UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM201 Linear Algebra and Differential Equations

6 Revision: ordinary differential equations

6.1 What you know already!

The mathematical model of many real-world problems describe relationships between changing quantities. You have learnt during your first year of studies, that rates of change can be mathematically described by derivatives and therefore the mathematical model often leads to equations relating an unknown function and its derivatives. Such equations are called **differential equations.**

In previous classes you will have encountered a number of ideas and techniques which will be used in this class to solve various types of differential equations. In this Section the most important parts of these previous classes will be summarised so that you can, if necessary, reacquaint yourself with this subject area.

During MM101 - 'Introduction to Calculus' you encountered the concepts of differentiation and integration. You saw that the derivative can be thought of as the 'slope' of a function and the integral as the 'area under the curve'. Differentiation and integration of common functions (polynomial, trigonometric, exponential functions etc.) was discussed and this class assumes that you are relatively happy with such calculations. One of the most important parts of MM101, certainly for the present class discussed the First and Second 'Fundamental Theorems of Calculus'. These theorems will be assumed and used extensively in this class and so are repeated here:

<u>The First Fundamental Theorem of Calculus:</u>

If f is continuous on $[a, b]$, then F defined on $[a, b]$ as $F(x) = \int_a^x f(s)\, ds$, is continuous and differentiable at all $x \in (a, b)$ and $F'(x) = f(x)$.

This theorem tells us that for every continuous function f there is an 'anti-derivative' F for which $F' = f$. This will be very useful when we are faced with solving an equation like $F' = f$, and want to find F.

<u>The Second Fundamental Theorem of Calculus:</u>

If $f(x)$ is continuous on $[a, b]$ and $f(x) = F'(x)$ for some function F, then

$$\int_a^b f(s)\, ds = F(b) - F(a).$$

This theorem means that if we know the anti-derivative of a function we can evaluate the integral by simply evaluating the anti-derivative at the end points.

During MM102 - 'Applications of Calculus' these theorems were reiterated and used to evaluate various derivatives and integrals. Also in MM101 and MM102, the following basic properties of differentiation and integration were introduced:

Linearity of differentiation and integration:

$$\begin{aligned} \frac{d}{dx}\left(a\,f(x) + b\,g(x)\right) &= a\frac{df}{dx}(x)0 + b\frac{dg}{dx}(x), \\ \int \left(a\,f(u) + b\,g(u)\right) du &= a\int f(u)du + b\int g(u)du. \end{aligned}$$

Note: This means that differentiation and integration are linear mappings.

Product, quotient and chain rules for differentiation:

$$\begin{aligned} \frac{d}{dx}\left(f(x)g(x)\right) &= g(x)\frac{df}{dx}(x) + f(x)\frac{dg}{dx}(x), \\ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{1}{g(x)^2}\left(g(x)\frac{df}{dx}(x) - f(x)\frac{dg}{dx}(x)\right), \\ \frac{d}{dx}\left(f(g(x))\right) &= \frac{dg}{dx}(x)\frac{df}{dg}(g(x)). \end{aligned}$$

Integration by parts and substitution:

$$\begin{aligned} \int f(x)\frac{dg}{dx}(x)\,dx &= f(x)g(x) - \int g(x)\frac{df}{dx}(x)\,dx, \\ \int f(u)\,du &= \int f(u(x))\frac{du}{dx}\,dx. \end{aligned}$$

You first met the concept of a differential equation in MM102 and we will review most of the relevant Chapters in the first part of this class.

6.2 Definitions and terminology

Definition 6.1 *A* ***differential equation*** *is a mathematical equation which involves unknown functions and their derivatives with respect to the variables upon which they depend.*

For instance, if y is a function of x, i.e. we can write $y(x)$, then a differential equation would be an equation which includes a derivative of y with respect to x. The following equation,

$$\frac{dy}{dx} = xy, \tag{6.1}$$

is an example of a differential equation.

These types of equations are important for many reasons, for example the concept of 'rate of change' is essential to understand behaviour of systems in the real-world. Since rates of change are expressed mathematically by derivatives (remember that the derivative dy/dx gives the rate of change of y with respect to x) they naturally play an important role in the application of mathematics to practical problems.

Examples include economics (rate of change of a commodity price with respect to time), chemical reactions (rate of change of chemical concentrations), radio-active decay (rate of change of the amount of an isotope with respect to time), temperature changes (air temperature as altitude changes), population dynamics (rate of change of population numbers with respect to time or location within a country), etc. Differential equations therefore provide a fundamental way of constructing mathematical models of real life phenomena.

When such models are being developed, the symbols used for functions and variables are usually chosen to remind us of the quantity being studied, e.g. t for time, v for velocity, T for temperature, etc. However, in these notes we will generally use the generic variables x and y, with y being the unknown function of x which is to be determined.

The unknown function, i.e. y, is often termed the *dependent variable*, while the quantity it depends on, i.e. x, is termed the *independent variable*. So, for example, if the temperature T of a metal bar was a function of time t and, in order to find the function $T(t)$, we had to solve an equation which included the unknown function $T(t)$ and its derivatives dT/dt, d^2T/dt^2, etc. then T is called the dependent variable and t is the independent variable. There may be other known or unknown constants in an equation, for example the length of the metal bar, and these are termed *parameters*. A solution to a differential equation will therefore be a function which depends on independent variables and parameters.

Definition 6.2 *For a differential equation such as (6.1), x is referred to as the* ***independent variable*** *and y is referred to as the* ***dependent variable****.*

In the above example, the unknown function T was only dependent on one independent variable t and in this case the equation involving T and the derivatives of T is called an *ordinary differential equation* or an *ODE* for short. However, the temperature in the metal bar may also be a function of the position along the bar x. In this case we need to find the function $T(x,t)$, which now depends on two independent variables, the position x and time t. The equation we need to solve will include the temperature T and the partial derivatives $\partial T/\partial t$, $\partial T/\partial x$, $\partial^2 T/\partial x^2$ etc. and this equation is then called a *partial differential equation* or a *PDE*. Partial differential equations will be studied in subsequent classes and in this class we will concentrate on techniques for the solution of ordinary differential equations.

In general, an ODE for an unknown function y of the variable x takes the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0, \tag{6.2}$$

i.e. the left hand side (LHS) is an expression involving the variable x, the unknown function $y(x)$ and its derivatives. If we assume that equation (6.5) can be rearranged to put the highest derivative $\dfrac{d^ny}{dx^n}$ in terms of everything else that appears in the equation, then we can consider,

$$\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right), \tag{6.3}$$

as a general form of an ODE. This form is often useful for theoretical discussions concerning existence and uniqueness of the solution.

It will be useful to group certain differential equations in to different classifications. We will use the following definitions to do this:

Definition 6.3 *The **order** of an ODE is the highest derivative appearing in the equation.*

Definition 6.4 *An ODE is **linear** if it is linear in the unknown function and its derivatives, that is it takes the form*

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots + a_1(x)\, y' + a_0(x)\, y = f(x). \tag{6.4}$$

Here we have used the notation $y^{(n)}$ for the n^{th} derivative $\dfrac{d^ny}{dx^n}$ — the brackets on the superscript are to distinguish this from the n^{th} power of y. For the first few derivatives we will also use the $'$ notation, i.e. $y' = dy/dx$ and $y'' = d^2y/dx^2$.

The functions of x, $a_i(x)$, $i = 1, 2, \ldots, n$ and $f(x)$ can take any form - they need not be linear functions.

Definition 6.5 *An equation that is not linear is **nonlinear**.*

We now introduce the definition of a linear mapping between two vector spaces:

Definition 6.6 *A mapping T is **linear** if*

$$T(\alpha \mathbf{v}) = \alpha\, T(\mathbf{v}) \qquad \text{and} \qquad T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})\ ,$$

where α is a scalar.

Differentiation (and integration) can be viewed of as a mapping. If we let $D = \dfrac{d}{dx}$ be a mapping from the space of differentiable functions to the space of all functions. Differentiation is linear because

$$D(ay_1(x) + by_2(x)) \;=\; a\frac{dy_1}{dx}(x) + b\frac{dy_2}{dx}(x) = aD(y_1(x)) + bD(y_2(x)),$$

where a and b are scalars and $y_1(x)$ and $y_2(x)$ are two functions in the space of differentiable functions.

Using this notation, i.e. using D to denote the mapping which is equivalent to differentiation, is often called 'operator notation'. With this notation we can also write multiple differentiation as multiple applications of the mapping,

$$D^2y := D(Dy) = \frac{d^2y}{dx^2}, \qquad \text{i.e.}\quad D^2 = \frac{d^2}{dx^2}$$
$$D^3y := D(D^2y) = \frac{d^3y}{dx^3}, \qquad \text{i.e.}\quad D^3 = \frac{d^3}{dx^3}$$

and in general

$$D^ny = \frac{d^ny}{dx^n}, \qquad \text{i.e.}\quad D^n = \frac{d^n}{dx^n}.$$

$D^2,\ D^3, \ldots$ are all linear mappings. Linear combinations of powers of D are again linear mappings.

This definition means that we say that an ODE is linear if it can be written in the following way,

$$\begin{aligned} L(y(x)) &= f(x), \text{where} \\ L &= a_n(x)\, D^{(n)} + a_{n-1}(x)\, D^{(n-1)} + \ldots + a_1(x)\, D^1 + a_0(x), \end{aligned}$$

so that L is a linear combination of linear mappings.

✓ Complete Quiz: Basic terminology

✓ Solve Exercise: Tutorial question 1.

6.3 The solution of an ODE

Consider an ODE

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0, \tag{6.5}$$

for some function F. A function $y = f(x)$ is called a **solution** of the ODE if

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right)$$

equals zero, when

$$y \text{ is replaced by } f(x), \quad \frac{dy}{dx} \text{ is replaced by } f'(x), \frac{d^2y}{dx^2} \text{ is replaced by } f''(x) \text{ and so on }.$$

Given an ODE for y and its derivatives in terms of independent variable x, our aim is to find a function $y = f(x)$ that satisfies the ODE.

Solving an ODE involves integration. Depending on the type of ODE the integration may be done explicitly (we can actually calculate appropriate integrals) or implicitly (we arrive at the solution without formally calculating integrals). In either case, we should expect arbitrary constants of integration to be involved.

In general, solving an n^{th} -order ODE (i.e. one involving the n^{th} derivative of the unknown function) requires n integrations and each integration produces a new arbitrary constant. Therefore, we must expect at most n independent arbitrary constants in the solution of an n^{th} -order ODE.

Definition 6.7 *A solution of an ODE which contains independent arbitrary constants is called* ***a general solution****.*

If we have found a general solution to an ODE it is natural to ask if this general form contains every possible solution to the differential equation. When we know that it does, we call it ***the*** general solution.

Definition 6.8 *A general solution of an ODE which includes every possible solution to the ODE is called* ***the general solution****.*

We may be given additional conditions on the solution of the ordinary differential equation, leading to specific values of the arbitrary constants:

Definition 6.9 *A solution obtained by assigning specific values to each of the arbitrary constants in the general solution is called a* ***particular solution****.*

The solution of a differential equation which is describing mathematically a real-life process has often to satisfy certain specified conditions corresponding to the values taken by physical quantities at particular instants. For instance, the initial temperature of the metal bar mentioned above might be known at some initial time.

Definition 6.10 *If the conditions are all given at the same value of the independent variable, (usually the value at which the process being modelled is starting, e.g. the position and velocity of a body), they are called* ***initial conditions****. The ODE and initial conditions together constitute an* ***initial value problem*** *(IVP).*

Definition 6.11 *When the conditions are given at two or more values of the independent variable, (often the smallest and largest values that you wish the independent variable to take, e.g. the concentration of a dissolved chemical at two different times), they are known as* ***boundary conditions****. With the ODE this results in a* ***boundary value problem*** *(BVP).*

To determine uniquely the n arbitrary constants in the general solution of an n^{th} -order ODE we normally need to specify n conditions.

Examples 6.1

(a) Verify that $y = Ce^{3x}$ is a general solution of the ODE $\dfrac{dy}{dx} = 3y$ for any constant C.

(b) What is a particular solution in the last example if we know that $y = 2$ when $x = 1$?

(c) Verify that $y = Ae^{x} + Be^{2x}$ is a general solution of the ODE $y'' - 3y' + 2y = 0$ for any constants A and B.

It is good practice to check any solution you obtain by substitution in the original differential equation.

✓ Watch Video: Example 6.1

✓ Solve Exercise: Tutorial question 2-4.

6.4 Solution curves

We have seen in the example above that the general solution of the ODE $\dfrac{dy}{dx} = 3y$ is $y = Ce^{3x}$ for any constant C.

This general solution, $y = Ce^{3x}$, represents a **family of solutions** or a set of solutions. Each value of C corresponds to a curve in this family. Using terminology you learnt in the first half of this module you can easily verify that the set of all solution of this ODE forms a vector space of functions. Indeed, the sum of two solutions is again a solution and if you multiply a solution with a constant then you will also get a solution. The basis of this vector space is the function e^{3x} and so the dimension of the vector space is 1.

Figure 6.1 shows some of the solution curves, for values $C = -1, 0, 1, 2, 3$, plotting just in the interval $x \in [0, 1]$.

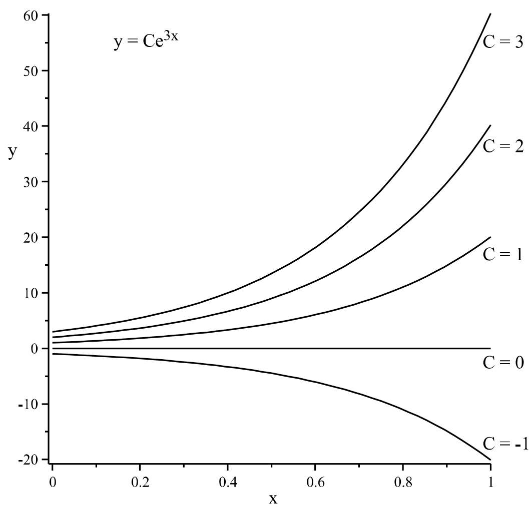

Figure 6.1: Plots of $y = Ce^{3x}$ for various values of C.

6.5 Summary

In this Chapter we have reviewed various concepts and definitions that you have seen in previous classes. This Chapter forms the basis of what is to come later (in this class and others) and it is extremely important that you are comfortable with all these concepts.

In the next few Chapters we will revisit a number of other topics that you encountered in previous classes (notably in MM102 'Applications of Calculus'), taking the mathematical analysis further and using various concepts from the linear algebra section of this class to further understand the area of differential equations.