13 Integrals

13.1 If $[0,1]$ is divided into n equal subintervals then each strip has width $1/n$. The height of the first strip is $(1/n)^3$, the height of the second one is $(2/n)^3$, and so on, so the total area covered by the strips is

$$A_n = \frac{1}{n}\left(\frac{1^3}{n^3} + \frac{2^3}{n^3} + \frac{3^3}{n^3} + \ldots + \frac{n^3}{n^3}\right) = \frac{1}{n^4}\sum_{k=1}^{n} k^3.$$

From Chapter 7, we know that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

so we have

$$A_n = \frac{n^2(n+1)^2}{4n^4} = \frac{(n+1)^2}{4n^2}.$$

In the limit as $n \to \infty$ we get

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2} = \lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{4}$$

as required.

13.2 Applying Theorem 13.1 to the interval $[a,d]$ for the point $b \in (a,d)$ shows that f is integrable on $[b,d]$. Similarly, applying Theorem 13.1 to the interval $[b,d]$ for the point $c \in (b,d)$ shows that f is integrable on $[b,c]$.

13.3 Let

$$I = \int_a^b \left(\int_c^d f(x)g(y)\,\mathrm{d}y\right)\mathrm{d}x.$$

First consider

$$\int_c^d f(x)g(y)\,\mathrm{d}y.$$

As $f(x)$ does not depend on y, we can apply Theorem 13.2 and write

$$\int_c^d f(x)g(y)\,\mathrm{d}y = f(x)\int_c^d g(y)\,\mathrm{d}y,$$

so

$$I = \int_a^b \left(f(x)\int_c^d g(y)\,\mathrm{d}y\right)\mathrm{d}x.$$

As $\int_c^d g(y)\,\mathrm{d}y$ does not depend on x, we can apply Theorem 13.2 again to get

$$I = \left(\int_c^d g(y)\,\mathrm{d}y\right)\left(\int_a^b f(x)\,\mathrm{d}x\right),$$

so

$$I = \left(\int_a^b f\right)\left(\int_c^d g\right).$$

13.4 Consider a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$. The lower sum of f for P is given by

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$$

where m_i is the minimum function value on the interval $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$. We have $\Delta x_i \geq 0$ for each i and, if $f(x) \geq 0$ for all $x \in [a, b]$, then $m_i \geq 0$ for every i, so $L(f, P) \geq 0$. But (by Definition 13.3) the definite integral of f on $[a, b]$ satisfies $L(f, P) \leq I$ so the result is proved.

13.5 If $f(x) \geq g(x)$ for all $x \in [a, b]$ then the function $h(x) = f(x) - g(x)$ satifies $h(x) \geq 0$ for all $x \in [a, b]$. Hence, by Exercise 13.4,

$$\int_a^b h \geq 0 \iff \int_a^b (f - g) \geq 0 \iff \int_a^b f - \int_a^b g \geq 0 \iff \int_a^b f \geq \int_a^b g$$

as required.