Properties of the Scalar Product (Example 1.2.4)

Let $\mathbf{u} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^T$ and $\mathbf{v} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$ be vectors in $\mathbb{R}^n$ (so that each $u_i, v_i \in \mathbb{R}$). The **scalar product** (or **dot product** or **inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is defined to be

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

It is important to note that the value of $\mathbf{u} \cdot \mathbf{v}$ is a real number. Geometrically, one can define the scalar product in the following way:

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta),$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

For any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ (of equal dimension) the following results hold.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$
4. $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$

Proof. Let u_i, v_i and w_i $(1 \leq i \leq n)$ be the components of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, respectively.

1. Since $u_i v_i = v_i u_i$ for all i, we have

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n = \mathbf{v} \cdot \mathbf{u}.$$

2. Since $u_i(v_i + w_i) = u_i v_i + u_i w_i$ for all i, we have

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n) \\
&= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + \cdots + u_n v_n + u_n w_n \\
&= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n + u_1 w_1 + u_2 w_2 + \cdots + u_n w_n \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.
\end{aligned}$$

3. Recall that $||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$. Then

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2 = ||\mathbf{u}||^2.$$

4. Using the geometric definition of the scalar product and the fact that $|\cos(\theta)| \leq 1$,

$$|\mathbf{u} \cdot \mathbf{v}| = \Big| ||\mathbf{u}||||\mathbf{v}|| \cos(\theta) \Big| = ||\mathbf{u}||||\mathbf{v}|| \Big| \cos(\theta) \Big| \leq ||\mathbf{u}||||\mathbf{v}||.$$

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