Properties of the Identity Matrix (Theorem 1.3.4)

The $n \times n$ identity matrix is defined to be the following diagonal matrix:

$$I_n = diag(\underbrace{1, 1, \ldots, 1}_{n \text{ times}}).$$

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem 1.3.4 For any $m \times n$ matrix A, $I_m A = A = I_n A$.

Proof. Let the entries of I_m be ε_{ij} so that $\varepsilon_{ij} = 0$ if $i \neq j$ and $\varepsilon_{ij} = 1$ if $i = j$. Let the entries of A be denoted by a_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$. Then, the $(i, j)^{th}$-entry of $I_m A$ equals

$$\sum_{k=1}^{m} \varepsilon_{ik} a_{kj}.$$

The only nonzero term in this sum is $\varepsilon_{ii} a_{ij}$, because $\varepsilon_{ij} = 0$ if $i \neq j$, and so

$$\sum_{k=1}^{m} \varepsilon_{ik} a_{kj} = \varepsilon_{ii} a_{ij} = a_{ij}.$$

It follows that the $(i, j)^{th}$-entry of $I_m A$ equals a_{ij}. In other words, $I_m A = A$.

Similarly, the $(i, j)^{th}$-entry of $A I_n$ is

$$\sum_{k=1}^{n} a_{ik} \varepsilon_{kj} = a_{ij} \varepsilon_{jj} = a_{ij},$$

and hence $I_n A = A$. $\square$