7 Sequences and series

7.1 Sequences

A **sequence** involves a set of objects arranged in a specific **order**, that is, arranged in such a way that we can identify a first object, a second object, a third object and so on. In this course we shall be concerned with sequences of real numbers. Some examples are

$$1,\ 2,\ 3$$
$$-1,\ -3,\ -5,\ -7,\ -9,\ -11,\ -13,\ \ldots$$
$$9,\ -11,\ \frac{2}{7},\ 22.7,\ 0$$

Sequences can be **finite** or **infinite** (in this case we use the $\ldots$ notation to indicate that the sequence goes on for ever). Also, there need not be an obvious rule relating the terms in a sequence.

An individual term in a sequence of real numbers is often denoted by a variable with a **subscript** to represent its position in the sequence. For example, we write u_n to represent the n^{th} number in the sequence, giving

$$u_1,\ u_2,\ u_3,\ u_4,\ \ldots,\ u_N$$

for a sequence with N terms, or

$$u_1,\ u_2,\ u_3,\ u_4,\ \ldots$$

for an infinite sequence. This notation is often shortened to $\{u_n\}_{n=1}^{N}$ for a finite sequence with N terms. For an infinite sequence, we may write $\{u_n\}_{n=1}^{\infty}$ or simply $\{u_n\}$.

Examples 7A

7.2 Series

If the terms in a sequence are added together, the result is known as a **series**. We usually represent the resulting sums using **sigma notation** which involves the Greek letter Σ. For example, the sums of the sequences above would be written as

$$u_1 + u_2 + u_3 + \ldots + u_N = \sum_{j=1}^{N} u_j$$

for the finite sequence $\{u_j\}_{j=1}^{N}$ and

$$u_1 + u_2 + u_3 + u_4 + \ldots = \sum_{j=1}^{\infty} u_j$$

for the infinite sequence. The values of the counter (j here) below and above the sigma give the first and last values of the subscripts of entries included in the sum, respectively. So we could write

$$u_4 + u_5 + u_6 = \sum_{j=4}^{6} u_j.$$

Furthermore, if the terms in a series are related by some definable pattern, that is, if we can write $u_j = f(j)$ for some function f, then we can also write the sum as

$$\sum_{j=1}^{N} f(j) \qquad \text{or} \qquad \sum_{j=1}^{\infty} f(j).$$

For example, the sum of the first n integers can be written as

$$1 + 2 + 3 + \ldots + n = \sum_{j=1}^{n} j$$

and the sum of their squares as

$$1 + 4 + 9 + \ldots + n^2 = \sum_{j=1}^{n} j^2.$$

Note that the letter j has nothing to do with the number denoted by $\sum_{j=1}^{n} j$, and can be replaced by any convenient symbol. That is, we could equally well write

$$1+2+3+\ldots+n = \sum_{i=1}^{n} i \quad \text{or} \quad 1+2+3+\ldots+n = \sum_{\theta=1}^{n} \theta \quad \text{or} \quad 1+2+3+\ldots+n = \sum_{j=2}^{n+1} j-1.$$

Other modifications of this notation are usually quite obvious. For example, we may write

$$\sum_{\substack{j=1 \\ j\neq 4}}^{n} u_j = u_1 + u_2 + u_3 + u_5 + u_6 + \ldots + u_n.$$

We usually denote the sum to n terms of a sequence by S_n.

Examples 7B

7.3 Arithmetic sequences and series

When the difference between any two successive terms in a sequence is always the same, the sequence is called an **arithmetic sequence** or **arithmetic progression**. For example, the sequence $\{1, 3, 5, 7, 9, 11, \ldots\}$ is an arithmetic sequence consisting of all the positive odd integers. Any general (infinite) arithmetic sequence can be written as

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

where a is the **first term** and d is the **common difference**. That is, the terms are given by the formula

$$u_j = a + (j-1)d \tag{7.1}$$

for $j = 1, 2, 3, \ldots,$ where a and d are fixed real numbers. For the sequence of odd positive integers above, the first term is $u_1 = 1$ $(= a)$ and the common difference is $d = 2$, so the j^{th} term is given by

$$u_j = a + (j-1)d = 1 + (j-1)2 = 2j - 1, \quad j = 1, 2, 3, \ldots$$

Similarly, for the arithmetic sequence $\{2, 4, 6, 8, \ldots\}$ consisting of all positive even integers, the j^{th} term is given by the formula

$$u_j = 2j, \quad j \in \mathbb{N}.$$

Theorem 7.1 *The sum to n terms for an arithmetic series with first term a and common difference d is given by*

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right].$$

Proof 7.1 *We may write*

$$\begin{aligned} S_n &= u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n \\ &= a + [a+d] + [a+2d] + \ldots + [a+(n-2)d] + [a+(n-1)d] \end{aligned} \tag{7.2}$$

(using the definition in (7.1)). Now reverse the order of the terms in (7.2) to get

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \ldots + [a+2d] + [a+d] + a. \tag{7.3}$$

Adding (7.2) and (7.3) together gives

$$\begin{aligned} 2S_n &= [2a+(n-1)d] + [2a+(n-1)d] + \ldots + [2a+(n-1)d] + [2a+(n-1)d] \\ &= n[2a+(n-1)d] \end{aligned}$$

and so $S_n = \frac{n}{2}[2a+(n-1)d]$ as required.

Examples 7C

7.4 Geometric sequences and series

A sequence in which each term is a constant multiple of the preceding term is called a **geometric sequence** or **geometric progression**. Examples of geometric progressions are $1, 2, 4, 8, \ldots$ and $3, -1, \frac{1}{3}, -\frac{1}{9} \ldots$ Any general (infinite) geometric progression can be written as

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

where a is the **first term** and r is the **common ratio**. That is, the terms are given by the formula

$$u_j = ar^{j-1} \tag{7.4}$$

for $j = 1, 2, 3, \ldots,$ where a and r are fixed real numbers. For the sample geometric sequences above, the first terms and common ratios are $a = 1$, $r = 2$ and $a = 3$, $r = -\frac{1}{3}$, respectively.

Theorem 7.2 *The sum to n terms for a geometric series with first term a and common ratio $r \neq 1$ is given by*

$$S_n = a\frac{1-r^n}{1-r}.$$

Proof 7.2 *We may write*

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \tag{7.5}$$

(using the definition in (7.4)). Now multiply both sides of the equation in (7.5) by the common ratio r to get

$$rS_n = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n. \tag{7.6}$$

Subtracting (7.5) from (7.6) gives

$$(r-1)S_n = ar^n - a \qquad \Leftrightarrow \qquad S_n = a\frac{1-r^n}{1-r}$$

as required.

Note: For $r = 1$, the sequence is simply an arithmetic progression with common difference $d = 0$.

Examples 7D

7.5 Other finite sequences

It is sometimes possible to find expressions for the sum to n terms of other types of sequence. Consider the following example:

Theorem 7.3 *The sum of the squares of the first n positive integers is given by*

$$\sum_{j=1}^{n} j^2 = \frac{n}{6}(n+1)(2n+1).$$

We will prove this in two different ways.

Proof 7.3 ***Method 1***
Our starting point is the identity

$$\sum_{j=1}^{n} \left[(j+1)^3 - j^3\right] = \sum_{j=1}^{n} [3j^2 + 3j + 1].$$

Expanding the left-hand side gives

$$[2^3 - 1^3] + [3^3 - 2^3] + [4^3 - 3^3] + \ldots + [n^3 - (n-1)^3] + [(n+1)^3 - n^3]$$

which, on cancelling like terms, becomes $(n+1)^3 - 1^3$. Hence we have

$$(n+1)^3 - 1 = \sum_{j=1}^{n} (3j^2 + 3j + 1).$$

The sum on the right-hand side can now be split into three separate sums to give

$$(n+1)^3 - 1 = 3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1.$$

Using the results $\displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{j=1}^{n} 1 = n$, we find

$$n^3 + 3n^2 + 3n + 1 - 1 = 3\sum_{j=1}^{n} j^2 + 3\frac{n(n+1)}{2} + n.$$

Rearranging gives

$$\begin{aligned}
3\sum_{j=1}^{n} j^2 &= n^3 + 3n^2 + 3n - \frac{3n^2}{2} - \frac{3n}{2} - n \\
&= n^3 + \frac{3}{2}n^2 + \frac{n}{2} \\
&= \frac{n}{2}(2n^2 + 3n + 1) \\
&= \frac{n}{2}(2n+1)(n+1)
\end{aligned}$$

so our final result is

$$\sum_{j=1}^{n} j^2 = \frac{n}{6}(n+1)(2n+1)$$

as required.

The same result can also be proved using **proof by induction**.

Proof 7.3 ***Method 2***
Step 1: *Check that the given result is true for* $n = 1$*.*

$$LHS = 1, \quad RHS = \frac{1 \times 2 \times 3}{6} = 1$$

Step 2: *Assume that the given result is true for* n *(i.e. for the sum of the first* n *squares). That is, assume*

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}. \tag{7.7}$$

Now try to prove the result for $n+1$*, that is, show that the sum of the first* $n+1$ *squares satisfies*

$$\sum_{j=1}^{n+1} j^2 = \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6} = \frac{(n+1)(n+2)(2n+3)}{6}. \tag{7.8}$$

We have

$$\begin{aligned}
\sum_{j=1}^{n+1} j^2 &= \left(\sum_{j=1}^{n} j^2\right) + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{(by assumption(7.7))} \\
&= \frac{(n+1)(2n^2+n+6n+6)}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

which shows that the theorem is true for $n+1$ *(see (7.8)). Hence, by induction, the theorem is true for all values of* n*.*

Note also the following theorem:

Theorem 7.4 *The sum of the cubes of the first n positive integers is given by*

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}.$$

Proof 7.4 *See exercises.*

More general sequences and series will be studied further later in your course. One other case we will illustrate here is the case of **telescoping series**, so called because of the cancellation of most of the terms (collapsing the length of the series like a telescope, as we saw in Proof 7.3 Method 1).

As an example, consider summing the sequence

$$\frac{1}{2}, \quad \frac{1}{6}, \quad \frac{1}{12}, \quad \frac{1}{20}, \quad \frac{1}{30}, \quad \cdots \frac{1}{n^2+n}.$$

We first note that this can be written as

$$\frac{1}{1\times 2}, \quad \frac{1}{2\times 3}, \quad \frac{1}{3\times 4}, \quad \frac{1}{4\times 5}, \quad \frac{1}{5\times 6}, \quad \cdots \frac{1}{n(n+1)}$$

so that

$$u_j = \frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}.$$

(You will learn how to split fractions into **partial fractions** in this way in a later course: for now, just check that the above result is true).

Then

$$\begin{aligned} S_n &= \sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1}. \end{aligned}$$

Examples 7E

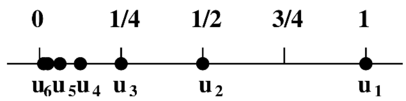
7.6 Infinite sequences

One important point about an infinite sequence is that there is a real number u_n corresponding to every natural number n. This means that an infinite sequence of real numbers can be thought of as a function whose domain is $\mathbb{N}$. In function notation, we would normally write $u(1), u(2), u(3), u(4), \ldots$ instead of using subscripts, but the subscript notation is much more common in practice.

Although we could plot the graph of an infinite sequence like any other function, it is more usual to plot the sequence of points on the real number line. As an illustration consider the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

which can be represented on a number line as follows:

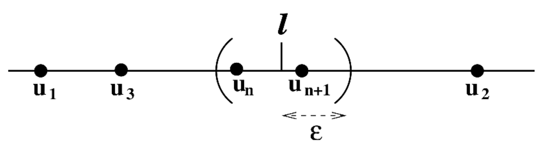


This sort of picture is helpful for identifying the long-term behaviour of a sequence. For this example, we say that the sequence **converges** to the **limit** 0, that is, as n increases, the terms in the sequence are getting closer and closer to 0. A more formal definition of this property is as follows:

Definition 7.5 *A sequence $\{u_n\}$ converges to a* ***limit*** *l if for every $\epsilon > 0$ there is a natural number N such that, for all natural numbers n, if $n \geq N$ then $|u_n - l| < \epsilon$. We write*

$$\lim_{n \to \infty} u_n = l.$$

Here ϵ is a positive number that can be chosen as small as we like. For any such ϵ, this definition is simply a mathematical way of saying that the distance between u_n and l (i.e. $|u_n - l|$) will be less than ϵ for all terms u_n in the sequence with n large enough (i.e. for u_n with $n \geq N$). That is, the integer N tells us how far down the sequence we must go before u_n is within a distance ϵ of l.



Note that $|u_n - l| < \epsilon \Leftrightarrow u_n \in (l - \epsilon, l + \epsilon)$.

If a sequence does not converge, it is said to **diverge**. Typical examples of divergent sequences are

$$1, 2, 3, 4, 5, \ldots$$

where the terms go off to infinity, or

$$1, -1, 1, -1, 1, -1, \ldots$$

where the terms jump back and forth between fixed values.

Examples 7F

Some standard rules for limits of sequences are listed in the following theorem.

Theorem 7.6 *Suppose $\{u_n\}$ and $\{v_n\}$ are convergent sequences. Then*

(i) $\lim_{n\to\infty}(u_n + v_n) = \lim_{n\to\infty} u_n + \lim_{n\to\infty} v_n$;

(ii) $\lim_{n\to\infty}(\alpha u_n) = \alpha \lim_{n\to\infty} u_n \qquad (\alpha \in \mathbb{R})$;

(iii) $\lim_{n\to\infty}(u_n v_n) = \left(\lim_{n\to\infty} u_n\right)\left(\lim_{n\to\infty} v_n\right)$;

(iv) $\lim_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{\lim_{n\to\infty} u_n}{\lim_{n\to\infty} v_n}$, *so long as the limit of $\{v_n\}$ is nonzero.*

Proof 7.6 *We prove only result (i) here.*
Suppose that

$$\lim_{n\to\infty} u_n = u \qquad \text{and} \qquad \lim_{n\to\infty} v_n = v.$$

Then, by Definition 7.5, we can choose natural numbers N_u and N_v such that

$$|u_n - u| < \frac{\epsilon}{2} \quad \text{for } n > N_u \text{ and} \qquad |v_n - v| < \frac{\epsilon}{2} \quad \text{for } n > N_v.$$

Now consider the sequence $\{w_n\}$ with $w_n = u_n + v_n$. If $N = \max(N_u, N_v)$, we have

$$|w_n - (u+v)| = |(u_n - u) + (v_n - v)| \leq |u_n - u| + |v_n - v| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $n > N$, that is, $\lim_{n\to\infty} w_n = u + v$ as required.

Note: the result

$$\lim_{n\to\infty}(u_n - v_n) = \lim_{n\to\infty} u_n - \lim_{n\to\infty} v_n$$

is an easily-proved corollary to this theorem.

The limits of many infinite sequences can be evaluated using the properties in Theorem 7.6. If the sequences do not match the conditions of the theorem then it may be possible to manipulate them so that they do. For example if $\{u_n\}$ and $\{v_n\}$ are both divergent then their ratio may still converge. If this limit is not immediately obvious we can find this limit by finding a third divergent sequence $\{w_n\}$ so that the terms u_n/w_n and v_n/w_n have an easily calculable limit. Care has to be taken so that $\{v_n/w_n\}$ does not converge to zero. A good choice for w_n is often n^p where p is the highest power in the denominator.

The limits when one sequence converges and the other diverges are not always obvious. But if $\{u_n\}$ converges and $\{v_n\}$ diverges then we can be sure that their sum diverges. Additional theory on combining infinite sequences will be provided in later courses.

Examples 7G

7.7 Infinite series

We can always obtain the sum of a finite series (simply add up the terms, even though it may take a long time!). Furthermore, for a finite sum, the associative law of addition (P1) tells us that we can add up the terms in any order. However, when we come to try to sum infinite sequences, it is not so clear whether or not (P1) is still applicable. It turns out that the order of terms can be rather important. Here we use the convention of summing the terms in an infinite sequence from left to right. For a specific sequence $\{u_j\}$, we do this by defining a new sequence of **partial sums**

$$S_1,\ S_2,\ S_3, \ldots$$

where S_n represents the sum to n terms of the sequence. For example, the sequence $\{1,\ 2,\ 3,\ 4,\ 5,\ \ldots\}$ has partial sums

$$\begin{array}{lllllllllll}
S_1 & = & \sum_{j=1}^{1} u_j & = & u_1 & = & 1 & = & 1 & = & S_1 \\
S_2 & = & \sum_{j=1}^{2} u_j & = & u_1 + u_2 & = & 1+2 & = & 3 & = & S_1 + u_2 \\
S_3 & = & \sum_{j=1}^{3} u_j & = & u_1 + u_2 + u_3 & = & 1+2+3 & = & 6 & = & S_2 + u_3 \\
S_4 & = & \sum_{j=1}^{4} u_j & = & u_1 + u_2 + u_3 + u_4 & = & 1+2+3+4 & = & 10 & = & S_3 + u_4 \\
S_5 & = & \sum_{j=1}^{5} u_j & = & u_1 + u_2 + u_3 + u_4 + u_5 & = & 1+2+3+4+5 & = & 15 & = & S_4 + u_5
\end{array}$$

etc. Note that in general

$$S_{n+1} = S_n + u_{n+1}.$$

The sum to infinity of a series

$$S_\infty = \sum_{j=1}^{\infty} u_j$$

is now defined in terms of the limit of the **sequence of partial sums**

$$S_1, S_2, S_3, \ldots$$

where S_n represents the sum to n terms of the associated infinite sequence. If this sequence of partial sums converges, the infinite series has a finite sum and is said be **summable** or **convergent**. Note that to study convergence of an infinite series we consider the sequence $\{S_n\}$, NOT the sequence $\{u_n\}$ itself!

One simple necessary condition for a series to be summable is the so-called **vanishing condition**.

$$\text{If } \{u_n\} \text{ is summable, then } \lim_{n\to\infty} u_n = 0.$$

This follows because if $\lim_{n\to\infty} S_n = s$, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = s - s = 0.$$

You will learn more on the convergence of infinite series later in your course. Here we look at only one special case, the sum to infinity of a geometric series.

Consider the geometric sequence $\{u_n\}$ with first term a and common ratio $r \neq 1$ so that $u_n = ar^{n-1}$, $n = 1, 2, 3, \ldots$ (from §7.4). The vanishing condition tells us that if $\lim_{n\to\infty} u_n \neq 0$, then $\{u_n\}$ is not summable. In other words, the only possiblity of a geometric series being summable is when $|r| < 1$. In this case, we can consider the sequence $\{S_n\}$ of partial sums given by

$$S_n = a\left(\frac{1-r^n}{1-r}\right) \tag{7.9}$$

(from Theorem 7.2). We have

$$\lim_{n\to\infty} S_n = a\frac{1 - \lim_{n\to\infty} r^n}{1-r} = \frac{a}{1-r},$$

so, for $|r| < 1$, we may write

$$S_\infty = \frac{a}{1-r}$$

to represent the sum of a geometric series with an infinite number of terms.

Note that in the case $|r| > 1$, S_n becomes unbounded as $n \to \infty$, and when $r = -1$, S_n does not settle down to a fixed value. So an infinite geometric series is convergent if and only if $|r| < 1$.

Examples 7H

8 The binomial theorem

8.1 The binomial expansion

We will now consider an important example of writing certain algebraic expressions as series. First, we note that we may write a general polynomial of degree n in x, $P_n(x)$, in series form using Sigma notation as

$$P_n(x) = \sum_{j=0}^{n} a_j\, x^j$$

where a_j, $j = 0, 1, 2, \ldots$ are the coefficients. In this section we will generalise this idea to obtain a series expansion of algebraic expressions of the form $(x + y)^n$.

We begin with n as a positive integer, and consider the first few expansions of $(x + y)^n$.

$$\begin{aligned}
(x+y)^2 &= (x+y)(x+y) = x^2 + 2xy + y^2 \\
(x+y)^3 &= (x+y)(x+y)^2 = x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= (x+y)(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x+y)^5 &= (x+y)(x+y)^4 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{aligned}$$

etc. The coefficients in these expressions follow a particular pattern, which we can see if we write them in a triangular array:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

This array is known as **Pascal's Triangle** and its entries are known as **binomial coefficients**. The outer coefficients are all equal to 1, and the inner coefficients are found by adding the two nearest numbers in the line above. Note that the array is also symmetric.

Using Pascal's Triangle, we can predict that the coefficients in the expansion of $(x + y)^6$ will be

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1,$$

and (by comparing with the other expressions) that the powers of x and y will be

$$x^6 \quad x^5y \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad xy^5 \quad y^6,$$

giving

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

which is correct (check it!).

Note that other expressions can be substituted for x and y in these expansions in an obvious way. For example,

$$(2t+t^2)^3 = (2t)^3 + 3(2t)^2t^2 + 3(2t)(t^2)^2 + (t^2)^3 = 8t^3 + 12t^4 + 6t^5 + t^6.$$

Examples 8A

Although this method is useful for small values of n, it becomes cumbersome as n grows. We will now look at a more convenient way of calculating the binomial coefficients in Pascal's Triangle (without having to write the whole thing out). To do this, we need some more notation.

8.1.1 Factorials

If n is a positive integer, we define $n!$ ("n factorial") to mean the product of the positive integers up to and including n. That is,

$$n! \quad = \quad 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n.$$

This can also be written more compactly using **Pi notation** as

$$n! = \prod_{j=1}^{n} j,$$

where Π plays a similar role to Σ in Sigma notation but represents a **product** of the terms as opposed to a sum.

Note: We define $0! = 1$. This means that we can state the general relation

$$(n+1)! \quad = \quad n! \times (n+1), \qquad n \in \mathbb{N}.$$

Examples 8B

8.1.2 Binomial coefficients and the Binomial Theorem

The binomial coefficients in Pascal's Triangle (except the 1's at the edges) are calculated as the sum of two coefficients from the previous row. However, each coefficient can also be calculated independently by looking at the construction of each term in the expansion.

Suppose $n \geq 0$ is an integer and $r \geq 0$ is an integer with $r \leq n$. Now consider the expansion of $(x+y)^n$ given by

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\dots(x+y)(x+y)}_{n \text{ times}}. \tag{8.1}$$

Clearly, choosing the x from each bracket gives a term of the form x^n (with coefficient 1). Similarly, choosing the y from each bracket gives y^n. For every other term, suppose we choose x from r of the brackets, and y from the remaining $n-r$ brackets. This will give us a term of the form $x^r y^{n-r}$, again with coefficient 1. However, we can do this choosing in several different ways, each of which will lead to a term of this form. The final coefficient of $x^r y^{n-r}$ in the full expansion will therefore be the total number of these terms, in other words, the **total number of ways in which we can choose r brackets out of n**.

To find the value of this number, we use the following argument. Recall that we have to choose an x from r of the brackets. We have n choices for the first bracket, $n-1$ for the second, $n-2$ for the third etc down to $(n-r+1)$ for the r^{th} bracket. However, as the order of the brackets in (8.1) doesn't matter, we must remove 'duplicates' in this sense. This means that we must discount $r!$ of these selections (there are $r!$ ways of ordering the chosen x-term brackets). In other words, the coefficient we seek is given by

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}.$$

In mathematics, the notation commonly used for this coefficient is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

The quantity $\binom{n}{r}$ is usually read as "n choose r" and is called a **binomial coefficient**. Sometimes the notation ${}^nC_r$ is used instead.

We now check that these expressions are consistent with Pascal's Triangle. First, note that

$$\binom{n}{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

This accounts for the symmetry in Pascal's Triangle (i.e. the coefficient of $x^r y^{n-r}$ is the same as the coefficient of $x^{n-r}y^r$). Also, it can be shown that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r} \tag{8.2}$$

(see exercises). This is the rule by which the coefficients in row $n+1$ of Pascal's Triangle are formed from those in row n.

Examples 8C

This definition of the binomial coefficients enables us to state the following important theorem.

Theorem 8.1 *For any positive integer n, we have*

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{r} x^{n-r}y^r + \ldots + \binom{n}{n} y^n.$$

In other words

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^r.$$

Proof 8.1 *We will give a proof by induction.*

Step 1: *Check the case $n = 1$.*

$$LHS = x + y, \quad RHS = \binom{1}{0} x^1 + \binom{1}{1} y^1 = x + y.$$

Step 2: *Assume that the given result is true for n, that is, assume*

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^r. \tag{8.3}$$

Now try to prove the result for $n+1$, that is, try to show that

$$(x+y)^{n+1} = \sum_{r=0}^{n+1} \binom{n+1}{r} x^{(n+1)-r}y^r.$$

We have

$$
\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\left[\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r\right] \qquad \textit{using assumption (8.3)} \\
&= x\left[\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r\right] + y\left[\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r\right] \\
&= x\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n\right] \\
&+ y\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n\right] \\
&= \binom{n}{0}x^{n+1} + \left[\binom{n}{1}+\binom{n}{0}\right]x^n y + \left[\binom{n}{2}+\binom{n}{1}\right]x^{n-1}y^2 + \ldots \\
&+ \left[\binom{n}{r}+\binom{n}{r-1}\right]x^{n+1-r}y^r + \ldots + \binom{n}{n}y^{n+1} \\
&= \binom{n+1}{0}x^{n+1} + \binom{n+1}{1}x^n y + \binom{n+1}{2}x^{n-1}y^2 + \ldots \\
&+ \binom{n+1}{r}x^{(n+1)-r}y^r + \ldots + \binom{n+1}{n}xy^n + \binom{n+1}{n+1}y^{n+1} \\
& \textit{(using result (8.2))} \\
\Rightarrow (x+y)^{n+1} &= \sum_{r=0}^{n+1}\binom{n+1}{r}x^{(n+1)-r}y^r
\end{aligned}
$$

so the given result holds for $n+1$. *Hence, by induction, the binomial expansion*

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

holds for all values $n = 1, 2, \ldots$.

Examples 8D