UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MATHEMATICS & STATISTICS

MM103 Geometry and Algebra

Chapter 2: The Straight Line

Q1.

(a) (i) $x - 2y = 0$, (ii) $y = x/2$, (iii) $\mathbf{r} = t\begin{bmatrix}2\\1\end{bmatrix}$

(b) (i) $x = 3$, (ii) No such form exists. (iii) $\mathbf{r} = \begin{bmatrix}3\\0\end{bmatrix} + t\begin{bmatrix}0\\1\end{bmatrix}$

(c) (i) $5x - y = 0$, (ii) $y = 5x$, (iii) $\mathbf{r} = t\begin{bmatrix}1\\5\end{bmatrix}$

(d) (i) $-x + 4y = 6$, (ii) $y = x/4 + 3/2$, (iii) $\mathbf{r} = \begin{bmatrix}-6\\0\end{bmatrix} + t\begin{bmatrix}4\\1\end{bmatrix}$

(e) (i) $2x - y = 0$, (ii) $y = 2x$, (iii) $\mathbf{r} = t\begin{bmatrix}1\\2\end{bmatrix}$

Q2. The required line must be of the form $\mathbf{r} = t\mathbf{u}$, where $\mathbf{u}$ is perpendicular to $\begin{bmatrix}3\\-1\end{bmatrix}$. Thus, $\mathbf{r} = t\begin{bmatrix}1\\3\end{bmatrix}$.

Q3. We have

$$\mathbf{r}\cdot\begin{bmatrix}b\\-a\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}\cdot\begin{bmatrix}b\\-a\end{bmatrix} = bx - ay$$

and

$$\mathbf{r}\cdot\begin{bmatrix}b\\-a\end{bmatrix} = \left(\begin{bmatrix}p\\q\end{bmatrix} + t\begin{bmatrix}a\\b\end{bmatrix}\right)\cdot\begin{bmatrix}b\\-a\end{bmatrix} = \begin{bmatrix}p\\q\end{bmatrix}\cdot\begin{bmatrix}b\\-a\end{bmatrix} = bp - aq.$$

The result now follows.

Q4. (a) $\frac{7}{13}$, (b) 0, (c) $\frac{8}{\sqrt{5}}$, (d) $\frac{|mx_0 - y_0 + c|}{\sqrt{1+m^2}}$

Q5. We have

$$|PS|^2 = |PQ|^2 + |QS|^2 \geq |PQ|^2$$

and so $|PS| \geq |PQ|$ with equality if and only if $Q = S$.

Q6. The lines are parallel if and only if P_0 and P_1 lie of the same side of L. In other words, if $ax_0 + by_0 \geq c$ and $ax_1 + by_1 \geq c$ or if $ax_0 + by_0 \leq c$ and $ax_1 + by_1 \leq c$.

Q7.

(a) $(3, 1)$

(b) $(1, 4)$

(c) The lines are parallel, so there are no points of intersection.

(d) $(2, 1)$

(e) Both lines are equal, so all points on the line $y = 2 - 3x$.

(f) $(0, 0)$

Chapter 2: Curves

Q1.

(a) There is precisely one polygon.

(b) There are four possible polygons.

(c) There are eleven possible polygons.

Q2.

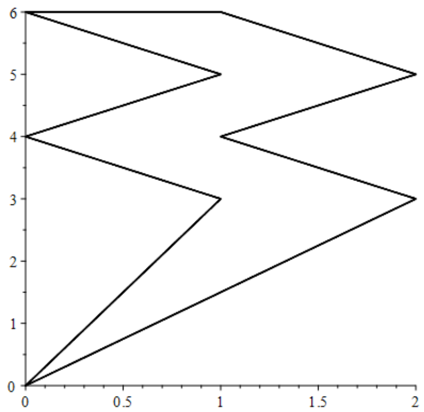

Figure 1: The polygon $OABCDEFGH$.

Q3. The possible fourth corners are $(4, 0)$, $(4, 2)$ and $(-2, 2)$.

Q4. Given that

$$x^2 + y^2 + 2gx + 2fy + c = (x + g)^2 + (y + f)^2 + c - g^2 - f^2 = 0$$

we have

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

and so the centre of the circle is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$.

Q5. The equations to solve are

$$8 - 4g + 4f + c = 0$$
$$20 + 8g + 4f + c = 0$$
$$34 - 6g - 10f + c = 0.$$

We find that $g = -1$, $f = 2$ and $c = -20$, and so the equation of the circle is

$$x^2 + y^2 - 2x + 4y - 20 = 0,$$

i.e.,

$$(x - 1)^2 + (y + 2)^2 = 25.$$

Q6.

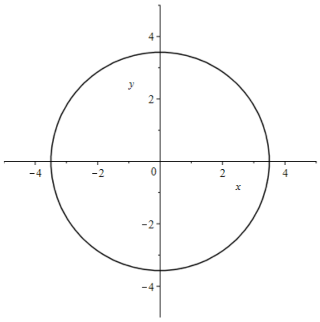

Figure 2: (a)

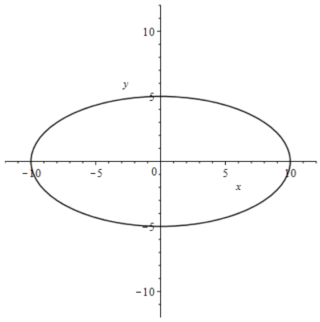

Figure 3: (b)

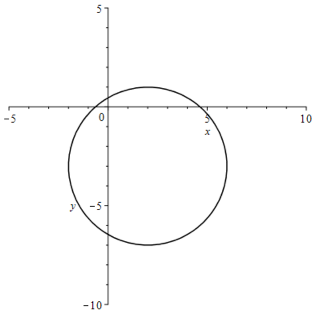

Figure 4: (c)

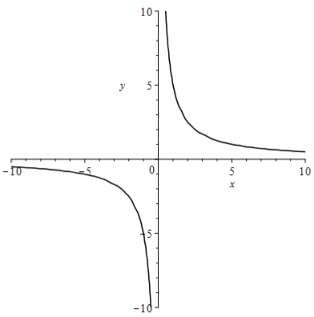

Figure 5: (d)

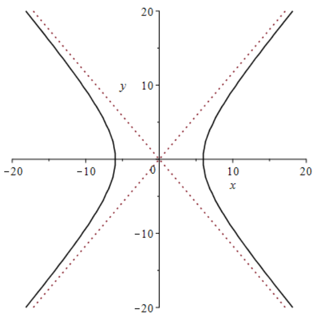

Figure 6: (e)

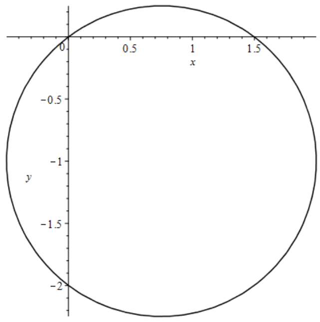

Figure 7: (f)

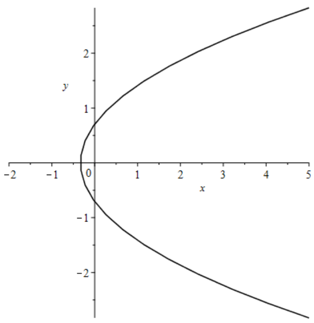

Figure 8: (g)

Q7. In each case, substitute $y = 2x + 1$ into the equation of the circle and solve the resulting quadratic equation for x.

(a) $(1/5, 7/5)$ and $(-1, -1)$.

(b) $(-2, -3)$

(c) There are no points of intersection.

Q8. Substitute $x^2 + y^2 = 8$ into the equation of the first circle:

$$2x - 2y - 8 = 0 \Longleftrightarrow y = x - 4.$$

Therefore, at the point of intersection,

$$x^2 + (x-4)^2 = 8$$

which has the unique solution $x = 2$. Hence, $(2, -2)$ is the unique point of intersection.

Q9. We have

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= \frac{1}{a^2}\frac{a^2(e^t + e^{-t})^2}{4} - \frac{1}{b^2}\frac{b^2(e^t - e^{-t})^2}{4} \\
&= \frac{1}{4}(e^{2t} + 2e^t e^{-t} + e^{-2t} - e^{2t} + 2e^t e^{-t} - e^{-2t}) \\
&= \frac{1}{4}(2+2) \\
&= 1
\end{aligned}$$

as required.

Q10. Recall that the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has asymptotes $y = \pm\frac{b}{a}x$.

(a) $a = 1/\sqrt{2}$ and $b = 1$, so the asymptotes are $y = \pm\sqrt{2}x$.

(b) $a = 2$ and $b = \sqrt{2}$. Given that the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{2} = 1$$

has asymptotes $y = \pm\frac{\sqrt{2}}{2}x = \pm\frac{x}{\sqrt{2}}$, the hyperbola

$$\frac{(x+2)^2}{4} - \frac{(y-3)^2}{2} = 1$$

has asymptotes $y - 3 = \pm\frac{(x+2)}{\sqrt{2}}$.

(c) $a = 3$ and $b = \sqrt{3}$, so the asymptotes are $y + 1 = \pm\frac{(x-1)}{\sqrt{3}}$.