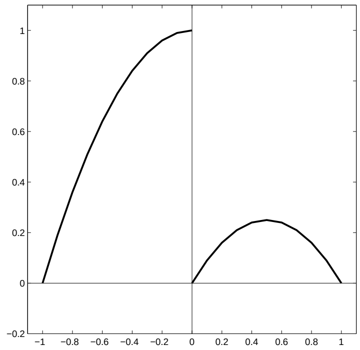
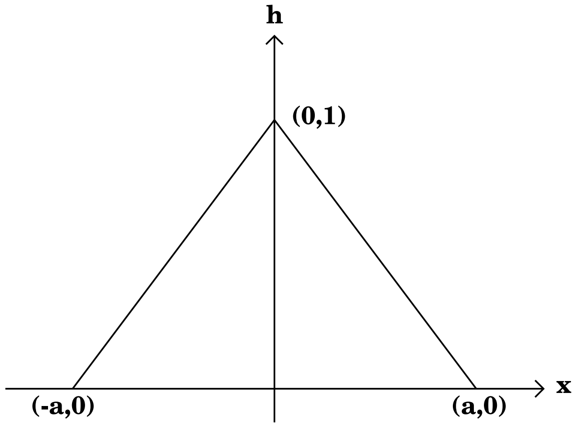
10 Continuity

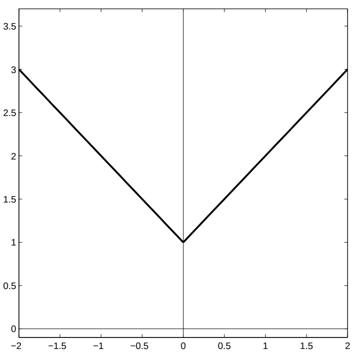
10.1 $u(-1) = u(0) = u(1) = 0, \quad 1 - x^2|_{x=0} = 1, \quad x(1-x)|_{x=0} = 0$



10.2 For $-a \leq x < 0, h(x)$ is a straight line: $h(-a) = 0$ and $\left(1 + \frac{x}{a}\right)\Big|_{x=0} = 1.$
For $0 \leq x \leq a, h(x)$ is a straight line: $h(a) = 0 \;\; h(0) = 1.$

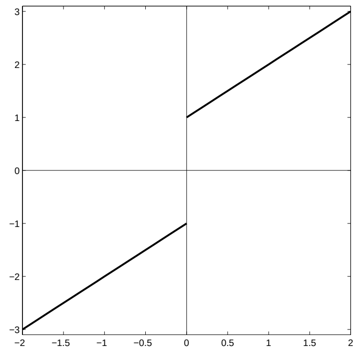


10.3 (a) $\lim_{x \to 0^-} f_a(x) = 1 = \lim_{x \to 0^+} f_a(x)$



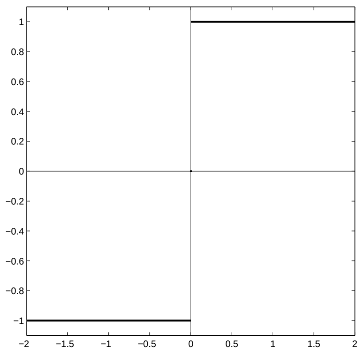
No jumps.

10.3 (b) $\lim_{x \to 0^-} f_b(x) = -1 \ \lim_{x \to 0^+} f_b(x) = 1$

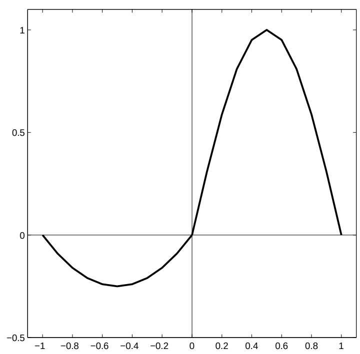


Jump of height 2 at $x = 0$.

(c) Two jumps of height 1 at $x = 0$.

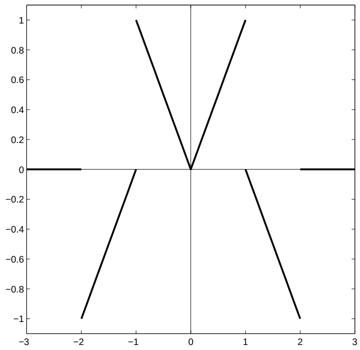


(d) $\lim_{x \to 0^-} f_d(x) = 0 = \lim_{x \to 0^+} f_d(x)$



No jumps.

10.3 (e) We have $f(-2^-) = 0$ and $f(-2^+) = -1$; $f(-1^+) = 0$ and $f(-1^+) = +1$; $f(1^-) = 1$ and $f(1^+) = 0$; $f(2^-) = -1$ and $f(2^+) = 0$.



Jumps of 1 at -2, 1 at -1, 1 at 1 and 1 at 2.

10.4 (a) $x = 5$. Can be 'patched' since the limit of the expression at $x = 5$ is 10.

(b) $x = -5$, $\quad x = 5$. The discontinuity at $x = -5$ cannot be 'patched'.

(c) $x = -4$. Can be 'patched' since the limit of the expression at $x = -4$ is -1

(d) $x = 8$, $\quad x = -3$. The discontinuity at $x = -3$ cannot be 'patched'.

10.5 (a) $\lim_{x\to -2} \dfrac{x^2+6x+8}{x+2} = 2$, so hole at $(-2, 2)$.

(b) $\lim_{x\to -3} \dfrac{x^3+27}{x+3} = \lim_{x\to -3} \dfrac{(x^2-3x+9)(x+3)}{(x+3)} = 9+9+9 = 27$, so hole at $(-3, 27)$.

(c) $\lim_{x\to 12} \sqrt{\dfrac{x^2-4}{x-2}} = \lim_{x\to 2} \sqrt{x+2} = 2$ so hole at $(2, 2)$.

10.6 At $x = 0$ we have $|f(0)| \leq |0| = 0$ and so $f(0) = 0$. But also $\lim_{x\to 0} f(x) = 0$ as for any $\epsilon > 0$ we have with $\delta = \epsilon$ that

$$|f(x) - 0| = |f(x)| \leq |x| < \epsilon,$$

and so whenever $0 < |x| < \delta = \epsilon$ also $|f(x) - 0| < \epsilon$. Hence f is continuous at 0.

10.7 We first compute $f(0)$: for all x, by definition $f(x) = f(x+0) = f(x) + f(0)$, and so $f(0) = 0$. Since f is continuous at 0, $\lim_{h\to 0} f(h) = f(0) = 0$. Then

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0}(f(a) + f(h)) = f(a) + \lim_{h\to 0} f(h) = f(a)$$

shows that f is continuous at a.