16 Integration

16.1 (a) $\displaystyle\int x^6\,\mathrm{d}x = \frac{1}{7}x^7$

(b) $\displaystyle\int x^{-2}\,\mathrm{d}x = -x^{-1}$

(c) $\displaystyle\int x^{1/3}\,\mathrm{d}x = \frac{3}{4}x^{4/3}$

(d) $\displaystyle\int x^{-3/2}\,\mathrm{d}x = -2x^{-1/2}$

(e) $\displaystyle\int \frac{1}{x^3}\,\mathrm{d}x = \int x^{-3}\,\mathrm{d}x = -\frac{1}{2}x^{-2} = -\frac{1}{2x^2}.$

16.2 (a) $\displaystyle\int_1^2 x^4\,\mathrm{d}x = \left[\frac{1}{5}x^5\right]_1^2 = \frac{1}{5}(32-1) = \frac{31}{5}.$

(b) $\displaystyle\int_2^5 x^{-3}\,\mathrm{d}x = \left[-\frac{1}{2x^2}\right]_2^5 = -\frac{1}{50} + \frac{1}{8} = \frac{-4+25}{200} = \frac{21}{200}.$

(c) $\displaystyle\int_4^9 x^{3/2}\,\mathrm{d}x = \left[\frac{2}{5}x^{5/2}\right]_4^9 = \frac{2}{5}(3^5 - 2^5) = \frac{2}{5}(243-32) = \frac{422}{5}.$

(d) $\displaystyle\int_1^4 \frac{1}{x^2\sqrt{x}}\,\mathrm{d}x = \int_1^4 x^{-5/2}\,\mathrm{d}x = -\frac{2}{3}\left[x^{-3/2}\right]_1^4 = -\frac{2}{3}\left(\frac{1}{8}-1\right) = \frac{7}{12}.$

16.3 $\displaystyle\int_{-a}^{a} x^3\,\mathrm{d}x = \left[\frac{1}{4}x^4\right]_{-a}^{a} = \frac{1}{4}(a^4 - a^4) = 0$

$\displaystyle\int_{-a}^{a} x^{2n-1}\,\mathrm{d}x = \left[\frac{1}{2n}x^{2n}\right]_{-a}^{a} = \frac{1}{2n}(a^{2n} - a^{2n}) = 0$

$\displaystyle\int_{-a}^{a} x^{2n}\,\mathrm{d}x = \frac{1}{2n+1}(a^{2n+1} + a^{2n+1}) = \frac{2a^{2n+1}}{2n+1}$

This cannot be extended to negative integers because the integrand is not continuous (or, indeed, integrable) at 0.

16.4 (a) $\displaystyle\int (x^2+1)\,\mathrm{d}x = \frac{1}{3}x^3 + x$

(b) $\displaystyle\int (2x^3+4x-2)\,\mathrm{d}x = \frac{1}{2}x^4 + 2x^2 - 2x$

(c) $\displaystyle\int (x+\sqrt{x})\,\mathrm{d}x = \int (x+x^{1/2})\,\mathrm{d}x = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2}$

(d) $\displaystyle\int (\sin x + 2\cos x)\,\mathrm{d}x = -\cos x + 2\sin x$

(e) $\displaystyle\int (4x^{-1/2} - 2\sec^2 x)\,\mathrm{d}x = 8x^{1/2} - 2\tan x = 8\sqrt{x} - 2\tan x$

(f) $\displaystyle\int (2x^2 - \sqrt{x^3} + 5\sin x)\,\mathrm{d}x = \int (2x^2 - x^{3/2} + 5\sin x)\,\mathrm{d}x = \frac{2}{3}x^3 - \frac{2}{5}x^{5/2} - 5\cos x$
$\displaystyle = \frac{2}{3}x^3 - \frac{2}{5}x^2\sqrt{x} - 5\cos x$

16.5 (a) $\displaystyle\int (x+2)(2x-1)\,\mathrm{d}x = \int (2x^2+3x-2)\,\mathrm{d}x = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$

(b) $\displaystyle\int \sqrt{x}\,(2x-1)\,\mathrm{d}x = \int (2x^{3/2} - x^{1/2})\,\mathrm{d}x = \frac{4}{5}x^{5/2} - \frac{2}{3}x^{3/2}$

(c) $\displaystyle\int \frac{2x-1}{\sqrt{x}}\,\mathrm{d}x = \int (2x^{1/2} - x^{-1/2})\,\mathrm{d}x = \frac{4}{3}x^{3/2} - 2x^{1/2}$

(d) $\displaystyle\int_{-3}^{1} \frac{x^2+5x-24}{x-3}\,\mathrm{d}x = \int_{-3}^{1} \frac{(x-3)(x+8)}{x-3}\,\mathrm{d}x = \int_{-3}^{1} (x+8)\,\mathrm{d}x = \left[\frac{1}{2}x^2 + 8x\right]_{-3}^{1}$
$\displaystyle = \frac{1}{2} + 8 - \frac{9}{2} + 24 = 28$

(e) $\displaystyle\int_{0}^{1} x^2(1-x)^2\,\mathrm{d}x = \int_{0}^{1} (x^4 - 2x^3 + x^2)\,\mathrm{d}x = \left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_{0}^{1}$
$\displaystyle = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} = \frac{6-15+10}{30} = \frac{1}{30}$

(f) $\displaystyle\int_{-2}^{1} x(x-1)(x+2)\,\mathrm{d}x = \int_{-2}^{1} (x^3 + x^2 - 2x)\,\mathrm{d}x = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_{-2}^{1}$
$\displaystyle = \frac{1}{4} + \frac{1}{3} - 1 - (4 - 8/3 - 4) = -\frac{3}{4} + \frac{9}{3} = \frac{9}{4}$

16.6 (a) $f'(x) = x^3 - x \iff f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$, $f(0) = C = 1$, so $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + 1$.

(b) $f'(x) = x^2 + x^{-2} \iff f(x) = \frac{1}{3}x^3 - \frac{1}{x} + C$, $f(1) = \frac{1}{3} - 1 + C = 2 \iff C = \frac{8}{3}$, so $f(x) = \frac{1}{3}x^3 - \frac{1}{x} + \frac{8}{3}$.

(c) $f''(x) = x^2 \iff f'(x) = \frac{1}{3}x^3 + C$
$f'(0) = C = 1$, so $f'(x) = \frac{1}{3}x^3 + 1$, and then
$f(x) = \frac{1}{12}x^4 + x + D$, $f(0) = D = 2$, and thus $f(x) = \frac{1}{12}x^4 + x + 2$.

(d) $f''(x) = \sin(x) \iff f'(x) = -\cos(x) + C$,
$f'(\pi) = -\cos(\pi) + C = 1 + C = -1 \iff C = -2$, and so $f'(x) = -\cos(x) - 2$.
$f(x) = -\sin(x) - 2x + D$, $f(2\pi) = -\sin(2\pi) - 4\pi + D = 2 \iff D = 2 + 4\pi$, hence $f(x) = 2 + 4\pi - 2x - \sin(x)$.

16.7 (a) $\int_0^\pi x\cos x\,\mathrm{d}x = [x\sin x]_0^\pi - \int_0^\pi \sin x\,\mathrm{d}x = 0 + [\cos x]_0^\pi = -2$

(b) $\int_0^a x^2\cos\left(\frac{\pi x}{a}\right)\,\mathrm{d}x = \left[\frac{a}{\pi}x^2\sin\left(\frac{\pi x}{a}\right)\right]_0^a - \frac{2a}{\pi}\int_0^a x\sin\left(\frac{\pi x}{a}\right)\,\mathrm{d}x$
$= 0 + \left[\frac{2a^2}{\pi^2}x\cos\left(\frac{\pi x}{a}\right)\right]_0^a - \frac{2a^2}{\pi^2}\int_0^a \cos\left(\frac{\pi x}{a}\right)\,\mathrm{d}x = \frac{2a^3}{\pi^2}\cos\pi - \frac{2a^3}{\pi^3}\left[\sin\left(\frac{\pi x}{a}\right)\right]_0^a$
$= -\frac{2a^3}{\pi^2}$

(c) $\int_0^1 x(1-x)\sin\pi x\,\mathrm{d}x = \left[-\frac{1}{\pi}x(1-x)\cos\pi x\right]_0^1 + \frac{1}{\pi}\int_0^1 (1-2x)\cos\pi x\,\mathrm{d}x$
$= 0 + \left[\frac{1}{\pi^2}(1-2x)\sin\pi x\right]_0^1 + \frac{2}{\pi^2}\int_0^1 \sin\pi x\,\mathrm{d}x = 0 + \left[-\frac{2}{\pi^3}\cos\pi x\right]_0^1$
$= -\frac{2}{\pi^3}(-1-1) = \frac{4}{\pi^3}$

(d) $\int x\ln x\,\mathrm{d}x = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{x^2}{4} = \frac{x^2}{4}(2\ln x - 1)$

16.7 (e) $\int (\ln x)^2 \,\mathrm{d}x = \ln x\,(x\ln x - x) - \int \frac{1}{x}\cdot(x\ln x - x)\,\mathrm{d}x = \ln x\,(x\ln x - x) - (x\ln x - x - x) = x(\ln x)^2 - 2x\ln x + 2x = x((\ln x - 1)^2 + 1) = x\left(\left(\ln\frac{x}{e}\right)^2 + 1\right)$

(f)

$$\begin{aligned}\int \sin^2 u\,\mathrm{d}u &= -\sin u\,\cos u + \int \cos^2 u\,\mathrm{d}u\\ &= -\sin u\,\cos u + \int (1-\sin^2 u)\,\mathrm{d}u\\ &= u - \sin u\,\cos u - \int \sin^2 u\,\mathrm{d}u,\end{aligned}$$

and so

$$\int \sin^2 u\,\mathrm{d}u = \frac{u - \sin u\,\cos u}{2}.$$

16.8 $\int_a^b (x-a)(b-x)\,f''(x)\,\mathrm{d}x = [(x-a)(b-x)f'(x)]_a^b - \int_a^b (a+b-2x)f'(x)\,\mathrm{d}x$
$= \int_a^b (2x-a-b)f'(x)\,\mathrm{d}x = [(2x-a-b)f(x)]_a^b - 2\int_a^b f(x)\,\mathrm{d}x = -2\int_a^b f(x)\,\mathrm{d}x.$

16.9 (a) $\int_0^1 xe^{-x}\,\mathrm{d}x = \big[-xe^{-x}\big]_0^1 + \int_0^1 e^{-x}\,\mathrm{d}x$
$= -e^{-1} + \big[-e^{-x}\big]_0^1 = 1 - 2e^{-1}$

(b) $\int_0^1 x^3\,e^{-2x}\,\mathrm{d}x = \left[-\frac{1}{2}\,x^3e^{-2x}\right]_0^1 + \frac{3}{2}\int_0^1 x^2\,e^{-2x}\,\mathrm{d}x$

$= -\frac{1}{2}\,e^{-2} + \left[-\frac{3}{4}\,x^2\,e^{-2x}\right]_0^1 + \frac{6}{4}\int_0^1 xe^{-2x}\,\mathrm{d}x$

$= -\frac{1}{2}\,e^{-2} - \frac{3}{4}\,e^{-2} + \left[-\frac{6}{8}\,xe^{-2x}\right]_0^1 + \frac{3}{4}\int_0^1 e^{-2x}\,\mathrm{d}x$

$= -\frac{1}{2}\,e^{-2} - \frac{3}{4}\,e^{-2} - \frac{3}{4}\,e^{-2} + \left[-\frac{3}{8}\,e^{-2x}\right]_0^1$

$= -2e^{-2} - \frac{3}{8}\,e^{-2} + \frac{3}{8} = \frac{3}{8} - \frac{19}{8}\,e^{-2}$

16.10 $I_n = \int_0^1 x^n\,e^{-x}\,\mathrm{d}x = \big[-x^n\,e^{-x}\big]_0^1 + n\int_0^1 x^{n-1}\,e^{-x}\,\mathrm{d}x$
$= -e^{-1} + n\,I_{n-1} = n\,I_{n-1} - \frac{1}{e}$

16.11 (a) $\displaystyle\int (x+1)^2\,\mathrm{d}x = \frac{1}{3}(x+1)^3$

(b) $\displaystyle\int (2x-1)^3\,\mathrm{d}x = \frac{1}{2}\cdot\frac{1}{4}(2x-1)^4 = \frac{1}{8}(2x-1)^4$

(c) $\displaystyle\int (3-4x)^{-3}\,\mathrm{d}x = -\frac{1}{4}\left(-\frac{1}{2}\right)(3-4x)^{-2} = \frac{1}{8}(3-4x)^{-2}$

(d) $\displaystyle\int \cos\pi x\,\mathrm{d}x = \frac{1}{\pi}\sin\pi x$

(e) $\displaystyle\int \frac{1}{\sqrt{1-(2x)^2}}\,\mathrm{d}x = \frac{1}{2}\arcsin 2x$

(f) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x = \frac{1}{3}\int \frac{\mathrm{d}x}{\sqrt{1-(x/3)^2}} = \frac{1}{3}\cdot 3\arcsin\frac{x}{3} = \arcsin\frac{x}{3}$

(g) $\displaystyle\int \frac{1}{1+(6x)^2}\,\mathrm{d}x = \frac{1}{6}\arctan 6x$

(h) $\displaystyle\int \frac{1}{49+x^2}\,\mathrm{d}x = \frac{1}{49}7\arctan\frac{x}{7} = \frac{1}{7}\arctan\frac{x}{7}$

16.12 (a) $\displaystyle\int_1^3 (2x-5)^2\,\mathrm{d}x = \left[\frac{1}{6}(2x-5)^3\right]_1^3 = \frac{1}{6}(1-(-3)^3) = \frac{1}{6}(1+27) = \frac{14}{3}$

(b) $\displaystyle\int_0^1 (3x+1)^{1/2}\,\mathrm{d}x = \left[\frac{1}{3}\cdot\frac{2}{3}(3x+1)^{3/2}\right]_0^1 = \frac{2}{9}(4^{3/2}-1) = \frac{14}{9}$

(c) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin\left(\frac{\pi}{8}-\frac{x}{2}\right)\,\mathrm{d}x = \left[(-2)\left(-\cos\left(\frac{\pi}{8}-\frac{x}{2}\right)\right)\right]_{-\pi/4}^{\pi/4}$
$\displaystyle = \left[2\cos\left(\frac{\pi}{8}-\tfrac{x}{2}\right)\right]_{-\pi/4}^{\pi/4} = 2\left(\cos 0-\cos\frac{\pi}{4}\right) = 2\left(1-\frac{1}{\sqrt{2}}\right) = 2-\sqrt{2}$

(d) $\displaystyle\int_1^2 [(2x-5)^3+x]\,\mathrm{d}x = \left[\frac{1}{8}(2x-5)^4+\frac{1}{2}x^2\right]_1^2 = \frac{1}{8}(1-(-3)^4)+\frac{1}{2}(4-1)$
$\displaystyle = \frac{1}{8}-\frac{81}{8}+\frac{3}{2} = -10+\frac{3}{2} = -\frac{17}{2}$

16.12 (e) $$\int_0^1 \left\{2\sin\left[\pi\left(x-\frac{1}{2}\right)\right] - 2\sec^2\left[\frac{\pi(2x-1)}{4}\right]\right\} \mathrm{d}x$$
$$= \left[-\frac{2}{\pi}\cos\left[\pi\left(x-\frac{1}{2}\right)\right] - \frac{4}{\pi}\tan\left[\frac{\pi}{4}(2x-1)\right]\right]_0^1$$
$$= -\frac{2}{\pi}\cos\frac{\pi}{2} - \frac{4}{\pi}\tan\frac{\pi}{4} + \frac{2}{\pi}\cos\left(-\frac{\pi}{2}\right) + \frac{4}{\pi}\tan\left(-\frac{\pi}{4}\right) = 0 - \frac{4}{\pi} + 0 - \frac{4}{\pi} = -\frac{8}{\pi}$$

16.13 (a) $$\int \frac{\mathrm{d}x}{(2x+5)^2+1} = \frac{1}{2}\arctan(2x+5)$$

(b) $$\int_2^5 \frac{\mathrm{d}x}{(x-2)^2+9} = \frac{1}{9}\int_2^5 \frac{\mathrm{d}x}{1+\left(\frac{x-2}{3}\right)^2} = \left[\frac{1}{9}\cdot 3\arctan\left(\frac{x-2}{3}\right)\right]_2^5$$
$$= \frac{1}{3}(\arctan 1 - \arctan 0) = \frac{\pi}{12}$$

(c) $$\int \frac{\mathrm{d}x}{\sqrt{1-(3x-1)^2}} = \frac{1}{3}\arcsin(3x-1)$$

(d) $$\int_{-\frac{1}{4}}^0 \frac{\mathrm{d}x}{\sqrt{1-(2x+1)^2}} = \left[\frac{1}{2}\arcsin(2x+1)\right]_{-\frac{1}{4}}^0 = \frac{1}{2}\left(\arcsin 1 - \arcsin\frac{1}{2}\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

(e) $$\int \frac{\mathrm{d}x}{x^2-6x+10} = \int \frac{\mathrm{d}x}{(x-3)^2+1} = \arctan(x-3)$$

(f) $$\int_0^4 \frac{\mathrm{d}x}{x^2-4x+8} = \int_0^4 \frac{\mathrm{d}x}{(x-2)^2+4} = \frac{1}{4}\int_0^4 \frac{\mathrm{d}x}{1+\left(\frac{x-2}{2}\right)^2}$$
$$= \left[\frac{1}{4}\cdot 2\arctan\left(\frac{x-2}{2}\right)\right]_0^4 = \frac{1}{2}\{\arctan 1 - \arctan(-1)\} = \frac{\pi}{4}$$

(g) $$\int \frac{\mathrm{d}x}{\sqrt{4x-x^2+5}} = \int \frac{\mathrm{d}x}{\sqrt{5-(x^2-4x)}} = \int \frac{\mathrm{d}x}{\sqrt{9-(x-2)^2}}$$
$$= \frac{1}{3}\int \frac{\mathrm{d}x}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}} = \frac{1}{3}\cdot 3\arcsin\left(\frac{x-2}{3}\right) = \arcsin\left(\frac{x-2}{3}\right)$$

16.13 (h) $$\int_1^2 \frac{\mathrm{d}x}{\sqrt{12x-4x^2-8}} = \int_1^2 \frac{\mathrm{d}x}{\sqrt{-8-(2x-3)^2+9}} = \int_1^2 \frac{\mathrm{d}x}{\sqrt{1-(2x-3)^2}}$$
$$= \left[\frac{1}{2}\arcsin(2x-3)\right]_1^2 = \frac{1}{2}\{\arcsin 1 - \arcsin(-1)\} = \pi/2$$

(i) $$\int \frac{\mathrm{d}x}{\sqrt{-x^2-x}} = \int \frac{\mathrm{d}x}{\sqrt{\frac{1}{4}-\left(x+\frac{1}{2}\right)^2}} = 2\int \frac{\mathrm{d}x}{\sqrt{1-(2x+1)^2}} = 2\cdot\frac{1}{2}\arcsin(2x+1)$$
$$= \arcsin(2x+1)$$

16.14 (a) $$\int \tan^2 x \sec^2 x \,\mathrm{d}x = \int \tan^2 x \frac{\mathrm{d}}{\mathrm{d}x}(\tan x)\,\mathrm{d}x = \frac{1}{3}\tan^3 x$$

(b) $$\int \frac{\cos x}{\sqrt{1+\sin x}}\,\mathrm{d}x = \int (1+\sin x)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\,\mathrm{d}x = 2(1+\sin x)^{1/2}$$

(c) $$\int x^2(1-x^3)^{1/3}\,\mathrm{d}x = -\frac{1}{3}\int (1-x^3)^{1/3}(-3x^2)\,\mathrm{d}x = -\frac{1}{3}\int (1-x^3)^{1/3}\frac{\mathrm{d}}{\mathrm{d}x}(-x^3)\,\mathrm{d}x$$
$$= -\frac{1}{3}\cdot\frac{3}{4}(1-x^3)^{4/3} = -\frac{1}{4}(1-x^3)^{4/3}$$

(d) $$\int_0^{\pi/2} \frac{\cos x}{(3+\sin x)^2}\,\mathrm{d}x = \left[-\frac{1}{(3+\sin x)}\right]_0^{\pi/2} = -\frac{1}{4}+\frac{1}{3} = \frac{1}{12}$$

(e) $$\int_0^{\pi/2} \cos^3\frac{x}{2}\,\mathrm{d}x = \int_0^{\pi/2}\left[1-\sin^2\frac{x}{2}\right]\cos\frac{x}{2}\,\mathrm{d}x$$
$$= 2\int_0^{\pi/2}\left[1-\sin^2\frac{x}{2}\right]\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\frac{x}{2}\right)\,\mathrm{d}x = 2\left[\sin\frac{x}{2}-\frac{1}{3}\sin^3\frac{x}{2}\right]_0^{\pi/2}$$
$$= 2\sin\frac{\pi}{4}-\frac{2}{3}\sin^3\frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}}-\frac{1}{6\sqrt{2}}\right) = 2\sqrt{2}\left(\frac{1}{2}-\frac{1}{12}\right) = \frac{5}{6}\sqrt{2}$$

(f) $$\int x\sin^3(x^2)\cos(x^2)\,\mathrm{d}x = \frac{1}{2}\int \sin^3(x^2)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\sin(x^2)\,\mathrm{d}x = \frac{1}{8}\sin^4(x^2)$$

(g) $$\int \tan^2 x\,\mathrm{d}x = \int (1+\tan^2 x)\,\mathrm{d}x - \int \mathrm{d}x = \tan x + x$$

16.14 (h) With $u = \sqrt{\tan x}$ we have $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1+\tan^2 x}{2\sqrt{\tan x}}$, that is, $\mathrm{d}x = \dfrac{2u}{1+u^4}\,\mathrm{d}u$. So

$$\int \sqrt{\tan x}\,\mathrm{d}x = \int \frac{2u^2}{1+u^4}\,\mathrm{d}u.$$

With $1+u^4 = (1+\sqrt{2}\,u+u^2)(1-\sqrt{2}\,u+u^2)$, this becomes

$$\int \frac{2u^2}{1+u^4}\,\mathrm{d}u = \frac{1}{\sqrt{2}}\int \frac{u}{1-\sqrt{2}\,u+u^2}\,\mathrm{d}u - \frac{1}{\sqrt{2}}\int \frac{u}{1+\sqrt{2}\,u+u^2}\,\mathrm{d}u.$$

We now write

$$\int \frac{u}{1+\sqrt{2}\,u+u^2}\,\mathrm{d}u = \frac{1}{2}\int \frac{2u+\sqrt{2}}{1+\sqrt{2}\,u+u^2}\,\mathrm{d}u - \frac{1}{\sqrt{2}}\int \frac{1}{1+\sqrt{2}\,u+u^2}\,\mathrm{d}u.$$

Then

$$\int \frac{2u+\sqrt{2}}{1+\sqrt{2}\,u+u^2}\,\mathrm{d}u = \ln(1+\sqrt{2}\,u+u^2).$$

Completing the square gives

$$\int \frac{1}{1+\sqrt{2}\,u+u^2}\,\mathrm{d}u = \int \frac{1}{(u+\frac{1}{\sqrt{2}})^2+\frac{1}{2}}\,\mathrm{d}u = \sqrt{2}\,\arctan(\sqrt{2}\,u+1).$$

Likewise,

$$\int \frac{u}{1-\sqrt{2}\,u+u^2}\,\mathrm{d}u = \frac{1}{2}\int \frac{2u-\sqrt{2}}{1-\sqrt{2}\,u+u^2}\,\mathrm{d}u + \frac{1}{\sqrt{2}}\int \frac{1}{1-\sqrt{2}\,u+u^2}\,\mathrm{d}u,$$

and then

$$\int \frac{2u-\sqrt{2}}{1-\sqrt{2}\,u+u^2}\,\mathrm{d}u = \ln(1-\sqrt{2}\,u+u^2)$$

and

$$\int \frac{1}{1-\sqrt{2}\,u+u^2}\,\mathrm{d}u = \int \frac{1}{(u-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}\,\mathrm{d}u = \sqrt{2}\,\arctan(\sqrt{2}\,u-1).$$

So

$$\int \frac{2u^2}{1+u^4}\,\mathrm{d}u = \frac{1}{\sqrt{2}}\left(\ln\sqrt{\frac{1-\sqrt{2}\,u+u^2}{1+\sqrt{2}\,u+u^2}} + \arctan(\sqrt{2}\,u-1) + \arctan(\sqrt{2}\,u+1)\right),$$

and substituting back

$$\int \sqrt{\tan x}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\Bigg(\ln\sqrt{\frac{1-\sqrt{2\tan x}+\tan x}{1+\sqrt{2\tan x}+\tan x}} + \arctan(\sqrt{2\tan x}-1) + \arctan(\sqrt{2\tan x}+1)\Bigg).$$

16.15 (a) $u = 4x+3 \iff x = \frac{1}{4}(u-3), \quad \mathrm{d}x = \frac{1}{4}\mathrm{d}u$

$$\int x(4x+3)^4 \mathrm{d}x = \int \frac{1}{4}(u-3)u^4 \cdot \frac{1}{4}\mathrm{d}u = \frac{1}{16}\int (u^5 - 3u^4)\mathrm{d}u = \frac{1}{16}\left(\frac{1}{6}u^6 - \frac{3}{5}u^5\right)$$
$$= \frac{u^5}{16}\left(\frac{1}{6}u - \frac{3}{5}\right) = \frac{u^5}{16}\frac{(5u-18)}{30} = \frac{(4x+3)^5}{480}(20x+15-18)$$
$$= \frac{1}{480}(4x+3)^5(20x-3)$$

(b) $u = x+3$, so $\mathrm{d}u = \mathrm{d}x$, when $x = 1$, then $u = 4$, when $x = 6$, then $u = 9$.

$$\int_1^6 \frac{x}{\sqrt{x+3}}\,\mathrm{d}x = \int_4^9 \frac{u-3}{\sqrt{u}}\,\mathrm{d}u = \int_4^9 (u^{1/2} - 3u^{-1/2})\,\mathrm{d}u = \left[\frac{2}{3}u^{3/2} - 6u^{1/2}\right]_4^9 =$$
$$18 - 18 - \frac{16}{3} + 12 = \frac{20}{3}$$

(c) $u = x^2$, so $\mathrm{d}u = 2x\,\mathrm{d}x \iff \frac{1}{2}\,\mathrm{d}u = x\,\mathrm{d}x$

$$\int \frac{x}{1+x^4}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{1+u^2}\,\mathrm{d}u = \frac{1}{2}\arctan u = \frac{1}{2}\arctan(x^2)$$

(d) $I = \displaystyle\int_0^{\pi/2} \frac{\cos x \sin x}{(2+\sin^2 x)^2}\,\mathrm{d}x$. Put $u = \sin x$, then $\mathrm{d}u = \cos x\,\mathrm{d}x$, when $x = 0$, then $u = 0$, when $x = \pi/2$, then $u = 1$. Now $I = \displaystyle\int_0^1 \frac{u}{(2+u^2)^2}\,\mathrm{d}u$. Put $v = u^2$, then $\mathrm{d}v = 2u\,\mathrm{d}u \iff u\,\mathrm{d}u = \frac{1}{2}\,\mathrm{d}v$; when $u = 0$, then $v = 0$, when $u = 1$, then $v = 1$. So

$$I = \int_0^1 \frac{\mathrm{d}v}{(2+v)^2} = \frac{1}{2}\left[-\frac{1}{2+v}\right]_0^1 = \frac{1}{2}\left(-\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{12}.$$

16.16 (a) $I = \displaystyle\int x(3x-1)^{1/3}\,\mathrm{d}x. \quad u = 3x-1 \iff x = \frac{1}{3}(u+1)$, and $\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$.

$$I = \int \frac{1}{3}(u+1)u^{1/3} \cdot \frac{1}{3}\,\mathrm{d}u = \frac{1}{9}\int (u^{4/3} + u^{1/3})\,\mathrm{d}u = \frac{1}{9}\left(\frac{3}{7}u^{7/3} + \frac{3}{4}u^{4/3}\right)$$
$$= \frac{1}{3}u^{4/3}\left(\frac{1}{7}u + \frac{1}{4}\right) = \frac{1}{3}u^{4/3}\left(\frac{4u+7}{28}\right) = \frac{1}{84}(3x-1)^{4/3}(12x-4+7)$$
$$= \frac{1}{28}(3x-1)^{4/3}(4x+1)$$

(b) $I = \displaystyle\int_1^2 \frac{x}{(2x-1)^3}\,\mathrm{d}x. \quad u = 2x-1 \iff x = \frac{1}{2}(u+1)$, so $\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$.

$$I = \int_1^3 \frac{1}{2}\cdot\frac{u+1}{u^3}\cdot\frac{1}{2}\,\mathrm{d}u = \frac{1}{4}\int_1^3 (u^{-2} + u^{-3})\,\mathrm{d}u = \frac{1}{4}\left[-\frac{1}{u} - \frac{1}{2u^2}\right]_1^3$$
$$= \frac{1}{4}\left(-\frac{1}{3} - \frac{1}{18} + 1 + \frac{1}{2}\right) = \frac{1}{4}\left(\frac{18+9-6-1}{18}\right) = \frac{1}{4}\cdot\frac{20}{18} = \frac{5}{18}$$

16.16 (c) $I = \displaystyle\int \frac{\tan x\ \sec^2 x}{\sqrt{1+\tan x}}\,\mathrm{d}x. \quad u = \tan x$, so $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$.

$$I = \int \frac{u}{\sqrt{1+u}}\,\mathrm{d}u. \quad v = 1+u \iff u = v-1, \text{ and } \mathrm{d}u = dv.$$

$$I = \int \frac{v-1}{\sqrt{v}} dv = \int (v^{1/2} - v^{-1/2}) dv = \frac{2}{3}v^{3/2} - 2v^{1/2} = \frac{2}{3}\sqrt{v}(v-3)$$

$$= \frac{2}{3}\sqrt{1+u}\,(u-2) = \frac{2}{3}(\tan x - 2)\,\sqrt{1+\tan x} = \frac{2}{3}(1+\tan x)^{3/2} - 2\sqrt{1+\tan x}$$

(d) $I = \displaystyle\int_0^1 x\sin(\pi x^2)\,\mathrm{d}x. \quad u = \pi x^2$, so $\mathrm{d}u = 2\pi x\,\mathrm{d}x \iff x\,\mathrm{d}x = \dfrac{1}{2\pi}\,\mathrm{d}u$.

$$I = \int_0^\pi \frac{1}{2\pi}\sin u\,\mathrm{d}u = \left[-\frac{1}{2\pi}\cos u\right]_0^\pi = \frac{1}{2\pi} + \frac{1}{2\pi} = \frac{1}{\pi}$$

16.17 With $x = \dfrac{1}{3}[(3x-1)+1]$, $\displaystyle\int x(3x-1)^{1/3}\,\mathrm{d}x = \frac{1}{3}\int [(3x-1)^{4/3} + (3x-1)^{1/3}]\,\mathrm{d}x$

$$= \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{3}{7}(3x-1)^{7/3} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{3}{4}(3x-1)^{4/3} = \frac{1}{21}(3x-1)^{7/3} + \frac{1}{12}(3x-1)^{4/3}$$

$$= \frac{1}{84}(3x-1)^{4/3}\{4(3x-1)+7\} = \frac{1}{84}(3x-1)^{4/3}(12x+3) = \frac{1}{28}(3x-1)^{4/3}(4x+1)$$

16.18 The natural domain of the integrand is $x \in (0,1)$.

(i) $u = \sqrt{x} \iff x = u^2$, so $\mathrm{d}u = \dfrac{1}{2\sqrt{x}}\,\mathrm{d}x \iff \dfrac{1}{\sqrt{x}}\,\mathrm{d}x = 2\,\mathrm{d}u$.

$$\int \frac{\mathrm{d}x}{\sqrt{x}\,\sqrt{1-x}} = 2\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = 2\arcsin u = 2\arcsin(\sqrt{x}).$$

(ii) $u = \sqrt{1-x} \iff u^2 = 1-x, \iff x = 1-u^2$, so $\mathrm{d}u = -\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{1-x}}\,\mathrm{d}x$.

$$\int \frac{\mathrm{d}x}{\sqrt{x}\,\sqrt{1-x}} = -2\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = -2\arcsin u = -2\arcsin(\sqrt{1-x}).$$

(iii) Since $0 < \sqrt{x} < 1$, $\quad 0 < \arcsin(\sqrt{x}) < 1$. Let $\arcsin(\sqrt{x}) = \alpha$. Then (from a right-angled triangle with angles α and β) $\alpha + \beta = \pi/2$ and $\sin(\beta) = \sqrt{1-x}$. Since $0 < \beta < \pi/2$,
$\beta = \arcsin(\sqrt{1-x})$. Hence $\alpha + \beta = \arcsin(\sqrt{x}) + \arcsin(\sqrt{1-x}) = \pi/2$. Finally, the two expressions for the integral from (i) and (ii) differ by

$$2\arcsin(\sqrt{x}) - (-2\arcsin(\sqrt{1-x})) = 2(\arcsin(\sqrt{x}) + \arcsin(\sqrt{1-x})) = \pi,$$

which is a constant. Hence both expressions are indeed primitives of the same function, namely $f(x) = \dfrac{1}{\sqrt{x}\,\sqrt{1-x}}$.

16.19 (a) $\displaystyle\int \frac{1}{\sqrt{8-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(2\sqrt{2})^2-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{2\sqrt{2}}\right).$

(b) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x = \int_0^2 \frac{1}{\sqrt{2^2-x^2}}\,\mathrm{d}x = \left[\arcsin\frac{x}{2}\right]_0^2 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2}.$

(c) $\displaystyle\int \frac{7}{1+x^2}\,\mathrm{d}x = 7\arctan x.$

(d) $\displaystyle\int \frac{1}{x^2+2}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right).$

16.20 (a) $\displaystyle\int \frac{\mathrm{d}x}{3x-1} = \frac{1}{3}\ln|3x-1|$

(b) $$\begin{aligned}\int \frac{2x}{x-1}\,\mathrm{d}x = \int \left(2+\frac{2}{x-1}\right)\mathrm{d}x &= 2x + 2\ln|x-1| \\ &= 2x + \ln[(x-1)^2]\end{aligned}$$

(c) $\displaystyle\int \frac{x^2}{x+7}\,\mathrm{d}x = \int\left(x-7+\frac{49}{x+7}\right)\mathrm{d}x = \frac{1}{2}x^2 - 7x + 49\ln|x+7|$

(d) $\displaystyle\int_0^2 \frac{x}{x^2+1}\,\mathrm{d}x = \left[\frac{1}{2}\ln(x^2+1)\right]_0^2 = \frac{1}{2}\ln 5$

(e) $\displaystyle\int_1^e \frac{1}{x}\ln x\,\mathrm{d}x = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}(\ln e)^2 = \frac{1}{2}$

(f) $$\begin{aligned}\int_{\pi/6}^{\pi/3} \frac{\sin x}{1-\cos x}\,\mathrm{d}x = \Big[\ln|1-\cos x|\Big]_{\pi/6}^{\pi/3} &= \ln\left(\frac{1}{2}\right) - \ln\left(1-\frac{\sqrt{3}}{2}\right) \\ &= \ln\left(\frac{1}{2-\sqrt{3}}\right)\end{aligned}$$

(g) $\displaystyle\int_1^2 x^3\ln x\,\mathrm{d}x = \left[\frac{1}{4}x^4\ln x\right]_1^2 - \frac{1}{4}\int_1^2 x^3\,\mathrm{d}x = 4\ln 2 - \left[\frac{1}{16}x^4\right]_1^2 = 4\ln 2 - \frac{15}{16}$

(h) $\displaystyle\int_0^{\pi/4} \tan x\,\mathrm{d}x = \int_0^{\pi/4}\frac{\sin x}{\cos x}\,\mathrm{d}x = [-\ln(|\cos x|)]_0^{\pi/4} = -\ln\left|\frac{1}{\sqrt{2}}\right| = \ln\sqrt{2}$

16.21 $u = 2x^2+1$, so $\mathrm{d}u = 4x\,\mathrm{d}x \iff x\,\mathrm{d}x = \frac{1}{4}\,\mathrm{d}u$. Then $\displaystyle\int \frac{x}{2x^2+1}\,\mathrm{d}x = \frac{1}{4}\int\frac{\mathrm{d}u}{u} = \frac{1}{4}\ln(2x^2+1).$

16.22 (a) $\displaystyle\int e^{7x+4}\,\mathrm{d}x = \frac{1}{7}\,e^{7x+4}$

(b) $\displaystyle\int e^{2-3x}\,\mathrm{d}x = -\frac{1}{3}\,e^{2-3x}$

(c) $\displaystyle\int e^{4\sin 2x}\cos 2x\,\mathrm{d}x = \frac{1}{8}\int 8e^{4\sin 2x}\cos 2x\,\mathrm{d}x = \frac{1}{8}\,e^{4\sin 2x}$

(d) $\displaystyle\int_0^1 \frac{1}{1+e^{-x}}\,\mathrm{d}x = \int_0^1 \frac{e^x}{e^x+1}\,\mathrm{d}x = [\ln|e^x+1|]_0^1 = \ln(e+1) - \ln 2 = \ln\left(\frac{e+1}{2}\right)$

(e) $\displaystyle\int a^x\,\mathrm{d}x = \int e^{x\ln a}\,\mathrm{d}x = \frac{1}{\ln a}\,e^{x\ln a} = \frac{1}{\ln a}\,a^x$

(f) $\displaystyle\int \frac{a^x}{b^x}\,\mathrm{d}x = \int \left(\frac{a}{b}\right)^x\,\mathrm{d}x = \int e^{x\ln(a/b)}\,\mathrm{d}x = \frac{1}{\ln(a/b)}e^{x\ln(a/b)} = \frac{1}{\ln(a/b)}\,\frac{a^x}{b^x}$

(g) $\displaystyle\int \log_a x\,\mathrm{d}x = \int \frac{\ln x}{\ln a}\,\mathrm{d}x = \frac{1}{\ln a}(x\ln x - x)$

16.23 (a) $\displaystyle\int \sinh(3x+2)\,\mathrm{d}x = \frac{1}{3}\,\cosh(3x+2)$

(b) $\displaystyle\int \tanh x\,\mathrm{d}x = \int \frac{\sinh x}{\cosh x}\,\mathrm{d}x = \ln|\cosh x| = \ln(\cosh x)$

(c) Put $u = \sqrt{x}$, so $\displaystyle \mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x \iff \frac{\mathrm{d}x}{\sqrt{x}} = 2\,\mathrm{d}u$. Then

$$\int_{[\ln 2]^2}^{[\ln 3]^2} \frac{1}{\sqrt{x}}\,\cosh(\sqrt{x})\,\mathrm{d}x = 2\int_{\ln 2}^{\ln 3}\cosh u\,\mathrm{d}u = 2\,[\sinh u]_{\ln 2}^{\ln 3}$$
$$= 2\left[\frac{e^u - e^{-u}}{2}\right]_{\ln 2}^{\ln 3} = \left(3 - \frac{1}{3}\right) - \left(2 - \frac{1}{2}\right) = \frac{7}{6}.$$