14 The Fundamental Theorem of Calculus

14.1 Find the derivative $F'(x)$ of $F(x) = \int_a^x \cos t \, \mathrm{d}t$

(a) by evaluating the integral via the Second Fundamental Theorem then differentiating;

(b) directly by applying the First Fundamental Theorem.

14.2 Find the derivatives of the following functions without performing any integrations.

(a) $F(x) = \int_a^x \cos^3 t \, \mathrm{d}t$

(b) $F(t) = \int_a^t \cos^3 x \, \mathrm{d}x$

(c) $F(x) = \int_a^b \cos^2 t \, \mathrm{d}t$

(d) $F(x) = \int_a^b x \cos^3 t \, \mathrm{d}t$

(e) $F(x) = \int_x^b \cos^3 t \, \mathrm{d}t$

(f) $F(x) = \int_2^x \left(\int_y^3 \cos^2 t \, \mathrm{d}t \right) \mathrm{d}y$

(g) $F(x) = \sin \left(\int_a^x \cos^2 t \, \mathrm{d}t \right)$

(h) $F(x) = \sin x \int_a^x \cos^2 t \, \mathrm{d}t$

(i) $F(x) = \int_a^{x^3} \cos^3 t \, \mathrm{d}t$ (Hint: look first at the function G with $F(x) = G(x^3)$.)

(j) $F(x) = \int_7^{\left(\int_6^x \sin^2 t \, \mathrm{d}t\right)} \frac{1}{1 + t^2 + \cos t} \, \mathrm{d}t$

(k) $F(x) = \int_x^{2x} \cos^2 t \, \mathrm{d}t$ (Hint: split up the integral.)

(l) Find the derivative of F^{-1} (expressed in terms of F^{-1}), where $F(x) = \int_1^x \frac{1}{t} \, \mathrm{d}t$

(m) Find the derivative of F^{-1} (expressed in terms of F^{-1}), where $F(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} \, \mathrm{d}t$.

14.3 Prove that if $F(x) = \int_{g(x)}^{h(x)} f(t) \, \mathrm{d}t$ for a continuous function f and differentiable functions g and h, then $F'(x) = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$.

14.4 Evaluate

(a) $\displaystyle\int_0^1 (x - x^2)\,\mathrm{d}x$ (b) $\displaystyle\int_{\pi/4}^{3\pi/4} (\sin x + \cos x)\,\mathrm{d}x$ (c) $\displaystyle\int_3^4 (2x^3 - 3x + 1)\,\mathrm{d}x$

(d) $\displaystyle\int_1^2 (x - 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})\,\mathrm{d}x$ (e) $\displaystyle\int_{\pi/6}^{\pi/3} (3\csc^2 x - \frac{1}{3}\sec^2 x)\,\mathrm{d}x.$

14.5 For each of the following functions, find the area between the x-axis and the graph of the function between the specified limits.

(a) $f(x) = \sin x$ between $x = 0$ and $x = 3\pi/2$;

(b) $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$;

(c) $f(x) = 1 - x^2$ between $x = 0$ and $x = 1$;

(d) $f(x) = 1 - x^2$ between $x = 0$ and $x = 2$;

(e) $f(x) = x - x^2$ between $x = 0$ and $x = 1$.

14.6 Find the area of the *finite regions* bounded by the following curves.

(a) $f(x) = \sqrt{x}$, the x-axis, and the vertical line through the point $(4, 0)$;

(b) $f(x) = (2 - x)(x + 1)$ and the x-axis;

(c) $f(x) = x^2$ and $g(x) = x^3$;

(d) $f(x) = 2 + x - x^2$ and $g(x) = x + 1$;

(e) $f(x) = x^3$, $g(x) = 1$, $h(x) = 8$, and the vertical line through the point $(0, 0)$;

(f) $f(x) = 4 - x^2$ and $g(x) = 1$.

14.7 Evaluate each of the following improper integrals, or show that it diverges.

(a) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^3}$ (b) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{1 + x^2}$ (c) $\displaystyle\int_{-\infty}^0 \frac{\mathrm{d}x}{4 + x^2}$ (d) $\displaystyle\int_{-\infty}^0 \cos x\,\mathrm{d}x$

(e) $\displaystyle\int_3^5 \frac{x}{\sqrt{x^2 - 9}}\,\mathrm{d}x$ (f) $\displaystyle\int_1^2 \frac{\mathrm{d}x}{(x - 2)^2}.$