Properties of Matrix Multiplication

Let A be an $m \times n$ matrix with entries a_{ij} and let B be an $n \times p$ matrix with entries b_{ij}. The product AB is defined to be the $m \times p$ matrix whose $(i,j)^{th}$-entry equals

$$\sum_{k=1}^{n} a_{ik} b_{kj}.$$

Matrix multiplication has the following properties.

1. Associativity: $A(BC) = (AB)C$

2. Homogeneity: $A(cB) = cAB$ and $c(bA) = cbA$ for all $b, c \in \mathbb{R}$

3. Distributivity: $A(B + C) = AB + BC$ and $(B + C)A = BA + CA$

We require two distributive rules because matrix multiplication is **not** commutative: $AB \neq BA$.

Proof. (Associativity) Let A and B be given as above and let C be of size $p \times q$ with entries c_{ij}. The $(i,j)^{th}$-entry of $A(BC)$ equals

$$\begin{aligned}
\sum_{k=1}^{n} a_{ik}((k,j)^{th}\text{-entry of } BC) &= \sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{p} b_{kl} c_{lj} \right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik}(b_{kl} c_{lj}) \\
&= \sum_{l=1}^{p} \sum_{k=1}^{n} (a_{ik} b_{kl}) c_{lj} \\
&= \sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj} \\
&= \sum_{l=1}^{p} ((i,l)^{th}\text{-entry of } AB) c_{lj},
\end{aligned}$$

where the last summation equals the $(i,j)^{th}$-entry of $(AB)C$. □

The above proof relies on the fact that two summation signs of the form $\sum_{s=1}^{m}$ and $\sum_{t=1}^{n}$, where m, n are finite constants, can be interchanged:

$$\sum_{s=1}^{m} \sum_{t=1}^{n} \alpha_{st} = \sum_{t=1}^{n} \sum_{s=1}^{m} \alpha_{st}.$$

Proof. (Homogeneity) Let $b, c \in \mathbb{R}$ and let $B = bI$ and $C = cI$. Then

$$A(cB) = A(cIB) = (AcI)B = (cIA)B = cAB$$

and

$$c(bA) = cI(bIA) = (cIbI)A = (cbII)A = cbIA = cbA.$$

$\square$

Proof. (Distributivity) See page 22 of Chapter 1 of the lecture notes.