13 Integrals

13.1 Show that $\displaystyle\int_0^1 t^3\,\mathrm{d}t = \frac{1}{4}$ by considering partitions of $[0,1]$ into n equal subintervals and using the formula for $\displaystyle\sum_{j=1}^{n} j^3$ from Chapter 7.

(Hint: the procedure is exactly the same as we used to find equation (13.1).)

13.2 If $a < b < c < d$ and f is integrable on $[a,d]$, prove that f is integrable on $[b,c]$.

(Hint: two careful applications of Theorem 13.1 will do the trick.)

13.3 Find $\displaystyle\int_a^b \left(\int_c^d f(x)g(y)\,\mathrm{d}y\right)\mathrm{d}x$ in terms of $\displaystyle\int_a^b f$ and $\displaystyle\int_c^d g$.

(Hint: two careful applications of Theorem 13.2 will do the trick.)

13.4 Show that if f is integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$ then $\displaystyle\int_a^b f \geq 0$.

(Hint: it is sufficient to look at the lower sums $L(f,P)$.)

13.5 Show that if f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for all $x \in [a,b]$ then $\displaystyle\int_a^b f \geq \int_a^b g$.

(Hint: this is hard if you start from scratch but easy if you use the result from Exercise 13.4 together with Theorem 13.3.)