

Anatomy of an Affine Transformation

Consider the following affine transformation:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(\mathbf{x}) = \begin{bmatrix} -1 & 4 \\ -12 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

We can understand the effect of this transformation on points in the plane by observing that it maps the unit triangle to the triangle with vertices at $P(3, 5)$, $Q(2, -7)$ and $R(7, 1)$. Let us consider the action of the matrix first.

The column vectors of the matrix are $\begin{bmatrix} -1 \\ -12 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$, and these are the (untranslated) images of the unit vectors \mathbf{e}_1 and \mathbf{e}_2 , respectively:

$$\begin{bmatrix} -1 & 4 \\ -12 & -4 \end{bmatrix} \mathbf{e}_1 = \begin{bmatrix} -1 & 4 \\ -12 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -12 \end{bmatrix} = \overrightarrow{PQ}$$

$$\begin{bmatrix} -1 & 4 \\ -12 & -4 \end{bmatrix} \mathbf{e}_2 = \begin{bmatrix} -1 & 4 \\ -12 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \overrightarrow{PR}.$$

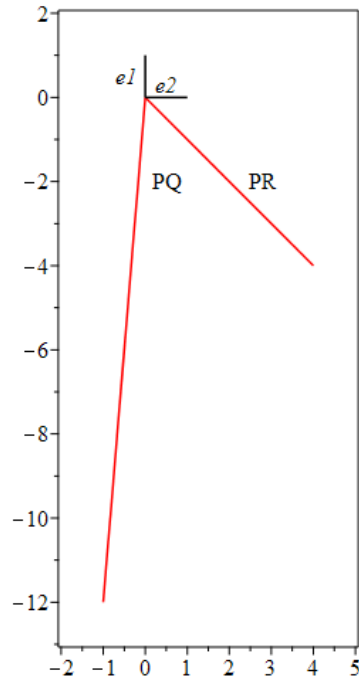


Figure 1: Rescaling the axes.

The effect of the vector $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is to move the origin to the point $P(3, 5)$. All other points are moved along the vector \overrightarrow{OP} .

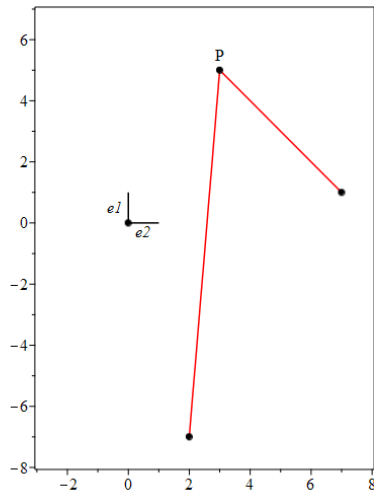


Figure 2: Moving the origin and rescaling the axes.

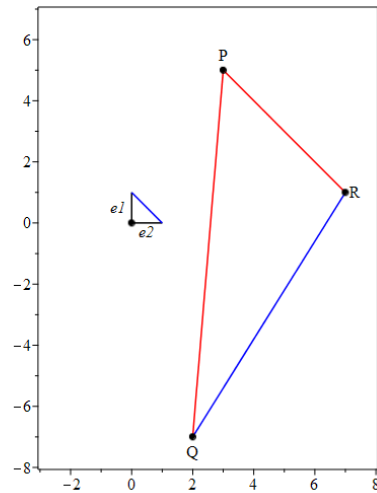


Figure 3: The image of the unit triangle.

The main features of the affine transformation are shown in figure 2. Finally, we can see the image of the unit triangle (figure 3) by completing the third edge of each triangle.