10 Continuity

10.1 Sketch $y = u(x)$ on $[-1, 1]$ if the function u is determined by

$$u(x) = \begin{cases} 1 - x^2, & \text{if} \quad -1 \le x < 0; \\ x(1-x), & \text{if} \quad 0 \le x \le 1. \end{cases}$$

10.2 Sketch the "hat" function $y = h(x)$, defined on $[-a, a]$ (with $a > 0$) by

$$h(x) = \begin{cases} 1 + \dfrac{x}{a}, & \text{if} \quad -a \le x < 0; \\[2ex] 1 - \dfrac{x}{a}, & \text{if} \quad 0 \le x \le a. \end{cases}$$

10.3 Sketch each of the following functions over the domain given, and identify any jump discontinuities.

(a) $f_a(x) = \begin{cases} 1 - x, & \text{if} \quad x < 0; \\ 1 + x, & \text{if} \quad x \ge 0. \end{cases}$ Sketch for $x \in [-2, 2]$.

(b) $f_b(x) = \begin{cases} x - 1, & \text{if} \quad x < 0; \\ x + 1, & \text{if} \quad x \ge 0. \end{cases}$ Sketch for $x \in [-2, 2]$.

(c) $f_c(x) = \begin{cases} -1, & \text{if} \quad x < 0; \\ 0, & \text{if} \quad x = 0; \\ 1, & \text{if} \quad x > 0. \end{cases}$ Sketch for $x \in [-2, 2]$.

(d) $f_d(x) = \begin{cases} x(x+1), & \text{if} \quad -1 \le x < 0; \\ \sin(\pi x), & \text{if} \quad 0 \le x \le 1. \end{cases}$ Sketch for $x \in [-1, 1]$.

(e) $f_e(x) = \begin{cases} 0, & \text{if} \quad -2 > x; \\ 1 + x, & \text{if} \quad -2 \le x < -1; \\ |x|, & \text{if} \quad -1 \le x \le 1; \\ 1 - x, & \text{if} \quad 1 < x \le 2; \\ 0, & \text{if} \quad x > 2. \end{cases}$ Sketch for $x \in [-3, 3]$.

10.4 Identify the values of x at which the following expressions are undefined.

(a) $\dfrac{x^2 - 25}{x - 5}$ (b) $\dfrac{x - 5}{x^2 - 25}$ (c) $\dfrac{x^2 + 7x + 12}{x + 4}$ (d) $\dfrac{x - 8}{x^2 - 5x - 24}$

For which of these expressions is it possible to define a function whose value agrees with that of the expression everywhere that it is defined, and which is continuous on $\mathbb{R}$?

10.5 Find the coordinates at which the graphs of the following expressions have holes.

(a) $\dfrac{x^2+6x+8}{x+2}$ (b) $\dfrac{x^3+27}{x+3}$ (c) $\sqrt{\dfrac{x^2-4}{x-2}}$

10.6 Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x.
Show that f is continuous at 0. (Hint: show first that $f(0) = 0$.)

10.7 Suppose that f satisfies $f(x+y) = f(x) + f(y)$ for all x, y and that f is continuous at 0.
Show that $f(0) = 0$ and that f is continuous everywhere.
(Hint: use the form of the limit given in Exercise 9.7.)