University of Strathclyde, Department of Mathematics and Statistics

MM102 Applications of Calculus
Exercises for Week 1
Solutions

Q1. In all cases the rational functions are proper. With constants $A, B, \ldots$ one has:

1(a) $\dfrac{2x+3}{(x-3)(x+5)} = \boxed{\dfrac{A}{x-3} + \dfrac{B}{x+5}}$

1(b) The factor x^2-1 is not irreducible ($\beta^2-4\gamma = 0^2-4\times(-1) = 4 \geq 0$) and can be factorised: $x^2-1 = (x+1)(x-1)$. Hence

$$\frac{2x+3}{(x^2-1)(x-1)} = \frac{2x+3}{(x+1)(x-1)^2} = \boxed{\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}}$$

1(c) $\dfrac{x^4+4x^3+2}{(x+2)^3(x-1)^2} = \boxed{\dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3} + \dfrac{D}{x-1} + \dfrac{E}{(x-1)^2}}$

1(d) The factor x^2+x+4 is irreducible since $\beta^2-4\gamma = 1^2-4\times 4 = -15 < 0$. Hence

$$\frac{5x^2+1}{(x^2+x+4)(x-2)(x+4)} = \boxed{\frac{Ax+B}{x^2+x+4} + \frac{C}{x-2} + \frac{D}{x+4}}$$

1(e) The factor x^2-2x+5 is irreducible since $\beta^2-4\gamma = (-2)^2-4\times 5 = -16 < 0$. Hence

$$\frac{3}{(x^2-2x+5)^2(x+3)^3}$$
$$= \boxed{\frac{Ax+B}{x^2-2x+5} + \frac{Cx+D}{(x^2-2x+5)^2} + \frac{E}{x+3} + \frac{F}{(x+3)^2} + \frac{G}{(x+3)^3}}$$

1(f) The factor x^2-x+4 is irreducible since $\beta^2-4\gamma = (-1)^2-4\times 4 = -15 < 0$. Hence

$$\frac{5x^5+4x^2+3}{(x^2-x+4)^3(x-1)(x+2)^2}$$
$$= \boxed{\frac{Ax+B}{x^2-x+4} + \frac{Cx+D}{(x^2-x+4)^2} + \frac{Ex+F}{(x^2-x+4)^3} + \frac{G}{x-1} + \frac{H}{x+2} + \frac{J}{(x+2)^2}}$$

Q2. 2(a) $\displaystyle\int \frac{2x+8}{x^2-1}\,\mathrm{d}x$

Solution:
The fraction is proper and the denominator can be factorised:
$x^2-1 = (x+1)(x-1)$. Hence

$$\frac{2x+8}{x^2-1} = \frac{2x+8}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

with some constants $A, B \in \mathbb{R}$. To find the constants, we multiply by the denominator:

$$2x+8 = A(x-1) + B(x+1).$$

We set x equal to certain values:

$$\begin{aligned} x = -1: &\qquad 6 = -2A \quad \Rightarrow \quad A = -3 \\ x = 1: &\qquad 10 = 2B \quad \Rightarrow \quad B = 5. \end{aligned}$$

Hence

$$\int \frac{2x+8}{x^2-1}\,\mathrm{d}x = \int \left(\frac{5}{x-1} - \frac{3}{x+1} \right) \mathrm{d}x = \boxed{5\ln|x-1| - 3\ln|x+1| + C}$$

2(b) $\displaystyle\int \frac{x^4+x^3-x^2+2x+3}{x^2+x-2}\,\mathrm{d}x$

Solution:
Since the fraction is improper, we have to do long division first:

$$\begin{array}{r|l} & \phantom{x^4 + x^3 -{}} x^2 \qquad\quad + 1 \\ \hline x^2+x-2 & x^4 + x^3 - x^2 + 2x + 3 \\ & x^4 + x^3 - 2x^2 \\ \hline & \phantom{x^4 + x^3 -{}} x^2 + 2x + 3 \\ & \phantom{x^4 + x^3 -{}} x^2 + x - 2 \\ \cline{2-2} & \phantom{x^4 + x^3 - x^2 + 2{}} x + 5 \end{array}$$

Hence

$$\frac{x^4+x^3-x^2+2x+3}{x^2+x-2} = x^2+1+\frac{x+5}{x^2+x-2} = x^2+1+\frac{x+5}{(x-1)(x+2)}.$$

For the last fraction we use partial fraction decomposition:

$$\frac{x+5}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

We multiply by the denominator:

$$x+5 = A(x+2) + B(x-1)$$

and set x equal to certain values:

$$\begin{aligned} x = 1: &\qquad 6 = 3A \quad \Rightarrow \quad A = 2 \\ x = -2: &\qquad 3 = -3B \quad \Rightarrow \quad B = -1. \end{aligned}$$

Hence

$$\int \frac{x^4+x^3-x^2+2x+3}{x^2+x-2}\,\mathrm{d}x = \int \left(x^2 + 1 + \frac{2}{x-1} - \frac{1}{x+2} \right) \mathrm{d}x$$

$$= \boxed{\frac{x^3}{3} + x + 2\ln|x-1| - \ln|x+2| + C}$$

2(c) $\displaystyle\int \frac{2x-11}{x^2-x-6}\,\mathrm{d}x$

Solution:

The fraction is proper and the denominator can be factorised:
$x^2 - x - 6 = (x+2)(x-3)$. Hence

$$\frac{2x-11}{x^2-x-6} = \frac{2x-11}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

with some constants $A, B \in \mathbb{R}$. To find the constants, we multiply by the denominator:

$$2x - 11 = A(x-3) + B(x+2).$$

We set x equal to certain values:

$$\begin{aligned} x = -2: &\qquad -15 = -5A \quad \Rightarrow \quad A = 3 \\ x = 3: &\qquad -5 = 5B \quad \Rightarrow \quad B = -1. \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^1 \frac{2x-11}{x^2-x-6}\,\mathrm{d}x = \int_0^1 \left(\frac{3}{x+2} - \frac{1}{x-3}\right)\mathrm{d}x \\ &= \Big[3\ln|x+2| - \ln|x-3|\Big]_0^1 \\ &= 3\ln|1+2| - \ln|1-3| - \big(3\ln|0+2| - \ln|0-3|\big) \\ &= 3\ln 3 - \ln 2 - 3\ln 2 + \ln 3 = 4(\ln 3 - \ln 2) = \boxed{4\ln\frac{3}{2}} \end{aligned}$$

2(d) $\displaystyle\int \frac{x^2-2x+10}{(x^2+4)(x-3)}\,\mathrm{d}x$

Solution:

Since the fraction is proper and x^2+4 is irreducible over $\mathbb{R}$, we have the following partial fraction decomposition:

$$\frac{x^2-2x+10}{(x^2+4)(x-3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-3}$$

with some constants $A, B, C \in \mathbb{R}$. To find the constants, we multiply by the denominator:

$$\begin{aligned} x^2 - 2x + 10 &= (Ax+B)(x-3) + C(x^2+4) \\ &= (A+C)x^2 + (B-3A)x - 3B + 4C. \end{aligned}$$

If we set $x = 3$, we obtain $13 = 13C$ and hence $C = 1$. If we compare the coefficients of x^2 and the constants terms, respectively, we obtain:

$$\begin{aligned} \text{coeff. of } x^2: &\qquad 1 = A + C \qquad \Rightarrow \quad A = 0 \\ \text{constant terms:} &\qquad 10 = -3B + 4C \quad \Rightarrow \quad B = -2. \end{aligned}$$

Hence

$$\int \frac{x^2-2x+10}{(x^2+4)(x-3)}\,\mathrm{d}x = \int\left(-\frac{2}{x^2+4}+\frac{1}{x-3}\right)\mathrm{d}x$$
$$= \boxed{-\arctan\frac{x}{2}+\ln|x-3|+C}$$

2(e) $\displaystyle\int_2^5 \frac{7x^2-15x+28}{(x^2-4x+13)(x-1)}\,\mathrm{d}x$

Solution:
The integrand is a proper rational function. Moreover, the factor $x^2-4x+13$ is irreducible over $\mathbb{R}$ since $\beta^2-4\gamma=(-4)^2-4\times 13=-49<0$. Hence we have the following partial fraction decomposition:

$$\frac{7x^2-15x+28}{(x^2-4x+13)(x-1)} = \frac{Ax+B}{x^2-4x+13}+\frac{C}{x-1}$$

with some constants $A, B, C \in \mathbb{R}$. To find the constants, we multiply by the denominator:

$$\begin{aligned}7x^2-15x+28 &= (Ax+B)(x-1)+C(x^2-4x+13)\\ &= Ax^2+Bx-Ax-B+Cx^2-4Cx+13C\\ &= (A+C)x^2+(B-A-4C)x+13C-B.\end{aligned}$$

Now

$$\begin{array}{lllll} x=1: & 20=10C & \Rightarrow & C=2 & \\ \text{constant terms:} & 28=13C-B & \Rightarrow & 28=26-B & \Rightarrow\quad B=-2\\ \text{coeff. of } x^2\text{:} & 7=A+C & \Rightarrow & 7=A+2 & \Rightarrow\quad A=5. \end{array}$$

Hence

$$\begin{aligned}\int_2^5 \frac{7x^2-15x+28}{(x^2-4x+13)(x-1)}\,\mathrm{d}x &= \int_2^5\left(\frac{5x-2}{x^2-4x+13}+\frac{2}{x-1}\right)\mathrm{d}x\\ &= \underbrace{\int_2^5 \frac{5x-2}{x^2-4x+13}\,\mathrm{d}x}_{=:I_1}+\underbrace{\int_2^5 \frac{2}{x-1}\,\mathrm{d}x}_{=:I_2}.\end{aligned}$$

For I_1 we complete the square in the denominator:

$$x^2-4x+13=(x-2)^2-2^2+13=(x-2)^2+9$$

and hence use the substitution

$$\begin{array}{lll} u=x-2, & & \mathrm{d}u=\mathrm{d}x,\\ x=2 & \Longrightarrow & u=0,\\ x=5 & \Longrightarrow & u=3, \end{array}$$

which yields

$$
\begin{aligned}
I_1 &= \int_2^5 \frac{5x-2}{(x-2)^2+9}\,\mathrm{d}x = \int_0^3 \frac{5(u+2)-2}{u^2+9}\,\mathrm{d}u = \int_0^3 \frac{5u+8}{u^2+9}\,\mathrm{d}u \\
&= \int_0^3 \frac{5u}{u^2+9}\,\mathrm{d}u + \int_0^3 \frac{8}{u^2+3^2}\,\mathrm{d}u \\
&\qquad \left[\begin{array}{lllll} \text{for the first integral use the substitution:} & & & & \\ v = u^2+9 & \Longrightarrow & \dfrac{\mathrm{d}v}{\mathrm{d}u} = 2u & \Longrightarrow & u\,\mathrm{d}u = \dfrac{1}{2}\,\mathrm{d}v \\ u = 0 & \Longrightarrow & v = 9 & & \\ u = 3 & \Longrightarrow & v = 18 & & \end{array}\right] \\
&= \frac{5}{2}\int_9^{18} \frac{1}{v}\,\mathrm{d}v + \int_0^3 \frac{8}{u^2+3^2}\,\mathrm{d}u \\
&= \frac{5}{2}\Big[\ln|v|\Big]_9^{18} + \frac{8}{3}\left[\arctan\Big(\frac{u}{3}\Big)\right]_0^3 \\
&= \frac{5}{2}\big(\ln 18 - \ln 9\big) + \frac{8}{3}\big(\arctan 1 - \arctan 0\big) \\
&= \frac{5}{2}\ln\frac{18}{9} + \frac{8}{3}\cdot\frac{\pi}{4} = \frac{5}{2}\ln 2 + \frac{2\pi}{3}\,.
\end{aligned}
$$

The second integral yields

$$
I_2 = \int_2^5 \frac{2}{x-1}\,\mathrm{d}x = 2\Big[\ln|x-1|\Big]_2^5 = 2\big(\ln 4 - \ln 1\big) = 2\ln(2^2) = 4\ln 2.
$$

Combining these two integrals we obtain

$$
\int_2^5 \frac{7x^2-15x+28}{(x^2-4x+13)(x-1)}\,\mathrm{d}x = I_1 + I_2 = \frac{5}{2}\ln 2 + \frac{2\pi}{3} + 4\ln 2 = \boxed{\frac{13}{2}\ln 2 + \frac{2\pi}{3}}
$$

2(f) $\displaystyle\int \frac{2x^2-3x}{(x-2)^2(x-1)}\,\mathrm{d}x$

Solution:

The integrand is a proper rational function. Hence we have the following partial fraction decomposition:

$$
\frac{2x^2-3x}{(x-2)^2(x-1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x-1}
$$

with some constants $A, B, C \in \mathbb{R}$. To find the constants, we multiply by the denominator:

$$
2x^2 - 3x = A(x-2)(x-1) + B(x-1) + C(x-2)^2.
$$

Now

$$
\begin{aligned}
x = 2: &\quad 2 = B \\
x = 1: &\quad -1 = C \\
x = 0: &\quad 0 = 2A - B + 4C \quad\Longrightarrow\quad 0 = 2A - 2 - 4 \quad\Longrightarrow\quad A = 3.
\end{aligned}
$$

Hence

$$\int \frac{2x^2-3x}{(x-2)^2(x-1)}\,\mathrm{d}x = \int\left(\frac{3}{x-2}+\frac{2}{(x-2)^2}-\frac{1}{x-1}\right)\mathrm{d}x$$
$$= \boxed{3\ln|x-2|-\frac{2}{x-2}-\ln|x-1|+C}$$

2(g) $\displaystyle\int \frac{3x^2+13x-2}{(x+1)^2(x-3)}\,\mathrm{d}x$

Solution:
The integrand is a proper rational function. Hence we have the following partial fraction decomposition:

$$\frac{3x^2+13x-2}{(x+1)^2(x-3)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-3}$$

with some constants $A, B, C \in \mathbb{R}$. To find the constants, we multiply by the denominator:

$$3x^2+13x-2 = A(x+1)(x-3)+B(x-3)+C(x+1)^2.$$

Now

$$\begin{aligned}
&x=3: && 64 = 16C && \Longrightarrow && C=4\\
&x=-1: && -12=-4B && \Longrightarrow && B=3\\
&x=0: && -2=-3A-3B+C && \Longrightarrow && -2=-3A-9+4 \quad\Longrightarrow\quad A=-1\\
&(\text{or coeff. of } x^2: && 3=A+C && \Longrightarrow && 3=A+4 \quad\Longrightarrow\quad A=-1)
\end{aligned}$$

Hence

$$\int \frac{3x^2+13x-2}{(x+1)^2(x-3)}\,\mathrm{d}x = \int\left(-\frac{1}{x+1}+\frac{3}{(x+1)^2}+\frac{4}{x-3}\right)\mathrm{d}x$$
$$= \boxed{-\ln|x+1|-\frac{3}{x+1}+4\ln|x-3|+C}$$

2(h) $\displaystyle\int \frac{3x^2+4x}{(x^2+2x+2)(x-2)}\,\mathrm{d}x$

Solution:
The integrand is a proper rational function. Moreover, the factor x^2+2x+2 is irreducible over $\mathbb{R}$ since $\beta^2-4\gamma = 2^2-4\times 2=-4<0$. Hence we have the following partial fraction decomposition:

$$\frac{3x^2+4x}{(x^2+2x+2)(x-2)} = \frac{Ax+B}{x^2+2x+2}+\frac{C}{x-2}$$

with some constants $A, B, C \in \mathbb{R}$. To find the constants, we multiply by the denominator:

$$\begin{aligned}
3x^2+4x &= (Ax+B)(x-2)+C(x^2+2x+2)\\
&= Ax^2+Bx-2Ax-2B+Cx^2+2Cx+2C\\
&= (A+C)x^2+(B-2A+2C)x-2B+2C.
\end{aligned}$$

Now

$$\begin{array}{lllllll}
x = 2: & 20 = 10C & \Longrightarrow & C = 2 & & \\
\text{constant terms:} & 0 = -2B + 2C & \Longrightarrow & 0 = -2B + 4 & \Longrightarrow & B = 2 \\
\text{coeff. of } x^2\text{:} & 3 = A + C & \Longrightarrow & 3 = A + 2 & \Longrightarrow & A = 1.
\end{array}$$

Hence

$$\begin{aligned}
\int \frac{3x^2 + 4x}{(x^2 + 2x + 2)(x - 2)} \, \mathrm{d}x &= \int \left(\frac{x + 2}{x^2 + 2x + 2} + \frac{2}{x - 2} \right) \mathrm{d}x \\
&= \underbrace{\int \frac{x + 2}{x^2 + 2x + 2} \, \mathrm{d}x}_{=:I_1} + \underbrace{\int \frac{2}{x - 2} \, \mathrm{d}x}_{=:I_2} .
\end{aligned}$$

For I_1 we complete the square in the denominator:

$$x^2 + 2x + 2 = (x + 1)^2 - 1^2 + 2 = (x + 1)^2 + 1$$

and hence use the substitution $u = x + 1$, $\mathrm{d}u = \mathrm{d}x$, which yields

$$\begin{aligned}
I_1 &= \int \frac{x + 2}{(x + 1)^2 + 1} \, \mathrm{d}x = \int \frac{(u - 1) + 2}{u^2 + 1} \, \mathrm{d}u = \int \frac{u + 1}{u^2 + 1} \, \mathrm{d}u \\
&= \int \frac{u}{u^2 + 1} \, \mathrm{d}u + \int \frac{1}{u^2 + 1} \, \mathrm{d}u \\
&\qquad \left[\begin{array}{lllll} \text{for the first integral use the substitution:} & & & & \\ v = u^2 + 1 & \Longrightarrow & \dfrac{\mathrm{d}v}{\mathrm{d}u} = 2u & \Longrightarrow & u \, \mathrm{d}u = \dfrac{1}{2} \, \mathrm{d}v \end{array} \right] \\
&= \frac{1}{2} \int \frac{1}{v} \, \mathrm{d}v + \int \frac{1}{u^2 + 1^2} \, \mathrm{d}u \\
&= \frac{1}{2} \ln |v| + \arctan u = \frac{1}{2} \ln(u^2 + 1) + \arctan u + C_1 \\
&= \frac{1}{2} \ln\big((x + 1)^2 + 1\big) + \arctan(x + 1) + C_1.
\end{aligned}$$

The second integral yields

$$I_2 = \int \frac{2}{x - 2} \, \mathrm{d}x = 2 \ln |x - 2| + C_2.$$

Combining these two integrals we obtain

$$\begin{aligned}
&\int \frac{3x^2 + 4x}{(x^2 + 2x + 2)(x - 2)} \, \mathrm{d}x = I_1 + I_2 \\
&= \boxed{\frac{1}{2} \ln\big(x^2 + 2x + 2\big) + \arctan(x + 1) + 2 \ln |x - 2| + C}
\end{aligned}$$