 3e^{3x}(-\cos x)\mathrm{d}x = -e^{3x}\cos x + 3\int e^{3x}\cos x\,\mathrm{d}x$$

$$\begin{bmatrix} f = e^{3x} & g' = \cos x, \\ f' = 3e^{3x}, & g = \sin x \end{bmatrix}$$

$$= -e^{3x}\cos x + 3\Big(e^{3x}\sin x - \int 3e^{3x}\sin x\,\mathrm{d}x\Big)$$

$$= -e^{3x}\cos x + 3e^{3x}\sin x - 9\underbrace{\int e^{3x}\sin x\,\mathrm{d}x}_{=I+C}$$

$$= -e^{3x}\cos x + 3e^{3x}\sin x - 9I - 9C.$$

Solving for I we obtain

$$10I = -e^{3x}\cos x + 3e^{3x}\sin x - 9C$$

$$\implies \quad I = \frac{1}{10}\Big(-e^{3x}\cos x + 3e^{3x}\sin x\Big) - \frac{9C}{10}$$

and hence

$$\int e^{3x}\sin x\,\mathrm{d}x = \boxed{\frac{1}{10}e^{3x}\Big(-\cos x + 3\sin x\Big) + C'}$$

with a new constant C'.

4(c) $\displaystyle\int e^{x}\cos x\,\mathrm{d}x$

Solution:
We denote the integral by I and integrate by parts. Then

$$I = \int e^{x}\cos x\,\mathrm{d}x$$

$$\begin{bmatrix} u = e^{x} & v' = \cos x, \\ u' = e^{x}, & v = \sin x \end{bmatrix}$$

$$= e^{x}\sin x - \int e^{x}\sin x\mathrm{d}x$$

$$\begin{bmatrix} f = e^{x} & g' = \sin x, \\ f' = e^{x}, & g = -\cos x \end{bmatrix}$$

$$= e^{x}\sin x - \left(e^{x}(-\cos x) - \int e^{x}(-\cos x)\,\mathrm{d}x\right)$$

$$= e^{x}\sin x + e^{x}\cos x - \underbrace{\int e^{x}\sin x\,\mathrm{d}x}_{=I+C}$$

$$= e^{x}\sin x + e^{x}\cos x - I - C.$$

Solving for I we obtain

$$2I = e^x \sin x + e^x \cos x - C$$

$$\implies \quad I = \frac{1}{2}\Big(e^x \sin x + e^x \cos x\Big) - \frac{C}{2}$$

and hence

$$\int e^x \cos x \, \mathrm{d}x = \boxed{\frac{1}{2}e^x\big(\sin x + \cos x\big) + C'}$$

with a new constant C'.

4(d) $\displaystyle\int e^x \cos(2x)\,\mathrm{d}x$

Solution:
We denote the integral by I and integrate by parts. Then

$$I = \int e^x \cos(2x)\,\mathrm{d}x$$

$$\begin{bmatrix} u = \cos(2x), & v' = e^x, \\ u' = -2\sin(2x), & v = e^x. \end{bmatrix}$$

$$= e^x \cos(2x) - \int e^x\big(-2\sin(2x)\big)\,\mathrm{d}x$$

$$= e^x \cos(2x) + 2\int e^x \sin(2x)\,\mathrm{d}x$$

$$\begin{bmatrix} u = \sin(2x), & v' = e^x \\ u' = 2\cos(2x), & v = e^x \end{bmatrix}$$

$$= e^x \cos(2x) + 2\left(e^x \sin(2x) - \int e^x \cdot 2\cos(2x)\,\mathrm{d}x\right)$$

$$= e^x \cos(2x) + 2e^x \sin(2x) - 4\underbrace{\int e^x \cos(2x)\,\mathrm{d}x}_{=I+C}$$

$$= e^x \cos(2x) + 2e^x \sin(2x) - 4I - 4C.$$

Solving for I we obtain

$$5I = e^x \cos(2x) + 2e^x \sin(2x) - 4C$$

$$\implies \quad I = \frac{1}{5}\Big(e^x \cos(2x) + 2e^x \sin(2x)\Big) - \frac{4C}{5}$$

and hence

$$\int e^x \cos(2x)\,\mathrm{d}x = \boxed{\frac{1}{5}\,e^x\Big(\cos(2x) + 2\sin(2x)\Big) + C'}$$

with a new constant C'.

Q5. Evaluate the following integrals.

5(a) $\displaystyle\int \frac{3x+2}{\sqrt{x-3}}\,\mathrm{d}x$

Solution:
We use the substitution

$$u = \sqrt{x-3} \qquad \Rightarrow \qquad x = u^2 + 3$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 2u \qquad \Rightarrow \qquad \mathrm{d}x = 2u\,\mathrm{d}u$$

Then

$$\int \frac{3x+2}{\sqrt{x-3}}\,\mathrm{d}x = \int \frac{3(u^2+3)+2}{u}\,2u\,\mathrm{d}u$$

$$= 2\int \big(3u^2 + 11\big)\mathrm{d}u = 2u^3 + 22u + C = \boxed{2(x-3)^{3/2} + 22(x-3)^{1/2} + C}$$

Alternative Solution:
We use the substitution

$$u = x - 3, \qquad \mathrm{d}u = \mathrm{d}x.$$

Then

$$\int \frac{3x+2}{\sqrt{x-3}}\,\mathrm{d}x = \int \frac{3(u+3)+2}{\sqrt{u}}\,\mathrm{d}u = \int \frac{3u+11}{u^{1/2}}\,\mathrm{d}u$$

$$= \int \big(3u^{1/2} + 11u^{-1/2}\big)\mathrm{d}u = 3 \times \frac{2}{3}u^{3/2} + 11 \times 2u^{1/2} + C$$

$$= \boxed{2(x-3)^{3/2} + 22(x-3)^{1/2} + C}$$

5(b) $\displaystyle\int \frac{\sqrt{2x-1}}{x}\,\mathrm{d}x$

Solution:
We use the substitution

$$u = \sqrt{2x-1} \qquad \Longrightarrow \qquad x = \frac{u^2+1}{2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = u \qquad \Longrightarrow \qquad \mathrm{d}x = u\,\mathrm{d}u$$

Then

$$\int \frac{\sqrt{2x-1}}{x}\,\mathrm{d}x = \int \frac{u}{\frac{u^2+1}{2}}\,u\,\mathrm{d}u$$

$$= 2\int \frac{u^2}{u^2+1}\,\mathrm{d}u = 2\int \left(1 - \frac{1}{u^2+1}\right)\mathrm{d}u \qquad \text{(long division)}$$

$$= 2(u - \arctan u) + C$$

$$= \boxed{2\left(\sqrt{2x-1} - \arctan\left(\sqrt{2x-1}\,\right)\right) + C}$$

Note that the substitution $u = 2x - 1$ is not useful for this integral.

5(c) $\displaystyle\int (x^2+2)\sqrt{x+1}\,\mathrm{d}x$

Solution:
We use the substitution

$$u = \sqrt{x+1} \qquad \Longrightarrow \qquad x = u^2 - 1$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 2u \qquad \Longrightarrow \qquad \mathrm{d}x = 2u\,\mathrm{d}u$$

Then

$$\int (x^2+2)\sqrt{x+1}\,\mathrm{d}x = \int \left((u^2-1)^2+2\right) u \cdot 2u\,\mathrm{d}u$$

$$= 2\int (u^4 - 2u^2 + 3)\,u^2\,\mathrm{d}u = 2\int (u^6 - 2u^4 + 3u^2)\,\mathrm{d}u$$

$$= \frac{2}{7}u^7 - \frac{4}{5}u^5 + 2u^3 + C$$

$$= \boxed{\frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + 2(x+1)^{3/2} + C}$$

Alternative Solution:
We use the substitution

$$u = x+1, \qquad \mathrm{d}u = \mathrm{d}x.$$

Then

$$\int (x^2+2)\sqrt{x+1}\,\mathrm{d}x = \int \left((u-1)^2+2\right)\sqrt{u}\,\mathrm{d}u$$

$$= \int (u^2 - 2u + 3)\sqrt{u}\,\mathrm{d}u = \int \left(u^{5/2} - 2u^{3/2} + 3u^{1/2}\right)\mathrm{d}u$$

$$= \frac{2}{7}u^{7/2} - 2 \times \frac{2}{5}u^{5/2} + 3 \times \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + 2(x+1)^{3/2} + C}$$

5(d) $\displaystyle\int_1^5 \frac{x-1}{\sqrt{2x-1}}\,\mathrm{d}x$

Solution:
We use the substitution

$$\begin{aligned} u &= \sqrt{2x-1} &&\implies && x = \frac{u^2+1}{2} \\ \frac{\mathrm{d}x}{\mathrm{d}u} &= u &&\implies && \mathrm{d}x = u\,\mathrm{d}u \\ x &= 1 &&\implies && u = 1 \\ x &= 5 &&\implies && u = 3 \end{aligned}$$

Then

$$\begin{aligned} \int_1^5 \frac{x-1}{\sqrt{2x-1}}\,\mathrm{d}x &= \int_1^3 \frac{\frac{u^2+1}{2}-1}{u}\,u\,\mathrm{d}u = \frac{1}{2}\int_1^3 (u^2-1)\,\mathrm{d}u \\ &= \frac{1}{2}\left[\frac{1}{3}u^3 - u\right]_1^3 = \frac{1}{2}\left(9-3-\left(\frac{1}{3}-1\right)\right) = \boxed{\frac{10}{3}} \end{aligned}$$

Alternative Solution:
We use the substitution

$$\begin{aligned} &u = 2x-1, \qquad \frac{1}{2}\mathrm{d}u = \mathrm{d}x, \qquad x = \frac{1}{2}(u+1)\,, \\ &x = 1 \implies u = 1 \\ &x = 5 \implies u = 9. \end{aligned}$$

Then

$$\begin{aligned} \int_1^5 \frac{x-1}{\sqrt{2x-1}}\,\mathrm{d}x &= \int_1^9 \frac{\frac{1}{2}(u+1)-1}{\sqrt{u}} \cdot \frac{1}{2}\,\mathrm{d}u = \frac{1}{4}\int_1^9 \frac{u-1}{u^{1/2}}\,\mathrm{d}u \\ &= \frac{1}{4}\int_1^9 \left(u^{1/2} - u^{-1/2}\right)\mathrm{d}u = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9 \\ &= \frac{1}{4}\left(\frac{2}{3}\times 27 - 2\times 3 - \left(\frac{2}{3}-2\right)\right) = \boxed{\frac{10}{3}} \end{aligned}$$

Q6. Evaluate the following integrals.

6(a) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$

Solution:
We use the substitution

$$x = \sec u, \qquad \frac{\mathrm{d}x}{\mathrm{d}u} = \sec u \cdot \tan u \implies \mathrm{d}x = \sec u \cdot \tan u\,\mathrm{d}u.$$

Using the relation $\sec^2 u = \tan^2 u + 1$ we can rewrite the square root:

$$\sqrt{x^2-1} = \sqrt{\sec^2 u - 1} = \sqrt{\tan^2 u} = \tan u.$$

Hence

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \int \frac{1}{\tan u}\sec u \cdot \tan u\,\mathrm{d}u = \int \sec u\,\mathrm{d}u$$

$$= \ln\left|\sec u + \tan u\right| + C = \ln\left|\sec u + \sqrt{\sec^2 u - 1}\right| + C$$

(We used the relation $\sec^2 u = \tan^2 u + 1$ again.)

$$= \boxed{\ln\left|x + \sqrt{x^2-1}\right| + C}$$

6(b) $\displaystyle\int_0^3 x^2\sqrt{9-x^2}\,\mathrm{d}x$

Solution:
The expression under the square root is of the form $3^2 - x^2$. Hence we use the substitution

$$x = 3\sin u, \qquad \frac{\mathrm{d}x}{\mathrm{d}u} = 3\cos u \quad \Longrightarrow \quad \mathrm{d}x = 3\cos u\,\mathrm{d}u,$$

$$u = \arcsin\left(\frac{x}{3}\right),$$

$$x = 0 \quad \Longrightarrow \quad u = \arcsin 0 = 0,$$

$$x = 3 \quad \Longrightarrow \quad u = \arcsin 1 = \frac{\pi}{2}.$$

The square root can be rewritten as

$$\sqrt{9-x^2} = \sqrt{9-(3\sin u)^2} = \sqrt{9\big(1-\sin^2 u\big)} = \sqrt{9\cos^2 u} = 3\cos u.$$

Hence

$$\int_0^3 x^2\sqrt{9-x^2}\,\mathrm{d}x = \int_0^{\pi/2} (3\sin u)^2 \times 3\cos u \times 3\cos u\,\mathrm{d}u$$

$$= 81\int_0^{\pi/2} \sin^2 u \cdot \cos^2 u\,\mathrm{d}u$$

$$= 81\int_0^{\pi/2} \frac{1}{2}\big(1-\cos(2u)\big)\frac{1}{2}\big(1+\cos(2u)\big)\,\mathrm{d}u$$

$$= \frac{81}{4}\int_0^{\pi/2} \big(1-\cos^2(2u)\big)\,\mathrm{d}u = \frac{81}{4}\int_0^{\pi/2} \sin^2(2u)\,\mathrm{d}u$$

$$= \frac{81}{8}\int_0^{\pi/2} \big(1-\cos(4u)\big)\,\mathrm{d}u = \frac{81}{8}\left[u - \frac{1}{4}\sin(4u)\right]_0^{\pi/2}$$

$$= \frac{81}{8}\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) - \left(0 - \frac{1}{4}\sin 0\right)\right) = \boxed{\frac{81\pi}{16}}$$

6(c) $\displaystyle\int \frac{1}{\sqrt{4x^2+4x+10}}\,\mathrm{d}x$

Solution:
First we complete the square of the expression under the square root:

$$4x^2+4x+10 = (2x+1)^2 - 1 + 10 = (2x+1)^2 + 9 = (2x+1)^2 + 3^2.$$

Hence the given integral is equal to

$$\int \frac{1}{\sqrt{(2x+1)^2+3^2}}\,\mathrm{d}x.$$

We can use the substitution

$$\begin{aligned} 2x+1 &= 3\tan u &&\Longrightarrow && x = \frac{3}{2}\tan u - \frac{1}{2} \\ \frac{\mathrm{d}x}{\mathrm{d}u} &= \frac{3}{2}\sec^2 u &&\Longrightarrow && \mathrm{d}x = \frac{3}{2}\sec^2 u\,\mathrm{d}u. \end{aligned}$$

The square root can be rewritten as

$$\begin{aligned} \sqrt{4x^2+4x+10} &= \sqrt{(2x+1)^2+3^2} = \sqrt{(3\tan u)^2+3^2} \\ &= \sqrt{3^2\left(\tan^2 u+1\right)} \\ &= \sqrt{3^2\sec^2 u} \qquad \text{(using the relation } \tan^2 u + 1 = \sec^2 u) \\ &= 3\sec u. \end{aligned}$$

The given integral is therefore equal to

$$\int \frac{1}{\sqrt{(2x+1)^2+3^2}}\,\mathrm{d}x = \int \frac{1}{3\sec u}\cdot\frac{3}{2}\sec^2 u\,\mathrm{d}u = \frac{1}{2}\int \sec u\,\mathrm{d}u$$

$$= \frac{1}{2}\ln\left|\sec u + \tan u\right| + C$$

$$= \frac{1}{2}\ln\left|\sqrt{\tan^2 u + 1} + \tan u\right| + C$$

(rewrite $\sec u$ in terms of $\tan u$)

$$= \boxed{\frac{1}{2}\ln\left|\sqrt{\left(\frac{2x+1}{3}\right)^2+1} + \frac{2x+1}{3}\right| + C}$$

6(d) $\displaystyle\int \frac{1}{\sqrt{x^2+6x+5}}\,\mathrm{d}x$

Solution:
Complete the square in the expression under the square root:

$$x^2+6x+5 = (x+3)^2 - 3^2 + 5 = (x+3)^2 - 4.$$

Hence we use the following substitution

$$x + 3 = 2\sec u, \qquad \mathrm{d}x = 2\sec u \cdot \tan u \,\mathrm{d}u.$$

The square root can be simplified as follows

$$\begin{aligned}\sqrt{x^2+6x+5} &= \sqrt{(x+3)^2-4} = \sqrt{4\sec^2 u - 4} = \sqrt{4(\sec^2 u - 1)}\\ &= \sqrt{4\tan^2 u} = 2\tan u.\end{aligned}$$

Hence

$$\int \frac{1}{\sqrt{x^2+6x+5}}\,\mathrm{d}x = \int \frac{1}{2\tan u}\, 2\sec u \cdot \tan u \,\mathrm{d}u$$

$$= \int \sec u \,\mathrm{d}u = \ln\left|\sec u + \tan u\right| + C = \ln\left|\sec u + \sqrt{\sec^2 u - 1}\right| + C$$

$$= \boxed{\ln\left|\frac{x+3}{2} + \sqrt{\left(\frac{x+3}{2}\right)^2 - 1}\right| + C}$$

One can simplify this to

$$\ln\left|x+3+\sqrt{x^2+6x+5}\right| + C'$$

with $C' = C + \ln\frac{1}{2}$.

6(e) $\displaystyle\int_2^5 \frac{x+4}{\sqrt{5+4x-x^2}}\,\mathrm{d}x$

Solution:
Complete the square in the expression under the square root:

$$5+4x-x^2 = 5-(x^2-4x) = 5-\left((x-2)^2-2^2\right) = 9-(x-2)^2.$$

Hence we use the following substitution

$$x-2 = 3\sin u, \qquad \mathrm{d}x = 3\cos u\,\mathrm{d}u, \qquad u = \arcsin\frac{x-2}{3}\,.$$

For the limits of the integral we have

$$\begin{aligned} x = 2 \quad &\Longrightarrow \quad u = \arcsin\frac{2-2}{3} = \arcsin 0 = 0,\\ x = 5 \quad &\Longrightarrow \quad u = \arcsin\frac{5-2}{3} = \arcsin 1 = \frac{\pi}{2}\,.\end{aligned}$$

The square root can be simplified as follows

$$\begin{aligned}\sqrt{5+4x-x^2} &= \sqrt{9-(x-2)^2} = \sqrt{9-9\sin^2 u} = \sqrt{9(1-\sin^2 u)}\\ &= \sqrt{9\cos^2 u} = 3\cos u.\end{aligned}$$

Hence

$$\int_2^5 \frac{x+4}{\sqrt{5+4x-x^2}}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{(3\sin u+2)+4}{3\cos u}\,3\cos u\,\mathrm{d}u = \int_0^{\frac{\pi}{2}} \big(3\sin u+6\big)\mathrm{d}u$$

$$= \Big[-3\cos u + 6u\Big]_0^{\frac{\pi}{2}} = -3\cos\frac{\pi}{2} + 6\times\frac{\pi}{2} - \Big(-3\cos 0 + 6\times 0\Big) = \boxed{3\pi+3}$$

6(f) $\displaystyle\int \frac{1}{\sqrt{4x^2-4x}}\,\mathrm{d}x$

Solution:
Complete the square in the expression under the square root:

$$4x^2-4x = (2x-1)^2 - 1.$$

Hence we use the following substitution:

$$2x-1 = \sec u, \qquad 2\mathrm{d}x = \sec u\cdot\tan u\,\mathrm{d}u.$$

Hence

$$\int \frac{1}{\sqrt{4x^2-4x}}\,\mathrm{d}x = \int \frac{1}{\sqrt{\sec^2 u - 1}}\cdot\frac{1}{2}\sec u\cdot\tan u\,\mathrm{d}u$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\tan^2 u}}\sec u\cdot\tan u\,\mathrm{d}u = \frac{1}{2}\int\frac{1}{\tan u}\sec u\cdot\tan u\,\mathrm{d}u$$

$$= \frac{1}{2}\int \sec u\,\mathrm{d}u = \frac{1}{2}\ln|\sec u + \tan u| + C = \frac{1}{2}\ln\Big|\sec u + \sqrt{\sec^2 u - 1}\Big| + C$$

$$= \frac{1}{2}\ln\Big|2x-1+\sqrt{(2x-1)^2-1}\Big| + C = \boxed{\frac{1}{2}\ln\Big|2x-1+\sqrt{4x^2-4x}\Big| + C}$$

6(g) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{3-4x-4x^2}}\,\mathrm{d}x$

Solution:
Complete the square in the expression under the square root:

$$3-4x-4x^2 = 3-(4x^2+4x) = 3-\big((2x+1)^2-1\big) = 4-(2x+1)^2.$$

Hence we use the following substitution:

$$2x+1 = 2\sin u, \qquad 2\mathrm{d}x = 2\cos u\,\mathrm{d}u,$$

$$x = \sin u - \frac{1}{2}, \qquad u = \arcsin\frac{2x+1}{2},$$

$$x = -\frac{1}{2} \qquad \Longrightarrow \qquad u = \arcsin 0 = 0,$$

$$x = \frac{1}{2} \qquad \Longrightarrow \qquad u = \arcsin 1 = \frac{\pi}{2}.$$

The square root simplifies as follows:

$$\begin{aligned}\sqrt{3-4x-4x^2} &= \sqrt{4-(2x+1)^2} = \sqrt{4-4\sin^2 u}\\ &= \sqrt{4(1-\sin^2 u)} = \sqrt{4\cos^2 u} = 2\cos u.\end{aligned}$$

Hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{3-4x-4x^2}}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\left(\sin u - \frac{1}{2}\right)^2}{2\cos u}\,\cos u\,\mathrm{d}u$$

$$= \frac{1}{2}\int_0^{\pi/2}\Big(\sin^2 u - \sin u + \frac{1}{4}\Big)\mathrm{d}u = \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{2}\Big(1-\cos(2u)\Big) - \sin u + \frac{1}{4}\right)\mathrm{d}u$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(-\frac{1}{2}\cos(2u) - \sin u + \frac{3}{4}\right)\mathrm{d}u = \frac{1}{2}\left[-\frac{1}{4}\sin(2u) + \cos u + \frac{3}{4}u\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(-\frac{1}{4}\sin\pi + \cos\frac{\pi}{2} + \frac{3}{4}\cdot\frac{\pi}{2} - \Big(-\frac{1}{4}\sin 0 + \cos 0 + 0\Big)\right)$$

$$= \frac{1}{2}\Big(\frac{3\pi}{8} - 1\Big) = \boxed{\frac{3\pi - 8}{16}}$$

6(h) $\displaystyle\int x^3\sqrt{1-x^2}\,\mathrm{d}x$

Solution:
We use the substitution

$$x = \sin u, \qquad \mathrm{d}x = \cos u\,\mathrm{d}u.$$

Hence

$$\int x^3\sqrt{1-x^2}\,\mathrm{d}x = \int \sin^3 u\cdot\sqrt{1-\sin^2 u}\cdot\cos u\,\mathrm{d}u$$

$$= \int \sin^3 u\cdot\cos^2 u\,\mathrm{d}u = \int (1-\cos^2 u)\cos^2 u\cdot\sin u\,\mathrm{d}u$$

$$\left[\quad t = \cos u, \qquad \mathrm{d}t = -\sin u\,\mathrm{d}u\right]$$

$$= \int (1-t^2)t^2(-1)\mathrm{d}t = \int (t^4 - t^2)\mathrm{d}t = \frac{1}{5}t^5 - \frac{1}{3}t^3 + C$$

$$= \frac{1}{5}\cos^5 u - \frac{1}{3}\cos^3 u + C = \frac{1}{5}\left(\sqrt{1-\sin^2 u}\right)^5 - \frac{1}{3}\left(\sqrt{1-\sin^2 u}\right)^3 + C$$

$$= \boxed{\frac{1}{5}\big(1-x^2\big)^{5/2} - \frac{1}{3}\big(1-x^2\big)^{3/2} + C}$$

6(i) $\displaystyle\int_1^3 \frac{x}{\sqrt{x^2-2x+5}}\,\mathrm{d}x$

Solution:
Complete the square in the expression under the square root:

$$x^2-2x+5=(x-1)^2-1+5=(x-1)^2+4.$$

Hence we use the following substitution:

$$\begin{aligned}
&x-1=2\tan u, && \mathrm{d}x=2\sec^2 u\,\mathrm{d}u,\\
&x=2\tan u+1, && u=\arctan\Big(\frac{x-1}{2}\Big),\\
&x=1 \implies && u=\arctan 0=0,\\
&x=3 \implies && u=\arctan 1=\frac{\pi}{4}\,.
\end{aligned}$$

The square root simplifies as follows:

$$\sqrt{(x-1)^2+4}=\sqrt{(2\tan u)^2+4}=\sqrt{4(\tan^2 u+1)}=\sqrt{4\sec^2 u}=2\sec u.$$

Hence

$$\int_1^3 \frac{x}{\sqrt{x^2-2x+5}}\,\mathrm{d}x=\int_0^{\pi/4}\frac{2\tan u+1}{2\sec u}\,2\sec^2 u\,\mathrm{d}u$$

$$=\int_0^{\pi/4}\big(2\tan u\cdot\sec u+\sec u\big)\mathrm{d}u=\Big[2\sec u+\ln\big|\sec u+\tan u\big|\Big]_0^{\pi/4}$$

$$=\frac{2}{\cos\frac{\pi}{4}}+\ln\left|\frac{1}{\cos\frac{\pi}{4}}+\tan\frac{\pi}{4}\right|-\left(\frac{2}{\cos 0}+\ln\left|\frac{1}{\cos 0}+\tan 0\right|\right)$$

$$=2\sqrt{2}+\ln\big(\sqrt{2}+1\big)-(2+\ln 1)=\boxed{2\sqrt{2}+\ln\big(\sqrt{2}+1\big)-2}$$

6(j) $\displaystyle\int_0^2 x^2\sqrt{4-x^2}\,\mathrm{d}x$

Solution:
We use the substitution

$$\begin{aligned}
&x=2\sin u, && \mathrm{d}x=2\cos u\,\mathrm{d}u,\\
&u=\arcsin\Big(\frac{x}{2}\Big), &&\\
&x=0 \implies && u=\arcsin 0=0,\\
&x=2 \implies && u=\arcsin 1=\frac{\pi}{2}\,.
\end{aligned}$$

The square root simplifies as follows:

$$\sqrt{4-x^2} = \sqrt{4-(2\sin u)^2} = \sqrt{4(1-\sin^2 u)} = \sqrt{4\cos^2 u} = 2\cos u.$$

Hence

$$\int_0^2 x^2\sqrt{4-x^2}\,\mathrm{d}x = \int_0^{\pi/2} (2\sin u)^2 \cdot 2\cos u \cdot 2\cos u\,\mathrm{d}u$$

$$= 16\int_0^{\pi/2} \sin^2 u \cdot \cos^2 u\,\mathrm{d}u = 16\int_0^{\pi/2} \frac{1}{2}\big(1-\cos(2u)\big)\frac{1}{2}\big(1+\cos(2u)\big)\,\mathrm{d}u$$

$$= 4\int_0^{\pi/2} \big(1-\cos^2(2u)\big)\,\mathrm{d}u = 4\int_0^{\pi/2} \sin^2(2u)\,\mathrm{d}u$$

$$= 2\int_0^{\pi/2} \big(1-\cos(4u)\big)\,\mathrm{d}u = 2\left[u - \frac{1}{4}\sin(4u)\right]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) - \left(0 - \frac{1}{4}\sin 0\right)\right) = \boxed{\pi}$$