UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Examples 7A

1 Write out the first six terms of the following sequences:

(i) $\left\{\left(-\frac{1}{2}\right)^n\right\}$, (ii) $\left\{\frac{\cos(n\pi/2)}{n}\right\}$, (iii) $u_1 = 1, u_2 = 1,\ u_{n+2} = u_n + u_{n+1},\ n = 1, 2, 3, \ldots$.

(i) $\left\{\left(-\frac{1}{2}\right)^n\right\} = -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, \ldots$.

(ii) $\left\{\frac{\cos(n\pi/2)}{n}\right\} = 0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{6}, \ldots$.

(iii) $u_1 = 1, u_2 = 1,\ u_{n+2} = u_n + u_{n+1},\ n = 1, 2, 3, \ldots$ so $\{u_n\} = 1, 1, 2, 3, 5, 8, \ldots$. This is called the **Fibonacci sequence**.

Examples 7B

1 Write the following sums using summation notation:

(i) $36 + 37 + 38 + \ldots + 51$;

(ii) $1 + 2 + 4 + 8 + \ldots$ to 10 terms;

(iii) $1 \times 5 + 2 \times 7 + 3 \times 9 + 4 \times 11 + \ldots + n(2n + 3)$.

(i)

$$36 + 37 + 38 + \ldots + 51 = \sum_{j=36}^{51} j.$$

Note: this answer is not unique, we could also write

$$\sum_{j=36}^{51} j = \sum_{j=0}^{15} (j + 36) = \sum_{j=2}^{17} (j + 34) = \ldots$$

(ii) We need ten terms, i.e.

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512.$$

Each term is a power of 2, starting from the first term $2^0 = 1$ and going up to the tenth term $2^9 = 512$. We can therefore write the sum as

$$\sum_{j=0}^{9} 2^j.$$

(iii) Here each term is of the form $f(j) = j(2j+3)$ for $j = 1, 2, \ldots, n$ so we can write the sum as

$$\sum_{j=1}^{n} j(2j+3).$$

Examples 7C

1 Evaluate $36 + 37 + 38 + \ldots + 51$.

This is an arithmetic series with first term $a = 36$ and common difference $d = 1$. Also, there are $n = 16$ terms in the sum so, from the formula derived in the notes,

$$S_n = S_{16} = \frac{16}{2}[2 \times 36 + 15 \times 1] = 696.$$

2 Find a formula for the sum of the first n integers.

We need to evaluate $\sum_{j=1}^{n} j$. This is an arithmetic series with first term 1, common difference 1 and n terms. Therefore

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

Note: we proved this by induction earlier.

Examples 7D

1 Evaluate the sum of the first 10 terms in the series $1 + 2 + 4 + 8 + \ldots$.

This is a geometric series with first term $a = 1$, common ratio $r = 2$ and $n = 10$ terms. So

$$S_{10} = 1\left(\frac{1 - 2^{10}}{1 - 2}\right) = 2^{10} - 1 = 1023.$$

2 Evaluate the sum of the first 10 terms in the series $1 - 2 + 4 - 8 + \ldots$.

Here $a = 1$, $r = -2$ and $n = 10$, so

$$S_{10} = 1\left(\frac{1-(-2)^{10}}{1-(-2)}\right) = \frac{1-2^{10}}{3} = -341.$$

3 Sum the series $3 + 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots$ to 6 terms.

Here $a = 3$, $r = 1/3$ and $n = 6$, so

$$\begin{aligned} S_6 &= 3\left(\frac{1-(1/3)^6}{1-1/3}\right) = 3\left(\frac{\frac{3^6-1}{3^6}}{2/3}\right) = \frac{3^6-1}{3^5} \times \frac{3}{2} \\ &= \left(\frac{3^6-1}{3^4}\right) \times \frac{1}{2} = \left(\frac{729-1}{81}\right)\frac{1}{2} = 4.49383\ldots \end{aligned}$$

Examples 7E

1 Evaluate $\displaystyle\sum_{j=1}^{n}(2j-1)^3$.

We have

$$\begin{aligned} \sum_{j=1}^{n}(2j-1)^3 &= \sum_{j=1}^{n}(8j^3 - 12j^2 + 6j - 1) \\ &= 8\sum_{j=1}^{n} j^3 - 12\sum_{j=1}^{n} j^2 + 6\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1 \\ &= 8\left[\frac{n^2(n+1)^2}{4}\right] - 12\left[\frac{n}{6}(n+1)(2n+1)\right] + 6\left[\frac{n}{2}(n+1)\right] - n \\ &= \ldots = n^2(2n^2-1). \end{aligned}$$

2 Use induction to prove that

$$1 \times 5 + 2 \times 7 + 3 \times 9 + 4 \times 11 + \ldots + n(2n+3) = \frac{n(n+1)(4n+11)}{6}$$

for all natural numbers n.

First we note that

$$1 \times 5 + 2 \times 7 + 3 \times 9 + 4 \times 11 + \ldots + n(2n+3) = \sum_{j=1}^{n} j(2j+3).$$

Step 1: Check the case $n = 1$.

$$LHS = 1 \times 5 = 5, \qquad RHS = \frac{1 \times 2 \times 15}{6} = 5$$

so proposition is true when $n = 1$.

Step 2: Assume that the given result is true for n, that is, assume that

$$\sum_{j=1}^{n} j(2j+3) = \frac{n(n+1)(4n+11)}{6}.$$

Now try to prove that this implies the result is true for $n+1$, that is, try to show that

$$\sum_{j=1}^{n+1} j(2j+3) = \frac{(n+1)((n+1)+1)(4(n+1)+11)}{6} = \frac{(n+1)(n+2)(4n+15)}{6}.$$

$$\begin{aligned}
LHS &= \sum_{j=1}^{n} j(2j+3) + (n+1)(2(n+1)+3) \\
&= \frac{n(n+1)(4n+11)}{6} + (n+1)(2n+5) \\
&= \frac{(n+1)}{6}(n(4n+11) + 6(2n+5)) \\
&= \frac{(n+1)}{6}(4n^2 + 23n + 30) \\
&= \frac{(n+1)(n+2)(4n+15)}{6}
\end{aligned}$$

so if the result holds for n, it also holds for $n+1$.
Hence, by the principle of mathematical induction, the result is true for all $n \in \mathbb{N}$.

Examples 7F

1 Show that the sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

We have

$$|u_n - l| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}$$

so, for any $\epsilon > 0$,

$$|u_n - l| < \epsilon \Leftrightarrow \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}.$$

By Definition 7.5 (with $N > 1/\epsilon$), the given sequence therefore converges to the limit 0.

2 Show that $\lim_{n\to\infty} \frac{c}{n^p} = 0$ for any real number c and any $p > 0$.

For any $\epsilon > 0$, we have

$$|u_n - l| = \left|\frac{c}{n^p} - 0\right| = \frac{|c|}{n^p}$$

so

$$|u_n - l| < \epsilon \Leftrightarrow \frac{|c|}{n^p} < \epsilon \Leftrightarrow n^p > \frac{|c|}{\epsilon} \Leftrightarrow n > \left(\frac{|c|}{\epsilon}\right)^{\frac{1}{p}}.$$

By Definition 7.5 (with $N > (|c|/\epsilon)^{1/p}$), the given sequence therefore converges to the limit 0.

3 Show that the sequence $\{u_n\}$ where $u_n = l \ \forall n \in \mathbb{N}$ converges to l.

We have

$$|u_n - l| = |0| = 0$$

so, for any $\epsilon > 0$,

$$|u_n - l| < \epsilon$$

for all $n \geq 1$. By Definition 7.5 (with $N = 1$), the sequence $\{u_n\}$ therefore converges to the limit l.

4 Show that the sequence $\{u_n\}$, $u_n = (-1)^n$ diverges.

Suppose that the sequence converges to a limit l. Then for any $\epsilon > 0$ we would have an N such that $|u_n - l| < \epsilon$ for $n > N$. When $\epsilon = 1$, this implies that

$$|1 - l| < 1 \qquad \text{and} \qquad |-1 - l| < 1 \Leftrightarrow |1 + l| < 1.$$

We would therefore have

$$2 = |1 + l + 1 - l| \leq |1 + l| + |1 - l| < 1 + 1 = 2,$$

that is, $2 < 2$, which is a contradiction. So the sequence must diverge.

Examples 7G

1 Show that the sequence $\{u_n\}$ where $u_n = \dfrac{n-1}{n}$ converges to 1.

Divide top and bottom of the fraction by the highest power of n on the denominator:

$$\frac{n-1}{n} = \frac{1 - \frac{1}{n}}{1}$$

so

$$\lim_{n\to\infty} \frac{n-1}{n} = \frac{1-0}{1} = 1.$$

[Note: here we have used results from Theorem 7.6 and Example 7F1.]

2 Find $l = \lim_{n\to\infty} \dfrac{4n^3+3n^2-n}{2n^4+n^3+1}$.

To find the limit we first divide top and bottom by n^4 (the highest power in the denominator) and

$$\frac{4n^3+3n^2-n}{2n^4+n^3+1} = \frac{\dfrac{4}{n}+\dfrac{3}{n^2}-\dfrac{1}{n^3}}{2+\dfrac{1}{n}+\dfrac{1}{n^4}}.$$

Then by Theorem 7.6 and Example 7F2

$$l = \frac{4\lim\limits_{n\to\infty}\dfrac{1}{n}+3\lim\limits_{n\to\infty}\dfrac{1}{n^2}-\lim\limits_{n\to\infty}\dfrac{1}{n^3}}{2+\lim\limits_{n\to\infty}\dfrac{1}{n}+\lim\limits_{n\to\infty}\dfrac{1}{n^4}} = \frac{4\times 0+3\times 0-0}{2+0+0} = 0.$$

3 Evaluate (i) $\lim_{n\to\infty}\dfrac{n^2}{n^3+1}$, (ii) $\lim_{n\to\infty}\dfrac{n^3+1}{n^2}$, (iii) $\lim_{n\to\infty}\dfrac{2n^2-n+4}{3n^2+n+3}$.

(i) $\dfrac{n^2}{n^3+1} = \dfrac{\dfrac{1}{n}}{1+\dfrac{1}{n^3}}$ so $\lim\limits_{n\to\infty}\dfrac{n^2}{n^3+1} = \dfrac{0}{1+0} = 0.$

(ii) $\dfrac{n^3+1}{n^2} = \dfrac{n+\dfrac{1}{n^2}}{1} = n+\dfrac{1}{n^2}$ and since the first term is divergent $\lim\limits_{n\to\infty}\dfrac{n^3+1}{n^2}$ diverges.

(iii) $\dfrac{2n^2-n+4}{3n^2+n+3} = \dfrac{2-\dfrac{1}{n}+\dfrac{4}{n^2}}{3+\dfrac{1}{n}+\dfrac{3}{n^2}}$ so $\lim\limits_{n\to\infty}\dfrac{2n^2-n+4}{3n^2+n+3} = \dfrac{2-0+0}{3+0+0} = \dfrac{2}{3}.$

Examples 7H

1 Sum the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

to infinity.

This is a geometric sequence with $a = 1$ and $r = \frac{1}{2}$. Therefore,

$$u_j = \left(\frac{1}{2}\right)^{j-1}, \quad j = 1, 2, 3, \ldots$$

and

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = \sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{j-1} = \frac{1}{1-\frac{1}{2}} = 2\,.$$

2 Sum the sequence

$$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8} \dots$$

to infinity.

This is a geometric sequence with $a = 1$ and $r = -\frac{1}{2}$ so

$$S_\infty = \sum_{j=1}^{\infty} \left(-\frac{1}{2}\right)^{j-1} = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

Examples 8A

1 Expand

$$\text{(i)} \quad (z+2)^4, \qquad \text{(ii)} \quad (s-3\theta)^3.$$

(i) We have $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ so put $x = z$, $y = 2$ to get

$$(z+2)^4 = z^4 + 4z^3(2) + 6z^2 2^2 + 4z 2^3 + 2^4 = z^4 + 8z^3 + 24z^2 + 32z + 16.$$

(ii) We have $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ so put $x = s$, $y = -3\theta$ to get

$$(s + (-3\theta))^3 = s^3 + 3s^2(-3\theta) + 3s(-3\theta)^2 + (-3\theta)^3 = s^3 - 9s^2\theta + 27s\theta^2 - 27\theta^3.$$

Examples 8B

1 Evaluate

$$\text{(i)} \quad 5!, \qquad \text{(ii)} \quad 5! - 4!, \qquad \text{(iii)} \quad \frac{25! - 24!}{23!}.$$

(i) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$

(ii) $5! - 4! = 5 \times 4! - 4! = (5-1)(4 \times 3 \times 2 \times 1) = 4 \times 24 = 96.$

(iii) $\dfrac{25! - 24!}{23!} = \dfrac{25 \times 24 \times 23! - 24 \times 23!}{23!} = (25-1) \times 24 = 24^2 = 576.$

Examples 8C

1 Evaluate

$$\text{(i)} \quad \binom{6}{5}, \qquad \text{(ii)} \quad \binom{21}{17}, \qquad \text{(iii)} \quad \binom{49}{6}.$$

$$\binom{6}{5} = \frac{6!}{5!1!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 1} = 6.$$

$$\binom{21}{17} = \frac{21!}{17!4!} = \frac{21 \times 20 \times 19 \times 18}{4 \times 3 \times 2} = 5985.$$

$$\binom{59}{6} = \frac{59!}{6!53!} = 45057474.$$

2 Evaluate $\binom{n+1}{3} \div \binom{n-1}{2}$.

$$\frac{\binom{n+1}{3}}{\binom{n-1}{2}} = \frac{(n+1)!}{3!(n-2)!}\,\frac{2!(n-3)!}{(n-1)!} = \frac{(n+1)n}{3(n-2)}.$$

3 Find s if

$$\text{(i)} \quad \binom{10}{s} = \binom{10}{s-2}, \qquad \text{(ii)} \quad \binom{s}{8} = \binom{s}{7}.$$

(i) $\binom{10}{s} = \binom{10}{10-s} = \binom{10}{s-2}$ so $10 - s = s - 2$, hence $s = 6$.

(ii) $\binom{s}{8} = \binom{s}{s-8} = \binom{s}{7}$ so $s - 8 = 7$, hence $s = 15$.

Examples 8D

1 Use the Binomial Theorem to expand $(1+x)^5$.

$$\begin{aligned}
(1+x)^5 &= \sum_{r=0}^{5} \binom{5}{r} x^r \\
&= \binom{5}{0} x^0 + \binom{5}{1} x^1 + \binom{5}{2} x^2 \\
&\quad + \binom{5}{3} x^3 + \binom{5}{4} x^4 + \binom{5}{5} x^5 \\
&= \frac{5!}{0!5!} + \frac{5!x}{1!4!} + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{0!5!}x^5 \\
&= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.
\end{aligned}$$

2 Use the Binomial Theorem to expand $(1+2x)^5$.

$$\begin{aligned}
(1+2x)^5 &= \binom{5}{0}(2x)^0 + \binom{5}{1}(2x)^1 + \binom{5}{2}(2x)^2 \\
&\quad + \binom{5}{3}(2x)^3 + \binom{5}{4}(2x)^4 + \binom{5}{5}(2x)^5 \\
&= 1 + 5 \times 2x + 10 \times 4x^2 + 10 \times 8x^3 + 5 \times 16x^4 + 1 \times 32x^5 \\
&= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.
\end{aligned}$$

3 Write down an expression for the general term in the expansion of $(2+x)^{11}$.

$$(2+x)^{11} = \sum_{r=0}^{11} \binom{11}{r} 2^{11-r} x^r,$$

so the general term is is $\binom{11}{r} 2^{11-r} x^r$.

4 Find the general term in $\left(2x^2 - \frac{3}{x}\right)^5$ and hence find the coefficient of x^4.

$$\left(2x^2 - \frac{3}{x}\right)^5 = \sum_{r=0}^{5} \binom{5}{r} (2x^2)^r \left(-\frac{3}{x}\right)^{5-r}.$$

The general term here is

$$\begin{aligned}
\binom{5}{r}(2x^2)^{5-r}\left(-\frac{3}{x}\right)^r &= \binom{5}{r} 2^{5-r} x^{10-2r} (-3)^r x^r \\
&= \binom{5}{r} 2^{5-r} (-3)^r x^{10-3r}.
\end{aligned}$$

The coefficient of x^4 is found by taking $10 - 3r = 4$, i.e. $r = 2$, giving

$$\binom{5}{3} 2^3 (-3)^2 = 720.$$