8.5 Second-Order Linear Equations with Constant Coefficients

We now consider a restricted type of second-order linear ODE which occurs very frequently in many different areas of mathematics (and physics, engineering etc.).

A second-order linear equation with constant coefficients has the general form:

$$a\,y'' \;+\; b\,y' \;+\; c\,y = f(x) \tag{8.27}$$

where $a,\ b,\ c$ are constants and $(a \neq 0)$.

So, rather than having functions $a_2(x)$, $a_1(x)$ and $a_0(x)$ multiplying y'', y' and y, we now have constants. An example of this type of equation was seen in the first example at the end of the last section, $y'' + 4y = 0$, so in that case $a = 1$, $b = 0$ and $c = 4$.

When we solved first-order linear ODEs, we found that the general solution of

$$by' + c\,y = f(x) \qquad (b,\, c \text{ constants}), \tag{8.28}$$

can be found using the integrating factor

$$\begin{aligned} m(x) &= e^{\int \frac{c}{b}\,dx} \\ &= e^{\frac{c}{b}x}, \end{aligned}$$

so that equation (8.28) becomes

$$\left(ye^{\frac{c}{b}x}\right)' = \frac{f(x)e^{\frac{c}{b}x}}{b},$$

which can be integrated to get

$$ye^{\frac{c}{b}x} = c_1 + \int \frac{f(x)e^{\frac{c}{b}x}}{b}\,dx,$$

and therefore the solution is given by

$$y = c_1 e^{-\frac{c}{b}x} + e^{-\frac{c}{b}x}\int \frac{f(x)e^{\frac{c}{b}x}}{b}\,dx,$$

where c_1 is an arbitrary constant.

This solution is made up of two parts. The first part, which contains the arbitrary constant, solves (8.28) when $f(x) \equiv 0$, i.e. the homogeneous equation associated with (8.28). The second part, containing $f(x)$, is a solution of the full nonhomogeneous equation (8.28) and does not contain any arbitrary constants.

Guided by this observation for first-order linear constant coefficient ODEs we seek a solution of (8.27) in two parts, say $y = Y_1 + Y_2$.

The first part of the solution (Y_1) of (8.27), being second-order, must contain two arbitrary constants and solve the homogeneous equation

$$a\,y'' \;+\; b\,y' \;+\; c\,y = 0 \tag{8.29}$$

associated with (8.27). Theorem 8.9 applies, hence there are two linearly independent solutions y_1 and y_2 of (8.29) giving the general solution $Y_1 = c_1\,y_1 + c_2\,y_2$.

The second contribution to the solution of (8.27) (Y_2) satisfies the full nonhomogeneous equation (8.27) and does not contain any arbitrary constants.

The sum of these two parts will then satisfy the original differential equation and contain two unknown constants which may be found using the initial or boundary conditions:

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= a\,(Y_1 + Y_2)'' + b\,(Y_1 + Y_2)' + c\,(Y_1 + Y_2) \\ &= [a\,Y_1'' + b\,Y_1' + c\,Y_1] + [a\,Y_2'' + b\,Y_2' + c\,Y_2] \\ &= 0 + f(x)\,. \end{aligned}$$

Definition 8.13 *The general solution of an n^{th}-order linear homogeneous differential equation is known as the **complementary function**. It comprises of a linear combination of the n linearly independent solutions of the homogeneous ODE. The coefficients in the sum being the arbitrary constants.*

Definition 8.14 *The solution of a linear nonhomogeneous differential equation, without any arbitrary constants, is known as a **particular integral**.*

In the example above, Y_1 is the complementary function (CF), and Y_2 is the particular integral (PI) for the ODE (8.27).

We still have to find methods for determining both the complementary function and the particular integral. We will now describe these methods for second-order equations with constant coefficients. First we look at how to solve the homogeneous equation (where $f(x) \equiv 0$) to obtain the complementary function and then, in the next section, we consider methods of solving the nonhomogeneous equation to determine the particular integral.

8.5.1 The Homogeneous Equation: Auxiliary Equation Method

To solve the second-order linear homogeneous constant coefficient differential equation

$$a\,y'' + b\,y' + \;c\,y = 0, \tag{8.30}$$

where a, b, c are constants and $(a \neq 0)$, we look for solutions of the form

$$\begin{aligned} & y &=& \; e^{mx} \qquad \text{where } m \text{ is a constant,} \\ \Longrightarrow \quad & y' &=& \; m\,e^{mx}, \\ \Longrightarrow \quad & y'' &=& \; m^2\,e^{mx}. \end{aligned}$$

Substituting these expressions into equation (8.30) we get:

$$(a\,m^2 + b\,m + c)e^{mx} = 0$$

$$\Longleftrightarrow \qquad a\,m^2 + b\,m + c = 0, \tag{8.31}$$

since $e^{mx} \neq 0$. So there are solutions of this type if m solves this quadratic equation.

Definition 8.15 *Equation (8.31) is called the* ***auxiliary equation (AE)*** *associated with (8.30).*

Because equation (8.31) is a quadratic equation we know that its solutions are

$$m = m_1 = \frac{-\,b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad m = m_2 = \frac{-\,b - \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two solutions $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ and can construct the general solution of the homogeneous equation, i.e. the complementary function, $y = c_1\,y_1 + c_2\,y_2 = c_1\,e^{m_1 x} + c_2\,e^{m_2 x}$.

However, we have to be careful because we have seen in previous classes that the solutions to a quadratic equation might be identical or complex.

There are three cases to consider:

1. Distinct Real Roots; $(b^2 - 4ac > 0)$
2. Equal Roots; $(b^2 - 4ac = 0)$
3. Complex Roots $(b^2 - 4ac < 0)$.

We will treat these three cases separately.

Case 1. $b^2 - 4ac > 0$

For this case the auxiliary equation $a\,m^2 + b\,m + c = 0$ has two distinct real roots

$$m_1 = \frac{-\,b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad m_2 = \frac{-\,b - \sqrt{b^2 - 4ac}}{2a},$$

and we know that $m_1 \neq m_2$.

So

$$y_1 = e^{m_1 x} \qquad \text{and} \qquad y_2 = e^{m_2 x},$$

are two solutions of equation (8.30). We can test if they are linearly independent using the Wronskian,

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1\, e^{m_1 x} & m_2\, e^{m_2 x} \end{vmatrix} = (m_2 - m_1) e^{m_1 x} e^{m_2 x},$$

which is not zero because we know that $m_1 \neq m_2$ in this case. The two solutions y_1 and y_2 are linearly independent and are therefore a fundamental set.

The general solution of (8.30) is therefore

$$Y_1 = c_1\, y_1 + c_2\, y_2 = c_1\, e^{m_1 x} + c_2\, e^{m_2 x},$$

where c_1 and c_2 are arbitrary constants.

Note: In the special case when $b = 0$ and $-4ac > 0$, we see that the solutions to the auxiliary equation are

$$m_1 = p \qquad \text{and} \qquad m_2 = -p,$$
$$\text{where} \;\; p = \frac{\sqrt{-ac}}{a},$$

and the general solution is therefore

$$Y_1 = c_1\, e^{px} + c_2\, e^{-px}.$$

In this case it is sometimes better to write the general solution in terms of hyperbolic functions:

$$e^x = \cosh(x) + \sinh(x) \qquad \text{and} \qquad e^{-x} \cosh(x) = \sinh(x).$$

The general solution is then

$$\begin{aligned} Y_1 &= c_1\, e^{px} + c_2\, e^{-px} \\ &= c_1\, [\cosh(px) + \sinh(px)] + c_2\, [\cosh(px) - \sinh(px)] \\ &= C\, \cosh(px) + D\, \sinh(px), \end{aligned}$$

where $C = c_1 + c_2$ and $D = c_1 - c_2$ are now the arbitrary constants.

Case 2. $\quad b^2 - 4ac = 0$

In this case the auxiliary equation has only one real root (repeated)

$$m = -\,\frac{b}{2a}.$$

So we obtain only one solution $y_1 = e^{mx}$. But this single solution cannot span the solution space and, by Theorem 8.9, there must be two linearly independent solutions of equation (8.30):

$$a\, y'' + b\, y' + c\, y = 0.$$

We need to try harder to find the second solution.

Consider the following function

$$y = v(x)\, e^{mx}, \tag{8.32}$$

where $v(x)$ is a new unknown function of x.

Now we can differentiate using the product rule to find

$$y' = (v' + m\, v)e^{mx}\ ,$$

$$y'' = (v'' + 2m\, v' + m^2\, v)e^{mx}\ .$$

Substituting into (8.30) we get

$$a(v'' + 2m\, v' + m^2\, v)e^{mx} + b(v' + m\, v)e^{mx} + c\, v\, e^{mx} = 0$$

$$\iff \qquad [a\, v'' + (2ma + b)v' + (am^2 + bm + c)v]e^{mx} = 0\ .$$

But $m = -\, b/(2a)$ so $2ma + b = 0$, and also $am^2 + bm + c = 0$, because m satisfies the auxiliary equation. Therefore, in order that the function in equation (8.32) satisfies the ODE, the function $v(x)$ must satisfy,

$$v'' = 0 \qquad \Rightarrow \qquad v = c_1 + c_2\, x.$$

The second solution to the original ODE is therefore

$$y_2 = (c_1 + c_2\, x)e^{mx},$$

and contains two arbitrary constants and also includes the solution y_1. A general solution can therefore be written as $Y_1 = c_1\, e^{mx} + c_2\, xe^{mx}$.

We should check that the two parts of this solution, e^{mx} and $x\, e^{mx}$ are linearly independent using the Wronskian,

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{mx} & xe^{mx} \\ m\, e^{mx} & e^{mx} + m\, x\, e^{mx} \end{vmatrix} = (1 + m\, x - m\, x)\, (e^{mx})^2 = e^{2mx} > 0,$$

so the general solution for this case is

$$Y_1 = c_1\, e^{mx} + c_2\, x\, e^{mx}.$$

The technique we have used above to find the general solution, when we already know one solution of the ODE, is called 'reduction of order' and can be used to solve more general ODEs (see later).

Case 3. $b^2 - 4ac < 0$

In this case the auxiliary equation has two complex roots

$$m = \frac{-\, b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\, b \pm\, i\, \sqrt{4ac - b^2}}{2a} = p + i\, q\ ,\ p - i\, q \quad (p, q \in \mathbb{R})\ .$$

i.e. a complex conjugate pair.

The general solution of (8.30) is

$$Y_1 = A\, e^{(p+i\, q)x} + B\, e^{(p-i\, q)x}$$

where A and B are arbitrary constants.

This is not in a very convenient form as it involves complex numbers. However, we can rewrite this general solution using

$$\begin{aligned} e^{i\,\theta} &= \cos(\theta) + i\, \sin(\theta) \\ e^{-i\,\theta} &= \cos(\theta) - i\, \sin(\theta). \end{aligned}$$

Then the solution can be rewritten

$$\begin{aligned} Y_1 &= A\, e^{(p+i\, q)x} + B\, e^{(p-i\, q)x} \\ &= e^{px}[A\, e^{i\, qx} + B\, e^{-i\, qx}] \\ &= e^{px}\, \{A\, [\cos(qx) + i\, \sin(qx)]\ +\ B\, [\cos(qx) - i\, \sin(qx)]\} \\ &= e^{px}[c_1\, \cos(qx) + c_2\, \sin(qx)] \\ &= c_1\, e^{px} \cos(qx) + c_2\, e^{px} \sin(qx), \end{aligned}$$

where $c_1 = A + B$ and $c_2 = i\,(A - B)$ are real arbitrary constants.

We should again check that the two parts of this solution, $e^{px}\cos(qx)$ and $e^{px}\sin(qx)$ are linearly independent using the Wronskian,

$$\begin{aligned} W(x) &= \begin{vmatrix} e^{px}\cos(qx) & e^{px}\sin(qx) \\ pe^{px}\cos(qx) - qe^{px}\sin(qx) & pe^{px}\sin(qx) + qe^{px}\cos(qx) \end{vmatrix} \\ &= \left[(p\cos(qx)\sin(qx) + q\cos^2(qx)) - (p\sin(qx)\cos(qx) - q\sin^2(qx))\right](e^{px})^2 \\ &= q[\cos^2(qx) + \sin^2(qx)]e^{2px} \\ &= q\,e^{2px}, \end{aligned}$$

which is never zero because we have assumed that $q = \sqrt{4ac - b^2} > 0$ at the beginning. So the general solution for this case is

$$Y_1 = c_1\,e^{px}\cos(qx) + c_2\,e^{px}\sin(qx).$$

Method

Homogeneous second-order linear ODE with constant coefficients $a\,y'' + b\,y' + c\,y = 0$

- Solve the auxiliary equation: $a\,m^2 + b\,m + c = 0$ for roots m_1 and m_2.
- If m_1 and m_2 are real and $m_1 \neq m_2$ then the general solution is

$$y = c_1\,e^{m_1 x} + c_2\,e^{m_2 x}$$

  Special case: When m_1 and m_2 are both real and $m_1 = p$, $m_2 = -p$ then the general solution is

$$y = c_1\,\cosh(px) + c_2\,\sinh(px)$$

- If $m_1 = m_2 = m$ then the solution is

$$y = (c_1 + c_2\,x)\,e^{mx}$$

- If m_1 and m_2 are complex and $m_1 = p + iq$, $m_2 = p - iq$ then the general solution is

$$y = e^{px}[c_1\,\cos(qx) + c_2\,\sin(qx)]\ .$$

Examples 8.6 Solve the following second-order linear ODEs

(a) $y'' - 3y' + 2y = 0$

(b) $y'' + 5y' = 0, \quad$ with $y(0) = 1$, $y'(0) = 1$

(c) $4y'' + 4y' + y = 0$

(d) $y'' + 2y' + 2y = 0$

✓ Watch Video: Example 8.6

✓ Solve Example: Tutorial question 1.

8.5.2 Homogeneous Linear Constant Coefficient Equations of Higher-Order

The auxiliary equation method also applies to higher-order homogeneous linear ODEs with constant coefficients,

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1\, y' + a_0\, y = 0,$$

where the a_i are constant coefficients.

The associated auxiliary equation is

$$a_n\, m^n + a_{n-1}\, m^{n-1} + \ldots + a_1\, m + a_0 = 0,$$

with solutions $m_1\ ,\ m_2\ ,\ \ldots\ ,\ m_n$.

These solutions for m might be distinct or there may be repeated roots (double, triple, $\ldots$), and also possibly complex conjugate pairs.

The same analysis as above can be carried out: if the root is a distinct real root we will have a term in the general solution of the form

$$c_1\, e^{mx}.$$

If the root is repeated twice we have a term in the solution of the form

$$(c_1\, x + c_2\,)e^{mx}.$$

If the root is repeated three times we have a term in the solution of the form

$$(c_1\, x^2 + c_2\, x + c_3\,)e^{mx},$$

(and so on for multiply repeated roots).

For complex conjugate pairs of roots ($m_{1,2} = p \pm i\,q$) we have a term in the general solution of the form

$$e^{px}[c_1\, \cos(qx) + c_2\, \sin(qx)].$$

Looking at each root, or group of repeated or complex roots, we can therefore construct the general solution to a homogeneous n^{th}-order linear ODE with constant coefficients.

Examples 8.7

Solve the following linear ODEs

(a) $y''' + 2y'' - y' - 2y = 0$

(b) $y^{(4)} - 16y = 0$

✓ Watch Video: Example 8.7

✓ Solve Example: Tutorial question 2.

8.5.3 The Nonhomogeneous Linear Equation

Remember that if Y_1 is the general solution (the complementary function) of the homogeneous equation

$$a\,y'' + b\,y' + c\,y = 0 \qquad \text{for } \; a, b, c \in \mathbb{R}\;, \quad a \neq 0$$

then the general solution of the nonhomogeneous equation

$$a\,y'' + b\,y' + c\,y = f(x), \tag{8.33}$$

where $f(x) \not\equiv 0$, is

$$y = Y_1 + Y_2$$

where Y_2 is a particular integral of equation (8.33).

However, we still need to find the particular integral Y_2. Unfortunately the calculation of a PI is only straightforward for certain types of function $f(x)$. Fortunately, these types of function often occur in real-life situations.

We start first by stating the principle of superposition for linear nonhomogeneous equations:

Theorem 8.16 *If* Y_{21} *is a solution of*

$$a\,y'' + b\,y' + c\,y = f_1(x),$$

and Y_{22} *is a solution of*

$$a\,y'' + b\,y' + c\,y = f_2(x),$$

then $Y_{21} + Y_{22}$ *is a solution of*

$$a\,y'' + b\,y' + c\,y = f_1(x) + f_2(x).$$

Proof 8.16 *To prove this theorem we can add the two ODEs involving* f_1 *and* f_2 *and use the linearity property of linear differential mappings.*∎

We can therefore construct the PI in stages. If the function $f(x)$ is a sum of basic functions such as combinations of polynomials, exponentials, and sines and cosines, the PI will be a sum of PIs solving an ODE for each of the components of $f(x)$.

For instance, if we want to find the particular integral for the following ODE

$$y'' + 2y' + y \;\; = \;\; x + \sin x, \tag{8.34}$$

we can find the particular integrals for the following two ODEs

$$\begin{aligned} y'' + 2y' + y &= x, \\ y'' + 2y' + y &= \sin x, \end{aligned}$$

let's call them Y_{21} and Y_{22}. Then the particular integral for the ODE (8.34) is $Y_2 = Y_{21} + Y_{22}$

Below we consider commonly occurring functions $f(x)$:

$f(x)$ is a polynomial of degree n

Suppose we want to find the particular integral of

$$a\,y'' + b\,y' + c\,y = p_0 + p_1\,x + \ldots + p_n\,x^n,$$

where $n \geq 0$ and $p_n \neq 0$ and where $p_0, p_1, \ldots\ p_n$ are known constants.

Note: We have included the possibility that $n = 0$ so that $f(x) = p_0$, a constant.

The particular integral in this situation will be a polynomial of degree of at most n

$$Y_2 = q_0 + q_1\,x + \ldots + q_n\,x^n.$$

If we put this PI into the left hand side of the ODE we find that

$$\begin{aligned} a\,Y_2'' + b\,Y_2' + c\,Y_2 &= (2a\,q_2 + b\,q_1 + c\,q_0) + \\ &\quad (6a\,q_3 + 2b\,q_2 + c\,q_1)\,x + \\ &\quad \ldots \\ &\quad (n(n-1)a\,q_n + (n-1)b\,q_{n-1} + c\,q_{n-2})\,x^{n-2} + \\ &\quad ((n-1)b\,q_n + c\,q_{n-1})\,x^{n-1} + \\ &\quad c\,q_n\,x^n. \end{aligned}$$

Comparing this to the right hand side of the ODE

$$a\,y'' + b\,y' + c\,y = p_0 + p_1\,x + \ldots + p_n\,x^n,$$

we can find the values of the constant coefficients $q_0,\ q_1,\ \ldots,\ q_n$ by comparing coefficients of powers of x

$$\begin{aligned} p_n &= c\,q_n, \\ p_{n-1} &= (n-1)b\,q_n + c\,q_{n-1}, \\ p_{n-2} &= n(n-1)a\,q_n + (n-1)b\,q_{n-1} + c\,q_{n-2}, \\ &\vdots \\ p_1 &= 6a\,q_3 + 2b\,q_2 + c\,q_1, \\ p_0 &= 2a\,q_2 + b\,q_1 + c\,q_0. \end{aligned}$$

We can solve the first of these equations to obtain q_n, then the next equation to find q_{n-1} etc., solving each successive equation, eventually finding q_0 from the last equation.

From these equations we see that if $c = 0$ then we have a contradiction as $p_n \neq 0$. In this case the particular integral will need to be a polynomial of degree $n+1$ obtained by multiplying by x to get

$$Y_2 = x[q_0 + q_1\,x + \ldots + q_n\,x^n].$$

If we have $b = c = 0$ then the particular integral will need to be a polynomial of degree $n+2$, i.e.

$$Y_2 = x^2[q_0 + q_1\,x + \ldots + q_n\,x^n].$$

$f(x)$ is an exponential

Suppose we want to find the particular integral of

$$a\,y'' + b\,y' + c\,y = p\,e^{qx}$$

where p and q are known constants.

For this situation the form of the particular integral depends on the value of q. For most cases we can try and find a PI which is of a similar form to the right hand side of the equation, i.e. an exponential function. However, we need to be careful when q is a root of the auxiliary equation.

If $m = q$ is not a root of the auxiliary equation $a\,m^2 + b\,m + c = 0$, then we consider the function

$$Y_2 = P\,e^{qx},$$

and substituting into the ODE gives

$$\begin{aligned} a\,q^2\,P\,e^{qx} + b\,q\,P\,e^{qx} + c\,P\,e^{qx} &= p\,e^{qx} \\ \Rightarrow \qquad P\,e^{qx}\left(a\,q^2 + b\,q + c\right) &= p\,e^{qx} \\ \Rightarrow \qquad\qquad\qquad P &= \frac{p}{(a\,q^2 + b\,q + c)}. \end{aligned}$$

(Note: this last step, dividing by $(a\,q^2 + b\,q + c)$, would not have been possible if q had been a root of the auxiliary equation.)

If, on the other hand, $m = q$ is a distinct root of the auxiliary equation then $aq^2 + bq + c = 0$ and e^{qx} is actually part of the complementary function.

In this situation we consider the function

$$Y_2 = v(x)\,e^{qx},$$

and substituting into the ODE gives

$$\begin{aligned} a\,v''(x)\,e^{qx} + (2a\,q+b)\,v'(x)\,e^{qx} + \left(a\,q^2+b\,q+c\right)v(x)\,e^{qx} &= p\,e^{qx} \\ \Rightarrow \qquad a\,v''(x) + (2a\,q+b)\,v'(x) &= p, \end{aligned} \tag{8.35}$$

since q satisfies the auxiliary equation. This last equation can now be solved using the substitution $w(x) = v'(x)$ (because $v(x)$ does not appear in the equation). This then gives

$$a\,w'(x) + (2a\,q+b)\,w(x) = p,$$

which is a first-order linear equation and can be solved to give $w(x) = p/(2a\,q+b)$, and therefore the solution for $v(x)$ is

$$v(x) = \left(\frac{p}{2a\,q+b}\right)x + c_1.$$

The particular integral in this case is then

$$Y_2 = \left(\frac{p}{2a\,q+b}\right)x\,e^{qx}.$$

The c_1 term has been dropped because this gives a term which is already in the complementary function.

Finally, if $m = q$ is actually a repeated root of the auxiliary equation $aq^2+bq+c = 0$ then it must also satisfy $(2a\,q+b) = 0$ (you can see this because the auxiliary equation is a complete square $aq^2+bq+c = a\left(q+\dfrac{b}{2a}\right)^2 = 0$).

Therefore equation (8.35) becomes

$$a\,v''(x) = p, \tag{8.36}$$

and we get the particular integral

$$Y_2 = \left(\frac{p}{2a}\right)x^2\,e^{qx}.$$

$f(x)$ is a sine and / or cosine

Suppose we want to find the particular integral of

$$a\,y'' + b\,y' + c\,y = p_1\,\cos(qx) + p_2\,\sin(qx),$$

or either of the equations

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= p_1\,\cos(qx), \\ a\,y'' + b\,y' + c\,y &= p_2\,\sin(qx), \end{aligned}$$

where p_1, p_2 and q are known constants.

In this case, if the roots of the auxiliary equation are not $m = \pm i\,q$, then we can find a particular integral of the form

$$Y_2 = P_1\,\cos(qx) + P_2\,\sin(qx),$$

where P_1 and P_2 are found by substituting this particular integral into the ODE.

Note: we always include both the cos and sin term in the particular integral even if $f(x)$ only involves a cos or a sin.

However, if $m = \pm i\,q$ are the roots of the auxiliary equation we can show, in a similar way as to the last case, that the particular integral will be

$$Y_2 = x[P_1\,\cos(qx) + P_2\,\sin(qx)],$$

where P_1 and P_2 are again found by substituting this particular integral into the ODE.

Method Nonhomogeneous n^{th}-order linear ODE with constant coefficients

- Find the complementary function
  - Consider $y = e^{mx}$ to obtain the auxiliary equation,
  - Find all roots of the auxiliary equation
  - Corresponding to
    (i) a distinct real root m, add a term of the form $c_1\,e^{m\,x}$
    (ii) an r-times repeated real roots m, add a term of the form $p_r(x)\,e^{m\,x}$
    where ($p_r(x)$ is a polynomial of degree r)
    (iii) complex conjugate roots $m = p \pm i\,q$, add a term of the form
    $e^{px}[c_1\,\cos(qx) + c_2\,\sin(qx)]$
- Examine $f(x)$ to determine the appropriate form for the particular integral Y_2
  [If terms in $f(x)$ already appear in the CF then multiply by x until it is distinct]
  Note that the process determines a unique set of constants in the PI.
  The 'constants' cannot involve x, be inconsistent, or have multiple values.
- Add the PI to the CF to get the general solution $y = Y_1 + Y_2$.
- Apply any initial or boundary conditions to determine the arbitrary constants.

Examples 8.8

Solve the following linear ODEs

(a) $y'' + 16\,y = e^{3x}$

(b) $2y'' + 4\,y' + 7\,y = x^2$

(c) $y'' + 4\,y' + 3\,y = 13\,\cos(2x)$

(d) $y''' - y = e^x + 7$

✓ Watch Video: Example 8.8

✓ Solve Example: Tutorial question 3.