Exercises and outline solutions for MM101 tutorial in week 9

1. Find $f'(x)$ for the following functions.

   (a) $f(x) = \ln x - \ln\sqrt{1+x^2} - \dfrac{1}{x}\arctan x$.

   (b) $f(x) = \ln(\tan x)$.

   (c) $f(x) = \ln\left(\dfrac{x+4}{(2x-7)^3}\right)$.

   (d) $f(x) = \exp(x\sin x)$.

   (e) $f(x) = \exp(\exp(x\sin x))$.

   (a) $f'(x) = \dfrac{1}{x} - \dfrac{x}{1+x^2} + \dfrac{\arctan x}{x^2} - \dfrac{1}{x(1+x^2)} = \dfrac{\arctan x}{x^2}$.

   (b) $f'(x) = \cot x \cdot (1+\tan^2 x) = \cot x + \tan x$.

   (c) $f(x) = \ln|x+4| - 3\ln|2x-7|$, and so $f'(x) = \dfrac{1}{x+4} - \dfrac{6}{2x-7}$.

   (d) $f'(x) = \exp(x\sin x)\cdot(\sin x + x\cos x)$.

   (e) $f'(x) = \exp(\exp(x\sin x))\cdot\exp(x\sin x)\cdot(\sin x + x\cos x)$.

2. Find the derivatives of the following functions.

   (a) $f(x) = 10^x$; (b) $f(x) = 10^{e^x}$; (c) $f(x) = e^{10^x}$;
   (d) $f(x) = \log_{(e^x)}(x)$; (e) $f(x) = \log_{(e^x)}(e^x)$; (f) $f(x) = \log_a(x^2)$;
   (g) $f(x) = \log_x(x^2)$.

   (a) $f(x) = e^{x\ln 10}$, so $f'(x) = \ln(10)\,10^x$.

   (b) $f'(x) = \ln(10)\,e^x\,10^{e^x}$.

   (c) $f'(x) = \ln(10)\,10^x\,e^{10^x}$.

   (d) $f'(x) = \dfrac{\frac{1}{x}x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$.

   (e) $f(x) = 1$, so $f'(x) = 0$.

   (f) $f(x) = \dfrac{2\ln x}{\ln a}$, so $f'(x) = \frac{2}{x\ln a}$.

   (g) $f(x) = 2\log_x x = 2$, so $f'(x) = 0$.

3. Find the derivatives of the functions f and g defined by

$$f(x) = \sinh(2\sinh^{-1}x), \qquad g(x) = \sinh^{-1}(2\sinh x).$$

$$f'(x) = \sinh'(2\sinh^{-1}x)\frac{\mathrm{d}}{\mathrm{d}x}\left(2\sinh^{-1}x\right) = 2\cosh(2\sinh^{-1}x)\frac{1}{\sqrt{x^2+1}}.$$

$$g'(x) = (\sinh^{-1})'(2\sinh x)\frac{\mathrm{d}}{\mathrm{d}x}\left(2\sinh x\right) = \frac{1}{\sqrt{(2\sinh x)^2+1}}2\cosh(x) = \frac{2\cosh(x)}{\sqrt{4\sinh^2 x+1}}.$$

4. Consider the function defined by $f(x) = \ln(\ln(\sin x))$ on its natural domain.

(a) Calculate the derivative $f'(x)$.

(b) Determine the domains of the functions f and f'. What do you notice?

(a) Applying the chain rule 'formally' we have

$$f'(x) = \frac{1}{\ln(\sin x)} \frac{\mathrm{d}}{\mathrm{d}x} \ln(\sin x) = \frac{\cos x}{\ln(\sin x) \sin x} = \frac{\cot x}{\ln(\sin x)}.$$

(b) Since $\sin(x) \leq 1$ for all $x \in \mathbb{R}$, $\ln(\sin x) \leq 0$ for all $x \in \mathbb{R}$, and thus $\ln(\ln(\sin x))$ is undefined for all $x \in \mathbb{R}$. Thus the natural domain of f is $\emptyset$. Since f' can only be defined at points where f is defined, we conclude that the domain of f' is also $\emptyset$. This is the case even though the *natural* domain of the function $g(x) = \dfrac{\cot x}{\ln(\sin x)}$ is not $\emptyset$; in fact it is $\{x \in \mathbb{R} \mid \sin x > 0\}$. There is in fact a sensible way to think about $f(x)$, but it involves complex numbers...