UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercise solutions: Chapter 3

1. (a) $x = 7 \quad \Rightarrow \quad x^2 = 49.$

   (b) $y = 90 \quad \Rightarrow \quad \cos y^\circ = 0.$

   (c) $x^3 = 64 \quad \Longleftrightarrow \quad x = 4.$

   (d) $ab = 0 \quad \Longleftrightarrow \quad a = 0$ or $b = 0.$

   (e) $x^4 = 16 \quad \Longleftrightarrow \quad x = \pm 2.$

2. (i) $=$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Rightarrow$ (v) $\Longleftrightarrow$ (vi) $\Longleftrightarrow$ (vii) $\Longleftrightarrow$ (viii) $\Longleftrightarrow$.

3. (i) This is not true because $x = -2$ also implies $x^2 = 4$.

   (ii) This is true: if $x = 16$ then $\log_4 x = \log_4 16 = 2$.

   (iii) This is true: $y + x^2 \leq 5$ implies $y \leq 5 - x^2$ which in turn implies that $y \leq 5$ (because x^2 is always greater than zero).

   (iv) This is not true: for example, take $x = 6$, $y = -5$. Then $x + y \geq 0$ is true but $y \geq 0$ is false.

   (v) This is true: $\log_{10} x = 2$ if and only if $x = 100$.

   (vi) This is true: if $x = -2 - y$ then $x + y = -2$ and $(x + y)^2 = 4$.

   (vii) This is false: for example, if $x = -10$ then $x \leq 1$ but $x^2 = 100 > 1$.

   (viii) This is true: $2^x = 4$ if and only if $x = 2$.

4. Going from line 3 to line 4 involves dividing both sides by $x - y$. If $x = y$ as initially assumed, then $x - y$ is zero, so this step is nonsense.

5. The product is

$$2 \times 4 \times \ldots \times 2n = 2^n(1 \times 2 \times 3 \times \ldots \times n) = 2^n(n!)$$

   (as $2 \times 4 \times \ldots \times 2n$ contains n terms).

6. The previous question showed that the product of the first n even integers is $2^n n!$. We note here that the product of the first n odd integers is equal to the product of the first $2n$ integers divided by the product of the first n even integers, i.e.

$$1 \times 3 \times 5 \times \ldots \times (2n-1) = \frac{1 \times 2 \times 3 \times \ldots \times (2n-1) \times (2n)}{2 \times 4 \times 6 \times \ldots \times 2n}.$$

The product of the first n odd integers is therefore $\dfrac{(2n)!}{2^n(n!)}$.

7. **Step 1:** Check the case $n = 1$.

$$LHS = 1 \times (1-x) = 1-x, \qquad RHS = 1-x$$

so proposition is true when $n = 1$.

Step 2: Assume that the given result is true for n, that is, assume that

$$(1 + x + x^2 + \ldots + x^{n-1})(1-x) = 1 - x^n.$$

Now try to prove the result for $n+1$, that is, try to show that

$$(1 + x + x^2 + \ldots + x^{n-1} + x^n)(1-x) = 1 - x^{n+1}.$$

We have

$$\begin{aligned}(1 + x + x^2 + \ldots + x^{n-1} + x^n)(1-x) &= (1 + x + x^2 + \ldots + x^{n-1})(1-x) + x^n(1-x) \\ &= 1 - x^n + x^n - x^{n+1} \qquad \text{(by the assumption above)} \\ &= = 1 - x^{n+1}\end{aligned}$$

so if the proposition is true for n, it is true for $n+1$. Hence, by the principle of mathematical induction, the proposition is true for all natural numbers n.

8. **Step 1:** Check the case $n = 1$.

$$LHS = 3, \qquad RHS = 1 \times (4 + 6 - 1)/3 = 3$$

so proposition is true when $n = 1$.

Step 2: Assume that the given result is true for n, that is, assume that

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \ldots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}.$$

Now try to prove the result for $n+1$, that is, try to show that

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \ldots + (2n-1)(2n+1) + (2(n+1)-1)(2(n+1)+1) =$$

$$\frac{(n+1)(4(n+1)^2 + 6(n+1) - 1)}{3} = \frac{(n+1)(4n^2 + 14n + 9)}{3}.$$

We have

$$\begin{aligned}
1\times 3 \quad + \quad & \ldots + (2n-1)(2n+1) + (2(n+1)-1)(2(n+1)+1) \\
= \quad & \frac{n(4n^2+6n-1)}{3} + (2n+1)(2n+3) \qquad \text{(by assumption)} \\
= \quad & \frac{1}{3}[4n^3+6n^2-n+3(4n^2+8n+3)] \\
= \quad & \frac{1}{3}[4n^3+18n^2+23n+9] \\
= \quad & \frac{1}{3}(n+1)(4n^2+14n+9)
\end{aligned}$$

so if the proposition is true for n, it is true for $n+1$. Hence, by the principle of mathematical induction, the proposition is true for all natural numbers n.

9. The first task here is to find an expression for the nth fraction. We see that the first factor in each denominator is 1, 4, 7, 10, etc, or 1, $1+3$, $1+2\times 3$, $1+3\times 3$ etc so for a general n it is $1+(n-1)3 = 3n-2$. The second factor in each case is the first factor plus 3, or $(3n-2)+3 = 3n+1$. Hence, the nth term can be written as $\dfrac{1}{(3n-2)(3n+1)}$.

   We now need to show that

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \frac{1}{10.13} + \ldots + \frac{1}{(3n+1)(3n-2)} = \frac{n}{3n+1}$$

   for all positive integers n.

   Step 1: Check the case $n=1$.

$$LHS = \frac{1}{1\times 4} = \frac{1}{4}, \qquad RHS = \frac{1}{3+1} = \frac{1}{4}$$

   so proposition is true when $n=1$.

   Step 2: Assume that the given result is true for n, that is, assume that

$$\frac{1}{1.4} + \ldots + \frac{1}{(3n+1)(3n-2)} = \frac{n}{3n+1}.$$

   Now try to prove the result for $n+1$, that is, try to show that

$$\frac{1}{1.4} + \ldots + \frac{1}{(3n+1)(3n-2)} + \frac{1}{(3(n+1)+1)(3(n+1)-2)} = \frac{(n+1)}{3(n+1)+1} = \frac{n+1}{3n+4}.$$

We have

$$\begin{aligned}
\frac{1}{1.4} &+ \ldots + \frac{1}{(3n+1)(3n-2)} + \frac{1}{(3(n+1)+1)(3(n+1)-2)} \\
&= \frac{n}{3n+1} + \frac{1}{(3n+4)(3n+1)} \qquad \text{(by assumption)} \\
&= \frac{1}{3n+1}\left[n + \frac{1}{3n+4}\right] \\
&= \frac{1}{3n+1}\left[\frac{n(3n+4)+1}{3n+4}\right] \\
&= \frac{3n^2+4n+1}{(3n+1)(3n+4)} \\
&= \frac{(3n+1)(n+1)}{(3n+1)(3n+4)} \\
&= \frac{n+1}{3n+4}
\end{aligned}$$

so if the proposition is true for n, it is true for $n+1$. Hence, by the principle of mathematical induction, the proposition is true for all natural numbers n.