UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MATHEMATICS & STATISTICS

MM103 Geometry and Algebra

Chapter 1: Cartesian Coordinates

Q1.

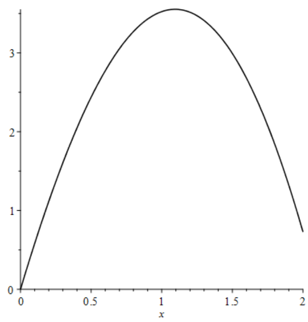

Figure 1: (a)

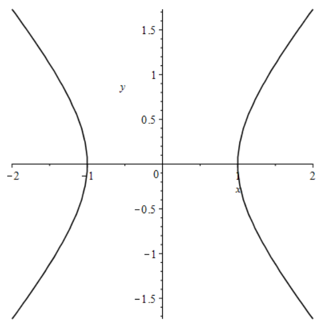

Figure 2: (b)

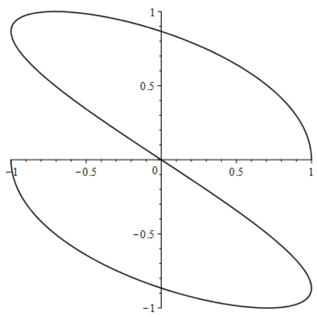

Figure 3: (c)

Q2.

(a) Anticlockwise rotation by a right angle around the origin.

(b) Translation 2 units right and then 2 units down.

(c) Clockwise rotation by a right angle around $(2, 1)$ followed by a translation 1 unit left and then 1 unit up.

(d) Anticlockwise rotation around $(2, 1)$ by $\dfrac{3\pi}{4}$ radians followed by a translation left by 2 units.

(e) Reflection through $y = \sqrt{3}x$.

Q3. (a) $(4, -\pi/3)$ (b) $(5, \tan^{-1}(4/3))$ (c) $(12, \pi)$ (d) $(128\sqrt{2}, -3\pi/4)$ (e) $(6\sqrt{2}, \pi/4)$ (f) $(4, \pi/2)$ (g) $(2, -\pi/2)$ (h) $(2\sqrt{3}, 5\pi/6)$

Q4. (a) $(3/\sqrt{2}, -3/\sqrt{2})$ (b) $(3\sqrt{3}, -3)$ (c) $(-2, 0)$

Q5.

(a) Set $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then

$$(x^2 + y^2)^2 = (r^2\cos^2(\theta) + r^2\sin^2(\theta))^2 = (r^2)^2 = r^4$$

and

$$x^2 - y^2 = r^2\cos^2(\theta) - r^2\sin^2(\theta) = r^2\cos(2\theta).$$

Therefore, the polar form of the lemniscate is

$$r^4 = r^2\cos(2\theta).$$

Note that

$$r^2 = \cos(2\theta)$$

defines the same curve.

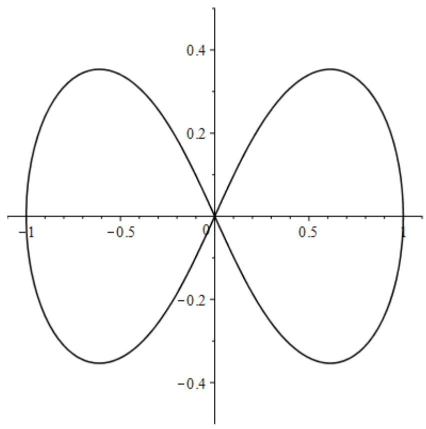

Figure 4: The lemniscate of Bernoulli.

(c) For the Cartesian form of the lemniscate, we have

$$\begin{aligned} x^2 + y^2 &= \frac{\cos^2(t)}{(1+\sin^2(t))^2} + \frac{\sin^2(2t)}{(2+2\sin^2(t))^2} \\ &= \frac{\cos^2(t)}{(1+\sin^2(t))^2} + \frac{4\sin^2(t)\cos^2(t)}{4(1+\sin^2(t))^2} \\ &= \cos^2(t)\left(\frac{1+\sin^2(t)}{(1+\sin^2(t))^2}\right) \\ &= \frac{\cos^2(t)}{1+\sin^2(t)} \end{aligned}$$

and

$$\begin{aligned} x^2 - y^2 &= \frac{\cos^2(t)}{(1+\sin^2(t))^2} - \frac{\sin^2(2t)}{(2+2\sin^2(t))^2} \\ &= \frac{\cos^2(t)}{(1+\sin^2(t))^2} - \frac{4\sin^2(t)\cos^2(t)}{4(1+\sin^2(t))^2} \\ &= \cos^2(t)\left(\frac{1-\sin^2(t)}{(1+\sin^2(t))^2}\right) \\ &= \frac{\cos^4(t)}{(1+\sin^2(t))^2}. \end{aligned}$$

Hence, $(x^2+y^2)^2 = x^2 - y^2$ as required.

In polar form,

$$r^2 = x^2 + y^2 = \frac{\cos^2(t)}{1+\sin^2(t)} = \frac{1-\sin^2(t)}{1+\sin^2(t)},$$

so we should look for a way of expressing $\sin(t)$ in terms of the polar coordinates r, θ. Now,

$$\tan(\theta) = \frac{y}{x} = \frac{\sin(2t)}{2+2\sin^2(t)} \times \frac{1+\sin^2(t)}{\cos(t)} = \frac{\sin(t)\cos(t)}{\cos(t)} = \sin(t)$$

and so

$$\begin{aligned} r^2 &= \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)} \\ &= \frac{\cos^2(\theta)-\sin^2(\theta)}{\cos^2(\theta)+\sin^2(\theta)} \\ &= \cos(2\theta), \end{aligned}$$

as required.

Chapter 1: Vectors

Q1.

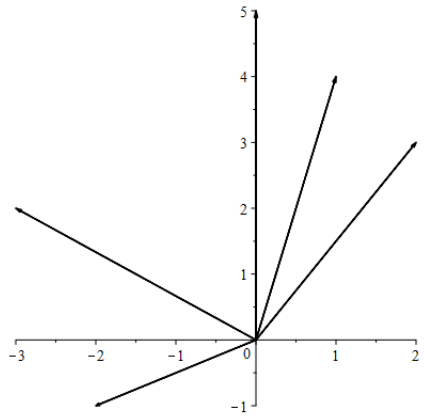

Figure 5: Vectors in the plane.

Q2. (a) $\begin{bmatrix} 12 \\ 0 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$ (c) 0 (d) 15 (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Q3.

(a) We are required to find $a, b, c \in \mathbb{R}$ that satisfy

$$\begin{bmatrix} 3a \\ a \\ a \end{bmatrix} + \begin{bmatrix} 3b \\ -b \\ b \end{bmatrix} + \begin{bmatrix} -2c \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

that is

$$\begin{bmatrix} 3a + 3b - 2c \\ a - b \\ a + b + c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that $b = a$ and $c = -2a$, and so $10a = 1$. Hence,

$$\frac{1}{10}\mathbf{a} + \frac{1}{10}\mathbf{b} - \frac{1}{5}\mathbf{c} = \mathbf{e}_1.$$

(b) Given that $2\mathbf{a} - \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$, we have

$$\mathbf{a} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{d}$$

and so

$$\frac{1}{10}\left(\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{d}\right) + \frac{1}{10}\mathbf{b} - \frac{1}{5}\mathbf{c} = \mathbf{e}_1.$$

Hence

$$\frac{3}{20}\mathbf{b} - \frac{1}{4}\mathbf{c} - \frac{1}{20}\mathbf{d} = \mathbf{e}_1.$$

Q4. It is easy to see that $\mathbf{a}+\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = 2\mathbf{f}$. Hence, $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \mathbf{f}$.

A linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{f}$ is a vector of the form

$$\begin{bmatrix} 3\alpha + 3\beta + 3\gamma \\ \alpha - \beta \\ \alpha + \beta + \gamma \end{bmatrix}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. If this equals $\mathbf{e}_1$, then we would require that $3(\alpha+\beta+\gamma) = 1$ and $\alpha+\beta+\gamma = 0$, which is impossible. Thus, no linear combination of these vectors can equal $\mathbf{e}_1$.

Q5. Let $\theta_1, \theta_2, \theta_3$ be the smallest angles between $\mathbf{a}$ and, respectively, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Then

$$\cos(\theta_1) = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||\,||\mathbf{b}||} = \frac{9}{11}$$

and so $\theta_1 = \cos^{-1}\left(\frac{9}{11}\right)$. Similarly, $\theta_2 = \pi - \cos^{-1}\left(\sqrt{\frac{5}{11}}\right)$ and $\theta_2 = \pi - \cos^{-1}\left(\frac{8}{\sqrt{154}}\right)$.

Q6. $||\mathbf{a}+\mathbf{b}-3\mathbf{c}|| = \sqrt{145}$, $\quad ||2\mathbf{d}-\mathbf{b}|| = \sqrt{75}$, $\quad ||2\mathbf{a}-\mathbf{b}+\mathbf{c}+\mathbf{d}|| = 0$.

Q7. $\frac{1}{\sqrt{11}}\mathbf{a}$, $\quad \frac{1}{\sqrt{11}}\mathbf{b}$, $\quad \frac{1}{\sqrt{5}}\mathbf{c}$, $\quad \frac{1}{\sqrt{14}}\mathbf{d}$.

Q8.

(a) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} = ||\mathbf{u}||\,||\mathbf{v}||\cos(\pi/6) + 0 = 6\sqrt{3}$.

$\mathbf{w}\cdot(\mathbf{u}-\mathbf{w}) = \mathbf{w}\cdot\mathbf{u} - \mathbf{w}\cdot\mathbf{v} = 0 - ||\mathbf{w}||\,||\mathbf{v}||\cos(2\pi/3) = 9/4$. Note that $2\pi/3$ is the angle between $\mathbf{v}$ and $\mathbf{w}$, and that $||\mathbf{w}|| = 3\sin(\pi/6)$.

(b) $||\mathbf{v}+\mathbf{w}|| = ||\mathbf{v}||\cos(\pi/6) = 3\sqrt{3}/2$.

Let $\mathbf{x}$ be such that $\mathbf{v}+\mathbf{x}+\mathbf{w} = \mathbf{u}$. Then, $\mathbf{u}-\mathbf{v} = \mathbf{x}+\mathbf{w}$. Moreover, since $\mathbf{x}\cdot\mathbf{w} = 0$,

$$||\mathbf{u}-\mathbf{v}|| = \sqrt{||\mathbf{x}||^2 + ||\mathbf{w}||^2}.$$

Now, $||\mathbf{x}|| = ||\mathbf{u}|| - ||\mathbf{v}+\mathbf{w}|| = 4 - 3\sqrt{3}/2$ and so

$$||\mathbf{u}-\mathbf{v}|| = \sqrt{(4-3\sqrt{3}/2)^2 + (3/2)^2} = \sqrt{25 - 12\sqrt{3}}.$$

Q9.

(a) False. Counterexample: $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = \mathbf{z} = \mathbf{e}_1$.

(b) False. Counterexample: $\mathbf{x} = \mathbf{e}_1 = -\mathbf{y}$ and $c = 1$, $d = 0$.

(c) False. Counterexample: $\mathbf{e}_1$ and -$\mathbf{e}_1$ are parallel and of the same length.

(d) True. Let x_i be the i-th component of $\mathbf{x}$. Then the i-th component of $a\mathbf{x}$ is ax_i and if this equals 0, then either $a = 0$ or $x_i = 0$. Since x_i was arbitrarily chosen, this must be true for all i. Hence, $a = 0$ or $\mathbf{x} = \mathbf{0}$.

(e) False. Counterexample: $\mathbf{x} = \mathbf{y} = \mathbf{e}_1$ and $c = d = 0$. The result is true if both c and d are non-zero.

Q10. Let $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$. Then $\mathbf{p} \cdot \mathbf{q} = 1000000$.

Q11. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$. Then $a\mathbf{u} = \begin{bmatrix} au_1 \\ \vdots \\ au_n \end{bmatrix}$ and so

$$||a\mathbf{u}|| = \sqrt{(au_1)^2 + \cdots + (au_n)^2} = \sqrt{a^2(u_1^2 + \cdots + u_n^2)} = \sqrt{a^2}\,||\mathbf{u}|| = |a|\,||\mathbf{u}||.$$

Q12. If $\mathbf{y} = k\mathbf{x}$, then

$$\begin{aligned} ||\mathbf{x} + \mathbf{y}|| &= ||\mathbf{x} + k\mathbf{x}|| \\ &= ||(1+k)\mathbf{x}|| \\ &= |1+k|\,||\mathbf{x}|| \\ &= (1+k)||\mathbf{x}|| \quad \text{(because } k > 0) \\ &= ||\mathbf{x}|| + k||\mathbf{x}|| \\ &= ||\mathbf{x}|| + ||k\mathbf{x}|| \\ &= ||\mathbf{x} + ||\mathbf{y}||. \end{aligned}$$

Q13. We have

$$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\mathbf{u} \cdot \mathbf{v}.$$

The triangle inequality states that $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$. Therefore,

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u}|| + ||\mathbf{v}|| &\iff ||\mathbf{u} + \mathbf{v}||^2 = (||\mathbf{u}|| + ||\mathbf{v}||)^2 \\ &\iff ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2||\mathbf{u}||\,||\mathbf{v}|| \\ &\iff \mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}|| \\ &\iff ||\mathbf{u}||\,||\mathbf{v}||\cos(\theta) = ||\mathbf{u}||\,||\mathbf{v}|| \\ &\iff \cos(\theta) = 1. \end{aligned}$$

It follows that θ, the angle between $\mathbf{u}$ and $\mathbf{v}$, equals 0 radians. Hence, $\mathbf{u}$ and $\mathbf{v}$ point in the same direction.

Q14. We have

$$||\mathbf{u}|| = ||\mathbf{u} - \mathbf{v} + \mathbf{v}|| \leq ||\mathbf{u} - \mathbf{v}|| + ||\mathbf{v}||$$

and so

$$||\mathbf{u}|| - ||\mathbf{v}|| \leq ||\mathbf{u} - \mathbf{v}||.$$

Similarly,

$$||\mathbf{v}|| \leq ||\mathbf{v} - \mathbf{u}|| + ||\mathbf{u}||$$

and so

$$||\mathbf{v}|| - ||\mathbf{u}|| \leq ||\mathbf{u} - \mathbf{v}||.$$

Now, either $|(||\mathbf{u} - ||\mathbf{v}||)| = ||\mathbf{u}|| - ||\mathbf{v}||$ or $|(||\mathbf{u} - ||\mathbf{v}||)| = ||\mathbf{v}|| - ||\mathbf{u}||$. Thus, in either case, the result follows.

Q15. Let $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. If $\mathbf{v} \cdot \mathbf{w} = 5$, then $2w_1 + w_2 = 5$ and so $w_2 = 5 - 2w_1$. Therefore,

$$\mathbf{w} = \begin{bmatrix} w_1 \\ 5 - 2w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + w_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

All such vectors correspond to the position vector of a point on the straight line $y = -2x + 5$:

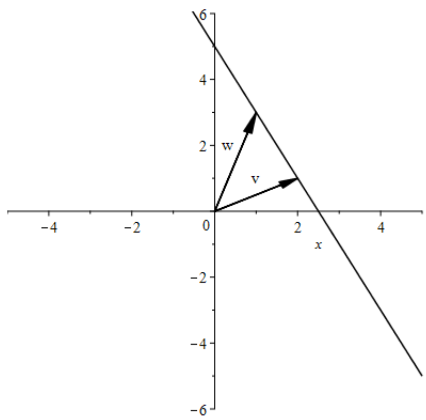

Figure 6: All vectors satisfying $\mathbf{v} \cdot \mathbf{w} = 5$

The shortest such $\mathbf{w}$ will be orthogonal to the straight line $y = -2x + 5$. This line is parallel to the vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so we need to find a vector $\mathbf{w}$ that satisfies

$$\mathbf{w} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0.$$

Clearly, $\mathbf{w} = \mathbf{v}$ will do.

Q16. (a) $\mathbf{p} = \begin{bmatrix} 3.3 \\ 1.1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1.3 \\ 3.9 \end{bmatrix}$.

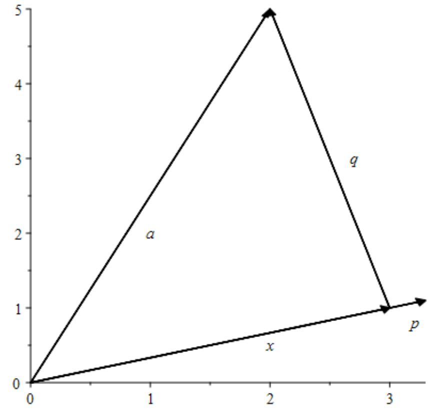

Figure 7: The projection of **a** onto **x**.

(b) Given that

$$(\mathbf{a} - \mathbf{p}) \cdot \mathbf{x} = \left(\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{x}}{||\mathbf{x}||^2} \mathbf{x} \right) \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{x} - \frac{\mathbf{a} \cdot \mathbf{x}}{||\mathbf{x}||^2} ||\mathbf{x}||^2 = 0,$$

the vectors $\mathbf{a} - \mathbf{p}$ and $\mathbf{x}$ must be orthogonal.

Chapter 1: Matrices

Q1. $A^T = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}$, $D^T = \begin{bmatrix} -1 & -3 \\ 2 & 0 \\ -2 & 2 \end{bmatrix}$, $F^T = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 2 & 0 & -1 & 4 \\ -1 & -1 & 0 & 5 \end{bmatrix}$.

Q2. $A+B = \begin{bmatrix} 1 & 6 \\ 3 & 1 \end{bmatrix}$, $3A-2B = \begin{bmatrix} 3 & 3 \\ 19 & 8 \end{bmatrix}$, $A+B^T+3C = \begin{bmatrix} 7 & 14 \\ 14 & 4 \end{bmatrix}$, $E-E^T = \begin{bmatrix} 0 & 3 & 3 \\ -3 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$.

Q3. $D(1, 1:3) = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$, $E(2:3, 1:3) = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $F(2:3, 1:3) = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

Q4. $G = \begin{bmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 3 \\ -2 & -1 \end{bmatrix}$, $\begin{bmatrix} D \\ E \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -3 & 0 & 2 \\ -3 & 2 & 1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 3 \\ 5 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \\ 3 & -1 & 2 & 1 \end{bmatrix}$,

$\begin{bmatrix} G & F \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 2 & -1 \\ 5 & 2 & 3 & 0 & -1 \\ 0 & 3 & 2 & -1 & 0 \\ -2 & -1 & -1 & 4 & 5 \end{bmatrix}$.

Q5. $AB = \begin{bmatrix} -6 & 0 \\ -4 & 13 \end{bmatrix}$, $CD = \begin{bmatrix} -5 & 4 & -2 \\ -5 & 4 & -2 \end{bmatrix}$, $D^T B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \\ -4 & -8 \end{bmatrix}$, $FE = \begin{bmatrix} -3 & 3 & 0 \\ -7 & 5 & 2 \\ -5 & 3 & 2 \\ -11 & 7 & 4 \end{bmatrix}$.

Q6. $A^2 = \begin{bmatrix} 16 & 9 \\ 15 & 19 \end{bmatrix}$, $C^k = 3^{k-1} C$ for $k \geq 1$.

Q7. $diag(A) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $diag(B) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $diag(C^3) = \begin{bmatrix} 18 & 0 \\ 0 & 9 \end{bmatrix}$.

Q8. $A^T A = \begin{bmatrix} 26 & 13 \\ 13 & 13 \end{bmatrix}$, $AA^T = \begin{bmatrix} 10 & 11 \\ 11 & 29 \end{bmatrix}$, $FF^T = \begin{bmatrix} 6 & 4 & 0 & -2 \\ 4 & 10 & 6 & 8 \\ 0 & 6 & 5 & -6 \\ -2 & -8 & -6 & 42 \end{bmatrix}$,

$FF^T = \begin{bmatrix} 15 & -4 & -9 \\ -4 & 21 & 18 \\ -9 & 18 & 27 \end{bmatrix}$.

Q9. Let $P = [p_{ij}]$ and $Q = [q_{ij}]$. Then $P^T = [p_{ji}]$ and so $(P^T)^T = [p_{ji}]^T = [p_{ij}] = P$. If $P + Q$ is defined, then

$$(P + Q)^T = [p_{ij} + q_{ij}]^T = [p_{ji} + q_{ji}] = [p_{ji}] + [q_{ji}] = P^T + Q^T.$$

Q10.

(a) True. Let A be of size $m \times n$. If A^2 is defined, then the number of columns of A must equal the numbers of rows of A, i.e., $n = m$. Thus, A is square.

(b) False. Counterexample: let A be of size 1×2 and B be of size 2×1. Then both AB and BA are defined.

(c) True. Let $C = AB$. Then $c_{i1} = \sum a_{ik}b_{k1} = \sum a_{ik}b_{k2}$ because $b_{k1} = b_{k2}$ for all k.

(d) False. Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$.

Q11. If A is skew-symmetric, then $A^T = -A$. In other words, $a_{ij} = -a_{ji}$ for all i, j. In particular, $a_{ii} = -a_{ii}$, which is only possible if $a_{ii} = 0$ for all i.

Q12. Let A and B be such that $A^T = -A$ and $B^T = -B$. Then

$$(A + B)^T = A^T + B^T = -A - B = -(A + B).$$

Q13.

(a) $A^2 - B^2 \neq (A - B)^2$. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -B$.

(b) $(B - A)^2 = (-(A - B))^2 = (-1)^2(A - B)^2 = (A - B)^2$.

(c) $A(A - B) - B(A - B) = (A - B)(A - B) = (A - B)^2$.

(d) $A^2 - 2AB - B^2 \neq (A - B)^2$. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -B$.

(e) $A^2 - AB - BA + B^2 = A(A - B) - B(A - B) = (A - B)^2$.

Q14. (a) $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \\ 1/6 & 1/7 & 1/8 & 1/9 & 1/10 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 0 \\ 16 & 8 & 4 & 2 & 1 \end{bmatrix}$.

Q15. Let $A = [a_{ij}]$ and $I_m = [\varepsilon_{ij}]$, where $\varepsilon_{ij} = 1$ if $i = j$ and $\varepsilon_{ij} = 0$ otherwise. Then $I_m A = [x_{ij}]$ where

$$x_{ij} = \sum \varepsilon_{ik} a_{kj} = 1 \times a_{ij} = a_{ij}.$$

Thus, $I_m A = [a_{ij}] = A$. Similarly, $AI_n = [y_{ij}]$ where

$$y_{ij} = \sum a_{ik} \varepsilon_{kj} = a_{ij} \times 1 = a_{ij}$$

and so $AI_n = A$.

Q16.

(a) $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(b) $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

(c) $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(d) $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(e) $\mathbf{p} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(f) $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}, \mathbf{y} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

Q17. If $A = \mathbf{u}\mathbf{v}^T$, then

$$A^2 = \mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T = \mathbf{u}(\mathbf{v} \cdot \mathbf{u})\mathbf{v}^T = (\mathbf{v} \cdot \mathbf{u})\mathbf{u}\mathbf{v}^T = (\mathbf{v} \cdot \mathbf{u})A.$$

Let $k = 1$. Then $A^1 = 1A = (\mathbf{v} \cdot \mathbf{u})^{1-1}A$. Assume that the result holds for $k = l$, i.e., that $A^l = (\mathbf{v} \cdot \mathbf{u})^{l-1}A$. Let $k = l + 1$. Then,

$$A^{l+1} = A^l A = (\mathbf{v} \cdot \mathbf{u})^{l-1}A^2 = (\mathbf{v} \cdot \mathbf{u})^{l-1}(\mathbf{v} \cdot \mathbf{u})A = (\mathbf{v} \cdot \mathbf{u})^l A.$$

Hence, by induction on k, $A^k = (\mathbf{v} \cdot \mathbf{u})^{k-1}A$ for all $k \in \mathbb{N}$.

If $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then $(\mathbf{u}\mathbf{v}^T)^k = 3^{k-1} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Q18. There are infinitely many such matrices. First note that

$$\begin{bmatrix} x-y \\ 2x+y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then if we choose B such that $B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and set $A = B + \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we have

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(B + \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ 2x+y \\ 0 \end{bmatrix}.$$

There are infinitely many choices of B. For example,

$$\alpha \begin{bmatrix} y & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

works for all $\alpha \in \mathbb{R}$.

Q19. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ where $a_{ij} = b_{ij} = 0$ if $i \neq j$ and $b_{ii} = 1/a_{ii}$. Then $AB = [c_{ij}]$ where

$$\begin{aligned} c_{ij} = \sum a_{ik} b_{kj} = a_{ii} b_{ij} &= \begin{cases} 0 & \text{if } i \neq j \\ a_{ii} b_{ii} & \text{if } i = j \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j \\ a_{ii} \frac{1}{a_{ii}} & \text{if } i = j \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}. \end{aligned}$$

It follows that $AB = I$.

Q20.

(a) $\dfrac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

(b) The inverse does not exist because the determinant of the matrix equals $1 \times 4 - 2 \times 2 = 0$.

(c) $\dfrac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\dfrac{1}{4} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

(e) $\dfrac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$