3 Logic, deduction and mathematical proof

3.1 Some basic ideas

Logic is the study of arguments, and in mathematics it is used to help decide what is true and what is false. In this chapter we introduce some of the vocabulary, notation and concepts which are commonly used in constructing mathematical arguments.

The building blocks of our arguments are **statements** (in our case, usually mathematical statements) or **propositions** which are arrangements of **terms** that can be sensibly said to be either **true** or **false**. In the course of mathematical reasoning, we will want to combine and transform these statements using **connectives** such as "and", "or", "if and only if" etc. As we are interested here in the underlying structure of arguments, we will often use letters to denote the statements without referring specifically to their content. For example, we may write the statements

p:	$4+6=10$.
q:	6 is a prime number.
r:	For all $a \in \mathbb{N}$, $2a$ is an even number.
s:	$x+y=10$.
t:	n is prime.

It is clear that statement p is true and statement q is false. Statement r is also true (as it holds for all possible values of a). However, we cannot decide anything about the truth (or otherwise) of s and t until we know the values of x, y and n.

The **negation** of a statement is written as $\neg p$ (read as '*not* p'). Note that $\neg p$ is false if p is true, and vice versa. For our examples above, the negations are

$\neg p$	$4+6 \neq 10$,
$\neg q$:	6 is not a prime number.
$\neg r$:	There exists an $a \in \mathbb{N}$ such that $2a$ is not an even number.

In mathematics it is important to be able to argue logically, that is, to start off with some **hypothesis**, or **premise** (a proposition which we assume (or know) to be true, such as '$x = 2$') and then draw some **conclusion** (such as '$x^2 = 4$'). A statement of the form

if *hypothesis* then *conclusion*,

which asserts that the conclusion is always true whenever the hypothesis is true, is often called a **conjecture**. This is a mathematical statement which appears likely to be true, but has not been formally proven to be true under the rules of mathematical logic. Once a conjecture is formally proven true it is elevated to the status of a **theorem**. A **proof** of a theorem consists of a chain of logical arguments between the hypothesis and the conclusion.

Note that a theorem usually has some sort of importance. For example, the proposition 'if $x = 2$ then $x^2 = 4$' would usually be regarded as too trivial to be called a theorem in its own right. Similarly, an auxiliary theorem which is used only as part of the proof of another theorem is usually called a **lemma**. Finally, a **corollary** is a statement which follows readily from a previously proved theorem.

In mathematics, although experiments and investigations play an important role in helping to form conjectures and potential theorems, without the certainty of proof, they mean nothing. As an example, consider the proposition that *every non-negative integer can be expressed as a sum of the squares of four other non-negative integers.* Suppose we check this up to 325, e.g.

$$1 = 1^2 + 0^2 + 0^2 + 0^2, \qquad 2 = 1^2 + 1^2 + 0^2 + 0^2, \quad \ldots \quad , 22 = 4^2 + 2^2 + 1^2 + 1^2, \quad \ldots,$$

$$196 = 9^2 + 9^2 + 5^2 + 3^2, \quad \ldots, \quad 302 = 17^2 + 3^2 + 2^2 + 0^2, \quad \ldots, 325 = 18^2 + 1^2 + 0^2 + 0^2.$$

Does this constitute a proof of the result? Of course not, even if we check the next 100,000 integers (the statement might be false for 100326!).

Confirming a result for a sample of cases (no matter how large the sample) is NEVER enough to constitute a proof!

Of course, not all statements turn out to be true. The easiest way to prove that a statement is false is by finding a **counterexample**, that is, an example which contradicts the conjecture. For example, the prime number 2 is a counterexample to the (false) statement *If a number is prime, then it is odd.* Note that just one counterexample is enough to show that a statement is false.

3.2 Elements of proof

Learning how to prove mathematical results can be tough. Once you have read and understood a lot of proofs, you will have at your disposal a range of methods and strategies which may be combined in attempts to prove new theorems. You will see many such proofs throughout your course: studying proofs helps you to remember results because you understand why the results are true.

The first step towards proving a result is to read carefully what is stated in the conjecture, identify what you are told (and can assume) and specify exactly what has to be proved. Then you must construct a chain of logical arguments to prove the result (or find a counterexample to disprove it!).

As an example, consider the following theorem.

Theorem 3.1 *If $n \in \mathbb{Z}$ is odd then n^2 is odd.*

We now construct a proof of this result. We are told that n is an odd integer and we have to produce a valid mathematical argument that shows that n^2 is also an odd integer. Note that we can make use of the following standard properties of integers: if a and b represent any two integers, then ab, $a+b$ and $a-b$ will also be integers. It follows that the square of any integer must also be an integer, so really our task is to show that n^2 must be odd whenever n is odd. The key is to use the fact that if (and only if) n is odd then it can be written as

$$n = 2k - 1, \text{ for some integer } k.$$

Consequently,

$$n^2 = (2k-1)^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k + 1) - 1 = 2l - 1$$

where $l = 2k^2 - 2k + 1$ is an integer (since it involves only sums, differences and products of integers). This shows that n^2 is odd and the proof is complete.

3.3 Some comments on notation

Usually, a mathematical argument such as the one above would be written in a more concise manner. To enable you to do this, and also to help you understand mathematical proofs of important results, it is essential that you become familiar with some notation and terminology. As stated in the introduction to these notes, all mathematics symbols have very precise meanings and hence must be used with care.

3.3.1 Symbols and letters

It is sensible to use symbols and letters which remind you of their role and relationship. For example, unknown quantities are usually represented by x, y and z with a, b, c reserved for known values. It is also usual to call integers k, m and n rather than e.g. P, α and $\hat{\mu}$. Angles are usually represented by Greek letters, e.g., α, β, γ. Physical quantities are often represented by their first letter, e.g., v for velocity, T for temperature, h for height.

3.3.2 Disjunction and conjunction

The symbols $\vee$ and $\wedge$ are used for the disjunction **or** and conjunction **and** (in a mathematical sense). The words **or** and **and** have very precise meanings in mathematics. When we write $x = 0 \vee y = 0$ (read as '$x = 0$ or $y = 0$') we mean that *either one or the other or both* of the statements '$x = 0$' and '$y = 0$' is true (which is different to the 'exclusive' way in which or is commonly used in the English language). If we write $x = 0 \wedge y = 0$ (read as '$x = 0$ and $y = 0$'), then *both* statements '$x = 0$' and '$y = 0$' must be true. These are the precise meanings represented by the symbols $\vee$ and $\wedge$.

These symbols are often useful when describing sets or intervals on the real line. For example, if we want to denote the **union** of two sets of real numbers, A and B say, then we may write

$$A \cup B = \{x \in \mathbb{R} \quad | \quad x \in A \vee x \in B\}$$

as $A \cup B$ consists of elements that are in set A or set B (or both). Note that, in mathematics, the meaning of "or" is different to that commonly used in the English language: in mathematics, "or" is always used in this inclusive sense of "one or the other, or both". In a similar way, the **intersection** of A and B may be written as

$$A \cap B = \{x \in \mathbb{R} \quad | \quad x \in A \wedge x \in B\}$$

as $A \cap B$ consists of all of the elements that are in both set A and set B.

As an example, suppose we are using combinations of inequalities to describe intervals on the real number line. For example, we can indicate that a number x satisfies $x > 1$ **and** $x \leq 5$ by writing

$$\{x \in \mathbb{R} \quad | \quad x > 1 \quad \wedge \quad x \leq 5\},$$

or that it satisfies $x < 1$ **or** $x \geq 5$ by writing

$$\{x \in \mathbb{R} \quad | \quad x < 1 \quad \vee \quad x \geq 5\}.$$

Warning: be careful to write exactly what you mean, as using the wrong conjunction can have a major effect. For example, the statement

$$\{x \in \mathbb{R} \quad | \quad x > 1 \quad \vee \quad x \leq 5\}$$

denotes the whole of the real number line, and

$$\{x \in \mathbb{R} \quad | \quad x < 1 \quad \wedge \quad x \geq 5\}$$

describes the empty set!

3.3.3 Quantifiers

Many mathematical statements involve the words ‘for each’, ‘for all’ and ‘there exist(s)’. To help with writing such statements clearly, we introduce the following notation:

- **The universal quantifier** $\forall$. This means ‘for all’. For example, we might write

  $$\forall n \in \mathbb{N} : \quad n < n + 1$$

  to state that it is true for all natural numbers that each number is one more than its predecessor.

- **The existential quantifier** $\exists$. This means ‘there exists’. For example, we might write

  $$\exists r \in \mathbb{R} : \quad r^2 = 2$$

  to state that there exists a real number whose square is 2.

These quantifiers often appear together. For example, we might write

$$\forall n \in \mathbf{Z}, \text{ if } n \text{ is even then } \exists m \in \mathbb{Z} \text{ such that } n = 2m$$

to state that all even integers can be written as the product of 2 and an integer.

Examples 3A

3.4 Implications

One very important set of relationships in mathematics are **implications**. They are assertions of the type

$$\text{if } p \text{ is true then } q \text{ is true}$$

where p and q are statements which are either true or false. Mathematically, such assertions are written using the **implication symbol** $\Rightarrow$, that is, we write

$$p \Rightarrow q.$$

Deciding whether or not an implication is true is a fundamental notion of logic. Here we simply state that **the implication** $p \Rightarrow q$ **is true if and only if it is impossible for** p **to be true and** q **to be false**.

Important note: if p is NOT true, then the implication $p \Rightarrow q$ is true but tells us absolutely nothing!

We will use the following symbols and terminology for implications.

- The **implication symbol** $\Rightarrow$ indicates that the right hand side follows from the left, so we write

  $$p \Rightarrow q$$

  when statement q follows from statement p. This is equivalent to writing

  - 'whenever p is true then q must also be true';
  - 'the truth of p **implies** the truth of q';
  - 'p **implies** q';
  - '**if** p is true **then** q is true';
  - 'p is a **sufficient** condition for q.'

  For example, writing

  'If Chris can play the guitar, then Chris can play a musical instrument.'

  as

  'Chris can play the guitar' $\Rightarrow$ 'Chris can play a musical instrument'

  is a correct use of the $\Rightarrow$ symbol (it is impossible that Chris can play the guitar but cannot play a musical instrument!). In other words, the fact that Chris can play the guitar is **sufficient** to tell you that Chris can play a musical instrument. Similarly, the mathematical implication

  $$x = 3 \Rightarrow x^2 = 9$$

  is true because it is impossible for $x = 3$ to be true and $x^2 = 9$ to be false.

- The **converse** of the implication $p \Rightarrow q$ is the implication

  $$p \Leftarrow q \qquad (\text{or } q \Rightarrow p).$$

  This is equivalent to writing

  - 'p **is implied by** q';
  - '**only if** p is true **then** q is true';
  - 'p is a **necessary** condition for q'.

Note that in general, we CANNOT assume that the implication $p \Rightarrow q$ can be reversed. That is, if $p \Rightarrow q$ is true, then the converse $p \Leftarrow q$ (or $q \Rightarrow p$) is not necessarily true. For our examples, the reverse implication

'Chris can play a musical instrument' $\Rightarrow$ 'Chris can play the guitar'

which reads as

'Only if Chris can play the guitar can Chris play a musical instrument.'

is NOT true. Here the fact that Chris can play the guitar is not **necessary** for her to be able to play a musical instrument (she could for example play the piano instead). Similarly, the implication

$$x^2 = 9 \Rightarrow x = 3$$

is false (because it is possible for the statement $x^2 = 9$ to be true while the statement $x = 3$ is false, namely when $x = -3$).

- The **equivalence symbol** $\Leftrightarrow$ is used when $p \Rightarrow q$ and $q \Rightarrow p$ are **both** true. We write

  '$p \Leftrightarrow q$' or 'p iff q'

  where 'iff' is short for **if and only if**. We say that

  - 'p is true **if and only if** q is true';
  - 'p is **necessary and sufficient** for q';
  - 'p is **equivalent** to q'.

  For example,

  $$x = \pm 2 \Leftrightarrow x^2 = 4$$

  is true but

  $$x = 2 \Leftrightarrow x^2 = 4$$

  is false (because truth of the right-hand side does not imply truth of the left-hand side).

- The **inverse** of the implication $p \Rightarrow q$ is the implication

  $$\neg p \Rightarrow \neg q.$$

  If $p \Rightarrow q$ is true, the inverse may be true or false.

- The **contrapositive** of the implication $p \Rightarrow q$ is the implication

  $$\neg q \Rightarrow \neg p.$$

  The **law of contraposition** states that an implication and its contrapositive are equivalent. In other words, if $p \Rightarrow q$ is true, then so is $\neg q \Rightarrow \neg p$.

Notes:

(a) It is important that the symbols '$\Rightarrow$', $\Leftarrow$ and '$\Leftrightarrow$' are not confused with '$=$'. The equals sign can only appear between two **expressions** which are (claimed to be) equal, such as $(-1)^2 = 1$ or $2x+1 = 7$. On the other hand, '$\Rightarrow$', $\Leftarrow$ and '$\Leftrightarrow$' can appear only between two **propositions** or **statements**, such as $x = 3 \Leftrightarrow 2x+1 = 7$. Writing something like $x \Rightarrow 2$ is completely meaningless, and as nonsensical as writing something like "your age implies 18"!

(b) If the truth of p does NOT imply that q must be true, we write '$p \not\Rightarrow q$'. We can often establish that $p \not\Rightarrow q$ by finding a **counterexample** that disproves the statement $p \Rightarrow q$.

(c) If $p \Rightarrow q$ and $q \Rightarrow r$ are both true then we can deduce that $p \Rightarrow r$ is also true.

Examples 3B

3.5 Common methods of proof

3.5.1 Direct proof

The first method we will discuss is using a so-called **direct proof** which means working directly from the assumption to the conclusion. For example, to prove $p \Rightarrow q$, we start with proposition p (assumed to be true) and produce a chain of true implications that results in the truth of q being established. In summary:

show that $p \Rightarrow r \Rightarrow s \Rightarrow t \Rightarrow \ldots \Rightarrow q$ (with each implication being true).

Examples 3C

3.5.2 Indirect proof

The most common example of an indirect proof is **proof by contradiction** or **reductio ad absurdum**. Its key step uses a special case of the law of contraposition which states that the implications $\neg p \Rightarrow q$ and $\neg q \Rightarrow p$ are equivalent.

The method of proof works as follows. Suppose we wish to prove that a given statement p is true.

1. Assume that p is false (that is, $\neg p$ is true).
2. Show that if $\neg p$ is true then q is true (i.e. show that $\neg p \Rightarrow q$) for a new statement q.
3. Show that q is false (that is, $\neg q$ is true).
4. Apply the **law of contraposition** to deduce that $\neg q \Rightarrow p$, i.e. p must be true.

Examples 3D

3.5.3 Proof by induction

One of the most useful methods of mathematical proof is **proof by induction**. It is based on the following principle:

Theorem 3.2 (The Principle of Mathematical Induction) *For each positive integer n, let $p(n)$ be a statement about (or property of) the number n. Then $p(n)$ is true for* ***all*** *positive integers n provided that*

(i) $p(1)$ is true;

(ii) if $p(n)$ is true, then $p(n+1)$ is true $\quad \forall\, n \in \mathbb{N} \quad$ (i.e. $p(n) \Rightarrow p(n+1), \quad \forall n \in \mathbb{N}$).

Helpful analogies for this process are the toppling of a row of dominoes (where the first domino tips over the second, the second domino tips over the third, etc., until all the dominoes fall) or a line of people playing Chinese whispers (if person 1 passes the message to person 2, person 2 passes the message to person 3, etc., the message will eventually pass all the way down the line). The steps in a proof by induction are:

Step 1: prove that the proposition is true when $n = 1$.
Step 2: Assume that the given result is true for an arbitrary n, and use this assumption to prove the result for $n + 1$.

If both of these two steps can be carried out, the principle of mathematical induction tells us that the proposition $p(n)$ is true for all natural numbers n.

This principle of mathematical induction can be generalised in many ways. Two examples are:

- Statement $p(n)$ is true for all positive integers n provided that

(i) $p(1)$ and $p(2)$ are true;

(ii) if $p(n)$ and $p(n+1)$ are true, $p(n+2)$ is true $\forall\ n \in \mathbb{N}$.

- The statement $p(n)$ is true for all positive integers $n \geq N$ for a given $N \in \mathbb{N}$ provided that

  (i) $p(N)$ is true;

  (ii) if $p(n)$ is true, $p(n+1)$ are true $\forall\ n \geq N$.

Examples 3E