12 Differentiation

12.1 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+4x+2)=2x+4$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(4x^3-2x^2+1)=12x^2-4x$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^9+7x^5-2x^4+3)=9x^8+35x^4-8x^3$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}(11x^{10}-10x^{11})=110x^9-110x^{10}=110x^9(1-x)$

12.2 (a) (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^2+x)(2x-1)\}=\dfrac{\mathrm{d}}{\mathrm{d}x}(2x^3+x^2-x)=6x^2+2x-1$

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^2+x)(2x-1)\}=(2x+1)(2x-1)+(x^2+x)2=4x^2-1+2x^2+2x=6x^2+2x-1$

(b) (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(2x^3+1)(x-1)\}=\dfrac{\mathrm{d}}{\mathrm{d}x}\{2x^4-2x^3+x-1\}=8x^3-6x^2+1$

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(2x^3+1)(x-1)\}=6x^2(x-1)+(2x^3+1)=8x^3-6x^2+1$

(c) (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^3-x+1)(x^2+3x-1)\}=\dfrac{\mathrm{d}}{\mathrm{d}x}\{x^5+3x^4-x^3-x^3-3x^2+x+x^2+3x-1\}$

$=\dfrac{\mathrm{d}}{\mathrm{d}x}\{x^5+3x^4-2x^3-2x^2+4x-1\}=5x^4+12x^3-6x^2-4x+4$

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^3-x+1)(x^2+3x-1)\}=(3x^2-1)(x^2+3x-1)+(x^3-x+1)(2x+3)$

$=3x^4+9x^3-3x^2-x^2-3x+1+2x^4-2x^2+2x+3x^3-3x+3$

$=5x^4+12x^3-6x^2-4x+4$

(d) (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x-1)(x^3+x^2+x+1)\}=\dfrac{\mathrm{d}}{\mathrm{d}x}\{x^4+x^3+x^2+x-x^3-x^2-x-1\}$

$=\dfrac{\mathrm{d}}{\mathrm{d}x}\{x^4-1\}=4x^3$

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x-1)(x^3+x^2+x+1)\}=1\cdot(x^3+x^2+x+1)+(x-1)(3x^2+2x+1)$

$=x^3+x^2+x+1+3x^3+2x^2+x-3x^2-2x+1$

$=4x^3$

12.3 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^2-x+1)(x^2+x+1)\} = (2x-1)(x^2+x+1)+(x^2-x+1)(2x+1)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^3-2x^2+1)(2x^4-x^2+5)\} = (3x^2-4x)(2x^4-x^2+5)+(x^3-2x^2+1)(8x^3-2x)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^3-5x+3)^2\} = \dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^3-5x+3)(x^3-5x+3)\}$

$= (3x^2-5)(x^3-5x+3)+(x^3-5x+3)(3x^2-5) = 2(3x^2-5)(x^3-5x+3)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(3x^3-2x^2+1)(x^7-4x^5+2x^2+3)\}$

$= (9x^2-4x)(x^7-4x^5+2x^2+3)+(3x^3-2x^2+1)(7x^6-20x^4+4x)$

12.4 $\{fgh\}' = \{f\,[gh]\}' = f'\,[gh]+f\,[gh]'$

$= f'gh+f\,[g'h+gh']$

$= f'gh+fg'h+fgh'$

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{x(x+1)(x+3)\} = (x+1)(x+3)+x(x+3)+x(x+1)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x+1)(x^2+2x+4)(x^3+3x^2+6x+9)\}$

$= (x^2+2x+4)(x^3+3x^2+6x+9) + (x+1)(2x+2)(x^3+3x^2+6x+9)$

$+ (x+1)(x^2+2x+4)(3x^2+6x+6)$

12.5 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^{-2}+2x^{-4}-7x^{-5}) = -2x^{-3}-8x^{-5}+35x^{-6}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^3-3x^{-1}+\dfrac{1}{2}x^{-2}) = 3x^2+3x^{-2}-x^{-3}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}(x^3+x+x^{-1}+x^{-3})$

$= 3x^2+1-x^{-2}-3x^{-4} = 3x^2+1-\dfrac{1}{x^2}-\dfrac{3}{x^4}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(1-\dfrac{2}{x^2}+\dfrac{4}{x^4}-\dfrac{8}{x^8}\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(1-2x^{-2}+4x^{-4}-8x^{-8}\right)$

$= 4x^{-3}-16x^{-5}+64x^{-9} = \dfrac{4}{x^3}-\dfrac{16}{x^5}+\dfrac{64}{x^9}$

12.6 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x-2}{x-3}\right) = \dfrac{(x-3)-(x-2)}{(x-3)^2} = -\dfrac{1}{(x-3)^2}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2x+1}{3x+2}\right) = \dfrac{2(3x+2)-(2x+1)3}{(3x+2)^2} = \dfrac{1}{(3x+2)^2}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^2-1}{x^2+1}\right) = \dfrac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x+1}{x^2+3x+6}\right) = \dfrac{x^2+3x+6-(x+1)(2x+3)}{(x^2+3x+6)^2}$

$$= \frac{x^2+3x+6-(2x^2+5x+3)}{(x^2+3x+6)^2} = \frac{-x^2-2x+3}{(x^2+3x+6)^2} = -\frac{x^2+2x-3}{(x^2+3x+6)^2}$$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x+1}{x^3}\right) = \dfrac{x^3-3x^2(x+1)}{x^6} = \dfrac{x^3-3x^3-3x^2}{x^6} = -\dfrac{(2x^3+3x^2)}{x^6} = -\dfrac{2x+3}{x^4}$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1-5x^4}{x+2}\right) = \dfrac{-20x^3(x+2)-(1-5x^4)}{(x+2)^2} = \dfrac{-20x^4-40x^3-1+5x^4}{(x+2)^2}$

$$= -\frac{15x^4+40x^3+1}{(x+2)^2}$$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{x-1}-\dfrac{1}{x+1}\right) = \dfrac{x-1-x}{(x-1)^2}+\dfrac{1}{(x+1)^2} = -\dfrac{1}{(x-1)^2}+\dfrac{1}{(x+1)^2}$

$$= \frac{-x^2-2x-1+x^2-2x+1}{(x^2-1)^2} = -\frac{4x}{(x^2-1)^2}$$

(h) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{(x-2)(x+3)}{x+4}\right) = \dfrac{[(x+3)+(x-2)](x+4)-(x-2)(x+3)}{(x+4)^2}$

$$= \frac{x^2+7x+12+x^2+2x-8-x^2-x+6}{(x+4)^2} = \frac{x^2+8x+10}{(x+4)^2}$$

12.7 The area of the circle is $A=\pi r^2 \implies r=\sqrt{\dfrac{A}{\pi}}$ (as r cannot be negative). So the circumference is $C=2\pi r = 2\pi\sqrt{\dfrac{A}{\pi}} = 2\sqrt{\pi}A^{\frac{1}{2}}$.
The rate of change with respect to A is therefore $\dfrac{\mathrm{d}C}{\mathrm{d}A} = 2\sqrt{\pi}\cdot\dfrac{1}{2}A^{-\frac{1}{2}} = \sqrt{\dfrac{\pi}{A}}$.

12.8 $y(x) = x^3+6x^2 \implies y'(x) = 3x^2+12x = 3x(x+4)$. This changes sign where $3x(x+4)=0 \implies x=-4$ or 0. When $x<-4$, $y'(x)$ is positive, so function is increasing. When $-4<x<0$, $y'(x)$ is negative so function is decreasing. When $x>0$, $y'(x)$ is positive so function is increasing.

12.9 $$\begin{aligned} f(x) &= \frac{4x}{x^2-7}, \quad x^2 \neq 7 \\ f'(x) &= \frac{4(x^2-7)-4x(2x)}{(x^2-7)^2} = \frac{-4x^2-28}{(x^2-7)^2} = \frac{-4(x^2+7)}{(x^2-7)^2}. \end{aligned}$$

Since $x^2+7 \geq 7 > 0$, $f'(x) \neq 0$ for all x such that $x^2 \neq 7$.

12.10 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(2x+1)^4\} = 4(2x+1)^3(2)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(3-2x)^3\} = 3(3-2x)^2(-2)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^2+3x+1)^6\} = 6(x^2+3x+1)^5(2x+3)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^2+1+x^{-2})^{-1}\} = -(x^2+1+x^{-2})^{-2}(2x-2x^{-3})$

12.11 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2x^4-3x^{1/2}) = 8x^3 - \dfrac{3}{2}\,x^{-1/2}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^{4/3}+x^{2/3}) = \dfrac{4}{3}\,x^{1/3} + \dfrac{2}{3}\,x^{-1/3}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2x^{5/2}+4x^{3/2}+6x^{1/2}) = 5x^{3/2}+6x^{1/2}+3x^{-1/2}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^{2/7}-x^{3/11}) = \dfrac{2}{7}\,x^{-5/7} - \dfrac{3}{11}\,x^{-8/11}$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(2-3x)^{1/3}\} = \dfrac{1}{3}(2-3x)^{-2/3}(-3) = -(2-3x)^{-2/3}$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{(5x+1)^{-1/4}\} = -\dfrac{1}{4}\,(5x+1)^{-5/4}(5) = -\dfrac{5}{4}\,(5x+1)^{-5/4}$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}\{\sqrt{x^2+2x}\} = \dfrac{\mathrm{d}}{\mathrm{d}x}\{(x^2+2x)^{1/2}\} = \dfrac{1}{2}\,(x^2+2x)^{-1/2}(2x+2) = \dfrac{x+1}{\sqrt{x^2+2x}}$

(h) $$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x^2-x+6}}\right\} &= \{(x^2-x+6)^{-1/2}\}' = -\frac{1}{2}\,(x^2-x+6)^{-3/2}(2x-1) \\ &= -\frac{2x-1}{2(x^2-x+6)^{3/2}} \end{aligned}$$

12.12 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x+\dfrac{1}{x}\right)^3 = 3\left(x+\dfrac{1}{x}\right)^2\left(1-\dfrac{1}{x^2}\right)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{2x^4-5x+2} = \dfrac{1}{2\sqrt{2x^4-5x+2}}\,(8x^3-5)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,(x^{-1}-x^{-2})^{-1/4} = -\dfrac{1}{4}(x^{-1}-x^{-2})^{-5/4}(-x^{-2}+2x^{-3})$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2x^2-1}{x^3+3}\right)^{1/3} = \dfrac{1}{3}\left(\dfrac{2x^2-1}{x^3+3}\right)^{-2/3}\;\dfrac{4x(x^3+3)-(2x^2-1)3x^2}{(x^3+3)^2}$

$= \dfrac{1}{3}\,\dfrac{(x^3+3)^{2/3}}{(2x^2-1)^{2/3}}\cdot\dfrac{4x^4+12x-6x^4+3x^2}{(x^3+3)^2} = -\dfrac{1}{3}\,\dfrac{(2x^4-3x^2-12x)}{(2x^2-1)^{2/3}(x^3+3)^{4/3}}$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(1+\sqrt{\dfrac{x-2}{3}}\right)^4 = 4\left(1+\sqrt{\dfrac{x-2}{3}}\right)^3\cdot\dfrac{1}{2}\left(\dfrac{x-2}{3}\right)^{-\frac{1}{2}}\cdot\dfrac{1}{3}$

$= \dfrac{2}{3}\left(1+\sqrt{\dfrac{x-2}{3}}\right)^3\left(\dfrac{x-2}{3}\right)^{-\frac{1}{2}}$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x+\left((3x)^5-2\right)^{-\frac{1}{2}}\right)^{-6}$

$= -6\left(x+\left((3x)^5-2\right)^{-\frac{1}{2}}\right)^{-7}\cdot\left(1+\left(-\dfrac{1}{2}\left((3x)^5-2\right)^{-\frac{3}{2}}\cdot 5\,(3x)^4\cdot 3\right)\right)$

$= -6\left(1-\dfrac{15}{2}\,(3x)^4\left((3x)^5-2\right)^{-\frac{3}{2}}\right)\left(x+\left((3x)^5-2\right)^{-\frac{1}{2}}\right)^{-7}$

12.13 These derivatives can be found by application of the quotient rule.
For $f(x)=\csc x=\dfrac{1}{\sin x}$ we have $f'(x)=\dfrac{-\cos x}{\sin^2 x}=-\csc x\cot x$.
For $g(x)=\sec x=\dfrac{1}{\cos x}$ we have $g'(x)=\dfrac{-(-\sin x)}{\cos^2 x}=\sec x\tan x$.
For $h(x)=\cot x=\dfrac{\cos x}{\sin x}$ we have
$h'(x)=\dfrac{-\sin^2 x-\cos^2 x}{\sin^2 x}=-\dfrac{1}{\sin^2 x}=-\csc^2 x=-(1+\cot^2 x)$.

12.14 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}3\sin x=3\cos x$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin 3x=3\cos 3x$

12.14 (c) $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin x^3 = 3x^2 \cos x^3$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin^3 x = 3\sin^2 x \ \cos x$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x} 3\sin^3 3x^3 = 3 \cdot 3\sin^2 3x^3 \cdot \cos 3x^3 \cdot 3 \cdot 3x^2 = 81x^2 \sin^2 3x^3 \cos 3x^3$

12.15 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x} (3\cos x + 4\tan x) = -3\sin x + 4\sec^2 x$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x} (\sin^2 x + \cos^2 x) = 2\sin x \cos x - 2\cos x \sin x = 0$, or simply
$\dfrac{\mathrm{d}}{\mathrm{d}x} (\sin^2 x + \cos^2 x) = \dfrac{\mathrm{d}}{\mathrm{d}x} 1 = 0.$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x} [\sec(4x-3)] = 4\sec(4x-3)\tan(4x-3)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x} [x^2 \sec 5x] = 2x \sec 5x + 5x^2 \sec 5x \ \tan 5x$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x} [\csc(\sqrt{x})] = -\dfrac{1}{2\sqrt{x}} \ \csc(\sqrt{x})\cot(\sqrt{x})$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x} \left[\cot \dfrac{1}{2x+1}\right] = -\csc^2 \dfrac{1}{2x+1} \cdot 2 \cdot \left(-\dfrac{1}{(2x+1)^2}\right) = \dfrac{2}{(2x+1)^2} \ \csc^2 \dfrac{1}{2x+1}$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x} [\sin 3x \ \cos 5x] = 3\cos 3x \ \cos 5x - 5\sin 3x \ \sin 5x$

(h) $\dfrac{\mathrm{d}}{\mathrm{d}x} \left[\dfrac{\sin x}{x^2+1}\right] = \dfrac{(x^2+1)\cos x - 2x\sin x}{(x^2+1)^2}$

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x} [\tan^3(2x^2+1)] = 3\tan^2(2x^2+1)\sec^2(2x^2+1) \cdot 4x$
$= 12x \tan^2(2x^2+1)\sec^2(2x^2+1).$

(j) $\dfrac{\mathrm{d}}{\mathrm{d}x} \left[\dfrac{1}{1+\sin x}\right] = -\dfrac{\cos x}{(1+\sin x)^2}$

12.15 (k) $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1+\sec x}{1-\sec x}\right] = \frac{\sec x\tan x(1-\sec x)-(1+\sec x)(-\sec x\tan x)}{(1-\sec x)^2} = \frac{2\sec x\tan x}{(1-\sec x)^2}$$

(l) $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x}\,\cos x\right] = \frac{1}{2\sqrt{x}}\,\cos x - \sqrt{x}\,\sin x.$$

(m) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\sin x) = \cos(\sin x)\cdot\cos x.$$

(n) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos(\sin x)) = \cos(\cos(\sin x))\cdot(-\sin(\sin x))\cdot\cos x.$$

(o) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{x}{\sin x}\right) = \cos\left(\frac{x}{\sin x}\right)\cdot\frac{\sin x - x\cos x}{\sin^2 x}.$$

(p) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{x}{\sin\left(\frac{x}{\sin x}\right)}\right) = \cos\left(\frac{x}{\sin\left(\frac{x}{\sin x}\right)}\right)\cdot\frac{\sin\left(\frac{x}{\sin x}\right) - x\cdot\cos\left(\frac{x}{\sin x}\right)\cdot\frac{\sin x - x\cos x}{\sin^2 x}}{\sin^2\left(\frac{x}{\sin x}\right)}.$$

12.16 (a) $2x, 2, 0$

(b) $8x^7, 56x^6, 336x^5$

(c) $-3x^{-4}, 12x^{-5}, -60x^{-6}$

(d) $-\frac{1}{2}\,x^{-3/2},\ \frac{3}{4}\,x^{-5/2},\ -\frac{15}{8}\,x^{-7/2}$

(e) $-\frac{1}{(x-3)^2},\quad \frac{2}{(x-3)^3},\quad -\frac{6}{(x-3)^4}.$

(f) $\frac{6}{(2-3x)^3},\ \frac{54}{(2-3x)^4},\ \frac{648}{(2-3x)^5}.$

(g) $\frac{x}{\sqrt{x^2+1}},\ \frac{1}{(x^2+1)^{3/2}},\ \frac{-3x}{(x^2+1)^{5/2}}.$

(h) $3\cos 3x,\ -9\sin 3x,\ -27\cos 3x$

(i) $-2x\sin x^2,\ -2\sin x^2 - 4x^2\cos x^2,\ -12x\cos x^2 + 8x^3\sin x^2$

12.17
$$\begin{aligned} f(x) &= \tan x \\ f'(x) &= \sec^2 x = 1+\tan^2 x = 1+(f(x))^2, \quad \text{so} \quad f' = 1+f^2 \\ f'' &= 2f\,f' = 2f(1+f^2) \\ f''' &= (2+6f^2)f' = 2(1+3f^2)(1+f^2) \end{aligned}$$

12.18 $u(t) = A\sin(\omega t+\phi) \implies \dfrac{\mathrm{d}u}{\mathrm{d}t} = \omega A\cos(\omega t+\phi) \implies \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} = -\omega^2 A\sin(\omega t+\phi)$

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + ku(t) = -\omega^2 A\sin(\omega t+\phi) + kA\sin(\omega t+\phi) = 0 \iff k=\omega^2.$$

12.19 $y(x) = \sin(\sqrt{x}) \quad y'(x) = \dfrac{1}{2\sqrt{x}}\cos(\sqrt{x}) \quad y''(x) = -\dfrac{1}{4}\dfrac{1}{x\sqrt{x}}\cos(\sqrt{x}) - \dfrac{1}{4x}\sin(\sqrt{x})$

$$4xy''(x)+ay'(x)+by(x) = -\frac{1}{\sqrt{x}}\cos(\sqrt{x}) - \sin(\sqrt{x}) + \frac{a}{2\sqrt{x}}\cos(\sqrt{x}) + b\sin(\sqrt{x}) = 0$$

$$\begin{aligned} -1+\frac{a}{2} = 0 &\implies a=2 \\ -1+b=0 &\implies b=1. \end{aligned}$$

The natural domain of f' is $(0,\infty)$; not the same as the natural domain of f.

12.20
$$\begin{aligned} u(x) &= \frac{1}{\sqrt{x}}\sin x \\ u'(x) &= -\frac{1}{2x\sqrt{x}}\sin x + \frac{1}{\sqrt{x}}\cos x \\ u''(x) &= \frac{3}{4x^2\sqrt{x}}\sin x - \frac{1}{x\sqrt{x}}\cos x - \frac{1}{\sqrt{x}}\sin x \end{aligned}$$

$$x^2u''(x)+xu'(x)+\left(x^2-\frac{1}{4}\right)u(x)$$
$$= \frac{3}{4\sqrt{x}}\sin x - \sqrt{x}\cos x - x\sqrt{x}\sin x - \frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x + x\sqrt{x}\sin x - \frac{1}{4}\frac{1}{\sqrt{x}}\sin x$$
$$= 0.$$

12.21 (a) For $n=1$, $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}x = \dfrac{\mathrm{d}}{\mathrm{d}x}1 = 0.$

Consider a given n and suppose that $\dfrac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}x^n = 0$, then

$$\frac{\mathrm{d}^{n+2}}{\mathrm{d}x^{n+2}}x^{n+1} = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}((n+1)x^n) = (n+1)\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}x^n = 0.$$

Hence the result is true for $n+1$ and by induction holds for all $n\in\mathbb{N}$.

12.21 (b) For $n=1$, $\dfrac{\mathrm{d}}{\mathrm{d}x}x = 1 = 1!$.

Assume that for some n we have $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}x^n = n!$. Then

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}x^{n+1} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}((n+1)x^n) = (n+1)\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^n = (n+1)n! = (n+1)!$$

and so whenever the result holds for n it holds for $n+1$ and so by induction it holds for all $n \in \mathbb{N}$.

12.22 For $n=1$, $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{-1/2} = -\dfrac{1}{2}\,x^{-3/2}$ while

$$(-1)^n\,\frac{(2n)!}{2^{2n}\,n!}\,x^{-(n+1/2)}\bigg|_{n=1} = -\frac{2!}{2^2(1)}\,x^{-3/2} = -\frac{2}{4}\,x^{-3/2} = -\frac{1}{2}\,x^{-3/2}.$$

Assume $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{-1/2}) = (-1)^n\,\dfrac{(2n)!}{2^{2n}n!}\,x^{-(n+1/2)}$ for some n. Then

$$\begin{aligned}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^{-1/2}) &= (-1)^n\,\frac{(2n)!}{2^{2n}n!}\,(-1)\left(n+\frac{1}{2}\right)x^{-(n+3/2)}\\ &= (-1)^{n+1}\,\frac{(2n)!(2n+1)}{2^{2n+1}n!}\,x^{-(n+3/2)} = (-1)^{n+1}\,\frac{(2n+1)!}{2^{2n+1}n!}\frac{(2n+2)}{2(n+1)}\,x^{-(n+3/2)}\\ &= \frac{(-1)^{n+1}\,(2n+2)!}{2^{2n+2}\,(n+1)!}\,x^{-(n+3/2)}\\ &= \frac{(-1)^{n+1}\,(2[n+1])!}{2^{2[n+1]}\,(n+1)!}\,x^{-([n+1]+1/2)},\end{aligned}$$

and so the formula is true for $n+1$ if it is true for n. Hence it is true for all $n \in \mathbb{N}$.

12.23 For any x,

$$\sin\left(x+\frac{\pi}{2}\right) = \sin x\cos\frac{\pi}{2} + \cos x\sin\frac{\pi}{2} = \cos x = \frac{\mathrm{d}\sin x}{\mathrm{d}x}, \tag{1}$$

so it is true for $n=1$.
Assume $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\sin x = \sin\left(x+\dfrac{n\pi}{2}\right)$ for a given n. Then

$$\begin{aligned}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\sin x &= \frac{\mathrm{d}}{\mathrm{d}x}\sin\left(x+\frac{n\pi}{2}\right) = \cos\left(x+\frac{n\pi}{2}\right)\\ &= \sin\left(\left[x+\frac{n\pi}{2}\right]+\frac{\pi}{2}\right) = \sin\left(x+\frac{(n+1)\pi}{2}\right),\end{aligned}$$

(from (1)). Thus true for $n+1$ if it is true for n.
Hence it is true for all $n \in \mathbb{N}$.

12.24 $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x = \sin\left(x + \dfrac{\pi}{2}\right)$

$$\implies \quad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin x = \frac{\mathrm{d}}{\mathrm{d}x}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right).$$

A similar proof to that in 12.23 shows that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\cos x = \cos\left(x + \frac{n\pi}{2}\right).$$

12.25 With $x = \sin y$ for $x \neq 0$ we have $1/x = 1/\sin y = \csc y$. Hence $y = \operatorname{arccsc} 1/x = \arcsin x$ and so

$$\operatorname{arccsc} x \;=\; \arcsin\frac{1}{x}.$$

Similarily,

$$\begin{aligned} \operatorname{arcsec} x &= \arccos\frac{1}{x} \quad \text{and} \\ \operatorname{arccot} x &= \arctan\frac{1}{x}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccsc} x &= \frac{\mathrm{d}}{\mathrm{d}x}\arcsin\frac{1}{x} = \frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right) \\ &= -\frac{1}{x^2}\sqrt{\frac{x^2}{x^2-1}} = -\frac{1}{x^2}\frac{|x|}{\sqrt{x^2-1}} = -\frac{1}{|x|\sqrt{x^2-1}}, \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsec} x &= \frac{\mathrm{d}}{\mathrm{d}x}\arccos\frac{1}{x} = -\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right) \\ &= \frac{1}{x^2}\sqrt{\frac{x^2}{x^2-1}} = \frac{1}{x^2}\frac{|x|}{\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot} x &= \frac{\mathrm{d}}{\mathrm{d}x}\arctan\frac{1}{x} = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) \\ &= -\frac{1}{1+x^2}. \end{aligned}$$

12.26 (a) $\dfrac{\mathrm{d}}{\mathrm{d}x} \arcsin 2x = \dfrac{2}{\sqrt{1-4x^2}}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x} \arccos \dfrac{x}{5} = -\dfrac{1}{5} \dfrac{1}{\sqrt{1-x^2/25}} = -\dfrac{1}{\sqrt{25-x^2}}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x} \arctan 7x = \dfrac{7}{1+49x^2}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x} \sqrt{\arctan(3x-2)} = \dfrac{1}{2\sqrt{\arctan(3x-2)}} \dfrac{3}{1+(3x-2)^2}$
$= \dfrac{3}{2(9x^2-12x+5)\sqrt{\arctan(3x-2)}}$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x} \{x \arcsin x + \sqrt{1-x^2}\} = \arcsin x + \dfrac{x}{\sqrt{1-x^2}} + \dfrac{1}{2} \dfrac{(-2x)}{\sqrt{1-x^2}} = \arcsin x$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x} \arccos(\sqrt{1-x^2}) = -\dfrac{1}{\sqrt{1-(1-x^2)}} \cdot \dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{x}{x\sqrt{1-x^2}} = \dfrac{1}{\sqrt{1-x^2}}$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x} \arcsin(\cos x) = \dfrac{1}{\sqrt{1-\cos^2 x}} (-\sin x) = -\dfrac{1}{|\sin x|} \sin x = -1, \qquad 0 < x < \pi$

(h) $\dfrac{\mathrm{d}}{\mathrm{d}x} \arcsin(\cos x) = \dfrac{1}{\sqrt{1-\cos^2 x}} (-\sin x) = -\dfrac{1}{|\sin x|} \sin x = 1, \qquad -\pi < x < 0$

12.27 Proof by induction on n: for $n=1$ we have

$$(f\cdot g)'(a)=\sum_{k=0}^{1}\binom{1}{k}f^{(k)}(a)\cdot g^{(1-k)}(a)=f(a)g'(a)+f'(a)g(a)$$

which is simply the product rule.
Suppose that the formula holds for some n. Then because $(f\cdot g)^{(n+1)}=((f\cdot g)^{(n)})'$ we have

$$\begin{aligned}
(f\cdot g)^{(n+1)} &= \sum_{k=0}^{n}\binom{n}{k}(f^{(k)}g^{(n-k)})'(a)\\
&= \sum_{k=0}^{n}\binom{n}{k}\left\{f^{(k+1)}(a)g^{(n-k)}(a)+f^{(k)}(a)g^{(n-k+1)}(a)\right\}\\
&= \sum_{k=0}^{n}\binom{n}{k}f^{(k+1)}(a)g^{(n-k)}(a)+\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(a)g^{(n-k+1)}(a)\\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}f^{(k)}(a)g^{(n-k+1)}(a)+\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(a)g^{(n-k+1)}(a)\\
&= f^{(n+1)}(a)g(a)+\left[\sum_{k=1}^{n}\left\{\binom{n}{k-1}+\binom{n}{k}\right\}f^{(k)}(a)g^{(n-k+1)}(a)\right]+f(a)g^{(n+1)}(a)\\
&= f^{(n+1)}(a)g(a)+\left[\sum_{k=1}^{n}\binom{n+1}{k}f^{(k)}(a)g^{(n-k+1)}(a)\right]+f(a)g^{(n+1)}(a)\\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}f^{(k)}(a)g^{(n-k+1)}(a).
\end{aligned}$$

So the formula holds for $n+1$ when it holds for n and hence it holds for all n. Note how this proofs mirrors the proof of the binomial theorem.

12.28 If $x=f(x)g(x)$ then $1=f'(x)g(x)+f(x)g'(x)$, and for $x=0$ this becomes $1=f'(0)g(0)+f(0)g'(0)$. If now also $f(0)=g(0)=0$, this leads to $1=0$, which is a contradiction.

12.29 $f'(0)=\lim_{h\to 0}\dfrac{f(h)-f(0)}{h}=\lim_{h\to 0}\dfrac{hg(h)-0g(0)}{h}=\lim_{h\to 0}g(h)=g(0)$ because g is continuous at 0.

12.30 If $g(x) = f(x)/x$ for $x \neq 0$, we have

$$\lim_{h\to 0} g(h) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f'(0).$$

Define

$$g(x) := \begin{cases} \dfrac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0, \end{cases}$$

then $\lim_{h\to 0} g(h) = g(0)$ and so g is continuous at 0. Furthermore, for all x we have that $f(x) = xg(x)$.