15 Logarithms and exponentials

In this chapter we look at logarithms and exponentials. So far, we can make sense of an expression like a^b only if b is rational, not if it is irrational. This is not sufficient if we want to differentiate, for example, a^x with respect to x. To do so, we need to find a new way of defining a^x, which is consistent with the old definition but also works when $x \neq \mathbb{Q}$.

We start by reviewing the basic properties of exponentials and logarithms. Before we can extend the domain of $f(x) = a^x$ to all real numbers, we will encounter new functions: the natural logarithm ln and *the* exponential function exp. We end the chapter with an introduction to hyperbolic functions.

15.1 Reminder: exponentials and logarithms in algebra

15.1.1 Exponentials

A function f of the form $f(x) = a^x$ for $a > 0$ is called an **exponential function** with **base** a; the variable x is called the **exponent**. This must not be confused with power functions, such as $f(x) = x^a$, where the base is variable and the exponent constant. Using elementary algebraic techniques, exponential functions can only be defined for *rational* arguments.

For all $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the number a^n is defined as

$$a^n := \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}.$$

The following identities follow directly from the definition.

For all $a \in \mathbb{R}^+$ and $n, m \in \mathbb{N}$,

$$a^n a^m = a^{n+m}, \tag{15.1}$$

$$(a^n)^m = a^{nm}. \tag{15.2}$$

To make (15.1) hold also when $m = 0$, we define

$$a^0 := 1 \tag{15.3}$$

so that $a^n a^0 = a^n \cdot 1 = a^n = a^{n+0}$.
To make (15.1) hold also when $m = -n$ we define

$$a^{-n} := \frac{1}{a^n}$$

so that $a^n a^{-n} = \dfrac{a^n}{a^n} = 1 = a^0 = a^{n-n}$. With this, we now have $a^n a^m = a^{n+m}$ for all $n, m \in \mathbb{Z}$.

To make sure that

$$\underbrace{a^{1/n} \times \cdots \times a^{1/n}}_{n \text{ times}} = a^{1/n+\cdots+1/n} = a^1 = a,$$

we define

$$a^{\frac{1}{n}} := \sqrt[n]{a}.$$

Then $(a^{\frac{1}{n}})^n = (\sqrt[n]{a})^n = a$.

Finally, to ensure that

$$\underbrace{a^{1/n} \times \cdots \times a^{1/n}}_{m \text{ times}} = a^{1/n+\cdots+1/n} = a^{m/n}$$

we define

$$a^{\frac{m}{n}} := (\sqrt[n]{a})^m.$$

With this, we now have $a^p a^q = a^{p+q}$ for all $p, q \in \mathbb{Q}$.

So far, we have defined a^q for all $a > 0$ and all $q \in \mathbb{Q}$. Unfortunately, this is the end of the road for this programme: there is no obvious way we can take to make a further extension to irrational exponents.

In a similar vein, we showed in Theorem 12.4, Section 12.1.3 and Section 12.5 that

$$\frac{\mathrm{d}x^p}{\mathrm{d}x} = px^{p-1}$$

first for $p \in \mathbb{N}$, then for $p \in \mathbb{Z}$, and finally for $p \in \mathbb{Q}$. Again, irrational exponents p are out of reach.

15.1.2 Logarithms

Logarithms are the inverse functions of exponential functions. The name derives from the Greek words logos ($\lambda \acute{o} \gamma o \sigma$) and arithmos ($\dot{\alpha} \rho \iota \theta \mu \acute{o} \sigma$) for ratio and number. The name "logarithm" was coined in 1614, in a book entitled *Mirifici Logarithmorum Canonis Descriptio* ('Description of the Wonderful Law of Logarithms'), by John Napier, Baron of Merchiston, in Edinburgh.

The inverse function of the exponential function $f(x) = a^x$ delivers $x = f^{-1}(a^x)$. This function is called $\log_a$, and it is defined via

$$\log_a b = x \iff b = a^x. \tag{15.4}$$

Note that from what we know so far, here x is restricted to be a rational number.

By its definition as the inverse of the exponential, the following rules for $\log_a$ follow from (15.1) and (15.2):

$$\begin{aligned} \log_a(xy) &= \log_a x + \log_a y, & (15.5)\\ \log_a x^m &= m \log_a x. & (15.6) \end{aligned}$$

It follows that

$$\begin{aligned} \log_a 1 &= 0 \qquad (\log_a x^0 = 0 \log_a x),\\ \log_a x^{-1} &= -\log_a x, \end{aligned}$$

and so

$$\log_a \frac{x}{y} = \log_a x - \log_a y.$$

So far, since logarithms defined by (15.4) are necessarily rational, all these identities only make sense when $\log_a x$ and $\log_a y$ are rational numbers, i.e. $\log_a x, \log_a y \in \mathbb{Q}$.

15.2 The natural logarithm

We are looking for functions that work for all numbers $x \in \mathbb{R}$ in exactly the same way that exponentials and logarithms work for $x \in \mathbb{Q}$.

We start by looking for a general exponential function f. Such a function would have to satisfy

$$f(x+y) = f(x) \cdot f(y)$$

for all $x, y \in \mathbb{R}$. (This is the basic property (15.1) of an exponential.) It follows that

$$f(0) = 1$$

because, for all x, $f(x) = f(x+0) = f(x) \cdot f(0)$.

The derivative of f is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} = f(x) \cdot \lim_{h\to 0} \frac{f(h)-1}{h}.$$

If we assume that $\alpha = \lim_{h\to 0} \frac{f(h)-1}{h}$ (in particular, we assume that the limit exists!), then

$$f'(x) = \alpha f(x) \quad \forall x.$$

At $x = 0$ we find

$$f'(0) = \alpha f(0) = \alpha,$$

and so α is the derivative of f at $x = 0$.

This in itself does not give us much useful information about f or f'. However, it allows us to compute explicitly the derivative of the inverse function f^{-1} of f (the derivative of the logarithm f^{-1} associated with f):

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\alpha f(f^{-1}(x))} = \frac{1}{\alpha x}.$$

This means that the derivative of our logarithm is simply $\dfrac{1}{\alpha x}$. This observation allows us to fill a gap: we know that $\dfrac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$, and this rule delivers all kinds of powers of x as derivatives, apart from $\dfrac{1}{x} = x^{-1}$. (Recall that x^0 has derivative 0.)

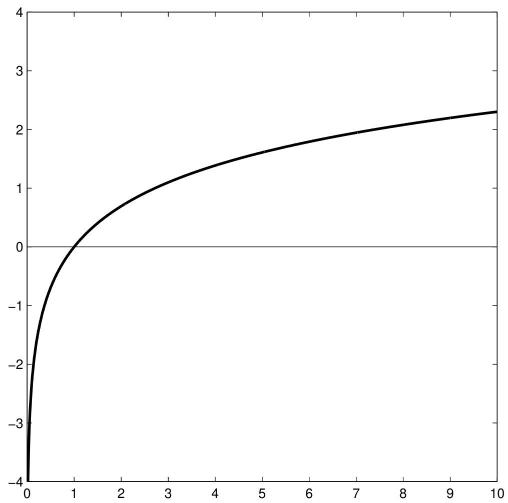
All this suggests that we *define* the logarithm in terms of the integral of $\dfrac{1}{\alpha x}$. Different values of α correspond to different exponential functions and associated logarithms. For simplicity, we choose (naturally…) $\alpha = 1$.

Definition 15.1 *The* ***natural logarithm*** ln *is defined by*

$$\ln(x) := \int_1^x \frac{1}{t}\,\mathrm{d}t, \qquad x > 0.$$

The lower limit of integration had to be chosen as 1 to ensure that $\ln(1) = 0$. Only positive values of x are allowed because of the obvious problem with the integrand at $x = 0$. The notation **ln** stems from **logarithmus naturalis**.[7]

The graph of ln is shown in the following figure.



[7]A couple of points about notation. (i) Write ln, not "In" or "in". (ii) As with the trig functions, we often omit the brackets and write $\ln x$ rather than $\ln(x)$. Only do this if there's no ambiguity about the argument of the ln function!

Notice that if $x > 1$ then $\ln x > 0$ because the integrand $\dfrac{1}{t}$ is positive, and so the integral gives the area under the curve of $\dfrac{1}{t}$ between 1 and x. If $0 < x < 1$, then $\ln x < 0$ because

$$\int_1^x \frac{1}{t}\,\mathrm{d}t = -\int_x^1 \frac{1}{t}\,\mathrm{d}t < 0.$$

Notice further that, by the First Fundamental Theorem, for $x > 0$ we have $\dfrac{\mathrm{d}\ln x}{\mathrm{d}x} = \dfrac{1}{x}$. Because $\dfrac{1}{x} > 0$ for $x > 0$, this implies that the ln function is always increasing.

Example 15.1 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(x^2 + x + 1)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x^2 + x + 1) = \frac{1}{x^2 + x + 1}\,\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + x + 1) = \frac{2x + 1}{x^2 + x + 1}.$$

Example 15.2 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\sin x)$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\cos x)$ at points x where $\sin x$ and $\cos x$ are positive.

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sin x) = \frac{\cos x}{\sin x} = \cot x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln(\cos x) = -\frac{\sin x}{\cos x} = -\tan x.$$

Example 15.3 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(ax)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}.$$

In this last example, negative values of a are allowed, provided that also x is negative so that the product ax is positive. For $a = -1$ we find a particularly interesting result.

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(-x) = \frac{1}{x}, \qquad x < 0.$$

Because $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(|x|) = \frac{1}{x}, \qquad x \neq 0. \tag{15.7}$$

The following theorem shows that ln is indeed a logarithm (see equation (15.5)).

Theorem 15.1 *If $x, y > 0$, then*

$$\ln(xy) = \ln x + \ln y.$$

Proof: Choose any $y > 0$ and define

$$f(x) := \ln(xy).$$

Then, by the chain rule,

$$f'(x) = \frac{1}{xy} \cdot y = \frac{1}{x}.$$

But this means that $f' = \ln'$, and so there is some $c \in \mathbb{R}$ such that

$$f(x) = \ln x + c \quad \forall x.$$

For $x = 1$ we get

$$\ln y = f(1) = \ln 1 + c = c,$$

and so

$$\ln(xy) = f(x) = \ln x + c = \ln x + \ln y.$$

$\square$

Corollary 15.1 *For $x, y > 0$ and $n \in \mathbb{N}$*

$$\ln \frac{x}{y} = \ln x - \ln y \tag{15.8}$$

and

$$\ln(x^n) = n \ln x. \tag{15.9}$$

Proof: The first part is easy:

$$\ln x = \ln\left(y \cdot \frac{x}{y}\right) = \ln y + \ln \frac{x}{y} \implies \ln \frac{x}{y} = \ln x - \ln y.$$

The second identity is proved by induction. It is obvious for $n = 1$. Assume that (15.9) holds for a given n. Then $\ln(x^{n+1}) = \ln(x \cdot x^n) = \ln x + \ln(x^n) = \ln x + n \ln x = (n+1) \ln x$, and so (15.9) also holds for $n + 1$. Hence by the principle of induction it is true for all n. $\square$

Remark: It is straightforward to show that $\ln(x^q) = q \ln x$ for $q \in \mathbb{Q}$ by using a proof similar to that of Theorem 15.1. However, we still cannot do anything about $\ln(x^y)$ with irrational y, simply because we do not know what x^y means in that case. (Even worse, if you try the proof you will see that we would also have to know how to differentiate x^y!) The result (15.9), valid for positive integer exponents, is all we need to prove the following corollary.

Corollary 15.2

$$\begin{array}{cccc} \ln x \to \infty & \text{as} & x \to \infty & \text{and} \\ \ln x \to -\infty & \text{as} & x \to 0. & \end{array}$$

Proof: $\ln 2 > 0$ because $2 > 1$. Consider 2^n, which tends to infinity as n tends to infinity. From (15.9), $\ln 2^n = n \ln 2$, which tends to infinity as n tends to infinity.

Likewise, $\frac{1}{2^n}$ tends to 0 as n tends to infinity. Now $\ln \frac{1}{2^n} = \ln 1 - \ln 2^n = -\ln 2^n = -n \ln 2$, which tends to minus infinity as n tends to infinity. $\square$

We already know that ln is monotonically increasing: $\ln'(x) = \dfrac{1}{x} > 0$ for all $x > 0$. This implies that ln is one-to-one on its natural domain $\mathbb{R}^+$. It tends to $-\infty$ near 0 and to $+\infty$ for $x \to \infty$, and so, because ln is continuous, its range is all of $\mathbb{R}$ and it takes every value exactly once. This means that the inverse of ln exists and has domain $\mathbb{R}$ and range $\mathbb{R}^+$.

15.3 The exponential function

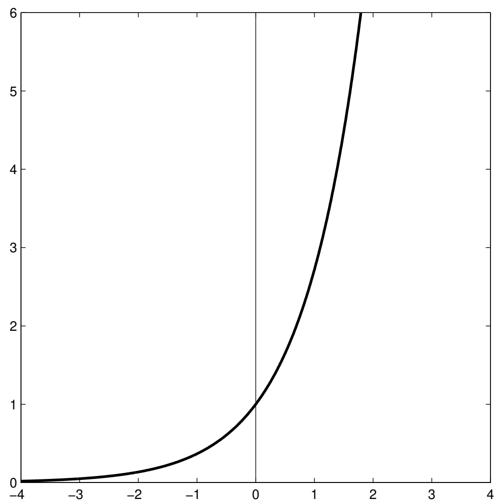
Definition 15.2 *The* ***exponential function***

$$\exp : \mathbb{R} \to \mathbb{R}^+$$

is defined as $\exp = \ln^{-1}$, *that is,*

$$\exp(x) = y \iff x = \ln y, \qquad \forall\, x \in \mathbb{R},\ y \in \mathbb{R}^+.$$

The following plot shows the graph of exp. It is, of course, a reflection of the graph of ln. The plot shows that $\exp(x) \to \infty$ as $x \to \infty$ and $\exp(x) \to 0$ as $x \to -\infty$; both these properties of exp follow directly from the corresponding properties of ln, Corollary 15.2.



Because it is the inverse function of ln, the derivative of exp is readily found.

Theorem 15.2

$$\exp'(x) = \exp(x) \qquad \forall\ x \in \mathbb{R}.$$

Proof: We have $\exp'(x) = (\ln^{-1})'(x) = \dfrac{1}{\ln'(\ln^{-1}(x))}$ using the rule for derivatives of inverse functions, Theorem 12.8. Thus

$$\exp'(x) = \frac{1}{\left(\frac{1}{\exp(x)}\right)} = \exp(x).$$

□

Example 15.4 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\exp(x^2+1)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(x^2+1) = \exp(x^2+1)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1) = 2x\exp(x^2+1).$$

Example 15.5 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\exp(\sin x)$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(\exp(x))$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(\sin x) = \cos x\ \exp(\sin x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\exp(x)) = \cos(\exp(x))\cdot\exp(x).$$

Theorem 15.3 *For any two numbers $x, y \in \mathbb{R}$,*

$$\exp(x+y) = \exp(x)\cdot\exp(y).$$

Proof: Let $\exp(x) = u$ and $\exp(y) = v$, then $x = \ln u$ and $y = \ln v$ so that

$$x + y = \ln(u) + \ln(v) = \ln(uv).$$

But this implies that

$$\exp(x+y) = \exp(\ln(uv)) = uv = \exp(x)\cdot\exp(y).$$

$\square$

Theorem 15.3 together with $\exp(0) = 1$, which follows from $\ln 1 = 0$, shows that exp does indeed behave like an 'ordinary' exponential. But what is its base? Since $a^1 = a$ for all a, we make the following definitions.

Definition 15.3

$$e := \exp(1).$$

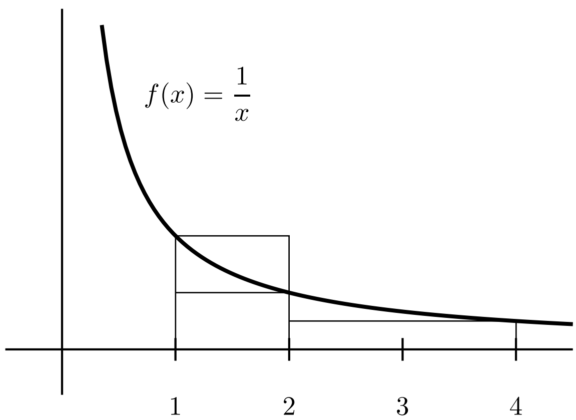
This implies that

$$1 = \ln e = \int_1^e \frac{1}{t}\,\mathrm{d}t. \tag{15.10}$$

The number e is arguably the most important number in mathematics, albeit less famous than its older relative π.

Both e and π are not only irrational, but *transcendental*. This means that they are not a zero of any polynomial with rational coefficients. None of these facts is trivial, and a proof that e is transcendental was first given by HERMITE in 1873. It took nine more years before it was also proved, by VON LINDEMANN in 1882, that π is transcendental. This result is rather spectacular: it implies that it is impossible to "square the circle".

Approximations for e can be found in numerous ways. For now, we can only resort to (15.10). We look at some simple upper and lower sums.



To find a lower bound for e, observe that

$$\int_1^2 \frac{1}{t}\,\mathrm{d}t < \int_1^2 \mathrm{d}t = 1.$$

We conclude that $e > 2$. For an upper bound, we consider

$$\int_1^4 \frac{1}{t}\,\mathrm{d}t = \int_1^2 \frac{1}{t}\,\mathrm{d}t + \int_2^4 \frac{1}{t}\,\mathrm{d}t > \frac{1}{2} + \frac{2}{4} = 1.$$

This implies $e < 4$, and so we have $2 < e < 4$. You will encounter more elegant (and far more efficient) ways to compute e later, so we stop here and note that

$$e = 2.71828\,18284\,59045\ldots$$

15.4 General exponential functions

We can now define powers with arbitrary (even irrational) exponents. We start with base e.

Definition 15.4 *For all $x \in \mathbb{R}$,*

$$e^x := \exp(x).$$

(A warning about notation: remember that e is a number and exp is a function. Do not write "$e(x)$" or "$\exp^x$", as neither of these is meaningful.)

The generalisation for arbitrary base a is straightforward, since $a = \exp(\ln a) = e^{\ln a}$. This means that, for rational x, we have $a^x = (e^{\ln a})^x = e^{x \ln a}$. But $e^{x \ln a}$ is defined for all $x \in \mathbb{R}$, and this gives rise to the following definition.

Definition 15.5 *For $a > 0$ and all $x \in \mathbb{R}$,*

$$a^x := e^{x \ln a}.$$

The following theorem shows that our definition of a^x is compatible with the 'old' concept of an exponential.

Theorem 15.4 *For all $a > 0$ and all $x, y \in \mathbb{R}$*

$$(a^x)^y = a^{xy},$$

$$a^0 = 1, \quad a^1 = a,$$

and

$$a^{x+y} = a^x \cdot a^y.$$

Furthermore,

$$\ln(a^x) = x \ln a.$$

Since this holds for all $a > 0$, it holds, in particular, for $a = e$. Hence also $\exp(x) = e^x$ is a 'proper' exponential.

Proof: All parts of the proof rely on Definition 15.5 and the fact that exp and ln are mutually inverse functions.

$$(a^x)^y = e^{y\ln(a^x)} = e^{y\ln(e^{x\ln a})} = e^{yx\ln a} = a^{xy}.$$

$$a^0 = e^{0\ln a} = e^0 = \exp(0) = 1.$$

$$a^1 = e^{1\ln a} = e^{\ln a} = a.$$

$$a^{x+y} = e^{(x+y)\ln a} = e^{x\ln a + y\ln a} = e^{x\ln a}\cdot e^{y\ln a} = a^x \cdot a^y.$$

$$\ln(a^x) = \ln(e^{x\ln a}) = x\ln a.$$

$\square$

Example 15.6 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}a^x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{x\ln a} = \ln(a)e^{x\ln a} = \ln(a)a^x.$$

Example 15.7 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}2^{x^2}$ (remember that a^{b^c} always means $a^{(b^c)}$).

$$\frac{\mathrm{d}}{\mathrm{d}x}2^{x^2} = \frac{\mathrm{d}}{\mathrm{d}x}e^{\ln(2)\,x^2} = e^{\ln(2)\,x^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\ln(2)\,x^2) = 2^{x^2}\cdot\ln(2)\cdot 2x = 2x\ln(2)\,2^{x^2}.$$

Example 15.8 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}10^{\arctan x}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}10^{\arctan x} = \frac{\mathrm{d}}{\mathrm{d}x}e^{\ln(10)\,\arctan x} = e^{\ln(10)\,\arctan x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\ln(10)\,\arctan x) = \frac{\ln 10}{1+x^2}\,10^{\arctan x}.$$

We can now also differentiate arbitrary power functions.

Theorem 15.5 *If $f(x) = x^a$ for any $a \in \mathbb{R}$, then*

$$f'(x) = ax^{a-1}.$$

Proof: With $f(x) = x^a = e^{a \ln x}$, we find

$$f'(x) = e^{a \ln x} \frac{\mathrm{d}}{\mathrm{d}x}(a \ln x) = \frac{a}{x} e^{a \ln x} = \frac{a}{x} x^a = a x^{a-1}.$$

$\square$

Example 15.9 Differentiate $x^{\sqrt{2}}$ with respect to x.

If we'd seen this in school we'd have done it without a second's thought, but we'd have been out of our depth if anyone had asked us what $x^{\sqrt{2}}$ meant. Having put in the hard work, we can now apply Theorem 15.5 and write $\frac{\mathrm{d}}{\mathrm{d}x} x^{\sqrt{2}} = \sqrt{2}\, x^{\sqrt{2}-1}$.

15.5 General logarithms

We have seen that a^x can be expressed in terms of exp as $a^x = e^{x \ln a} = \exp(x \ln a)$. Similarly, the logarithm to base a, $\log_a$, can be expressed in terms of ln. We have

$$\log_a x = y \iff x = a^y$$

and so

$$\ln x = \ln(a^y) = y \ln a,$$

which means that

$$\log_a x = y = \frac{\ln x}{\ln a}.$$

This is of course nothing else but the familiar rule

$$\log_a x = \frac{\log_b x}{\log_b a}$$

with

$$\ln x = \log_e x.$$

(Note that in mathematical contexts it is common to write simply log for $\log_e$; however, some scientists and engineers prefer to write log for $\log_{10}$. If the base of a logarithm is not specified, you may have to think carefully about the context.)

This last logarithm rule is very important in practice: pocket calculators, computer programming languages, and mathematical software packages usually only have built-in functions for ln and $\log_{10}$ (which they may abbreviate to log), and at best for $\mathrm{ld} = \log_2$. Should you ever need to compute $\log_3 27$, your calculator will happily give you the answer if you ask it for $\frac{\ln 27}{\ln 3}$. (You should probably know the answer anyway.)

Example 15.10 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\log_a x$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\log_x a$.

We have
$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x}{\ln a} = \frac{1}{x\ln a}$$
and
$$\frac{\mathrm{d}}{\mathrm{d}x}\log_x a = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln a}{\ln x} = -\frac{\ln a}{x(\ln x)^2}.$$

Example 15.11 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\log_{10} x^2$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\log_{10} x^2 = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x^2}{\ln 10} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{2\ln x}{\ln 10} = \frac{2}{x\ln 10}.$$

Example 15.12 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\log_x x$.

Since $\log_x x = 1$ for all x, its derivative is zero. Formally,
$$\frac{\mathrm{d}}{\mathrm{d}x}\log_x x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x}{\ln x} = \frac{\mathrm{d}}{\mathrm{d}x}1 = 0.$$

Example 15.13 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\log_{2x} x$.

This is surprisingly different from the previous Example!
$$\frac{\mathrm{d}}{\mathrm{d}x}\log_{2x} x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x}{\ln 2x} = \frac{\frac{1}{x}\ln 2x - \frac{1}{x}\ln x}{(\ln 2x)^2} = \frac{\ln 2}{x(\ln 2x)^2}.$$

Sometimes a bit of simplifying helps...

Example 15.14 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln\left[\dfrac{x\sqrt{x^2+1}}{2+\sin x}\right]$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\frac{x\sqrt{x^2+1}}{2+\sin x}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x) + \frac{1}{2}\ln(x^2+1) - \ln(2+\sin x)\right] \\ &= \frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - \frac{\cos x}{2+\sin x} \\ &= \frac{1}{x} + \frac{x}{x^2+1} - \frac{\cos x}{2+\sin x}.\end{aligned}$$

15.6 Logarithmic differentiation

We can sometimes differentiate complicated expressions more easily by taking the natural logarithm of the expression that we want to differentiate, and using the properties of the logarithm to simplify the expression before differentiating. This is called **logarithmic differentiation**. It relies on the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x)) = \frac{f'(x)}{f(x)},$$

and so

$$f'(x) = f(x)\frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x)).$$

Example 15.15 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}a^x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\frac{\mathrm{d}}{\mathrm{d}x}\ln(a^x) = a^x\frac{\mathrm{d}}{\mathrm{d}x}(x\ln a) = \ln(a)\,a^x$$

as before.

Example 15.16 Differentiate

$$(5x^2-1)^{\frac{1}{2}}(3-2x)^{\frac{1}{3}}$$

with respect to x.

With $f(x) = (5x^2-1)^{\frac{1}{2}}(3-2x)^{\frac{1}{3}}$ we have

$$\begin{aligned}\ln(f(x)) &= \ln\left[(5x^2-1)^{\frac{1}{2}}(3-2x)^{\frac{1}{3}}\right]\\ &= \frac{1}{2}\ln(5x^2-1) + \frac{1}{3}\ln(3-2x).\end{aligned}$$

Differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x)) = \frac{1}{2}\cdot\frac{1}{5x^2-1}\cdot(10x) + \frac{1}{3}\cdot\frac{1}{3-2x}\cdot(-2)$$

and so

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = f(x)\left[\frac{5x}{5x^2-1} - \frac{2}{3(3-2x)}\right] = (5x^2-1)^{\frac{1}{2}}(3-2x)^{\frac{1}{3}}\left[\frac{5x}{5x^2-1} - \frac{2}{3(3-2x)}\right].$$

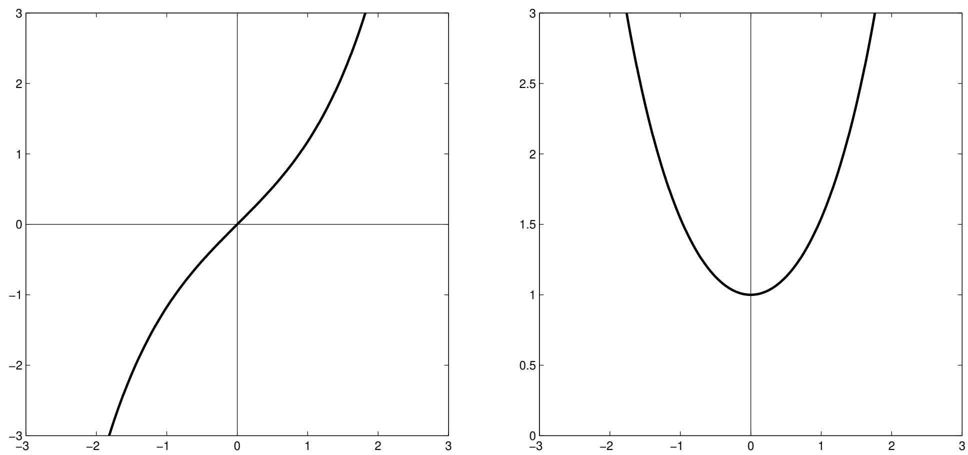
15.7 Hyperbolic functions

Certain combinations of exponential functions occur so frequently that they are given special names.

Definition 15.6 *We define the* ***hyperbolic sine****,* sinh *and the* ***hyperbolic cosine****,* cosh*, by*

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}).$$

Note that sinh is an odd function and cosh is an even function. For large positive values of x, $\sinh x \approx \cosh x \approx \frac{1}{2}e^x$, while for large negative values of x, $\sinh x \approx -\cosh x \approx -\frac{1}{2}e^{-x}$.

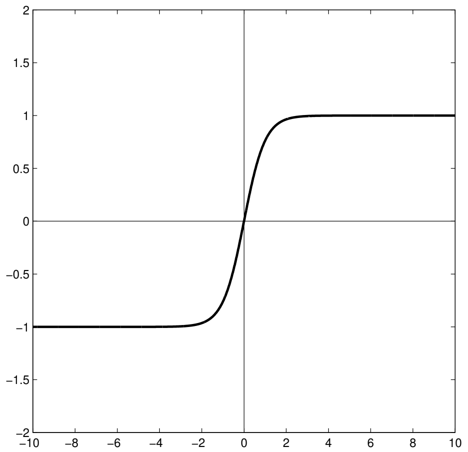


These can be used to define hyperbolic functions corresponding to each of the trigonometric functions:

Definition 15.7 *We define further hyperbolic functions as follows.*

$$\begin{aligned}
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\operatorname{sech} x &= \frac{1}{\cosh x}, \\
\operatorname{cosech} x &= \frac{1}{\sinh x}, \\
\coth x &= \frac{1}{\tanh x}.
\end{aligned}$$

Note that tanh is an odd function, and that $\tanh x \to 1$ as $x \to \infty$ and $\tanh x \to -1$ as $x \to -\infty$.



The hyperbolic sine and cosine satisfy identities similar to those satisfied by the sine and cosine.

Theorem 15.6 *For all $x \in \mathbb{R}$,*

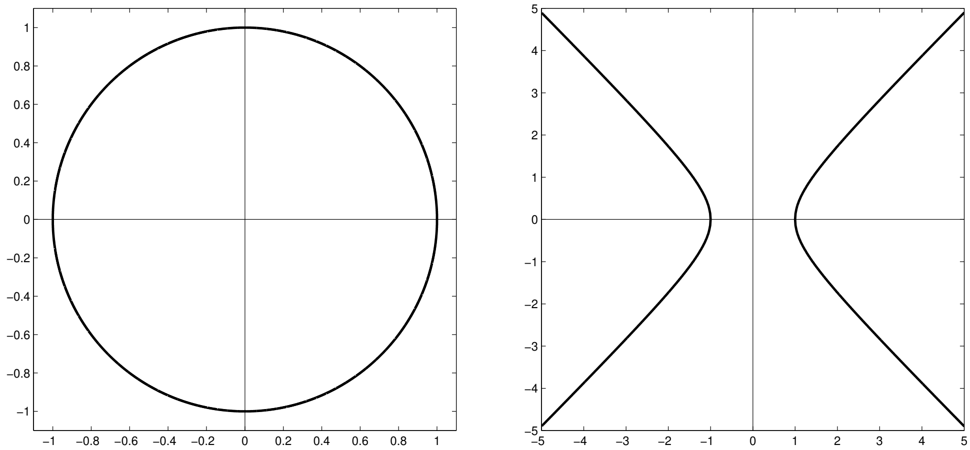
$$\cosh^2 x - \sinh^2 x = 1.$$

Proof:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{1}{2}(e^x + e^{-x})\right)^2 - \left(\frac{1}{2}(e^x - e^{-x})\right)^2 \\ &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\ &= 1.\end{aligned}$$

□

Note: The sine and cosine functions are called "circular functions" because they are related to the circle $x^2 + y^2 = 1$ via the parametric representation $x = \cos t$, $y = \sin t$: for each real t we have $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, that is, all points lie on a circle with radius 1. Similarly, using the above identity means that sinh and cosh are related to the curve $x^2 - y^2 = 1$ via the parametric representation $x = \cosh t$, $y = \sinh t$. This curve, shown in the right-hand plot below, is called a **hyperbola**, hence the name "hyperbolic functions".



15.7.1 Derivatives of hyperbolic functions

The derivatives of hyperbolic functions are easy to obtain.

Theorem 15.7

$$\sinh' = \cosh, \qquad \cosh' = \sinh, \qquad \tanh' = \mathrm{sech}^2.$$

Proof:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x;$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \sinh x;$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sinh x}{\cosh x} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \mathrm{sech}^2 x.$$

$\square$

Example 15.17 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sinh x^2)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x^2) = 2x\cosh x^2$$

Example 15.18 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}[e^x(\cosh x + \sinh x)]$.

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x(\cosh x + \sinh x)] = e^x(\cosh x + \sinh x) + e^x(\sinh x + \cosh x) = 2e^x(\cosh x + \sinh x).$$

15.8 Inverse hyperbolic functions

The functions sinh and tanh are one-to-one. Their inverse functions $\sinh^{-1}$ and $\tanh^{-1}$ are defined on $\mathbb{R}$ and $(-1,1)$, respectively. These functions are sometimes denoted by $\arg\sinh$ and $\arg\tanh$ (where 'arg' stands for 'argument') or simply by asinh and atanh. Provided that we restrict the domain of the hyperbolic cosine to $\mathbb{R}^+$, it also has an inverse, denoted by $\arg\cosh$, or simply $\cosh^{-1}$.

The inverse hyperbolic functions can be written in terms of natural logarithms.

Theorem 15.8

$$\begin{aligned} \sinh^{-1} x &= \ln(x+\sqrt{x^2+1}), && x \in \mathbb{R}; \\ \cosh^{-1} x &= \ln(x+\sqrt{x^2-1}), && x \geq 1; \\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), && -1 < x < 1. \end{aligned}$$

Proof: For example, put

$$y = \sinh^{-1} x \iff x = \sinh y = \frac{1}{2}(e^y - e^{-y}).$$

From $\cosh^2 y - \sinh^2 y = 1$, we see that

$$\cosh y = \frac{1}{2}(e^y + e^{-y}) = \sqrt{x^2+1}.$$

Adding these gives

$$x + \sqrt{x^2+1} = e^y \implies y = \sinh^{-1} x = \ln(x+\sqrt{x^2+1}).$$

The inverses of cosh and tanh follow in the same way. □

Derivatives of inverse hyperbolic functions We can obtain the derivatives either from the logarithmic representation or in the same way as for inverse trigonometric functions.

Theorem 15.9

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \qquad \textit{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x = \frac{1}{1-x^2}.$$

Proof: For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}x = \frac{\mathrm{d}}{\mathrm{d}x}\ln(x+\sqrt{x^2+1}) = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}\,\frac{\sqrt{x^2+1}+x}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}.$$

Alternatively, this result can be derived without resorting to the logarithmic representation of $\sinh^{-1}$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}x &= \frac{1}{\sinh'(\sinh^{-1}(x))} = \frac{1}{\cosh(\sinh^{-1}(x))} = \frac{1}{\sqrt{\sinh^2(\sinh^{-1}(x))+1}}\\ &= \frac{1}{\sqrt{x^2+1}}.\end{aligned}$$

The results for $\cosh^{-1}$ and $\tanh^{-1}$ follow in the same way. □

Example 15.19 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}(x^2)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}(x^2) = \frac{1}{\sqrt{(x^2)^2+1}}\cdot 2x = \frac{2x}{\sqrt{x^4+1}}.$$

Example 15.20 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(6x+5)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(6x+5) = \frac{1}{1-(6x+5)^2}\cdot 6 = \frac{6}{-36x^2-60x-24} = -\frac{1}{2(3x^2+5x+2)}.$$