UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MATHEMATICS & STATISTICS

MM103 Geometry and Algebra

Chapter 3: Affine Transformations

Q1. Let $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$ and $g(\mathbf{x}) = B\mathbf{x} + \mathbf{q}$. Then

$$(f+g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}) = A\mathbf{x} + \mathbf{p} + B\mathbf{x} + \mathbf{q} = (A+B)\mathbf{x} + (\mathbf{p}+\mathbf{q}),$$

which proves that $f+g$ is an affine transformation.

Q2. If $O(0,0)$, $X(1,0)$ and $Y(0,1)$ are the vertices of the unit triangle, then, in each case, we have to construct the affine transformation that maps $O \to A$, $X \to B$ and $Y \to C$. The map will be of the form

$$f(\mathbf{x}) = [\overrightarrow{AB}\ \overrightarrow{AC}]\mathbf{x} + [\overrightarrow{OA}].$$

(a) $f(\mathbf{x}) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(b) $f(\mathbf{x}) = \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(c) $f(\mathbf{x}) = \begin{bmatrix} -3 & -2 \\ -1 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(d) $f(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}$

Q3. An affine transformation that maps the unit triangle onto itself corresponds to a permutation of the vertices $O(0,0)$, $X(1,0)$, $Y(0,1)$ of the unit triangle. For example,

$$O \to X,\ X \to Y,\ Y \to O.$$

There are six permutations of the set $\{O, X, Y\}$, so there are six affine transformations. They are each of the form

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$$

where:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

4. $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Q4. First construct affine transformations f and g that map the unit triangle onto the triangles ABC and PQR, respectively. The affine transformation that maps ABC onto PQR is then given by $g \circ f^{-1}$.

(a) $f(\mathbf{x}) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and, since PQR is the unit triangle, $g(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$. Therefore,

$$(g \circ f^{-1})(\mathbf{x}) = f^{-1}(\mathbf{x}) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}^{-1} \left(\mathbf{x} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(b) $f(\mathbf{x}) = \begin{bmatrix} 2 & -4 \\ -4 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $g(\mathbf{x}) = \begin{bmatrix} 2 & -13 \\ -4 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. Then

$$f^{-1}(\mathbf{x}) = -\frac{1}{18} \begin{bmatrix} -1 & 4 \\ 4 & 2 \end{bmatrix} \left(\mathbf{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

and so

$$\begin{aligned} (g \circ f^{-1})(\mathbf{x}) &= \begin{bmatrix} 2 & -13 \\ -4 & -1 \end{bmatrix} \left(-\frac{1}{18} \begin{bmatrix} -1 & 4 \\ 4 & 2 \end{bmatrix} \left(\mathbf{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \right) + \begin{bmatrix} 7 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \end{aligned}$$

(c) $(g \circ f^{-1})(\mathbf{x}) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(d) $(g \circ f^{-1})(\mathbf{x}) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Q5.

(a) $ABCD$ can be mapped onto $PQRS$ because both are parallelograms. The map

$$f(\mathbf{x}) = \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

works.

(b) $ABCD$ is a parallelogram but $PQRS$ is not, so no such map exists.

(c) $ABCD$ is not a parallelogram while $PQRS$ is, so no such map exists.

(d) Both $ABCD$ and $PQRS$ are parallelograms, so such a map does exist. The transformation

$$f(\mathbf{x}) = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

works.

Q6.

(a) Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ be the position vector of a point (x, y) that lies on the unit circle. So, $x^2 + y^2 = 1$. Let (u, v) be the image of the point (x, y), i.e.,

$$f(\mathbf{x}) = \begin{bmatrix} 12 & 5 \\ -5 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = f^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \frac{1}{169}\begin{bmatrix} 12 & -5 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{169}\begin{bmatrix} 12u - 5v \\ 5u + 12v \end{bmatrix}.$$

It follows that $x = \dfrac{1}{169}(12u - 5v)$ and $y = \dfrac{1}{169}(5u + 12v)$, and hence

$$\left(\frac{1}{169}(12u - 5v)\right)^2 + \left(\frac{1}{169}(5u + 12v)\right)^2 = 1.$$

Simplifying gives

$$\frac{u^2}{169} + \frac{v^2}{169} = 1 \iff u^2 + v^2 = 13^2.$$

Thus, the image of the unit circle is a circle of centre $(0, 0)$ and radius 13.

(b) The inverse of the given affine transformation is

$$f^{-1}(\mathbf{x}) = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \mathbf{x} - \frac{1}{25}\begin{bmatrix} -2 \\ 11 \end{bmatrix}.$$

If (u, v) is the image of a point (x, y) on the unit circle, then $\begin{bmatrix} x \\ y \end{bmatrix} = f^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$ and so we find that

$$x = \frac{1}{25}(4u - 3v + 2) \text{ and } y = \frac{1}{25}(3u + 4v - 11).$$

Substituting these expressions into $x^2 + y^2 = 1$ and then simplifying gives

$$(u - 1)^2 + (v - 2)^2 = 5^2.$$

Hence, the image of the unit circle is a circle of centre $(1, 2)$ and radius 5.

(c) The image is an ellipse given by

$$5u^2 + 29v^2 - 24uv + 48u - 116v + 116 = 1.$$

(d) The image is an ellipse given by

$$u^2 - 2uv + 2v^2 = 1.$$

Q7. We follow the same method as in Q6. First, find the inverse of the given transformation f and then compute

$$\begin{bmatrix} x \\ y \end{bmatrix} = f^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right).$$

Then, substitute these expressions for x and y into (i) $x^2 - y^2 = 1$ and (ii) $xy = 1$.

(a) (i) $\dfrac{119u^2}{169^2} - \dfrac{240}{169^2}uv - \dfrac{119}{169^2}v^2 = 1$ (ii) $\dfrac{60u^2}{169^2} + \dfrac{119}{169^2}uv - \dfrac{60}{169^2}v^2 = 1$.

(b) (i) $\frac{7}{625}u^2 - \frac{48}{625}uv - \frac{7}{625}v^2 + \frac{82}{625}u + \frac{76}{625}v - \frac{117}{625} = 1$
(ii) $\frac{12}{757}u^2 + \frac{7}{757}uv - \frac{12}{757}v^2 - \frac{8}{757}u + \frac{81}{757}v = 1$.

(c) (i) $v^2 - u^2 = 1$ (ii) $uv = 1$.

(d) (i) $u^2 - 2uv = 1$ (ii) $uv - v^2 = 1$.

Q8. We have

$$cu + dv = (au + bv)^2 + 1 = a^2u^2 + 2abuv + b^2v^2 + 1$$

and so

$$-a^2u^2 - 2abuv - b^2v^2 + cu + dv = 1.$$

We can write this as

$$Au^2 + Buv + Cv^2 + cu + dv = 1,$$

where $A = -a^2$, $B = -2ab$ and $C = -b^2$. Given that $B^2 - 4AC = 4a^2b^2 - 4a^2b^2 = 0$, it follows that this expression defines a parabola.

Chapter 3: Matrix Transformations

Q1. $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_{\pi/6} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$, $A_{2\pi/3} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$, $A_{-3\pi/4} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$.
$A_{\pi/6} \left(A_{2\pi/3}\right)^{-1} \left(A_{-3\pi/4}\right)^2$ corresponds to a rotation matrix A_α where

$$\alpha = \frac{\pi}{6} - \frac{2\pi}{3} + 2 \times \frac{-3\pi}{4} = -2\pi.$$

Hence, $A_\alpha = I$.

Q2.

(a) $A_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and so if (x, y) is a point on the hyperbola $xy = 1$, then the image (u, v) with respect to the rotation satisfies

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left(A_{\pi/2}\right)^{-1} \begin{bmatrix} u \\ v \end{bmatrix}.$$

It follows that $x = v$ and $y = -u$, and hence the image of the hyperbola is given by $-uv = 1$.

(b) $A_{2\pi/3} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$ and so if (x, y) is a point on the hyperbola $xy = 1$, then the image (u, v) with respect to the rotation satisfies

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left(A_{2\pi/3}\right)^{-1} \begin{bmatrix} u \\ v \end{bmatrix}.$$

It follows that $x = \frac{1}{2}(-u + \sqrt{3}v)$ and $y = \frac{1}{2}(-\sqrt{3}u - v)$. Hence, the image of the hyperbola is given by

$$\frac{\sqrt{3}}{4}u^2 - \frac{1}{2}uv - \frac{\sqrt{3}}{4}v^2 = 1.$$

Q3. In each case, express the line in the form $ax + by = 0$. The reflection matrix is then given by $A = I - \dfrac{2}{\mathbf{w}^T\mathbf{w}}\mathbf{w}\mathbf{w}^T$, where $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$.

(a) $A = \begin{bmatrix} 1/3 & 2\sqrt{2}/3 \\ 2\sqrt{2}/3 & -1/3 \end{bmatrix}$

(b) $A = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} -12/13 & 5/13 \\ 5/13 & 12/13 \end{bmatrix}$

Q4. Follow the same method as in Q2, where the appropriate reflection matrix is used in place of the rotation matrix.

(a) The reflection matrix is $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the image of hyperbola is $u^2 - v^2 = 1$. Thus, the hyperbola is an *invariant* with respect to the reflection.

(b) The reflection matrix is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the image of hyperbola is $-u^2 + v^2 = 1$.

(c) The reflection matrix is $A = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}$ and the image of hyperbola is

$$\frac{7}{25}u^2 - \frac{48}{25}uv - \frac{7}{25}v^2 = 1.$$

Q5. The matrix that corresponds to a reflection through the x-axis is $A_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the matrix that corresponds to a reflection through the line $y = \tan(\theta)x$ is $A_\theta = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$. Thus, the combination of the two reflections gives

$$A_\theta A_x = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix},$$

which corresponds to an anticlockwise rotation through an angle of 2θ.

Q6. Below, A_H, A_V and A_S denote matrices that correspond to horizontal stretches, vertical stretches and shears, respectively.

(a) The matrix representation is

$$A_H A_{\pi/2} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}.$$

(b) The matrix representation is

$$A_{2x} A_V A_H = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -6 & 12 \\ 8 & 9 \end{bmatrix}.$$

(c) The matrix representation is

$$A_{\pi/4}A_S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -1\\ 2 & 1\end{bmatrix}.$$

Q7. $\begin{bmatrix}1 & m\\ 0 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & -m\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}h & m\\ 0 & v\end{bmatrix}^{-1} = \dfrac{1}{hv}\begin{bmatrix}v & -m\\ 0 & h\end{bmatrix}$.

Q8. (a) The vertices of the unit square are given by the position vectors $\mathbf{0}$, $\mathbf{e_1} = \begin{bmatrix}1\\ 0\end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix}0\\ 1\end{bmatrix}$ and $\mathbf{e_1} + \mathbf{e_2} = \begin{bmatrix}1\\ 1\end{bmatrix}$. The linear transformation induced by the matrix $A = \begin{bmatrix}2 & 3\\ 0 & 3\end{bmatrix}$ has the following effect:

$$A\mathbf{0} = \mathbf{0}, \quad A\mathbf{e_1} = \begin{bmatrix}2\\ 0\end{bmatrix}, \quad A\mathbf{e_2} = \begin{bmatrix}3\\ 3\end{bmatrix}, \quad A(\mathbf{e_1} + \mathbf{e_2}) = \begin{bmatrix}5\\ 3\end{bmatrix}$$

Similar calculations are carried out in (b), (c) and (d).

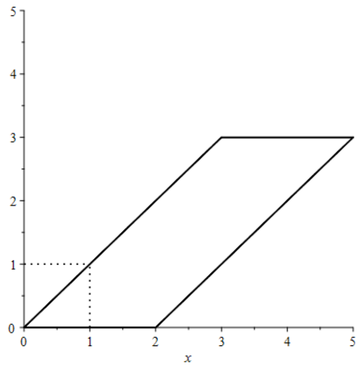

Figure 1: (a)

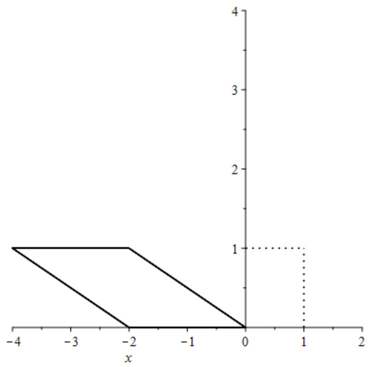

Figure 2: (b)

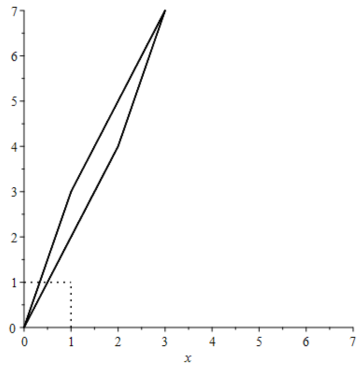

Figure 3: (a)

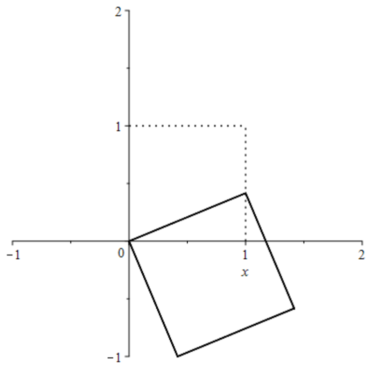

Figure 4: (b)