## University of Strathclyde
## Department of Mathematics and Statistics

# MM102: Applications f Calculus

## Lecture Notes for Week 1

Differentiation and integration play central roles in mathematics because they are used in so many different areas of applications. In the class MM101 you learnt the definition and the meaning of derivatives and integrals and learnt how to differentiate and integrate simple functions. It is vital that you know these techniques from the first semester for this class and also for the rest of the degree course.

In the current class MM102 you will learn further techniques of differentiation and integration and then apply these techniques and those you have learnt in the class MM101 to various problems: optimisation, approximation of functions, curve sketching, solving differential equations, and calculation of volumes, curve lengths and surface areas. Since for some of these techniques complex numbers are needed, we will also discuss these in a separate section.

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# 1 Further Techniques of Integration

In this section we shall discuss various further techniques to determine integrals of more complicated functions. In general, the process of integration is not so straightforward as differentiation, and there exist functions whose integrals cannot be expressed in terms of elementary functions. Sometimes one has to try various techniques before the integral can be successfully evaluated.

## 1.1 Definite and Indefinite Integrals

Let us first recall some basic facts about definite and indefinite integrals. The definite integral was introduced in the class MM101 in Section 13. For a continuous function $f$ defined on the interval $[a, b]$, the definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

gives the area between the graph of a function $f$ and the $x$-axis and between the vertical lines $x = a$ and $x = b$ (at least if $f$ is positive, otherwise the part of the area where $f(x) < 0$ has to be subtracted). It is defined by a limit process using lower and upper sums. There is an important connection between integration and differentiation: the two fundamental theorems of calculus, first found by Sir Isaac Newton and Gottfried Wilhelm Leibniz in the 17$^{\text{th}}$ century.

**Theorem 1.1. (The First Fundamental Theorem of Calculus)** [MM101, Theorem 14.1] *If $f$ is continuous on $[a, b]$, then $F$ defined on $[a, b]$ as*

$$F(x) := \int_a^x f(u)\,\mathrm{d}u$$

*is differentiable at all $c \in (a, b)$ and*

$$F'(c) = f(c).$$

This theorem tells us that for every continuous function $f$ there exists an *anti-derivative*, or *primitive* or *indefinite integral*, i.e. a function $F$ for which $F' = f$. Note that any two anti-derivatives of a given function $f$ differ by a constant:

$$F_1' = f \quad \text{and} \quad F_2' = f \qquad \Longrightarrow \qquad F_1(x) = F_2(x) + C$$

for some constant $C$. An anti-derivative (or the set of all anti-derivatives) is denoted by

$$\int f \qquad \text{or} \qquad \int f(x)\,\mathrm{d}x.$$

The process of finding a primitive is particularly important when we want to solve differential equations or find the displacement of a particle if the velocity is given (see later sections).

**Theorem 1.2. (The Second Fundamental Theorem of Calculus)**
[MM101, Theorem 14.2]
*If $f$ is continuous on $[a, b]$ and $f = g'$ for some differentiable function $g$, then*

$$\int_a^b f(x)\,\mathrm{d}x = g(b) - g(a).$$

The second fundamental theorem of calculus gives us a powerful tool to evaluate definite integrals efficiently without taking the limit of lower and upper sums. In order to apply this theorem, we have to find an anti-derivative.

## 1.2 Standard Integrals and Basic Rules

You will need to know the following integrals for this class. An arbitrary constant $C$ should be added to the functions on the right-hand side. The relations in this table can be proved by differentiating the right-hand sides of each line (see MM101).

| $f(x)$ | $\int f(x)\,\mathrm{d}x$ | |
|---|---|---|
| $a$ | $ax$ | ($a$ ... constant) |
| $x^\alpha$ | $\dfrac{x^{\alpha+1}}{\alpha+1}$ | $(\alpha \neq -1)$ |
| $\dfrac{1}{x}$ | $\ln\lvert x\rvert$ | |
| $\sin x$ | $-\cos x$ | |
| $\cos x$ | $\sin x$ | |
| $\tan x$ | $\ln\lvert\sec x\rvert$ | |
| $\sec x$ | $\ln\lvert\sec x + \tan x\rvert$ | |
| $\dfrac{1}{\sqrt{a^2-x^2}}$ | $\arcsin\left(\dfrac{x}{a}\right)$ | $(a > 0)$ |
| $\dfrac{1}{x^2+a^2}$ | $\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$ | $(a > 0)$ |
| $e^x$ | $e^x$ | |
| $\sinh x$ | $\cosh x$ | |
| $\cosh x$ | $\sinh x$ | |

Let us also recall some rules of integration from MM101.

### Linearity.

$$\int cf(x)\,\mathrm{d}x = c \cdot \int f(x)\,\mathrm{d}x \qquad (c\ \ldots\ \text{constant}),$$

$$\int \big(f(x)+g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

## Integration by parts.

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x.$$

Note that there is *no* product rule for integration.

## Substitution.

If $F$ is an anti-derivative (or primitive) of $f$, i.e. $F' = f$, and $g$ continuously differentiable, then

$$\int f\big(g(x)\big)g'(x)\,\mathrm{d}x = F\big(g(x)\big).$$

In practice, one introduces a new variable, say $u$, which is a function of $x$, i.e. $u = g(x)$, and replaces **all** occurrences of $x$ and the differential $\mathrm{d}x$ by expressions in term of $u$ and $\mathrm{d}u$. For the latter, one has to differentiate $g$: $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$, which yields (at least formally) $\mathrm{d}u = g'(x)\mathrm{d}x$. Sometimes one expresses $x$ in terms of $u$, e.g. $x = h(u)$ and differentiates with respect to $u$, $\frac{\mathrm{d}x}{\mathrm{d}u} = h'(u)$, which yields $\mathrm{d}x = h'(u)\mathrm{d}u$.

**Important Note.** If one uses a substitution in an indefinite integral, then after finding the anti-derivative, one has to substitute back and write the result in terms of the original variable. In the case of a definite integral one has to change the limits of the integral as well; then the integration is performed with respect to the new variable, i.e. one should **not** go back to the original variable.

As a special case, we obtain the following rule.

**Linear substitution.**
If $F$ is an anti-derivative of $f$, then

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) \qquad a \neq 0.$$

The following examples illustrate some of the ideas you have learnt in MM101.

**Examples 1.1.**

(i) Evaluate

$$\int \sin(2x+3)\,\mathrm{d}x.$$

We can use the substitution

$$u = 2x+3.$$

Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \Longrightarrow \quad \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u,$$

which yields

$$\int \sin(2x+3)\,\mathrm{d}x = \int \sin u \cdot \frac{1}{2}\,\mathrm{d}u = \frac{1}{2}\int \sin u\,\mathrm{d}u = \frac{1}{2}\big(-\cos u\big) + C$$
$$= -\frac{1}{2}\cos(2x+3) + C$$

(ii) Evaluate

$$\int x^2 \cos x\,\mathrm{d}x.$$

We apply integration by parts twice:

$$\begin{aligned}
\int x^2 \cos x\,\mathrm{d}x &= \\
&\qquad \begin{bmatrix} f = x^2 & g' = \cos x, \\ f' = 2x, & g = \sin x \end{bmatrix} \\
&= x^2 \sin x - \int 2x \sin x\,\mathrm{d}x \\
&\qquad \begin{bmatrix} f = 2x & g' = \sin x, \\ f' = 2, & g = -\cos x \end{bmatrix} \\
&= x^2 \sin x - \left(2x(-\cos x) - \int 2(-\cos x)\mathrm{d}x\right) && (*) \\
&= x^2 \sin x + 2x\cos x - \int 2\cos x\,\mathrm{d}x && (**) \\
&= x^2 \sin x + 2x\cos x - 2\sin x + C \\
&= (x^2-2)\sin x + 2x\cos x + C.
\end{aligned}$$

It is highly recommended to simplify double and triple minus signs before evaluating integrals, like in the step from $(*)$ to $(**)$.

(iii) Evaluate

$$\int_{-2}^{1} \frac{\mathrm{d}x}{(2x+1)^2+9}\,.$$

Solution in video

(iv) Evaluate

$$\int \sqrt{x}\cos\big(\sqrt{x}\big)\,\mathrm{d}x.$$

Solution in video

(v) Evaluate

$$\int_1^e x \ln x \, \mathrm{d}x.$$

Solution in video

## Integration of even and odd functions.

Recall that a function $f$ is called **even** if

$$f(-x) = f(x) \qquad \text{for all } x \in \operatorname{dom}(f);$$

it is called **odd** if

$$f(-x) = -f(x) \qquad \text{for all } x \in \operatorname{dom}(f);$$

see MM101, Section 4.4.

The following proposition is often useful when we integrate even or odd functions over an interval of the form $[-a, a]$ with $a > 0$.

**Proposition 1.3.** *Let $a > 0$.*

(i) *If $f$ is an even function defined on $[-a, a]$, then*

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 2 \int_0^a f(x) \, \mathrm{d}x. \tag{1.1}$$

(ii) *If $f$ is an odd function defined on $[-a, a]$, then*

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 0. \tag{1.2}$$

*Proof.* (i) We can write

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = \int_{-a}^{0} f(x) \, \mathrm{d}x + \int_0^a f(x) \, \mathrm{d}x. \tag{1.3}$$

For the first integral we use the substitution $x = -t$, $\mathrm{d}x = -\mathrm{d}t$, which yields

$$\int_{-a}^{0} f(x) \, \mathrm{d}x = \int_a^0 f(-t)(-1) \, \mathrm{d}t = \int_0^a f(-t) \, \mathrm{d}t = \int_0^a f(t) \, \mathrm{d}t,$$

where in the last step we used the fact that $f$ is even. Combining this with (1.3) we obtain (1.1).

Item (ii) is proved in a similar way. $\square$

**Example 1.2.**
Evaluate
$$\int_{-2}^{2} x^4 \,\mathrm{d}x.$$
The integrand ($x \mapsto x^4$) is an even function. Hence
$$\int_{-2}^{2} x^4 \,\mathrm{d}x = 2\int_{0}^{2} x^4 \,\mathrm{d}x = \frac{2}{5}\Big[x^5\Big]_0^2 = \frac{2}{5}\big(2^5 - 0\big) = \frac{64}{5}\,.$$
We can check the result by a direct (but slightly lengthier) calculation:
$$\int_{-2}^{2} x^4 \,\mathrm{d}x = \frac{1}{5}\Big[x^5\Big]_{-2}^2 = \frac{1}{5}\big(2^5 - (-2)^5\big) = \frac{1}{5}\big(32 - (-32)\big) = \frac{1}{5}\big(32 + 32\big) = \frac{64}{5}\,.$$

## 1.3 Integration of Rational Functions

Recall that a rational function is the quotient of two polynomials:
$$f(x) = \frac{p_1(x)}{p_2(x)} \tag{1.4}$$
where $p_1$ and $p_2$ are polynomials.

**Proper and improper rational functions**

Denote by $\deg p$ the **degree** of a polynomial $p$, i.e. $\deg p = n$ if
$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$
where $a_0, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Recall that a rational function as in (1.4) is called

**proper** if $\quad \deg p_1 < \deg p_2$

and

**improper** if $\quad \deg p_1 \geq \deg p_2$.

If a given rational function, which we have to integrate, is improper, then we must apply long division in a first step to write $\frac{p_1}{p_2}$ as
$$\frac{p_1(x)}{p_2(x)} = q(x) + \frac{r(x)}{p_2(x)}$$
where $q$ and $r$ are polynomials with $\deg r < \deg p_2$, i.e. the rational function $\frac{r}{p_2}$ on the right-hand side is proper. The polynomial $q$ can easily be integrated. To integrate the term $\frac{r}{p_2}$ we have to use partial fraction decomposition.

**Partial fraction decomposition**

Let now

$$\frac{p_1(x)}{p_2(x)}$$

be a proper rational function, i.e. $\deg p_1 < \deg p_2$. In a first step we have to factorise the denominator into a product of powers of linear and irreducible quadratic factors. We assume, without loss of generality, that the leading coefficient of $p_2$ is $1$. Then

$$p_2(x) = (x-\alpha_1)^{r_1}\cdots(x-\alpha_k)^{r_k}(x^2+\beta_1x+\gamma_1)^{s_1}\cdots(x^2+\beta_lx+\gamma_l)^{s_l}, \tag{1.5}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ and $r_i, s_i \in \mathbb{N}$. That a quadratic factor $(x^2+\beta_ix+\gamma_i)$ is irreducible means that it cannot be factorised into a product of two linear factors with real coefficients, which is equivalent to the inequality $\beta_i^2 - 4\gamma_i < 0$. We assume that the factors $(x-\alpha_i)$ are pairwise distinct and also that the factors $(x^2+\beta_ix+\gamma_i)$ are pairwise distinct. The fact that $p_2$ can be factorised as in (1.5) follows from the **Fundamental Theorem of Algebra**, which we do not prove here. To actually find the factorisation can be very difficult since there is no formula for $\alpha_i, \beta_i, \gamma_i, r_i, s_i$ when $\deg p_2 \geq 5$.

Now the aim is to write $\frac{p_1}{p_2}$ as a sum of simpler fractions. Each factor on the right-hand side of (1.5) contributes certain terms in the partial fraction decomposition:

- for each factor $(x-\alpha_i)^{r_i}$ we take

  $$\frac{A_{i,1}}{x-\alpha_i} + \frac{A_{i,2}}{(x-\alpha_i)^2} + \ldots + \frac{A_{i,r_i}}{(x-\alpha_i)^{r_i}}$$

  with some real constants $A_{i,j}$; in particular, if $r_i = 1$, then we have only one term:

  $$\frac{A_{i,1}}{x-\alpha_i};$$

- for each factor $(x^2+\beta_ix+\gamma_i)^{s_i}$ we take

  $$\frac{B_{i,1}x+C_{i,1}}{x^2+\beta_ix+\gamma_i} + \frac{B_{i,2}x+C_{i,2}}{(x^2+\beta_ix+\gamma_i)^2} + \ldots + \frac{B_{i,s_i}x+C_{i,s_i}}{(x^2+\beta_ix+\gamma_i)^{s_i}}.$$

  with some real constants $B_{i,j}, C_{i,j}$.

One can prove that there exist unique constants $A_{i,j}, B_{i,j}, C_{i,j}$ such that

$$\frac{p_1(x)}{p_2(x)} = \sum_{i=1}^{k}\left(\frac{A_{i,1}}{x-\alpha_i} + \frac{A_{i,2}}{(x-\alpha_i)^2} + \ldots + \frac{A_{i,r_i}}{(x-\alpha_i)^{r_i}}\right)$$

$$+ \sum_{i=1}^{l}\left(\frac{B_{i,1}x+C_{i,1}}{x^2+\beta_ix+\gamma_i} + \frac{B_{i,2}x+C_{i,2}}{(x^2+\beta_ix+\gamma_i)^2} + \ldots + \frac{B_{i,s_i}x+C_{i,s_i}}{(x^2+\beta_ix+\gamma_i)^{s_i}}\right)$$

for all $x$ for which $p_2(x) \neq 0$. One then has to determine the constants $A_{i,j}, B_{i,j}, C_{i,j}$ by multiplying both sides by the common denominator and then either setting $x$ equal to special values or comparing coefficients of different powers of $x$, or a combination of both.

**Example 1.3.**
Find the partial fraction decomposition for the following proper rational functions without determining the constants:

(i) $\dfrac{x+9}{x^2-2x-3} = \dfrac{x+9}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$

(two distinct linear factors)

(ii) $\dfrac{5x+2}{x^2-2} = \dfrac{5x+2}{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)} = \dfrac{A}{x+\sqrt{2}} + \dfrac{B}{x-\sqrt{2}}$

(two distinct linear factors)

(iii) $\dfrac{3x^2+1}{(x-1)(x+2)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

(three distinct linear factors)

(iv) $\dfrac{x^2+5}{(x^2-1)(x+1)} = \dfrac{x^2+5}{(x+1)(x-1)(x+1)} = \dfrac{x^2+5}{(x+1)^2(x-1)}$

$= \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-1}$

(repeated linear factors; note that one has to fully factorise the denominator into linear and irreducible quadratic factors and then combine common factors)

(v) $\dfrac{5}{(x-2)^4(x+2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3} + \dfrac{D}{(x-2)^4} + \dfrac{E}{x+2} + \dfrac{F}{(x+2)^2}$

(repeated linear factors)

(vi) $\dfrac{2x+5}{(x^2+2x+3)(x-3)} = \dfrac{Ax+B}{x^2+2x+3} + \dfrac{C}{x-3}$

($x^2+2x+3$ is irreducible since $\beta^2-4\gamma = 2^2-4\times 3 = -8 < 0$)

(vii) $\dfrac{x^3+3x-1}{(x^2+x+4)^3(x-1)^2}$

$= \dfrac{Ax+B}{x^2+x+4} + \dfrac{Cx+D}{(x^2+x+4)^2} + \dfrac{Ex+F}{(x^2+x+4)^3} + \dfrac{G}{x-1} + \dfrac{H}{(x-1)^2}$

(repeated linear and irreducible quadratic factors;
note that $x^2+x+4$ is irreducible since $\beta^2-4\gamma = 1^2-4\times 4 = -15 < 0$)

The total number of constants in the partial fraction decomposition is always equal to the degree of the polynomial in the denominator. One can use this as a good sanity check so that one does not forget any terms.

### Integrating the terms

Once we have obtained the partial fraction decomposition, we can integrate each term separately.
For the **linear factors** this is relatively straightforward:

$$\int \frac{1}{x-\alpha}\,\mathrm{d}x = \ln|x-\alpha|,$$

and for $r \geq 2$:

$$\int \frac{1}{(x-\alpha)^r}\,\mathrm{d}x = \int (x-\alpha)^{-r}\mathrm{d}x = \frac{1}{-r+1}(x-\alpha)^{-r+1} = -\frac{1}{r-1}\cdot\frac{1}{(x-\alpha)^{r-1}}\,.$$

**Examples 1.4.**

(i) Evaluate the integral

$$\int \frac{x^3-4x-7}{x^2-2x-3}\,\mathrm{d}x.$$

Solution in video

(ii) Evaluate the integral

$$\int \frac{x^2+3x-2}{(x-1)^2(x+1)}\,\mathrm{d}x.$$

Solution in video

Next let us next consider a single **quadratic factor**. First we complete the square in the denominator:

$$\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{Bx+C}{(x+b)^2+a^2}$$

with some $b \in \mathbb{R}$ (namely $b = \beta/2$) and $a > 0$. Note that we have $+a^2$ and not $-a^2$ because the quadratic factor is irreducible. Next we make the substitution

$$u = x+b, \qquad \mathrm{d}u = \mathrm{d}x,$$

which leads to

$$\int \frac{Bx+C}{(x+b)^2+a^2}\,\mathrm{d}x = \int \frac{B(u-b)+C}{u^2+a^2}\,\mathrm{d}u,$$

i.e. the new integral is of the form

$$\int \frac{Bu+D}{u^2+a^2}\,\mathrm{d}u = B\int \frac{u}{u^2+a^2}\,\mathrm{d}u + D\int \frac{1}{u^2+a^2}\,\mathrm{d}u$$

with some $D \in \mathbb{R}$. The second integral is a standard integral:

$$\int \frac{1}{u^2+a^2}\,\mathrm{d}u = \frac{1}{a}\arctan\Big(\frac{u}{a}\Big).$$

For the first integral we use the substitution

$$v = u^2 + a^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = 2u \qquad \Longrightarrow \qquad u\,\mathrm{d}u = \frac{1}{2}\,\mathrm{d}v,$$

which yields

$$\int \frac{u}{u^2+a^2}\,\mathrm{d}u = \frac{1}{2}\int \frac{1}{v}\,\mathrm{d}v = \frac{1}{2}\ln|v| = \frac{1}{2}\ln(u^2+a^2).$$

Note that the modulus is not needed in the last expression since $u^2 + a^2 > 0$. Finally, if the integral is an indefinite integral, one has to write the result again in terms of $x$ by replacing $u$ by $x + b$.

**Example 1.5.**
Evaluate the integral

$$\int \frac{3x-11}{(x^2-2x+5)(x-2)}\,\mathrm{d}x.$$

Solution in video

Let us finally mention how to deal with **powers of quadratic factors**. These appear very rarely. After completing the square and making a linear substitution as above one splits the fraction:

$$\int \frac{Bu+D}{\left(u^2+a^2\right)^s}\,\mathrm{d}u = \frac{B}{2}\int \frac{2u}{\left(u^2+a^2\right)^s}\,\mathrm{d}u + D\int \frac{1}{\left(u^2+a^2\right)^s}\,\mathrm{d}u.$$

The first integral can be integrated with the substitution $v = u^2 + a^2$. For the second integral one can use integration by parts to reduce the exponent. Let us consider only a simple example (the general case can be treated in a similar way):

$$\int \frac{1}{(u^2+1)^2}\,\mathrm{d}u = \int \frac{u^2+1-u^2}{(u^2+1)^2}\,\mathrm{d}u = \int \frac{u^2+1}{(u^2+1)^2}\,\mathrm{d}u - \frac{1}{2}\int u\frac{2u}{(u^2+1)^2}\,\mathrm{d}u$$

$$\left[\begin{array}{ll} \text{for the second integral use integration by parts:} & \\ f = u, & g' = \dfrac{2u}{(u^2+1)^2} \\ f' = 1, & g = -\dfrac{1}{u^2+1} \end{array}\right]$$

$$\begin{aligned}
&= \int \frac{1}{u^2+1}\,\mathrm{d}u - \frac{1}{2}\left(-\frac{u}{u^2+1} - \int \frac{-1}{u^2+1}\,\mathrm{d}u\right) \\
&= \int \frac{1}{u^2+1}\,\mathrm{d}u + \frac{1}{2}\cdot\frac{u}{u^2+1} - \frac{1}{2}\int \frac{1}{u^2+1}\,\mathrm{d}u \\
&= \frac{1}{2}\int \frac{1}{u^2+1}\,\mathrm{d}u + \frac{u}{2(u^2+1)} = \frac{1}{2}\arctan u + \frac{u}{2(u^2+1)} + C.
\end{aligned}$$