Examples 3A

1 Write out the members of the following sets:

$$\text{(i)} \quad \{a \in \mathbb{Z} : a^2 = 4\} \vee \{b \in \mathbb{Z} : b^2 = 9\};$$

$$\text{(ii)} \quad \{x \in \mathbb{R} : 0 < x < 20\} \wedge \{n \in \mathbb{N} : \ n \text{ is divisible by } 5\}.$$

(i) $\{-2, 2, -3, 3\}$ (ii) $\{5, 10, 15\}$

2 Write out a version of the proof of Theorem 3.1 using appropriate symbols.

If n is odd then $\exists k \in \mathbb{Z}$ such that $n = 2k - 1$ so

$$n^2 = (2k-1)^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k + 1) - 1 = 2l - 1$$

which is also an odd integer.

Examples 3B

1 Consider the statements

$$p : x = 2 \qquad q : 2x + 4 = 8\,.$$

Prove that $p \Leftrightarrow q$.

We have that

$$x = 2 \Rightarrow 2x + 4 = 4 + 4 = 8$$

so $p \Rightarrow q$. Also,

$$2x + 4 = 8 \Rightarrow 2x = 4 \Rightarrow x = 2$$

so $q \Rightarrow p$ (or $p \Leftarrow q$). Hence $p \Leftrightarrow q$.

2 Consider the statements $p : x = 2$ and $q : x^2 = 4$. Prove that p is sufficient but not necessary for q.

If p is sufficient for q, then $p \Rightarrow q$. This is true as $x = 2 \Rightarrow x^2 = 4$. However, if p is necessary for q, then $q \Rightarrow p$ (or $p \Leftarrow q$) which is NOT true as $x^2 = 4 \not\Rightarrow x = 2$ (as x could also be -2).

3 Let p and q be the statements

$$p : n \in \mathbb{N} \text{ is divisible by 3}, \qquad q : n \in \mathbb{N} \text{ is divisible by 6}.$$

Prove that p is necessary but not sufficient for q.

We must show that $q \Rightarrow p$ but $p \not\Rightarrow q$.

If q is true then $n = 6k$ for some $k \in \mathbb{N}$ so $n = 3(2k)$ where $2k \in \mathbb{N}$ and n is divisible by 3. That is, $q \Rightarrow p$, and p is necessary for q.

However, the counterexample of $n = 9$ (which is divisible by 3 and not by 6) shows that it is NOT true that $p \Rightarrow q$ (or $q \Leftarrow p$), so p is NOT sufficient for q.

Examples 3C

1 Prove that the sum of the first n natural numbers is given by $\frac{1}{2}n(n+1)$.

We must prove that

$$1 + 2 + \ldots + (n-1) + n = \frac{1}{2}n(n+1).$$

Denote the LHS by S_n. Then

$$\begin{aligned}
S_n &= 1 + 2 + \ldots + (n-1) + n \\
\Leftrightarrow S_n &= n + (n-1) + \ldots + 2 + 1 \qquad \text{(reversing the order of the numbers)} \\
\Leftrightarrow 2S_n &= \underbrace{(n+1) + (n+1) + \ldots + (n+1) + (n+1)}_{n \text{ times}} \quad \text{(adding the two lines above)} \\
\Leftrightarrow 2S_n &= n(n+1) \\
\Leftrightarrow S_n &= \frac{1}{2}n(n+1)
\end{aligned}$$

as required.

Examples 3D

1 Use proof by contradiction to prove that there are infinitely many prime numbers.

Let p be the statement "There are infinitely many prime numbers."

1. *Assume that p is false (i.e. $\neg p$ is true).*
   Assume that there must be finitely many primes a_1, a_2,…,a_n say.

2. *Show that $\neg p \Rightarrow q$ for a new statement q.*
   It follows that that there is a largest prime number (a_n in our notation).

3. *Show that q is false.*
   Consider the number
   $$A = a_1 a_2 a_3 \dots a_n + 1.$$
   This is not divisible by any of the numbers a_1, a_2,…,a_n, so A is a new prime which is bigger than a_n.

4. *Apply the law of contrapostion to deduce that p must be true.*
   As our assumption that p is false (that there are finitely many primes) has led with correct reasoning to a false statement (that there is a largest prime), the original statement p must be true.

2 Use proof by contradiction to prove that $\forall n \in \mathbb{N}$, if n^2 is even then n is even.

Let p be the statement "$\forall n \in \mathbb{N}$, if n^2 is even then n is even".

1. Assume p is false, that is, suppose that "$\exists n \in \mathbb{N}$ such that n^2 is even but n is odd".
2. Call this new statement q.
3. From Theorem 3.1 we know that if $n \in \mathbb{Z}$ is odd, then n^2 is odd, so q is false.
4. As our assumption that p is false has led to a false conclusion (contradiction), the original statement p must be true.

3 Use proof by contradiction to prove that $\sqrt{2}$ is an irrational number.

Assume that $\sqrt{2}$ is a rational number, so $\sqrt{2} = a/b$ where integers a and b have no common factor greater than 1.

Now observe that
$$(\sqrt{2})^2 = 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$$
so a^2 must be an even number. But we know from Example 3D.2 that this means a is also even, so we can write $a = 2k$ for some $k \in \mathbb{Z}$, giving
$$a^2 = 2b^2 \Leftrightarrow (2k)^2 = 2b^2 \Leftrightarrow 4k^2 = 2b^2 \Rightarrow 2k^2 = b^2.$$
This shows that b is also an even number, which is a contradiction to the fact that a and b have no common factor greater than 1. Hence our original assumption that $\sqrt{2}$ is rational must be false, and the result is proved.

4 Combination of contradiction and direct proof

Find all pairs of integers a, b such that $a + b = a \cdot b$.

(a) Suppose that a is an odd integer. Then $a = 2n + 1$ for some $n \in \mathbb{Z}$. So

$$a + b = a \cdot b \Leftrightarrow 2n + 1 + b = (2n + 1)b \Leftrightarrow 2n + 1 = 2nb.$$

There are no solutions to this equation as the LHS is odd and the RHS is even.

(b) Suppose that a is an even integer. Then $a = 2n$ for some $n \in \mathbb{Z}$. So

$$a + b = a \cdot b \Leftrightarrow 2n + b = 2nb \Leftrightarrow 2n = (2n - 1)b.$$

As the LHS is even, b must be even, so $b = 2k$ for some $k \in \mathbb{Z}$. Then

$$2n = (2n - 1)b \Leftrightarrow 2n = 2k(2n - 1) \Leftrightarrow k = \frac{n}{2n - 1}.$$

(i) Suppose $a \geq 0$ so that $n \geq 0$. For k to be an integer we require $n \geq 2n - 1$, so $n \leq 1$. This gives two pairs of solutions:

$$n = 0 \Rightarrow k = 0 \quad \text{so} \quad a = b = 0;$$
$$n = 1 \Rightarrow k = 1 \quad \text{so} \quad a = b = 2.$$

(ii) Suppose $a < 0$ so that $n < 0$. As $2n < 0$ and $2n - 1 < -1 < 0$, for k to be an integer we require $n \leq 2n - 1$, so $n \geq 1$. So there are no solutions with negative n.

Examples 3E

1 Use induction to prove that the sum of the first n natural numbers is given by $\frac{1}{2}n(n + 1)$.

We must prove that $p(n)$ is true $\forall n \in \mathbb{N}$ where

$$p(n) \quad \equiv \quad 1 + 2 + \ldots + (n - 1) + n = \frac{1}{2}n(n + 1).$$

Step 1: Check the case $n = 1$.

$$LHS = 1, \qquad RHS = \frac{1}{2} \times (1 \times 2) = 1$$

so proposition is true when $n = 1$.

Step 2: Assume that the given result is true for n, that is, assume that

$$1 + 2 + 3 + \ldots + (n - 1) + n = \frac{1}{2}n(n + 1).$$

Now try to prove the result for $n + 1$, that is, try to show that

$$1 + 2 + 3 + \ldots + (n - 1) + n + (n + 1) = \frac{1}{2}(n + 1)(n + 2).$$

We have

$$\begin{aligned} 1+2+3+\ldots+(n-1)+n+(n+1) &= \frac{1}{2}n(n+1)+(n+1) \\ &= \frac{1}{2}[n(n+1)+2(n+1)] \\ &= \frac{1}{2}[n^2+3n+2] \\ &= \frac{1}{2}(n+1)(n+2) \end{aligned}$$

so if the proposition is true for n, it is true for $n+1$.

Hence, by the principle of mathematical induction, the proposition is true for all natural numbers n.

2 Use induction to prove that $n < 2^n$ for all $n \in \mathbb{N}$.

Here $p(n) \quad \equiv \quad n < 2^n$.

Step 1: Check the case $n = 1$.

$$LHS = 1, \qquad RHS = 2^1 = 2$$

so proposition is true when $n = 1$.

Step 2: Assume that the given result is true for n, that is, assume that

$$n < 2^n.$$

Now try to prove the result for $n+1$, that is, try to show that

$$n+1 < 2^{n+1}.$$

We have

$$2^{n+1} = 2 \times 2^n > 2n \geq n+1$$

as $n \geq 1$, so if the proposition is true for n, it is true for $n+1$.

Hence, by the principle of mathematical induction, the proposition is true for all natural numbers n.

3 Use induction to prove that every natural number is either even or odd.

Step 1: Check the case $n = 1$.

Proposition is true when $n = 1$, as 1 is either even or odd (in fact it is odd).

Step 2: Assume that the given result is true for n, that is, assume that n is either even or odd. Now try to prove the result for $n+1$, that is, try to show that that $n+1$ is either even or odd.

Treat the two cases separately:

1. if n is even, then $n = 2m$ for some natural number m and $n+1 = 2m+1$ is odd;
2. if n is odd, then $n = 2m-1$ for some natural number m and $n+1 = (2m-1)+1 = 2m$ is even;

so if the proposition is true for n, it is true for $n+1$.

Hence, by the principle of mathematical induction, the proposition is true for all natural numbers n.

4 Use induction to prove that the inequality $3n^2 \geq 3n+2$ holds $\forall n \in \mathbb{N}, n \geq 2$.

Step 1: Check the case $n = 2$.

$$3 \times 2^2 > 3 \times 2 + 2.$$

Step 2: Assume that the given result is true for n, that is, assume that $3n^2 \geq 3n+2$. Now try to prove the result for $n+1$.

$$3(n+1)^2 = 3n^2 + 6n + 3 \geq (3n+2) + 6n + 3 = 9n + 5 \geq 3n + 5 = 3(n+1) + 2,$$

so if the proposition is true for n, it is true for $n+1$.

Hence, by the principle of mathematical induction, the proposition is true for all natural numbers $n \geq 2$.

5 Use induction to prove that the inequality $n^3 > 3n^2$ holds $\forall n \in \mathbb{N}, n \geq 4$.

Step 1: Check the case $n = 4$.

$$4^3 = 64 > 48 = 3 \times 4^2.$$

Step 2: Assume that the given result is true for n, and try to prove the result for $n+1$.

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 > 3n^2 + 3n^2 + 3n + 1 = 3(n+1)^2 + 3n^2 - 3n - 2.$$

so the proposition is true for $n+1$ so long as $3n^2 > 3n+2$. But from the previous example we know this is the case for $n \geq 2$.

Hence, by the principle of mathematical induction, the proposition is true for all natural numbers $n \geq 4$.