Exercises and outline solutions for MM101 tutorial in week 10

1. Find $f(x)$ if $f'(x) = \dfrac{1}{x} + \dfrac{3}{2\sqrt{x}}$ and $f(1) = 2$.

$\int f'(x)\,\mathrm{d}x = \ln x + 3\sqrt{x} + C$, so any f of the form $f(x) = \ln x + 3\sqrt{x} + C$ is a primitive for f'. With $f(1) = 3 + C = 2$, we find the solution $f(x) = \ln x + 3\sqrt{x} - 1$.

2. Evaluate the following integrals.

   (a) $\displaystyle\int 3^x\,\mathrm{d}x$.

   (b) $\displaystyle\int \log_{10} x\,\mathrm{d}x$.

(a) $\displaystyle\int 3^x\,\mathrm{d}x = \int e^{x\ln 3}\,\mathrm{d}x = \frac{1}{\ln 3}e^{x\ln 3} + C = \frac{1}{\ln 3}3^x + C$.

(b) $\displaystyle\int \log_{10} x\,\mathrm{d}x = \int \frac{\ln x}{\ln 10}\,\mathrm{d}x = \frac{1}{\ln 10}(x\ln x - x) + C$.

3. Use integration by parts to evaluate the following integrals.

   (a) $\displaystyle\int \sqrt{x}\,\ln x\,\mathrm{d}x$.

   (b) $\displaystyle\int \sin^2 x\,\mathrm{d}x$.

(a) With $f(x) = \ln x$ and $g'(x) = x^{1/2}$ we have $f'(x) = x^{-1}$ and $g(x) = \frac{2}{3}x^{3/2}$. So $\displaystyle\int \sqrt{x}\,\ln x\,\mathrm{d}x = \frac{2}{3}x^{3/2}\,\ln x - \int \frac{2}{3}\,x^{1/2}\,\mathrm{d}x = \frac{2}{3}x^{3/2}\,\ln x - \frac{4}{9}\,x^{3/2} + C$.

(b) We have $f(x) = g'(x) = \sin x$ and so $f'(x) = \cos x$ and $g(x) = -\cos x$. Hence $\displaystyle\int \sin^2 x\,\mathrm{d}x = -\sin x\,\cos x + \int \cos^2 x\,\mathrm{d}x$.

Another integration by parts leads nowhere (more precisely, $\displaystyle\int \cos^2 x\,\mathrm{d}x = \sin x\,\cos x + \int \sin^2 x\,\mathrm{d}x$, and so we obtain $\displaystyle\int \sin^2 x\,\mathrm{d}x = \int \sin^2 x\,\mathrm{d}x$.

However, $\cos^2 x = 1 - \sin^2 x$, so we get $\displaystyle\int \sin^2 x\,\mathrm{d}x = -\sin x\,\cos x + \int \mathrm{d}x - \int \sin^2 x\,\mathrm{d}x$. This means that $\displaystyle\int \sin^2 x\,\mathrm{d}x = \frac{1}{2}(x - \sin x\,\cos x) + C$.

4. Use appropriate substitutions to evaluate the following integrals.

   (a) $\displaystyle\int x\sqrt{3x^2+1}\,\mathrm{d}x$.

   (b) $\displaystyle\int e^x \sin(e^x)\,\mathrm{d}x$.

   (c) $\displaystyle\int \ln(|\sin x|)\,\cot x\,\mathrm{d}x$.

(a) With $u = 3x^2 + 1$ we have $\mathrm{d}u = 6x\,\mathrm{d}x$ and so $\displaystyle\int x\sqrt{3x^2+1}\,\mathrm{d}x = \frac{1}{6}\int \sqrt{u}\,\mathrm{d}u = \frac{1}{9}u^{3/2} + C = \frac{1}{9}\left(\sqrt{3x^2+1}\right)^3 + C.$

(b) With $u = e^x$ we have $\mathrm{d}u = e^x\,\mathrm{d}x$ and so $\displaystyle\int e^x \sin(e^x)\,\mathrm{d}x = \int \sin u\,\mathrm{d}u = -\cos(e^x).$

(c) With $u = \sin x$ we have $\mathrm{d}u = \cos x\,\mathrm{d}x$ and so $\displaystyle\int \ln(|\sin x|)\cot x\,\mathrm{d}x = \int \frac{\ln|u|}{u}\,\mathrm{d}u = \frac{1}{2}(\ln|u|)^2 + C = \frac{1}{2}\{\ln(|\sin x|)\}^2 + C.$

5. Use appropriate methods to evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/4} \frac{\cos x}{2+\sin x}\mathrm{d}x.$

(b) $\displaystyle\int e^x \cos x\,\mathrm{d}x.$

(c) $\displaystyle\int_{-\infty}^{0} e^x \arctan(e^x)\,\mathrm{d}x.$

(a) With the substitution $u = 2 + \sin x$ we have $\mathrm{d}u = \cos x\mathrm{d}x$ and so

$$\int_0^{\pi/4} \frac{\cos x}{2+\sin x}\mathrm{d}x = \int_2^{2+1/\sqrt{2}} \frac{1}{u}\mathrm{d}u = [\ln u]_2^{2+1/\sqrt{2}} = \ln\left(\frac{2\sqrt{2}+1}{\sqrt{2}}\right) - \ln 2 = \ln\left(\frac{2\sqrt{2}+1}{2\sqrt{2}}\right)$$

(b) Integrate by parts twice:

$$\int e^x \cos x\,\mathrm{d}x = e^x\sin(x) - \int e^x \sin x\,\mathrm{d}x = e^x \sin x + e^x \cos x - \int e^x \cos x\,\mathrm{d}x,$$

and thus $\displaystyle\int e^x \cos x\,\mathrm{d}x = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C.$

(c) First deal with the improper integral: $\displaystyle\int_{-\infty}^{0} e^x \arctan(e^x)\,\mathrm{d}x = \lim_{a\to-\infty}\int_a^0 e^x\arctan(e^x)\,\mathrm{d}x.$ Now make the substitution $u = e^x$ to find

$$\lim_{a\to-\infty}\int_a^0 e^x \arctan(e^x)\,\mathrm{d}x = \lim_{a\to-\infty}\int_{e^a}^1 \arctan u\,\mathrm{d}u = \lim_{b\to0}\int_b^1 \arctan u\,\mathrm{d}u = \int_0^1 \arctan u\,\mathrm{d}u.$$

Now we can use IBP followed by substitution to find

$$\int_0^1 \arctan u\,\mathrm{d}u = \Big[u\arctan u\Big]_0^1 - \int_0^1 \frac{u}{1+u^2}\,\mathrm{d}u = \left[u\arctan u - \frac{1}{2}\ln(1+u^2)\right]_0^1,$$

and thus, putting the pieces together,

$$\int_{-\infty}^{0} e^x \arctan(e^x)\,\mathrm{d}x = \arctan(1) - \frac{1}{2}\ln 2 - 0 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

6. Find $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x$

(a) by using the substitution $u = \ln x$,

(b) by using the substitution $x = e^u$, and

(c) by using integration by parts.

(a) $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$, so $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$

(b) $\mathrm{d}x = e^u\,\mathrm{d}u$, so $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \int \frac{u}{e^u}\cdot e^u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$

(c) $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = (\ln x)^2 - \int \frac{\ln x}{x}\,\mathrm{d}x$, so $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{1}{2}(\ln x)^2 + C.$