8.6 Variation of Parameters

The previous section gave a method of finding the particular integral for some common cases: namely when $f(x)$ is a combination of polynomials, exponentials, sines or cosines.

If the right hand side of the equation is not of that form there is another method that may produce the particular integral.

We start, as before, with the nonhomogeneous second-order linear constant coefficient ODE

$$a\,y'' + b\,y' + c\,y = f(x), \tag{8.37}$$

where a, b, c are constants and $a \neq 0$. We can always solve the associated homogeneous equation for the complementary function, say $Y_1 = c_1\,y_1 + c_2\,y_2$, where c_1 and c_2 are unknown constants and $\{y_1,\, y_2\}$ is a fundamental set of solutions of the homogeneous equation.

We now search for a particular integral of the form

$$y = c_1(x)\,y_1 + c_2(x)\,y_2, \tag{8.38}$$

i.e. we let the parameters c_1 and c_2 become variable functions of x. This method is therefore called 'Variation of Parameters'. If we differentiate this function (8.38) we obtain

$$y' = [c_1\,y_1' + c_2\,y_2'] + [c_1'\,y_1 + c_2'\,y_2].$$

Now if we differentiated again to get y'' we would end up with a very complicated expression. However, because we are have some freedom to choose $c_1(x)$ and $c_2(x)$ we decide to choose them so that

$$c_1'\,y_1 + c_2'\,y_2 = 0 \tag{8.39}$$

Then we obtain

$$y'' = c_1\,y_1'' + c_2\,y_2'' + c_1'\,y_1' + c_2'\,y_2'.$$

Substituting these expressions for the derivatives into (8.37) gives

$$c_1(a\,y_1'' + b\,y_1' + c\,y_1) + c_2(a\,y_2'' + b\,y_2' + c\,y_2) + a(c_1'\,y_1' + c_2'\,y_2') = f(x)$$

$$\text{i.e.} \qquad c_1'\,y_1' + c_2'\,y_2' = f(x)/a, \tag{8.40}$$

because y_1 and y_2 are solutions of the associated homogeneous ODE.

Equations (8.39) and (8.40) are simultaneous equations for c_1' and c_2' which can be put into matrix form

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x)/a \end{bmatrix}.$$

The determinant of the matrix on the left hand side is just the Wronskian, $W[y_1, y_2]$.

Now, as $\{y_1, y_2\}$ is a fundamental set of solutions they are linearly independent and so $W[y_1, y_2] \neq 0$ for all x. Hence we can solve this matrix equation for c_1' and c_2'. If we finally integrate these solutions we obtain the functions $c_1(x)$ and $c_2(x)$, and then the particular integral $Y_2 = c_1(x)\, y_1 + c_2(x)\, y_2$.

Notes:

1. The final step, of integrating $c_1'(x)$ and $c_2'(x)$, introduces integration constants and so the particular integral, $Y_2 = c_1(x)\, y_1 + c_2(x)\, y_2$, will in fact include the complementary function.
2. The method also works for linear equations with variable coefficients (i.e. a, b, c could be functions of x) but we still need to find two linearly independent solutions of the associated homogeneous equation (and we can't use the auxiliary equation method because a, b, c are not constant).
3. Integrating c_1' and c_2' to find $c_1(x)$ and $c_2(x)$ may be difficult.
4. An extension of the method works for higher-order linear equations.

Method Variation of Parameters for $a\, y'' + b\, y' + c\, y = f(x)$

- Find the complementary function $Y_1 = c_1\, y_1 + c_2\, y_2$
- Let $y = c_1(x)\, y_1 + c_2(x)\, y_2$
- Differentiate y and set $c_1'\, y_1 + c_2'\, y_2 = 0 \qquad (1)$
- Differentiate y again and then substitute for y into the full ODE to get the equation
$$c_1'\, y_1' + c_2'\, y_2' = f(x)/a \qquad (2)$$
[as c_1, c_2 terms must vanish.]
- Solve (1) and (2) for c_1' and c_2'
- Integrate c_1' and c_2' to obtain c_1 and c_2 - remembering the arbitrary constants.
- General solution is $y = c_1(x)\, y_1 + c_2(x)\, y_2$.

Examples 8.9

Use the method of variation of parameters to find the general solution of the following equations: solution to the following linear ODEs

(a) $y'' + y = 2\sin x$

(b) $xy'' - (x+1)y' + y = x^2$, given that two solutions to the associated homogeneous equation are $y_1(x) = e^x$ and $y_2(x) = x + 1$.

✓ Watch Video: Example 8.9

✓ Solve Example: Tutorial question 1.

8.7 Reduction of Order

One of the methods we used in the previous sections can actually be used for more general ODEs. Consider the second-order linear equation with coefficients which are not necessarily constants

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{8.41}$$

Sometimes one solution $y = y_1(x)$ of the associated homogeneous equation (i.e. where $f(x) \equiv 0$) is known, or can be easily spotted.We proceed by considering the following function $y = v(x)y_1(x)$ where $v(x)$ is a new unknown function. Substituting into equation (8.41) gives

$$a(x)[v''y_1 + 2v'y_1' + vy_1''] + b(x)[v'y_1 + vy_1'] + c(x)\, vy_1 = f(x).$$

We know that y_1 is a solution to the associated homogeneous ODE so $ay_1'' + by_1' + cy_1 = 0$. Therefore

$$a(x)y_1\, v'' + [2a(x)y_1' + b(x)y_1]v' = f(x), \tag{8.42}$$

which is a second-order ODE in $v(x)$ with no explicit v term.

We have already seen how to solve this type of ODE at the beginning of this Chapter. We set $w = v'$ in equation (8.42) which then reduces the equation to a first-order linear ODE for the new unknown w,

$$a(x)y_1\, w' + [2a(x)y_1' + b(x)y_1]w = f(x)\ . \tag{8.43}$$

We now need to solve this equation (8.43) for w and then integrate w with respect to x to get v. Finally, the solution is $y = v(x)\, y_1(x)$.

Method Reduction of Order $a(x)y'' + b(x)y' + c(x)y = f(x)$

- Find a solution $y = y_1(x)$ of the associated homogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

- Put $y = v(x)y_1(x)$
- Substitute into full ODE to get $a(x)y_1\, v'' + [2a(x)y_1' + b(x)y_1]v' = f(x)$
  [Note that the v term must vanish]
- Solve this for v using the method of substituting $w = v'$
- Finally, the general solution is $y = v(x)\, y_1(x)$.

Examples 8.10

Use the reduction of order method to find general solutions to the following

(a) $2x^2y'' + xy' - 3y = 0$, given that $y_1(x) = \dfrac{1}{x}$ is a solution.

(b) $x^2y'' - xy' + y = 0$, given that $y_1(x) = x$ is a solution.

✓ Watch Video: Example 8.10

✓ Solve Example: Tutorial question 2.

8.8 Linear ODEs and systems of equations

Consider the second-order nonhomogeneous linear ODE with constant coefficients:

$$ay'' + by' + cy = f(x). \tag{8.44}$$

If we let $z_1(x) = y(x)$ and define a new variable $z_2(x) = y'(x)$. Equation (8.44) can now be rewritten as two equations:

$$\begin{aligned} z_2 &= z_1' , \\ a\, z_2' + b\, z_2 + c\, z_1 &= f(x) , \end{aligned}$$

which can then be rearranged as

$$\begin{aligned} z_1' &= && z_2 \\ z_2' &= \left(-\frac{c}{a}\right) z_1 &+ \left(-\frac{b}{a}\right) z_2 &+ \frac{f(x)}{a} \end{aligned} .$$

This can then be written in matrix form as,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{f(x)}{a} \end{bmatrix},$$

or equivalently,

$$\boldsymbol{z}' = \boldsymbol{A}\, \boldsymbol{z} + \boldsymbol{f}, \tag{8.45}$$

where we have defined

$$\boldsymbol{z}(x) = \begin{bmatrix} z_1(x) \\ z_2(x) \end{bmatrix}, \quad \boldsymbol{A}(x) = \begin{bmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{bmatrix}, \quad \boldsymbol{f}(x) = \begin{bmatrix} 0 \\ \dfrac{f(x)}{a} \end{bmatrix}.$$

One way of solving the matrix equation (8.45) is to calculate the eigenvalues λ_1, λ_2 of the system matrix $\boldsymbol{A}$ (We assume here that these eigenvalues are distinct and discuss repeated eigenvalues later). Recall that we do this by solving the following equation

$$\det(\boldsymbol{A} - \boldsymbol{\lambda I}) = \det \begin{bmatrix} -\lambda & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} - \lambda \end{bmatrix} = \lambda \left(\frac{b}{a} + \lambda\right) + \frac{c}{a} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0,$$

Note that this is equivalent to solving the auxiliary equation $a\, m^2 + b\, m + c = 0$, identifying m with λ. Having found the eigenvalues of $\boldsymbol{A}$ we can then construct the diagonal matrix

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

and the matrix $\boldsymbol{P}$ of eigenvectors so that $\boldsymbol{\Lambda} = \boldsymbol{P^{-1}}\,\boldsymbol{A}\,\boldsymbol{P}$ or $\boldsymbol{A} = \boldsymbol{P}\,\Lambda\,\boldsymbol{P^{-1}}$. Therefore, equation (8.45) becomes

$$\begin{aligned} \boldsymbol{z}' &= \boldsymbol{P}\,\Lambda\,\boldsymbol{P^{-1}}\,\boldsymbol{z} + \boldsymbol{f}, \\ \Rightarrow \qquad \boldsymbol{P^{-1}}\,\boldsymbol{z}' &= \boldsymbol{\Lambda}\,\boldsymbol{P^{-1}}\,\boldsymbol{z} + \boldsymbol{P^{-1}}\,\boldsymbol{f}. \end{aligned}$$

If we now define $\boldsymbol{w} = \boldsymbol{P^{-1}}\,\boldsymbol{z}$ and $\boldsymbol{g} = \boldsymbol{P^{-1}}\boldsymbol{f}$, then we have a new equation

$$\boldsymbol{w}' = \boldsymbol{\Lambda}\,\boldsymbol{w} + \boldsymbol{g}. \tag{8.46}$$

Note that $\boldsymbol{g}$ is determined by solving $\boldsymbol{P}\boldsymbol{g} = \boldsymbol{f}$ via EROs on the augmented matrix $(\boldsymbol{P}|\boldsymbol{f})$. Do not workout $\boldsymbol{P^{-1}}$ (except in the 2×2 case) in order to calculate $\boldsymbol{g}$ – this is very inefficient.

Recall that in the 2×2 case where $\boldsymbol{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\boldsymbol{P^{-1}} = \frac{1}{a\,d - b\,c}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $\boldsymbol{w} = [w_1(x),\, w_2(x)]^T$ and $\boldsymbol{g} = [g_1(x),\, g_2(x)]^T$ then this equation (8.46) is equivalent to the following two equations

$$\begin{aligned} w_1' &= \lambda_1 w_1 + g_1, \\ w_2' &= \lambda_2 w_2 + g_2, \end{aligned}$$

each of which is a first-order linear ODE which we can solve using an integrating factor. Hence,

$$\begin{aligned} w_1 &= c_1\,e^{\lambda_1 x} + e^{\lambda_1 x}\int g_1(x)\,e^{-\lambda_1 x}\,dx, \\ w_2 &= c_2\,e^{\lambda_2 x} + e^{\lambda_2 x}\int g_2(x)\,e^{-\lambda_2 x}\,dx. \end{aligned}$$

However, this is a solution for w_1, w_2 and not of the original variables $z_1 = y$ and $z_2 = y'$. We can recover y $(= z_1)$ (and z_2) because we know that $\boldsymbol{z} = \boldsymbol{P}\,\boldsymbol{w}$.

The solution of

$$\boldsymbol{z}' = \boldsymbol{A}\,\boldsymbol{z}$$

is

$$\boldsymbol{z} = \boldsymbol{P}\begin{bmatrix} c_1\,e^{\lambda_1\,x} \\ c_2\,e^{\lambda_2\,x} \end{bmatrix}.$$

Using this method we can solve systems of n linear first-order equations. Note that this includes n^{th}-order nonhomogeneous linear ODEs as we can introduce new variables.

$$\begin{aligned}
z_1(x) &= y(x) \\
z_2(x) &= y'(x) = z_1'(x) \\
z_3(x) &= y''(x) = z_2'(x) \\
\cdots &= \cdots \\
z_n(x) &= y^{(n-1)}(x) = z_{n-1}^{(n-1)}(x) \ .
\end{aligned}$$

In order to carry out the diagonalisation of the $n \times n$ system matrix $\boldsymbol{A}$ all the eigenvalues of $\boldsymbol{A}$ must be distinct (real or complex). This is usually the case for 'real' problems. (Repeated roots are a little more difficult and will be looked at in future classes.) This results in the associated eigenvectors being linearly independent and so the matrix $\boldsymbol{P}$ of eigenvectors has rank n and $\boldsymbol{P^{-1}}$ exists (though $\boldsymbol{P^{-1}}$ is not usually required explicitly).