UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

Geometry and Linear Algebra
Chapter 2: 2D Objects

Geometry begins to get "interesting" in two dimensions. The original formalisation of two dimensional geometry was in terms of **axioms**. If we accept these axioms, then everything else follows as a consequence. Euclidean geometry in two dimensions is built on the following axioms.

1. A straight line can be drawn between any two points.
2. A finite line can be extended infinitely in both directions.
3. A circle can be drawn with any centre and any radius.
4. All right angles are equal to each other.
5. Given a line and a point not on the line, only one line can be drawn through the point parallel to the line.

We can boil geometry down to four essential elements: two objects and two measurements. The fundamental objects are the point (which has zero dimensions) and the straight line (which has one). The measurements are distance and angle. More complex objects can be built from the fundamental elements. For example, circles are sets of points that share a common distance from a centre and polygons can be built from line segments.

By fixing an origin and working in Cartesian coordinates, we need not work with the axioms directly. We can use an algebraic approach to geometry instead. Points can be represented by their coordinates, or by a position vector from the origin; and curves can be described as a set of points satisfying a functional relationship. Our two dimensional world is known as the **Euclidean plane** (often abbreviated to "the plane"). Since all points in the plane can be described in terms of two real numbers (the x- and y-coordinates) it is also known as $\mathbb{R}^2$.

In this brief chapter we study the basic building blocks of two dimensional geometry to prepare us for the next chapter where you will learn how to manipulate shapes and curves. While some of the ideas we look at are specific to two dimensions, we will also be interested in concepts that generalise to higher dimensions, preparing us for the later chapters of the course.

2.1 The Straight Line

Definition 2.1.1 *Given three scalars $a, b, c \in \mathbb{R}$, a* **straight line** *is the set of points*

$$L = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}. \tag{2.1.1}$$

This is simply a formalisation of the familiar implicit formula of a straight line. Notice that it means that a straight line is of infinite length. We now collect together some well known properties of straight lines which you are assumed to have seen before.

Remarks

1. Straight lines have a constant slope, known as the **gradient**. Given two points on a line, $P(x_1, y_1)$ and $Q(x_2, y_2)$, this can be measured by the ratio $m = (y_2 - y_1)/(x_2 - x_1)$. In the degenerate case ($x_1 = x_2$) the line is vertical and is said to have infinite gradient.

2. Using the implicit definition means that we don't have to treat vertical lines as a special case: we let $b = 0$. If $b \neq 0$, the gradient of L in (2.1.1) is $-a/b$.

3. The x-axis is the line $\{(x, y) \in \mathbb{R}^2 | y = 0\}$ and the y-axis is the line $\{(x, y) \in \mathbb{R}^2 \mid x = 0\}$.

4. Two lines are **parallel** if they have the same gradient. Two lines are **perpendicular** if one is horizontal and the other vertical or if their gradients m_1 and m_2 satisfy the equality $m_1 m_2 = -1$.

5. All lines (except vertical lines) cross the y-axis at a unique point known as the **intercept**. The intercept of L in (2.1.1) has coordinates $(0, c/b)$.

6. The **explicit form** of a straight line with gradient m and intercept $(0, c)$ is given by the expression $y = mx + c$. Formally, this is the line L in (2.1.1) with $a = -m$, $b = 1$ (and $c = c$).

7. There is a unique straight line that passes through every pair of distinct points. The line through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$L = \{(x, y) \in \mathbb{R}^2 \mid (y_1 - y_2)x + (x_2 - x_1)y = y_1 x_2 - x_1 y_2\}.$$

   If the line is not vertical, it is more normally expressed in the explicit form

$$y - y_1 = m(x - x_1),$$

   where $m = (y_2 - y_1)/(x_2 - x_1)$ is the gradient. If the line through P and Q is vertical it can be written as $x = x_1$.

8. The line $y = mx + c$ makes an angle θ with the horizontal line $y = c$ at its intercept where $m = \tan\theta$.

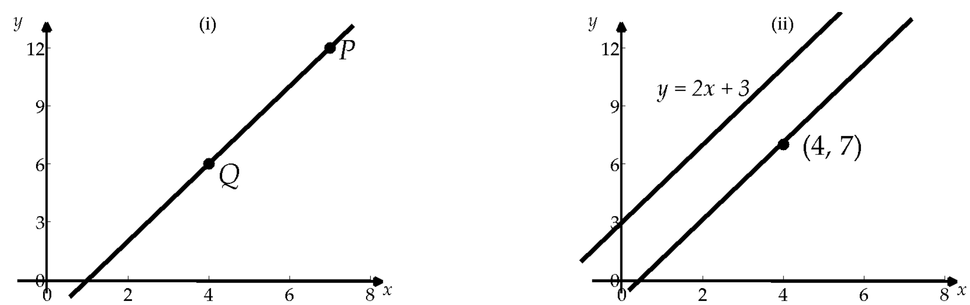
Examples 2.1.1

(i) Find the equation of the line which passes through the points $P(7,12)$ and $Q(4,6)$.

The gradient of the line is $(12-6)/(7-4) = 2$, so the line has equation $y = 2x + c$. Substituting $(x,y) = (7,12)$ gives $12 = 14 + c$ hence $c = -2$ and L can be written $y = 2x - 2$.

(ii) Find the line, L, parallel to $y = 2x + 3$ which passes through the point $(4,7)$.

L must have gradient $m = 2$, so it can be written in the form $y = 2x + c$. Substituting $(x,y) = (4,7)$ gives $7 = 8 + c$ hence $c = -1$ and L can be written $y = 2x - 1$.



Examples 2.1.2

Vector Form

Looking at our pictures of vectors in Chapter 1 it should come as no great surprise that straight lines can be defined in terms of vectors. We can represent any straight line as a linear combination of two vectors: one being a position vector of a point in the line and one representing the direction.

If $P(x,y)$ lies on the line L, as defined in (2.1.1) then

$$\overrightarrow{OP} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = ax + by = c.$$

Now notice that if $\mathbf{r}$ is a vector that is perpendicular to $\begin{bmatrix} a \\ b \end{bmatrix}$,

$$(\overrightarrow{OP} + \mathbf{r}) \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \overrightarrow{OP} \cdot \begin{bmatrix} a \\ b \end{bmatrix} + \mathbf{r} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = c + 0 = c,$$

so the point with position vector $\overrightarrow{OP} + \mathbf{r}$ lies on L, too. We can choose $\mathbf{r} = \begin{bmatrix} b \\ -a \end{bmatrix}$ and since $(t\mathbf{u}) \cdot \mathbf{v} = t(\mathbf{u} \cdot \mathbf{v})$, the line

$$L = \{(x,y) \in \mathbb{R}^2 \mid ax + by = c\}$$

can be written in vector form as

$$L = \left\{ \mathbf{p} + t \begin{bmatrix} b \\ -a \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\},$$

where $\mathbf{p}$ is the position vector of any point on L.

Definition 2.1.2 *Given two vectors* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$*, a straight line is the set of points with position vectors*

$$L = \{\mathbf{a} + t\mathbf{b} \mid t \in \mathbb{R}\}. \tag{2.1.2}$$

An advantage of (2.1.2) over (2.1.1) is that it immediately generalises to n dimensions by letting $\mathbf{a}$ and $\mathbf{b}$ have n components.

Examples 2.1.3

(i) Find the vector form of the straight line, L, which goes through the points $P(3, 2)$ and $Q(5, 7)$.

To write L in vector form we need vectors $\mathbf{a}$ and $\mathbf{b}$. For $\mathbf{a}$ we can use the position vector $\overrightarrow{OP}$. For $\mathbf{b}$ we can use $\overrightarrow{PQ}$. So,

$$L = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

(ii) The line $y = mx + c$ goes through the points $(0, c)$ and $(1, m + c)$. In vector form we can write it as

$$\mathbf{r} = \begin{bmatrix} 0 \\ c \end{bmatrix} + t \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

(iii) The line $y = mx$ can be written in vector form as $\mathbf{r} = t \begin{bmatrix} 1 \\ m \end{bmatrix}$.

(iv) The vector form of the straight line is in **parametric** form. Using the notation of Chapter 1, we can write (2.1.2) as

$$x = a_1 + b_1 t, \;\; y = a_2 + b_2 t, \;\; t \in \mathbb{R}.$$

Examples 2.1.4

2.1.1 Distance Formulae

The **distance formula** was introduced in Chapter 1. If P has coordinates (x, y) and Q has coordinates (ξ, η) then the Euclidean distance between the two points is given by the formula

$$d_{PQ} = \sqrt{(x - \xi)^2 + (y - \eta)^2}. \tag{2.1.3}$$

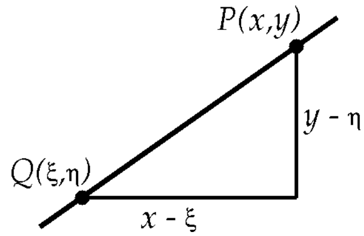


Figure 2.1: Distance between points.

Thus the Euclidean distance is the **straight line** distance between two points. Recall that we can write $d_{PQ} = \|\overrightarrow{PQ}\|$.

Now consider the distance from a general point in the plane to the straight line

$$L = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c.\}$$

We are particularly interested in the minimum distance. Figure 2.2 illustrates the fact that if Q is the nearest point on L to P then the line PQ meets L at right angles (the proof of this fact is left as an exercise). This observation is enough for us to come up with a formula for the distance from P to Q.

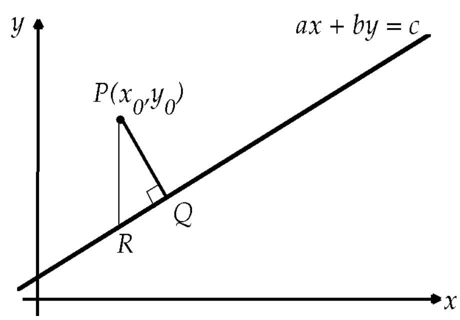


Figure 2.2: Distance between point and line.

There are many ways of doing this. We proceed as follows. We draw a vertical line from P to R. Since R lies on L its coordinates are $(x_0, (c - ax_0)/b)$ so PR has length $|(ax_0 + by_0 - c)/b|$. Therefore the distance from P to Q is given by

$$d = \frac{|ax_0 + by_0 - c|}{|b|} \cos\theta,$$

where $\theta = \angle RPQ$. It is straightforward to show that θ is the angle that L makes with the x-axis and hence $\tan\theta = -a/b$ (see Remark 8, earlier).

Finally, we use the identity $\sec^2\theta = 1 + \tan^2\theta$ to establish that

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + a^2/b^2}} = \frac{|b|}{\sqrt{a^2 + b^2}},$$

and hence

$$d = \frac{|ax_0 + by_0 - c|}{|b|} \times \frac{|b|}{\sqrt{a^2 + b^2}} = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}.$$

Note that we have used modulus signs to ensure that the distance from P to R is positive and so that we measure θ as an acute angle.

Example 2.1.5

- The minimum distance between the line $3x + 4y = 7$ and the point $(-5, -2)$ is

$$d = \frac{|3 \times (-5) + 4 \times (-2) - 7|}{\sqrt{3^2 + 4^2}} = \frac{|-30|}{5} = 6,$$

whereas the minimum distance from the point $(5, -2)$ to the line is

$$d = \frac{|15 - 8 - 7|}{5} = 0,$$

so $(5, -2)$ lies on the line.

- Find the minimum distance from the point $P(2, -1)$ to the line $y = 3x + 1$.

Since $y = 3x + 1$ can be written as $-3x + y = 1$, we have

$$d = \frac{|-6 - 1 - 1|}{\sqrt{9 + 1}} = \frac{8}{\sqrt{10}}.$$

Example 2.1.6

- Another distance we can consider is the distance between two parallel lines. Suppose our parallel lines are expressed as $ax + by = c_1$ and $ax + by = c_2$. Let $P(x_0, y_0)$ be a point on the first line. Then we compute its perpendicular distance to the second line as

$$\frac{|ax_0 + by_0 - c_2|}{\sqrt{a^2 + b^2}}.$$

But since $ax_0 + by_0 = c_1$, this simplifies to

$$\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}.$$

2.1.2 The Section Formula

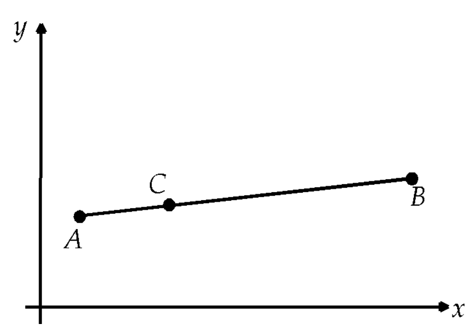


Figure 2.3: Intermediate points.

Take two points A and B and consider a third point C that lies along the line connecting A and B, as illustrated in Figure 2.3 . Clearly $\overrightarrow{OC} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$, where $0 < \lambda < 1$ if C is to lie strictly between A and B. Letting $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$ gives

$$\mathbf{c} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}. \tag{2.1.4}$$

Notice that

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$

and

$$\overrightarrow{CB} = \mathbf{b} - \mathbf{c} = (1 - \lambda)(\mathbf{b} - \mathbf{a}),$$

so C splits AB in the ratio $\lambda : (1 - \lambda)$. Thus to split AB in the ratio $m : n$ we let $\lambda = m/(m + n)$ in (2.1.4).

Example 2.1.7

- Find the midpoint of the line between $A(3, 7)$ and $B(-1, 5)$.

  To split AB in the ratio $1 : 1$ we let $\lambda = 1/2$ in (2.1.4) to give

  $$\mathbf{c} = \frac{1}{2}\begin{bmatrix} 3 \\ 7 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix},$$

  so the midpoint is $C(1, 6)$.

- Find the position vector of the point on AB which lies 3 times as close to $A(-1, -3)$ as it does to $B(7, 1)$.

  If C lies three times as close to A as it does to B then $AC : CB = 1 : 3$ and $\lambda = 1/4$ in (2.1.4). Hence

  $$\mathbf{c} = \frac{3}{4}\begin{bmatrix} -1 \\ -3 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

2.1.3 Intersecting Lines

A direct consequence of Euclid's fifth axiom (the parallel postulate) is that if you take two straight lines then they will intersect at a unique point so long as the lines are not parallel.

Consider the lines denoted by $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$. At the unique intersection point $P(\xi, \eta)$, the equations $a_1\xi + b_1\eta = c_1$ and $a_2\xi + b_2\eta = c_2$ are solved simultaneously. That is,

$$\begin{cases} a_1\xi + b_1\eta = c_1, \\ a_2\xi + b_2\eta = c_2, \end{cases}$$

or,

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{2.1.5}$$

Thus intersection can be phrased algebraically by simultaneous equations or by a matrix equation. It is straightforward to show by substitution that

$$\xi = \frac{b_2c_1 - b_1c_2}{a_1b_2 - b_1a_2}, \quad \eta = \frac{c_2a_1 - c_1a_2}{a_1b_2 - b_1a_2},$$

or

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{2.1.6}$$

Writing (2.1.5) as $A\mathbf{x} = \mathbf{b}$, (2.1.6) simply says $\mathbf{x} = A^{-1}\mathbf{b}$. Recall that the inverse is undefined if $a_1b_2 = b_1a_2$, but in this case $a_1/b_1 = a_2/b_2$ so the lines have the same slope and are parallel.

If two lines are parallel they either never intersect or they intersect everywhere (when the lines are identical).

Examples 2.1.8

(i) Find the point $P(\xi, \eta)$ where the lines $2x - 3y = 4$ and $3x - 5y = 7$ meet.

At the intersection point $6\xi - 9\eta = 12$ and $6\xi - 10\eta = 14$. Taking the difference of these two equations gives $\eta = -2$. Substituting back into the equation for either of the two lines gives $\xi = -1$.

Alternatively, using the matrix equation given above,

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{-10+9} \begin{bmatrix} -5 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

(ii) By equating for y, we see that the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ meet when $x = (c_1 - c_2)/(m_2 - m_1)$. Substitution then gives $y = (m_2c_1 - m_1c_2)/(m_2 - m_1)$.

The intersection point is undefined if $m_1 = m_2$, that is, if the lines are parallel.

Exercises: The Straight Line

1. Write the lines through the following pairs of points in

   (i) implicit form (ii) the form $y = mx + c$ (where possible) (iii) vector form.

   Try and make the constants involved as simple as possible.

   (a) $(0,0),\ (2,1)$ (b) $(3,4),\ (3,5)$ (c) $(1,5),\ (2,10)$ (d) $(6,3),\ (-2,1)$ (e) $(1,2),\ (-1,-2)$
   (f) $(63,126),\ (-1204,-2408)$

2. Find, in vector form, the line which passes through the origin and is perpendicular to the line $\mathbf{r} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

3. Show that the implicit form of the straight line $\mathbf{r} = \begin{bmatrix} p \\ q \end{bmatrix} + t\begin{bmatrix} a \\ b \end{bmatrix}$ can be written $bx - ay = c$ where $c = pb - aq$.

4. Find the minimum distance between the following points and lines.

   (a) $(0,0)$ and $5x - 12y = 7$

   (b) $(-1,3)$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 13 \end{bmatrix} + t\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

   (c) $(5,1)$ and $y = 2x - 1$

   (d) (x_0, y_0) and $y = mx + c$

5. Copy Figure 2.2 and draw a point S on the line $ax + by = c$ away from Q.

   Use Pythagoras' theorem to show that PS is longer than PQ and hence confirm that Q is the nearest point on the line to P.

6. Suppose the points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ are the same distance from the line $L : ax + by = c$. What are the conditions which ensure that the line through P_0 and P_1 is parallel to L?

   You may find a sketch helpful.

7. Find the intersection points of the following pairs of lines.

   (a) $2x + 3y = 9$ and $x + 3y = 6$ (b) $y = 3x + 1$ and $y = 4x$ (c) $x - 4y = 7$ and $-3x + 12y = 0$
   (d) $y = 4x - 7$ and $2x - y = 3$ (e) $6x + 2y = 4$ and $y = 2 - 3x$ (f) $-4x + 5y = 0$ and $7x - 3y = 0$

2.2 Shapes and Curves

In Chapter 3 we will look at transformations which map one shape into another. In particular, we will look at how to map one triangle into another and quadrilaterals into other quadrilaterals. Before doing that, we review the ways of expressing various shapes algebraically.

2.2.1 Polygons

A triangle can be defined unambiguously by the coordinates of its vertices: there is only one way to join them together. For polygons with more than three sides, there is more than one way of "connecting the dots", so we need to know more than just the locations of the vertices. Given the position vector of one corner of the polygon, one way of defining the polygon is to then list the direction vectors that take us from one corner to the next.

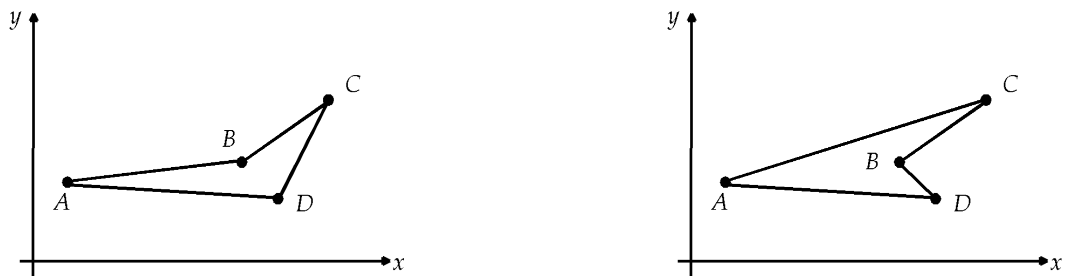


Figure 2.4: Quadrilaterals with the same vertices.

If we know that a polygon is **convex**, it is uniquely defined by its vertices. It can also be defined in terms of a sequence of inequalities (see examples).

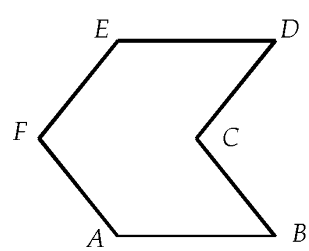
Parallelograms can be described by knowing the position of one corner and its two adjacent vertices (the fourth corner is then prescribed).

Example 2.2.1

- Draw the hexagon $ABCDEF$ given that

  $\overrightarrow{OA} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\overrightarrow{AB} = \overrightarrow{ED} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$,

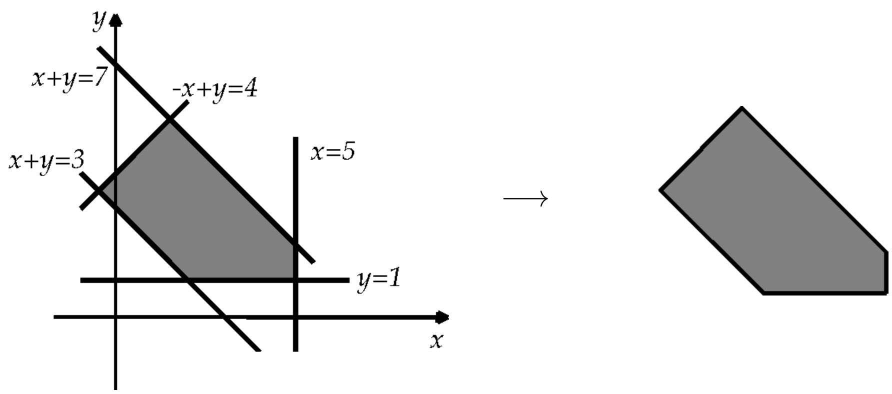
  $\overrightarrow{BC} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\overrightarrow{CD} = \overrightarrow{FE} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.



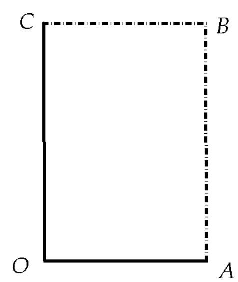
Example 2.2.2

- Sketch the area of the plane where all of the following inequalities hold

$$x + y \geq 3, \;\; x \leq 5, \;\; y \geq 1, \;\; x + y \leq 7, \;\; -x + y \leq 4.$$



- Sketch the parallelogram $OABC$ given that
$\overrightarrow{OA} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\overrightarrow{OC} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.



2.2.2 Circles and Ellipses

The circle is one of Euclid's axiomatic shapes. To describe it algebraically we note that a circle is the set of points a common distance from a given centre. If the centre is $C(a, b)$ then the distance of a point $P(x, y)$ from C is $\sqrt{(x-a)^2 + (y-b)^2}$. The set of points a common distance r from C is then

$$\left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \sqrt{(x-a)^2 + (y-b)^2} = r \right\},$$

or

$$\left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, (x-a)^2 + (y-b)^2 = r^2 \right\}.$$

this is our equation of a circle centre C with radius r.

Alternatively, by expanding the brackets, a circle can be written

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

from which the centre and radius can be inferred (see exercises).

The circle can be described parametrically as the set

$$\{(a + r\cos\theta, b + r\sin\theta) | 0 \leq \theta \leq 2\pi\}.$$

Example 2.2.3

- Find the radius and centre of the circle

$$x^2 + y^2 - 4x + 3y - 36 = 0.$$

$$x^2 + y^2 - 4x + 3y - 36 = (x-2)^2 - 4 + (y+3/2)^2 - \frac{9}{4} - 36 = (x-2)^2 + (y+3/2)^2 - \frac{169}{4},$$

a circle with centre $(2, -3/2)$ and radius $13/2$.

Examples 2.2.4

A circle with centre at the origin, radius r, can be written

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1.$$

by letting the denominators in this expression vary we get the **canonical form of an ellipse**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.2.1}$$

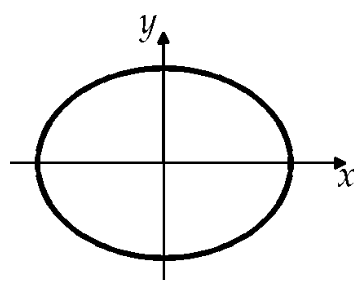


Figure 2.5: A canonical ellipse.

A typical canonical ellipse is show in Figure 2.5. Note that it intersects the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm b)$, as is easily seen by substitution into (2.2.1). The ellipse is centred on the origin but can easily be translated to move the centre to (p, q) with the expression

$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1. \tag{2.2.2}$$

While a circle is perfectly symmetric and remains unchanged when reflected through any diameter, there are only two lines of symmetry in an ellipse. These are called the semi-axes. For the ellipse described in (2.2.2), the semi-axes are the lines $x = p$ and $y = q$, but by reflecting through other lines we get ellipses with different semi-axes, meaning that (2.2.2) does not give the most general form of an ellipse.

Figure 2.6 shows the effect of reflecting the ellipse represented by $x^2/4 + 4y^2/9 = 1$ through the line $y = \sqrt{3}x$ (effectively the same as a rotation through $\pi/3$ radians). The semi-axes of the image are the lines $y = \sqrt{3}x$ and $y = -x/\sqrt{3}$ (which are perpendicular). It can be shown[1] that this ellipse has the equation

$$\frac{19}{48}x^2 + \frac{43}{144}y^2 - \frac{7\sqrt{2}}{72}xy = 1.$$

[1] We'll see how later on.

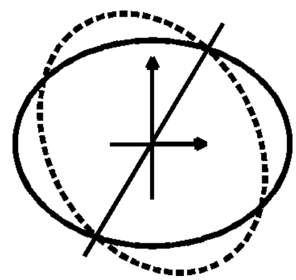


Figure 2.6: A rotated ellipse.

The general algebraic expression for an ellipse is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 1, \tag{2.2.3}$$

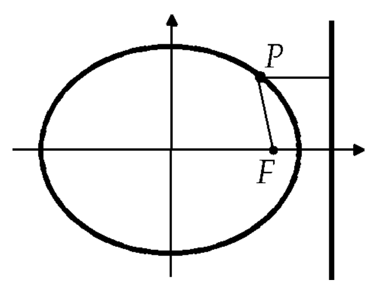
where $B^2 < 4AC$. It is much harder to picture the relationship between this algebraic expression and the curve in the plane it represents than (2.2.2). However (2.2.3) is an expression you'll see again in many places during your mathematics career.

The parametric form of the canonical ellipse is given by

$$\{(a\cos\theta, b\sin\theta) | 0 \le \theta \le 2\pi\}.$$

Examples 2.2.5

(i) Suppose $P(x, y)$ lies on the ellipse $9x^2 + 25y^2 = 225$. Let F have coordinates $(4, 0)$ and L be the line $4x = 25$. Find the distances d_{PF} and d_{PL} in terms of x.



$d_{PF} = \sqrt{(x-4)^2 + y^2}$ and since $y^2 = 9 - 9x^2/25$,

$$d_{PF} = \sqrt{(x-4)^2 + 9 - 9x^2/25} = \sqrt{16x^2/25 - 8x + 25} = \sqrt{(4x/5 - 5)^2} = |4x/5 - 5|.$$

By using the formula for the distance of a point from a line, or observing that we only need to look at the change in the x-coordinate in this case, we find that $d_{PL} = |x - 25/4|$. Notice that $d_{PF} : d_{PL} = 4/5$.

For any ellipse it is possible to find a point F and a line L so that the ratio $e = d_{PF} : d_{PL}$ is fixed (and less than one) for all points P on the ellipse. In fact the classical way of defining an ellipse geometrically is in terms of P, L and e.

(ii) An ellipse with centre (p, q) and semi axes $x = p$, $y = q$ as given in (2.2.2) can be written in parametric form as

$$\{(p + a\cos\theta, q + b\sin\theta)|0 \leq \theta \leq 2\pi\}.$$

(iii) If we rotate the ellipse (2.2.2) around its centre through the angle ϕ then its coordinates can be written parametrically as

$$x = p + a\cos\theta\cos\phi - b\sin\theta\sin\phi, \ \ y = q + a\cos\theta\sin\phi + b\sin\theta\cos\phi,$$

where $\theta \in [0, 2\pi]$.

2.2.3 Hyperbolae

Changing the sign in (2.2.1) from $+$ to $-$ defines a curve that is clearly closely related algebraically to the ellipse. The **hyperbola** has canonical form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

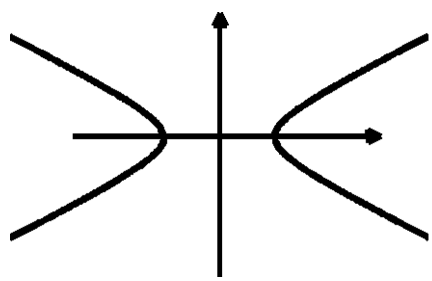


Figure 2.7: A hyperbola.

A typical canonical hyperbola, centred on the origin, is illustrated in Figure 2.7. In contrast to the circle and ellipse, a hyperbola is unbounded but it has the same symmetry properties as the ellipse. Notice that as you move further away from the origin, the four ends of the hyperbola look like straight lines. It can be shown that the hyperbola has asymptotes $y = \pm bx/a$. A hyperbola can be sketched by marking its centre and its asymptotes.

The hyperbola defined by

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = 1$$

is a simple translation of the canonical form. To describe all possible orientations of the hyperbola we can use the same expression we used for a general ellipse, namely (2.2.3), but now with the condition $B^2 > 4AC$.

The hyperbola can be described parametrically using the hyperbolic functions cosh and sinh, but we will not be using these functions in this course.

If $e > 1$ in Example 2.2.5(i), we get a geometric description of the hyperbola.

We get a special case of a hyperbola when we let $A = C = 0$ in (2.2.3). For example, when $D = E = 0$, too, we get $Bxy = 1$, which you may recognise better when written $y = 1/(Bx)$. This is known as a rectangular hyperbola. The lines of symmetry for this curve are $y = x$ and $y = -x$ and the asymptotes are the x- and y-axes.

Examples 2.2.6

2.2.4 Parabolas

Consider the general quadratic expression

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 1. \tag{2.2.4}$$

By varying the constants we can recover all the curves we gave seen so far. For example, if $B^2 < 4AC$ we have an ellipse (and a circle if $A = C > 0$). If $B^2 > 4AC$ we get a hyperbola. If $A = B = C = 0$ we get the equation for a straight line. The remaining case to consider is when $B^2 = 4AC$. The curve this describes is a **parabola**. This is easiest to see when $B = 0$ (which means either $A = 0$ or $C = 0$, too). For example, if $A = -a/c$, $B = C = 0$, $D = -b/c$ and $E = 1/c$ we get $y = ax^2 + bx + c$, the familiar expression for a quadratic which has a typical parabolic shape. We can't let $c = 0$ in this expression, but letting the right hand side of (2.2.4) equal zero overcomes this problem, without altering the theory.

For historical reasons, the canonical form for the parabola is $y^2 = 4ax$. This has the single line of symmetry $y = 0$.

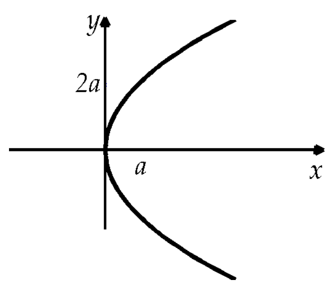


Figure 2.8: A canonical parabola.

By varying the parameter values in the general expression we can rotate and translate the parabola to orient it however we wish. The canonical form of the parabola can be written parametrically as

$$x = at^2, \;\; y = 2at,$$

where t ranges from $-\infty$ to $+\infty$.

Of course, there are many other curves than the ones we have looked at in this section. We have collected together the curves that can be expressed using a generalised quadratic for their historical importance. Parabolas, circles, ellipses and hyperbolae are known as **conic sections**. They are the cross sectional curves you get when a cone intersects a plane and have intrigued geometers for thousands of years. The conic sections are illustrated in Figure 2.9. Note that section 7 is simply a circle.

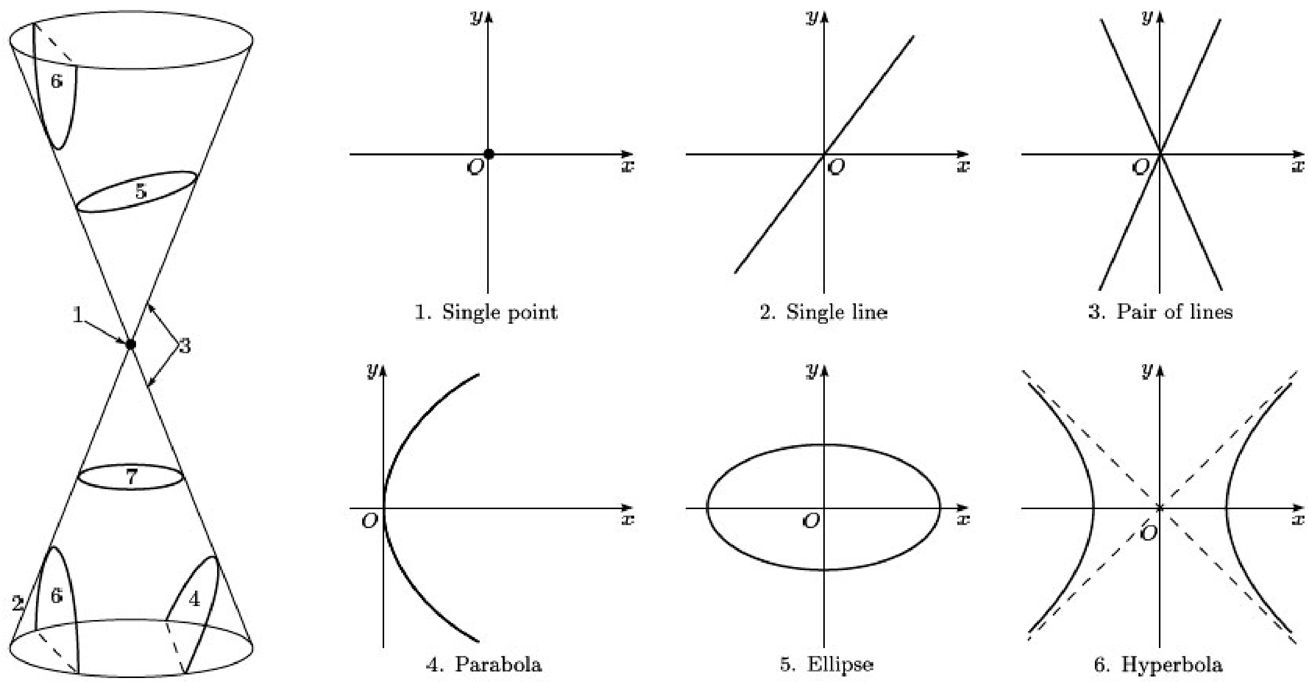


Figure 2.9: Conic sections.

Exercises: Curves

1. Sketch all the polygons with the following vertices.

   (a) $(0,0)$, $(3,0)$, $(4,2)$, $(2,4)$, $(-1,2)$

   (b) $(0,0)$, $(2,0)$, $(1,1)$, $(0,2)$, $(2,2)$

   (c) $(0,0)$, $(2,0)$, $(1,1)$, $(0,3)$, $(1,2)$, $(2,3)$

2. Sketch the polygon $OABCDEFGH$ given that

$$\overrightarrow{OA} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ \overrightarrow{OH} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ \overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{FE} = \overrightarrow{HG} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \overrightarrow{BC} = \overrightarrow{GF} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. If three corners of a parallelogram are $(1,1)$, $(4,2)$ and $(1,3)$, what are all the possible fourth corners? Draw two of them.

4. Find the centre and radius of the circle given by the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

   given that $g^2 + f^2 > c$.

5. By substituting the coordinates into the formula for a circle given in the previous question, find the circle through the points $(-2,2)$, $(4,2)$ and $(-3,-5)$.

6. Sketch the following conic sections, showing relevant facts (e.g., centre, radius, asymptotes, lines of symmetry).

   (a) $x^2 + y^2 = \dfrac{49}{4}$ (b) $x^2 + 4y^2 = 100$ (c) $x^2 + y^2 - 4x + 6y - 3 = 0$ (d) $xy = 5$

   (e) $\dfrac{x^2}{36} - \dfrac{y^2}{49} = 1$ (f) $2x^2 - 3x + 4y + 2y^2 = 0$ (g) $2y^2 - 3x = 1$

7. Find the points of intersection (if any) of the line $2x - y + 1 = 0$ with the following circles.

   (a) $x^2 + y^2 - 4x + 2y - 4 = 0$

   (b) $x^2 + y^2 + 8y + 11 = 0$

   (c) $x^2 + y^2 - 4x + 2y + 4 = 0$

8. Find the coordinates of the point at which the circle with equation $x^2 + y^2 + 2x - 2y - 16 = 0$ touches the circle with equation $x^2 + y^2 = 8$.

9. Show that the point P with co-ordinates $x = a(e^t + e^{-t})/2$ and $y = b(e^t - e^{-t})/2$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

   for any value of t.

10. Find the asymptotes of the following hyperbolae.

    (a) $2x^2 - y^2 = 1$ (b) $2(x+2)^2 - 4(y-3)^2 = 8$ (c) $x^2 - 3y^2 - 2x - 6y - 11 = 0$

2.3 Summary

Here is a list of skills you are expected to have picked up from this chapter.

1. Understand all terms in **bold face** (all sections).

2. Understand all the numbered definitions and statement of all numbered theorems and recognise the relevance of the numbered equations (all sections).

3. Be able to write any straight line in implicit form, vector form or (where possible) explicit form and convert between the three (§2.1).

4. Solve simple problems involving straight lines (§2.1).

5. Appreciate the derivation of the distance formula and the section formula (§2.1).

6. Use the distance formula
$$d = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}},$$
to solve simple problems (§2.1).

7. Find the intersection points of pairs of straight lines (§2.1).

8. Describe polygons using position vectors (§2.2).

9. Be familiar with the algebraic definition of conic sections (§2.2).

10. Be able to find key features (e.g., centre, radius, asymptotes, lines of symmetry) of conic sections: **translations of canonical forms only** (§2.2).

11. Solve simple intersection problems involving conic sections (§2.2).

12. Manipulate the algebraic definition to sketch conic sections (§2.2).