4 Functions

4.1 Defining a function

The concept of a **function** is one of the most important ideas in mathematics.

Definition 4.1 *Given two sets A and B, a function f from A to B is a rule which assigns to each member of A a single (unique) member of B.*

The idea of a function is very general, and the "rule" in Definition 4.1 can be of many different types. Functions encompass not just rules which can be expressed using algebraic formulas, or rules which apply the same condition to all numbers in A, or even rules which can be applied in practice: all sorts of rules can define functions. Examples might be "the rule which assigns to each number in A the square of that number" or "the rule which assigns to each number $a \in A$ the number 0 if a is rational, and the number 1 if a is irrational".

It is important to remember that a function is defined not only by the rule, but also by the sets A and B. As we will see later, changing the set of possible input values to a function can significantly change its properties. In this class, A and B will be sets of real numbers. If no domain is specified, we will assume that the domain of the function is the set of all real numbers, $\mathbb{R}$.

In terms of notation, the standard practice is to represent a function (rule) by a letter, often f or g, and the numbers in sets A and B by different letters, such as x or a. For example, we can write

$$f(x) = x^2 \text{ for all } x \in \mathbb{R}$$

and

$$g(a) = \begin{cases} 0, & a \text{ is rational,} \\ 1, & a \text{ is irrational.} \end{cases}$$

We read $f(x)$ as "f of x" or "the value of f at x". The number in set A (x or a in our examples), which is the input to the function, is called the **argument** or the **independent variable**. The number in set B ($f(x)$ or $g(a)$ here), which is output by the function, is called the **value** of the function. If the output is given a new name by writing e.g. $y = f(x)$, y is called the **dependent variable**.

It is very important to appreciate the difference between a function and its value. In our examples, f and g are functions, and $f(x)$ and $g(a)$ are values of the functions (usually numbers for us).

Functions are also sometimes referred to as **mappings**, with the function f from A to B defined by a statement like $f : A \to B,\ x \mapsto f(x)$ ("f maps x in A to $f(x)$ in B"). With this notation, the first of our example functions above could be written as $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$. This notation helps to emphasise the fact that f is a function, not a number. Note that the expressions

$$f(x) = x^2, \qquad f(t) = t^2, \qquad f(\theta) = \theta^2 \quad \text{etc.}$$

and

$$f : x \mapsto x^2, \qquad f : t \mapsto t^2, \qquad f : \theta \mapsto \theta^2 \quad \text{etc.}$$

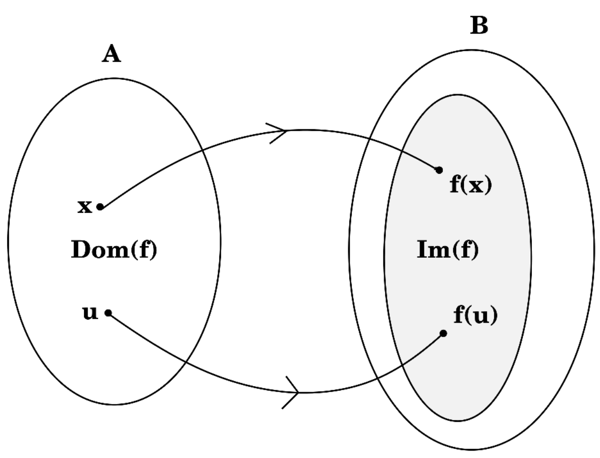
all represent the same function, that is, "the rule which assigns to each real number the square of that number".

Examples 4A

4.2 Domains, images and graphs

Let f be a function from A to B.

- The set A is called the **domain** (or **initial set**) of f, written $\mathrm{Dom}(f)$.
- The set B is called the **codomain** (or **final set**) of f, written $\mathrm{Cod}(f)$.
- The set of numbers $y \in B$ such that $y = f(x)$ for some $x \in A$ is called the **image** (or **range**) of f and is denoted by $\mathrm{Im}(f)$ or $f(A)$. Clearly $f(A) \subseteq B$.



Definition 4.2 *The largest set of numbers for which a function f can be defined is called its **natural domain**.*

Examples 4B

Recall that a function f from A to B assigns **exactly one** member of B to **every** member of A.

Definition 4.3 *A function f from A to B is said to be **injective** (or **one-to-one**) if and only if no two distinct points in $Dom(f)$ have the same image under f, that is,*

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \qquad x_1, x_2 \in Dom(f).$$

In other words, an injective function (or **injection**) never maps two elements of the domain to the same element of the image.

Definition 4.4 *A function f from A to B is said to be **surjective** (or **onto**) if $Im(f) = B$.*

In other words, for a surjective function (or **surjection**) every element of B can be obtained by applying f to at least one element of A.

Definition 4.5 *A function f from A to B is said to be **bijective** if it is both injective and surjective.*

A bijective function (or **bijection**) establishes a **one-to-one correspondence** between A and B.

Definition 4.6 *The **graph** of f is the set of all points $P(x, f(x))$ in the plane, where $x \in Dom(f)$.*

A rough sketch can usually be made by plotting several points $P(x, f(x))$ and joining them with a curve. Note that, by definition, for any function f, there can only be one y-value (where $y = f(x)$) for each x-value. If f is injective, the reverse is also true: that is, there can be only one x-value associated with each y-value.

Examples 4C

4.3 The modulus function

One very important function in mathematics is the **modulus function**

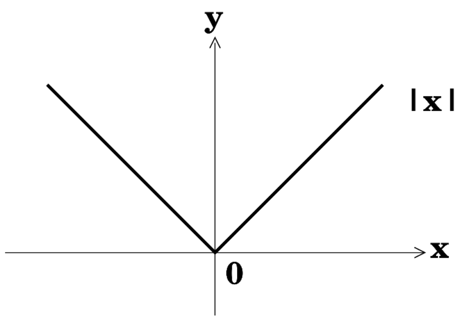
$$f : x \mapsto |x|.$$

Its domain is the set of all real numbers, and its image is the set of all **positive** real numbers. It is formally defined as follows (recall Definition 2.4):

Definition 4.7 *For any real number x, the **modulus function** is given by*

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

The graph of $|x|$ is



(that is, $y = x$ when $x \geq 0$ and $y = -x$ when $x < 0$).

We now prove some important properties of the modulus function.

Theorem 4.8 *For all $x \in \mathbb{R}$,*

$$(a)\ |x| = |-x|, \qquad (b)\ x \leq |x|, \qquad (c)\ -x \leq |x|.$$

Proof 4.8 *These results can be verified by checking the cases x positive and x negative separately.*

Theorem 4.9 *For all $x, y \in \mathbb{R}$, (a) $|xy| = |x|\,|y|$,* $\qquad$ *(b)* $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|} \qquad (y \neq 0).$

Proof 4.9 *We prove only part (a) here as the proof of (b) is similar. We have*

$$|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 .$$

On taking the square root of both sides, we obtain $|xy| = |x|\,|y|$.

Theorem 4.10 ***The Triangle Inequality***

$$|x + y| \le |x| + |y| \quad \textit{for} \quad \textit{all } x, y \in \mathbb{R}.$$

Proof 4.10 *We have*

$$|x + y|^2 = (x + y)^2 = x^2 + 2xy + y^2 = |x|^2 + 2xy + |y|^2.$$

Now using Theorem 4.8(b) and Theorem 4.9(a) we get

$$xy \le |xy| = |x||y|$$

so

$$|x + y|^2 \le |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$$

and taking the square root of both sides gives the required result.

Note that the triangle inequality can be extended to

$$|x_1 + x_2 + x_3 + \cdots + x_n| \le |x_1| + |x_2| + |x_3| + \cdots + |x_n| .$$

Examples 4D

Other examples of basic functions include polynomial, rational and trigonometric functions, all of which we will study in detail later.

4.4 Even and odd Functions

Definition 4.11 *The function f is **even** if $f(-x) = f(x)$ for all $x \in \mathrm{Dom}(f)$.*

For example, f is even if $f(x) = x^2$, $f(x) = \cos(x)$, $f(x) = |x|$, $f(x) = 1 + x^2 + x^4$, $f(x) = x\sin(x)$. The graph of an even function is symmetric about the y-axis.

Definition 4.12 *The function f is **odd** if $f(-x) = -f(x)$ for all $x \in \mathrm{Dom}(f)$.*

For example, f is odd if $f(x) = x$, $f(x) = x^3$, $f(x) = \sin(x)$, $f(x) = \tan(x)$, $f(x) = x^3 + x\cos(x)$. The graph of an odd function is symmetric about the origin.

Some functions are neither odd nor even. Examples of this type are $f(x) = 1 + x + x^2$ and $f(x) = \sin(x) + \cos(x)$.

Examples 4E

4.5 Combining functions

4.5.1 Algebraic combinations

The sum, difference, product and quotient of two functions f and g (with suitable definitions) are defined in a natural way in terms of the sum, difference, product and quotient of real numbers:

sum	$(f+g)(x) = f(x) + g(x)$	
difference	$(f-g)(x) = f(x) - g(x)$	
product	$(fg)(x) = f(x)g(x)$	
quotient	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	where $g(x) \neq 0$.

Note that any function $f(x)$ can be written in a unique way as

$$f(x) = f_e(x) + f_o(x)$$

where $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ is an **even** function and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ is an **odd** function.

Examples 4F

4.5.2 Composition of functions

We also often want to combine two functions, f and g, say, to obtain the value $f(g(x))$. In diagrammatic terms, functions are combined via

$$x \mapsto g(x) \mapsto f(g(x)).$$

We can also write $f(g(x))$ as $(f \circ g)(x)$, or 'f after g' of x. The domain of $f(g(x))$ is all or part of the domain of g: the values of $g(x)$ must lie in the domain of f.

Note that three or more functions can be composed in a similar way.

Examples 4G

4.6 Linear functions

One of the most important concepts in all of mathematics and its applications is the idea of **linearity**. Here we introduce the idea of **linear** functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 4.13 *A function $f : \mathbb{R} \mapsto \mathbb{R}$ is **linear** if it satisfies*

$$\begin{array}{lrcl} \text{(i)} & f(x+y) & = & f(x)+f(y); \\ \text{(ii)} & f(cx) & = & cf(x) \quad \forall c \in \mathbb{R}. \end{array}$$

In general, both of these properties are needed, although the second can in fact be proved from the first.

Using property (ii), we see that every linear function on $\mathbb{R}$ satisfies $f(0) = 0$: putting $c = 0$ in property (ii) gives

$$f(0) = f(0 \cdot x) = 0f(x) = 0.$$

In fact, every linear function on $\mathbb{R}$ is of the form

$$\boxed{f(x) = ax, \qquad a = f(1) \in \mathbb{R}.}$$

This can be seen as follows:

$$f(x) = f(x \cdot 1) = xf(1) \quad \text{(from property (ii))} = ax \quad (\text{as } a = f(1)).$$

Very important note: this means that

$$\boxed{\textbf{no other functions satisfy } f(x+y) = f(x) + f(y) \quad \forall x, y, \in \mathbb{R}.}$$

In particular, normally

$$\begin{array}{lrcl} f(x) = \sqrt{x}: & \sqrt{(x+y)} & \neq & \sqrt{x} + \sqrt{y} \\ f(x) = x^2: & (x+y)^2 & \neq & x^2 + y^2 \\ f(x) = \dfrac{1}{x}: & \dfrac{1}{x+y} & \neq & \dfrac{1}{x} + \dfrac{1}{y} \\ f(x) = \sin(x): & \sin(x+y) & \neq & \sin(x) + \sin(y) \\ f(x) = \ln(x): & \ln(x+y) & \neq & \ln(x) + \ln(y) \\ f(x) = \exp(x): & \exp(x+y) & \neq & \exp(x) + \exp(y) \end{array}$$

4.7 Inverse functions

An **inverse function** is a function which reverses or cancels out the effect of another function. For example, suppose we have a function $f : A \to B$ such that $f(x) = y$. Then the inverse function, which we will denote by $f^{-1} : B \to A$, is the function such that $f^{-1}(y) = x$. Note that we can do this if and only if there is a unique relationship between x and y, that is, if and only if f is a **bijection**.

Note that the equations involving f and f^{-1} above have the roles of x and y reversed, so the graph of f^{-1} can be obtained from the graph of f by swapping the x and y axes. This is equivalent to reflecting the graph across the line $y = x$.

If f is an invertible function from A to B then

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x \qquad \text{for every } x \in A.$$

Examples 4H

4.8 Implicit and parametric functions

The examples we have seen so far have involved so-called **explicit** functions of x where the number $y = f(x)$ is completely defined in terms of x, e.g. $y = 17x + 3$ or $y = \sqrt{25x^2 - 1}$. This is not always the case.

Implicit functions
Sometimes the relationship between x and y is more involved and it may not be possible to separate y completely on the left-hand side, e.g. $xy + \cos y = 0$ or $x^2 + 3xy - y^3 = 5$. In this case, y is said to be an **implicit** function of x, with the exact relationship $y = f(x)$ implied by the defining equation.

Parametric functions
Sometimes it is more convenient to represent a function by expressing x and y separately in terms of a third variable, e.g. $x = \sin t$, $y = \cos 2t$ or $x = 3(1 - \cos\theta)$, $y = 3(\theta - \sin\theta)$. The third variable ($t$ or θ here) is called a **parameter** and the expressions for x and y are the **parametric equations** which describe a function $y = f(x)$, as any value which we give to the parameter will fix a pair of values $(x, y = f(x))$. To graph parametric functions, we can produce pairs of points by varying the value of the parameter in a specified range.

Examples 4I