University of Strathclyde
Department of Mathematics and Statistics
MM101 Introduction to Calculus

Exercises for chapters 9 to 16

9 Limits

9.1 For each of the following, first find the limit $l = \lim_{x\to a} f(x)$ for the given a, and then prove that it is the limit by finding, for an arbitrary $\epsilon > 0$, a suitable $\delta > 0$ such that
$|f(x) - l| < \epsilon$ whenever x satisfies $0 < |x - a| < \delta$.
(a) $f(x) = 7$, $a = 42$ (b) $f(x) = x$, $a = 0$ (c) $f(x) = x$, $a = 3$ (d) $f(x) = 4x$, $a = \frac{2}{3}$
(e) $f(x) = x^2$, $a = 3$ (f) $f(x) = 7x - 5$, $a = 3$ (g) $f(x) = x^2 + 7x - 5$, $a = 3$

9.2 For each of the following, decide whether the limit exists and, if it does, evaluate it.
(a) $\lim_{y\to 4} \dfrac{y^2 - 16}{y - 4}$ (b) $\lim_{f\to 4} \dfrac{f^2 - 2f - 8}{f - 4}$ (c) $\lim_{x\to 4} \dfrac{x^2 - 2x - 8}{x + 3}$
(d) $\lim_{x\to 1} \dfrac{x^2 + 6x - 7}{x^2 + 4x - 5}$ (e) $\lim_{x\to 3/2} \dfrac{2x^2 + 5x - 12}{4x^2 + 4x - 15}$ (f) $\lim_{x\to c} \dfrac{x^2 - c^2}{x - c}$
(g) $\lim_{x\to c} \dfrac{x^3 - c^3}{x - c}$ (h) $\lim_{x\to -2} \dfrac{x^3 + 8}{x + 2}$ (i) $\lim_{x\to 25} \dfrac{x - 25}{\sqrt{x} - 5}$
(j) $\lim_{x\to 7} \left(\dfrac{1}{x} - \dfrac{1}{7}\right) \dfrac{1}{x - 7}$ (k) $\lim_{x\to 2} \dfrac{x^2 + 5x - 14}{x^2 + 5x + 6}$ (l) $\lim_{x\to 2} \dfrac{x^2 - x - 6}{x^2 + 3x - 10}$.

9.3 Evaluate the following limits.
(a) $\lim_{h\to 0} \dfrac{\sqrt{a + h} - \sqrt{a}}{h}$ (b) $\lim_{x\to 1} \dfrac{1 - \sqrt{x}}{1 - x}$ (c) $\lim_{x\to 0} \dfrac{1 - \sqrt{1 - x^2}}{x}$ (d) $\lim_{x\to 0} \dfrac{1 - \sqrt{1 - x^2}}{x^2}$.

9.4 Evaluate the following limits.
(a) $\lim_{x\to 0} \dfrac{\sin 3x}{x}$ (b) $\lim_{x\to 0} \dfrac{\sin x}{3x}$ (c) $\lim_{x\to 0} \dfrac{\sin 5x}{7x}$ (d) $\lim_{x\to 0} \dfrac{12x}{\sin 25x}$
(e) $\lim_{x\to 0} \dfrac{\sin 3x \, \tan 5x}{4x^2}$ (f) $\lim_{x\to 0} \dfrac{1 - \cos^2 x}{x^2}$ (g) $\lim_{x\to \pi/2} \dfrac{\cos x}{\pi - 2x}$.

9.5 Suppose $|f(x)| \le g(x)$ for all x. What can you conclude about $\lim_{x\to a} f(x)$ if $\lim_{x\to a} g(x) = 0$? What if $\lim_{x\to a} g(x) = 5$?

9.6 (a) Suppose that $f(x) \le g(x) \le h(x)$ for all x and that $l = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$.
Show that $\lim_{x\to a} g(x) = l$.
(b) What changes if $f(x) < g(x) < h(x)$?

9.7 Prove that $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$.
(Hint: define $g(h) := f(a+h)$ and consider $\lim_{h \to 0} g(h)$.)

9.8 Prove that if $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| = |l|$.

9.9 Prove that $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.