7.3 Change of variables

A number of equations which do not at first seem to be separable, i.e. of the form (7.3), may be transformed into a more convenient form through a change of variables. Here we consider two of the more common examples of this method. However, it is not always easy to spot how to do these transformations.

If the right hand side of the ODE (7.1) can be expressed in the form $f(x,y) = g\left(\frac{y}{x}\right)$, so that

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right), \tag{7.4}$$

then we can use the substitution,

$$u(x) = \frac{y}{x}. \tag{7.5}$$

[Note: A function $f(x,y)$ is called ***homogeneous*** if $f(x,y) = f(\lambda x, \lambda y), \quad \forall \lambda \in \mathbb{R}$. Since functions of the form $g\left(\frac{y}{x}\right)$ are certainly homogeneous, ODEs given by (7.4) are called ***homogeneous*** ODEs. However this can be a little confusing because this term has a different definition later.]

Using the substitution from equation (7.5) we see that

$$\begin{aligned} y &= x\,u(x) \\ \text{and so} \quad \frac{dy}{dx} &= \frac{d}{dx}(x\,u(x)) = x\,\frac{du}{dx} + u, \end{aligned}$$

by the product rule, and then equation (7.4) becomes

$$x\,\frac{du}{dx} + u = g(u).$$

Therefore

$$\frac{du}{dx} = \frac{g(u) - u}{x}$$

which is a separable equation for u and can be solved by the same method as in Section 7.2 by integration,

$$\int \frac{1}{g(u) - u}\,du = \int \frac{1}{x}\,dx. \tag{7.6}$$

Once equation (7.6) is solved for u we reintroduce y by eliminating u using equation (7.5). As before, the integrations in equation (7.6) introduce an integration constant which can be determined using the initial condition (if it is known).

Method Substitution: $u = y/x$ (for homogeneous equations)

- Write the ODE in the form $\dfrac{dy}{dx} = g\left(\dfrac{y}{x}\right)$.
- Substitute $y = x\,u(x)$, so that $\dfrac{dy}{dx} = x\,\dfrac{du}{dx} + u \;=\; g(u)$.
- Rearrange the ODE to make it separable and then solve.
- Reintroduce y using $u = y/x$.
- Use the initial condition to determine the integration constant.

Another useful substitution can be used when the right hand side of the ODE (7.1) can be written as $f(x, y) = g(ax + by)$ so that

$$\frac{dy}{dx} = g(ax + by). \tag{7.7}$$

For this case we can use the substitution $u = ax + by$ which, when differentiated, leads to,

$$\begin{aligned} \frac{d}{dx}(u(x)) &= \frac{d}{dx}(ax + by), \\ \Rightarrow \qquad \frac{du}{dx} &= a + b\,\frac{dy}{dx}. \end{aligned}$$

Rearranging this equation leads to

$$\frac{dy}{dx} = \frac{\dfrac{du}{dx} - a}{b},$$

which, with the original ODE (7.7), leads to the separable equation

$$\frac{du}{dx} = a + b\,g(u) = h(u).$$

This equation can then be solved using the method in Section 7.2 (or 7.1).

Method Substitution: $u = ax + by$

- Substitute $u = ax + by$ so that $\dfrac{du}{dx} = a + b\,\dfrac{dy}{dx}$,
- ODE then becomes $\dfrac{du}{dx} = a + b\,g(u) = h(u)$ and can be solved using separation of variables,
- Write the solution in terms of y and x by substituting $u = ax + by$.

Examples 7.3

(a) Solve: $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x} + 1.$

(b) Solve: $\dfrac{dy}{dx} = \dfrac{x^2 + 2y^2}{xy}.$

(c) Solve: $\dfrac{dy}{dx} = \dfrac{1}{x + y}.$

(d) Solve: $\dfrac{dy}{dx} - 4y(x + y) = x^2.$

✓ Watch Video: Example 7.3

✓ Solve Exercise: Tutorial question 1.

7.4 Exact equations

The second type of equation that we consider are termed ***exact equations***. As we will see below, this type of equation can be rearranged to form an equation which is similar to the directly integrable equations in Section 7.1. To understand how these equations occur we first consider the solution and derive the ODE.

Consider the equation

$$G(x,y) = y\,P(x) - F(x) = C, \tag{7.8}$$

where C is a constant and $P(x)$ and $F(x)$ are known functions of x. It is in fact possible to consider more general forms of the function $G(x,y)$ (see later classes involving partial differentiation) but we will restrict our attention to forms of $G(x,y)$ which are linear in y, such as equation (7.8).

Equation (7.8) explicitly defines $y = (F(x)+C)/P(x)$ as a function of x. However, we will keep it in the form in equation (7.8) for now.

Differentiating equation (7.8) with respect to x, using the product rule, gives

$$P(x)\,\frac{dy}{dx} + y\,\frac{dP}{dx} = \frac{dF}{dx}, \tag{7.9}$$

and (7.8) defines a solution of the ODE (7.9).

[Note: The ODE (7.2) is of this form with $P(x) = 1$ and $\dfrac{dF}{dx} = f(x)$.]

These results leads us to the following definition.

Definition 7.2 *A linear ODE of the form*

$$P(x)\,\frac{dy}{dx} + Q(x)\,y = f(x), \tag{7.10}$$

*is called **exact** if*

$$Q = \frac{dP}{dx}. \tag{7.11}$$

Therefore, if we have an exact equation, we can write it as

$$\begin{aligned}
P(x)\,\frac{dy}{dx} + y\,\frac{dP}{dx} &= f(x),\\
\Rightarrow \qquad \frac{d}{dx}(y\,P(x)) &= f(x),\\
\Rightarrow \qquad y\,P(x) &= F(x) + C,
\end{aligned}$$

where C is a constant and $F(x)$ is the anti-derivative of $f(x)$. A solution of the exact equation (7.10) is therefore given by

$$y = \frac{F(x) + C}{P(x)}. \tag{7.12}$$

Two questions present themselves at this point:

- Can we test a linear ODE of the form (7.10) to see if it is exact?
- If the ODE is exact, can we always find y?

The answer to both these questions is YES! If we have a linear equation such as (7.10) we just need to check that

$$\frac{dP}{dx} = Q. \tag{7.13}$$

Furthermore, if (7.13) holds, then it is straightforward to find the solution by integrating $f(x)$ to get $F(x)$ and the solution is then given by equation (7.12).

Method Exact equations

- Put the ODE into the form
$$P(x)\,\frac{dy}{dx} + Q(x)\,y = f(x).$$
- Check that $\dfrac{dP}{dx} = Q(x)$.
- Write the equation as
$$\frac{d}{dx}\left(y\,P(x)\right) = f(x).$$
- Integrate both sides to obtain $y\,P(x) = F(x) + C$.
- Rearrange to find the solution $y(x) = (F(x) + C)/P(x)$.
- Apply an initial condition (if given) to find C.

Examples 7.4 Solve the following differential equations:

(a) $x\,\dfrac{dy}{dx} + y = 0$, (b) $\sin x\,\dfrac{dy}{dx} + y\cos x = x$, (c) $\dfrac{dy}{dx} - y = xe^x$

✓ Watch Video: Example 7.4

7.5 Integrating factors

It might seem like the chance of coming across an exact equation, with the functions $P(x)$ and $Q(x)$ in equation (7.10) being related by equation (7.13) is small.

However, as we saw in the last example above, it is sometimes the case that an equation of the form (7.10) which is not exact, can be made exact by multiplying the whole equation by an ***integrating factor***.

What if we can find a function $m(x)$ (the integrating factor) such that the equation,

$$m(x)\,P(x)\,\frac{dy}{dx} + m(x)\,Q(x)\,y = m(x)\,f(x), \tag{7.14}$$

is exact? We could then proceed as before to find the solution to this equation, which is also the solution to equation (7.10).

It is more usual to consider the 'standard' form of the linear ODE in equation (7.10), which results from dividing both sides by $P(x)$ so that we have an equation of the form,

$$\frac{dy}{dx} + p(x)\,y = q(x). \tag{7.15}$$

Then the question becomes: What function $m(x)$ is needed so that the equation,

$$m(x)\,\frac{dy}{dx} + m(x)\,p(x)\,y = m(x)\,q(x), \tag{7.16}$$

is exact?

We proceed as follows: If equation (7.16) is exact then, by the condition (7.13), we must have,

$$\frac{dm}{dx} = m\,p(x),$$

which is a separable equation and can be solved thus,

$$\int \frac{1}{m}\,dm = \int p(x)\,dx.$$

Therefore

$$\ln|m(x)| = \int p(x)\,dx,$$

i.e.

$$m(x) = e^{\int p(x)\,dx}. \tag{7.17}$$

Equation (7.16) now becomes

$$\frac{d}{dx}\left(y\,e^{\int p(x)\,dx}\right) = q(x)\,e^{\int p(x)\,dx},$$

which may be integrated to give

$$y\, e^{\int p(x)\,dx} = \int q(x)\, e^{\int p(x)\,dx}\, dx + C,$$

so that

$$y = e^{-\int p(x)\,dx} \left(\int q(x)\, e^{\int p(x)\,dx}\, dx + C \right).$$

You may have noticed that we did not introduce an integration constant when we found the integrating factor in equation (7.17). In fact it turns out that if we do add an integrating factor it cancels out in the final solution.

It is good practice to check your integrating factor $m(x)$ by seeing that $\dfrac{d(my)}{dx}$ really does give $m\dfrac{dy}{dx} + mpy$.

Method Integrating factor

- Put the ODE into the form $\dfrac{dy}{dx} + p(x)\, y = q(x)$.
- Calculate the integrating factor $m(x) = e^{\int p(x)\,dx}$.
- Write the equation as
$$\frac{d}{dx}(m(x)y) = m(x)q(x).$$
- Integrate to find the solution,
$y = \dfrac{1}{m(x)} \left(\int m(x)q(x)\, dx + C \right) = e^{-\int p(x)\,dx} \left(\int q(x)\, e^{\int p(x)\,dx}\, dx + C \right).$
- Use the initial condition (if given) to find the constant C.

Examples 7.5 Solve the following differential equations:

(a) $\dfrac{dy}{dx} + 2y = 1,$ (b) $x\dfrac{dy}{dx} - y = x^2 e^{2x},$ (c) $x(x-1)y' - y = (x-1)$

✓ Watch Video: Example 7.5

✓ Solve Exercise: Tutorial question 2.

7.6 Existence and uniqueness

In our search for solutions to ODEs it might be useful to consider if there are **any** solutions, and if there are more than one solution.

Two fundamental questions in ODE theory are therefore:

- **Existence:** Does a given IVP or BVP have a solution?
  (If the IVP/BVP is modelling some real-life phenomenon, there should be a solution that describes what is happening! If no solution to the ODE exists then this indicates that the model requires re-examination.)
- **Uniqueness:** Is there only one solution?
  (It would normally be unacceptable for a mathematical model in the form of an IVP to have more than one solution. If you know where you start from (the initial values), you should know where you are going (i.e. where the solution tells you to go) - there shouldn't be a choice. BVPs are a little different as we will see in examples.)

It is certainly not always true that a differential equation has a solution, or that a solution is unique. For instance, there are two relatively simple examples where existence and uniqueness fail:

- The initial value problem
  $$\frac{dy}{dx} = \frac{1}{x}, \quad \text{with} \quad y(0) = 0,$$
  has no solution, because the integration of this equation,
  $$y = \int \frac{1}{x}\,dx = \ln|x| + C,$$
  is not defined at $x = 0$ so we cannot find the constant C.
- The initial value problem
  $$\frac{dy}{dx} = 2\sqrt{y}, \quad \text{with} \quad y(0) = 0,$$
  has two solutions: $y = 0$ and $y = x^2$. Note: the solution $y = 0$ is a singular solution to the equation which the initial value happens to allow.

However, with some conditions on the right hand side of the ODE we are able to show that there is a unique solution.

For first-order linear equations we can prove the following theorem.

Theorem 7.3 *Let $p(x)$ and $q(x)$ be defined and continuous for $x \in I,\ y \in I_0$, where I and I_0 are open intervals in $\mathbb{R}$. Then, for any $(x_0, y_0) \in I \times I_0$, the first-order linear initial value problem*

$$\frac{dy}{dx} + p(x)\,y = q(x), \ \text{ with } y(x_0) = y_0, \tag{7.18}$$

has a unique solution on the interval I.

Proof 7.3 *Let us assume that there are two solutions $y_1(x)$ and $y_2(x)$ (we will eventually be able to show that these two solutions are the same). Now define the function $Y(x) = y_1(x) - y_2(x)$. Substituting $Y(x)$ into the left hand side of equation (7.18) gives*

$$\begin{aligned}
\frac{dY}{dx} + p(x)\,Y &= \frac{dy_1}{dx} - \frac{dy_2}{dx} + p(x)\,y_1 - p(x)\,y_2 \\
&= \left(\frac{dy_1}{dx} + p(x)\,y_1\right) - \left(\frac{dy_2}{dx} + p(x)\,y_2\right) \\
&= q(x) - q(x) \\
&= 0,
\end{aligned}$$

because both y_1 and y_2 satisfy equation (7.18).

We also know that

$$Y(x_0) = y_1(x_0) - y_2(x_0) = y_0 - y_0 = 0.$$

Therefore, Y satisfies the associated initial value problem,

$$\frac{dY}{dx} + p(x)\,Y = 0, \quad \text{with } Y(x_0) = 0,$$

which we can solve using the integrating factor method to give,

$$\begin{aligned}
\frac{d}{dx}\left(Y e^{\int p(x)dx}\right) &= 0, \\
\Rightarrow \qquad Y e^{\int p(x)dx} &= C, \\
\Rightarrow \qquad Y &= C e^{-\int p(x)dx}.
\end{aligned}$$

However, since $Y(x_0) = 0$, the constant C must be zero. We have therefore proved that $Y(x) = 0$ for all x, in other words we have shown that $y_1(x) = y_2(x)$ and that there is a unique solution to the original initial value problem. ∎

For more general initial value problems (i.e. nonlinear) there is a similar theorem about existence and uniqueness of the solution:

Theorem 7.4 *Let $f(x,y)$ be defined and continuous for $x \in I,\ y \in I_0$, where I and I_0 are open intervals in $\mathbb{R}$, with the partial derivative $\dfrac{\partial f}{\partial y}(x,y)$ defined and continuous on $I \times I_0$. Then, for any $(x_0, y_0) \in I \times I_0$, there is an interval J, with $x_0 \in J \subset I$, such that the first-order initial value problem*

$$\frac{dy}{dx} = f(x,y), \quad \textit{with} \ \ y(x_0) = y_0, \tag{7.19}$$

has a unique solution defined on the interval J.

If the interval $I_0 = \mathbb{R}$, then the unique solution exists on all of I.

Although the proof of this theorem is highly technical and will not be attempted in this class we will briefly outline some of the main points in the proof.

We first transform the differential equation (7.19) by integrating both sides,

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds.$$

We can then prove that a solution exists by constructing better and better approximations to the solution using the following definition of a sequence of functions,

$$\begin{aligned}
\bar{y}_0(x) &= y_0, \\
\bar{y}_1(x) &= y_0 + \int_{x_0}^{x} f(s, \bar{y}_0(s))\, ds, \\
\bar{y}_2(x) &= y_0 + \int_{x_0}^{x} f(s, \bar{y}_1(s))\, ds, \\
&\ \vdots \\
\bar{y}_{k+1}(x) &= y_0 + \int_{x_0}^{x} f(s, \bar{y}_k(s))\, ds.
\end{aligned}$$

This is known as ***Picard iteration***.

It can then be shown (but we will not attempt it) that this sequence of functions $\{\bar{y}_k(x)\}$ is convergent and that the limit is a solution to the problem. This proves that a solution exists; uniqueness is more complicated.

All we need is an understanding that, under reasonably simple conditions on the function $f(x,y)$ close to the point (x_0, y_0), the initial value problem has a unique solution that exists close to x_0. [In fact, given that we could linearise the nonlinear function $f(x,y)$, close to (x_0, y_0), to give a linear first-order problem, we might have guessed that existence and uniqueness might follow.]

We can therefore be confident that: (a) a solution always exists to a first-order ODE with an associated initial condition (if the function $f(x,y)$ is defined and smooth enough, and we consider some region close to (x_0, y_0)); and (b) if we find a solution to the problem, it will be the only solution (again at least close to the initial condition $y(x_0) = y_0$).

Method

Picard iteration for the initial value problem $\dfrac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$,

- Start a sequence of functions with $\bar{y}_0 = y_0$.
- Create successive terms in the sequence using $\bar{y}_{k+1}(x) = y_0 + \displaystyle\int_{x_0}^{x} f(s, \bar{y}_k(s))\, ds$.
- (Possibly: recognise the sequence of functions $\{\bar{y}_k\}$ is tending towards some analytic function.)

Examples 7.6

Use Picard iteration to find solutions to the following initial value problems

(a) Solve: $\dfrac{dy}{dx} = y, \quad y(0) = 1.$

(b) Solve: $\dfrac{dy}{dx} = 2x(1 + y), \quad y(0) = 0.$

✓ Watch Video: Example 7.6

✓ Solve Exercise: Tutorial question 3.

7.7 Links to linear algebra

In the linear algebra section of this class we saw that the set of all real-valued functions, which we denote by $\mathcal{F}$, was a vector space over the field $\mathbb{R}$ because the space is closed under addition and scalar multiplication,

$$\begin{aligned}(f+g)(x) &= f(x)+g(x),\\ (\alpha f)(x) &= \alpha f(x),\end{aligned}$$

for $\alpha \in \mathbb{R}$.

We will then call a subspace of $\mathcal{F}$ a ***function space***. For instance, we previously saw that the set of all polynomials of degree at most n, $\mathcal{P}_n$, was a vector space, and is clearly a subspace of all real-valued functions.

Another function space that we have encountered in this part of the course is the space of all continuous functions on the interval $[a,b]$, denoted by $C[a,b]$.

In this Chapter we have seen that the solution to a first-order ODE involves an arbitrary constant. For instance, the solution to

$$\frac{dy}{dx} = 2y,$$

is $y(x) = Ce^{2x}$, where C is a constant. This definition of a general solution defines a ***solution space***, $\mathcal{S}$, and it is relatively easy to see that this forms a vector space since the linear combination of two functions in the space $y(x) = a\,C_1 e^{2x} + b\,C_2 e^{2x} = (a\,C_1 + b\,C_2)e^{2x}$ is also a solution to the ODE.

This solution space has one arbitrary constant and is spanned by the set of all multiples of e^{2x} so that $\mathcal{S} = \text{span}(\{e^{2x}\})$ and in fact $\{e^{2x}\}$ is a basis for the space.

If we now rearrange the equation above to be

$$\frac{dy}{dx} - 2y = 0,$$

we can use operator notation to write

$$L(y) = D(y) - 2(y) = 0,$$

where we have defined the linear mapping $L = D - 2$. This equation tells us that the linear mapping L maps a function in the space of all real-valued differentiable functions to the zero function.

7.8 Summary

In this Chapter we have considered first-order ordinary differential equations. We found solutions to special types of ODEs, namely: ***separable equations***; equations which can be made separable through a ***change of variables***; ***exact equations***; and equations that can be made exact by using an ***integrating factor***. We also discussed the existence and uniqueness of solutions of first-order ODEs and the link between the general solutions and vector spaces. In the next Chapter we will consider higher-order ODEs, concentrating mainly on second-order ODEs.

✓ Complete Quiz: First-order differential equations