15 Logarithms and Exponentials

15.1 (a) $\ln 16 = \ln 2^4 = 4\ln 2$

(b) $\ln 12 = \ln(2^2 \cdot 3) = 2\ln 2 + \ln 3$

(c) $\ln 36 = \ln(2^2 \cdot 3^2) = 2\ln 2 + 2\ln 3$

(d) $\ln(2\sqrt{2}) = \ln\left(2^{3/2}\right) = \frac{3}{2}\ln 2$

(e) $\ln\left(\frac{9}{8}\right) = \ln\left(\frac{3^2}{2^3}\right) = 2\ln 3 - 3\ln 2$

(f) $\ln\sqrt{13.5} = \ln\sqrt{\frac{27}{2}} = \ln\left(\frac{3^{3/2}}{2^{1/2}}\right) = \frac{3}{2}\ln 3 - \frac{1}{2}\ln 2.$

15.2 (a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x^2+2x)\right) = \frac{2x+2}{x^2+2x}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(-\ln(\cos x)) = -\frac{(-\sin x))}{\cos x} = \tan x$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x[\sin(\ln x)+\cos(\ln x)]) = [\sin(\ln x)+\cos(\ln x)]+x\left[\frac{1}{x}\cos(\ln x) - \frac{1}{x}\sin(\ln x)\right]$

$= \sin(\ln x) + \cos(\ln x) + \cos(\ln x) - \sin(\ln x) = 2\cos(\ln x)$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(\,x\ln x) = \ln x + x\cdot\frac{1}{x} = 1 + \ln x$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln x - \ln\sqrt{1+x^2} - \frac{1}{x}\arctan x\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln x - \frac{1}{2}\ln(1+x^2) - \frac{1}{x}\arctan x\right\}$

$= \frac{1}{x} - \frac{2x}{2(1+x^2)} + \frac{1}{x^2}\arctan x - \frac{1}{x}\cdot\frac{1}{1+x^2}$

$= \frac{1}{x^2}\arctan x + \frac{1+x^2-x^2-1}{x(1+x^2)} = \frac{1}{x^2}\arctan x$

(f) $\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tan x)] = \frac{1}{\tan x}\cdot\sec^2 x = \frac{1}{\sin x\cos x}$

15.3 (a) $\frac{\mathrm{d}}{\mathrm{d}x}\ln[(5x+1)^3] = 3\frac{\mathrm{d}}{\mathrm{d}x}\ln(5x+1) = \frac{15}{5x+1}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\ln[(3x^3+1)^{1/3}] = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}\ln(3x^3+1) = \frac{1}{3}\frac{9x^2}{3x^3+1} = \frac{3x^2}{3x^3+1}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\ln[(x-1)(x-2)(x-3)] = \frac{\mathrm{d}}{\mathrm{d}x}\{\ln|x-1| + \ln|x-2| + \ln|x-3|\}$

$= \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{x+4}{x^2-7}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\{\ln|x+4| - \ln|x^2-7|\} = \frac{1}{x+4} - \frac{2x}{x^2-7}$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{(2x+1)^{1/3}}{(3x-2)^{1/4}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\ln(2x+1) - \frac{1}{4}\ln(3x-2)\right) = \frac{2}{3(2x+1)} - \frac{3}{4(3x-2)}$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ \ln\left(\dfrac{x\sqrt{2x^2+3}}{(x-1)\sqrt{1-3x^2}}\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}\ \left\{\ln|x| + \dfrac{1}{2}\ \ln(2x^2+3) - \ln|x-1| - \dfrac{1}{2}\ \ln(1-3x^2)\right\}$

$= \dfrac{1}{x} + \dfrac{2x}{2x^2+3} - \dfrac{1}{x-1} + \dfrac{3x}{1-3x^2}$

15.4 (a)

$$\begin{aligned}
f(x) &= \frac{(x-1)^3(x+2)^2}{x+1} \\
\Longrightarrow \ln f(x) &= \ln\left(\frac{(x-1)^3(x+2)^2}{x+1}\right) = 3\ln|x-1| + 2\ln|x+2| - \ln|x+1| \\
\Longrightarrow \frac{1}{f(x)}\frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \frac{3}{x-1} + \frac{2}{x+2} - \frac{1}{x+1} \\
\Longrightarrow \frac{\mathrm{d}f}{\mathrm{d}x} &= f\left(\frac{3}{x-1} + \frac{2}{x+2} - \frac{1}{x+1}\right) \\
&= \frac{(x-1)^3(x+2)^2}{x+1}\left(\frac{3}{x-1} + \frac{2}{x+2} - \frac{1}{x+1}\right).
\end{aligned}$$

(b)

$$\begin{aligned}
g(x) &= \frac{(3x-2)^{\frac{1}{2}}(x+1)^3}{1+2x} \\
\Longrightarrow \ln g(x) &= \ln\left(\frac{(3x-2)^{\frac{1}{2}}(x+1)^3}{1+2x}\right) = \frac{1}{2}\ln(3x-2) + 3\ln|x+1| - \ln|1+2x| \\
\Longrightarrow \frac{1}{g(x)}\frac{\mathrm{d}g(x)}{\mathrm{d}x} &= \frac{3}{2(3x-2)} + \frac{3}{x+1} - \frac{1}{1+2x} \\
\Longrightarrow \frac{\mathrm{d}g(x)}{\mathrm{d}x} &= g(x)\left(\frac{3}{2(3x-2)} + \frac{3}{x+1} - \frac{1}{1+2x}\right) \\
&= \frac{(3x-2)^{\frac{1}{2}}(x+1)^3}{1+2x}\left(\frac{3}{2(3x-2)} + \frac{3}{x+1} - \frac{1}{1+2x}\right)
\end{aligned}$$

(c)

$$\begin{aligned}
h(x) &= \frac{(x+2)^{\frac{1}{2}}}{(3x^2+1)^{\frac{3}{2}}} \\
\Longrightarrow \ln h(x) &= \ln\left(\frac{(x+2)^{\frac{1}{2}}}{(3x^2+1)^{\frac{3}{2}}}\right) = \frac{1}{2}\ln(x+2) - \frac{3}{2}\ln(3x^2+1) \\
\Longrightarrow \frac{1}{h(x)}\frac{\mathrm{d}h(x)}{\mathrm{d}x} &= \frac{1}{2(x+2)} - \frac{9x}{3x^2+1} \\
\Longrightarrow \frac{\mathrm{d}h(x)}{\mathrm{d}x} &= h(x)\left(\frac{1}{2(x+2)} - \frac{9x}{3x^2+1}\right) \\
&= \frac{(x+2)^{\frac{1}{2}}}{(3x^2+1)^{\frac{3}{2}}}\left(\frac{1}{2(x+2)} - \frac{9x}{3x^2+1}\right)
\end{aligned}$$

15.5 (a) i. Direct approach: $x^x = e^{x\ln x}$, and so $\dfrac{\mathrm{d}}{\mathrm{d}x}x^x = e^{x\ln x}\cdot(\ln x+1) = (\ln x+1)x^x$.

ii. Using logarithmic differentiation with $f(x) = x^x$: $\ln f(x) = x\ln x \implies \dfrac{1}{f(x)}f'(x) = \ln x + \dfrac{x}{x} = 1+\ln x \implies f'(x) = f(x)(1+\ln x) = x^x(1+\ln x)$.

(b) $x^{x^x} = e^{x^x\ln x}$, so $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{x^x} = e^{x^x\ln x}(\dfrac{\mathrm{d}}{\mathrm{d}x}[x^x\ln x]) = x^{x^x}((\ln x+1)x^x\ln x + x^{x-1})$ using the result from part (a).

(c) $(x^x)^x = x^{x^2} = e^{x^2\ln x}$, so $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^x)^x = e^{x^2\ln x}(2x\ln x + x) = (2\ln x+1)x^{x^2+1}$.

(d) From part (a) we have $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}x^x = \dfrac{\mathrm{d}}{\mathrm{d}x}x^x = \dfrac{\mathrm{d}}{\mathrm{d}x}(1+\ln x)x^x = [\dfrac{1}{x} + (1+\ln x)^2]x^x$.

15.6 (a) $\ln(e^{3x}) = 3x$

(b) $\ln\left(\dfrac{1}{e^x}\right) = \ln(e^{-x}) = -x$

(c) $e^{\ln x+\ln y} = e^{\ln x}\,e^{\ln y} = xy$

(d) $\ln(x^2\,e^{2x}) = \ln(x^2) + \ln(e^{2x}) = 2\ln x + 2x$

(e) $\ln(e^{\ln(e^x)}) = \ln(e^x) = x$

15.7 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2\,e^x) = 2xe^x + x^2\,e^x = (x^2+2x)e^x$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,e^{\sin x} = \cos x\,e^{\sin x}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,e^{1+\tan 2x} = 2\sec^2 2x\,e^{1+\tan 2x}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\arctan(e^x) = \dfrac{e^x}{1+e^{2x}}$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,e^{x\sin x} = (\sin x + x\cos x)e^{x\sin x}$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,e^{e^x} = e^x e^{e^x}$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,e^{e^{e^x}} = e^x e^{e^x} e^{e^{e^x}}$

15.8 All these derivatives can be found by writing expressions of the form a^x as $e^{x\ln a}$.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^2a^x = 2xa^x + x^2\ln a\,a^x = (2x + x^2\ln a)a^x$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}10^{\cos x} = 10^{\cos x}\cdot(-\ln 10\,\sin x) = -\ln 10\,\sin x\,10^{\cos x}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}2^{1+\ln 2x} = 2^{1+\ln 2x}\cdot\ln 2\,\dfrac{1}{x} = \dfrac{\ln 2}{x}2^{1+\ln 2x}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\arctan a^x = \dfrac{1}{1+(a^x)^2}\cdot\ln a\,a^x = \dfrac{\ln a\,a^x}{1+a^{2x}}$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}3^{x\cos x} = 3^{x\cos x}\cdot\ln 3\,(\cos x - x\sin x)$

(f) i. Direct approach: $\frac{\mathrm{d}}{\mathrm{d}x}a^{a^x} = a^{a^x} \cdot \ln a \cdot \frac{\mathrm{d}}{\mathrm{d}x}a^x = a^{a^x} \cdot \ln a \cdot \ln a\, a^x = (\ln a)^2\, a^x\, a^{a^x}$

ii. Using logarithmic differentiation: $\frac{\mathrm{d}}{\mathrm{d}x}a^{a^x} = a^{a^x}\frac{\mathrm{d}}{\mathrm{d}x}\ln(a^{a^x})$. With $\ln(a^{a^x}) = a^x \ln a$ we find $\frac{\mathrm{d}}{\mathrm{d}x}a^{a^x} = (\ln a)^2 a^x a^{a^x}$.

(g) i. Direct approach: $\frac{\mathrm{d}}{\mathrm{d}a}a^{a^x} = \frac{\mathrm{d}}{\mathrm{d}a}e^{\ln a\, a^x} = e^{\ln a\, a^x} \cdot \frac{\mathrm{d}}{\mathrm{d}a}(\ln a\, a^x) = a^{a^x} \cdot (\frac{1}{a}a^x + \ln a\, x a^{x-1}) = (1 + x\ln a)a^{x-1}a^{a^x}$

ii. Using logarithmic differentiation: $\frac{\mathrm{d}}{\mathrm{d}a}a^{a^x} = a^{a^x}\frac{\mathrm{d}}{\mathrm{d}a}\ln(a^{a^x})$. With $\ln(a^{a^x}) = a^x \ln a$ we find $\frac{\mathrm{d}}{\mathrm{d}a}a^{a^x} = (xa^{x-1}\ln a + a^{x-1})a^{a^x} = (1 + x\ln a)a^{x-1}a^{a^x}$.

(h) i. Direct approach: $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)^{\ln x} = \frac{\mathrm{d}}{\mathrm{d}x}e^{\ln(\ln x)\,\ln x} = (\ln x)^{\ln x} \cdot (\frac{1}{x\ln x} \cdot \ln x + \ln(\ln x) \cdot \frac{1}{x}) = \frac{1 + \ln(\ln x)}{x}(\ln x)^{\ln x}$

ii. Using logarithmic differentiation: $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)^{\ln x} = (\ln x)^{\ln x}\frac{\mathrm{d}}{\mathrm{d}x}\ln((\ln x)^{\ln x})$. With $\ln((\ln x)^{\ln x}) = \ln x \ln(\ln(x))$ we have $\frac{\mathrm{d}}{\mathrm{d}x}\ln((\ln x)^{\ln x}) = \frac{1}{x}\ln(\ln x) + \ln x\frac{1}{x\ln x}$ and so $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)^{\ln x} = \frac{1}{x}(\ln(\ln x) + 1)(\ln x)^{\ln x}$.

15.9

- $e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(e^{-x^2}) = e^{x^2}(-2x\, e^{-x^2}) = -2x$
- $e^{x^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{-x^2}) = e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(-2x\, e^{-x^2}) = e^{x^2}(4x^2\, e^{-x^2} - 2e^{-x^2}) = 4x^2 - 2$
- $$\begin{aligned} e^{x^2}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(e^{-x^2}) &= e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(4x^2\, e^{-x^2} - 2e^{-x^2}) \\ &= e^{x^2}(8x\, e^{-x^2} - 8x^3\, e^{-x^2} + 4x\, e^{-x^2}) = -8x^3 + 12x \end{aligned}$$

15.10 (a) Differentiating $\int_0^x f = e^x$ shows that $f(x) = e^x$. But this f does not work since $\int_0^x e^t\,\mathrm{d}t = e^x - e^0 = e^x - 1 \neq e^x$. Hence there is no such function f. This can also be seen directly by setting $x = 0$: $0 = \int_0^0 f \neq 1 = e^0$.

(b) Differentiating gives (see 14.2 (i)) $2xf(x^2) = -4xe^{2x^2}$. This suggests that $f(t) = -2e^{2t}$ with $t \geq 0$. This works: $\int_0^{x^2}(-2e^{2t})\,\mathrm{d}t = [-e^{2t}]_0^{x^2} = 1 - e^{2x^2}$.

15.11

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}F_{A,B}(x) &= Ae^x\cos x - Ae^x\sin x + Be^x\sin x + Be^x\cos x \\ &= (A+B)e^x\cos x + (B-A)e^x\sin x \\ &= F_{A+B,B-A}(x). \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dx^2}F_{A,B}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[F_{A+B,B-A}(x)\right] \\ &= F_{A+B+B-A,B-A-(A+B)}(x) \\ &= F_{2B,-2A}(x). \end{aligned}$$

15.12 All these derivatives can be found by using $\log_a x = \dfrac{\ln x}{\ln a}$.

(a) $f'(x) = \dfrac{1}{\ln 10} \cdot \dfrac{\cos x}{\sin x} = \dfrac{\cot x}{\ln 10}$

(b) $g'(x) = \dfrac{1}{\ln 2}$

(c) $h'(x) = \cos(\log_a x) \cdot \dfrac{1}{x \ln a}$

(d) $i'(x) = \dfrac{x \cot x - \ln(\sin x)}{x^2}$

(e) $j'(x) = -\dfrac{\ln a}{x(\ln x)^2}$

(f) $k'(a) = \dfrac{1}{a \ln x}$

15.13 First note that $a^x = e^{x \ln a}$. Since $\ln a < 0$ for $0 < a < 1$, $x \ln a \to -\infty$ as $x \to \infty$. Thus $\lim\limits_{x\to\infty} a^x = \lim\limits_{y\to-\infty} e^y = 0$ with $y = x \ln a$.

15.14 (a) Assume that $\log_3 2$ is rational. As $\log_3 2 > 0$, there would then exist $r, s \in \mathbb{N}$, $r, s \neq 0$, such that $\log_3 2 = \dfrac{r}{s}$. This means that

$$3^{r/s} = 2 \iff 3^r = 2^s.$$

The left hand side is an odd integer and the right hand side is an even integer, which is a contradiction.

(b) As $q > 1$, $\log_p q > 0$. Assume that $\log_p q = \dfrac{r}{s}$ with $r, s \in \mathbb{N}$ and $r, s \neq 0$. Then

$$p^{r/s} = q \iff p^r = q^s.$$

But this is impossible because of the uniqueness of the prime factorisation.

15.15 (a) $\sinh(x)\cosh(y) + \cosh(x)\ \sinh(y)$

$$= \frac{1}{4}\left[(e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})\right]$$

$$= \frac{1}{4}\left[e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}\right]$$

$$= \frac{1}{4}\left(2e^{x+y} - 2e^{-x-y}\right) = \frac{1}{2}\left(e^{x+y} - e^{-(x+y)}\right) = \sinh(x+y)$$

(b) $\cosh(x)\ \cosh(y) + \sinh(x)\ \sinh(y)$

$$= \frac{1}{4}\left[(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})\right]$$

$$= \frac{1}{4}\left[e^{x+y} + e^{x-y} + e^{-x+y} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}\right]$$

$$= \frac{1}{2}\left[e^{x+y} + e^{-x-y}\right] = \cosh(x+y)$$

Analogous results:

$$\begin{aligned}\sinh(x-y) = \sinh[x+(-y)] &= \sinh(x)\cosh(-y) + \cosh(x)\sinh(-y)\\ &= \sinh(x)\cosh(y) - \cosh(x)\sinh(y)\end{aligned}$$

$$\begin{aligned}\cosh(x-y) = \cosh[x+(-y)] &= \cosh(x)\cosh(-y) + \sinh(x)\sinh(-y)\\ &= \cosh(x)\cosh(y) - \sinh(x)\sinh(y)\end{aligned}$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$
$$\sinh(2x) = 2\sinh(x)\cosh(x).$$

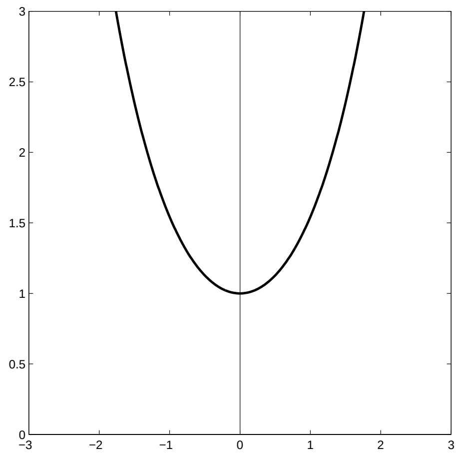
15.16 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ \cosh(\sqrt{1-x^2}) = -\dfrac{x}{\sqrt{1-x^2}}\ \sinh(\sqrt{1-x^2})$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ \{x^2\sinh(3x^5)\} = 2x\sinh(3x^5) + 15x^6\cosh(3x^5)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ \ln(\tanh x) = \dfrac{\mathrm{sech}^2 x}{\tanh x} = \dfrac{1}{\cosh^2 x}\ \dfrac{\cosh x}{\sinh x} = \dfrac{1}{\sinh x\ \cosh x}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ \{\ln[\sinh(x^3+3x)]\} = (3x^2+3)\ \dfrac{\cosh(x^3+3x)}{\sinh(x^3+3x)} = 3(x^2+1)\coth(x^3+3x)$

15.17 (a) dom (cosh) $= \mathbb{R}$ range (cosh) $= [1,\infty)$.



(b) For each $y \in [1,\infty)$ there is a unique $x \in [0,\infty)$ such that $y = \cosh(x)$: this is not true for $x \in \mathbb{R}$. Hence there is an inverse function for the restricted domain only.

(c) Let $z = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$ so $x = \cosh^{-1}(z)$. Then

$$\cosh^2 x - \sinh^2 x = 1 \iff \sinh^2 x = \cosh^2 x - 1 = z^2 - 1 \iff \sinh x = \sqrt{z^2-1}.$$

But

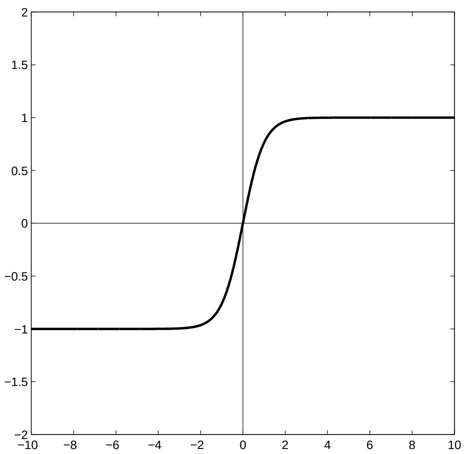
$$\cosh x + \sinh x = e^x \iff z + \sqrt{z^2-1} = e^x$$

so (taking the natural logarithm of each side) $x = \ln(z + \sqrt{z^2 - 1})$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh^{-1} x) = \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x + \sqrt{x^2 - 1})) = \frac{1}{x + \sqrt{x^2 - 1}}.\frac{\mathrm{d}}{\mathrm{d}x}\left(x + \sqrt{x^2 - 1}\right)$$

$$= \frac{1}{x + \sqrt{x^2 - 1}}.\left(1 + \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}.2x\right) = \frac{1}{\sqrt{x^2 - 1}}.$$

15.18 (a) dom (tanh) $= \mathbb{R}$ range (tanh) $= (-1, 1)$.



(b) For each $y \in (-1, 1)$ there is a unique $x \in \mathbb{R}$ such that $y = \tanh(x)$. Hence there is an inverse function.
Let $y = \tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ then

$$e^{2x}y + y = e^{2x} - 1 \implies e^{2x}(y - 1) = -(y + 1) \to e^{2x} = \frac{1 + y}{1 - y}$$

and

$$x = \tanh^{-1}(y) = \frac{1}{2}\ln\left(\frac{1 + y}{1 - y}\right), \; y \in (-1, 1).$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\{\ln(1 + x) - \ln(1 - x)\}$$

$$= \frac{1}{2}\left\{\frac{1}{1 + x} + \frac{1}{1 - x}\right\} = \frac{1}{1 - x^2}.$$

15.19 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1}(2x) = \dfrac{1}{\sqrt{(2x)^2 - 1}}.2 = \dfrac{2}{\sqrt{4x^2 - 1}}.$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(\sin x) = \dfrac{1}{1 - \sin^2 x}.\cos x = \dfrac{1}{\cos x} = \sec x.$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}(\sqrt{x}) = \dfrac{1}{\sqrt{(\sqrt{x})^2 + 1}}.\dfrac{1}{2}\dfrac{1}{\sqrt{x}} = \dfrac{1}{2\sqrt{x(x + 1)}}.$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(e^{5x^2}) = \dfrac{1}{1 - (e^{5x^2})^2}.e^{5x^2}.10x = \dfrac{10xe^{5x^2}}{1 - e^{10x^2}}.$

15.20

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\tanh^{-1}\left[\tan\frac{x}{2}\right]\right\}=\frac{1}{1-\tan^2\left(\frac{x}{2}\right)}.\sec^2\left(\frac{x}{2}\right).\frac{1}{2}=\frac{1}{2}.\frac{\cos^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}.\frac{1}{\cos^2\left(\frac{x}{2}\right)}$$

$$=\frac{1}{2\cos x}=\frac{1}{2}\sec x.$$

15.21

$$y'=2\sinh^{-1}x.\frac{1}{\sqrt{x^2+1}}$$

$$y''=\frac{2}{\sqrt{x^2+1}}.\frac{1}{\sqrt{x^2+1}}+2\sinh^{-1}x.-\frac{1}{2}\frac{1}{(x^2+1)^{\frac{3}{2}}}.2x=\frac{2}{x^2+1}-\frac{2x\sinh^{-1}x}{(x^2+1)^{\frac{3}{2}}}$$

So

$$(1+x^2)y''+xy'=2-\frac{2x\sinh^{-1}x}{\sqrt{x^2+1}}+\frac{2x\sinh^{-1}x}{\sqrt{x^2+1}}=2.$$