University of Strathclyde, Department of Mathematics and Statistics

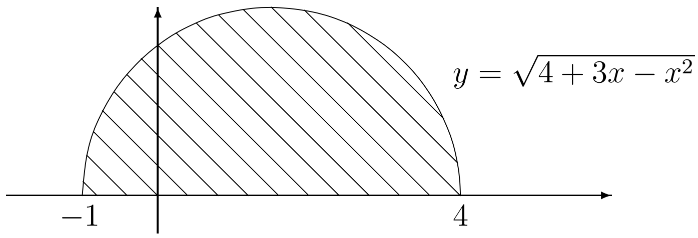
MM102 Applications of Calculus
Exercises for Week 3
Solutions

Q1. Sketch the finite region bounded by following curves and the x-axis. Hence find the volume generated when this region is rotated through 360° about the $\boldsymbol{x}$**-axis**:

1(a) $y = \sqrt{4 + 3x - x^2}$

Solution:
The curve $y = \sqrt{4 + 3x - x^2}$ intersects the x-axis when

$$4 + 3x - x^2 = 0 \quad \iff \quad x = -1 \quad \text{or} \quad x = 4.$$



Let $f(x) = \sqrt{4 + 3x - x^2}$. Then the volume is equal to

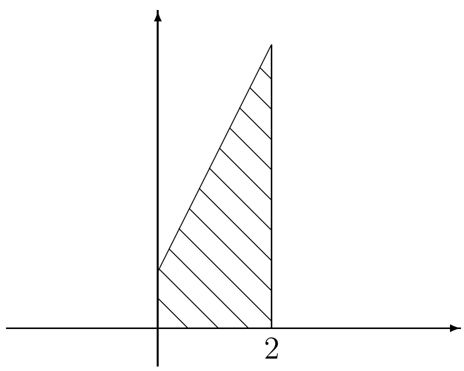
$$V = \pi \int_{-1}^{4} \big(f(x)\big)^2 \,\mathrm{d}x = \pi \int_{-1}^{4} \big(4 + 3x - x^2\big)\mathrm{d}x$$

$$= \pi \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{4}$$

$$= \pi\left(16 + \frac{3}{2} \times 16 - \frac{1}{3} \times 64 - \left(-4 + \frac{3}{2} + \frac{1}{3}\right)\right) = \boxed{\frac{125}{6}\pi}$$

1(b) $y = 2x + 1, \qquad x = 0, \qquad x = 2$

Solution:



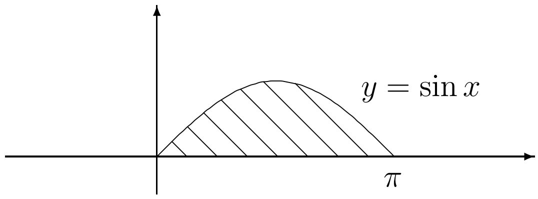
Let $f(x) = 2x + 1$. Then the volume is equal to

$$V = \pi \int_0^2 \big(f(x)\big)^2 \, \mathrm{d}x = \pi \int_0^2 (2x+1)^2 \mathrm{d}x = \pi \int_0^2 (4x^2 + 4x + 1) \mathrm{d}x$$

$$= \pi \left[\frac{4}{3}x^3 + 2x^2 + x \right]_0^2 = \pi \left(\frac{32}{3} + 8 + 2 \right) = \boxed{\frac{62\pi}{3}}$$

1(c) $y = \sin x, \qquad x = 0, \qquad x = \pi$

Solution:



Let $f(x) = \sin x$. Then the volume is equal to

$$V = \pi \int_0^\pi \big(f(x)\big)^2 \, \mathrm{d}x = \pi \int_0^\pi \sin^2 x \, \mathrm{d}x = \pi \int_0^\pi \frac{1}{2}\big(1 - \cos(2x)\big) \mathrm{d}x$$

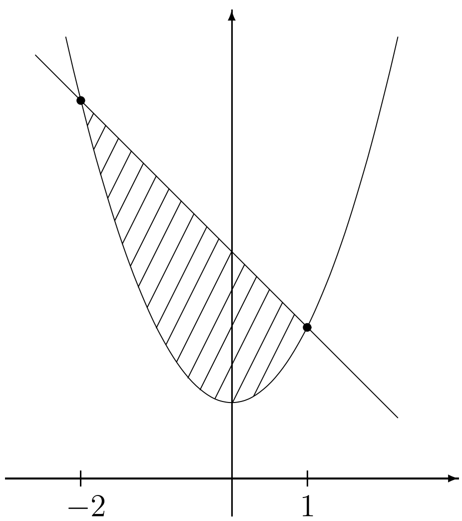
$$= \frac{\pi}{2} \left[x - \frac{1}{2}\sin(2x) \right]_0^\pi = \frac{\pi}{2} \left(\pi - \frac{1}{2}\sin(2\pi) - \left(0 - \frac{1}{2}\sin 0\right) \right) = \boxed{\frac{\pi^2}{2}}$$

Q2. Find the points of intersection of the following two curves; hence find the volume generated when this region is rotated through 360° about the ***x*-axis**:

2(a) $y = x^2 + 1 \quad$ and $\quad y = 3 - x$

Solution:
The graphs intersect when

$$x^2 + 1 = 3 - x \quad \Longleftrightarrow \quad x^2 + x - 2 = 0 \quad \Longleftrightarrow \quad x = 1 \quad \text{or} \quad x = -2.$$



On the interval $(-2, 1)$, the graph of $y = 3 - x$ lies above the graph of $y = x^2 + 1$. We set $f_1(x) = 3 - x$, $f_2(x) = x^2 + 1$. Hence the volume is equal to

$$V = \pi \int_{-2}^{1} \Big(\big(f_1(x)\big)^2 - \big(f_2(x)\big)^2 \Big) \mathrm{d}x = \pi \int_{-2}^{1} \Big((3-x)^2 - (x^2+1)^2 \Big) \mathrm{d}x$$

$$= \pi \int_{-2}^{1} \Big(9 - 6x + x^2 - \big(x^4 + 2x^2 + 1\big) \Big) \mathrm{d}x$$

$$= \pi \int_{-2}^{1} \big(-x^4 - x^2 - 6x + 8 \big) \mathrm{d}x = \pi \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 - 3x^2 + 8x \right]_{-2}^{1}$$

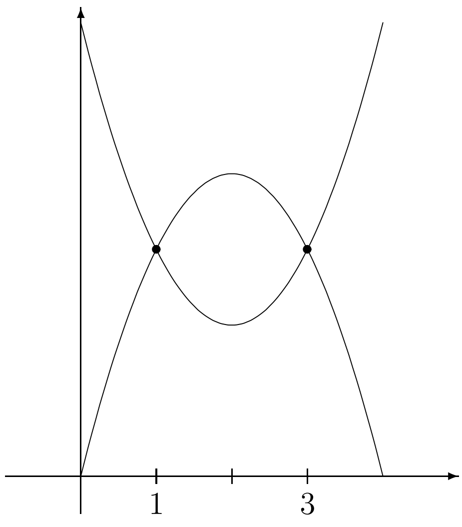
$$= \pi \left(-\frac{1}{5} - \frac{1}{3} - 3 + 8 - \left(\frac{32}{5} + \frac{8}{3} - 12 - 16 \right) \right) = \boxed{\frac{117}{5}\pi}$$

2(b) $y = x^2 - 4x + 6 \quad$ and $\quad y = 4x - x^2$

Solution:
The graphs intersect when

$$x^2 - 4x + 6 = 4x - x^2 \quad \Longleftrightarrow \quad x^2 - 4x + 3 = 0 \quad \Longleftrightarrow \quad x = 1 \quad \text{or} \quad x = 3.$$

In both cases the y-coordinate is $y = 3$.



On the interval $(1, 3)$ the graph of $y = 4x - x^2$ lies above the graph of $y = x^2 - 4x + 6$. We set $f_1(x) = 4x - x^2$, $f_2(x) = x^2 - 4x + 6$.

Hence the volume is equal to

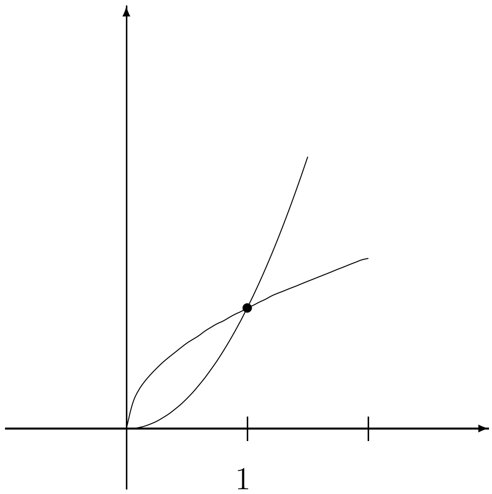
$$
\begin{aligned}
V &= \pi \int_1^3 \Big(\big(f_1(x)\big)^2 - \big(f_2(x)\big)^2 \Big) \mathrm{d}x \\
&= \pi \int_1^3 \Big((4x - x^2)^2 - (x^2 - 4x + 6)^2 \Big) \mathrm{d}x \\
&= \pi \int_1^3 \Big(16x^2 - 8x^3 + x^4 - \big(x^4 + 16x^2 + 36 - 8x^3 + 12x^2 - 48x \big) \Big) \mathrm{d}x \\
&= \pi \int_1^3 \big(-12x^2 + 48x - 36 \big) \mathrm{d}x = \pi \Big[-4x^3 + 24x^2 - 36x \Big]_1^3 \\
&= \pi \Big(-4 \times 27 + 24 \times 9 - 36 \times 3 - \big(-4 + 24 - 36 \big) \Big) = \boxed{16\pi}
\end{aligned}
$$

2(c) $y = x^2 \quad$ and $\quad y = \sqrt{x}$

Solution:
The graphs intersect when

$$x^2 = \sqrt{x} \quad \Longleftrightarrow \quad x^4 = x, \quad x \geq 0 \quad \Longleftrightarrow \quad x = 0 \quad \text{or} \quad x = 1.$$

When $x = 0$, then $y = 0$; when $x = 1$, then $y = 1$.



On the interval $(0, 1)$ the graph of $y = \sqrt{x}$ lies above the graph of $y = x^2$. Hence the volume is equal to

$$
\begin{aligned}
V &= \pi \int_0^1 \Big(\big(f_1(x)\big)^2 - \big(f_2(x)\big)^2 \Big) \mathrm{d}x = \pi \int_0^1 \Big(\big(\sqrt{x}\,\big)^2 - \big(x^2\big)^2 \Big) \mathrm{d}x \\
&= \pi \int_0^1 \big(x - x^4 \big) \mathrm{d}x = \pi \left[\frac{1}{2}x^2 - \frac{1}{5}x^5 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} - 0 \right) = \boxed{\frac{3\pi}{10}}
\end{aligned}
$$

2(d) $y = 2x + 3 \quad$ and $\quad y = x^2$

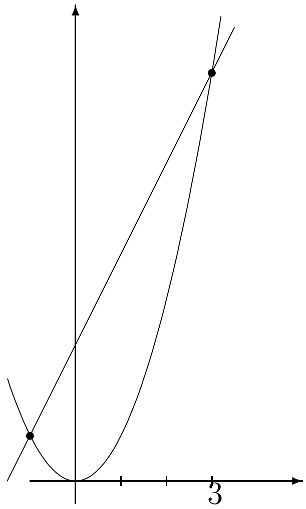
Solution:
The points of intersection are found by solving the equation

$$2x + 3 = x^2,$$

which gives

$$x^2 - 2x - 3 = 0 \quad \Longleftrightarrow \quad x = -1 \quad \text{or} \quad x = 3.$$

The corresponding y-coordinates are $y = 1$ and $y = 9$.



On the interval $(-1, 3)$, the curve $y = 2x + 3$ lies above the curve $y = x^2$. We set $f_1(x) = 2x + 3$, $f_2(x) = x^2$. Hence the volume is equal to

$$V = \pi \int_{-1}^{3} \Big(\big(f_1(x)\big)^2 - \big(f_2(x)\big)^2 \Big) \mathrm{d}x = \pi \int_{-1}^{3} \Big((2x+3)^2 - (x^2)^2 \Big) \mathrm{d}x$$

$$= \pi \int_{-1}^{3} \Big(4x^2 + 12x + 9 - x^4 \Big) \mathrm{d}x = \pi \left[\frac{4}{3}x^3 + 6x^2 + 9x - \frac{1}{5}x^5 \right]_{-1}^{3}$$

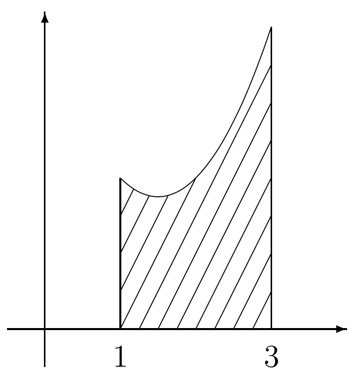
$$= \pi \Bigg[\frac{4}{3} \times 3^3 + 6 \times 3^2 + 9 \times 3 - \frac{1}{5} \times 3^5$$

$$- \left(\frac{4}{3} \times (-1)^3 + 6 \times (-1)^2 + 9 \times (-1) - \frac{1}{5} \times (-1)^5 \right) \Bigg] = \boxed{\frac{1088\pi}{15}}$$

Q3. Sketch the finite region bounded by the following curves. Hence find the volume generated when this region is rotated through 360° about the ***y*-axis**:

3(a) $y = x^2 - 3x + 4, \qquad y = 0, \qquad x = 1, \qquad x = 3$

Solution:



Set $f(x) = x^2 - 3x + 4$. Then the volume is equal to

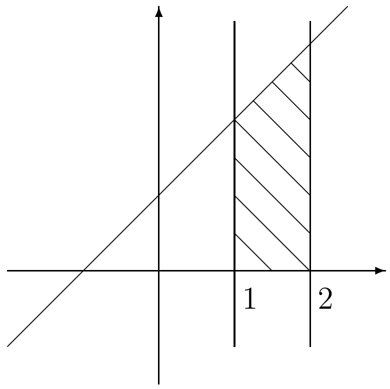
$$V = 2\pi \int_1^3 x f(x)\,\mathrm{d}x = 2\pi \int_1^3 x\big(x^2 - 3x + 4\big)\mathrm{d}x$$

$$= 2\pi \int_1^3 \big(x^3 - 3x^2 + 4x\big)\mathrm{d}x$$

$$= 2\pi \left[\frac{1}{4}x^4 - x^3 + 2x^2\right]_1^3 = 2\pi\left(\frac{81}{4} - 27 + 18 - \left(\frac{1}{4} - 1 + 2\right)\right) = \boxed{20\pi}$$

3(b) $y = x + 1, \qquad y = 0, \qquad x = 1, \qquad x = 2$

Solution:



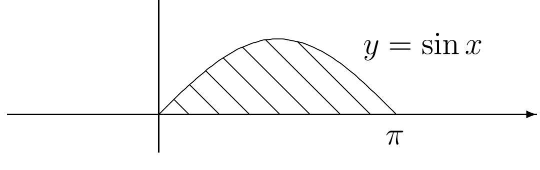
We set $f(x) = x + 1$. Then the volume is equal to

$$V = 2\pi \int_1^2 x f(x)\mathrm{d}x = 2\pi \int_1^2 x(x+1)\mathrm{d}x = 2\pi \int_1^2 (x^2 + x)\mathrm{d}x$$

$$= 2\pi \left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_1^2 = 2\pi\left(\frac{8}{3} + 2 - \left(\frac{1}{3} + \frac{1}{2}\right)\right) = \boxed{\frac{23\pi}{3}}$$

3(c) $y = \sin x, \qquad y = 0, \qquad x = 0, \qquad x = \pi$

Solution:



We set $f(x) = \sin x$. Then the volume is equal to

$$V = 2\pi \int_0^{\pi} x f(x) \mathrm{d}x = 2\pi \int_0^{\pi} x \sin x \, \mathrm{d}x$$

$$\begin{bmatrix} u = x & v' = \sin x, \\ u' = 1, & v = -\cos x \end{bmatrix}$$

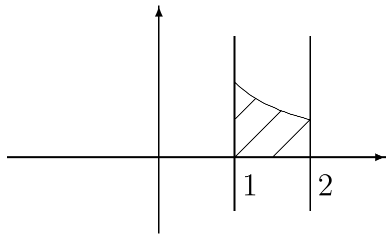
$$= 2\pi \left(\Big[x(-\cos x) \Big]_0^{\pi} - \int_0^{\pi} (-\cos x) \mathrm{d}x \right)$$

$$= 2\pi \big(-\pi \cos \pi - (-0 \cos 0) \big) + 2\pi \int_0^{\pi} \cos x \, \mathrm{d}x$$

$$= 2\pi^2 + 2\pi \Big[\sin x \Big]_0^{\pi} = 2\pi^2 + 2\pi \Big(\sin \pi - \sin 0 \Big) = \boxed{2\pi^2}$$

3(d) $y = \dfrac{1}{x}$, $\quad y = 0$, $\quad x = 1$, $\quad x = 2$

Solution:

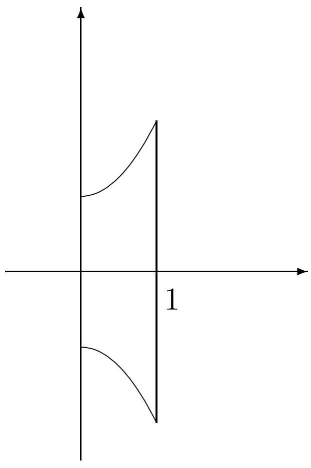


We set $f(x) = \dfrac{1}{x}$. Then the volume is equal to

$$V = 2\pi \int_1^2 x f(x) \mathrm{d}x = 2\pi \int_1^2 x \cdot \frac{1}{x} \, \mathrm{d}x = 2\pi \int_1^2 \mathrm{d}x = 2\pi \Big[x \Big]_1^2 = 2\pi \big(2 - 1 \big) = \boxed{2\pi}$$

3(e) $y = x^2 + 1$, $\quad y = -x^2 - 1$, $\quad x = 0$, $\quad x = 1$

Solution:



The graph of $y = x^2 + 1$ lies above the graph of $y = -x^2 - 1$. If we set

$$f_1(x) = x^2 + 1, \qquad f_2(x) = -x^2 - 1,$$

then the volume is equal to

$$V = 2\pi \int_0^1 x\big(f_1(x) - f_2(x)\big)\mathrm{d}x = 2\pi \int_0^1 x\big(x^2 + 1 - (-x^2 - 1)\big)\mathrm{d}x$$

$$= 2\pi \int_0^1 \big(2x^3 + 2x\big)\mathrm{d}x = 2\pi \left[\frac{1}{2}x^4 + x^2\right]_0^1 = 2\pi\left(\frac{1}{2} + 1 - 0\right) = \boxed{3\pi}$$

Q4. Find the arc length of the following curves:

4(a) $y = \dfrac{1}{8}x^2 - \ln x, \qquad x \in [1, 4]$

Solution:
The derivative of the function $f(x) := \dfrac{1}{8}x^2 - \ln x$ is

$$f'(x) = \frac{1}{4}x - \frac{1}{x}.$$

We need the following expression

$$1 + \big(f'(x)\big)^2 = 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2 = 1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}$$

$$= \frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2} = \left(\frac{x}{4} + \frac{1}{x}\right)^2.$$

Hence the length of the curve is equal to

$$s = \int_1^4 \sqrt{1 + \big(f'(x)\big)^2}\,\mathrm{d}x = \int_1^4 \left(\frac{x}{4} + \frac{1}{x}\right)\mathrm{d}x$$

$$= \left[\frac{x^2}{8} + \ln|x|\right]_1^4 = 2 + \ln 4 - \left(\frac{1}{8} + \ln 1\right) = \boxed{\frac{15}{8} + \ln 4}$$

4(b) $y = e^x, \qquad x \in \left[0, \dfrac{1}{2}\ln 3\right]$

Solution:
Set $f(x) := e^x$. Its derivative is $f'(x) = e^x$. Hence the length of the curve is equal to

$$s = \int_0^{\frac{1}{2}\ln 3} \sqrt{1 + \big(f'(x)\big)^2}\,\mathrm{d}x = \int_0^{\frac{1}{2}\ln 3} \sqrt{1 + \big(e^x\big)^2}\,\mathrm{d}x = \int_0^{\frac{1}{2}\ln 3} \sqrt{1 + e^{2x}}\,\mathrm{d}x.$$

We use the substitution

$$u = \sqrt{1 + e^{2x}}.$$

If we solve for x, we obtain (with $u \geq 0$)

$$\begin{aligned} u^2 = 1 + e^{2x} \quad &\Longleftrightarrow \quad u^2 - 1 = e^{2x} \\ &\Longleftrightarrow \quad \ln(u^2 - 1) = 2x \\ &\Longleftrightarrow \quad x = \frac{1}{2}\ln(u^2 - 1) \end{aligned}$$

We use the last equality to obtain a relation between the differentials:

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2} \cdot \frac{1}{u^2 - 1} \cdot 2u = \frac{u}{u^2 - 1} \quad \Longrightarrow \quad \mathrm{d}x = \frac{u}{u^2 - 1}\,\mathrm{d}u.$$

For the limits we obtain

$$\begin{aligned} x = 0 \qquad &\Longrightarrow \quad u = \sqrt{1 + e^0} = \sqrt{2} \\ x = \frac{1}{2}\ln 3 \quad &\Longrightarrow \quad u = \sqrt{1 + e^{\ln 3}} = \sqrt{1 + 3} = 2 \end{aligned}$$

Hence

$$s = \int_{\sqrt{2}}^{2} u \frac{u}{u^2 - 1}\,\mathrm{d}u = \int_{\sqrt{2}}^{2} \frac{u^2}{u^2 - 1}\,\mathrm{d}u.$$

With long division we can write

$$\frac{u^2}{u^2 - 1} = 1 + \frac{1}{u^2 - 1} = 1 + \frac{1}{(u+1)(u-1)}\,.$$

For the last fraction we use partial fractional decomposition:

$$\frac{1}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1}$$

with some constants A, B. Multiply both sides by the common denominator

$$1 = A(u-1) + B(u+1)$$

and set

$$\begin{aligned} u = -1 \quad &\Longrightarrow \quad A = -\frac{1}{2}\,, \\ u = 1 \quad &\Longrightarrow \quad B = \frac{1}{2}\,. \end{aligned}$$

Hence

$$\frac{u^2}{u^2 - 1} = 1 + \left(-\frac{1/2}{u+1} + \frac{1/2}{u-1}\right) = 1 - \frac{1}{2} \cdot \frac{1}{u+1} + \frac{1}{2} \cdot \frac{1}{u-1}$$

and we obtain

$$s = \int_{\sqrt{2}}^{2} \left(1 - \frac{1}{2} \cdot \frac{1}{u+1} + \frac{1}{2} \cdot \frac{1}{u-1}\right) \mathrm{d}u = \left[u - \frac{1}{2} \ln|u+1| + \frac{1}{2} \ln|u-1|\right]_{\sqrt{2}}^{2}$$

$$= 2 - \frac{1}{2} \ln 3 + \frac{1}{2} \ln 1 - \left(\sqrt{2} - \frac{1}{2} \ln\big(\sqrt{2}+1\big) + \frac{1}{2} \ln\big(\sqrt{2}-1\big)\right)$$

$$= \boxed{2 - \frac{1}{2} \ln 3 - \sqrt{2} + \frac{1}{2} \ln\big(\sqrt{2}+1\big) - \frac{1}{2} \ln\big(\sqrt{2}-1\big)}$$

$$\approx 0.917854$$

4(c) $y = 2x^{3/2}, \qquad x \in [0, 1]$

Solution:
Set $f(x) = 2x^{3/2}$. The derivative is equal to $f'(x) = 3x^{1/2}$.
Hence the length of the curve is equal to

$$s = \int_0^1 \sqrt{1 + \big(f'(x)\big)^2}\, \mathrm{d}x = \int_0^1 \sqrt{1 + \big(3x^{1/2}\big)^2}\, \mathrm{d}x = \int_0^1 \sqrt{1+9x}\, \mathrm{d}x.$$

We use the substitution

$$u = 1 + 9x, \qquad \mathrm{d}u = 9\, \mathrm{d}x,$$

$$x = 0 \implies u = 1,$$

$$x = 1 \implies u = 10.$$

Hence

$$s = \int_1^{10} \sqrt{u}\, \frac{1}{9}\, \mathrm{d}u = \frac{1}{9}\left[\frac{2}{3} u^{3/2}\right]_1^{10} = \frac{2}{27}\Big(10^{3/2} - 1\Big) = \boxed{\frac{2}{27}\Big(10\sqrt{10} - 1\Big)} \approx 2.268$$

4(d) $y = \ln(\cos x), \qquad x \in \left[0, \frac{\pi}{4}\right]$

Solution:
We need the first derivative of f:

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x.$$

Hence the arc length is equal to

$$s = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\, \mathrm{d}x = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, \mathrm{d}x$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, \mathrm{d}x = \int_0^{\pi/4} \sec x\, \mathrm{d}x = \Big[\ln\big|\sec x + \tan x\big|\Big]_0^{\pi/4}$$

$$= \ln\big|\sqrt{2} + 1\big| - \ln\big|1 + 0\big| = \boxed{\ln(\sqrt{2}+1)} \approx 0.88137$$

Q5. Find the surface area when the following curve is rotated through 360° about the x-axis:

5(a) $y = \sqrt{2x+1}, \qquad x \in [1,7]$

Solution:
The derivative of the function $f(x) := \sqrt{2x+1}$ is

$$f'(x) = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = \frac{1}{\sqrt{2x+1}}\,.$$

The surface area is equal to

$$\begin{aligned} S &= 2\pi \int_1^7 f(x)\sqrt{1+\big(f'(x)\big)^2}\,\mathrm{d}x = 2\pi \int_1^7 \sqrt{2x+1}\cdot\sqrt{1+\frac{1}{2x+1}}\,\mathrm{d}x \\ &= 2\pi \int_1^7 \sqrt{2x+1}\sqrt{\frac{2x+2}{2x+1}}\,\mathrm{d}x = 2\pi \int_1^7 \sqrt{2x+2}\,\mathrm{d}x \\ &= 2\pi \left[\frac{1}{2}\cdot\frac{2}{3}(2x+2)^{3/2}\right]_1^7 = \frac{2\pi}{3}\left(16^{3/2} - 4^{3/2}\right) = \frac{2\pi}{3}(64-8) = \boxed{\frac{112\pi}{3}} \end{aligned}$$

5(b) $y = \sqrt{x}, \qquad x \in [0,1]$

Solution:
Set $f(x) := \sqrt{x}$. The derivative is equal to

$$f'(x) = \frac{1}{2\sqrt{x}}\,.$$

Hence the surface area is equal to

$$\begin{aligned} S &= 2\pi \int_0^1 \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,\mathrm{d}x = 2\pi \int_0^1 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,\mathrm{d}x \\ &= 2\pi \int_0^1 \sqrt{x\left(1+\frac{1}{4x}\right)}\,\mathrm{d}x = 2\pi \int_0^1 \sqrt{x+\frac{1}{4}}\,\mathrm{d}x. \end{aligned}$$

We use the substitution

$$\begin{aligned} u &= x + \frac{1}{4}, \qquad \mathrm{d}x = \mathrm{d}u, \\ x &= 0 \implies u = \frac{1}{4} \\ x &= 1 \implies u = \frac{5}{4} \end{aligned}$$

Hence

$$\begin{aligned} S &= 2\pi \int_0^1 \sqrt{x+\frac{1}{4}}\,\mathrm{d}x = 2\pi \int_{1/4}^{5/4} u^{1/2}\,\mathrm{d}u = 2\pi \cdot \frac{2}{3}\left[u^{3/2}\right]_{1/4}^{5/4} \\ &= \frac{4\pi}{3}\left(\frac{5^{3/2}}{4^{3/2}} - \frac{1}{4^{3/2}}\right) = \frac{4\pi}{3}\cdot\frac{5\sqrt{5}-1}{8} = \boxed{\frac{(5\sqrt{5}-1)\pi}{6}} \approx 1.69672 \end{aligned}$$