Exercises and solutions for MM101 Tutorial in Week 4

1. Find the inverse functions of the following one-to-one functions:

$$\text{(a)}\quad f(x) = 7x - 4, \qquad\qquad \text{(b)}\quad f(x) = (1-2x)^3,$$

$$\text{(c)}\quad f(x) = \frac{1-2x}{1+x}, \qquad\qquad \text{(d)}\quad f(x) = \frac{x}{\sqrt{x^2+1}}.$$

(a) $y = 7x - 4 \Leftrightarrow 7x = y + 4 \Leftrightarrow x = \frac{1}{7}(y+4)$ so $f^{-1}(x) = \dfrac{1}{7}(x+4)$.

(b) $y = (1-2x)^3 \Leftrightarrow y^{\frac{1}{3}} = 1 - 2x \Leftrightarrow 2x = 1 - y^{\frac{1}{3}} \Leftrightarrow x = \dfrac{1}{2}(1 - y^{\frac{1}{3}})$ so $f^{-1}(x) = \dfrac{1}{2}(1 - x^{\frac{1}{3}})$.

(c) $y = \dfrac{1-2x}{1+x} \Leftrightarrow (1+x)y = 1 - 2x \Leftrightarrow xy + 2x = 1 - y \Leftrightarrow x = \dfrac{1-y}{2+y}$ so $f^{-1}(x) = \dfrac{1-x}{2+x}$.

(d) $y = \dfrac{x}{\sqrt{x^2+1}} \Leftrightarrow y\sqrt{x^2+1} = x \Rightarrow y^2(x^2+1) = x^2 \Leftrightarrow x^2(y^2-1) = 1 \Leftrightarrow$ $x^2 = \dfrac{1}{y^2-1} \Leftrightarrow x = \dfrac{y}{\sqrt{1-y^2}}$ so $f^{-1}(x) = \dfrac{x}{\sqrt{1-x^2}}$.

2. Simplify the following improper rational function and identify the quotient and the remainder:

$$\frac{-x^4 - 2x^3 + 2x^2 + 3x - 57}{x^2 + x - 6}.$$

$$\begin{array}{ccc|ccccc}
 & & & & & -x^2 & -x & -3 \\
\hline
x^2 & +x & -6 & -x^4 & -2x^3 & +2x^2 & +3x & -57 \\
 & & & -x^4 & -x^3 & +6x^2 & & \\
 & & & & -x^3 & -4x^2 & +3x & \\
 & & & & -x^3 & -x^2 & +6x & \\
 & & & & & -3x^2 & -3x & -57 \\
 & & & & & -3x^2 & -3x & +18 \\
\hline
 & & & & & & & -75
\end{array}$$

The quotient is $-x^2 - x - 3$ and the remainder is -75.

3. In each case, factorise the given polynomial $p(x)$ and find all real roots of the equation $p(x) = 0$.

$$\text{(a)} \quad p(x) = x^4 + 6x^3 + 9x^2, \qquad \text{(b)} \quad p(x) = x^5 - x^4 - 16x + 16.$$

(a) $p(x) = x^4 + 6x^3 + 9x^2 = x^2(x^2 + 6x + 9) = x^2(x + 3)^2$. Real roots of $p(x) = 0$ are therefore $x = 0$ (twice) and $x = -3$ (twice).

(b) Integer roots must be factors of 16 or -16.
Try $x = 1$: $p(1) = 0 \Leftrightarrow x - 1$ is a factor of $p(x)$.
Divide by $x - 1$:

$$\begin{array}{cc|cccc}
 & & & x^4 & & -16 \\ \hline
x & -1 & x^5 & -x^4 & -16x & +16 \\
 & & x^5 & -x^4 & & \\
 & & & & -16x & +16 \\
 & & & & -16x & +16 \\ \hline
 & & & & & 0
\end{array}$$

So $p(x) = (x - 1)(x^4 - 16)$.
Try $x = 2$: $p(2) = 0 \Leftrightarrow x - 2$ is a factor of $x^4 - 16$. Divide by $x - 2$:

$$\begin{array}{cc|ccccc}
 & & & x^3 & +2x^2 & +4x & +8 \\ \hline
x & -2 & x^4 & & & & -16 \\
 & & x^4 & -2x^3 & & & \\
 & & & 2x^3 & & & \\
 & & & 2x^3 & -4x^2 & & \\
 & & & & 4x^2 & -8x & \\
 & & & & & 8x & -16 \\
 & & & & & 8x & -16 \\ \hline
 & & & & & & 0
\end{array}$$

So $p(x) = (x - 1)(x - 2)(x^3 + 2x^2 + 4x + 8)$.
Try $x = -2$: $p(-2) = 0 \Leftrightarrow x + 2$ is a factor of $x^3 + 2x^2 + 4x + 8$. Divide by $x + 2$:

$$\begin{array}{cc|cccc}
 & & & x^2 & & +4 \\ \hline
x & +2 & x^3 & +2x^2 & +4x & +8 \\
 & & x^3 & +2x^2 & & \\
 & & & & 4x & +8 \\
 & & & & 4x & +8 \\ \hline
 & & & & & 0
\end{array}$$

So $p(x) = (x-1)(x-2)(x+2)(x^2+4)$. As x^2+4 is irreducible (i.e. can't be factorised), this is a full factorisation of $p(x)$. Real roots of $p(x) = 0$ are therefore $x = 1$, $x = 2$

and $x = -2$.

Note: this could have been done more directly by noting that the factor $x^4 - 16$ is the difference of two squares, i.e.

$$p(x) = (x-1)(x^4-16) = (x-1)(x^2-4)(x^2+4) = (x-1)(x-2)(x+2)(x^2+4).$$

4. Show that $x - 1$ is a factor of a polynomial P of positive degree if and only if the sum of the coefficients of P is zero.

Let P be the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

of positive degree n. We know that $x - 1$ is a factor of P if and only if 1 is a root, i.e. if and only if

$$P(1) = a_n + a_{n-1} + \ldots + a_1 + a_0 = 0.$$

That is, the sum of the coefficients of P is zero if and only if $x - 1$ is a factor of P.

5. Solve the following inequalities for x:

$$\text{(a)} \quad |x-2| < 4, \qquad \text{(b)} \quad \frac{2}{x-1} \geq 5.$$

(a) x must satisfy two inequalities:

$$x - 2 < 4 \qquad \text{and} \qquad -(x-2) < 4 \Leftrightarrow x - 2 > -4.$$

This gives $x < 6$ and $x > -2$, that is, $x \in (-2, 6)$.

(b) $\dfrac{2}{x-1} \geq 5 \Leftrightarrow \dfrac{2}{x-1} - 5 \geq 0 \Leftrightarrow \dfrac{2-5(x-1)}{x-1} \geq 0 \Leftrightarrow \dfrac{7-5x}{x-1} \geq 0.$
Factors change sign when $x = 1$, $x = 7/5$. Table of signs:

x values	$x < 1$	$x = 1$	$1 < x < 7/5$	$x = 7/5$	$x > 7/5$
$7 - 5x$	$+$	$+$	$+$	0	-
$x - 1$	-	0	$+$	$+$	$+$
$\dfrac{7-5x}{x-1}$	-	nd	$+$	0	-

So the solution is $x \in (1, 7/5]$.

6. Solve the simultaneous equations $\quad x^2+y^2=1, \quad x+y=\frac{1}{2}.$

Second equation gives $y=\frac{1}{2}-x$. Substitute this into first equation:

$$x^2+(\frac{1}{2}-x)^2-1=0 \Leftrightarrow 2x^2-x-\frac{3}{4}=0.$$

This is a quadratic equation in x : solutions are

$$x=\frac{1\pm\sqrt{1+6}}{4}=\frac{1}{4}\pm\frac{\sqrt{7}}{4}.$$

If $x=\frac{1}{4}+\frac{\sqrt{7}}{4}$, $y=\frac{1}{2}-x=\frac{1}{4}-\frac{\sqrt{7}}{4}$. If $x=\frac{1}{4}-\frac{\sqrt{7}}{4}$, $y=\frac{1}{2}-x=\frac{1}{4}+\frac{\sqrt{7}}{4}$.