UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercise solutions: Chapter 4

1. $1,\ 10,\ 26,\ t^2+1,\ 4x^2+1,\ a^4+1,\ x^2-2x+2,\ x^4+2x^2+2.$

2. (a) 45, 125, 0, -55.

   (b) $h(10)=0$ indicates the projectile crashes to the ground after 10 seconds. The value of $h(11)$ no longer indicates the height of the projectile.

   (c) $0 \le t \le 10$.

   (d) $h(t)=0$, $t<0$.

3. (i) $\{x \in \mathbb{R}\}$ (ii) $\{x \in \mathbb{R} \mid x \neq \pm 2\}$ (iii) $\{x \in \mathbb{R} \mid x \geq 4\}$
   (iv) $\{x \in \mathbb{R} \mid x > 4\}$ (v) $\{x \in \mathbb{R} \mid x \neq 3 \text{ and } x \neq -4\}$ (vi) $\{x \in \mathbb{R}\}$.

4. (a) $\{x : x-2 \geq 0\} = [2,\infty)$
   (b) $\{x : x+3 \geq 0\} = [-3,\infty)$
   (c) $\{x : (x-2)(x+3) \geq 0\} = \{x : x \leq -3 \text{ or } x \geq 2\} = (-\infty,-3] \cup [2,\infty)$
   (d) $\{x : 1-2x \geq 0\} = \{x; x \leq 1/2\} = (-\infty, 1/2]$
   (e) $\{x : 1-2x > 0\} = \{x; x < 1/2\} = (-\infty, 1/2)$
   (f) $\{x : x+3 \geq 0 \quad \text{and} \quad 2-x > 0\} = \{x : x \geq -3 \quad \text{and} \quad x < 2\} = [-3,2)$

5. Ranges are
   (i) $\{y \in \mathbb{R}\}$ (ii) $\{y \in \mathbb{R} \mid y \geq 3\}$ (iii) $\{y \in \mathbb{R} \mid y \leq 4\}$
   (iv) $\{y \in \mathbb{R} \mid y \geq 0\}$ (v) $\{y \in \mathbb{R} \mid y > 0\}$ (vi) $\{y \in \mathbb{R} \mid 0 < y \leq 1\}$.

6. (a) If $x>0$, $|x|=x$ and $|-x| = -(-x) = x$. If $x<0$, $|x|=-x$ and $|-x|=-x$. If $x=0$, $|x|=0$ and $|-x|=0$. So result is true for all $x \in \mathbb{R}$.
   (b) If $x>0$, $|x|=x$. If $x<0$, $|x|=-x>x$. If $x=0$, $|x|=0$. So result is true for all $x \in \mathbb{R}$.
   (c) If $x>0$, $|x|=x \geq -x$. If $x<0$, $|x|=-x$. If $x=0$, $-x=|x|$. So result is true for all $x \in \mathbb{R}$.

7. We have
$$\left|\frac{x}{y}\right|^2 = \frac{|x|^2}{|y|^2} = \frac{x^2}{y^2} = \frac{|x|^2}{|y|^2} \qquad (y \neq 0).$$
   Result follows on taking square root of both sides.

8. (i) $f(-x) = (-x)^3 + (-x) = -x^3 - x = -f(x)$ so $f(x)$ is odd.
(ii) $g(-x) = 1 - (-x)^2 = 1 - x^2 = g(x)$ so $g(x)$ is even.
(iii) $h(-x) = (-x)^2 + (-x) + 1 = x^2 - x + 1$ so $h(x)$ is neither even nor odd.
(iv) $p(-x) = (-x)^2((-x)^3 - 4(-x)) = -x^2(x^3 - 4x) = -p(x)$ so $p(x)$ is odd.
(v) $q(-x) = \frac{(-x)}{(-x)^2+1} = -\frac{x}{x^2+1} = -q(x)$ so $q(x)$ is odd.
(vi) $r(-x) = ((-x)^2 + 1)^3 = (x^2 + 1)^3 = r(x)$ so $r(x)$ is even.

9. $f_1(-x) + f_2(-x) = f_1(x) + f_2(x) = (f_1 + f_2)(x)$: even.

$(f_2 + g_1)(-x) = f_2(-x) + g_1(-x) = f_2(x) - g_1(x) = (f_2 - g_1)(x)$: neither.

$(f_1 f_2)(-x) = f_1(-x)f_2(-x) = f_1(x)f_2(x) = (f_1 f_2)(x)$: even.

$(f_1 g_1)(-x) = f_1(-x)g_1(-x) = f_1(x)[-g_1(x)] = -(f_1 g_1)(x)$: odd.

$(g_1 g_2)(-x) = g_1(-x)g_2(-x) = [-g_1(x)][-g_2(x)] = (g_1 g_2)(x)$: even.

10. Let $\phi(x) = f(x) + f(-x)$ and $\psi(x) = f(x) - f(-x)$. Then

$$\phi(-x) = f(-x) + f(x) = \phi(x), \quad \psi(-x) = f(-x) - f(x) = -\psi(x)$$

so ϕ is even and ψ is odd. Finally,

$$f(x) = \frac{1}{2}\{\phi(x) + \psi(x)\}$$

as required.

11. If f is even, $f(-x) = f(x)$ for all $x \in D$, and if f is odd, $f(-x) = -f(x)$ for all $x \in D$. So if it is both even and odd, then $f(x) = -f(x)$ for all $x \in D$ so $f(x) = 0$ for all $x \in D$.

12.

$$(f + g)(x) = f(x) + g(x) = 1 - \frac{1}{x} + 1 + \frac{1}{x} = 2.$$

Noting that the domains of f and g are $\mathbb{R}\backslash\{0\}$, the domain of $f + g$ is also $\mathbb{R}\backslash\{0\}$.

$$(fg)(x) = f(x)g(x) = 1 - \frac{1}{x^2}.$$

The domain is $\mathbb{R}\backslash\{0\}$.

$$\left(\frac{f}{g}\right)(x) = \frac{1 - \frac{1}{x}}{1 + \frac{1}{x}}.$$

We require $x \neq 0$ and $1 + \frac{1}{x} \neq 0$, hence domain is $\mathbb{R}\backslash\{0, -1\}$.

13. (i)

$$f(g(x)) = f(x^2 + 2x + 5) = 1 + \frac{1}{x^2 + 2x + 5} = \frac{x^2 + 2x + 6}{x^2 + 2x + 5}.$$

Since $x^2+2x+5=(x+1)^2+4\geq 4$, the domain of $f\circ g$ is $\mathbb{R}$.

$$g(f(x))=g\left(1+\frac{1}{x}\right)=\left(1+\frac{1}{x}\right)^2+2\left(1+\frac{1}{x}\right)+5=1+\frac{2}{x}+\frac{1}{x^2}+2+\frac{2}{x}+5=8+\frac{4}{x}+\frac{1}{x^2}.$$

The domain of $g\circ f$ is $\mathbb{R}\backslash\{0\}$.

(ii)

$$f(g(x))=f(\sqrt{x})=(\sqrt{x})^2=x.$$

Since the domain of g is $[0,\infty)$, the domain of $f\circ g$ is also $[0,\infty)$.

$$g(f(x))=g(x^2)=\sqrt{x^2}=|x|.$$

Since the domain of f is $\mathbb{R}$ and $x^2\geq 0$ for all $x\in\mathbb{R}$, the domain of $g\circ f$ is $\mathbb{R}$.

(iii)

$$f(g(x))=f(x^2+2x+3)=\sqrt{x^2+2x+4}=\sqrt{(x+1)^2+3}.$$

Since $(x+1)^2+3\geq 3>0$ for all x, the domain of $f\circ g$ is $\mathbb{R}$.

$$g(f(x))=g(\sqrt{x+1})=\left(\sqrt{x+1}\right)^2+2\left(\sqrt{x+1}\right)+3=x+4+2\left(\sqrt{x+1}\right).$$

The domain of f is $[-1,\infty)$ and this is also the domain of $g\circ f$.

14. (i) $f(g(x))=\dfrac{1}{x^2}+1$, $\quad g(f(x))=\dfrac{1}{x^2+1}$.
    (ii) $f(g(x))=(1-x^2)^3$, $\quad g(f(x))=1-x^6$.
    (iii) $f(g(x))=\sqrt{\dfrac{3}{x^2}+4}$, $\quad g(f(x))=\dfrac{3}{x+4}$.
    (iv) $f(g(x))=1+\dfrac{4}{x^2}$, $\quad g(f(x))=\dfrac{4(x^2-4x+7)}{(x-3)^2}$.

15. $f(g(h(x)))=\dfrac{1}{(x^2+1)^3}$, $\quad h(g(f(x)))=\dfrac{1}{x^6}+1$, $\quad h(f(g(x)))=\dfrac{1}{x^6}+1$.

16. (i) $f^{-1}(x)=\dfrac{x-3}{5}$ $\quad$ (ii) $f^{-1}(x)=\sqrt[3]{\dfrac{x+5}{4}}$ $\quad$ (iii) $f^{-1}(x)=\dfrac{1}{2x}-\dfrac{3}{2}$.

17. (a) For $f:x\mapsto 3x-1$, first note that $\text{Dom}f=\mathbb{R}$ and $\text{Im}(f)=\mathbb{R}$. Let $y=f(x)=3x-1$. Then for each $y\in\text{Im}(f)=\mathbb{R}$,

$$y=f(x)=3x-1\Rightarrow x=\frac{y}{3}+\frac{1}{3}\in\text{Dom}(f)=\mathbb{R}.$$

Hence $f^{-1}:y\mapsto y/3+1/3$ or

$$f^{-1}:x\mapsto\frac{x}{3}+\frac{1}{3},$$

with $\text{Dom}(f^{-1})=\mathbb{R}$.

(b) $f : x \mapsto 2x + 3$.

$$y = f(x) = 2x + 3 \Rightarrow x = \frac{y}{2} - \frac{3}{2} \in \mathrm{Dom}(f)$$

for each $y \in \mathrm{Im}(f) = \mathbb{R}$. Hence

$$f^{-1} : x \mapsto \frac{x}{2} - \frac{3}{2},$$

with $\mathrm{Dom}(f^{-1}) = \mathbb{R}$.

(c) $f : x \mapsto 1/(x - 7)$. $\mathrm{Dom}(f) = \mathbb{R}\backslash\{7\}$, $\mathrm{Im}(f) = \mathbb{R}\backslash\{0\}$.

$$y = f(x) = \frac{1}{x - 7} \Rightarrow x = 7 + \frac{1}{y} = f^{-1}(y),$$

hence

$$f^{-1} : x \mapsto 7 + \frac{1}{x}$$

and $\mathrm{Dom}(f^{-1}) = \mathbb{R}\backslash\{0\}$.

(d) $f : x \mapsto \sqrt{x - 5}$. $\mathrm{Dom}(f) = [5, \infty)$ and $\mathrm{Im}(f) = [0, \infty)$.

$$y = f(x) = \sqrt{x - 5} \Rightarrow x = 5 + y^2 = f^{-1}(y),$$

for the domain $[0, \infty)$. Hence

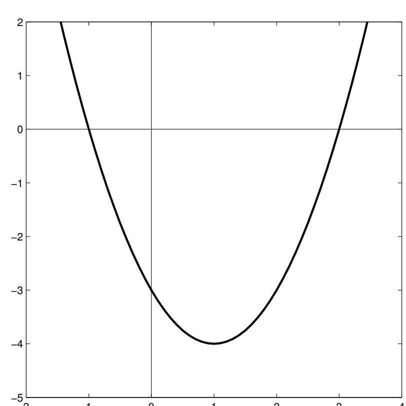
$$f^{-1} : x \mapsto 5 + x^2$$

and $\mathrm{Dom}(f^{-1}) = [0, \infty)$.

18. (i) $f_1^{-1}(x) = \sqrt{x - 4}$ (ii) $f_2^{-1}(x) = -\sqrt{x - 4}$.

19. (a) $y(x) = x^2 - 2x - 3 = (x - 1)^2 - 4 \geq -4$, so $y(x)$ has a minimum of -4 at $x = 1$.

(b) $y(-1) = 1 + 2 - 3 = 0$, $y(3) = 9 - 6 - 3 = 0$.



(c) Suppose $x^2 - 2x - 3 = y_0$, $y_0 \in (-4, \infty)$. Then

$$(x - 1)^2 = y_0 + 4 \Rightarrow x_1 = 1 - \sqrt{y_0 + 4}, \quad x_2 = 1 + \sqrt{y_0 + 4}.$$

(d) Let $f_1 : x \mapsto x^2 - 2x - 3$, $x \in (-\infty, 1]$. For each $y \in (-4, \infty)$ there is a unique $x \in (-\infty, 1]$ with $y = x^2 - 2x - 3$, namely $x = 1 - \sqrt{y+4}$, so

$$f_1^{-1} : y \mapsto 1 - \sqrt{y+4}, \quad y \in [-4, \infty).$$

Similarly, for $f_2 : x \mapsto x^2 - 2x - 3$, $x \in [1, \infty)$ we get

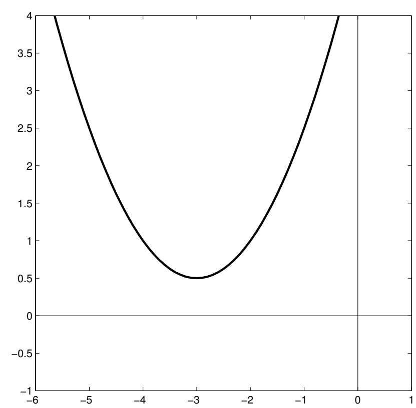
$$f_1^{-1} : y \mapsto 1 + \sqrt{y+4}, \quad y \in [-4, \infty).$$

20. This is similar to the previous question. Briefly:

(a) $y = 1/2[(x+3)^2 + 1] \geq 1/2$, so $y = 1/2$ is the minimum (at $x = -3$).

(b) For $y \in (1/2, \infty)$ we find two roots

$$x_1 = -3 - \sqrt{2y-1}, \quad x_2 = -3 + \sqrt{2y-1}.$$



Finally,

$$f_1^{-1} : x \mapsto 3 - \sqrt{2x-1}, \quad x \in [1, \infty),$$
$$f_2^{-1} : x \mapsto 3 + \sqrt{2x-1}, \quad x \in [1, \infty).$$

21. (i)

$$x = t + 1 \Rightarrow t = x - 1 \Rightarrow y = 2t - 4 = 2(x-1) - 4 = 2x - 6$$

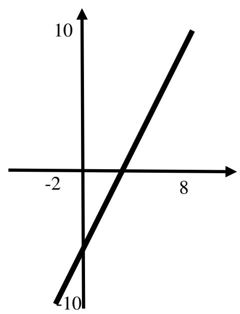
As t ranges over $\mathbb{R}$, the point $(x(t), y(t))$ traces out the straight line $y = 2x - 6$.

(ii)

$$x = 2t \Rightarrow t = \frac{x}{2} \Rightarrow y = t^2 = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$$

As t ranges over $[0, \infty)$, the point $(x(t), y(t))$ traces out the parabola $y = x^2/4$ in the right-half plane.

(i)



(ii)