12 Differentiation

12.1 Find the derivatives of the following functions of x.

(a) $x^2 + 4x + 2$ (b) $4x^3 - 2x^2 + 1$ (c) $x^9 + 7x^5 - 2x^4 + 3$ (d) $11x^{10} - 10x^{11}$.

12.2 For each of the following expressions:

(i) multiply out the polynomials and differentiate the result with respect to x;

(ii) use the product rule to differentiate and then, by multiplying out, write the answer as a polynomial.

(The answers should be identical.)

(a) $(x^2 + x)(2x - 1)$ (b) $(2x^3 + 1)(x - 1)$ (c) $(x^3 - x + 1)(x^2 + 3x - 1)$

(d) $(x - 1)(x^3 + x^2 + x + 1)$.

12.3 Use the product rule to differentiate the following with respect to x (without simplifying the answer).

(a) $(x^2 - x + 1)(x^2 + x + 1)$ (b) $(x^3 - 2x^2 + 1)(2x^4 - x^2 + 5)$

(c) $(x^3 - 5x + 3)^2$ (d) $(3x^3 - 2x^2 + 1)(x^7 - 4x^5 + 2x^2 + 3)$.

12.4 Show by repeated application of the product rule that

$(fgh)' = f'gh + fg'h + fgh'$.

Hence find the derivatives with respect to x of

(a) $x(x + 1)(x + 3)$ (b) $(x + 1)(x^2 + 2x + 4)(x^3 + 3x^2 + 6x + 9)$.

12.5 Find the derivatives of the following functions of x.

(a) $x^{-2} + 2x^{-4} - 7x^{-5}$ (b) $x^3 - 3x^{-1} + \frac{1}{2}x^{-2}$ (c) $x^3 + x + \frac{1}{x} + \frac{1}{x^3}$

(d) $1 - \frac{2}{x^2} + \frac{4}{x^4} - \frac{8}{x^8}$.

12.6 Use the quotient rule to differentiate the following functions of x.

(a) $\dfrac{x-2}{x-3}$ (b) $\dfrac{2x+1}{3x+2}$ (c) $\dfrac{x^2-1}{x^2+1}$ (d) $\dfrac{x+1}{x^2+3x+6}$

(e) $\dfrac{x+1}{x^3}$ (f) $\dfrac{1-5x^4}{x+2}$ (g) $\dfrac{x}{x-1} - \dfrac{1}{x+1}$ (h) $\dfrac{(x-2)(x+3)}{x+4}$.

12.7 What is the rate of change of the circumference C of a circle with respect to the area A of the circle?

12.8 On what intervals is the function $y(x) = x^3 + 6x^2$ increasing and decreasing?

12.9 Show that if $f(x) = \dfrac{4x}{x^2-7}$ then $f'(x) \neq 0$ for all x for which $f(x)$ is defined.

12.10 Use the chain rule to differentiate the following with respect to x (without simplifying your answers).

(a) $(2x+1)^4$ (b) $(3-2x)^3$ (c) $(x^2+3x+1)^6$ (d) $(x^2+1+x^{-2})^{-1}$.

12.11 Find the derivatives with respect to x of the following expressions.

(a) $2x^4 - 3x^{\frac{1}{2}}$ (b) $x^{\frac{4}{3}} + x^{\frac{2}{3}}$ (c) $2x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + 6x^{\frac{1}{2}}$ (d) $x^{\frac{2}{7}} - x^{\frac{3}{11}}$

(e) $(2-3x)^{\frac{1}{3}}$ (f) $(5x+1)^{-\frac{1}{4}}$ (g) $\sqrt{x^2+2x}$ (h) $\dfrac{1}{\sqrt{x^2-x+6}}$.

12.12 Find the derivatives with respect to x of

(a) $\left(x + \dfrac{1}{x}\right)^3$ (b) $\sqrt{2x^4 - 5x + 2}$ (c) $(x^{-1} - x^{-2})^{-1/4}$ (d) $\left(\dfrac{2x^2-1}{x^3+3}\right)^{1/3}$

(e) $\left(1 + \sqrt{\dfrac{x-2}{3}}\right)^4$ (f) $\left(x + \left((3x)^5 - 2\right)^{-\frac{1}{2}}\right)^{-6}$.

12.13 Find the derivatives of csc, sec, and cot.

12.14 Find the derivatives with respect to x of

(a) $3\sin x$ (b) $\sin 3x$ (c) $\sin x^3$ (d) $\sin^3 x$ (e) $3\sin^3 3x^3$.

12.15 Find the derivatives with respect to x of

(a) $3\cos x + 4\tan x$ (b) $\sin^2 x + \cos^2 x$ (c) $\sec(4x-3)$ (d) $x^2 \sec 5x$

(e) $\csc(\sqrt{x})$ (f) $\cot \dfrac{1}{2x+1}$ (g) $\sin 3x\, \cos 5x$ (h) $\dfrac{\sin x}{x^2+1}$

(i) $\tan^3(2x^2+1)$ (j) $\dfrac{1}{1+\sin x}$ (k) $\dfrac{1+\sec x}{1-\sec x}$ (l) $\sqrt{x}\cos x$

(m) $\sin(\sin x)$ (n) $\sin(\cos(\sin x))$ (o) $\sin\left(\dfrac{x}{\sin x}\right)$ (p) $\sin\left(\dfrac{x}{\sin\left(\frac{x}{\sin x}\right)}\right)$.

12.16 Find the first, second and third derivatives with respect to x of

(a) x^2 (b) x^8 (c) x^{-3} (d) $x^{-\frac{1}{2}}$ (e) $\dfrac{1}{x-3}$ (f) $\dfrac{1}{(2-3x)^2}$

(g) $\sqrt{x^2+1}$ (h) $\sin 3x$ (i) $\cos x^2$.

12.17 Let $f(x) = \tan x$. Prove that $f' = 1 + f^2$, $f'' = 2f(1+f^2)$, and $f''' = 2(1+f^2)(1+3f^2)$.

12.18 Let $u(t) = A\sin(\omega t + \phi)$. Determine k if $u'' + ku = 0$.

12.19 Let $f(x) = \sin(\sqrt{x})$, $\quad x > 0$. Determine a and b if $4xf''(x) + af'(x) + bf(x) = 0$.
Why did we require $x > 0$ rather than $x \geq 0$ (the natural domain of f)?

12.20 Verify that, for $x > 0$, $u(x) = \dfrac{1}{\sqrt{x}}\sin x$ satisfies the equation
$x^2\,u''(x) + x\,u'(x) + \left(x^2 - \dfrac{1}{4}\right)u(x) = 0.$

12.21 Prove by induction that for $f(x) = x^n$, $n \in \mathbb{N}$,
(a) $f^{(n+1)}(x) = 0$, $\quad$ (b) $f^{(n)}(x) = n!$ for all x.

12.22 Prove that the nth derivative of $f(x) = x^{-\frac{1}{2}}$ is $f^{(n)} = (-1)^n \dfrac{(2n)!}{2^{2n}n!} x^{-(n+\frac{1}{2})} \quad (n \in \mathbb{N})$.

12.23 Verify that $\dfrac{\mathrm{d}\sin x}{\mathrm{d}x} = \sin\left(x + \dfrac{\pi}{2}\right)$ and hence prove by induction that
$\dfrac{\mathrm{d}^n \sin x}{\mathrm{d}x^n} = \sin\left(x + \dfrac{n\pi}{2}\right) \quad (n \in \mathbb{N})$.

12.24 From the result of 12.23 guess a similar formula for $\dfrac{\mathrm{d}^n \cos x}{\mathrm{d}x^n}$ and prove it by induction.

12.25 Find the derivatives of arccsc, arcsec, and arccot.

12.26 Find the derivatives with respect to x of
(a) $\arcsin 2x$; $\quad$ (b) $\arccos\dfrac{x}{5}$; $\quad$ (c) $\arctan 7x$; $\quad$ (d) $\sqrt{\arctan(3x-2)}$;
(e) $x\arcsin x + \sqrt{1-x^2}$; $\quad$ (f) $\arccos(\sqrt{1-x^2})$, $\quad 0 \leq x < 1$;
(g) $\arcsin(\cos x)$, $\quad 0 < x < \pi$; $\quad$ (h) $\arcsin(\cos x)$, $\quad -\pi < x < 0$.

12.27 Suppose that $f^{(n)}$ and $g^{(n)}$ exist. Prove that $(f \cdot g)^{(n)}(a) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) \cdot g^{(n-k)}(a)$.
(This is sometimes called the general Leibniz rule.)

12.28 Prove that it is impossible to have $x = f(x)g(x)$ where f and g are differentiable and $f(0) = g(0) = 0$. (Hint: differentiate.)

12.29 Suppose that $f(x) = xg(x)$ for some function g that is continuous at 0.
Prove that f is differentiable at 0, and find $f'(0)$ in terms of g.

12.30 Suppose that f is differentiable at 0 and that $f(0) = 0$.

Prove that $f(x) = xg(x)$ for some function g which is continuous at 0.

(Hint: define $g(x) = f(x)/x$ for $x \neq 0$ and find a suitable value for $g(0)$.)