9 Limits

9.1 Introduction

The collection of ideas that form "calculus" (more formally *the differential and integral calculus*) originate in geometrical problems to do with curves and areas. The roots of these ideas can be found in work more than two thousand years ago by mathematicians including ARCHIMEDES of Syracuse; the framework we recognise as "calculus" today was constructed around 300–350 years ago, and put into its modern form around 200 years ago. If you find it heavy going in places, remember that these ideas took a very long time to develop — they are not obvious!

The common approach to problems in calculus is that we start by approximating the quantity we want to describe, and as these approximations become more and more accurate we approach the exact value of that quantity.

To talk sensibly about this process, we need to consider the **limit of a function**.

9.2 Intuitive concept of a limit

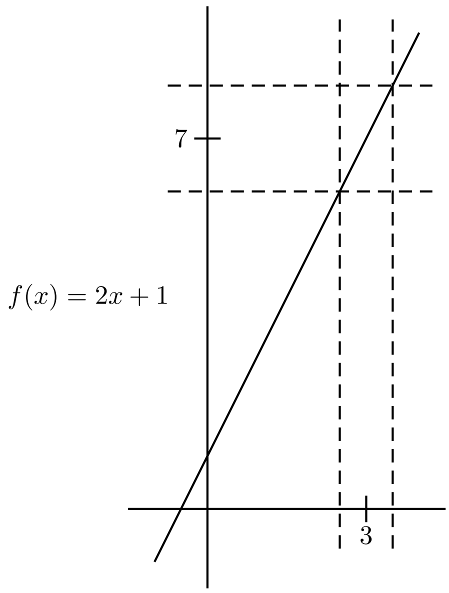
As with other concepts we have encountered, we will start by formulating an intuitive idea of what we mean by a limit, and we will then consider how to make it more logically rigorous.

We consider a function f in the vicinity of a value a of its argument. We say that f approaches the limit l near a if its value $f(x)$ can be made as close to l as we like by making x close enough to a.

9.2.1 Some examples

Example 9.1 Consider the function f defined by $f(x) = 2x+1$. Show that $f(x)$ approaches 7 as x approaches 3.

This is a fairly harmless example to get us started. We know that $f(3) = 7$, so we should be able to show that $f(x)$ gets as close to 7 as we like provided that x is close enough to 3.



The figure shows the basic idea: for a given distance between the horizontal lines, we want to know how close the vertical lines have to be to guarantee that $f(x)$ lies between the horizontal lines.

The subtlety is that we are interested in the value of $f(x)$ for values of x *close to* $x = 3$ but not *at* $x = 3$. (The reason for this will become clearer as this chapter goes on.)

Suppose, for example, that we want the value of f to lie within a distance $\frac{1}{10}$ of 7, that is,

$$7 - \frac{1}{10} < 2x + 1 < 7 + \frac{1}{10}.$$

Subtracting 7 gives

$$-\frac{1}{10} < 2x - 6 < \frac{1}{10},$$

that is, we require that $|f(x) - 7| = |2x - 6| < \frac{1}{10}$.

Next, dividing the inequalities by 2 shows that they are equivalent to

$$-\frac{1}{20} < x - 3 < \frac{1}{20}.$$

What we have found is that whenever $|x-3| < \frac{1}{20}$, we have $|f(x)-7| < \frac{1}{10}$.

In the argument above, there is nothing special about $\frac{1}{10}$. Given any positive number ϵ, we can ensure that $|f(x)-7| < \epsilon$ by restricting x to the interval $(3-\frac{\epsilon}{2}, 3+\frac{\epsilon}{2})$, that is, $|x-3| < \frac{\epsilon}{2}$.

Example 9.2 Consider the function f defined by $f(x) = x^2$. Show that $f(x)$ approaches 0 as x approaches 0.

Given $\epsilon > 0$, we want to find the conditions under which

$$x^2 = |x| \cdot |x| < \epsilon.$$

We see that if $|x| < \sqrt{\epsilon}$ then $x^2 < \epsilon$, so if $-\sqrt{\epsilon} < x < \sqrt{\epsilon}$ then $|f(x)| < \epsilon$.

In other words, given any positive number ϵ, we can ensure that $|f(x) - 0| < \epsilon$ by restricting x to the interval defined by $|x-0| < \sqrt{\epsilon}$.

Example 9.3 Consider the function f defined by $f(x) = x^2$. Show that $f(x)$ approaches 4 as x approaches 2.

We want to find how close x needs to be to 2 to ensure that

$$|x^2 - 4| < \epsilon \tag{9.1}$$

for any $\epsilon > 0$.

To start, we rewrite

$$|x^2-4| = |x-2| \cdot |x+2|,$$

which already features a factor $|x-2|$. If we can estimate $|x+2|$, we will be able to find a condition for $|x-2|$ that will ensure that (9.1) is satisfied.

We know that $|x-2|$ will have to become very small as ϵ does. If we restrict ourselves to the interval $|x-2| < 1$, then we have $-1 < x-2 < 1$ and so $1 < x < 3$. But then $3 < x+2 < 5$, and so clearly $|x+2| < 5$. Thus

$$|x^2-4| = |x-2| \cdot |x+2| < 5\,|x-2|,$$

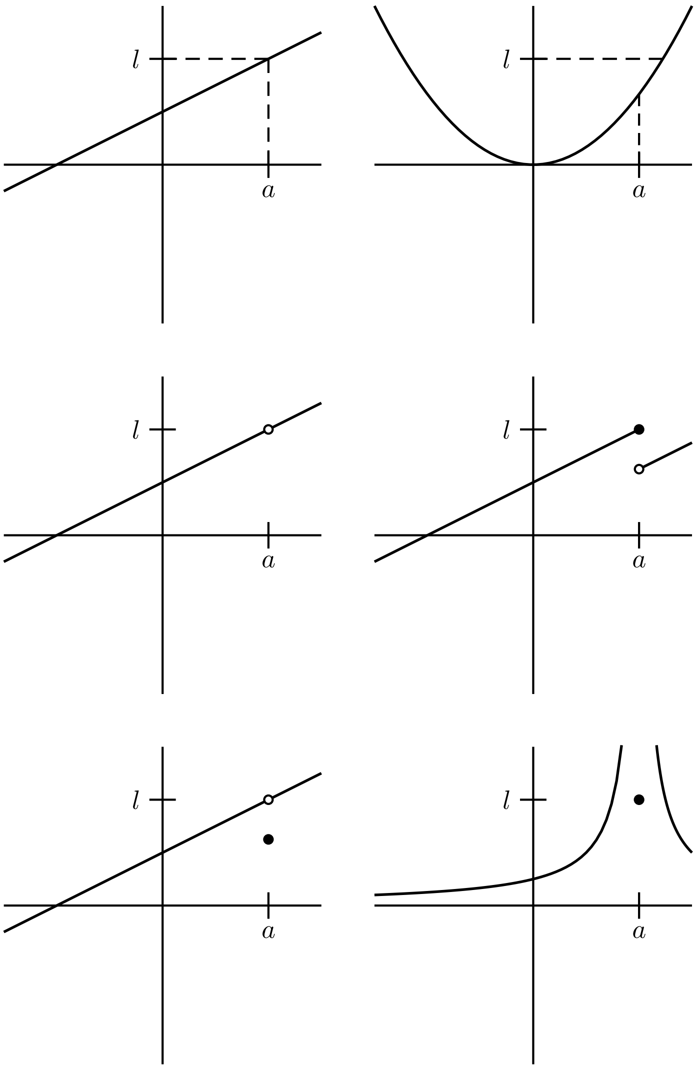
and so if $|x-2| < \min(\frac{\epsilon}{5}, 1)$ then

$$|x^2-4| = |x-2| \cdot |x+2| < 5\,|x-2| < 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

9.2.2 Some more complicated examples

Note that in the examples above we have not referred to the value of the function at the point a. In fact, our concept of a limit was designed so it did not involve $f(a)$. This becomes useful when we consider functions that are 'badly behaved' at a (in a sense that we will make more precise in Chapter 10).

Example 9.4 Of the following six functions only three of them approach l as the argument approaches a.

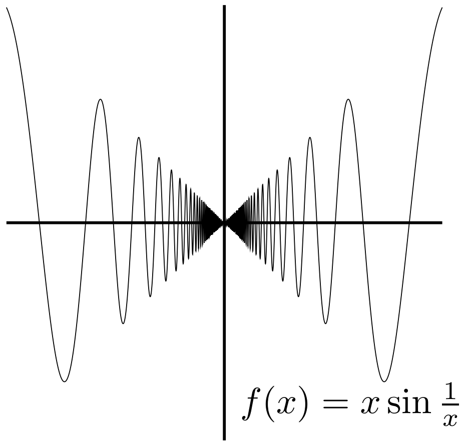


All the functions in the left-hand column approach l as the argument approaches a. This is the case even though in the second row the function isn't defined at a and in the third row it is defined to be a different value at a.

None of the functions in the right-hand column approach l as the argument approaches a. In the second row, the function approaches l only if we approach from the left (we will return to this idea in section 9.6.1), and in the third row the function increases unboundedly as we approach a, even though the function has been given a value at a.

The next example shows that we can still discuss limits even when it is not obvious what is going on at the point in question.

Example 9.5 What is the limit of the function f defined by $f(x) = x \sin \dfrac{1}{x}$ as x approaches $a = 0$?



Obviously, f is not defined at zero, and because $\frac{1}{x}$ goes to $\pm\infty$ as x goes to zero, f oscillates wildly. Nevertheless, since the sine function only takes values between minus one and one, we expect intuitively that f approaches zero as x approaches zero.

How can we show that f approaches zero as x approaches zero? We must show we can make f as close to zero as we like by making x sufficiently close to zero.

We start again with a specific bound: suppose we want to get f to be within $1/100$ of zero, that is we want

$$-\frac{1}{100} < x \sin \frac{1}{x} < \frac{1}{100},$$

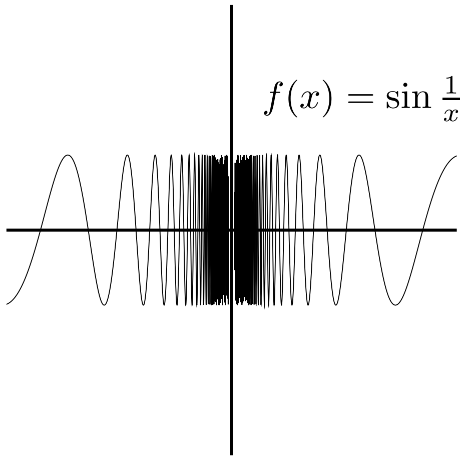
or simply

$$\left| x \sin \frac{1}{x} \right| < \frac{1}{100}.$$

We can easily satisfy this condition: since $|\sin \frac{1}{x}| \leq 1$, we know that $|x \sin \frac{1}{x}| \leq |x|$, and so whenever $|x| < \frac{1}{100}$ we have $|x \sin \frac{1}{x}| = |x| \cdot |\sin \frac{1}{x}| < \frac{1}{100}$ as required.

Clearly, there is nothing special about the number $\frac{1}{100}$, and the same argument works for any positive number ϵ, however small: if we require $|f(x)| < \epsilon$, all we need to ensure is that $|x| < \epsilon$. So whenever $-\epsilon < x < \epsilon$, we have $|x \sin \frac{1}{x}| < \epsilon$.

Example 9.6 What is the limit of the function f defined by $f(x) = \sin\dfrac{1}{x}$ as x approaches $a = 0$?



We now look at $f(x) = \sin\frac{1}{x}$ close to zero. This function *does not* approach zero as x approaches zero. Formally, this means that it is not true for every $\epsilon > 0$ that we have $|f(x) - 0| < \epsilon$ for x sufficiently close to zero.

How can we show this? It is sufficient to find a single number ϵ for which $|f(x) - 0| < \epsilon$ cannot be guaranteed. Any number $0 < \epsilon < 1$ will do.

Take for example $\epsilon = 1/2$: any interval $(-\delta, \delta)$ containing zero also contains a number $x = 1/(\pi/2 + 2\pi n)$ (for a very small interval, we just choose n big enough so that $x < \delta$), and then $f(x) = \sin(\pi/2 + 2\pi n) = 1$, and so $|f(x) - 0| = 1 > \epsilon$.

Could it be that f approaches 1 (or any other value) near zero? No, because every interval containing zero also contains a number $y = 1/(3\pi/2 + 2\pi m)$ (again, just choose m big enough!), and $f(y) = -1$.

9.3 Formal definition of a limit

We now give a mathematically precise definition based on the intuitive concept of a limit that we discussed in the preceding section. This precise definition will enable us to prove theorems relating to limits.

From what was said before, the function f approaches the limit l near a when f gets arbitrarily close to l as x gets close to a. We know how to measure distances by using the modulus function: f close to l means that $|f(x) - l|$ must be small. We can expect this only when x is close to a, that is, when $|x - a|$ is small. When it comes to limits, the Greek letters ϵ and δ (**epsilon** and **delta**) are conventionally used to measure these distances.[1]

Definition 9.1 *f approaches the limit l near a if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $0 < |x - a| < \delta \implies |f(x) - l| < \epsilon$.*

That is, the function f approaches the limit l near a when for every $\epsilon > 0$, however small, there is some $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$ and $x \neq a$.

Note that in this definition $x \neq a$: as mentioned before, the value of the function f at a is irrelevant and does not even have to be defined.

9.4 Some notation and theorems on limits

The following theorem might seem obvious, but it is very important: it ensures that the limit (if it exists at all) is a well-defined number.

Theorem 9.1 *If f approaches both m and l near a then $l = m$.*

Proof: To prove the theorem, note that if f approaches l near a then this implies that for any number $\epsilon > 0$ there is a number $\delta_1 > 0$ such that $|f(x) - l| < \epsilon$ when $0 < |x - a| < \delta_1$. At the same time, if f approaches m near a then this implies that for the same $\epsilon > 0$ there is a number $\delta_2 > 0$ such that $|f(x) - l| < \epsilon$ when $0 < |x - a| < \delta_2$. If we choose

$$\delta := \min(\delta_1, \delta_2),$$

then if $0 < |x - a| < \delta$ then both

$$|f(x) - l| < \epsilon \quad \text{and} \quad |f(x) - m| < \epsilon.$$

If $l \neq m$, this cannot possibly be true for all values of ϵ. Consider $\epsilon = |l - m|/2$ (half the distance between m and l); then there would have to be a number δ such that for $0 < |x - a| < \delta$ we would have

$$|l - m| = |l - f(x) + f(x) - m| \leq |l - f(x)| + |f(x) - m| < \frac{|l - m|}{2} + \frac{|l - m|}{2} = |l - m|,$$

which is a contradiction. $\square$

[1]The choice of ϵ goes back to Cauchy, who used it to represent the "ϵrror" in the value of f; δ is simply adjacent to ϵ in the Greek alphabet.

Now that we know that it is unique, we can introduce a convenient notation for *the* number l that f approaches near a. We write it as

$$\lim_{x \to a} f(x) = l.$$

This means literally that the limit of f as x approaches a is l. Note that this makes sense only if the limit actually exists, that is, if there is some l that f tends to near a. Otherwise, when f does not approach any number l near a we simply say that $\lim_{x \to a} f(x)$ does not exist.

There is an alternative notation which is sometimes more convenient: the statement

$$f(x) \to l \text{ as } x \to a$$

expresses the same as $\lim_{x \to a} f(x) = l$.

Example 9.7 Show that $\lim_{x \to 3} \dfrac{1}{x} = \dfrac{1}{3}$.

We need to show that for any given $\epsilon > 0$ we can find a $\delta > 0$ such that if $0 < |x - 3| < \delta$, then

$$\left| \frac{1}{x} - \frac{1}{3} \right| < \epsilon.$$

The first step is to write

$$\left| \frac{1}{x} - \frac{1}{3} \right| = \left| \frac{3 - x}{3x} \right| = \frac{1}{3} \cdot \frac{1}{|x|} \cdot |x - 3|.$$

We now need to find an estimate for the factor $\frac{1}{|x|}$. Any *finite* estimate will do, so we need to ensure that $|x|$ stays away from zero. If we assume that $|x - 3| < 1$, then $-1 < x - 3 < 1$ and $2 < x < 4$ and so $|x| > 2$. This implies that $\frac{1}{|x|} < \frac{1}{2}$, and so

$$\left| \frac{1}{x} - \frac{1}{3} \right| = \frac{1}{3} \cdot \frac{1}{|x|} \cdot |x - 3| < \frac{|x - 3|}{6}.$$

This shows that with the choice $\delta = \min(6\epsilon, 1)$,

$$0 < |x - 3| < \delta \implies \left| \frac{1}{x} - \frac{1}{3} \right| < \frac{|x - 3|}{6} < \epsilon.$$

The following theorem allows us to compute many limits without having to resort to the ϵ–δ definition. This will make life considerably easier.

Theorem 9.2 *Let* $\lim_{x \to a} f(x) = l$ *and* $\lim_{x \to a} g(x) = m$. *Then*

1.
$$\lim_{x \to a}[f(x) + g(x)] = l + m;$$

2. *For any constant* α,
$$\lim_{x \to a} \alpha f(x) = \alpha l;$$

3. *For any constants* α *and* β,
$$\lim_{x \to a}[\alpha f(x) + \beta g(x)] = \alpha l + \beta m;$$

4.
$$\lim_{x \to a} f(x) g(x) = lm;$$

5. *If* $m \neq 0$ *then*
$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m};$$

6. *If* $m \neq 0$ *then*
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}.$$

Proof: All the statements of the theorem should be intuitively clear. However, a formal proof requires us to phrase all statements in terms of the ϵ–δ definition of the limit and then show that they indeed hold.

1. The premiss implies that for any numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$ we can find numbers δ_1 and δ_2 such that $|f(x) - l| < \epsilon_1$ when $0 < |x - a| < \delta_1$ and $|g(x) - m| < \epsilon_2$ when $0 < |x - a| < \delta_2$. The trick of the proof is to start with $\epsilon_1 = \epsilon_2 = \epsilon/2$ (which is just another positive number). By the premiss we can then find numbers δ_1 and δ_2 such that $|f(x) - l| < \epsilon/2$ when $0 < |x - a| < \delta_1$ and $|g(x) - m| < \epsilon/2$ when $0 < |x - a| < \delta_2$. Now we choose $\delta := \min(\delta_1, \delta_1)$ and it follows that whenever $0 < |x - a| < \delta$,
$$\begin{aligned} |(f(x) + g(x)) - (l + m)| &= |(f(x) - l) + (g(x) - m)| \\ &\leq |f(x) - l| + |g(x) - m| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$
which means that we do indeed have $\lim_{x \to a}[f(x) + g(x)] = l + m$.

2. To prove this, first notice that the statement is trivial when $\alpha = 0$. When $\alpha \neq 0$ we need a trick similar to that used to prove the first statement. Instead of starting with

$\epsilon > 0$, we start with $\epsilon/|\alpha|$ which is again a positive number. Because $\lim_{x\to a} f(x) = l$, we can find a $\delta > 0$ such that when $0 < |x-a| < \delta$ we have $|f(x) - l| < \epsilon/|\alpha|$. But then

$$\begin{aligned} 0 < |x-a| < \delta \implies |\alpha f(x) - \alpha l| &= |\alpha(f(x) - l)| \\ &= |\alpha||f(x) - l| \\ &< |\alpha|\frac{\epsilon}{|\alpha|} = \epsilon \end{aligned}$$

as required.

3. This follows by combining parts 1 and 2.

4. This follows from the fact that if f is close to l and g is close to m, then fg is close to lm. The formal proof is rather technical and is omitted here. It can be found, for example, in the Calculus book by SPIVAK.

5. Again a rather technical affair; for the proof see SPIVAK.

6. This follows immediately from parts 4 and 5.

□

9.5 Some more examples and results

Example 9.8 Given that $\lim_{x\to a} x = a$, find $\lim_{x\to a}(x^2 + 3x)$.

The point of this example is to demonstrate how Theorem 9.2 can be employed.

$$\begin{aligned} \lim_{x\to a}(x^2 + 3x) &= \lim_{x\to a} x^2 + \lim_{x\to a} 3x && \text{using Theorem 9.2(1)} \\ &= \left(\lim_{x\to a} x\right) \cdot \left(\lim_{x\to a} x\right) + 3\left(\lim_{x\to a} x\right) && \text{using Theorem 9.2(4) and (2)} \\ &= a^2 + 3a && \text{using the information in the question.} \end{aligned}$$

Example 9.9 If $g(x) = \dfrac{x^2 + 3x - 4}{x^2 + x - 12}$, find $\lim_{x\to -4} g(x)$.

The function g is not defined at -4. However, note that

$$g(x) = \frac{x^2 + 3x - 4}{x^2 + x - 12} = \frac{(x-1)(x+4)}{(x-3)(x+4)}$$

and so

$$g(x) = \frac{x-1}{x-3} \qquad \text{when} \qquad x \neq -4.$$

This is all we need to compute the limit:

$$\lim_{x \to -4} \frac{x^2+3x-4}{x^2+x-12} = \lim_{x \to -4} \frac{x-1}{x-3} = \frac{-5}{-7} = \frac{5}{7}.$$

It's tempting to think that we should always be able to do this. However, a function may be indeterminate at $x = a$ and not have a limit there. This is illustrated by the next example.

Example 9.10 Examine the behaviour of $r(x)$ as $x \to 1$ where

$$r(x) = \frac{x-1}{x^2-2x+1}.$$

Let

$$\rho(x) = \frac{1}{x-1}.$$

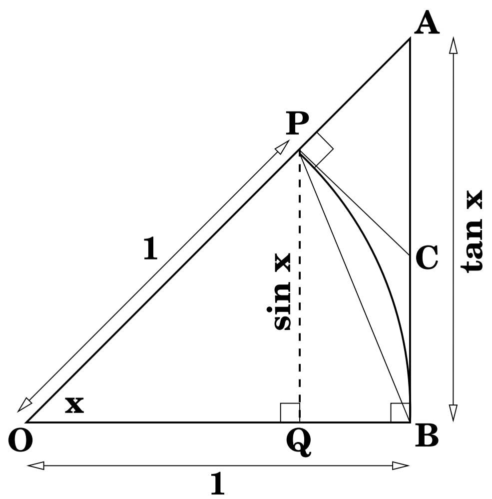
Then $r(x) = \rho(x)$ for $x \neq 1$. However $\rho(x)$ is large and negative for x slightly less than 1, and large and positive for x slightly bigger than 1. Thus the values $\rho(x)$ do not approach a finite value as x approaches 1. Hence "$\lim_{x \to 1} r(x)$" does not exist.

We will look in a later section at how to describe this behaviour.

To evaluate many limits involving trigonometric functions, the next theorem (and the following Lemma) are very useful.

Theorem 9.3 $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Proof: Consider the following diagram.



In the diagram, OBP is a sector of a circle of radius 1, and the angle BOA is $x > 0$ radians, i.e., $\operatorname{arc} \mathrm{BP} = x$. The line PQ is perpendicular to OB, and AB is the tangent to the circle at B so that AB is perpendicular to OB. Then

$$\sin x = |\mathrm{QP}| < |\mathrm{BP}| < \operatorname{arc} \mathrm{BP} = x$$

and so $\dfrac{\sin x}{x} < 1$.

Also, because the circumference of a polygon is bigger than the circumference of its inscribed circle,

$$x = \operatorname{arc} \mathrm{BP} < |\mathrm{BC}| + |\mathrm{CP}| < |\mathrm{BC}| + |\mathrm{CA}| = |\mathrm{BA}| = \tan x,$$

whence $\dfrac{\sin x}{x} > \cos x$.

Thus, we have

$$\cos x < \frac{\sin x}{x} < 1$$

and since $\lim_{x \to 0} \cos x = 1$ (think how you might prove this!) it follows that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{9.2}$$

$\square$

It is easy to generalise this theorem.

Lemma 9.1 *For all numbers m, $n \neq 0$*

$$\lim_{x \to 0} \frac{\sin mx}{nx} = \frac{m}{n}.$$

Proof: We first show that $\lim_{x \to 0} \dfrac{\sin mx}{mx} = 1$. Define $y = mx$. We know that $\lim_{y \to 0} \dfrac{\sin y}{y} = 1$, so for any $\epsilon > 0$ there is a number $\delta_1 > 0$ such that

$$|y| < \delta_1 \implies \left| \frac{\sin y}{y} - 1 \right| < \epsilon.$$

Now

$$|y| = |mx| = |m| \cdot |x|$$

and so $|y| < \delta_1$ exactly when $|x| < \frac{\delta_1}{|m|}$. This means that if we define $\delta := \frac{\delta_1}{|m|}$ then

$$|x| < \delta \implies \left| \frac{\sin y}{y} - 1 \right| = \left| \frac{\sin mx}{mx} - 1 \right| < \epsilon$$

and so $\lim_{x \to 0} \dfrac{\sin mx}{mx} = 1$.

Now

$$\lim_{x\to 0}\frac{\sin mx}{nx} = \lim_{x\to 0}\left(\frac{m}{n}\cdot\frac{\sin mx}{mx}\right) = \frac{m}{n}\lim_{x\to 0}\frac{\sin mx}{mx} = \frac{m}{n}$$

which proves the Lemma. □

Example 9.11 Find

$$\lim_{x\to 0}\frac{\sin 5x}{x} \qquad \text{and} \qquad \lim_{x\to 0}\frac{7x}{\sin 3x}.$$

Using the Lemma, we have

$$\lim_{x\to 0}\frac{\sin 5x}{x} = 5,$$

and using the Lemma together with Theorem 9.2 part 5,

$$\lim_{x\to 0}\frac{7x}{\sin 3x} = \frac{7}{3}.$$

Example 9.12 Find $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$.

Here we need a trick:

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x} = 1\cdot 0 = 0. \tag{9.3}$$

9.6 Other types of limit

Sometimes it is useful to describe the behaviour of a function when it does not approach a (finite) limit in the manner described by Definition 9.1. In these situations we may be able to extend the definition appropriately.

9.6.1 One-sided limits

Sometimes we need to compute what are called **one-sided limits**. For example, $f(x) = \sqrt{x}$ is defined only for positive arguments. It still makes sense to speak of the limit of f as x approaches zero, but x can approach zero only from the right. Two different notations are in common use.

A **right-hand limit** is defined as

$$\lim_{x\to a^+} f(x) = \lim_{x\searrow a} f(x) = l_1$$

when for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - l_1| < \epsilon$ whenever $0 < x - a < \delta$.

Similarily, a **left-hand limit** is defined as

$$\lim_{x\to a^-} f(x) = \lim_{x\nearrow a} f(x) = l_2$$

when for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - l_2| < \epsilon$ whenever $0 < a - x < \delta$.

Clearly, the ordinary limit $\lim_{x\to a} f(x)$ exists if and only if both the left- and the right-hand limits exist and they are the same, $l_1 = l_2$.

Example 9.13 Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} 3 & \text{if} \quad x < 0, \\ x^2 & \text{if} \quad x \geq 0. \end{cases}$$

We have $\lim_{x\to 0^+} f(x) = 0$ but $\lim_{x\to 0^-} f(x) = 3$, so the limit does not exist.

9.6.2 Limits 'at infinity'

Another type of limit arises when we consider the behaviour of a function as its argument approaches infinity. We write

$$\lim_{x\to\infty} f(x) = l$$

to state that the limit of $f(x)$ as x approaches ∞ is l. Formally, this means that for every $\epsilon > 0$ there is a number N such that $|f(x) - l| < \epsilon$ for all $x > N$. The notation $\lim_{x\to-\infty}$ has an exactly analogous meaning.

Example 9.14 Show that $\lim_{x\to\infty} 1/x = 0$.

To see this, given an ϵ, simply choose $N = 1/\epsilon$. Then $x > N = 1/\epsilon \iff 1/x < \epsilon$ and so $|1/x - 0| = 1/x < \epsilon$ as required.

9.6.3 Functions that 'tend to infinity'

Finally, it is sometimes useful to have a way of talking about functions that do not approach a finite value in some limit, but instead 'tend to infinity' in some limit. Since infinity is not a number, we need to define this carefully.

Definition 9.2 *The statement*

$$f(x) \to \infty \quad \textit{as} \quad x \to a$$

means that $\forall\ M\ \exists\ \delta > 0$ *such that* $0 < |x - a| < \delta \implies f(x) > M$*.*

That is, the function f tends to infinity near a when for every M, however large, there is some $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$ and $x \neq a$.

Similarly, the statement

$$f(x) \to -\infty \quad \textit{as} \quad x \to a$$

means that $\forall\ M\ \exists\ \delta > 0$ *such that* $0 < |x - a| < \delta \implies f(x) < M$*.*

Example 9.15 Show that $\dfrac{1}{x^2} \to \infty \qquad \text{as} \qquad x \to 0.$

To see this, given a value of $M > 0$, choose $\delta = 1/\sqrt{M}$. Then $0 < |x| < \delta = 1/\sqrt{M} \iff 1/x^2 > M$ as required.

Note that Theorem 9.2 does not apply if l or m is infinite. For example, if $f(x) \to l$ and $g(x) \to l$ as $x \to a$ ($l \in \mathbb{R}$) then we know that $(f - g)(x) \to 0$ as $x \to a$. However, if we are told that $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, this tells us nothing about the behaviour of $f - g$: this could tend to infinity, to minus infinity, or to a constant, as $x \to a$. [Consider, for example, $f(x) = 1/x$ and $g(x) = 1/x^2$ in the limit $x \to 0$.]

10 Continuity

The basic idea of **continuity** is simple and intuitive: a function is continuous if you can draw its graph with a single stroke without lifting your pencil off the paper. In other words, the graph of a continuous function has no jumps and no holes. Building on the previous chapter, we can give a very concise definition of this.

We start by defining continuity of a function at a given point a.

Definition 10.1 *A function f is* ***continuous*** *at a if*

$$\lim_{x\to a} f(x) = f(a).$$

Example 10.1 Consider the six functions in Example 9.4.

Only the two functions in the top row are continuous at a. (Note that the one at the top right is continuous because $\lim_{x\to a} f(x) = f(a)$ even though the value of this limit isn't l.)

The other functions in the left-hand column are not continuous at a because $f(a)$ is not defined (middle row) or is defined to be something other than $\lim_{x\to a} f(x)$.

The other functions in the right-hand column cannot be continuous at a because $\lim_{x\to a} f(x)$ does not exist.

Remark: It is often useful to use an equivalent form of the limit when checking continuity:

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h),$$

(see Exercise 9.6). A function f is then continuous at a when

$$\lim_{h\to 0} f(a+h) = f(a).$$

The following two theorems tell us how we can construct new continuous functions from known continuous functions.

Theorem 10.1 *If f and g are continuous at a then:*

1. *$f+g$ is continuous at a;*
2. *$f \cdot g$ is continuous at a; and*
3. *if $g(a) \neq 0$ then $1/g$ is continuous at a.*

Proof: All three parts of the theorem are direct consequences of Theorem 9.2 on limits. For example, since f and g are continuous at a, we have $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. But then $\lim_{x\to a}(f+g)(x) = f(a) + g(a) = (f+g)(a)$. Parts 2 and 3 are proved similarly. $\Box$

Theorem 10.2 *If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

Proof: We need to show that $\lim_{x\to a} f(g(x)) = f(g(a))$. This means that for a given $\epsilon > 0$ we need to find $\delta > 0$ such that

$$\text{if} \quad |x-a| < \delta \quad \text{then} \quad |f(g(x)) - f(g(a))| < \epsilon. \tag{10.1}$$

(Note that we can get away without requiring $0 < |x-a|$, since for a continuous function there is no problem for $x = a$.) First, because of continuity of f at $g(a)$, there is a number δ_1 such that

$$|f(y) - f(g(a))| < \epsilon \quad \text{if} \quad |y - g(a)| < \delta_1.$$

This implies that (use $y = g(x)$)

$$|f(g(x)) - f(g(a))| < \epsilon \quad \text{if} \quad |g(x) - g(a)| < \delta_1.$$

But from the continuity of g at a we know (if we use our δ_1 as 'ϵ' in the definition of continuity) that there is some δ such that

$$|g(x) - g(a)| < \delta_1 \quad \text{if} \quad |x-a| < \delta.$$

Combining the last two equations shows that (10.1) is true and so $f \circ g$ is indeed continuous at a. $\Box$

The concept of continuity at a point leads naturally to the concept of continuity in an interval. (Drawing a point without lifting your pencil off the paper is not much of an achievement.)

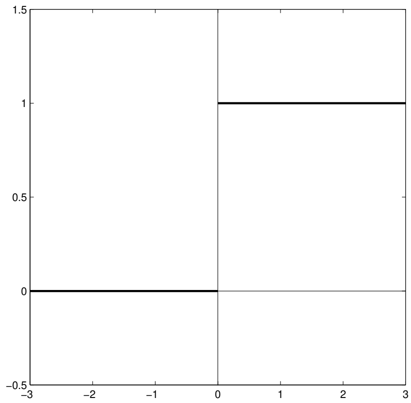
Definition 10.2

1. *A function f is* ***continuous in an open interval*** *(a, b) if it is continuous at all points $x \in (a, b)$.*

2. *A function f is* ***continuous in a closed interval*** *$[a, b]$ if it is continuous in (a, b) and if $\lim_{x \searrow a} f(x) = f(a)$ and $\lim_{x \nearrow b} f(x) = f(b)$.*

Example 10.2 Consider HEAVISIDE's **step function**, which has value 0 to the left of $x = 0$ and value 1 to the right:

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0; \\ 1 & \text{if} \quad x \geq 0. \end{cases}$$

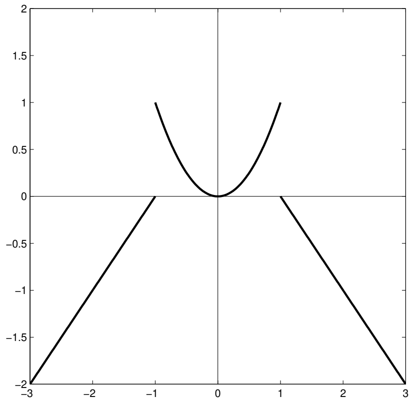


It is easy to show that $\lim_{x \to 0^-} H(x) = 0$ but $\lim_{x \to 0^+} H(x) = 1$. Hence the Heaviside step function is discontinuous at 0; more specifically, it has a **jump discontinuity** at $x = 0$. (It is not important whether the value at $x = 0$ is 1 or 0, or any other value. However, if it is undefined, then the resulting function is continuous on its entire domain, because the domain then does not include the point 0. We sometimes refer to a "punctured domain" in situations like this.)

Example 10.3 Consider the function

$$a(x) = \begin{cases} 1 + x & \text{if} \quad x \leq -1; \\ x^2 & \text{if} \quad -1 < x < 1; \\ 1 - x & \text{if} \quad x \geq 1. \end{cases}$$

A sketch shows that this function is discontinuous (has jumps) at $x = -1$ and $x = 1$.



Note that a function isn't necessarily discontinuous at a point where the formula changes!

Example 10.4 Consider the function

$$p(x) = \begin{cases} 0 & \text{if} \quad x < 0; \\ \sin x & \text{if} \quad x \geq 0; \end{cases}$$

We have $\lim_{x\to 0^-} p(x) = \lim_{x\to 0^-} 0 = 0$, while $\lim_{x\to 0^+} p(x) = \lim_{x\to 0^+} \sin x = 0$, so $\lim_{x\to 0} p(x) = 0$ (it's a two-sided limit). Since $p(0) = 0 = \lim_{x\to 0} p(x)$, the function p is continuous at 0.

The next example shows how the idea of continuity may be used to 'patch a hole' in the definition of a function.

Example 10.5 Consider the function

$$f(x) = \frac{x^2 - 8x + 15}{x - 3} = \frac{(x-3)(x-5)}{x-3}.$$

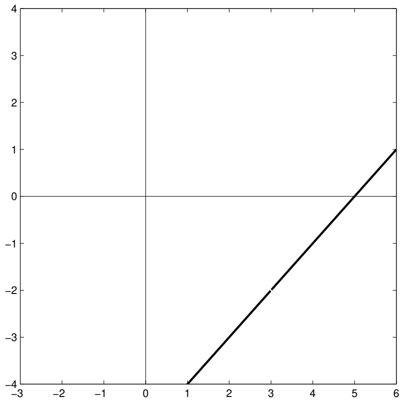
The value of $f(x)$ is undefined at $x = 3$, but we have

$$\lim_{x\to 3} f(x) = -2.$$

Since f is not defined at 3, it does not make sense to speak about continuity there. However, since 3 is not in the domain of f, f is continuous on every point in its domain. We can extend f by defining

$$F(x) := \begin{cases} f(x) & \text{if} \quad x \neq 3 \\ \alpha & \text{if} \quad x = 3. \end{cases}$$

The domain of F is all of $\mathbb{R}$, and F is everywhere continuous if $\alpha = -2$. For any other value $\alpha \neq -2$ F is discontinuous at 3. Of course, for the choice $\alpha = -2$ we simply have $F(x) = x - 5$.



Unfortunately, some holes are beyond repair!

Example 10.6 The function

$$g(x) = \frac{1}{x-5}, \qquad x \in (-\infty, 5) \cup (5, \infty)$$

is undefined at $x = 5$.

As in the preceding example, g is continuous on every point in its domain, but here we cannot extend it to a continuous function that is defined on all of $\mathbb{R}$. Even if we assign a value to an extended function G at 5, the limit $\lim_{x \to 5} g(x)$ simply does not exist.