UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercises: Chapter 5

1. State the degree of the following polynomials:

$$\text{(i)}\quad 6x^3 - 5x^2 + 10x - 15\,, \qquad \text{(ii)}\quad 3 + x - 15x^5 + x^7\,.$$

2. Classify the following as proper or improper rational functions:

$$\text{(i)}\quad \frac{x}{x^2+x+1}, \qquad \text{(ii)}\quad \frac{1}{x^2-4}, \qquad \text{(iii)}\quad \frac{x^3}{x^2-1},$$

$$\text{(iv)}\quad \frac{x^3+1}{x^3-1}, \qquad \text{(v)}\quad \frac{x}{(x-3)(x+4)}, \qquad \text{(vi)}\quad \frac{x(x+1)(x^2+4)}{x^3+2}.$$

3. Simplify the following rational functions by factorising the numerator and/or denominator and cancelling common factors:

$$\text{(i)}\quad \frac{x}{x^2+x}, \qquad \text{(ii)}\quad \frac{x-2}{x^2-4}, \qquad \text{(iii)}\quad \frac{x^3}{x^4+x^2},$$

$$\text{(iv)}\quad \frac{x^2+2x+1}{x^2-4x-4}, \qquad \text{(v)}\quad \frac{2-x}{x^4-16}, \qquad \text{(vi)}\quad \frac{x^3-3x^2+2x}{x^2+2x-8}.$$

4. Use long division to simplify these improper rational functions and identify the quotient and the remainder in each.

$$\text{(i)}\quad \frac{2x^2+x+7}{x-4}, \quad \text{(ii)}\quad \frac{3x^3+4x^2+2x+1}{x^2+2x+2}, \quad \text{(iii)}\quad \frac{4x^2+4x-2}{x-5}, \quad \text{(iv)}\quad \frac{x^3+2x^2-x-3}{-x^2+2x+1}.$$

5. Sketch the graphs of the following linear functions:
(i) $f(x) = 2x - 1$, (ii) $g(x) = 4 - x$, (iii) $h(x) = \dfrac{1}{5}x + 1$.

6. Solve the following linear equations:

$$\text{(i)}\quad 5x + 9 = 0, \qquad \text{(ii)}\quad 17 - 3x = 4, \qquad \text{(iii)}\quad \frac{3}{7}x + \frac{1}{5} = \frac{2}{3}.$$

7. Complete the square in the following quadratics and hence find the minimum value taken by each quadratic over all x values.

$$\text{(i)}\ x^2 + 4x - 7, \qquad \text{(ii)}\ -2x^2 + 3x + 1.$$

8. Solve the following quadratic equations:

$$\begin{array}{llll} \text{(i)} & x^2 - 8x + 15 = 0, & \text{(ii)} & 12 - 4x - x^2 = 0, \\ \text{(iii)} & 2x^2 - x - 3 = 0, & \text{(iv)} & x^2 - 6x + 6 = 0, \\ \text{(v)} & 3x^2 - 4x - 2 = 0, & \text{(vi)} & 2x^2 - x + 3 = 0. \end{array}$$

9. Find all real solutions of the following polynomial equations:

$$\text{(i)} \quad x^3 - 6x^2 - 9x + 14 = 0, \qquad \text{(ii)} \quad x^4 - 7x^3 + 8x^2 + 28x - 48 = 0.$$

10. Factorise the following polynomials:

(a) $x^3 + 5x^2 - 2x - 24$, (b) $x^3 - x^2 - 5x - 3$, (c) $x^3 + x^2 + x + 1$, (d) $x^3 - x$.

11. Show that $(x + y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$, and that $(x + y)^3 = x^3 + y^3$ if and only if $x = 0$ or $y = 0$ or $x = -y$.

12. Show that $x - 1$ is a factor of a polynomial P of positive degree if and only if the sum of the coefficients of P is zero. What conditions should the coefficients of a polynomial satisfy to ensure that $x + 1$ is a factor of that polynomial?

13. Solve the following inequalities for x

$$\begin{array}{llllll} \text{(a)} & 2x + 3 > 1 - 4x, & \text{(b)} & x^2 - 11x + 24 < 0, & \text{(c)} & 6 + x \le 4x + 3, \\ \text{(d)} & 0 \le 1 - 2x \le 3, & \text{(e)} & \dfrac{1}{2x - 1} > 1, & \text{(f)} & \dfrac{1}{2x - 1} > \dfrac{1}{3x + 2}. \end{array}$$

14. Express the solution sets of the following inequalities using interval notation

$$\begin{array}{llllll} \text{(a)} & |x - 4| < 2, & \text{(b)} & |4x + 3| \le 1, & \text{(c)} & |2x + 5| \ge 7, \\ \text{(d)} & |x - 2| < 3, & \text{(e)} & |4x + 3| \ge 1, & \text{(f)} & |2 - 3x| \le 1. \end{array}$$

15. Solve the following simultaneous linear equations:

$$\text{(i)} \begin{array}{rcl} 5x + 3y & = & 1, \\ x - 4y & = & 14. \end{array} \qquad \text{(ii)} \begin{array}{rcl} 3x - 7y & = & -9, \\ 2x + 3y & = & 17. \end{array} \qquad \text{(iii)} \begin{array}{rcl} 4x + 5y & = & 3, \\ 3x - \frac{5}{2}y & = & 1. \end{array}$$

16. Solve the following simultaneous nonlinear equations:

$$\text{(i)} \begin{array}{rcl} (x + 1)(y - 2) & = & 0, \\ x(y - 1) & = & 0. \end{array} \qquad \text{(ii)} \begin{array}{rcl} x^2 + 2y^2 & = & 8, \\ x - y^2 & = & 4. \end{array} \qquad \text{(iii)} \begin{array}{rcl} 4x^2 - y^2 & = & 0, \\ x^2 + y & = & 1. \end{array}$$

Exercises: Chapter 6

1. Without using a calculator, complete the following table, giving *exact* values (using surds where necessary).

Angle in degrees	0°	30°	45°	60°	90°
Fraction of circle	0	$\frac{1}{12}$			
Angle θ in radians					
$\sin(\theta)$					
$\cos(\theta)$					
$\tan(\theta)$					

2. Evaluate each of the following *exactly*, i.e as surds:

(i) $\cos(210^\circ)$, $\sin(135^\circ)$, $\tan(330^\circ)$, $\cot(225^\circ)$, $\sec(-60^\circ)$, $\operatorname{cosec}(-240^\circ)$.

(ii) $\sin\left(\frac{5\pi}{2}\right)$, $\tan\left(\frac{5\pi}{4}\right)$, $\cos\left(\frac{13\pi}{4}\right)$, $\cos\left(\frac{7\pi}{4}\right)$, $\cos\left(-\frac{2\pi}{3}\right)$.

3. Find all values of θ (in radians) such that

   (i) $\tan(\theta) = -2$; (ii) $\sin\theta = 0.1$; (iii) $\cos\theta = -0.9$.

4. Find all values of $x \in \mathbb{R}$ such that

$$\text{(i)} \quad \tan(x) = -\frac{1}{\sqrt{3}}, \qquad \text{(ii)} \quad \cos(x) = -\frac{\sqrt{3}}{2}.$$

5. For acute angles θ, use suitably labelled right-angled triangles to find

   (i) $\tan(\theta)$ given $\sin(\theta) = 5/13$;

   (ii) $\operatorname{cosec}(\theta)$ given $\cos(\theta) = 1/2$.

6. If $\tan(\alpha) = 3$ verify that $\dfrac{\sin(\alpha) - \cos(\alpha)}{\sec(\alpha) - \operatorname{cosec}(\alpha)} = \dfrac{3}{10}$.

7. Express each of the following as a trigonometric ratio of an acute angle with appropriate sign (e.g. $\cos(170^\circ) = \cos(180-10)^\circ = -\cos(10^\circ)$):

(i) $\cos(252^\circ)$, $\sin(116^\circ)$, $\sin(-10^\circ)$, $\tan(187.5^\circ)$.

(ii) $\cos\left(\frac{7\pi}{12}\right)$, $\sin\left(\frac{9\pi}{8}\right)$, $\tan\left(-\frac{11\pi}{12}\right)$, $\sec\left(\frac{7\pi}{5}\right)$, $\operatorname{cosec}\left(-\frac{15\pi}{8}\right)$.

8. Sketch on the same diagram the graphs of

   (i) $\sin(x)$ and $\sin(2x)$, (ii) $\cos x()$ and $\cos(2x)$,

   (iii) $\cos(x)$ and $\cos\left(x - \frac{\pi}{4}\right)$, (iv) $\cos(x)$ and $4\cos\left(3x - \frac{\pi}{2}\right)$.

9. Sketch on the same diagram the graphs of

   (i) $\sin(x)$ and $\operatorname{cosec}(x)$, (ii) $\cos(x)$ and $\sec(x)$, (iii) $\tan(x)$ and $\cot(x)$.

10. If $f(x) = \sqrt{1 - x^2}$ and $g(x) = \sin(x)$, determine $f \circ g$ and $g \circ f$, and their domains.

11. Determine if the following functions are even, odd or neither even nor odd.

    (a) $x^3 + \sin(x)$, (b) $x + \cos(x)$, (c) $x\sin(x) + x + 1$,

    (d) $x^2\tan(x) + \sin(2x)$, (e) $\cos(\pi x) + \sin^2(x) + 3$.

12. Show that $\sin\left(\frac{\pi}{2} - \alpha\right)\cot\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha)$ for $0 < \alpha < \frac{\pi}{2}$.

13. Show that $\dfrac{\cos\left(\frac{\pi}{2} - \beta\right)}{\sec\left(\frac{\pi}{2} - \beta\right)} = \sin^2(\beta)$ for $0 < \beta < \frac{\pi}{2}$.

14. Simplify

    (i) $\dfrac{1}{\sin^2(\theta)} - 1$, (ii) $\sin^3(A) + \sin(A)\cos^2(A)$,

    (iii) $\dfrac{\sec(\alpha) - \cos(\alpha)}{\sin(\alpha)}$, (iv) $\dfrac{\cot(\theta)}{1 + \cot^2(\theta)}$.

15. Simplify (i) $\dfrac{1}{1 + \sin(x)} + \dfrac{1}{1 - \sin(x)}$, (ii) $\dfrac{\sin^2(A)}{\tan(A)} - \dfrac{\cos^2(A)}{\cot(A)}$.

16. Prove that

    (i) $\tan(A) + \cot(A) = \sec(A)\operatorname{cosec}(A)$,

    (ii) $\sec^2(A) + \operatorname{cosec}^2(A) = \sec^2(A)\operatorname{cosec}^2(A)$,

    (iii) $\sin^2(A)\cos^2(B) - \cos^2(A)\sin^2(B) = \sin^2(A) - \sin^2(B)$,

    (iv) $\sin^4(A) - \cos^4(A) = 1 - 2\cos^2(A)$,

    (v) $\dfrac{1 + \sin(x)}{1 - \sin(x)} = (\sec(x) + \tan(x))^2$

    (vi) $\dfrac{1 - \cos(x)}{1 + \cos(x)} = (\operatorname{cosec}(x) - \cot(x))^2$.

17. For each of the following find

 a) **all** values of θ which solve the equation, and

 b) list those solutions that lie in the interval $[0,\, 2\pi]$:

 (i) $\cos^2(\theta) - \sin(\theta) - \dfrac{1}{4} = 0$,

 (ii) $2\sin^2(\theta) + 3\cos(\theta) = 0$,

 (iii) $2\sqrt{3}\cos^2(\theta) = \sin(\theta)$,

 (iv) $\tan^2(\theta) + \cot^2(\theta) = 2$ *(Hint: write in terms of* $\tan(\theta)$ *only).*

 (v) $\tan^2(\theta)^\circ - 4\tan(\theta)^\circ + 1 = 0$ *(give answers to 3 decimal places).*

 (vi) $\sin(\theta) + \tan(\theta) = 0$.

18. Given that α and β are acute with $\sin(\alpha) = \dfrac{3}{5}$ and $\cos(\beta) = \dfrac{9}{41}$, find the values of $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.

19. Show that $\sin(105^\circ) + \cos(105^\circ) = \dfrac{1}{\sqrt{2}}$.

20. Show that $\sin(A+B)\sin(A-B) = \sin^2(A) - \sin^2(B)$.

21. Show that $\cos(A)\cos(B-A) - \sin(A)\sin(B-A) = \cos(B)$ — in one line!

22. Show that for any number n:

$$\cos[(n+1)A]\cos[(n-1)A] + \sin[(n+1)A]\sin[(n-1)A] = \cos(2A).$$

23. Show that

 (i) $\dfrac{\sin(2A)}{1+\cos(2A)} = \tan(A)$, (ii) $\dfrac{1-\cos(2A)}{1+\cos(2A)} = \tan^2(A)$.

24. Write the following as sums:

 (i) $2\cos(3x)\ \cos(x)$, (ii) $2\sin(3x)\ \cos(5x)$, (iii) $\sin(4x)\ \sin(x)$, (iv) $\cos(5x)\ \sin(2x)$.

25. Write the following as products:

 (i) $\cos(6x) + \cos(4x)$, (ii) $\sin(3x) + \sin(5x)$, (iii) $\sin(x+\alpha) - \sin(x-\alpha)$.

26. Find $R > 0$ and α $(0 \le \alpha < 2\pi)$ such that

 (i) $3\sin(x) + 4\cos(x) = R\sin(x+\alpha)$;

 (ii) $\cos(x) - 3\sin(x) = R\cos(x-\alpha)$;

 (iii) $\sin(3x) - \cos(3x) = R\sin(3x+\alpha)$;

 (iv) $2\sin(\omega x) + 3\cos(\omega x) = R\cos(\omega x+\alpha)$.

27. Express $f(x) = \cos(x) - \sin(x)$ in the form $A\cos(x + \alpha)$ and hence determine the value of $x \in [0, \pi]$ for which $f(x)$ has its minimum value.

28. Express $g(t) = \sqrt{3}\cos(t) - \sin(t)$ in the form $A\cos(t + \alpha)$. Hence find the values of $t \in [-\pi, \pi]$ where $g(t)$ has its maximum and minimum values.

29. Express $g(t) = \sqrt{3}\sin(2t) - 3\cos(2t)$ in the form $A\sin(2t + \alpha)$. Hence find the values of $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $g(t)$ has its maximum and minimum values.

30. Solve the equation $\cos(2\theta) - 3\sin(\theta) + 1 = 0$ for $0 \le \theta \le 2\pi$.

31. Solve the equation $6\sin^2(x) + \cos(x) = 5$ for $0 \le x \le \pi$. Express the answers as decimals.

32. Express $\sin(A+B+C)$ and $\cos(A+B+C)$ in terms of $\sin(A)$, $\sin(B)$, $\sin(C)$, $\cos(A)$, $\cos(B)$ and $\cos(C)$.

33. Calculate the following (where defined).

$$\text{(a)}\quad \arcsin\left(\sin\frac{\pi}{8}\right), \qquad \text{(b)}\quad \arcsin\left(\sin\frac{7\pi}{3}\right), \qquad \text{(c)}\quad \sin\left(\arcsin 2\right).$$

34. Without using a calculator, evaluate $\sin(\theta)$ given that $\theta = \arcsin\left(\frac{1}{3}\right) + \arccos\left(\frac{1}{3}\right)$.

35. Without using a calculator, show that if $\phi = 2\arcsin\left(\frac{3}{5}\right) - \arccos\left(\frac{12}{13}\right)$, then $\sin(\phi) = \frac{253}{325}$.

36. Let $\arctan(x) = \alpha$ and $\arctan(y) = \beta$. Show that, provided $-\frac{\pi}{2} < \alpha + \beta < \frac{\pi}{2}$, then $\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$. Verify the result in the case $\alpha = \beta = \frac{\pi}{6}$; and show that the result is false in the case $\alpha = \beta = \frac{\pi}{3}$ when the condition $-\frac{\pi}{2} < \alpha + \beta < \frac{\pi}{2}$ is violated.

37. By considering the tangent of both sides, find a value of x that satisfies the equation $\arctan(3x) + \arctan(2x) = \frac{\pi}{4}$. [*Hint*: Use the result in the previous question.]