University of Strathclyde
Department of Mathematics and Statistics

MM102: Applications of Calculus

Lecture Notes for Week 4

Contents

3 Further Techniques of Differentiation

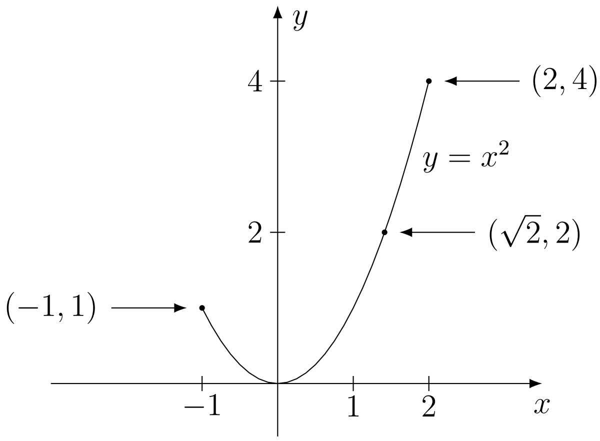
In this section we find tangents and normals to graphs of functions that are given explicitly, implicitly or parametrically. For this we have to learn about implicit differentiation and parametric differentiation. Moreover, we shall determine the length of a parametrically given curve and the surface area when such a curve is rotated.

3.1 Tangents and Normals

Recall that the equation of a straight line with slope m through a point (a, b) is

$$y - b = m(x - a). \tag{3.1}$$

Let us consider a differentiable function f with domain $\mathrm{dom}(f)$. The **graph** of the function f is the set of points (x, y) in the plane for which $x \in \mathrm{dom}(f)$ and $y = f(x)$, or in other words, it is the set of all points $(x, f(x))$ with $x \in \mathrm{dom}(f)$. For instance, the graph of the function $f(x) = x^2$ defined on $\mathrm{dom}(f) = [-1, 2]$ is



Let $(a,b) = (a, f(a))$ be one point on the graph. The **tangent** at this point is the line touching the graph of f at (a,b) and having the same slope as the graph. The slope is equal to the derivative at the point a, i.e. $m = f'(a)$; see the class MM101. Hence the equation of the tangent is given by (3.1) with

$$b = f(a) \qquad \text{and} \qquad m = f'(a).$$

The **normal** at (a,b) is the line that is perpendicular to the tangent and also passes through (a,b). Its equation is given again by (3.1) with the same a and b but with

$$m = -\frac{1}{f'(a)}$$

provided that $f'(a) \neq 0$. We can also write this as

$$m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}}\,. \tag{3.2}$$

If $f'(a) = 0$, then the tangent is a horizontal line: $y = b$; the normal is a vertical line and given by the equation $x = a$.

Example 3.1.
Find the equations of the tangent and the normal to the graph of the function $f(x) = x^2$ with domain $\mathrm{dom}(f) = \mathbb{R}$ at $x = 3$. We have

$$a = 3, \qquad b = f(3) = 3^2 = 9,$$

$$f'(x) = 2x \qquad \Longrightarrow \qquad f'(3) = 6.$$

For the tangent we have $m = f'(3) = 6$ and hence the equation of the tangent is

$$y - 9 = 6(x-3) \qquad \text{or} \qquad y = 6x - 9.$$

For the normal we have $m = -\dfrac{1}{6}$ and hence the equation is

$$y - 9 = -\frac{1}{6}(x-3) \qquad \text{or} \qquad y = -\frac{1}{6}x + \frac{19}{2}\,.$$

3.2 Implicit Differentiation

Sometimes the relation $y = f(x)$ is not given explicitly by a formula but only by an equation relating x and y; we say that the function f is given **implicitly**; see also MM101, Section 4.8. The solutions of certain differential equations are sometimes given only in implicit form; see the forthcoming chapter on differential equations.

Let us consider the following equation

$$y^3 + x^2 e^y = 2; \tag{3.3}$$

it cannot be solved explicitly for y. Nevertheless, it can be shown that for every $x \in \mathbb{R}$, there exists exactly one y that solves the equation and hence gives the value $f(x) = y$, i.e. the equation

$$\big(f(x)\big)^3 + x^2 e^{f(x)} = 2 \tag{3.4}$$

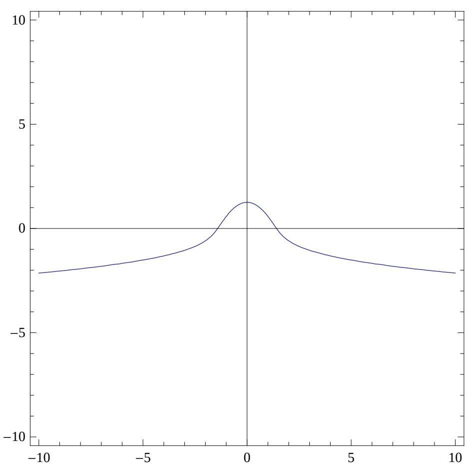
is valid. Although we do not have an explicit formula for $f(x)$, we can still find its derivative by differentiating both sides of (3.4) with respect to x. Using chain and product rules we get

$$3\big(f(x)\big)^2 f'(x) + 2xe^{f(x)} + x^2 e^{f(x)} f'(x) = 0.$$

This equation can be solved for $f'(x)$:

$$\begin{aligned} f'(x)\Big(3\big(f(x)\big)^2 + x^2 e^{f(x)}\Big) &= -2xe^{f(x)} \\ \Rightarrow \quad f'(x) &= -\frac{2xe^{f(x)}}{3\big(f(x)\big)^2 + x^2 e^{f(x)}}\,. \end{aligned}$$

The curve defined by equation (3.4) looks as follows.



In practice it is better to write y instead of $f(x)$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ or y' instead of $f'(x)$.

Implicit Differentiation

(a) Differentiate both sides with respect to x bearing in mind that y is a function of x. (Apply chain and product rules!)

(b) Put all terms with $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ on one side and the other terms on the other side.

(c) Take out the common factor $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(d) Solve for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Example 3.2.

Let us consider the example at the beginning of the section again but now using shorter notation:

$$
\begin{aligned}
& y^3 + x^2 e^y = 2 \\
\Longrightarrow \quad & 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^y + x^2 e^y \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\
\Longrightarrow \quad & \frac{\mathrm{d}y}{\mathrm{d}x}\big(3y^2 + x^2 e^y\big) = -2xe^y \\
\Longrightarrow \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xe^y}{3y^2 + x^2 e^y}
\end{aligned}
$$

Example 3.3.

Calculate the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ of the function $y = f(x)$ defined by the equation

$$\ln(x^2 + y) + \sin x \cdot \cos y + y = 1.$$

Solution in video

Sometimes an implicit equation does not determine y uniquely for given x, as the following example shows. However, often the following situation is true (sufficient conditions are provided by Implicit Function Theorem; see second year): if for some x_0 one has a y_0 that solves the implicit equation, there exists an interval (a, b) such that $x_0 \in (a, b)$ and that for every $x \in (a, b)$ there exists a unique y so that the function that is defined by the mapping $x \mapsto y$ is continuous (and even differentiable).

Example 3.4.
The equation

$$x^2 + y^2 = 1 \tag{3.5}$$

defines two functions:

$$y = f_1(x) = \sqrt{1-x^2}, \qquad \mathrm{dom}(f_1) = [-1, 1];$$

$$y = f_2(x) = -\sqrt{1-x^2}, \quad \mathrm{dom}(f_2) = [-1, 1].$$

Using implicit differentiation we can calculate the derivatives of both functions. Differentiate both sides of (3.5) with respect to x:

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\Rightarrow \quad 2y\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \qquad \text{if } y \neq 0 \ \text{ (i.e. } x \in (-1, 1)\text{)}.$$

We can check this result by, e.g. evaluating $f_2'(x)$:

$$f_2'(x) = -\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x) = \frac{x}{\sqrt{1-x^2}} = -\frac{x}{-\sqrt{1-x^2}} = -\frac{x}{y}$$

for $x \in (-1, 1)$.

Tangents at curves given by implicit equations
If we know one point (a, b) on a curve that is given by an implicit equation, then we can determine the equation of the tangent at this point: in the expression for the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is usually given in terms of both x and y, we replace x by a and y by b; this gives m. Then we can plug a, b and m into equation (3.1). The normal can be found using relation (3.2).

Example 3.5.
Verify that the point $P = (2, -1)$ lies on the curve

$$x^2 + xy + y^3 = 1. \tag{$*$}$$

Hence find the equation of the tangent and the normal to this curve at the point P.

Solution in video

Second derivative

Once the first derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ has been found and expressed in terms of x and y, one can differentiate this expression with respect to x to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Occurrences of $\frac{\mathrm{d}y}{\mathrm{d}x}$ that might appear should be replaced by the expression already found for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Alternatively, one can differentiate the equation one has got after one differentiation a second time and then solve for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Again, occurrences of $\frac{\mathrm{d}y}{\mathrm{d}x}$ have to be replaced.

Example 3.6.

Find the first and second derivatives, $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, of the function implicitly defined by the equation

$$\sin y + y = x^2.$$

Solution in video

Example 3.7.

Find the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ of the function implicitly defined by the equation

$$x^2 + y^2 = 1. \tag{3.6}$$

In Example 3.4 we have found the first derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}, \qquad y \neq 0.$$

Differentiate this with respect to x:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}(x)y - x\frac{\mathrm{d}}{\mathrm{d}x}(y)}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y + \frac{x^2}{y}}{y^2}$$

$$= -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3},$$

where in the last step we used (3.6).

Alternatively, we can differentiate the relation

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

which gives

$$1 + \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

If we replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $-\frac{x}{y}$ and solve for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, we obtain

$$1 + \frac{x^2}{y^2} + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \Rightarrow \quad y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{y^2 + x^2}{y^2} \quad \Rightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^3}.$$

Note that we have not discussed the question whether, for a given equation in x and y, there exists a function $y = f(x)$ that satisfies this equation. This question is, in general, not easy to answer. Sometimes, one function is not enough, but one needs several functions to describe all branches of the curve as for the circle in Example 3.4. Existence of local solutions is sometimes guaranteed by the Implicit Function Theorem, which will not be discussed in this lecture. In the examples and exercises in this lecture there always exists at least one such function that solves the equation, and you will have to calculate its derivative.

3.3 Parametric Differentiation

Sometimes a curve in the x, y-plane is given by two functions u and v depending on some parameter t varying in some interval I; see MM101, Section 4.1. The functions u and v describe the x and y coordinates of points on the curve; so the curve is given by all points (x, y) with

$$\begin{aligned} x &= u(t) \\ y &= v(t) \end{aligned} \qquad t \in I.$$

For example, the functions

$$\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned} \qquad t \in [0, 2\pi]$$

describe the unit circle. One could also let $t \in \mathbb{R}$; then all points on the circle would be obtained infinitely many times. A parametric representation is particularly useful if $(u(t), v(t))$ is the position of a particle at time t.

We want to calculate the slope and hence the tangent at a point of the curve. Assume that the curve (at least locally) is also given by the graph of a function f, i.e. $y = f(x)$. This means that the equality

$$v(t) = f\big(u(t)\big)$$

is satisfied for $t \in I$. If we differentiate both sides of this equality with respect to t and apply the chain rule, we obtain the relation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = f'\big(u(t)\big)\frac{\mathrm{d}u(t)}{\mathrm{d}t}\,. \tag{3.7}$$

Observing that we can write $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}x}{\mathrm{d}t}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$ for $f'(u(t))$, $\frac{\mathrm{d}u(t)}{\mathrm{d}t}$, $\frac{\mathrm{d}v(t)}{\mathrm{d}t}$, respectively, and solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$, we obtain

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\;\frac{\mathrm{d}y}{\mathrm{d}t}\;}{\frac{\mathrm{d}x}{\mathrm{d}t}}} \tag{3.8}$$

Often the notation $\dot{x}$ is used for $\dfrac{\mathrm{d}x}{\mathrm{d}t}$; then relation (3.8) becomes

$$\boxed{\textbf{Parametric Differentiation:} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}}}$$

Examples 3.8.

(i) Find the first derivative of the function given by the parametric description

$$\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned} \qquad t \in \mathbb{R}.$$

We have

$$\begin{aligned} \dot{x} &= -\sin t \\ \dot{y} &= \cos t \end{aligned}$$

and hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{\cos t}{-\sin t} = -\cot t.$$

(ii) Find the equation of the tangent to the parametric curve

$$\begin{aligned} x &= t^2 \\ y &= t + \frac{1}{t} \end{aligned} \qquad t > 0$$

at $t = \frac{1}{2}$.

Solution in video

Second derivative

To find the second derivative $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, set

$$z(t) := \frac{\mathrm{d}y}{\mathrm{d}x}(t) = \frac{\dot{v}(t)}{\dot{u}(t)}$$

and consider again the relation (3.7), which we write as

$$f'\big(u(t)\big) = \frac{\dot{v}(t)}{\dot{u}(t)} = z(t).$$

Differentiating both sides with respect to t we obtain

$$f''\big(u(t)\big)\dot{u}(t) = \dot{z}(t)$$

and hence

$$f''\big(u(t)\big) = \frac{\dot{z}(t)}{\dot{u}(t)}\,.$$

This means, one differentiates the first derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (which is a function of t) with respect to t and divides by $\dot{x}$.

Parametric Differentiation (Second Derivative):

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\dot{z}}{\dot{x}} \qquad \text{where} \quad z(t) = \frac{\mathrm{d}y}{\mathrm{d}x}(t) = \frac{\dot{y}(t)}{\dot{x}(t)}$$

Examples 3.9.

(i) Find the first and second derivatives of the function defined by the parametric representation:

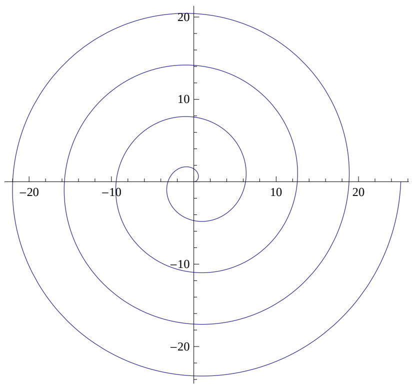
$$\begin{aligned} x &= 3t^2 \\ y &= 4t - 3t^2 \end{aligned} \qquad (t > 0).$$

Solution in video

(ii) Find the first and second derivatives of the the function defined by the parametric representation:

$$\begin{aligned} x &= t\cos t \\ y &= t\sin t \end{aligned} \qquad (t > 0).$$

This parametric curve is a spiral:



Differentiate with respect to t:

$$\begin{aligned} \dot{x} &= \cos t - t\sin t \\ \dot{y} &= \sin t + t\cos t. \end{aligned}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{\sin t + t\cos t}{\cos t - t\sin t}\,.$$

Set

$$z(t) := \frac{\sin t + t\cos t}{\cos t - t\sin t}\,.$$

The derivative of z with respect to t is

$$\begin{aligned}
\dot{z} &= \frac{\big(\cos t + \cos t + t(-\sin t)\big)(\cos t - t\sin t) - (\sin t + t\cos t)(-\sin t - \sin t - t\cos t)}{(\cos t - t\sin t)^2} \\
&= \frac{(2\cos t - t\sin t)(\cos t - t\sin t) - (\sin t + t\cos t)(-2\sin t - t\cos t)}{(\cos t - t\sin t)^2} \\
&= \frac{2\cos^2 t - 3t\sin t\cos t + t^2\sin^2 t - (-2\sin^2 t - 3t\sin t\cos t - t^2\cos^2 t)}{(\cos t - t\sin t)^2} \\
&= \frac{2\cos^2 t + t^2\sin^2 t + 2\sin^2 t + t^2\cos^2 t}{(\cos t - t\sin t)^2} \\
&= \frac{2(\cos^2 t + \sin^2 t) + t^2(\cos^2 t + \sin^2 t)}{(\cos t - t\sin t)^2} \\
&= \frac{2 + t^2}{(\cos t - t\sin t)^2}\,.
\end{aligned}$$

Hence

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\dot{z}}{\dot{x}} = \frac{2 + t^2}{(\cos t - t\sin t)^3}\,.$$

The tangent to a parametric curve can be interpreted as follows. Suppose that a particle moves along a parametric curve according to some external forces. If these external forces are suddenly switched off, the particle moves along the tangent.

3.4 Arc Lengths of Parametric Curves

We want to determine the arc length of a curve that is given in parametric form

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \qquad t \in [a, b].$$

As in Section 2.2 of this lecture we can approximate the arc length by a sequence of straight lines. For a partition $P = \{t_0, t_1, \ldots, t_n\}$ of the parameter interval $[a, b]$, the approximation for the arc length is

$$\sum_{j=1}^{n} \sqrt{\big(x(t_j) - x(t_{j-1})\big)^2 + \big(y(t_j) - y(t_{j-1})\big)^2}$$

$$= \sum_{j=1}^{n} \sqrt{\left(\frac{x(t_j) - x(t_{j-1})}{t_j - t_{j-1}}\right)^2 + \left(\frac{y(t_j) - y(t_{j-1})}{t_j - t_{j-1}}\right)^2}\, \Delta t_j \qquad \text{(where } \Delta t_j = t_j - t_{j-1}\text{)}.$$

One can show that this converges to an integral and hence the arc length of the parametric curve is equal to

$$\boxed{s = \int_a^b \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\, \mathrm{d}t}$$

where $\dot{}$ denotes again the derivative with respect to the parameter t.

Examples 3.10.

(i) A parametric curve is given by

$$x = \frac{4}{3}\, t^{3/2} \qquad y = \frac{1}{2}(t-1)^2 \qquad t \in [1, 2].$$

Find the arc length of the curve.

The derivatives with respect to t are

$$\dot{x} = 2\, t^{1/2}, \qquad \dot{y} = t - 1.$$

Hence the arc length is

$$s = \int_1^2 \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\, \mathrm{d}t = \int_1^2 \sqrt{4t + (t-1)^2}\, \mathrm{d}t$$

$$= \int_1^2 \sqrt{4t + t^2 - 2t + 1}\, \mathrm{d}t = \int_1^2 \sqrt{t^2 + 2t + 1}\, \mathrm{d}t$$

$$= \int_1^2 (t+1)\, \mathrm{d}t = \left[\frac{1}{2}t^2 + t\right]_1^2$$

$$= \frac{1}{2} \cdot 4 + 2 - \left(\frac{1}{2} + 1\right) = \frac{5}{2}\,.$$

(ii) A parametric curve is given by

$$\begin{aligned} x &= e^t \cos t \\ y &= e^t \sin t \end{aligned} \qquad t \in [0, \pi].$$

Find the arc length of the curve.

Solution in video

3.5 Areas of surfaces obtained when parametric curves are rotated

When a parametric curve is rotated about the x-axis, a surface is obtained whose area is given by the following formula

$$\boxed{S = 2\pi \int_a^b y(t) \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\, \mathrm{d}t.}$$

Example 3.11.
Find the surface area of a sphere with radius r that is obtained from rotating the parametric curve

$$\begin{aligned} x &= r \cos t \\ y &= r \sin t \end{aligned} \qquad t \in [0, \pi]$$

about the x-axis through $360°$.

Solution in video