UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Exercises: Chapter 4

1. For the function f defined by $f(x) = x^2 + 1, \; x \in \mathbb{R}$ evaluate

$$f(0), \quad f(-3), \quad f(5), \quad f(t), \quad f(2x), \quad f(a^2), \quad f(1-x), \quad f(x^2+1).$$

2. The function h defined by $h(t) = 50t - 5t^2$, gives the height in metres of a projectile fired vertically upwards t seconds after launch.

(i) Evaluate

$$h(1), \quad h(5), \quad h(10), \quad h(11).$$

(ii) What meaning can you give to the values $h(10)$ and $h(11)$?

(iii) State a sensible domain for the function h.

(iv) What would be a sensible definition of the function h for $t \le 0$?

3. State the natural domain for each of the following functions:

(i) $f_1(x) = x^2 - 4\,,$ (ii) $f_2(x) = \dfrac{1}{x^2-4}\,,$

(iii) $f_3(x) = \sqrt{x-4}\,,$ (iv) $f_4(x) = \dfrac{1}{\sqrt{x-4}}\,,$

(v) $f_5(x) = \dfrac{x}{(x-3)(x+4)}\,,$ (vi) $f_6(x) = \sqrt{x^2+3}.$

4. Determine the natural domain of each of the following functions.

(i) $f : x \mapsto \sqrt{x-2}$ (ii) $f : x \mapsto \sqrt{x+3}$ (iii) $f : x \mapsto \sqrt{(x-2)(x+3)}$

(iv) $f : x \mapsto \sqrt{1-2x}$ (v) $f : x \mapsto \dfrac{1}{\sqrt{1-2x}}$ (vi) $f : x \mapsto \sqrt{x+3} + \dfrac{1}{\sqrt{2-x}}$

5. Make a rough sketch of the graph of each of the following functions and hence state the range of the function:

(i) $f(x) = 3x + 2,$ (ii) $g(x) = x^2 + 3,$

(iii) $h(x) = 4 - x^2,$ (iv) $p(x) = \sqrt{x},$

(v) $q(x) = \dfrac{1}{x^2},$ (vi) $r(x) = \dfrac{1}{x^2+1}.$

6. Verify the results in Theorem 4.6 in the notes.

7. Prove the result in Theorem 4.7 (b).

8. Determine whether the following expressions define functions that are odd or even or neither:

   (i) $f(x) = x^3 + x$, (ii) $g(x) = 1 - x^2$,

   (iii) $h(x) = x^2 + x + 1$, (iv) $p(x) = x^2(x^3 - 4x)$,

   (v) $q(x) = \dfrac{x}{x^2+1}$, (vi) $r(x) = (x^2+1)^3$.

9. Let f_1 and f_2 be even functions, and g_1 and g_2 be odd functions, with a common domain D. What can you say about $f_1 + f_2$, $f_2 + g_1$, $f_1 f_2$, $f_1 g_1$ and $g_1 g_2$?

10. Prove that every function is the sum of an even and an odd function.
    Hint: start by verifying that if $\phi(x) = f(x) + f(-x)$ then ϕ is even.

11. If f is both an even and an odd function, show that $f(x) = 0$ at every point of its domain.

12. Given that $f(x) = 1 - \dfrac{1}{x}$ and $g(x) = 1 + \dfrac{1}{x}$, write down $(f+g)(x)$, $(fg)(x)$ and $\left(\dfrac{f}{g}\right)(x)$. Determine the domains of $f+g$, fg and $\dfrac{f}{g}$.

13. For each of the following pairs of functions determine $(f(g(x))$ and $(g(f(x))$, and in each case determine the domain of the composite function.

    (i) $f(x) = 1 + \dfrac{1}{x}$, $g(x) = x^2 + 2x + 5$;

    (ii) $f(x) = x^2$, $g(x) = \sqrt{x}$;

    (iii) $f(x) = \sqrt{x+1}$, $g(x) = x^2 + 2x + 3$.

14. Find $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$ when

    (i) $f(x) = x^2 + 1$, $g(x) = \dfrac{1}{x}$;

    (ii) $f(x) = x^3$, $g(x) = 1 - x^2$;

    (iii) $f(x) = \sqrt{x+4}$, $g(x) = \dfrac{3}{x^2}$;

    (iv) $f(x) = \dfrac{x+1}{x-3}$, $g(x) = x^2 + 3$.

15. Find $f\Big(g(h(x))\Big)$, $h\Big(g(f(x))\Big)$ and $h\Big(f(g(x))\Big)$ when

    $$f(x) = x^3, \qquad g(x) = \frac{1}{x} \qquad \text{and} \qquad h(x) = x^2 + 1.$$

16. Find $f^{-1}(x)$ when $f(x)$ is given by:

$$\text{(i)}\quad 5x+3; \qquad \text{(ii)}\quad 4x^3-5; \qquad \text{(iii)}\quad \frac{1}{2x+3}.$$

In each case, verify that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

17. Determine the inverse function of each of the following functions and state their domains.

(a) $f : x \mapsto 3x-1$ (b) $f : x \mapsto 2x+3$ (c) $f : x \mapsto \dfrac{1}{x-7}$ (d) $f : x \mapsto \sqrt{x-5}$.

18. (i) Find the inverse of the function f_1 defined by

$$f_1(x) = x^2+4, \quad x \geq 0.$$

Sketch, on the same diagram, the graphs of $f_1(x)$ and $f_1^{-1}(x)$.
(ii) Find the inverse of the function f_2 defined by

$$f_2(x) = x^2+4, \quad x \leq 0.$$

Sketch, on the same diagram, the graphs of $f_2(x)$ and $f_2^{-1}(x)$.

19. Let $y(x) = x^2-2x-3$.
(a) Complete the square in x and show that $y(x)$ has a minimum of -4 when $x=1$.
(b) Verify that $y(-1) = y(3) = 0$ and sketch $y(x)$.
(c) Confirm from your sketch that for each $y_0 \in (-4,\infty)$ there are two values of x for which $y(x) = y_0$. Show that these are $x = 1-\sqrt{y_0+4}$ and $x = 1+\sqrt{y_0+4}$.
(d) Deduce the inverse functions of
(i) $f_1 : x \mapsto x^2-2x-3, \quad x \in (-\infty, 1]$,
(ii) $f_2 : x \mapsto x^2-2x-3, \quad x \in [1,\infty)$.

20. (a) Show, by completing the square, that the curve $y = \dfrac{1}{2}(x^2+6x+10)$ has a minimum at $(x,y) = (-3, \dfrac{1}{2})$. Sketch the curve for $-6 \leq x \leq 0$.
(b) Determine the inverse functions of
(i) $f_1 : x \mapsto \frac{1}{2}(x^2+6x+10), \quad x \in (-\infty,-3]$ (ii) $f_2 : x \mapsto \frac{1}{2}(x^2+6x+10), \quad x \in [-3,\infty)$.

21. Sketch the graphs of the functions given by the following parametric equations.

(i) $x(t) = t+1$, $y(t) = 2t-4$, $t \in (-\infty,\infty)$.

(ii) $x(t) = 2t$, $y(t) = t^2$, $t \in [0,\infty)$.