UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercise solutions: Chapter 7

1. (i) $\sum_{j=1}^{20} j$. (ii) $\sum_{j=1}^{15} 2j$. (iii) $\sum_{j=1}^{10} j^3$. (iv) $\sum_{j=15}^{25} j$.

2. (i) $1^4 + 2^4 + 3^4 + 4^4 + 5^4 = 979$.
   (ii) $1^2 + 3^2 + 5^2 + 7^2 + 9^2 = 165$.
   (iii) $2^2 + 4^2 + 6^2 + 8^2 + \ldots + (2n)^2$.
   (iv) $3^2 + 6^2 + 9^2 + 12^2 = 270$.
   (v) $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6$.
   (vi) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!}$.

3. (a) Putting $n = 20$ into the formula $n(n+1)(2n+1)/6$ produces 2870.

   (b) We have

$$\begin{aligned} 2^2 + 4^2 + 6^2 + 8^2 + \ldots + (2n)^2 &= 2^2[1^2 + 2^2 + 3^2 + \ldots + n^2] \\ &= 4\sum_{r=1}^{n} r^2 \\ &= \frac{4n(n+1)(2n+1)}{6} \\ &= \frac{2n(n+1)(2n+1)}{3}. \end{aligned}$$

4. Expanding the left-hand side gives

$$[2^4 - 1^4] + [3^4 - 2^4] + [4^4 - 3^4] + \ldots + [n^4 - (n-1)^4] + [(n+1)^4 - n^4]$$

which, on cancelling like terms, becomes $(n+1)^4 - 1^4$. Hence we have

$$(n+1)^4 - 1 = \sum_{j=1}^{n}(4j^3 + 6j^2 + 4j + 1) = 4\sum_{j=1}^{n} j^3 + 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1.$$

Using our previously proved results for the sum of the first n integers and first n squares, we have

$$n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 = 4\sum_{j=1}^{n} j^3 + 6\left(\frac{n}{6}(n+1)(2n+1)\right) + 4\left(\frac{n(n+1)}{2}\right) + n.$$

Rearranging gives

$$\begin{aligned} 4\sum_{j=1}^{n} j^3 &= n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n \\ &= n^4 + 2n^3 + n^2 \\ &= n^2(n+1)^2 \end{aligned}$$

so our final result is

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

as required.

5. (a) We have

$$\begin{aligned} \sum_{r=1}^{n} \big(2r^3 + (r+1)^2\big) &= \sum_{r=1}^{n} \big(2r^3 + r^2 + 2r + 1\big) \\ &= 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \\ &= 2\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} + n. \end{aligned}$$

This can be simplified, e.g. to

$$\frac{n(3n^3 + 8n^2 + 12n + 13)}{6}.$$

(b) We have

$$\begin{aligned} \sum_{r=1}^{n} \big(3r^3 + 2r^2 + 3r + 5\big) &= 3\sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + 5\sum_{r=1}^{n} 1 \\ &= 3\frac{n^2(n+1)^2}{4} + 2\frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2} + 5n. \end{aligned}$$

This can be simplified, e.g. to

$$\frac{n(9n^3 + 26n^2 + 39n + 82)}{12}.$$

6. We have

$$\left|\frac{n-1}{n} - 1\right| = \left|1 - \frac{1}{n} - 1\right| = \left|-\frac{1}{n}\right| = \frac{1}{n}$$

so if we choose an integer $N > \dfrac{1}{\epsilon}$ $(\epsilon > 0)$ then

$$\left|\frac{n-1}{n} - 1\right| < \epsilon$$

for all $n \geq N$ and, using Definition 7.5,

$$\lim_{n\to\infty} \frac{n-1}{n} = 1.$$

7. (a)

$$\frac{1-2n}{1+2n} = \frac{\frac{1}{n}-2}{\frac{1}{n}+2}$$

so

$$\lim_{n\to\infty}\frac{1-2n}{1+2n} = \frac{\lim_{n\to\infty}\left(\frac{1}{n}-2\right)}{\lim_{n\to\infty}\left(\frac{1}{n}+2\right)} = \frac{-2}{2} = -1$$

as $\lim_{n\to\infty}\frac{1}{n} = 0$. So $\{u_n\}$ converges and has limit -1.

(b)

$$\frac{3+4n^4}{n^4+3n^3} = \frac{\frac{3}{n^4}+4}{1+\frac{3}{n}}$$

so

$$\lim_{n\to\infty}\frac{3+4n^4}{n^4+3n^3} = \frac{\lim_{n\to\infty}\left(\frac{3}{n^4}+4\right)}{\lim_{n\to\infty}\left(1+\frac{3}{n}\right)} = \frac{4}{1} = 4$$

as $\lim_{n\to\infty}\frac{3}{n} = 0$ and $\lim_{n\to\infty}\frac{3}{n^4} = 0$. So $\{u_n\}$ converges and has limit 4.

(c)

$$\frac{n^2-2n+1}{n-1} = \frac{1-\frac{2}{n}+\frac{1}{n^2}}{\frac{1}{n}-\frac{1}{n^2}}$$

so $\{u_n\}$ is divergent as $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to\infty}\frac{1}{n^2} = 0$.

8. This is a geometric progression with first term $a = 2/3$ and common ratio $r = 2/3$. Since $|r| < 1$, the inifinite series converges and has the sum

$$\frac{a}{1-r} = \frac{2/3}{1/3} = 2.$$

9. Let

$$S_n = 1\times 3 + 2\times 3^2 + 3\times 3^3 + 4\times 3^4 + \cdots + n\times 3^n, \qquad \text{(A)}.$$

Then

$$3S_n = 1\times 3^2 + 2\times 3^3 + 3\times 3^4 + \cdots + (n-1)\times 3^n + n\times 3^{n+1}, \qquad \text{(B)}.$$

Subtracting, (A) - (B) gives

$$-2S_n = \left(1 \times 3 + 1 \times 3^2 + 1 \times 3^3 + 1 \times 3^4 + \cdots + 1 \times 3^n\right) - n \times 3^{n+1},$$

The term in brackets on the RHS is sum of a geometric progression. Using the formula we get

$$-2S_n = 3\frac{1-3^n}{1-3} - n3^{n+1}.$$

This simplifies to, e.g.,

$$S_n = 3^{n+1}\frac{2n-1}{4} + \frac{3}{4}.$$

For the general case, let

$$S_n = ar + a2r^2 + a3r^3 + a4r^4 + \cdots + anr^n. \qquad \text{(C)}.$$

Then

$$rS_n = ar^2 + a2r^3 + a3r^4 + \cdots + a(n-1)r^n + anr^{n+1}, \qquad \text{(D)}.$$

Subtracting, (C) - (D) gives

$$(1-r)S_n = \left(ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n\right) - anr^{n+1}.$$

The term in brackets on the RHS is sum of a geometric progression. Using the formula we get

$$(1-r)S_n = ar\frac{1-r^n}{1-r} - anr^{n+1}.$$

This simplifies to, e.g.,

$$S_n = ar\frac{1-(1+n)r^n + nr^{n+1}}{(1-r)^2}.$$

Exercise solutions: Chapter 8

1. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 20 \times 6 = 120$
   $6! = 6 \times 5! = 720$
   $7! = 7 \times 6! = 5040$
   $8! = 8 \times 7! = 40320$
   $9! = 9 \times 8! = 362880$
   $10! = 10 \times 9! = 3628800$

2. $20! = 2432902008176640000$,
   $30! = 265252859812191058636308480000000$,
   $40! = 815915283247897734345611269596115894272000000000$.

3. (a) $5! + 4! = (5+1)4! = 6 \times 4!$
   (b) $100! - 98! = (9900-1)98! = 9899 \times 98!$
   (c) $(n+1)! - n! = \Big((n+1) - 1\Big)n! = n \times n!$

4. (a)

$$\begin{aligned}\frac{15! - 13!}{11!2!} &= \frac{15.14.13.12.(11!) - 13.12.(11!)}{11!2!} \\ &= \frac{13.12[15.14 - 1]}{2!} \\ &= 13.6.(15.14 - 1) = 16302\end{aligned}$$

   (b)

$$\begin{aligned}\frac{12! + 11!}{8!3!} &= \frac{12.11.10.9.(8!) + 11.10.9.(8!)}{8!3!} \\ &= \frac{(11.10.9).(12+1)}{3!} \\ &= 11.5.3.(12+1) = 13.11.5.3 = 2145\end{aligned}$$

5.

$$\begin{aligned}\frac{1}{n!} + \frac{3}{(n-1)!} + \frac{1}{(n-2)!} &= \frac{1 + 3n + n(n-1)}{n!} \\ &= \frac{1 + 2n + n^2}{n!} \\ &= \frac{(n+1)^2}{n!}\end{aligned}$$

6. (a)

$$\binom{7}{2} = \frac{7!}{5!2!} = \frac{7.6}{2} = 21.$$

(b)

$$\binom{6}{0} = 1, \quad \binom{6}{1} = 6, \quad \binom{6}{2} = \frac{6.5}{2} = 15$$

$$\binom{6}{3} = \frac{6.5.4}{3.2.1} = 20, \quad \binom{6}{4} = 15, \quad \binom{6}{5} = 6, \quad \binom{6}{6} = 1.$$

7. We have

$$\binom{14}{r} = \binom{14}{14-r},$$

and

$$\binom{14}{14-r} = \binom{14}{r-4}$$

so $14 - r = r - 4$, giving $r = 9$.

8. (a) $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

(b) $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$,

9. (i)
$$\begin{aligned}(x-y)^4 &= x^4 + 4x^3(-y) + 6x^2(-y)^2 + 4x(-y)^3 + (-y)^4 \\ &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.\end{aligned}$$

(ii)
$$\begin{aligned}(2x+y)^5 &= (2x)^5 + 5(2x)^4y + 10(2x)^3y^2 + 10(2x)^2y^3 + 5(2x)y^4 + y^5 \\ &= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5.\end{aligned}$$

(iii)
$$\begin{aligned}(2p+3q)^4 &= (2p)^4 + 4(2p)^3(3q) + 6(2p)^2(3q)^2 + 4(2p)(3q)^3 + (3q)^4 \\ &= 16p^4 + 96p^3q + 216p^2q^2 + 216pq^3 + 81q^4.\end{aligned}$$

(iv)
$$\begin{aligned}(x-2y)^6 &= x^6 + 6x^5(-2y) + 15x^4(-2y)^2 + 20x^3(-2y)^3 \\ &\quad +15x^2(-2y)^4 + 6x(-2y)^5 + (-2y)^6 \\ &= x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6.\end{aligned}$$

(v)
$$\begin{aligned}(4r-3s)^5 &= ((4r)^5 + 5(4r)^4(--3s) + 10(4r)^3(-3s)^2 + 10(4r)^2(-3s)^3 \\ &\quad +5(4r)(-3s)^4 + (-3s)^5 \\ &= 1024r^5 - 3840r^4s + 5760r^3s^2 - 4320r^2s^3 + 1620rs^4 - 243s^5.\end{aligned}$$

(vi)
$$\begin{aligned}\left(x + \frac{1}{x}\right)^5 &= x^5 + 5x^4\left(\frac{1}{x}\right) + 10x^3\left(\frac{1}{x}\right)^2 + 10x^2\left(\frac{1}{x}\right)^3 + 5x\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5 \\ &= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}.\end{aligned}$$

(vii)
$$\begin{aligned}\left(2y^2 - \frac{1}{3y}\right)^4 &= (2y^2)^4\left(-\frac{1}{3y}\right) + 6(2y^2)^2\left(-\frac{1}{3y}\right)^2 + 4(2y^2)\left(-\frac{1}{3y}\right)^3 + \left(-\frac{1}{3y}\right)^4 \\ &= 16y^8 - \frac{32y^5}{3} + \frac{8y^2}{3} - \frac{8}{27y} + \frac{1}{81y^4}.\end{aligned}$$

10. (i) General term is

$$\binom{9}{r}(1)^{9-r}(2x)^r = \binom{9}{r}2^r x^r.$$

For term in x^5, set $r = 5$ to get

$$\begin{pmatrix} 9 \\ 5 \end{pmatrix} 2^5 = \frac{9!}{5!4!}2^5 = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}2^5 = 3 \times 7 \times 6 \times 2^5 = 4032.$$

(ii) General term is

$$\begin{pmatrix} 7 \\ r \end{pmatrix} x^{7-r} \left(\frac{3}{x}\right)^r = \begin{pmatrix} 7 \\ r \end{pmatrix} 3^r x^{7-2r}.$$

For term in x^3, set $7 - 2r = 3 \Rightarrow 2r = 4 \Rightarrow r = 2$ to get

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix} 3^2 = \frac{7!}{5!2!}3^2 = \frac{7 \times 6}{2} \times 9 = 189.$$

(iii) General term is

$$\begin{pmatrix} 12 \\ r \end{pmatrix} (3x)^{12-r} \left(-\frac{2}{x^2}\right)^r = \begin{pmatrix} 12 \\ r \end{pmatrix} 3^{12-r}(-2)^r x^{12-3r}.$$

For constant term, set $12 - 3r = 0 \Rightarrow r = 4$ to get

$$\begin{pmatrix} 12 \\ 4 \end{pmatrix} 3^8(-2)^4 = \frac{12!}{4!8!}3^8 \times 2^4 = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1}3^8 \times 2^4 = 11 \times 9 \times 5 \times 3^8 \times 2^4 = 51963120.$$

11. (i) First four terms are

$$1^9 + \begin{pmatrix} 9 \\ 1 \end{pmatrix} 1^8(2x) + \begin{pmatrix} 9 \\ 2 \end{pmatrix} 1^7(2x)^2 + \begin{pmatrix} 9 \\ 3 \end{pmatrix} 1^6(2x)^3 = 1 + 18x + 144x^2 + 672x^3.$$

(ii) First three terms are

$$1^7 + \begin{pmatrix} 7 \\ 1 \end{pmatrix} 1^6 \left(\frac{3}{x^2}\right) + \begin{pmatrix} 7 \\ 2 \end{pmatrix} 1^5 \left(\frac{3}{x^2}\right)^2 = 1 + \frac{21}{x^2} + \frac{189}{x^4}.$$

(iii) First four terms are

$$1^8 + \begin{pmatrix} 8 \\ 1 \end{pmatrix} 1^7 \left(-\frac{x^2}{3}\right) + \begin{pmatrix} 8 \\ 2 \end{pmatrix} 1^6 \left(-\frac{x^2}{3}\right)^2 + \begin{pmatrix} 8 \\ 3 \end{pmatrix} 1^5 \left(-\frac{x^2}{3}\right)^3 = 1 - \frac{8}{3}x^2 + \frac{28}{9}x^4 - \frac{56}{27}x^6.$$

12.

$$\begin{aligned}
\begin{pmatrix} n \\ r-1 \end{pmatrix} + \begin{pmatrix} n \\ r \end{pmatrix} &= \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!} \\
&= \frac{n!}{(n-r)!(r-1))!} \left[\frac{1}{n-r+1} + \frac{1}{r}\right] \\
&= \frac{n!}{(n-r)!(r-1))!} \left[\frac{r+n-r+1}{r(n-r+1)}\right] \\
&= \frac{n!}{(n-r)!(r-1)!} \left[\frac{n+1}{r(n-r+1)}\right] \\
&= \frac{(n+1)!}{r!(n-r+1)!} = \begin{pmatrix} n+1 \\ r \end{pmatrix}.
\end{aligned}$$

13. (a) Binomial expansion is $(x+y)^n = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} x^{n-r} y^r$.

(i) Choose $x = 1$ and $y = 1$. Expansion gives $2^n = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix}$.

(ii) Choose $x = 1$ and $y = -1$. Expansion gives $0 = \sum_{r=0}^{n} (-1)^r \begin{pmatrix} n \\ r \end{pmatrix}$.

(b) <u>Step 1</u>: check the case $n = 1$.

$$\text{LHS} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2, \ \ \text{RHS} = 2^n = 2$$

<u>Step 2</u>: Assume result is time for n, i.e. $\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} = 2^n$.

Now try to prove result for $n+1$, i.e. $\sum_{r=0}^{n+1} \begin{pmatrix} n+1 \\ r \end{pmatrix} = 2^{n+1}$.

$$\begin{aligned}
\text{LHS} &= \sum_{r=0}^{n+1} \begin{pmatrix} n+1 \\ r \end{pmatrix} = \begin{pmatrix} n+1 \\ 0 \end{pmatrix} + \begin{pmatrix} n+1 \\ n+1 \end{pmatrix} + \sum_{r=1}^{n} \begin{pmatrix} n+1 \\ r \end{pmatrix} \\
&= 2 + \sum_{r=1}^{n} \left\{ \begin{pmatrix} n \\ r-1 \end{pmatrix} + \begin{pmatrix} n \\ r \end{pmatrix} \right\} \quad \text{from exercise 12} \\
&= 2 + \sum_{r=1}^{n} \begin{pmatrix} n \\ r-1 \end{pmatrix} + \sum_{r=1}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \\
&= \left\{ \begin{pmatrix} n \\ n \end{pmatrix} + \sum_{s=0}^{n-1} \begin{pmatrix} n \\ s \end{pmatrix} \right\} + \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix} + \sum_{r=1}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \right\} \\
&= \sum_{s=0}^{n} \begin{pmatrix} n \\ s \end{pmatrix} + \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \\
&= 2 \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} = 2 \times 2^n = 2^{n+1}
\end{aligned}$$

so given result holds for $n+1$.

Hence by the principle of mathematical induction the result holds for all $n = 1, 2, 3, ...$

14. (a) General term is

$$\binom{7}{r} (3x)^{7-r} (-2)^r = \binom{7}{r} 3^{7-r} (-2)^r x^{7-r}.$$

We require the coefficient of x^5, so $7 - r = 5 \Rightarrow r = 2$. The coefficient of x^5 is

$$\binom{7}{2} 3^5 (-2)^2 = 21.3^5.4 = 20412.$$

(b) General term is

$$\binom{9}{r} (2x^2)^{9-r} \left(\frac{-1}{x} \right)^r = \binom{9}{r} 2^{9-r} (-1)^r x^{18-3r}.$$

We require the coefficient of x^3, so $18 - 3r = 3$, giving $r = 5$. The coefficient of x^3 is

$$\binom{9}{5} 2^4(-1)^5 = \frac{9!}{5!4!}2^4(-1) = -\frac{9.8.7.6}{4.3.2.1}2^4 = -2016.$$

15.

$$\left(x - \frac{3}{x}\right)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \left(\frac{-3}{x}\right)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{2k-n}(-3)^{n-k}.$$

So term involving x^r arises when $2k - n = r$, that is, $k = (r+n)/2$, giving

$$\binom{n}{(r+n)/2} x^r(-3)^{(n-r)/2}.$$

(Note: x^r term only arises when $r + n$ is even.)

16. General term is

$$\left(\begin{array}{c} 38 \\ r \end{array}\right) (x^2)^{38-r} \left(-\frac{1}{2x}\right)^r = \left(\begin{array}{c} 38 \\ r \end{array}\right) \left(-\frac{1}{2x}\right)^r x^{76-3r}.$$

For term in x^{-17}, we require $76 - 3r = -17 \Rightarrow 3r = 93 \Rightarrow r = 31$ which gives

$$\left(\begin{array}{c} 38 \\ 31 \end{array}\right) \left(-\frac{1}{2}\right)^{31} = -\frac{38 \times 37 \times 36 \times 35 \times 34 \times 33 \times 32}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \frac{1}{2^{31}} = \frac{394383}{67108864}.$$