# UNIVERSITY OF STRATHCLYDE
# DEPARTMENT OF MATHEMATICS AND STATISTICS

**Geometry and Linear Algebra**
**Chapter 3: 2D Geometry and Transformations**

In this chapter we focus on geometric transformations: how to map one shape into another. In particular, we will look at **affine** and **linear transformations**. We will see that these can be used to preserve certain geometrical features and that if we work with Cartesian coordinates then they can be represented using matrix algebra. We will finish by cataloguing these transformations.

When we talk about transformations we are concerned with one thing turning into another. In mathematics, a transformation is a rule (usually a function) for turning one mathematical set into another. Geometrically we can describe transformations as rules for turning one shape into another, such as "rotation", "reflection" or "stretch". Using our algebraic representation of geometry we can look for functions that can perform these operations. In particular, in two dimensional geometry a transformation can be represented by a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that takes points $(x, y)$ and maps them to other points such that $(u, v) = f(x, y)$. Note that we can also define functions $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ such that $u = f_1(x, y)$ and $v = f_2(x, y)$.

## 3.1 Affine Transformations

The set of geometric transformations is far too big for us to consider as a whole. What is more interesting to a mathematician is transformations that preserve particular properties of the space we are working in. For example, we may be interested in transformations that preserve shapes, or angles, or distances. In this section we will restrict ourselves to the set of **affine transformations**. We will see that this set of transformations can be represented very neatly using matrices and vectors.

**Definition 3.1.1** *An affine transformation in Euclidean space is a transformation that preserves* **collinearity** *and* **ratios of distances between collinear points**.

In other words if $A$, $B$ and $C$ are points on a straight line and are mapped by an affine transformation onto $P$, $Q$ and $R$ then $P$, $Q$ and $R$ lie on a straight line and $d_{AB} : d_{BC} = d_{PQ} : d_{QR}$.

**Example 3.1.1**

- Figure **??** illustrates the effect of three transformations. The collinear points $A$, $B$, $C$ are mapped to $P$, $Q$, $R$.

  Clearly transformation (i) is not affine as $P$, $Q$ and $R$ are not collinear.

  Transformation (ii) is not affine. Although $P$, $Q$ and $R$ are collinear, $d_{AB} < d_{BC}$ but $d_{PQ} > d_{QR}$ so distance ratios are not preserved.

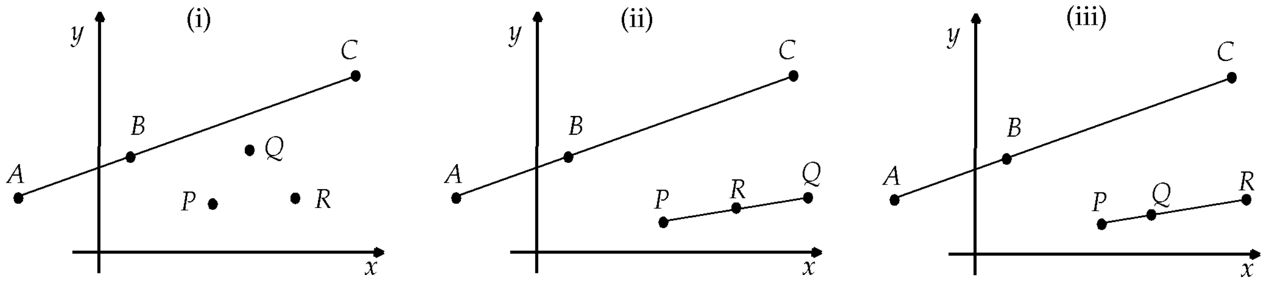



Figure 3.1: Transformations of $\mathbb{R}^2$.

> Transformation (iii) appears to preserve straight lines and relative distances, but we need to know the effect of the transformation on the whole of $\mathbb{R}^2$ before we can determine whether or not it is affine.

Examples of affine transformations of Euclidean space are **translations**, **rotations**, **reflections** and **stretches**. An affine transformation can be thought of as a way of viewing an object from a different perspective. Alternatively, we can think of it in terms of transformations of the coordinate axes. An affine transformation moves the origin and replaces the $x$- and $y$-axes with another pair of straight lines, as illustrated in Figure **??**

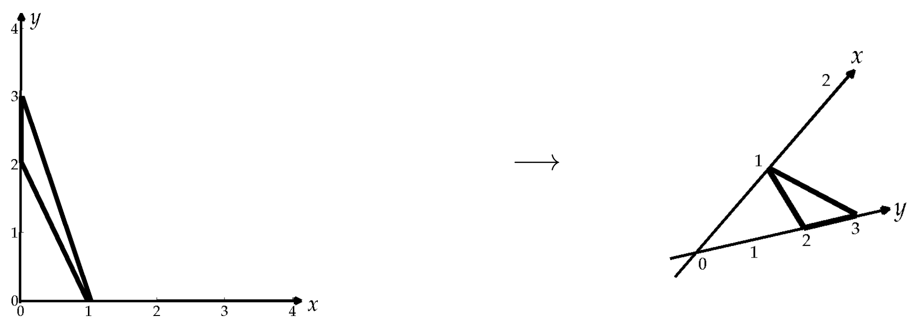
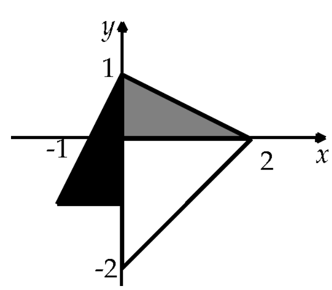



Figure 3.2: Moving the axes.

<u>**Example 3.1.2**</u>

- Look at the three triangles in the figure below. We can describe affine transformations to move one triangle onto another. For example, we can move the grey triangle onto the black triangle by rotating it around the origin anticlockwise through a right angle and translating down by one unit. To move the grey triangle onto the white triangle we reflect it through the $x$-axis and then stretch it vertically by a factor of 2.



In order to work systematically with affine transformations we will describe them algebraically. To do this we exploit the following theorem.

**Theorem 3.1.2** *The map $(u, v) = f(x, y)$ is an affine transformation if and only if there are scalars $a$, $b$, $c$, $d$, $p$ and $q$ such that $u = ax + by + p$ and $v = cx + dy + q$ for all $(x, y) \in \mathbb{R}^2$.*

Before we prove this theorem we make a couple of observations. First, note that the maps $u = f_1(x, y)$ and $v = f_2(x, y)$ involve straight lines. An affine transformation maps straight lines onto straight lines. Next, note that if $\mathbf{x}$ and $\mathbf{u}$ are the position vectors of $(x, y)$ and $(u, v)$, respectively, then

$$\mathbf{u} = f(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} + \begin{bmatrix} p \\ q \end{bmatrix}.$$

We can represent our affine transformations using vectors and matrices and then we can manipulate them via matrix algebra. This observation allows us to rewrite Theorem 3.**??** as follows.

**Theorem 3.1.3** *The map $\mathbf{u} = f(\mathbf{x})$ is an affine transformation if and only if there is a matrix $A$ and vector $\mathbf{p}$ such that $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$ for all $\mathbf{x} \in \mathbb{R}^2$.*

**Proof.** We only prove the "if" part. We need to show the effect of $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$ on three collinear points. Suppose, then, that $B$, $C$ and $D$ are collinear and that $d_{BC} : d_{CD} = m : n$. Then if $B$, $C$ and $D$ have position vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, respectively, by the section formula (2.1.4),

$$\mathbf{c} = (1 - \lambda)\mathbf{b} + \lambda\mathbf{d},$$

where $\lambda = m/(m + n)$.

By the distributivity and homogeneity of matrix multiplication (see §1.3),

$$\begin{aligned} f(\mathbf{c}) &= A\mathbf{c} + \mathbf{p} = A[(1 - \lambda)\mathbf{b} + \lambda\mathbf{d}] + \mathbf{p} \\ &= (1 - \lambda)A\mathbf{b} + \lambda A\mathbf{d} + \mathbf{p} \\ &= (1 - \lambda)(A\mathbf{b} + \mathbf{p}) + \lambda(A\mathbf{d} + \mathbf{p}) = (1 - \lambda)f(\mathbf{b}) + \lambda f(\mathbf{d}). \end{aligned}$$

Hence the image of the point $C$ satisfies (2.1.4) with respect to the images of $B$ and $D$ and so collinearity and the distance ratio are preserved. ∎

Thus to understand and catalogue affine transformations we simply need to understand the effect of transformations of the form $A\mathbf{x} + \mathbf{p}$. An affine transformation will affect the whole of the Euclidean plane by reorienting the axes. To view the effects, it is easier to consider the action of an affine transformation on a simple shape from which we can extrapolate to the whole plane.

**Examples 3.1.3**

(i) Suppose $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$ and $g(\mathbf{x}) = B\mathbf{x} + \mathbf{q}$ are affine transformations. Then $f \circ g$ is the affine transformation effected by carrying out $g$ followed by $f$. It can be represented algebraically as follows.

$$(f \circ g)(\mathbf{x}) = f(B\mathbf{x} + \mathbf{q}) = A(B\mathbf{x} + \mathbf{q}) + \mathbf{p} = AB\mathbf{x} + \mathbf{p} + A\mathbf{q} = C\mathbf{x} + \mathbf{r},$$

where $C = AB$ and $\mathbf{r} = \mathbf{p} + A\mathbf{q}$.

(ii) To translate an object 4 units to the right we use the affine transformation

$$r(\mathbf{x}) = I\mathbf{x} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

To undo this transformation we translate 4 units to the left with

$$l(\mathbf{x}) = I\mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

So $l$ inverts $r$, or $r^{-1}(\mathbf{x}) = l(\mathbf{x})$. Observe that

$$(l \circ r)(\mathbf{x}) = l\left(I\mathbf{x} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = l\left(\mathbf{x} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = I\left(\mathbf{x} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) - \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \mathbf{x}.$$

We will investigate how to invert a general affine transformation shortly.

### 3.1.1 Triangle Maps

Given any pair of triangles, we can use affine transformations to map one onto the other. To see this first consider Figure **??**.

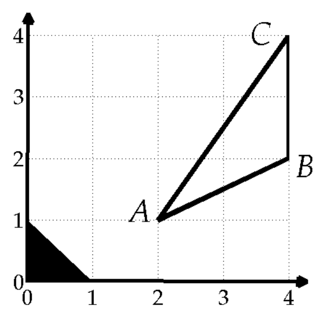



Figure 3.3: One triangle to another.

How do we move the black triangle (which we'll call the **unit triangle** from now on) so that it relocates to $ABC$? As we've already stated, an affine transformation can be thought of as a relocation and realignment of the axes. Two of the unit triangle's sides lie along the axes so let us choose two of the white triangle's sides to be directions for our new axes. Any pair of sides will do. Let's choose $AB$ to be the direction of our new $x$-axis and $AC$ to be the direction of the new $y$-axis. Then the sides of the original triangle that lie along the axes will now lie along the new axes and our triangle should be in place.

In order to accomplish our transformation, then, we need the origin to move to $A$ and the new axes to have directions $\overrightarrow{AB}$ and $\overrightarrow{AC}$ rather than $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We use the affine transformation

$$f(\mathbf{x}) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

To see that this works, first consider what happens to the corners of the unit triangle.

$$\begin{aligned}
f(\mathbf{0}) &= \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \overrightarrow{OA}, \\
f(\mathbf{e}_1) &= \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \overrightarrow{OB}, \\
f(\mathbf{e}_2) &= \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \overrightarrow{OC}.
\end{aligned}$$

So the corners have moved to the correct places. Next, consider the effect of $f$ on points along the sides of the unit triangle. Along the base, the points have position vectors $b\mathbf{e}_1$ $(0 \le b \le 1)$. Now

$$f(b\mathbf{e}_1) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This is the position vector of a point along $AB$. Similarly,

$$f(c\mathbf{e}_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

a point along $AC$. Points on the diagonal of the unit triangle have coordinates $(a, 1-a)$ and

$$f\left(\begin{bmatrix} a \\ 1-a \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ 1-a \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2a + 2(1-a) \\ a + 3(1-a) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + a\begin{bmatrix} 0 \\ -2 \end{bmatrix},$$

confirming that all three sides have ended up in the right places.

We could go further and show that the interiors of the triangles match up, too, but this would be overkill (it's a direct consequence of continuity, anyway). In fact, all we really need to do is show the vertices are mapped correctly and the rest follows by the collinearity properties of affine transformations.

We can generalise the process we've seen here to map the unit triangle onto any other triangle in Euclidean space. We can identify a triangle uniquely by giving the coordinates of one corner and the position vectors from this corner to the other two. If we map the origin to a given corner and $\mathbf{e}_1$ and $\mathbf{e}_2$ to the new edges then we're in business. The properties of an affine transformation ensure that the unit triangle as a whole moves into position.

To map onto the triangle $PQR$ we use the transformation

$$f(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{x} + \overrightarrow{OP},$$

where $\overrightarrow{PQ} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} b \\ d \end{bmatrix}$. For,

$$f(\mathbf{0}) = \overrightarrow{OP}, \;\; f(\mathbf{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix} + \overrightarrow{OP} = \overrightarrow{PQ} + \overrightarrow{OP} = \overrightarrow{OQ}, \;\; f(\mathbf{e}_2) = \overrightarrow{PR} + \overrightarrow{OP} = \overrightarrow{OR}.$$

The map isn't unique as we could choose to map the origin to any one of the vertices of the new triangle.

**Examples 3.1.4**

(i) To transform the unit triangle into $PQR$ with vertices $P(-3,2)$, $Q(4,1)$, $R(2,-5)$ we can use the map

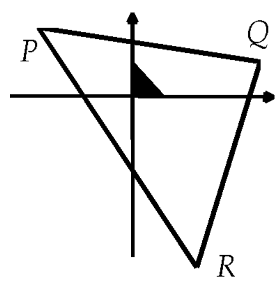
$$f(\mathbf{x}) = \begin{bmatrix} 7 & 5 \\ -1 & -7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -3 \\ 2 \end{bmatrix},$$

since the columns of the matrix are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and the translation is $\overrightarrow{OP}$.

Alternatively, we could choose $\overrightarrow{OQ}$ as the translation and $\overrightarrow{QP}$ and $\overrightarrow{QR}$ as the matrix columns giving

$$g(\mathbf{x}) = \begin{bmatrix} -7 & -2 \\ 1 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

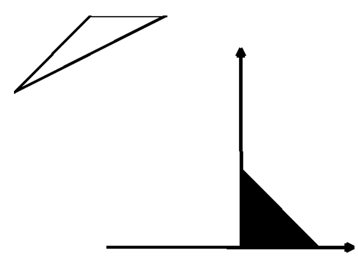
There are several over alternatives as well.




(ii) Find the effect of the transformation

$$h(\mathbf{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -3 \\ 2 \end{bmatrix},$$

on the unit triangle.



$$h(\mathbf{0}) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad h(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad h(\mathbf{e}_2) = \begin{bmatrix} -2 \\ 3 \end{bmatrix},$$

so $OAB$ maps onto the triangle with vertices $(-3,2)$, $(-1,3)$, $(-2,3)$.

Examples 3.1.5

Having dealt with transformations to the unit triangle, how do we deliver on our aim for this subsection, namely, to be able to transform between any pair of triangles? We can proceed as follows. Suppose we want to transform triangle $T_1$ into triangle $T_2$. We can now find maps $f$ and $g$ so that $f(U) = T_1$ and $g(U) = T_2$, where $U$ is the unit triangle. If we can find the map that reverses the action of $f$ we can then transform $T_1$ to $T_2$ via $U$. And how do we reverse a map? We **invert** it. In summary, we can achieve our goal with the map $g \circ f^{-1}$. To invert $f$ note that if $\mathbf{y} = f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$ then

$$\mathbf{x} = f^{-1}(\mathbf{y}) = A^{-1}(\mathbf{y} - \mathbf{p}),$$

so the inverse of an affine transformation is also an affine transformation. The inverse transformation exists so long as $A^{-1}$ exists, and $A$ is invertible so long as its columns are not a multiple of each other.

**Examples 3.1.6**

(i) The inverse map of $f$ in Example 3.1.4(i) is

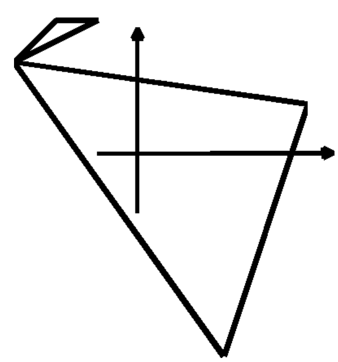
$$\begin{aligned} f^{-1}(\mathbf{x}) &= \begin{bmatrix} 7 & 5 \\ -1 & -7 \end{bmatrix}^{-1} \left(\mathbf{x} - \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) = -\frac{1}{44}\begin{bmatrix} -7 & -5 \\ 1 & 7 \end{bmatrix}\left(\mathbf{x} - \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) \\ &= \frac{1}{44}\begin{bmatrix} 7 & 5 \\ -1 & -7 \end{bmatrix}\mathbf{x} + \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

To confirm, we look at the effect of the map on $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$.

$$\begin{aligned} f^{-1}\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) &= \frac{1}{44}\begin{bmatrix} 7 & 5 \\ -1 & -7 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{44}\begin{bmatrix} -11 \\ -11 \end{bmatrix} + \frac{1}{4} = \mathbf{0}, \\ f^{-1}\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) &= \frac{1}{44}\begin{bmatrix} 7 & 5 \\ -1 & -7 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 33 \\ -11 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e}_1, \\ f^{-1}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right) &= \frac{1}{44}\begin{bmatrix} 7 & 5 \\ -1 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e}_2. \end{aligned}$$

(ii) To map the triangle with vertices $(-3,2)$, $(4,1)$, $(2,-5)$ onto that with vertices $(-3,2)$, $(-1,3)$, $(-2,3)$ we combine the map from the last example with that from Example 3.1.4(ii). The required map $t = h \circ f^{-1}$ is

$$t(\mathbf{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{44}\begin{bmatrix} 7 & 5 \\ -1 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}\right) + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 13 & 3 \\ 6 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{9}{4} \\ \frac{10}{4} \end{bmatrix}.$$



For example, $t\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = \frac{1}{44}\begin{bmatrix} 13 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{9}{4} \\ \frac{10}{4} \end{bmatrix} = \begin{bmatrix} \frac{55}{44} \\ \frac{22}{44} \end{bmatrix} + \begin{bmatrix} -\frac{9}{4} \\ \frac{10}{4} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$

Examples 3.1.7

**Example 3.1.8**

- The map from Example 3.1.5(iii) mapped the unit triangle onto a line. Consider its effect on $P(a, b)$.

$$f(\overrightarrow{OP}) = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = (a + 2b) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

  This point lies on the line with vector form

$$\mathbf{r} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

  which is the same line the unit triangle ended up on. We've shown that all points on the Euclidean plane are mapped onto this line, so any triangle is mapped to a straight line.

  We cannot invert this map because $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ is singular.

  The general points to take from this example are that mappings involving singular matrices are degenerate. They remove a dimension from our triangle. And we can't undo these transformations: a straight line can't be mapped onto a triangle via an affine transformation.

### 3.1.2 Mapping Polygons

Having successfully dealt with triangles a natural next step is to consider what we can do with more complex shapes. Affine transformations are somewhat limited in what they can accomplish. Figure **??** gives an idea of what's possible. Imagine you have pinned a piece of string to a garden trellis.

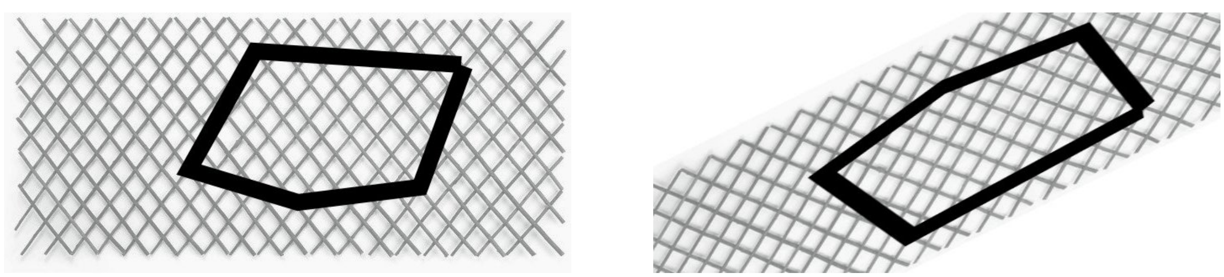


Figure 3.4: Transformations on a trellis

Affine transformations can do little more than you can achieve by squashing and stretching this trellis. One thing you can't do is change the number of sides in your shape (beyond compressing the whole thing onto a straight line) so when we attempt to apply affine transformations to polygons we must bear this restriction in mind. Quadrilaterals can only be mapped to quadrilaterals, pentagons to pentagons, and so on. But we are further restricted. Whilst we could transform between any two triangles, we can only map edgier polygons to subsets of their class. Basically, we can only prescribe the destination of two edges with an affine map. While this is enough for a triangle (given two sides, the third is predestined) it does not suffice for the general case.

However, there's an important class of quadrilaterals that we can work within. We can use affine transformations to map between any two parallelograms. Furthermore, all affine transformations of parallelograms produce parallelograms as a consequence of the following theorem.

**Theorem 3.1.4** *Affine transformations preserve parallel lines.*

**Proof.** If the lines $L_1$ and $L_2$ are parallel then they can be written in vector form as $\mathbf{r}_1 = \mathbf{a}_1 + t\mathbf{b}$ and $\mathbf{r}_2 = \mathbf{a}_2 + t\mathbf{b}$, for some common vector $\mathbf{b}$. Suppose $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$ is an affine transformation. Then,

$$f(\mathbf{r}_1) = A(\mathbf{a}_1 + t\mathbf{b}) + \mathbf{p} = (A\mathbf{a}_1 + \mathbf{p}) + tA\mathbf{b},$$

and

$$f(\mathbf{r}_2) = A(\mathbf{a}_2 + t\mathbf{b}) + \mathbf{p} = (A\mathbf{a}_2 + \mathbf{p}) + tA\mathbf{b}$$

are parallel, too. ∎

As we mentioned in §2.2.1, a parallelogram is uniquely defined by the position of one of its corners and the two edges emerging from it—the same information needed to describe a triangle. So we can map from parallelogram to parallelogram as we did for triangles. We use the **unit square** with vertices at $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ as our reference shape and go from one parallelogram to another via the unit square.

**Examples 3.1.9**

(i) Find an affine transformation to map the unit square onto the rhombus with a corner at $P(3,2)$ connected to two other vertices $Q(2,-1)$ and $R(4,-1)$.

We use the same procedure as in Example 3.1.4(i). The required map is $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$ where $\mathbf{p} = \overrightarrow{OP}$ and the columns of $A$ are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. So

$$f(\mathbf{x}) = \begin{bmatrix} -1 & 1 \\ -3 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

It is easy to see that $f(\mathbf{0}) = \overrightarrow{OP}$, $f(\mathbf{e}_1) = \overrightarrow{OQ}$ and $f(\mathbf{e}_2) = \overrightarrow{OR}$. Finally,

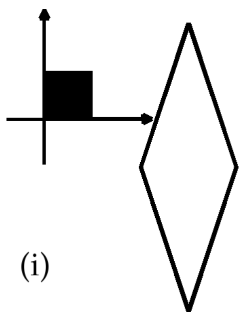
$$f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix},$$

which is indeed the position vector of the fourth corner of the rhombus.

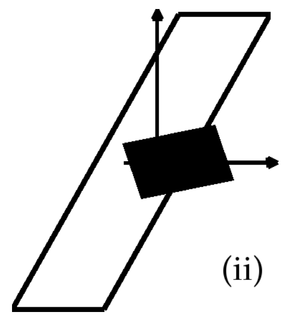
(ii) Sketch the effect of the affine transformation $f(\mathbf{x}) = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ on the parallelogram $PQRS$ with corners $P(-1,-2)$, $Q(4,-1)$, $R(3,2)$, $S(-2,1)$.

Applying $f$ to the corners gives $f\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$, $f\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$,

$f\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$, $f\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -8 \\ -8 \end{bmatrix}$.



(i)



(ii)

**Example 3.1.10**

- Find an affine transformation that takes the parallelogram with corners $(-3,2)$, $(4,1)$, $(2,-5)$, $(-5,-4)$ onto the parallelogram with corners $(-3,2)$, $(-1,3)$, $(-2,3)$, $(-4,2)$.

  Compare this problem with Example 3.1.6(ii). Three of the corners of each parallelogram match those of the triangles involved, so we can try $t$ on the fourth corner.

$$t\left(\begin{bmatrix} -5 \\ -4 \end{bmatrix}\right) = \frac{1}{44}\begin{bmatrix} 13 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} -5 \\ -4 \end{bmatrix} + \begin{bmatrix} -\frac{9}{4} \\ \frac{10}{4} \end{bmatrix} = -\begin{bmatrix} \frac{77}{44} \\ \frac{22}{44} \end{bmatrix} + \begin{bmatrix} -\frac{9}{4} \\ \frac{10}{4} \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

  It works! Beware, though. Given three corners of a parallelogram, there is more than one way to pick a fourth. If we'd picked fourth corners that didn't match, we'd have needed to find a new transformation.

### 3.1.3 Mapping Curves

Finally in this section, let us consider the effects of affine transformations on conic sections. Hopefully by now you can picture, to some extent, what distortions an affine transformation can make. We can squeeze and squash but not split so we shouldn't expect to be able to turn a circle into a hyperbola, for example. But by approaching the problem algebraically we can rapidly establish everything that's possible.

**Example 3.1.11**

- Find the effect of the affine transformation

$$f(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

  on the unit circle.

  A point on the unit circle has coordinates $(x,y) = (\cos\theta, \sin\theta)$ where $0 \leq \theta \leq 2\pi$. The effect of $f$ on this is to map $(x,y)$ to

$$(u,v) = (\cos\theta + 2\sin\theta - 1, \sin\theta - 2\cos\theta + 2).$$

Note that

$$
\begin{aligned}
(u+1)^2+(v-2)^2 &= (\cos\theta+2\sin\theta)^2+(\sin\theta-2\cos\theta)^2 \\
&= (\cos^2\theta+4\cos\theta\sin\theta+4\sin^2\theta)+(\sin^2\theta-4\sin\theta\cos\theta+4\cos^2\theta) \\
&= 5\cos^2\theta+5\sin^2\theta=5,
\end{aligned}
$$

and we conclude that $f$ maps the unit circle onto a circle centred on $(-1,2)$ radius $\sqrt{5}$.

Examples 3.1.12

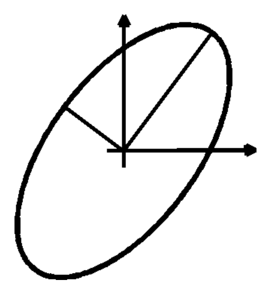
In the last example we used the simplified map $f(\mathbf{x}) = A\mathbf{x}$. If we are only concerned with the shapes we can make with affine transformations, this is a general enough map to work with because all the "$+\mathbf{p}$" does is translate the shape.

If we plug the transformation $(x,y) = (qu+rv, su+tv)$ into (2.2.4) it should be clear that the resulting expression is still that of a conic section. The discriminant can be shown to be $(qt-rs)^2(B^2-4AC)$, so it cannot change sign meaning that, in general, invertible transformations map hyperbolae into hyperbolae and ellipses (including circles) into ellipses.

**Examples 3.1.13**

(i) Find the image of the unit circle under the affine map

$$f(\mathbf{x}) = \frac{1}{5}\begin{bmatrix} 6 & -4 \\ 8 & 3 \end{bmatrix}\mathbf{x}.$$



To write the image algebraically we first invert $f$.

$$f^{-1}(\mathbf{x}) = \frac{1}{10}\begin{bmatrix} 3 & 4 \\ -8 & 6 \end{bmatrix}\mathbf{x},$$

so if $(u,v) = f(x,y)$ then $(x,y) = f^{-1}(u,v) = (0.3u+0.4v, 0.6v-0.8u)$.
Substituting into $x^2+y^2=1$, gives $(0.3u+0.4v)^2+(0.6v-0.8u)^2=1$, or

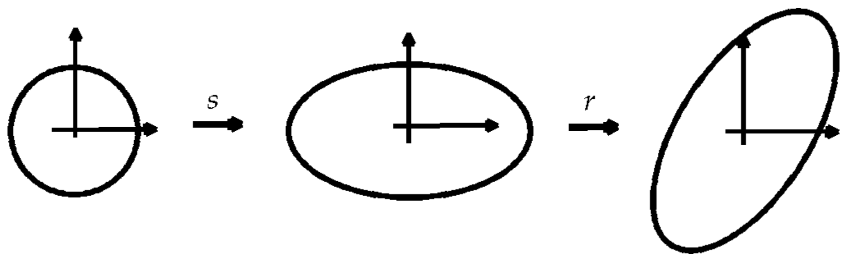
$$0.73u^2-0.72uv+0.52v^2=1.$$

What does this look like? Well, it's not at all obvious from the algebraic expression. By applying $f$ to the parametrised unit circle $(\cos\theta, \sin\theta)$ we can trace the image of $f$. The semi-axes of the transformed ellipse are shown. The effect of $f$ is to stretch the circle parallel to the axes and then to rotate it.

(ii) The transformation $s(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}$ stretches the $x$-axis by a factor of 2 (since $s(x,y) = (2x,y)$).

The transformation $r(\mathbf{x}) = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\mathbf{x}$ rotates the axes (see Example 3.1.4(iii)) anticlockwise through an angle $\theta$ where $\tan\theta = 3/4$.

So applying $r \circ s$ to the unit circle will first stretch it into the ellipse $x^2/4 + y^2 = 1$ before rotating it through an angle $\theta$.




Note that

$$(r \circ s)(\mathbf{x}) = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x} = \frac{1}{5}\begin{bmatrix} 6 & -4 \\ 8 & 3 \end{bmatrix}\mathbf{x},$$

precisely the map from the last example. By decomposing it into its constituent parts, it is much easier to interpret the transformation.

## Exercises: Affine Transformations

1. If $f(\mathbf{x})$ and $g(\mathbf{x})$ are affine maps, show that $h = f + g$ defines an affine map, too.

2. Find affine transformations that map the unit triangle onto triangles with the following vertices.

   (a) $A(2,0)$, $B(2,2)$, $C(4,0)$

   (b) $A(-1,3)$, $B(3,-1)$, $C(2,5)$

   (c) $A(3,1)$, $B(0,0)$, $C(1,3)$

   (d) $A(0,0)$, $B(-1,0)$, $C(0,1)$

3. Find all the affine transformations which map the unit triangle onto itself.

4. Find affine transformations which map $ABC$ onto $PQR$ for the following sets of vertices.

   (a) $A(2,0)$, $B(2,2)$, $C(4,0)$ and $P(0,0)$, $Q(1,0)$, $R(0,1)$

   (b) $A(1,2)$, $B(3,-2)$, $C(-3,1)$ and $P(7,2)$, $Q(9,-2)$, $R(-6,1)$

   (c) $A(1,2)$, $B(3,-2)$, $C(-3,1)$ and $P(0,4)$, $Q(-6,2)$, $R(3,-1)$

   (d) $A(0,0)$, $B(2,0)$, $C(0,2)$ and $P(-1,1)$, $Q(-3,3)$, $R(1,3)$

5. By sketching the quadrilaterals $ABCD$ and $PQRS$, determine whether it is possible to map one onto the other with an affine transformation.

   Where it is possible, find the map in question.

   (a) $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$ and $P(2,1)$, $Q(3,1)$, $R(2,5)$, $S(1,5)$

   (b) $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$ and $P(0,0)$, $Q(2,0)$, $R(2,1)$, $S(-1,1)$

   (c) $A(2,1)$, $B(3,2)$, $C(5,3)$, $D(6,0)$ and $P(0,0)$, $Q(2,1)$, $R(3,3)$, $S(1,2)$

   (d) $A(0,0)$, $B(2,1)$, $C(3,3)$, $D(1,2)$ and $P(0,3)$, $Q(4,0)$, $R(3,0)$, $S(-1,3)$

6. Find the algebraic form of the ellipses (or circles) that result from applying the following affine transformations to the unit circle.

   (a) $\begin{bmatrix} 12 & 5 \\ -5 & 12 \end{bmatrix}\mathbf{x}$ (b) $\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{x}$

7. Find the algebraic form of the hyperbolae that result from applying the following affine transformations to (i) the hyperbola $x^2 - y^2 = 1$ (ii) the rectangular hyperbola $xy = 1$.

   (a) $\begin{bmatrix} 12 & 5 \\ -5 & 12 \end{bmatrix}\mathbf{x}$ (b) $\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$ (d) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{x}$

8. Suppose that $x = au + bv$ and $y = cu + dv$. Given that $y = x^2 + 1$, find the generalised quadratic form relating $u$ and $v$.

   Confirm that this generalised quadratic represents a parabola.

## 3.2 Linear Transformations

By moving and reshaping the axes we can use affine transformations, $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$, to carry out a wide variety of geometric manipulations. We have seen, for example, how we can map between any two triangles. In this section we adopt a different approach. We look for explicit representations of transformations that carry out particular changes to shapes. For example, we will describe transformations that can rotate a shape through any angle, or reflect a shape through any line. The "$+\mathbf{p}$" in our affine transformations is simply a translation and has no effect in shape changing. In this section we will drop the translation and look at a subset of affine transformations known as **linear transformations**. All linear transformations in Euclidean space can be written $f(\mathbf{x}) = A\mathbf{x}$. Thus every matrix can be thought of as a linear transformation and we can use matrix algebra to carry out manipulations of linear transformations.

**Example 3.2.1**

- Suppose $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{x}) = B\mathbf{x}$ are linear transformations. Then we can compose $f$and $g$ by multiplying $A$ and $B$. That is,
$$(f \circ g)(\mathbf{x}) = AB\mathbf{x}.$$
Such manipulations are only possible because of our definition of matrix multiplication. See Example 1.3.9 for more detail.

**Definition 3.2.1** *A transformation $f$ on $\mathbb{R}^2$ is a linear transformation if for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ and all $c$ and $d$ in $\mathbb{R}$,*
$$f(c\mathbf{u} + d\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v}).$$

That the map $A\mathbf{x}$ is a linear transformation is a direct consequence of the homogeneity of matrix multiplication and the distributive law (see §1.3.3).

Examples 3.2.2

**Example 3.2.3**

- Linear transformations are not restricted to Euclidean space. For example, let
$$T(f) = \int_a^b f(x)\,dx.$$
Then $T$ is a linear transformation over a space of functions because
$$\int_a^b (cf(x) + dg(x))dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx.$$
You'll see the idea of linearity developed in future course, but we'll limit ourselves to linearity in two and three dimensional space.

Recall that affine transformations can be viewed as manipulations of the coordinate axes. By restricting ourselves to linear transformations we are simply leaving the origin as a fixed point. In order to carry out the transformations we are interested in, we focus again on the unit triangle. As one vertex of this triangle (the origin) is now pinned in place, we simply need to consider how to move the two sides that emerge from this corner. And these sides are simply $\mathbf{e}_1$ and $\mathbf{e}_2$.

### 3.2.1 Special Transformations

**Rotations**

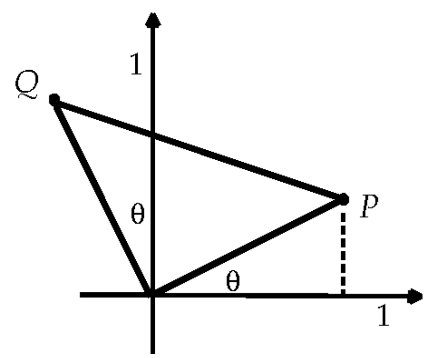



Figure 3.5: Rotation through $\theta$.

The first transformation we'll consider is one we've looked at briefly before, a rotation. Suppose, as in Figure **??**, we rotate the unit triangle anticlockwise around the origin through the angle $\theta$. This sends $A(1,0)$ to $P(x_0, y_0)$ and $B(0,1)$ to $Q(x_1, y_1)$ while leaving the origin fixed.

Using the process we adopted earlier for triangle maps, we can write $r_\theta(\mathbf{x}) = A_\theta(\mathbf{x})$ where

$$A_\theta = \begin{bmatrix} x_0 & x_1 \\ y_0 & y_1 \end{bmatrix}.$$

Since rotations do not affect lengths,

$$\|\overrightarrow{OA}\| = \|\overrightarrow{OB}\| = \|\overrightarrow{OP}\| = \|\overrightarrow{OQ}\| = 1,$$

and simple trigonometry gives

$$x_0 = \cos\theta, \;\; y_0 = \sin\theta, \;\; x_1 = -\sin\theta, \;\; y_1 = \cos\theta.$$

Hence $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the **rotation matrix** such that $A_\theta\mathbf{x}$ rotates the vector $\mathbf{x}$ through an angle $\theta$ anticlockwise.

Note that

$$\begin{aligned} B = A_\phi A_\theta &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix}. \end{aligned}$$

So $B = A_{\phi+\theta}$, which is an obvious geometric fact that we have established algebraically. Again, it relied on the definition of matrix multiplication we've chosen to work with. For once, the geometric approach is easier.

**Examples 3.2.4**

(i) If we were to apply successive rotations, geometrically we'd expect it to make no difference which order we did it in. That is, $r_\theta \circ r_\phi = r_\phi \circ r_\theta$. This seems to contradict the noncommutativity of matrix multiplication, but

$$\begin{aligned} B = A_\theta A_\phi &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix}. \end{aligned}$$

So for this particular class of matrices, multiplication is commutative. However, if $A$ is a rotation matrix but $B$ is not, do not expect $AB = BA$.

(ii) We can use rotation matrices to derive affine transformations which will have the effect of rotating objects around any point in the plane. Notice that we can achieve our goal, to rotate through an angle $\theta$ around the point $P(a, b)$, by first translating points so that $P$ moves to the origin, carrying out the rotation, and then translating back. Thus the appropriate transformation is

$$A_\theta(\mathbf{x} - \mathbf{a}) + \mathbf{a} = A_\theta\mathbf{x} + (I - A_\theta)\mathbf{a},$$

where $\mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}$.

**Example 3.2.5**

- According to the formula, $A_{\pi/4} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $A_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

  Note that
  $$A^2_{\pi/4} = \frac{1}{2}\begin{bmatrix} 1-1 & -1-1 \\ 1+1 & -1+1 \end{bmatrix} = A_{\pi/2}.$$

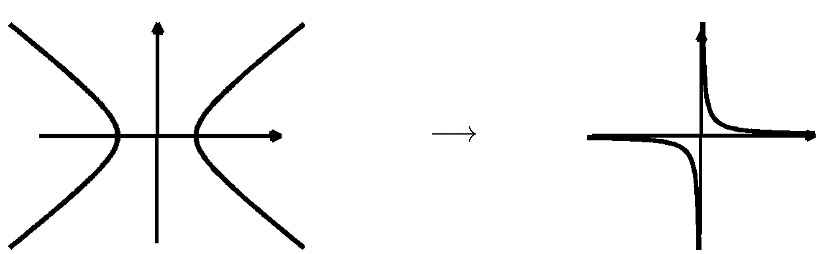
- If we rotate anticlockwise through an angle $\theta$ and then clockwise through the same angle, the overall result is to leave things as they were. In matrix algebra, we have $A_{-\theta}A_\theta = I$, or $A_\theta^{-1} = A_{-\theta}$.

  Now,
  $$A_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A_\theta^T,$$
  so for rotation matrices, $A_\theta^{-1} = A_\theta^T$.

- By definition,
  $$\|A_\theta \mathbf{u}\|^2 = (A_\theta\mathbf{u})^T A_\theta \mathbf{u} = \mathbf{u}^T A_\theta^T A_\theta \mathbf{u} = \mathbf{u}^T\mathbf{u} = \|\mathbf{u}\|^2,$$
  so vector lengths are preserved by rotations. Also,
  $$(A_\theta\mathbf{u})\cdot(A_\theta\mathbf{v}) = \mathbf{u}^T A_\theta^T A_\theta \mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v},$$
  so scalar products are preserved and hence so are angles between vectors (why?). These two results confirm obvious geometric properties of rotations.

**Example 3.2.6**

- Suppose we want to find the general quadratic formula for the hyperbola we get when we rotate $x^2 - y^2 = 1$ through $\pi/4$ radians. Following the approach of Examples 3.1.11 we write
  $$\mathbf{x} = r^{-1}_{\pi/4}\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} u+v \\ v-u \end{bmatrix}.$$
  So the rotated hyperbola can be written as
  $$\left(\frac{u+v}{\sqrt{2}}\right)^2 - \left(\frac{v-u}{\sqrt{2}}\right)^2 = 1,$$
  which simplifies to the rectangular hyperbola $2uv = 1$.



### Reflections

Next we consider reflections through a line. In Figure **??** we illustrate the effect of a reflection through the line through the origin which makes an angle $\theta$ with the $x$-axis, $y = (\tan\theta)x$. In the diagram $A(1,0)$ is mapped to $P$ and $B(0,1)$ to $Q$. Notice that $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ meet the line of symmetry at right angles.

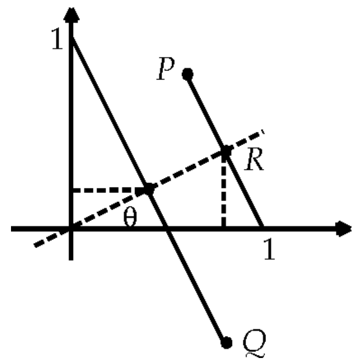



Figure 3.6: Reflection through a line.

To derive the matrix representation of this transformation we need to find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. To do this, first note that $\overrightarrow{OP} = \overrightarrow{OA} + 2\overrightarrow{AR}$. Simple trigonometry gives $\|\overrightarrow{AR}\| = \sin\theta$. Now look at the right angled triangle with hypotenuse $AR$. The length of the vertical is $\sin\theta\cos\theta$ and the horizontal is $\sin^2\theta$ and thus

$$\overrightarrow{OP} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -\sin^2\theta \\ \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} 1-2\sin^2\theta \\ 2\sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}.$$

A similar analysis of $BQ$ gives

$$\overrightarrow{OQ} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} \sin\theta\cos\theta \\ -\cos^2\theta \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}.$$

We can conclude that we can reflect objects through $y = (\tan\theta)x$ with the symmetric matrix

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

**Examples 3.2.7**

(i) The line $y = x$ makes an angle of $\pi/4$ radians with the $x$-axis, so the reflection matrix for this transformation is
$$\begin{bmatrix} \cos\pi/2 & \sin\pi/2 \\ \sin\pi/2 & -\cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
To reflect through the $x$-axis ($\theta = 0$) and $y$-axis ($\theta = \pi/2$) we use
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$
respectively.

(ii) Since
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A_{-2\theta},$$
so we can achieve a reflection through the line $y = (\tan\theta)x$ by rotating clockwise through an angle $2\theta$ and then reflecting through the $x$-axis.

(iii) Notice that in our analysis above,
$$\left[\ \overrightarrow{OP}\ \middle|\ \overrightarrow{OQ}\ \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} -\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & -\cos^2\theta \end{bmatrix} = I - 2\mathbf{w}\mathbf{w}^T,$$
where $\mathbf{w} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$ (and $\|\mathbf{w}\| = 1$).

Looking back to Example 1.3.12(iii), we can conclude that reflection matrices are Householder matrices. In fact, the converse is true, too. All Householder matrices are reflection matrices.

To undo the effect of a reflection, we simply need to repeat it. That is, a reflection is its own inverse. This tallies with reflection matrices being Householder matrices, which we've already seen are their own inverse. Matrices which are their own inverses are known as **involutions**.

**Examples 3.2.8**

(i) Suppose $A$ is a reflection matrix. Then since $A = A^T = A^{-1}$,
$$\|A\mathbf{u}\|^2 = (A\mathbf{u})^T A\mathbf{u} = \mathbf{u}^T A^2 \mathbf{u} = \|\mathbf{u}\|^2,$$
and $(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u}.\mathbf{v}$. This confirms that reflections also preserve lengths and angles, as we'd expect.

(ii) Find the matrix $A$ such that the mapping $f(\mathbf{x}) = A\mathbf{x}$ reflects a vector through the line $ax + by = 0$.

To find $A$, note that $\tan\theta = -a/b$ and we can use the trigonometric identities

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}, \quad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta},$$

to write $A = \dfrac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$.

Look again at the last example.

$$A = \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix} = I - \frac{2}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} = I - \frac{2}{\mathbf{w}^T\mathbf{w}}\mathbf{w}\mathbf{w}^T,$$

where $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$.

Thus we can generate the reflection through the line $ax + by = 0$ with the Householder matrix generated by $\mathbf{w}$, avoiding the need to work with $\theta$.

Examples 3.2.9

### Stretch

One particularly straightforward transformation to express is the stretch, where we simulate the effect of pulling the axes away from the origin, or squashing them inwards. This can be achieved with a diagonal matrix

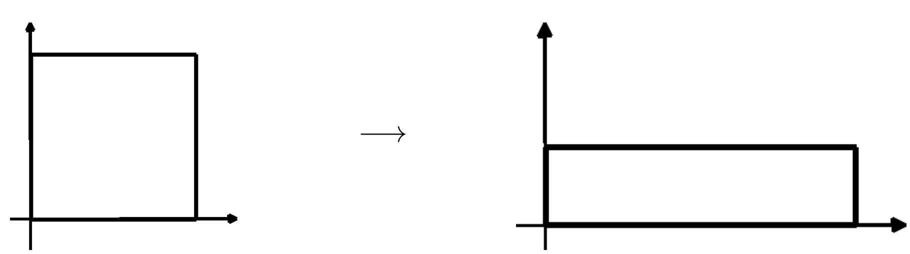
$$S = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Notice that $S\mathbf{e}_1 = a\mathbf{e}_1$ and $S\mathbf{e}_2 = b\mathbf{e}_2$. If $a$ is bigger than one horizontal distances are stretched, and if $0 < a < 1$ they shrink. If $a < 0$ vectors flip horizontally, effectively reflected as they are stretched/squashed. Analogous observations can be made about the effects of $b$ on vertical directions.

### Examples 3.2.10

(i) Consider the effect of $S = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ on the square with vertices $O(0,0)$, $A(1,0)$, $B(1,1)$, $C(0,1)$.

The vertices are mapped to $S\mathbf{0} = \mathbf{0}$, $S\mathbf{e}_1 = 2\mathbf{e}_1$, $S\mathbf{e}_2 = \mathbf{e}_2/2$, $S\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$.



(ii) As stretches are represented by diagonal matrices, we can apply a succession of stretches in any order as **diagonal matrices commute**.

(iii) Combination of stretches and rotations/reflections do not commute.

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}, \quad \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & \frac{3}{5} \\ 3 & \frac{4}{5} \end{bmatrix}.$$

Clearly stretches do not preserve distances. While Example 3.2.10(i) may lead you to suspect that angles are preserved, a moment's thought shows this is not generally true. Simply consider the effect of $S$ on the unit triangle. Its image has vertices $O$, $(2,0)$ and $(0,1/2)$. This is still a right angled triangle but it is no longer isosceles.

**Shear**

Imagine your feet are glued to the floor and you lean your body to the side (while trying to keep your head at the same vertical level). Clearly the top of your body will move further to the side than your legs. The action of swaying to the side is similar to the geometrical transformation called the shear. A typical shear is illustrated in Figure **??**.

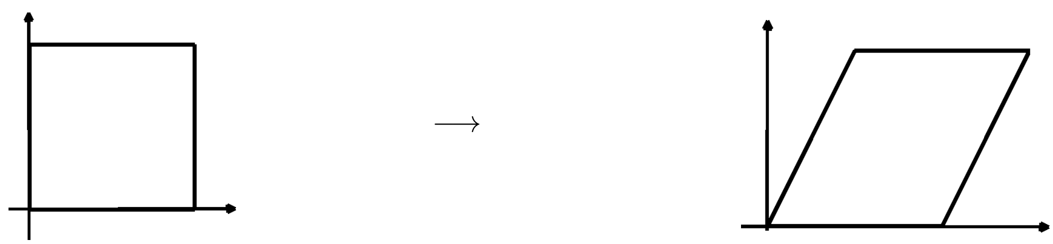


Figure 3.7: A typical shear.

Notice that the base of the square is left invariant by the shear, and vertical coordinates are preserved. The matrix which induces this linear transformation can be written

$$H = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}.$$

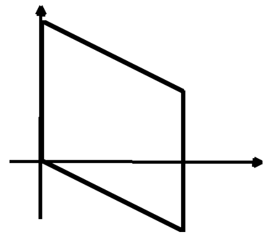
If $m$ is positive things are pushed to the right, and to the left if $m$ is negative. The square is transformed into a parallelogram by a shear, so angles aren't preserved, and most distances are changed. But the area of the parallelogram is found by multiplying the base length by height, and since neither of these are altered under the transformation we can conclude that **shears are area preserving transformations**.

**Example 3.2.11**

- The matrix

$$V = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$$

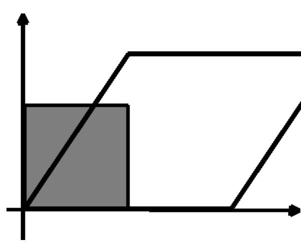
induces a vertical shear. We illustrate the effect of $V$ on a square with $m = -1/2$.

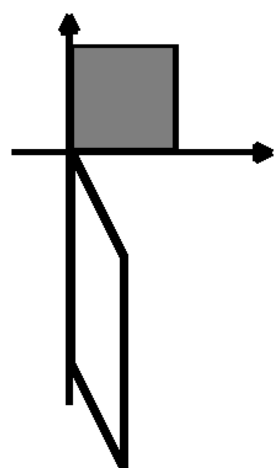


**Example 3.2.12**

- It is often useful to combine a stretch and a shear together. This can be achieved by the transformation induced by the matrix
$$R = \begin{bmatrix} h & m \\ 0 & v \end{bmatrix},$$
where $h$ and $v$ are the horizontal and vertical stretch factors and $m$ can be controlled to give an appropriate degree of horizontal shearing. Alternatively,
$$L = \begin{bmatrix} h & 0 \\ m & v \end{bmatrix},$$
gives a vertical shear.

  We illustrate the effects of the stretch-shears induced by $R = \begin{bmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{bmatrix}$ and $L = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & -2 \end{bmatrix}$ on the unit square.





**Degenerate Transformations**

The transformations we have concentrated on so far all map triangles into triangles and quadrilaterals into quadrilaterals. But there are certain transformations which fail to do this. These involve singular matrices. If $A$ is a $2 \times 2$ singular matrix then its columns are multiples of each other, so we can write $A = \left[\begin{array}{c|c} a\mathbf{u} & b\mathbf{u} \end{array}\right]$, or $A = \mathbf{u}\mathbf{v}^T$.

Consider the effect of the transformation induced by $A$ on a vector $\mathbf{x}$.

$$A\mathbf{x} = \mathbf{u}\mathbf{v}^T\mathbf{x} = (\mathbf{v}^T\mathbf{x})\mathbf{u}.$$

Since $\mathbf{v}^T\mathbf{x}$ is simply a scalar, $A\mathbf{x}$ is just a multiple of $t\mathbf{u}$. So $A\mathbf{x}$ lies on the line with vector definition $\mathbf{r} = t\mathbf{u}$. Such a transformation is called **degenerate**. Everything is squashed onto a line and we lose a dimension. Notice that since $A$ is singular, this mapping is not invertible: we cannot undo this dimension reduction. This makes sense, as once we've lost a dimension we've thrown away vital information. The transformation induced by the matrix $O$ is doubly degenerate, as all of space is mapped to a single point and we lose two dimensions at once.

**Example 3.2.13**

- Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T$ and the transformation $A\mathbf{x}$ maps all points onto the line $\mathbf{r} = t\mathbf{u}$, or $y = 2x$.

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \mathbf{0}, \quad A\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A\mathbf{e}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

## Exercises: Matrix Transformations

1. Find the rotation matrices $A_\theta$ for $\theta = 0$, $\theta = \pi/6$, $\theta = 2\pi/3$ and $\theta = -3\pi/4$.

   Evaluate $A_{\pi/6}A_{2\pi/3}^{-1}A_{-3\pi/4}^2$.

2. Find the generalised quadratic formula for the hyperbola we get if we rotate the rectangular hyperbola $xy = 1$ anticlockwise through (a) a right angle (b) $2\pi/3$ radians.

3. Find the matrices that induce the reflections through the following lines and sketch the image of the unit square under the action of these reflections.

   (a) $y = x/\sqrt{2}$ (b) $y = 2x$ (c) $y = -x$ (d) $y = -5x$

4. Find the generalised quadratic formula for the hyperbola we get if we reflect $x^2 - y^2 = 1$ through the lines (a) $y = 0$ (b) $y = x$ (c) $y = 3x$.

5. Show that a reflection through the line $y = (\tan\theta)x$ can be achieved by a reflection through the $x$-axis followed by an anticlockwise rotation through an angle $2\theta$.

6. By multiplying together the appropriate matrices, find matrix representations of linear transformations to carry out the following geometric effects.

   Confirm your results by applying the transformations to the unit square.

   (a) Rotate through a right angle (anticlockwise) then stretch horizontally by a factor of 4.

   (b) Stretch in the horizontal direction by a factor of 2 and in the vertical direction by a factor of 3 then reflect through the line $y = 2x$.

   (c) Shear with the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then rotate anticlockwise through $\pi/4$ radians.

7. Find the inverse of the shear matrix $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ and the stretch-shear matrix $\begin{bmatrix} h & m \\ 0 & v \end{bmatrix}$.

8. Illustrate the effect of the linear transformations induced by the following matrices on the unit square.

   (a) $\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{5}{12} & 1 \\ -1 & \frac{5}{12} \end{bmatrix}$

## 3.3 Summary

Here is a list of skills you are expected to have picked up from this chapter.

1. Understand all terms in **bold face** (all sections).
2. Understand all the numbered definitions and statement of all numbered theorems and recognise the relevance of the numbered equations (all sections).
3. Be able to apply affine transformations to vectors to understand their geometric effects (§3.1).
4. Be able to invert affine transformations and to recognise those affine transformations which cannot be inverted (§3.1).
5. Find an affine transformation that maps the unit triangle onto any given triangle (§3.1 and Exercises).
6. Find the affine transformation that maps between any two given triangles or any two parallelograms (§3.1 and Exercises).
7. Understand the limitations of affine transformations (§3.1).
8. Apply affine transformations to the unit circle and canonical hyperbolae (§3.1 and Exercises).
9. Find the algebraic form of transformed circles/hyperbolae (§3.1 and Exercises).
10. Recognise rotation, reflection, stretch and shear matrices (§3.2).
11. Find the rotation matrix to rotate a vector through a given angle (§3.2 and Exercises).
12. Find the reflection matrix to reflect a vector through a given line (§3.2 and Exercises).
13. List the invariant properties of reflections and rotations (§3.2).
14. Be able to invert rotations, reflections, stretches and shears (§3.2 and Exercises).
15. Be able to generate a matrix to carry out a given sequence of geometric operations (§3.2 and Exercises).