Exercises and solutions for MM101 Tutorial in Week 3

1. Show that for any two numbers x and y

$$\frac{1}{2}(x^2+y^2) \geq xy.$$

Deduce that the *arithmetic mean* of any two positive numbers a and b is greater than or equal to their *geometric mean*.
(Note: the arithmetic mean is defined as $\frac{1}{2}(a+b)$, the geometric mean is defined as $\sqrt{ab}$.)

$$\frac{1}{2}(x^2+y^2) \geq xy \Leftrightarrow x^2+y^2 \geq 2xy \Leftrightarrow x^2-2xy+y^2 \geq 0 \Leftrightarrow (x-y)^2 \geq 0.$$

Applying this result to the two numbers $\sqrt{a}$ and $\sqrt{b}$ gives

$$\frac{1}{2}(a+b) \geq \sqrt{ab}$$

(for any positive a, b) as required.

2. Use induction to prove that the sum of the first n positive odd integers is n^2 for any $n \in \mathbb{N}$.

We need to prove that $p(n)$ is true where

$$p(n): \qquad 1+3+5+7+\ldots+(2n-1) = n^2.$$

Step 1: Check the case $n=1$.
Sum of the first integer is $1 = 1^2$ so proposition is true when $n=1$.

Step 2: Assume that $p(n)$ is true, that is, assume that

$$1+3+5+7+\ldots+(2n-1) = n^2.$$

Now try to prove that $p(n) \Rightarrow p(n+1)$, where

$$p(n+1): \qquad 1+3+5+7+\ldots+(2n-1)+(2(n+1)-1) = (n+1)^2.$$

We have

$$\begin{aligned} 1+3+5+7+\ldots+(2n-1)+(2(n+1)-1) &= n^2+(2n+2-1) \\ &= n^2+2n+1 \\ &= (n+1)^2 \end{aligned}$$

so if the proposition is true for n, it is true for $n+1$. Hence, by the principle of mathematical induction, the proposition is true for all natural numbers n.

3. Find the natural domain of the following functions:

   (a) $f(x) = \sqrt{1-x^2}$,

   (b) $f(x) = \sqrt{1-\sqrt{1-x^2}}$,

   (c) $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$,

   (d) $f(x) = \sqrt{1-x} + \sqrt{x-2}$.

(a) We require $1-x^2 \geq 0 \Leftrightarrow |x| \leq 1$ so domain is $\{x : |x| \leq 1\}$.

(b) We require $1-x^2 \geq 0 \Leftrightarrow |x| \leq 1$ so domain is $\{x : |x| \leq 1\}$. (Note that $1-\sqrt{1-x^2} \geq 0$ is always satisfied.)

(c) We require $1-x^2 \geq 0 \Leftrightarrow |x| \leq 1$ and $x^2-1 \geq 0 \Leftrightarrow |x| \geq 1$ so domain is $\{1, -1\}$.

(d) We require $x \leq 1$ and $x \geq 2$ so the domain is the empty set $\emptyset$.

4. Use induction to show that $2n+1 < 2^n$ for all integers $n \geq 3$.

Step 1: Check the case $n = 3$.

$$LHS = 7, \qquad RHS = 2^3 = 8$$

so proposition is true when $n = 3$.

Step 2: Assume that the given result is true for n, that is, assume that

$$2n+1 < 2^n.$$

Now try to prove the result for $n+1$, that is, try to show that

$$2(n+1)+1 < 2^{(n+1)} \Leftrightarrow 2n+3 < 2^{(n+1)}.$$

We have

$$2n+3 = (2n+1)+2 < 2^n+2 < 2^n+2^n = 2(2^n) = 2^{(n+1)}$$

so if the proposition is true for n, it is true for $n+1$. Hence, by the principle of mathematical induction, the proposition is true for all natural numbers $n \geq 3$.