UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercises: Chapter 7

1. Use sigma notation to write down
   (i) the sum of the first 20 positive integers;
   (iii) the sum of the first 15 even positive integers;
   (iii) the sum of the cubes of the first 10 positive integers;
   (iv) the sum of the integers between 15 and 25, inclusive.

2. Write out in full (i.e. without using sigma notation):

$$\text{(i)}\quad \sum_{r=1}^{5} r^4 \qquad \text{(ii)}\quad \sum_{r=1}^{5}(2r-1)^2 \qquad \text{(iii)}\quad \sum_{r=1}^{n}(2r)^2$$

$$\text{(iv)}\quad \sum_{j=1}^{4}(3j)^2 \qquad \text{(v)}\quad \sum_{k=1}^{6} kx^k \qquad \text{(vi)}\quad \sum_{r=0}^{n} \frac{x^r}{r!}\ .$$

3. Given that

$$\sum_{r=1}^{n} r^2 = n(n+1)(2n+1)/6,$$

   determine

$$\text{(a)}\quad 1+4+9+16+25+\cdots+400, \qquad \text{(b)}\quad 2^2+4^2+6^2+\ldots+(2n)^2.$$

4. Use the identity

$$\sum_{j=1}^{n}\left((j+1)^4 - j^4\right) = \sum_{j=1}^{n}(4j^3+6j^2+4j+1)$$

   to show that the sum of the first n cubes is given by

$$\sum_{j=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}.$$

5. Use the formulas for $\sum_{r=1}^{n} r^p$ for $p=1,2,3$ in the lecture notes to find expressions for the following.

$$\text{(a)}\quad \sum_{r=1}^{n}\left(2r^3+(r+1)^2\right), \qquad \text{(b)}\quad \sum_{r=1}^{n}\left(3r^3+2r^2+3r+5\right).$$

6. Apply Definition 7.5 directly to verify that $\lim_{n\to\infty} \frac{n-1}{n} = 1$. (That is, do not use Theorem 7.6, but find N for a given ϵ as in Examples 7F).

7. Determine which of the following sequences $\{u_n\}$ converge. Find the limit of each convergent sequence. (You may use the results in Theorem 7.6).

$$\text{(a)} \quad u_n = \frac{1-2n}{1+2n}, \qquad \text{(b)} \quad u_n = \frac{3+4n^4}{n^4+3n^3}, \qquad \text{(c)} \quad u_n = \frac{n^2-2n+1}{n-1}.$$

8. Determine the value of

$$2/3 + 2^2/3^2 + 2^3/3^3 + \dots$$

9. In the notes, we derive the following formula for the sum to n terms of a geometric series (for $r \neq 1$):

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}.$$

Use a similar method to derive a formula for the sum

$$1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + 4 \times 3^4 + \cdots + n \times 3^n,$$

then generalise this to get a general formula for

$$\sum_{k=1}^{n} akr^k.$$

Exercises: Chapter 8

1. Without using a calculator, evaluate $5!$, $6!$ $7!$, ... $10!$.

2. Use your calculator to find $20!$, $30!$ and $40!$. (Your calculator may have a "factorial function key".) What is the smallest value of n for which your calculator produces an E (for ERROR)? Why has this happened?

3. Factorise (a) $5! + 4!$ (b) $100! - 98!$ (c) $(n+1)! - n!$.

4. Without using a calculator, evaluate
$$\text{(a)} \quad \frac{15! - 13!}{11!2!}, \qquad \text{(b)} \quad \frac{12! + 11!}{8!3!}.$$

5. By putting the LHS over a common denominator, show that for any positive integer n (with $n > 2$)
$$\frac{1}{n!} + \frac{3}{(n-1)!} + \frac{1}{(n-2)!} = \frac{(n+1)^2}{n!}.$$

6. Compute the values of
$$\text{(a)} \quad \binom{7}{2}, \qquad \text{(b)} \quad \binom{6}{r} \quad \text{for } r = 0, 1, 2, \ldots, 6.$$

7. Find r if
$$\binom{14}{r} = \binom{14}{r-4}.$$

8. Extend Pascal's Triangle by two more rows and hence write out the binomial expansion of
$$\text{(i)} \quad (a+b)^7 \qquad \text{(ii)} \quad (a+b)^8.$$

9. Use the binomial expansion to expand the following expressions, simplifying the terms in the expansion where possible.

(i)	$(x-y)^4$	*(ii)*	$(2x+y)^5$	*(iii)*	$(2p+3q)^4$
(iv)	$(x-2y)^6$	*(v)*	$(4r-3s)^5$	*(vi)*	$\left(x+\frac{1}{x}\right)^5$
(vii)	$\left(2y^2-\frac{1}{3y}\right)^4.$				

10. Find

(i) the coefficient of x^5 in the expansion of $(1+2x)^9$.

(ii) the coefficient of x^3 in the expansion of $\left(x+\dfrac{3}{x}\right)^7$.

(iii) the constant term in the expansion of $\left(3x - \dfrac{2}{x^2}\right)^{12}$.

11. Write out

(i) the first four terms in the expansion of $(1+2x)^9$.

(ii) the first three terms in the expansion of $\left(1 + \dfrac{3}{x^2}\right)^7$.

(iii) first four terms in the expansion of $\left(1 - \dfrac{x^2}{3}\right)^8$.

12. Show that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

This is the rule for calculating the numbers in Pascal's Triangle.

13. (a) By choosing particular values of x and y in the binomial expansion of $(x+y)^n$, show that

$$\text{(i)} \quad \sum_{r=0}^{n} \binom{n}{r} = 2^n, \qquad \text{(ii)} \quad \sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0.$$

(b)* Use induction to prove the first result in part (a) by induction.

14. Use the Binomial Theorem to find

(a) the coefficient of x^5 in the expansion of $(3x-2)^7$,

(b) the coefficient of x^3 in the expansion of $\left(2x^2 - \dfrac{1}{x}\right)^9$.

15. Find the general x^r term in the expansion of $\left(x - \dfrac{3}{x}\right)^n$.

16. Write down the general term in the expansion of $\left(x^2 - \dfrac{1}{2x}\right)^{38}$ and hence evaluate the coefficient of the x^{-17} term in this expansion.