University of Strathclyde
Department of Mathematics and Statistics

MM102: Applications of Calculus

Lecture Notes for Week 2

Contents

1.4 Integration of Trigonometric Functions

In this section we consider some strategies how one can integrate functions that involve trigonometric functions.

$\int \sin^m x \cdot \cos^n x \,\mathrm{d}x$ **where m or n or both are odd non-negative integers.**

(a) If m is odd, substitute $u = \cos x$ and use $\sin^2 x = 1 - \cos^2 x$ to replace $(\sin x)^{m-1}$; the remaining $\sin x$ is used for the substitution of the differential.

(b) If n is odd, substitute $u = \sin x$ and proceed in a similar way as in (a).

(c) If both n and m are odd, use either (a) or (b).

In other words: if at least one power of $\sin x$, $\cos x$ is odd, substitute for the other trigonometric function.

Examples 1.6.

(i) Evaluate

$$\int \sin^4 x \cdot \cos^5 x \,\mathrm{d}x.$$

Since the integrand contains an odd power of $\cos x$, we use the substitution

$$u = \sin x, \qquad \mathrm{d}u = \cos x \,\mathrm{d}x.$$

Hence

$$\begin{aligned}\int \sin^4 x \cdot \cos^5 x \,\mathrm{d}x &= \int \sin^4 x \cdot (\cos^2 x)^2 \cos x \,\mathrm{d}x = \int \sin^4 x \cdot (1-\sin^2 x)^2 \cos x \,\mathrm{d}x \\ &= \int u^4(1-u^2)^2 \,\mathrm{d}u = \int u^4(1-2u^2+u^4)\,\mathrm{d}u \\ &= \int \left(u^4 - 2u^6 + u^8\right)\mathrm{d}u = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C \\ &= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C.\end{aligned}$$

(ii) Evaluate

$$\int_0^{\pi/2} \sin^3 x \cdot \cos^9 x \,\mathrm{d}x.$$

Solution in video

(iii) Evaluate

$$\int \sin^5(2x)\,\mathrm{d}x.$$

Solution in video

$\displaystyle\int \sin^m x \cdot \cos^n x \,\mathrm{d}x$ **where m <u>and</u> n are even non-negative integers.**

(a) Replace $\sin^2 x$ and $\cos^2 x$ using the double angle formulae

$$\sin^2 x = \frac{1}{2}\big(1-\cos(2x)\big), \qquad \cos^2 x = \frac{1}{2}\big(1+\cos(2x)\big).$$

(b) Continue using this or the previous technique until the integral can be evaluated.

Examples 1.7.

(i) Evaluate $\displaystyle\int \sin^2 x \,\mathrm{d}x.$

With the double-angle formula we obtain

$$\begin{aligned}\int \sin^2 x \,\mathrm{d}x &= \int \frac{1}{2}\big(1-\cos(2x)\big)\mathrm{d}x = \frac{1}{2}\Big(x - \frac{1}{2}\sin(2x)\Big) + C \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

(ii) Evaluate the following integral

$$\int_0^{\pi} \sin^2 x \cdot \cos^2 x \, \mathrm{d}x.$$

In this example we have to apply the double angle formulae a couple of times:

$$\int_0^{\pi} \sin^2 x \cdot \cos^2 x \, \mathrm{d}x = \int_0^{\pi} \frac{1}{4}\big(1 - \cos(2x)\big)\big(1 + \cos(2x)\big)\mathrm{d}x$$

$$= \frac{1}{4}\int_0^{\pi} \big(1 - \cos^2(2x)\big)\mathrm{d}x \qquad \text{(using } (a+b)(a-b) = a^2 - b^2\text{)}$$

$$= \frac{1}{4}\int_0^{\pi} \sin^2(2x)\, \mathrm{d}x \qquad \text{(using } \cos^2(2x) + \sin^2(2x) = 1\text{)}$$

$$= \frac{1}{4}\int_0^{\pi} \frac{1}{2}\big(1 - \cos(4x)\big)\, \mathrm{d}x \qquad \text{(using the double angle formula again)}$$

$$= \frac{1}{8}\left[x - \frac{1}{4}\sin(4x)\right]_0^{\pi}$$

$$= \frac{1}{8}\left(\pi - \frac{1}{4}\sin(4\pi) - \left(0 - \frac{1}{4}\sin 0\right)\right) = \frac{\pi}{8}\,.$$

(iii) Evaluate the definite integral

$$\int_0^{2\pi} \cos^2 x \, \mathrm{d}x.$$

Solution in video

(iv) Evaluate the following integral

$$\int \sin^6 x \, \mathrm{d}x.$$

Solution in video

Integrals involving
$\sin(mx) \cdot \cos(nx)$, $\sin(mx) \cdot \sin(nx)$ **or** $\cos(mx) \cdot \cos(nx)$

Use one of the following formulae

$$\sin\alpha \cdot \cos\beta = \frac{1}{2}\Big[\sin(\alpha-\beta) + \sin(\alpha+\beta)\Big],$$
$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\Big[\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big],$$
$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\Big[\cos(\alpha-\beta) + \cos(\alpha+\beta)\Big].$$

Examples 1.8.

(i) Evaluate the following integral

$$\int \sin(3x) \cdot \cos(4x)\,\mathrm{d}x.$$

We have

$$\int \sin(3x) \cdot \cos(4x)\,\mathrm{d}x = \frac{1}{2}\int \Big(\sin(3x-4x) + \sin(3x+4x)\Big)\mathrm{d}x$$
$$= \frac{1}{2}\int \big(-\sin x + \sin 7x\big)\mathrm{d}x = \frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C.$$

(ii) Evaluate

$$\int_0^{\pi/2} \cos(2x) \cdot \cos x\,\mathrm{d}x.$$

Solution in video

Other integrals involving $\sin x$ and $\cos x$
If the integrand is a rational function of $\sin x$ and $\cos x$, one can use the following substitution:

$$t = \tan\frac{x}{2}\,, \qquad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t,$$

$$\cos x = \frac{1-t^2}{1+t^2}\,, \qquad \sin x = \frac{2t}{1+t^2}\,,$$

which leads to a rational integrand in t.

Let us see why for the substitution $t = \tan\frac{x}{2}$ the other three formulae above are valid. If $x \in (-\pi, \pi)$, we can reformulate the relation $t = \tan\frac{x}{2}$ as

$$x = 2\arctan t.$$

(Otherwise, we just have to add a constant on the right-hand side.) From this we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\frac{1}{1+t^2} \qquad \Longrightarrow \qquad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.$$

Let us find a relation between t and $\cos x$:

$$t^2 = \tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{\frac{1}{2}\big(1-\cos x\big)}{\frac{1}{2}\big(1+\cos x\big)} = \frac{1-\cos x}{1+\cos x}\,.$$

Hence

$$\begin{aligned} t^2 + t^2\cos x = 1 - \cos x \quad &\Longleftrightarrow \quad t^2\cos x + \cos x = 1 - t^2 \\ &\Longleftrightarrow \quad (t^2+1)\cos x = 1 - t^2 \\ &\Longleftrightarrow \quad \cos x = \frac{1-t^2}{1+t^2}\,. \end{aligned}$$

Finally, we can also express $\sin x$ in terms of t:

$$\begin{aligned} \sin x &= \pm\sqrt{1-\cos^2 x} = \pm\sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)^2} = \pm\sqrt{1-\frac{1-2t^2+t^4}{(1+t^2)^2}} \\ &= \pm\sqrt{\frac{1+2t^2+t^4-(1-2t^2+t^4)}{(1+t^2)^2}} = \pm\sqrt{\frac{4t^2}{(1+t^2)^2}} = \frac{2t}{1+t^2} \end{aligned}$$

where the plus signs have to be chosen for $x \in [0, \pi)$ and the minus signs for $x \in (-\pi, 0)$.

Examples 1.9.

(i) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate the integral

$$\int \frac{1}{\sin x}\,\mathrm{d}x.$$

With the formulae from above we obtain

$$\begin{aligned}\int \frac{1}{\sin x}\,\mathrm{d}x &= \int \frac{1}{\dfrac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int \frac{1}{t}\,\mathrm{d}t = \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C.\end{aligned}$$

(ii) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate the integral

$$\int \frac{1}{1+2\cos x}\,\mathrm{d}x.$$

Solution in video

$$\int e^{ax}\cos(mx)\,\mathrm{d}x, \qquad \int e^{ax}\sin(mx)\,\mathrm{d}x$$

1. Set I to be the integral.
2. Use integration by parts twice. Either differentiate e^{ax} in both steps or integrate e^{ax} in both steps, but do not mix.
3. The right-hand side now involves I. Solve the equation for I.

Example 1.10.
Evaluate the following integral

$$\int e^{-2x}\cos(3x)\mathrm{d}x.$$

Solution in video

1.5 Integrals Involving Square Roots

Integrals involving $\sqrt{ax+b}$

Use the substitution $u = \sqrt{ax+b}$.
Solve this equation for x and differentiate with respect to u to obtain the connection between $\mathrm{d}x$ and $\mathrm{d}u$.

Sometimes one can also use the substitution $u = ax+b$, but there are cases where it doesn't help; see Example 1.11 (ii).

Examples 1.11.

(i) Evaluate the following integral

$$\int (x+2)\sqrt{2x+1}\,\mathrm{d}x.$$

We use the substitution

$$u = \sqrt{2x+1}, \quad \Longrightarrow \quad x = \frac{1}{2}\big(u^2-1\big).$$

$$\Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}u} = u \quad \Longrightarrow \quad \mathrm{d}x = u\,\mathrm{d}u.$$

Hence

$$\int (x+2)\sqrt{2x+1}\,\mathrm{d}x = \int \left(\frac{1}{2}\big(u^2-1\big)+2\right)u\cdot u\,\mathrm{d}u = \frac{1}{2}\int (u^2+3)u^2\,\mathrm{d}u$$

$$= \frac{1}{2}\int \big(u^4+3u^2\big)\mathrm{d}u = \frac{1}{2}\Big(\frac{1}{5}u^5+u^3\Big)+C = \frac{1}{10}(2x+1)^{5/2}+\frac{1}{2}(2x+1)^{3/2}+C.$$

Alternative Solution: With the substitution

$$u = 2x+1, \qquad x = \frac{1}{2}\big(u-1\big), \qquad \mathrm{d}u = 2\,\mathrm{d}x$$

we obtain

$$\begin{aligned}\int (x+2)\sqrt{2x+1}\,\mathrm{d}x &= \int \left(\frac{1}{2}\big(u-1\big)+2\right)\sqrt{u}\,\frac{1}{2}\,\mathrm{d}u \\ &= \frac{1}{4}\int (u+3)u^{\frac{1}{2}}\mathrm{d}u = \frac{1}{4}\int \Big(u^{\frac{3}{2}}+3u^{\frac{1}{2}}\Big)\mathrm{d}u \\ &= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}}+3\cdot\frac{2}{3}u^{\frac{3}{2}}\right)+C = \frac{1}{10}u^{\frac{5}{2}}+\frac{1}{2}u^{\frac{3}{2}}+C \\ &= \frac{1}{10}(2x+1)^{5/2}+\frac{1}{2}(2x+1)^{3/2}+C.\end{aligned}$$

(ii) Evaluate the following integral

$$\int \frac{\sqrt{x+1}}{x}\,\mathrm{d}x.$$

Solution in video

Integrals involving $\sqrt{Ax^2+Bx+C}$

By completing the square one can write Ax^2+Bx+C as

$$(bx+c)^2+a^2, \qquad (bx+c)^2-a^2 \qquad \text{or} \qquad a^2-(bx+c)^2.$$

(Two minus signs cannot occur unless the expression under the square root is negative for all x.) We can assume that $a \neq 0$ because otherwise, the expression under the square root is a perfect square. Without loss of generality we assume that $a > 0$. We can also assume that $b \neq 0$ because otherwise, the expression under the square root is a constant. Again without loss of generality we assume that $b > 0$.

In all three cases we set $bx+c$ equal to a times some trigonometric function of u.

Integrals involving $\sqrt{Ax^2+Bx+C}$

1. By completing the square, express Ax^2+Bx+C in the form $\pm(bx+c)^2 \pm a^2$ with $a > 0$ and $b > 0$.
2. Make the following substitution according to the signs:
   (i) $\sqrt{(bx+c)^2+a^2}$: $\quad bx+c = a\tan u$
   (ii) $\sqrt{(bx+c)^2-a^2}$: $\quad bx+c = a\sec u$
   (iii) $\sqrt{a^2-(bx+c)^2}$: $\quad bx+c = a\sin u$
3. Use the trigonometric identities

$$\cos^2 u + \sin^2 u = 1 \qquad \text{or} \qquad \tan^2 u + 1 = \sec^2 u$$

   to write the expression under the square root as a complete square.

Remark 1.4.

- In the special case when $b = 1, c = 0$ we have the following substitutions:

$$\sqrt{x^2+a^2}: \quad x = a\tan u;$$
$$\sqrt{x^2-a^2}: \quad x = a\sec u;$$
$$\sqrt{a^2-x^2}: \quad x = a\sin u.$$

- In case (i) one takes

$$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad \text{so that} \qquad u = \arctan\frac{bx+c}{a};$$

 for such u one has $\sec u > 0$.

 In case (ii) one usually takes

$$u \in \left[0, \frac{\pi}{2}\right) \qquad \text{when } \ bx+c \geq a$$

 and

$$u \in \left[\pi, \frac{3\pi}{2}\right) \qquad \text{when } \ bx+c \leq -a;$$

 in both cases one has $\tan u \geq 0$.

 In case (iii) one takes

$$u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \text{so that} \qquad u = \arcsin\frac{bx+c}{a};$$

 for such u one has $\cos u \geq 0$.

- Sometimes it is better to use the substitutions

$$bx + c = a\sinh u \quad \text{in case (i)}$$

 and

$$bx + c = a\cosh u \quad \text{in case (ii).}$$

Example 1.12.
Let us consider a couple of examples that show which substitution should be used.

- Integrand contains $\sqrt{3^2 - (2x+5)^2}$:
 use the substitution $2x + 5 = 3\sin u$.
- Integrand contains $\sqrt{(2x-3)^2 + 4^2}$:
 use the substitution $2x - 3 = 4\tan u$.
- Integrand contains $\sqrt{(6x+1)^2 - 5^2}$:
 use the substitution $6x + 1 = 5\sec u$.
- Integrand contains $\sqrt{9x^2+4} = \sqrt{(3x)^2 + 2^2}$:
 use the substitution $3x = 2\tan u$.
- Integrand contains $\sqrt{1-(2x)^2}$:
 use the substitution $2x = \sin u$.
- Integrand contains $\sqrt{x^2 - 4^2}$:
 use the substitution $x = 4\sec u$.

Examples 1.13.

(i) Let $r > 0$ be a positive constant and evaluate the definite integral

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\,\mathrm{d}x.$$

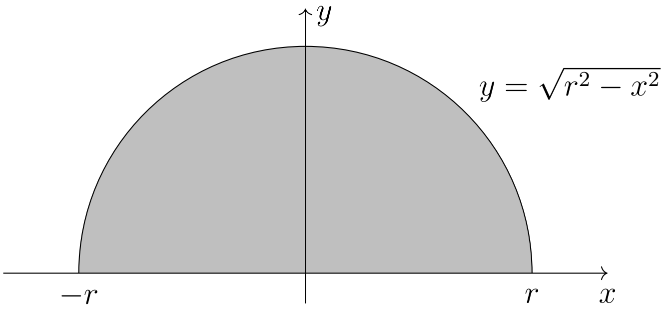
We can use the substitution

$$\begin{aligned}
&x = r\sin u \quad \left(u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right),\\
&\frac{\mathrm{d}x}{\mathrm{d}u} = r\cos u \qquad \Longrightarrow \qquad \mathrm{d}x = r\cos u\,\mathrm{d}u,\\
&u = \arcsin\left(\frac{x}{r}\right),\\
&x = -r \qquad\qquad \Longrightarrow \qquad u = \arcsin(-1) = -\frac{\pi}{2}\,,\\
&x = r \qquad\qquad\;\; \Longrightarrow \qquad u = \arcsin(1) = \frac{\pi}{2}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{-r}^{r} \sqrt{r^2 - x^2}\,\mathrm{d}x &= \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - (r\sin u)^2} \cdot r\cos u\,\mathrm{d}u\\
&= r\int_{-\pi/2}^{\pi/2} \sqrt{r^2\big(1 - \sin^2 u\big)} \cdot \cos u\,\mathrm{d}u\\
&= r\int_{-\pi/2}^{\pi/2} \sqrt{r^2\cos^2 u} \cdot \cos u\,\mathrm{d}u = r^2\int_{-\pi/2}^{\pi/2} \cos^2 u\,\mathrm{d}u\\
&= 2r^2\int_{0}^{\pi/2} \cos^2 u\,\mathrm{d}u \qquad \text{(since the integrand is even)}\\
&= 2r^2\int_{0}^{\pi/2} \frac{1}{2}\big(1 + \cos(2u)\big)\mathrm{d}u\\
&= r^2\left[u + \frac{1}{2}\sin(2u)\right]_0^{\pi/2} = r^2\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right)\\
&= \frac{r^2\pi}{2}
\end{aligned}$$

The graph of the integrand is the upper half of the circle $x^2 + y^2 = r^2$.



The area under this curve is exactly half the area of the disc with radius r.

(ii) Evaluate the following integral

$$\int \frac{x}{\sqrt{x^2 - 4x - 5}}\,\mathrm{d}x.$$

First we have to complete the square for the expression under the square root:

$$x^2 - 4x - 5 = (x-2)^2 - 2^2 - 5 = (x-2)^2 - 9 = (x-2)^2 - 3^2.$$

We use the following substitution

$$\begin{aligned} & x - 2 = 3\sec u \\ \implies \quad & x = 3\sec u + 2 \\ \implies \quad & \frac{\mathrm{d}x}{\mathrm{d}u} = 3\sec u \cdot \tan u \\ \implies \quad & \mathrm{d}x = 3\sec u \cdot \tan u\,\mathrm{d}u. \end{aligned}$$

The square root can be written as

$$\sqrt{x^2 - 4x - 5} = \sqrt{9\sec^2 u - 9} = \sqrt{9(\sec^2 u - 1)} = \sqrt{9\tan^2 u} = 3\tan u,$$

where we used the fact that we can choose u in an interval such that $\tan u \geq 0$; see Remark 1.4. Now we have

$$\begin{aligned} \int \frac{x}{\sqrt{x^2 - 4x - 5}}\,\mathrm{d}x &= \int \frac{3\sec u + 2}{3\tan u}\,3\sec u \cdot \tan u\,\mathrm{d}u \\ &= \int \big(3\sec u + 2\big)\sec u\,\mathrm{d}u = \int \big(3\sec^2 u + 2\sec u\big)\mathrm{d}u \\ &= 3\tan u + 2\ln\big|\sec u + \tan u\big| + C \\ &= 3\sqrt{\sec^2 u - 1} + 2\ln\Big|\sec u + \sqrt{\sec^2 u - 1}\Big| + C \\ &= 3\sqrt{\left(\frac{x-2}{3}\right)^2 - 1} + 2\ln\left|\frac{x-2}{3} + \sqrt{\left(\frac{x-2}{3}\right)^2 - 1}\right| + C. \end{aligned}$$

One can simplify this expression as follows

$$
\begin{aligned}
&= 3\sqrt{\frac{(x-2)^2-9}{9}} + 2\ln\left|\frac{x-2}{3} + \sqrt{\frac{(x-2)^2-9}{9}}\right| + C \\
&= \sqrt{(x-2)^2-9} + 2\ln\left|\frac{1}{3}\left(x-2+\sqrt{(x-2)^2-9}\right)\right| + C \\
&= \sqrt{x^2-4x-5} + 2\ln\frac{1}{3} + 2\ln\left|x-2+\sqrt{x^2-4x-5}\right| + C \\
&= \sqrt{x^2-4x-5} + 2\ln\left|x-2+\sqrt{x^2-4x-5}\right| + C'
\end{aligned}
$$

with a new constant $C' = C - 2\ln 3$.

(iii) Evaluate the following integral

$$\int \sqrt{9-4x^2}\,\mathrm{d}x.$$

Solution in video

(iv) Evaluate the definite integral

$$\int_0^2 \frac{\mathrm{d}x}{(x^2+2x+4)^{3/2}}\,.$$

Solution in video