UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Contents

1 Preliminaries

Learning mathematics is like learning a language: you have to know the vocabulary and the rules of "mathematical grammar" which enable you to put correct mathematical statements together. However, one difference with mathematics is that is is a very **precise** and **exact** language. All mathematics symbols have very precise meanings, and hence must be used with care. Careless or sloppy notation is at best confusing and ambiguous, and usually downright wrong. One of your challenges on this course is to not just learn mathematics, but to learn to think (and write) like a mathematician.

In studying for a mathematics degree, you will learn in two distinct ways:

- learning **facts**: this is equivalent to "knowing that" something is true (e.g. knowing that for two numbers a and b, $a + b = b + a$);
- learning **skills**: this is equivalent to "knowing how" something is done (e.g. knowing how to solve a quadratic equation).

Learning facts is something which is best done by connecting them to other facts and things that you already know, in particular, other mathematical ideas. The lectures (and these notes) are full of facts, but most of these do not need to be learnt. The most important facts will be highlighted during the course. In general, remembering **definitions** and **theorems** is important, but memorising **examples** is not.

Learning skills usually involves two stages: first having the rules in front of you and following them, then being able to complete the task without reference to your notes. The main hint on how to achieve this is to **do lots of examples**, not just the ones set in lectures. Watching someone else do examples and copying their solutions is completely useless for learning a skill, you must practice yourself. If you are having problems, then you should try to identify what your problem is by trying some examples and seeing how far you get: there is no point in simply asking someone else to do more examples for you.

Mathematics is sometimes called a **linear** discipline, because it builds from one idea to the next. This means that it is very important to get any problems you have sorted out as soon as possible: if you don't understand today's topic, then you may not be able to understand how it applies to tomorrow's topic. The recommended text books contain many extra examples for the topics we will cover in this module. Some revision of Higher material can be found in the Red Book. Similar material is also available online at `outreach.mathstat.strath.ac.uk/basicmaths` and `mathcentre.ac.uk` (these links can also be accessed via the MM101 *Web Links* page on MyPlace). We will NOT be covering such basic material in lectures: **it is your responsibility to make sure that you are comfortable with this material before beginning the course**.

2 Sets and number systems

Basic mathematics involves counting and measurement, which is usually described in terms of **numbers**. More abstract mathematical ideas are described in terms of **set theory**. This section contains an explanation of some of the basic properties of sets and numbers, most of which should be familiar to you. However, this is not just a review: we will introduce these properties in a formal and organised way which will highlight some significant ideas and structures which you will meet repeatedly on your mathematics course.

2.1 Sets

2.1.1 Definition and notation

Definition 2.1

(i) *A **set** is a collection of objects where there is a well-defined way of determining whether a given object is included or excluded.*

(ii) *An object in a set is called an **element** or **member** of that set. If the element x is a member of the set A, we write $x \in A$. Alternatively, if x is not a member of the set A, we write $x \notin A$.*

Some examples of sets include the set of students registered for MM101, the set of Munros (mountains in Scotland with a height over 3000 feet), or the set of prime numbers. All of these examples have clear rules that allow someone with enough information to determine whether a given object is included in the collection or not. The clear rule is of key importance here: we might talk about our 'set' of friends, but this is NOT a set in the sense of Definition 2.1 as there is not a clear rule for determining when people are friends. In contrast, the people listed as someone's 'Friends' on a social networking site such as Facebook, do form a set.

There are two common ways of describing sets:

1. by listing all of the elements of a set between curly brackets $\{\cdots\}$. For example, the set of all integers x which satisfy $2 \leq x < 5$ can be written as $S = \{2, 3, 4\}$. Note that the order of the elements in the set does not matter. Also, each entry is listed only once: repetitions are ignored.

2. by explicitly stating a defining property of the form

$$\{x \mid x \text{ satisfies property } P\}.$$

The $|$ symbol is read as 'such that'; sometimes a colon (:) is used instead. For example, for set S above we may write $S = \{x|x \text{ is an integer and } 2 \leq x < 5\}$.

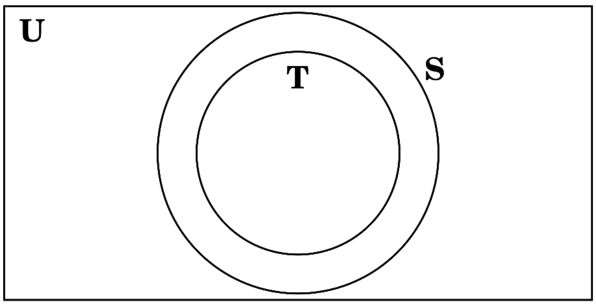
Two sets S and T are considered equal when they have exactly the same elements: we write this as $S = T$. If a set has a finite number of members, it is called a **finite set**; otherwise, it is an **infinite set**. The number of members in a finite set S is called the **order** of S, and written as $|S|$. The **empty set**, denoted by $\emptyset$, is the (unique) set which has no elements. That is, $\emptyset = \{\}$. Note that the empty set $\emptyset$ is NOT the same as $\{\emptyset\}$, which is 'the set which contains the empty set'.

2.1.2 Subsets and set operations

Definition 2.2 *A set T is called a* ***subset*** *of a set S if every element of T is an element of S. We write this relationship as $T \subset S$. When T is NOT a subset of S, we write $T \not\subset S$.*

Note: Some authors use slightly different notation, with $T \subseteq S$ used to represent a subset, and $T \subset S$ reserved for a so-called **proper subset** (where $T \subseteq S$ and $T \neq S$). We will not make that distinction in this class.

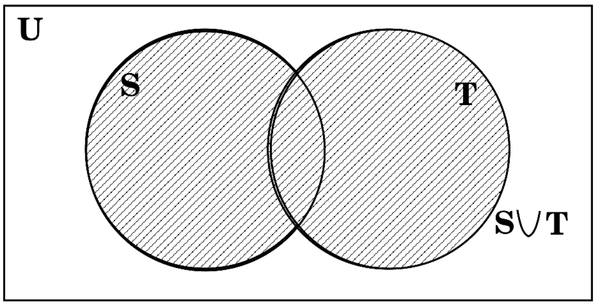
One visualisation technique which can be useful for depicting sets is a **Venn diagram**. Given two sets S and T of elements from a *universal set* U, we use a rectangle to represent U, then add circles within that rectangle to represent the relationship between sets S and T, assuming that elements of U in S are inside the circle for S, and likewise for T. For example, the concept $T \subset S$ can be represented by



We will now use Venn diagrams to illustrate some standard relationships between subsets of a universal set U.

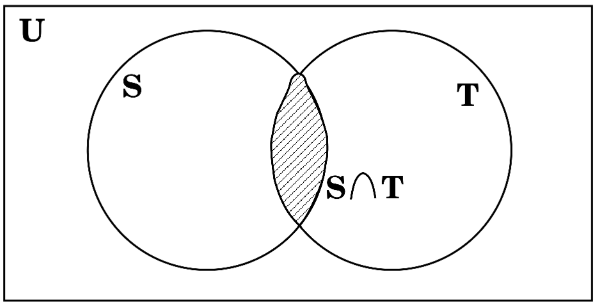
The **union** of two subsets S and T consists of the elements which are in either S or T or both. We write

$$S \cup T = \{x \in U | x \in S \text{ or } x \in T\}.$$



The **intersection** of two subsets S and T consists of the elements which are in both S and T.

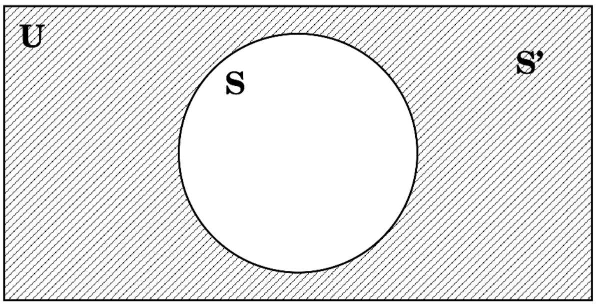
$$S \cap T = \{x \in U | x \in S \text{ and } x \in T\}.$$



If S and T have no elements in common, they are said to be **disjoint** or **mutually exclusive**, that is, $S \cap T = \emptyset$.

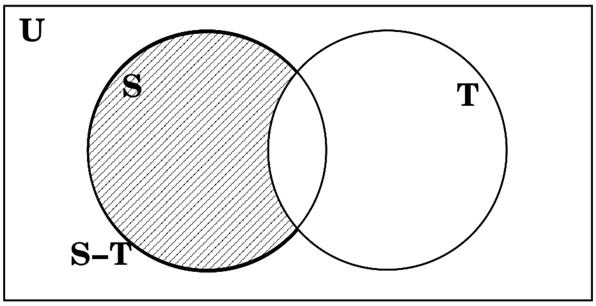
The **complement** of a subset S is the set of all elements of U which are NOT in S.

$$S' = \{x \in U | x \notin S\}.$$



The **set difference** of T from S consists of the elements in S but NOT in T.

$$S - T = \{x | x \in S \text{ and } x \notin T\}.$$



An alternative notation for $S - T$ is $S \backslash T$.

Examples 2A

2.2 Algebraic properties of numbers

We now discuss some of the basic properties of numbers. Although some of these properties may seem almost too obvious to mention, understanding their significance is vital to the understanding of mathematics. We begin by assuming the existence of two fundamental operations,

addition and **multiplication**.

We consider addition to mean that, for any two given numbers a and b (which may be the same), the **sum** $a + b$ exists; similarly, for multiplication, we say that the **product** $a \cdot b$ (or ab) exists. Note that although these operations can in fact be performed on several numbers at once, e.g.

$$a_1 + a_2 + a_3 + \ldots + a_n, \qquad a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n,$$

it is more convenient for us to consider adding and multiplying pairs of numbers only, and to define other sums in terms of sums of pairs of numbers.

The other fundamental idea used in the following is that of an **equation**. This is an expression where two equal quantities are displayed either side of an **equals sign**, =. If an equation is known to be true, applying any operation to both sides of the equation will produce another true equation. In particular,

1. any quantity can be added to both sides;
2. any quantity can be subtracted from both sides;
3. both sides can be multiplied by any quantity;
4. both sides can be divided by any nonzero quantity.

2.2.1 Basic properties of addition

- **Associative law of addition.** This states that the placement of brackets in a sum is irrelevant.

> (P1) *If a, b and c are any numbers, then*
> $$a + (b + c) = (a + b) + c.$$

- **Existence of an additive identity.** The number 0 is called the **additive identity**, that is, adding it to a number leaves the number unchanged.

> (P2) *If a is any number,*
> $$a + 0 = 0 + a = a.$$

- **Existence of additive inverses.** When added, a number and its **additive inverse** sum to zero (the additive identity).

> (P3) *For every number a, there is number $-a$ such that*
> $$a + (-a) = (-a) + a = 0.$$

- **Commutative law for addition.** This property shows that the order in which the two numbers are added does not matter.

> (P4) *If a and b are any numbers, then*
> $$a + b = b + a.$$

One important observation here is that we can regard **subtraction** as an operation derived from addition, and think of $a - b$ as an abbreviated form of $a + (-b)$.

These basic properties are enough to let us start solving simple equations. For example, suppose we want to solve $x + 3 = 6$ for x. We may write the following:

If	$x + 3 = 6$	
then	$(x + 3) + (-3) = 6 + (-3)$	adding -3 to both sides
hence	$x + (3 + (-3)) = 6 - 3 = 3$	using (P1)
hence	$x + 0 = 3$	using (P3)
hence	$x = 3$	using (P2).

Obviously writing out all of the steps is not usually necessary but it is important to realise that we are implicitly using these basic properties of addition in such situations.

Note that there is one important result that we cannot yet show, namely, that the order of the numbers **is** important for subtraction. That is, we cannot derive the result that $a - b = b - a$ only when $a = b$ from the properties stated so far. Surprisingly, it turns out that we need a property involving multiplication to do this!

2.2.2 Basic properties of multiplication

- **Associative law of multiplication.** This states that the placement of brackets is irrelevant.

> (P5) *If a, b and c are any numbers, then*
> $$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

- **Existence of a multiplicative identity.** The number 1 is called the **multiplicative identity**, that is, multiplying a number by 1 leaves the number unchanged.

> (P6) *If a is any number,*
> $$a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$$

- **Existence of multiplicative inverses.** A number and its **multiplicative inverse** multiply to give 1 (the multiplicative identity).

> (P7) *For every $a \neq 0$, there is a number a^{-1} such that*
> $$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

- **Commutative law for multiplication.** This property shows that the order in which the two numbers are multiplied does not matter.

> (P8) *If a and b are any numbers, then*
> $$a \cdot b = b \cdot a.$$

Note that we have to state that $1 \neq 0$ in (P6) because we cannot prove it on the basis of the properties we already have (we have seen nothing yet which says we can't use 0 here instead of 1: we will see why this is the case shortly). One other point to be emphasised is that the condition 'for $a \neq 0$' is needed in (P7) because there is no number 0^{-1} such that $0 \cdot 0^{-1} = 0^{-1} \cdot 0 = 1$. This has consequences for the operation of **division**. Just as we defined subtraction in terms of addition, we will define division in terms of multiplication, so that the symbol a/b means $a \cdot b^{-1}$. This means that as 0^{-1} is meaningless, so is $a/0$, and **division by zero is always undefined**.

Two further important consequences of (P7) are

(i) if $a \cdot b = a \cdot c$ with $a \neq 0$, then $b = c$;

(ii) if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

The second of these is frequently used in solving equations. For example, if a number x is known to satisfy $(x-3)(x-5) = 0$, then it follows that either $x - 3 = 0$ or $x - 5 = 0$, so $x = 3$ or $x = 5$.

These properties allow us to solve a further range of equations. For example:

If	$x \cdot 3 = 6$	
then	$(x \cdot 3) \cdot 3^{-1} = 6 \cdot 3^{-1} = 2$	multiplying both sides by $\frac{1}{3}$
hence	$x \cdot (3 \cdot 3^{-1}) = 2$	using (P5)
hence	$x \cdot 1 = 2$	using (P7)
hence	$x = 2$	using (P6).

Again, in practice we would not normally detail all of these steps.

2.2.3 The distributive law

So far we have listed four properties of addition and four properties of multiplication which allow us to prove a very limited range of results. The inclusion of the **distributive law**, which combines both operations, changes this situation drastically.

(P9) *If a, b and c are any numbers, then*

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Note that although brackets do not change anything in a sum or a product, here they play a crucial role.

We can use this result to verify many important results. In addition, it is the basis of almost all algebraic manipulations.

Examples 2B

2.3 Sets of numbers

2.3.1 Natural numbers

The simplest set of numbers is the set of *counting numbers*

$$\boxed{\mathbb{N} = \{1, 2, 3, 4, \ldots\}.}$$

It is represented by the symbol $\mathbb{N}$, for *natural numbers*. Notice that not all of the basic properties in Section 2.2 hold in $\mathbb{N}$: for example, properties (P2) and (P3) don't make sense here.

2.3.2 Integers

Early civilisations soon realised that counting numbers were not enough to deal with some everyday situations. The concept of zero was first used by the Greeks to represent the absence of a number, and the Hindus introduced negative numbers to represent debt. This extension of $\mathbb{N}$ leads to the set of **integers**

$$\boxed{\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}.}$$

Here (P2) and (P3) do hold (additive identity is 0, additive inverse is $-a$), but (P7) still fails. The label $\mathbb{Z}$ comes from *Zahl*, which is German for *number*. Note that $\mathbb{N}$ is sometimes denoted by $\mathbb{Z}^+$ (and called the set of positive integers).

2.3.3 Rational numbers

We can obtain a larger set of numbers by taking ratios or **quotients** of the form a/b where a and b are integers with $b \neq 0$. This gives

$$\boxed{\mathbb{Q} = \left\{ \frac{a}{b} : a, b, \in \mathbb{Z}, b \neq 0 \right\}}$$

which is the set of **rational numbers** (denoted by $\mathbb{Q}$ for *quotient*). Note that a given rational number may be written in many ways. For example

$$\frac{1}{2} = \frac{2}{4} = \frac{-3}{-6} = \frac{5}{10}.$$

If necessary, we can specify that the integers a and b have no common factor greater than 1 and that b is positive to produce a standard way of representing a rational number. Note that all natural numbers and integers are rational, e.g.

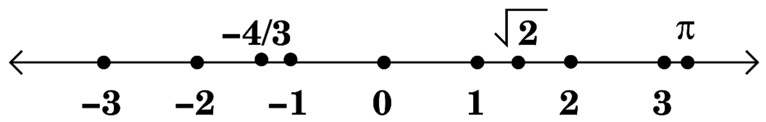
$$-8 = \frac{-8}{1} = \frac{8}{-1}.$$

All of the numbers $\frac{a}{b} \in \mathbb{Q}$ satisfy properties (P1)-(P9). We have

- **additive identity** $\frac{0}{b}$ (see (P2))
- **additive inverse** $\frac{-a}{b}$ (see (P3))
- **multiplicative identity** $\frac{1}{1}$ (see (P6))
- **multiplicative inverse** $\frac{b}{a}$ (see (P7)).

2.3.4 Real numbers

In addition to $\mathbb{Q}$, there is an even larger set of numbers for which our algebraic properties hold, namely the set of all **real numbers** (denoted by $\mathbb{R}$). As well as containing all rational numbers (i.e. all members of $\mathbb{Q}$), $\mathbb{R}$ also contains **irrational** numbers. These are numbers which CANNOT be written as a ratio of integers. We can think of the real numbers as lying on the **real number line**



where any real number can be represented by a point on the line.

We observe that all real numbers can be represented by infinite decimal expansions: e.g.

$$\begin{aligned}
2 &= 2.000000000000000\ldots \\
2 &= 1.999999999999999\ldots \\
-\frac{4}{3} &= -1.333333333333333\ldots \\
\sqrt{2} &= 1.414213562373095\ldots \\
\pi &= 3.141592653589793\ldots
\end{aligned}$$

If a number is rational, its decimal expansion either terminates (i.e. ends in a string of zeros) or is repeating (i.e. ends with a string of digits repeating endlessly). Irrational numbers such as $\sqrt{2}$ and π do not satisfy either of these conditions, their decimal expansions never end or repeat (for example, currently the value of π is known to 10^{12} decimal places but no further!). We will see a proof that $\sqrt{2}$ is irrational later in the course.

As mentioned above, all numbers $a \in \mathbb{R}$ satisfy properties (P1)-(P9). We have

- **additive identity** 0 (see (P2))
- **additive inverse** $-a$ (see (P3))
- **multiplicative identity** 1 (see (P6))
- **multiplicative inverse** $\frac{1}{a}$ (see (P7)).

The property possessed by $\mathbb{R}$ but not by $\mathbb{Q}$ is called **completeness**. This is more subtle than (P1)-(P9) so we will not define it in a rigorous way here. Roughly speaking, it says that there are no holes or gaps in the real line, that is, every point corresponds to a real number.

Note that we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

2.3.5 Complex numbers

Not even the set of real numbers is large enough to contain solutions to some relatively simple equations. For example, no real number x satisfies $x^2 + 1 = 0$. Historically, this caused many difficulties for mathematicians, and led to the introduction of **complex numbers**. We will not discuss these numbers in this class, but you will meet them in MM102. sometimes they are denoted names can be a bit $\mathbb{R} \subset \mathbb{C}$, difficulty by part). learn

2.4 Order properties of real numbers

The final three basic properties of real numbers which we will discuss on this course will enable us to make statements about the **order** of numbers on the real line. These three basic **order properties** of real numbers are: reverse of

- **Trichotomy law.** Any real number is either positive, negative or zero.

> (P10) *For every real number a, only one of the following properties holds:*
> (i) $a = 0$; (ii) a is positive; (iii) $-a$ is positive.

- **Closure under addition.** Adding two positive real numbers gives a positive real number.

> (P11) *If a and b are positive, then $a + b$ is positive.*

- **Closure under multiplication.** Multiplying two positive real numbers gives a positive real number.

> (P12) *If a and b are positive, then $a \cdot b$ is positive.*

Ordering of real numbers is usually denoted using **inequalities**, that is, expressions involving $<$, $\leq$, $>$, $\geq$.

Definition 2.3 *For any two real numbers a and b, we have that*

(i) $a > b$ *if* $a - b$ *is positive;*

(ii) $a < b$ *if* $b > a$*;*

(iii) $a \geq b$ *if* $a > b$ *or* $a = b$*;*

(iv) $a \leq b$ *if* $a < b$ *or* $a = b$*;*

Many other elementary properties of inequalities follow from these. For example, suppose we have two real numbers $a < 0$ and $b < 0$. As $a < 0$, $0 - a = -a$ is positive, and a similar argument shows that $-b$ is also positive. Thus (P12) means that the product $(-a)(-b) = ab$ must be positive. That is, we have shown that the product of two negative real numbers is always positive.

Note that inequalities can be combined provided that they **all** involve **either** $>$ and $\geq$ **or** $<$ and $\leq$. For example,

- $a < b \leq c$ means $a < b$ and $b \leq c$, from which we can deduce that $a < c$;
- $a \geq b \geq c > d \geq e$ implies $a > e$ ($>$ because of the strict inequality between c and d).

However, we CANNOT deduce anything from something like $a < b > c$, because knowing that $a < b$ and $b > c$ does not let us compare a and c (we could have $a < c$, $a = c$ or $a > c$).

Examples 2C

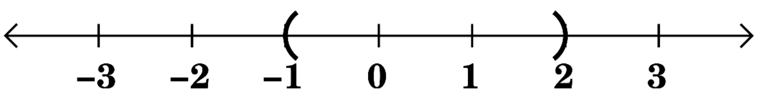
2.5 Intervals on the real line

In calculus, we deal mostly with sets of real numbers. Often the sets can be expressed in terms of **intervals** on the real line. An interval of the real line is a set of numbers lying between two values. We use brackets to denote the ends of the interval.

Round brackets denote that the endpoints are not included:

$$(a, b) \text{ denotes the set } \{x \in \mathbb{R} : a < x < b\}.$$

This is the set of points lying strictly between a and b, not including a and b (i.e. the intersection of the set of points greater than a and the set of points less than b). For example, $(-1, 2)$ denotes the set $\{x \in \mathbb{R} : -1 < x < 2\}$.

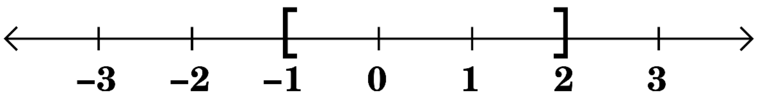


Intervals of the form (a, b) are called **open intervals**.

Square brackets denote that the endpoints of the interval are included:

$$[a, b] \text{ denotes the set } \{x \in \mathbb{R} : a \leq x \leq b\}.$$

This is the set of points lying between a and b, including a and b (i.e. the intersection of the set of points greater than or equal to a and the set of points less than or equal to b). So $[-1, 2]$ denotes the set $\{x \in \mathbb{R} : -1 \leq x \leq 2\}$.



Intervals of the form $[a, b]$ are called **closed intervals**.

We can also have a combination of round and square brackets:

$$[a, b) \text{ denotes the set } \{x \in \mathbb{R} : a \leq x < b\}$$

and

$$(a, b] \text{ denotes the set } \{x \in \mathbb{R} : a < x \leq b\}.$$

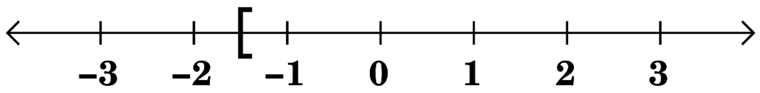
We use the symbol ∞ (for infinity) and a round bracket to specify unbounded intervals:

$$(-\infty, a] \text{ denotes the set } \{x \in \mathbb{R} : x \leq a\}.$$

and

$$(a, \infty) \text{ denotes the set } \{x \in \mathbb{R} : a < x\}.$$

For example, the interval $[-1.5, \infty)$ can be represented as



Examples 2D

2.6 Modulus

One concept which plays an extremely important role in mathematics is the **modulus** (or **absolute value** or **magnitude**) of a number.

Definition 2.4 *For any real number a, we define the* ***modulus*** *$|a|$ of a as*

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0. \end{cases}$$

That is, **taking the modulus of a number removes the minus sign if there is one**. Note that $|a| \geq 0$ for all $a \in \mathbb{R}$ and $|a| = 0$ if and only if $a = 0$. In other words, **the modulus (or absolute value or magnitude) of a number is always positive**.

In fact, the modulus is the (positive) **distance** between the real number a and the point 0 on the real number line. More generally, for any $a, b \in \mathbb{R}$,

$$|a - b| = \text{ distance between } a \text{ and } b \text{ on the real number line.}$$

This interpretation of the modulus will be vital when we solve inequalities later in the course.

We can also give an alternative definition of $|a|$ as

$$|a| = \sqrt{a^2}.$$

Important note: For every positive number a, we use the notation $\sqrt[n]{a}$ to mean the unique **nonnegative** nth root of a. In particular, that means that $\sqrt{a}$ always means the **nonnegative** square root of a.

Examples 2E

Summary

(P1)	**Associative law for addition**	$a + (b + c) = (a + b) + c$
(P2)	**Existence of an additive identity**	$a + 0 = 0 + a = a$
(P3)	**Existence of additive inverses**	$a + (-a) = (-a) + a = 0$
(P4)	**Commutative law for addition**	$a + b = b + a$

(P5)	**Associative law for multiplication**	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
(P6)	**Existence of a multiplicative identity**	$a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0$
(P7)	**Existence of multiplicative inverses**	$a \cdot a^{-1} = a^{-1} \cdot a = 1$ for $a \neq 0$
(P8)	**Commutative law for multiplication**	$a \cdot b = b \cdot a$

(P9)	**Distributive law**	$a \cdot (b + c) = a \cdot b + a \cdot c$

(P10) **Trichotomy law**

For every real number a, only one of the following properties holds:

(i) $a = 0$; (ii) a is positive; (iii) $-a$ is positive.

(P11) **Closure under addition** If a and b are positive, then $a + b$ is positive.

(P12) **Closure under multiplication** If a and b are positive, then $a \cdot b$ is positive.