UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercises and solutions for MM101 Tutorial in Week 5

1. Without using a calculator, evaluate

(a) $\arcsin\left(\sin\left(-\frac{11\pi}{4}\right)\right)$, (b) $\arccos\left(\cos\left(-\frac{11\pi}{4}\right)\right)$, (c) $\arctan\left(\tan\left(-\frac{11\pi}{4}\right)\right)$.

We observe that $-\frac{11\pi}{4} = -2\pi - \frac{3\pi}{4}$ so

(a) $\arcsin\left(\sin\left(-\frac{11\pi}{4}\right)\right) = \arcsin\left(\sin\left(-\frac{3\pi}{4}\right)\right) = \theta$ where $\sin(\theta) = \sin\left(-\frac{3\pi}{4}\right)$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, that is, $\theta = -\frac{\pi}{4}$;

(b) $\arccos\left(\cos\left(-\frac{11\pi}{4}\right)\right) = \arccos\left(\cos\left(-\frac{3\pi}{4}\right)\right) = \theta$ where $\cos(\theta) = \cos\left(-\frac{3\pi}{4}\right)$ and $0 \leq \theta \leq \pi$, that is, $\theta = \frac{3\pi}{4}$;

(c) $\arctan\left(\tan\left(-\frac{11\pi}{4}\right)\right) = \arctan\left(\tan\left(-\frac{3\pi}{4}\right)\right) = \theta$ where $\tan(\theta) = \tan\left(-\frac{3\pi}{4}\right)$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, that is, $\theta = \frac{\pi}{4}$.

2. Use Definition 7.5 to show that the sequence $\left\{\frac{1+n}{2n}\right\}$ converges to $\frac{1}{2}$.

We have

$$|u_n - l| = \left|\frac{1+n}{2n} - \frac{1}{2}\right| = \frac{1+n-n}{2n} = \frac{1}{2n}$$

so, for any $\epsilon > 0$,

$$|u_n - l| < \epsilon \Leftrightarrow \frac{1}{2n} < \epsilon \Leftrightarrow n > \frac{1}{2\epsilon}.$$

By Definition 7.5 (with $N > 1/2\epsilon$), the given sequence therefore converges to the limit $\frac{1}{2}$.

3. Evaluate (i) $\lim_{n\to\infty} \dfrac{2n^2-2}{n^5-2}$, (ii) $\lim_{n\to\infty} \dfrac{n^6}{6-n^2}$.

(i) $\dfrac{2n^2-2}{n^5-2} = \dfrac{\dfrac{2}{n^3} - \dfrac{2}{n^5}}{1-\dfrac{1}{n^5}}$ so $\lim_{n\to\infty} \dfrac{2n^2-2}{n^5-2} = \dfrac{0-0}{1-0} = 0.$

(ii) $\dfrac{n^6}{6-n^2} = \dfrac{n^4}{\dfrac{6}{n^2}-1}$ so $\lim_{n\to\infty} \dfrac{n^6}{6-n^2}$ does not exist.

4. Write the following series using Sigma notation. Is the series convergent? If so, find its sum to infinity.

$$\pi - e + \frac{e^2}{\pi} - \frac{e^3}{\pi^2} + \dots$$

This is a geometric sequence with first term $a = \pi$ and common ratio $r = -\dfrac{e}{\pi}$ so we may write it as

$$\sum_{j=1}^{\infty} \pi \left(-\frac{e}{\pi}\right)^{j-1}.$$

As $|r| < 1$ the series is convergent with

$$S_\infty = \frac{a}{1-r} = \frac{\pi}{1-\left(-\dfrac{e}{\pi}\right)} = \frac{\pi^2}{\pi+e}.$$