16 Integration

16.1 Evaluate

(a) $\displaystyle\int x^6\,\mathrm{d}x$ (b) $\displaystyle\int x^{-2}\,\mathrm{d}x$ (c) $\displaystyle\int x^{\frac{1}{3}}\,\mathrm{d}x$ (d) $\displaystyle\int x^{-\frac{3}{2}}\,\mathrm{d}x$ (e) $\displaystyle\int \frac{1}{x^3}\,\mathrm{d}x$.

16.2 Evaluate

(a) $\displaystyle\int_1^2 x^4\,\mathrm{d}x$ (b) $\displaystyle\int_2^5 x^{-3}\,\mathrm{d}x$ (c) $\displaystyle\int_4^9 x^{\frac{3}{2}}\,\mathrm{d}x$ (d) $\displaystyle\int_1^4 \frac{1}{x^2\sqrt{x}}\,\mathrm{d}x$.

16.3 (a) Show that $\displaystyle\int_{-a}^{a} x^3\,\mathrm{d}x = 0$.

(b) Verify that $\displaystyle\int_{-a}^{a} x^{2n-1}\,\mathrm{d}x = 0$ for all $n \in \mathbb{N}$, and evaluate $\displaystyle\int_{-a}^{a} x^{2n}\,\mathrm{d}x$.

(c) Can you extend these results to negative integers n?

16.4 Evaluate

(a) $\displaystyle\int (x^2+1)\,\mathrm{d}x$ (b) $\displaystyle\int (2x^3+4x-2)\,\mathrm{d}x$ (c) $\displaystyle\int (x+\sqrt{x})\,\mathrm{d}x$

(d) $\displaystyle\int (\sin x + 2\cos x)\,\mathrm{d}x$ (e) $\displaystyle\int \left(\frac{4}{\sqrt{x}} - 2\sec^2 x\right)\mathrm{d}x$ (f) $\displaystyle\int (2x^2 - \sqrt{x^3} + 5\sin x)\,\mathrm{d}x$.

16.5 By first multiplying or dividing out the integrands, evaluate the following.

(a) $\displaystyle\int (x+2)(2x-1)\,\mathrm{d}x$ (b) $\displaystyle\int \sqrt{x}(2x-1)\,\mathrm{d}x$ (c) $\displaystyle\int \frac{2x-1}{\sqrt{x}}\,\mathrm{d}x$

(d) $\displaystyle\int_{-3}^{1} \frac{x^2+5x-24}{x-3}\,\mathrm{d}x$ (e) $\displaystyle\int_0^1 x^2(1-x)^2\,\mathrm{d}x$ (f) $\displaystyle\int_{-2}^{1} x(x-1)(x+2)\,\mathrm{d}x$.

16.6 Determine $f(x)$ from the information given in each case.

(a) $f'(x) = x^3 - x, \quad f(0) = 1$.

(b) $f'(x) = \dfrac{x^4+1}{x^2}, \quad f(1) = 2$.

(c) $f''(x) = x^2, \quad f'(0) = 1, \quad f(0) = 2$.

(d) $f''(x) = \sin x, \quad f'(\pi) = -1, \quad f(2\pi) = 2$.

16.7 Use integration by parts to evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi} x\cos x\,\mathrm{d}x$ (b) $\displaystyle\int_0^{a} x^2\cos\left(\frac{\pi x}{a}\right)\mathrm{d}x$ (c) $\displaystyle\int_0^1 x(1-x)\sin(\pi x)\,\mathrm{d}x$

(d) $\displaystyle\int x\ln x\,\mathrm{d}x$ (e) $\displaystyle\int (\ln x)^2\,\mathrm{d}x$ (f) $\displaystyle\int \sin^2 x\,\mathrm{d}x$.

16.8 If f is twice differentiable on $[a,b]$ and $f(a) = f(b) = 0$, use integration by parts to show that $\displaystyle\int_a^b (x-a)(b-x)f''(x)\,\mathrm{d}x = -2\int_a^b f(x)\,\mathrm{d}x$.

16.9 Use integration by parts to determine the values of the following integrals.

(a) $\displaystyle\int_0^1 xe^{-x}\,\mathrm{d}x$ (b) $\displaystyle\int_0^1 x^3e^{-2x}\,\mathrm{d}x$.

16.10 Let $I_n = \displaystyle\int_0^1 x^n e^{-x}\,\mathrm{d}x$, $n \in \mathbb{N}$. Use integration by parts to show that $I_n = nI_{n-1} - \dfrac{1}{e}$.

16.11 Evaluate

(a) $\displaystyle\int (x+1)^2\,\mathrm{d}x$ (b) $\displaystyle\int (2x-1)^3\,\mathrm{d}x$ (c) $\displaystyle\int (3-4x)^{-3}\,\mathrm{d}x$ (d) $\displaystyle\int \cos(\pi x)\,\mathrm{d}x$

(e) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$ (f) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$ (g) $\displaystyle\int \frac{1}{1+36x^2}\,\mathrm{d}x$ (h) $\displaystyle\int \frac{1}{49+x^2}\,\mathrm{d}x$.

16.12 Evaluate

(a) $\displaystyle\int_1^3 (2x-5)^2\,\mathrm{d}x$ (b) $\displaystyle\int_0^1 (3x+1)^{\frac{1}{2}}\,\mathrm{d}x$ (c) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin\left(\frac{\pi}{8}-\frac{x}{2}\right)\mathrm{d}x$

(d) $\displaystyle\int_1^2 \left[(2x-5)^3+x\right]\mathrm{d}x$ (e) $\displaystyle\int_0^1 \left\{2\sin\left[\pi\left(x-\frac{1}{2}\right)\right] - 2\sec^2\left[\frac{\pi(2x-1)}{4}\right]\right\}\mathrm{d}x$.

16.13 Evaluate the following integrals, assuming that the quantities inside the square roots are positive. [*Hint:* For (e)—(i) you will need to complete the square in x first.]

(a) $\displaystyle\int \frac{\mathrm{d}x}{(2x+5)^2+1}$ (b) $\displaystyle\int_2^5 \frac{\mathrm{d}x}{(x-2)^2+9}$ (c) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-(3x-1)^2}}$

(d) $\displaystyle\int_{-\frac{1}{4}}^0 \frac{\mathrm{d}x}{\sqrt{1-(2x+1)^2}}$ (e) $\displaystyle\int \frac{\mathrm{d}x}{x^2-6x+10}$ (f) $\displaystyle\int_0^4 \frac{\mathrm{d}x}{x^2-4x+8}$

(g) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{4x-x^2+5}}$ (h) $\displaystyle\int_1^2 \frac{\mathrm{d}x}{\sqrt{12x-4x^2-8}}$ (i) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{-x^2-x}}$.

16.14 Evaluate each of the following integrals.

(a) $\displaystyle\int \tan^2 x\sec^2 x\,\mathrm{d}x$ (b) $\displaystyle\int \frac{\cos x}{\sqrt{1+\sin x}}\,\mathrm{d}x$ (c) $\displaystyle\int x^2(1-x^3)^{1/3}\,\mathrm{d}x$

(d) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(3+\sin x)^2}\,\mathrm{d}x$ (e) $\displaystyle\int_0^{\pi/2} \cos^3\left(\frac{x}{2}\right)\mathrm{d}x$ (f) $\displaystyle\int x\sin^3(x^2)\cos(x^2)\,\mathrm{d}x$

(g) $\displaystyle\int \tan^2 x\,\mathrm{d}x$ (h) $\displaystyle\int \sqrt{\tan x}\,\mathrm{d}x$.

16.15 Evaluate each of the following integrals using the substitution given. (Express the answers to (a) and (c) in terms of x.)

(a) $\displaystyle\int x(4x+3)^4\,\mathrm{d}x, \quad u=4x+3;$ (b) $\displaystyle\int_1^6 \frac{x}{\sqrt{x+3}}\,\mathrm{d}x, \quad u=x+3;$

(c) $\displaystyle\int \frac{x}{1+x^4}\,\mathrm{d}x, \quad u=x^2;$ (d) $\displaystyle\int_0^{\pi/2} \frac{\cos x\sin x}{(2+\sin^2 x)^2}\,\mathrm{d}x, \quad u=\sin x \quad \text{then} \quad v=u^2.$

16.16 Evaluate each of the following integrals by using suitable substitutions. (Express the answers to (a) and (c) in terms of x.)

(a) $\displaystyle\int x(3x-1)^{1/3}\,\mathrm{d}x$ (b) $\displaystyle\int_1^2 \frac{x}{(2x-1)^3}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{\tan x\sec^2 x}{\sqrt{1+\tan x}}\,\mathrm{d}x$

(d) $\displaystyle\int_0^1 x\sin(\pi x^2)\,\mathrm{d}x$

16.17 An alternative for 16.16 (a) is to write $x=\dfrac{1}{3}[(3x-1)+1]$. Try this.

16.18 Consider $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}\sqrt{1-x}}$.

(i) Use the substitution $u=\sqrt{x}$ to show that $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}\sqrt{1-x}} = 2\arcsin(\sqrt{x})$.

(ii) Use the substitution $u=\sqrt{1-x}$ to show that $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}\sqrt{1-x}} = -2\arcsin(\sqrt{1-x})$.

(iii) Verify that $\arcsin(\sqrt{x})+\arcsin(\sqrt{1-x})=\dfrac{\pi}{2}$. Use this to show that the results obtained in (i) and (ii) are consistent.

16.19 Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{8-x^2}}\,\mathrm{d}x$ (b) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{7}{1+x^2}\,\mathrm{d}x$ (d) $\displaystyle\int \frac{1}{x^2+2}\,\mathrm{d}x.$

16.20 Evaluate the following integrals.

(a) $\displaystyle\int \frac{\mathrm{d}x}{3x-1}$ (b) $\displaystyle\int \frac{2x}{x-1}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{x^2}{x+7}\,\mathrm{d}x$ (d) $\displaystyle\int_0^2 \frac{x}{x^2+1}\,\mathrm{d}x$

(e) $\displaystyle\int_1^e \frac{1}{x}\ln(x)\,\mathrm{d}x$ (f) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x}{1-\cos x}\,\mathrm{d}x$ (g) $\displaystyle\int_1^2 x^3\ln x\,\mathrm{d}x$ (h) $\displaystyle\int_0^{\pi/4} \tan x\,\mathrm{d}x.$

16.21 Use the substitution $u=2x^2+1$ to evaluate $\displaystyle\int \frac{x}{2x^2+1}\,\mathrm{d}x.$

16.22 Evaluate the following integrals.

(a) $\displaystyle\int e^{7x+4}\,\mathrm{d}x$ (b) $\displaystyle\int e^{2-3x}\,\mathrm{d}x$ (c) $\displaystyle\int e^{4\sin(2x)}\cos(2x)\,\mathrm{d}x$ (d) $\displaystyle\int_0^1 \frac{1}{1+e^{-x}}\,\mathrm{d}x$

(e) $\displaystyle\int a^x\,\mathrm{d}x$ (f) $\displaystyle\int \frac{a^x}{b^x}\,\mathrm{d}x$ (g) $\displaystyle\int \log_a x\,\mathrm{d}x.$

16.23 Evaluate the following integrals.

(a) $\displaystyle\int \sinh(3x+2)\,\mathrm{d}x$ (b) $\displaystyle\int \tanh x\,\mathrm{d}x$ (c) $\displaystyle\int_{[\ln 2]^2}^{[\ln 3]^2} \frac{1}{\sqrt{x}}\cosh(\sqrt{x})\,\mathrm{d}x$.