UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM101 Introduction to Calculus

Examples 4A

1 Let $f(x) = 4 + x^2$. Evaluate (i) $f(5)$ (ii) $f(x^2)$ (iii) $f(4 + x^2)$.

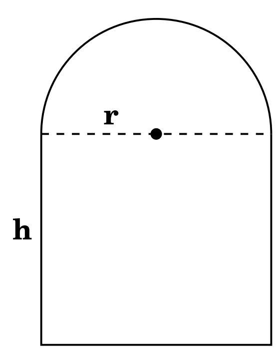
(i) $f(5) = 4 + 5^2 = 29$.
(ii) $f(x^2) = 4 + (x^2)^2 = 4 + x^4$.
(iii) $f(4 + x^2) = 4 + (4 + x^2)^2 = 4 + (16 + 8x^2 + x^4) = 20 + 8x^2 + x^4$.

2 Find the numbers which map to -2 under the function $h : t \mapsto t(t + 3)$.

The function can be written as $h(t) = t^2 + 3t$ and if t maps to -2 then $h(t) = -2$, that is,

$$t^2 + 3t = -2 \Leftrightarrow t^2 + 3t + 2 = 0 \Leftrightarrow (t + 1)(t + 2) = 0 \Leftrightarrow t = -1 \text{ or } t = -2.$$

3 An archway is in the form of a rectangle of height h surmounted by a semi-circle of radius r. Find the area as a function of r if the perimeter is 15 metres.



Working in metres we have

$$(i) \qquad 15 = 2h + 2r + \pi r$$

and

$$(ii) \qquad A = \frac{1}{2}\pi r^2 + 2rh.$$

Equation (i) gives

$$h = \frac{1}{2}(15 - 2r - \pi r)$$

so from (ii)

$$A = \frac{1}{2}\pi r^2 + r(15 - 2r - \pi r) = 15r - 2r^2 - \frac{1}{2}\pi r^2.$$

Examples 4B

1 Let $f(x) = 1 + x^2$, $A = \mathrm{Dom}(f) = [-1, 10]$ and $B = [-1000, 1000]$. What is the image of f?

The image of f is the set of all possible values which can be obtained by applying f to the values in $\mathrm{Dom}(f)$. Since $x^2 \geq 0$ we have $f(x) \geq 1$. The largest possible value of $f(x)$ is $1 + 10^2 = 101$ so

$$\mathrm{Im}(f) = [1, 101].$$

2 If $g(x) = \dfrac{1}{x-2}$, find the natural domain and image of g.

The function $g(x)$ is defined unless $x = 2$. The natural domain of g is therefore $(-\infty, 2) \cup (2, \infty)$. If $y \in \mathrm{Im}(g)$ then for some x we have

$$y = \frac{1}{x-2} \Rightarrow (x-2)y = 1 \Rightarrow x - 2 = \frac{1}{y} \Rightarrow x = \frac{1}{y} + 2$$

so that x is defined for all $y \neq 0$. It follows that $\mathrm{Im}(g) = (-\infty, 0) \cup (0, \infty)$.

3 Find the natural domains of the following functions:

(i) $f(x) = x - 3$, (ii) $f(x) = \dfrac{1}{x-3}$, (iii) $f(x) = \sqrt{x-3}$, (iv) $f(x) = \dfrac{1}{\sqrt{x-3}}$.

(i) $x \in \mathbb{R}$ (all values of x are OK).
(ii) $x \neq 3$ (as we cannot divide by zero).
(iii) We need a non-negative number under the sqare root sign, i.e. we need $x - 3 \geq 0$ which means $x \geq 3$.
(iv) We need a positive number under the sqare root sign (as we cant divide by 0), i.e. we need $x - 3 > 0$ which means $x > 3$.

4 State the natural domain and image of the functions r_1 and r_2 given by

$$r_1(x) = x + 1, \qquad r_2(x) = \frac{x^2 + 3x + 2}{x + 2} = \frac{(x+1)(x+2)}{x+2}.$$

Note that $r_1(x) = r_2(x)$ except when $x = -2$.

For r_1, the natural domain is $(-\infty, \infty)$. If $y \in \mathrm{Im}(r_1)$ then

$$y = x + 1 \Rightarrow x = y - 1$$

so that x is defined for all y. Hence the image of r_1 is $(-\infty, \infty)$.

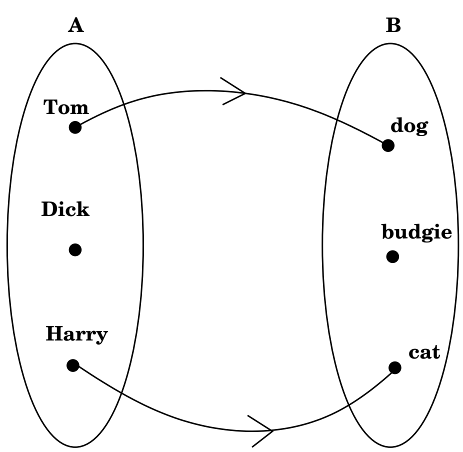
For r_2, $r_2(-2)$ is not defined so its natural domain is $(-\infty, -2) \cup (-2, \infty)$. We might therefore also expect its image to exclude $y = r_1(-2) = -1$. Check:

$$r_2(x) = \frac{(x+1)(x+2)}{x+2} = -1 \Rightarrow (x+1)(x+2) = -(x+2) \Rightarrow (x+2)^2 = 0 \Rightarrow x = -2$$

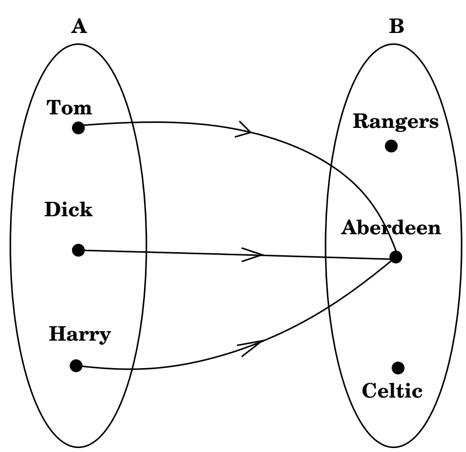
which is not possible so the image of r_2 is $(-\infty, -1) \cup (-1, \infty)$.

Examples 4C

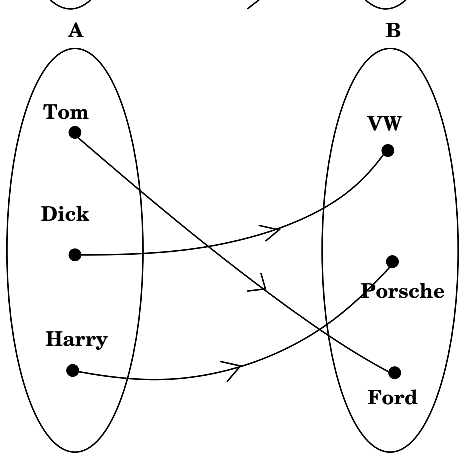
1 Do the following diagrams represents functions and, if so, are they injective, surjective or bijective?



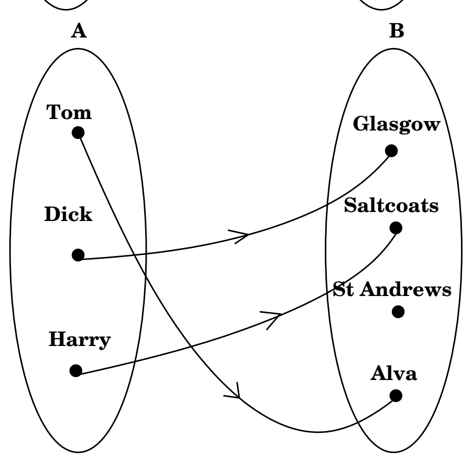
(i)



(ii)



(iii)

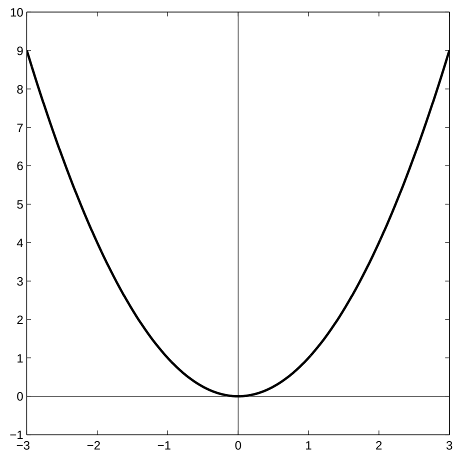


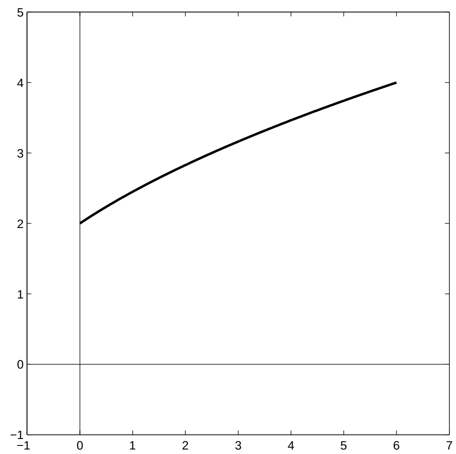
(iv)

(i) This is not a function (some elements of A are not mapped to anything).
(ii) This is a function but is not injective (more than one element of A is mapped to the same element of B). It is not surjective as the image is not the same as the codomain B.
(iii) This function is injective and surjective (hence bijective).
(iv) This function is injective but not surjective (B contains an element which is not the image of any point in A).

2 Plot the graphs of the following functions and state whether or not they are bijective.

$$\text{(i) } f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2, \qquad \text{(ii) } g(x) = \sqrt{2x+4},\ x \in [0,6].$$

(i) 

(ii) 

Function (i) is not injective, so is not bijective.
Function (ii) is injective, but it is not surjective (as we have to assume the codomain is $\mathbb{R}$).

3 Determine whether or not the following functions are bijective:

$$\text{(i) } f(x) = \frac{1}{(x-2)^2} \qquad \text{(ii) } g(x) = ax + b \text{ for } a, b \neq 0.$$

(i) It is easy to see that $f(0)$ and $f(4)$ both equal $1/4$, so this function is not bijective.
(ii)

$$g(x) = g(y) \Leftrightarrow ax + b = ay + b \Leftrightarrow ax = ay \Leftrightarrow x = y$$

so $g(x)$ is injective.
For any $y \in \mathbb{R}$,

$$y = g(x) \Leftrightarrow y = ax + b \Leftrightarrow x = \frac{1}{a}(y - b)$$

so $g(x)$ is also surjective. Hence $g(x)$ is bijective.

Examples 4D

1 Use the triangle inequality to prove that $|x-y| \leq |x|+|y|$ for all $x, y \in \mathbb{R}$.

Using the triangle inequality,

$$|x-y| = |x+(-y)| \leq |x| + |-y| = |x|+|y|$$

as required.

2 Give a counter-example to show that it is incorrect to write

$$|x-y| \leq |x|-|y| \text{ for all } x, y \in \mathbb{R}.$$

Let $x=0$. Then the statement reads $|-y| \leq -|y|$ which is obviously false for any value of $y \neq 0$ so the result does not hold.

3 Prove that $\big||x|-|y|\big| \leq |x-y|$ for all $x, y \in \mathbb{R}$.

We have

$$\begin{aligned}
\big||x|-|y|\big|^2 &= (|x|-|y|)^2 \\
&= |x|^2 - 2|x||y| + |y|^2 \\
&= x^2 - 2|x||y| + y^2 \\
&\leq x^2 - 2xy + y^2 \qquad \text{as } xy \leq |xy| \\
&= (x-y)^2 \\
&= |x-y|^2
\end{aligned}$$

so result follows on taking the square root of both sides.

Examples 4E

1 Determine whether the following expressions define functions that are odd or even or neither:

$$f(x) = 3x^4 - 5x^2 - 1, \qquad g(x) = 4x^3 - \frac{2}{x}, \qquad h(x) = x^2 - 2x.$$

$$f(-x) = 3(-x)^4 - 5(-x)^2 - 1 = f(x) \Rightarrow f(x) \text{ is an even function.}$$

$$g(-x) = 4(-x)^3 - \frac{2}{(-x)} = -4x^3 + \frac{2}{x} = -g(x) \Rightarrow g(x) \text{ is an odd function.}$$

$$h(-x) = (-x)^2 - 2(-x) = x^2 + 2x \Rightarrow h(x) \text{ is neither even nor odd.}$$

Examples 4F

1 If $f(x) = 2x - 3$ and $g(x) = 2 - x$, find $f + g$, $f - g$, fg and f/g and give the natural domain of each.

$(f + g)(x) = f(x) + g(x) = 2x - 3 + 2 - x = x - 1, \quad \mathrm{dom}(f + g) = \mathbb{R}.$

$(f - g)(x) = f(x) - g(x) = 2x - 3 - (2 - x) = 3x - 5, \quad \mathrm{dom}(f - g) = \mathbb{R}.$

$(fg)(x) = f(x)g(x) = (2x-3)(2-x) = 4x - 2x^2 - 6 + 3x = 7x - 2x^2 - 6, \quad \mathrm{dom}(fg) = \mathbb{R}.$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{(2x-3)}{(2-x)}, \quad \mathrm{dom}(f/g) = \{x \in \mathbb{R}, x \neq 2\}.$

2 Write $h(x) = x^2 - 2x$ as a sum of even and odd functions.

$$h(x) = h_e(x) + h_o(x)$$

where

$$h_e(x) = \frac{1}{2}(h(x) + h(-x)) = \frac{1}{2}(x^2 - 2x + (-x)^2 - 2(-x)) = \frac{1}{2}(2x^2) = x^2$$

and

$$h_o(x) = \frac{1}{2}(h(x) - h(-x)) = \frac{1}{2}(x^2 - 2x - [(-x)^2 - 2(-x)]) = \frac{1}{2}(-4x) = -2x.$$

Examples 4G

1 Suppose $f(x) = 2 + x^2$ and $g(x) = 3 + x$. Find $f(g(x))$ and $g(f(x))$.

$$\begin{aligned} f(g(x)) &= 2 + g(x)^2 = 2 + (3 + x)^2 = x^2 + 6x + 11; \\ g(f(x)) &= 3 + f(x) = 5 + x^2. \end{aligned}$$

Note that $f(g(x)) \neq g(f(x))$.

2 If $f(x) = x^2 - 1$ and $g(x) = \sqrt{3 - x}$, find $f(g(x))$ and $g(f(x))$ and state the natural domain in each case.

$f(g(x)) = f(\sqrt{3-x}) = (\sqrt{3-x})^2 - 1 = 3 - x - 1 = 2 - x$, $\text{dom}(f(g(x)) = (-\infty, 3]$ (as $3 - x \geq 0$ for the square root to make sense).

$g(f(x)) = g(x^2 - 1) = \sqrt{3 - (x^2 - 1)} = \sqrt{4 - x^2}$, $\text{dom}(g(f(x)) = [-2, 2]$ (as $4 - x^2 \geq 0$ for the square root to make sense).

3 If $f(x) = \sin x$, $g(x) = 1 - x^2$ and $h(x) = \sqrt{x}$, find $f(g(h(x)))$ and $h(g(f(x)))$ and state the domain of each.

We note that

$$\text{Dom}(f) = (-\infty, \infty) = \text{Dom}(g) \quad \text{and} \quad \text{Dom}(h) = [0, \infty).$$

We have

$$f(g(h(x))) = f[g(h(x))] = f(g(\sqrt{x})) = f(1 - (\sqrt{x})^2) = f(1 - x) = \sin(1 - x).$$

$$h(g(f(x))) = h[g(f(x))] = h(g(\sin x))) = h(1 - \sin^2 x) = \sqrt{1 - \sin^2 x} = |\cos x|.$$

Now $\text{Dom}(h) = [0, \infty)$ so $\text{Dom}(g(h)) = [0, \infty)$ and $\text{Dom}(f(g(h))) = [0, \infty)$. Also, since $|f(x)| \leq 1$ for all x, then $0 \leq g(f(x)) \leq 1$ and $h(g(f(x)))$ is defined for all x, that is, $\text{Dom}(h(g(f))) = (-\infty, \infty)$.

Examples 4H

1 If $g(x) = x^3, -\infty < x < \infty$, find $g^{-1}(x)$.

For each $y \in (-\infty, \infty)$ there is a unique x with $y = x^3$, that is, $x = y^{1/3} = g^{-1}(y)$.

2 If $h_1(x) = x^2$, find $h_1^{-1}(x)$ when the domain of h is

$$\text{(i) } x \in \mathbb{R}, \qquad \text{(ii) } x \in [0, \infty), \qquad \text{(iii) } x \in (-\infty, 0].$$

(i) For each $y \in [0, \infty)$ there are two values of x such that $x^2 = y$, namely $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$. So we cannot construct an inverse function as $h(x)$ is not a bijection.

(ii) For each $y \in [0, \infty)$ there is a unique $x \in [0, \infty)$ such that $y = h_1(x) = x^2$, namely $x = \sqrt{y}$. Hence $h_1^{-1}(y) = \sqrt{y}$.

(iii) For each $y \in [0, \infty)$ there is a unique $x \in (-\infty, 0]$ such that $y = h_2(x) = x^2$, namely $x = -\sqrt{y}$. Hence $h_2^{-1}(y) = -\sqrt{y}$.

3 If $\phi(x) = \dfrac{4x+1}{3x-2}$, $\quad x \in (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$, find $\phi^{-1}(x)$.

Put
$$y = \phi(x) = \frac{4x+1}{3x-2}.$$
Solving for x,
$$3xy - 2y = 4x + 1 \Rightarrow x(3y-4) = 1 + 2y$$
hence
$$x = \frac{1+2y}{3y-4} = \phi^{-1}(y), \quad y \in \left(-\infty, \frac{4}{3}\right) \cup \left(\frac{4}{3}, \infty\right).$$
So we have $\phi^{-1}(x) = \dfrac{1+2x}{3x-4}, x \in (-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$.

Examples 4I

1 What curve is represented by the parametric equations $\quad x = t+1, \qquad y = t^2 - 2t, \quad$ for $t \in \mathbb{R}$?

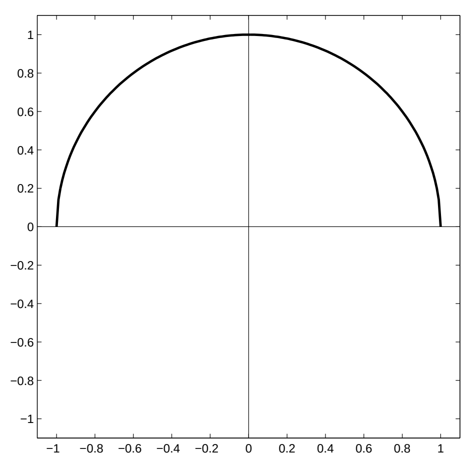
Eliminate the parameter: $t = x - 1$ so
$$y = t^2 - 2t = (x-1)^2 - 2(x-1) = x^2 - 4x + 3.$$
This is a parabola.

2 Plot the graph defined by the parametric equations $x(t) = \cos t$, $y(t) = \sin t$ for

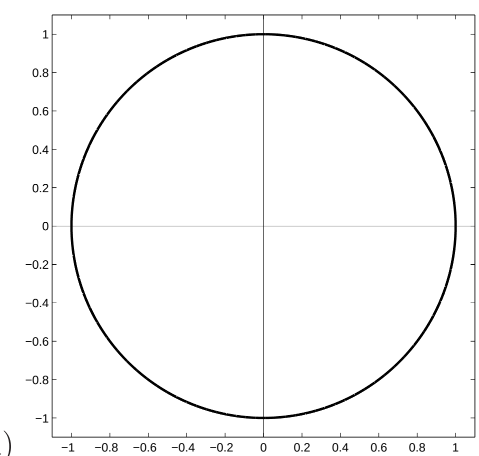
(i) $t \in [0, \pi]$, $\qquad$ (ii) $t \in [0, 2\pi]$.

For each real t we have $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, i.e. all points lie on a circle radius 1.

(i)



(ii)



Note that with domain (ii), these equations do not define a function!