UNIVERSITY OF STRATHCLYDE
MATHEMATICS & STATISTICS

MM103 Part II: Applications

4. Simply the best!

4.1. Optimization

Optimization is the name given to a class of mathematical problems involving finding the "best possible" value of something (e.g. the **lowest** cost, **biggest** profit, **least** toxic waste produced from a manufacturing process, etc.). As such it is a very important part of applied mathematics, and is used throughout industry and business to boost yields and profits.

Simple examples of optimization problems usually involve finding the global maximum or minimum values of a function, and you should have already seen examples of this in MM102 or MM112.

Example 4.1 Find the maximum and minimum values of the function $f(x) = 2\,x^3 - 3\,x^2 - 12\,x + 1$ for $x \in [-2, 4]$.

You should also have seen some more practical problems which involve more than one variable and a **constraint**. The aim with these is to use the constraint to rewrite the problem as an optimization problem in one variable. It is usually helpful to start with a sketch, and then write down mathematical equations for all the pieces of information in the problem.

Example 4.2 The aim is to design a circular cylindrical drink can of volume 375 ml. The metal used to make the top of the can costs twice as much per square centimetre as the metal used to make the sides and bottom. What height and radius should the can have to minimize the costs? (Note 1 ml $=$ 1 cm^3.)

In this example the constraint is an **equation:** $\pi r^2 h = 375$, but in many problems of practical interest the constraint is an **inequality**. For example

- Maximize the rate of a chemical reaction subject to the concentration of chemical A being non-negative.
- Maximize the profit of company X subject to the factory area being less than the land available.
- Minimize the cost of production of 1 tonne of steel, subject to the finished product having tensile strength greater than some prescribed value.

Linear programming is the name given to solving optimization problems in which the function to be optimized is a linear combination of variables, and it is subject to constraints which are all inequalities involving linear combinations of variables. Real life linear programming problems typically involve a very large number of variables, but in MM103 we shall just look at problems in two variables, x and y. Before this we shall investigate linear functions of **one** variable.

4.2. Linear functions of one variable

If $f(x)$ is a linear function, then its graph is a straight line and it must have the form $f(x) = a\,x + b$ for some constants a and b. The various possibilities are illustrated below.

Result - 1D:

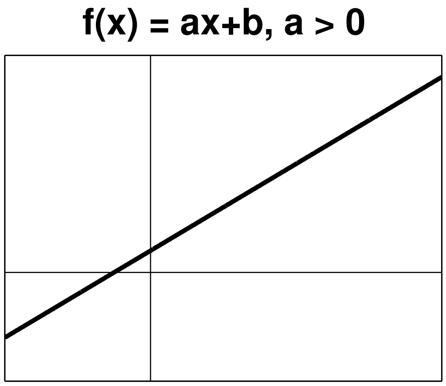
If $f(x)$ is a **linear function** defined for $x \in [x_0, x_1]$ then the maximum value of f occurs at one of the interval endpoints, i.e. it is $f(x_0)$ or $f(x_1)$. The minimum value of f occurs at the other endpoint.

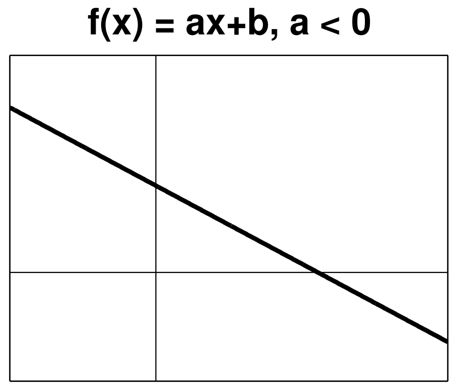
Examples 4.3

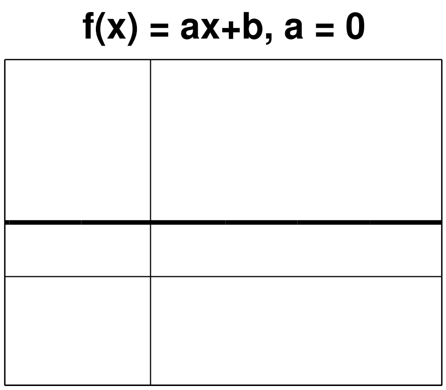
(a) Find the maximum and minimum values of $f(x) = 5\,x - 7$ for $x \in [-2, 4]$.

(b) Find the maximum and minimum values of $f(x) = 8 - 3\,x$ for $x \in [1, 5]$.

This simple 1D result is the key to linear programming in two (or more) variables.







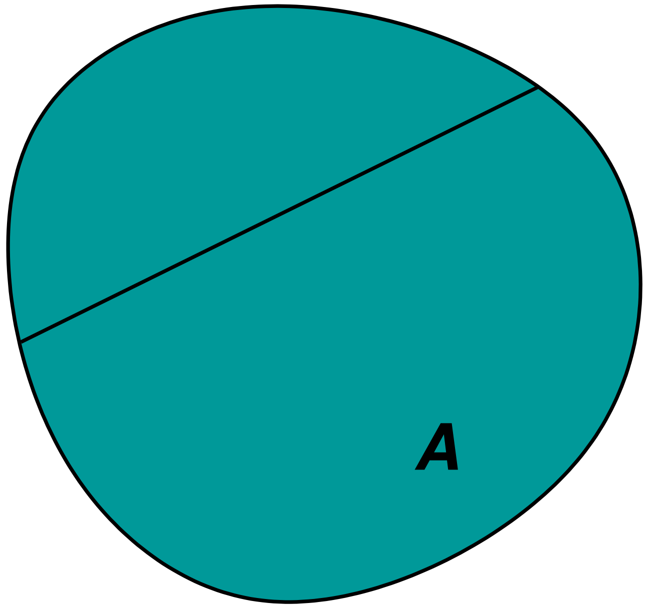
4.3. Linear functions of two variables

A linear function in the two variables x and y has the form $f(x, y) = a\,x + b\,y + c$ for some constants a, b, c.

Suppose that the linear function $f(x, y)$ is defined for $(x, y) \in A$ where A is a bounded region in $\mathbb{R}^2$, as shown below.

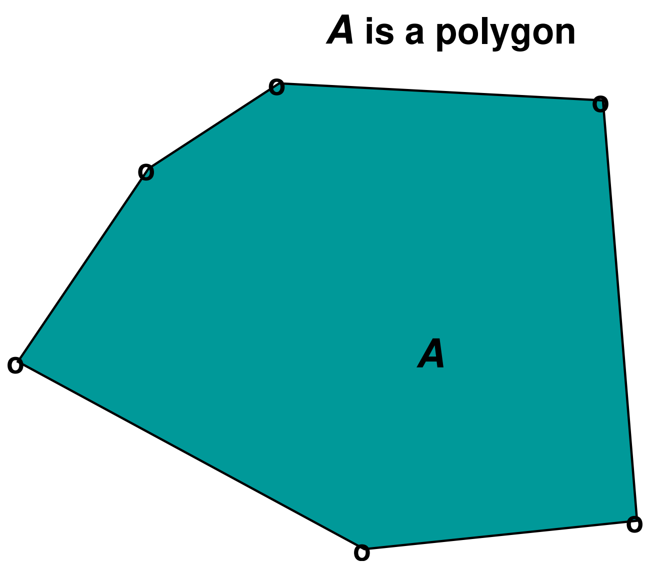
On any straight line through A, $f(x, y)$ is a linear function of one variable, and so by the 1D result, it will attain its maximum and minimum values at the ends of this straight line – i.e. on the boundary of A. This is true for any straight line through the region A, which means that the maximum and minimum values of $f(x, y)$ must occur on the **boundary** of A.

In theory this means that we can find the maximum and minimum values of the 2D linear function f on A just by checking the value of f at all boundary points of A, but in practice this may not be easy. One special case which is easy is when A is a polygon, and we state this as our 2D result.



Result - 2D:

If $f(x, y)$ is a **linear function** defined for $(x, y) \in A$ where A is a bounded **polygon** in $\mathbb{R}^2$, then the maximum value of $f(x, y)$ on A occurs at one of the corner points on the boundary of the region A. The minimum value of $f(x, y)$ on A also occurs at a corner point on the boundary of A.



This result is true because the boundary of A is now formed of straight lines, and the maximum and minimum values of f on a line occurs at the endpoints, i.e. at one of the corners on the boundary of A.

Examples 4.4

Let $f(x,y) = 5\,x - 3\,y + 12$.

(a) Find the maximum and minimum values of $f(x,y)$ on the square with corners at $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.

(b) Find the maximum and minimum values of $f(x,y)$ on the triangle with corners at $(-1,0)$, $(2,1)$ and $(1,5)$.

We now use this result to solve linear programming problems in two variables.

4.4. Linear programming in two variables

We are going to look at optimizing (i.e. finding the maximum or minimum values of) a **linear function** $f(x,y)$ which is subject to **linear constraints**. The function f is called the **objective function** and the constraints define the region $A \in \mathbb{R}^2$ over which we want to optimize f. It is called the **feasible region**, and because the constraints are all linear, it will be a polygon. It is easiest to illustrate the method by looking at some concrete examples.

Example: A company makes one product and sells x units of it per day in the UK and y units of it per day in Europe. The profit is £200 per unit sold in the UK and £100 per unit sold in Europe.

Suppose that the total production capacity is at most 100 units per day, and at most 60 units can be sold per day in the UK.

(a) Find the total sales profit f per day, in terms of x and y.

(b) Write down all the constraints on f and draw the feasible region.

(c) Find the values of x and y which maximise f, and calculate the maximum possible daily profit.

Solution

(a) The objective function is the total profit per day: $f(x,y) = 200\,x + 100\,y$.

(b) There are **four** constraints:

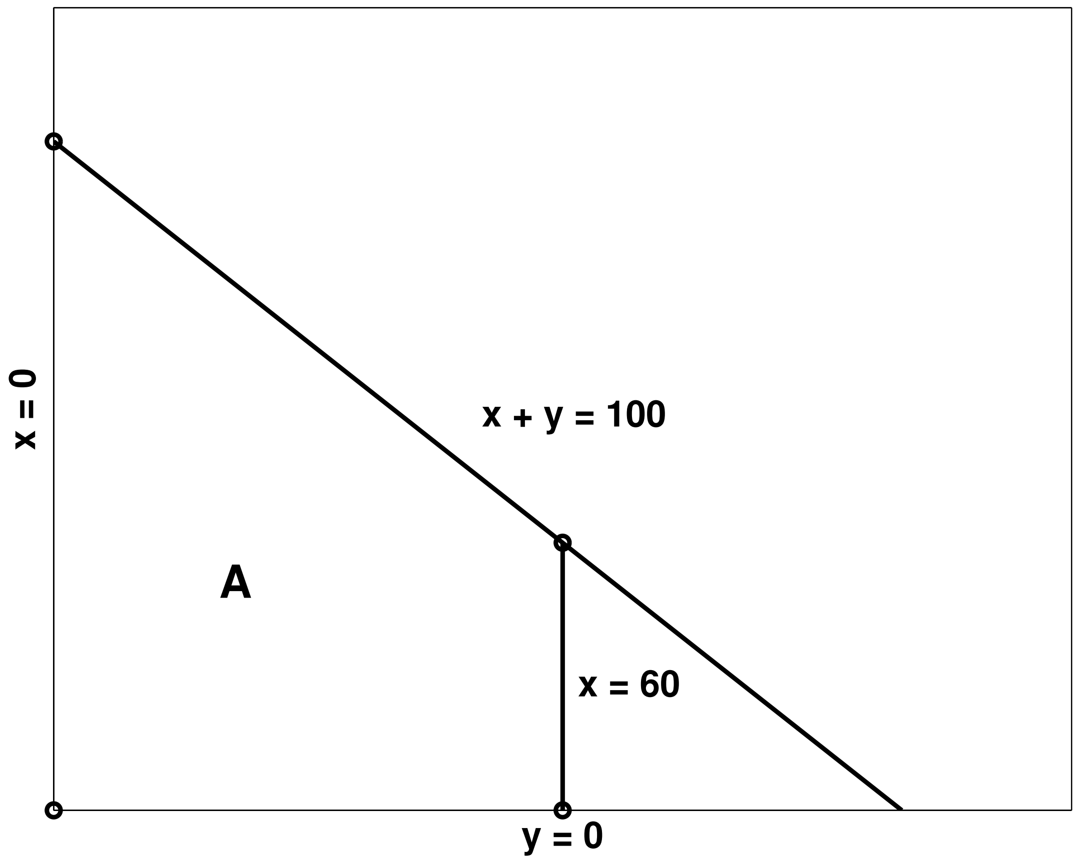
$x + y \leq 100$ (production is at most 100 units per day);

$x \leq 60$ (UK sales are at most 60 units per day);

$x \geq 0$ (you can't sell a negative number of units);

$y \geq 0$ (you can't sell a negative number of units).

These four constraints determine the feasible region, A, which is illustrated below. The four corners of A are at: $(0,0)$, $(60,0)$, $(60,40)$ and $(0,100)$.



(c) The maximum value of $f(x,y)$ must occur at one of these four corner values. We have $f(x,y) = 200\,x + 100\,y$ and so:

$f(0,0) = 0$

$f(60,0) = 12,000$

$f(60,40) = 16,000$

$f(0,100) = 10,000$

Hence the maximum daily sales profit is £16,000 and occurs when 60 units are sold per day in the UK and 40 units are sold per day in Europe.

In linear programming problems it is usual to use Z rather than f to denote the objective function, and we shall do this from now on.

Example 4.5 Consider the following linear programming problem:

$$\text{maximise } Z = x + y$$

subject to

$$\begin{aligned} 2x + y &\leq 8 \\ 2x + 3y &\leq 12 \\ y &\leq 3 \\ x,\ y &\geq 0\,. \end{aligned}$$

(a) Sketch the feasible region for the problem.

(b) Find the optimal solution to the problem.

Example 4.6 The Belgian chocolate linear programming problem is:

$$\text{maximise } Z = 55M + 89H$$

subject to

$$\begin{aligned} 4M + 18H &\leq 1296 \\ 12M + 6H &\leq 1824 \\ M,\ H &\geq 0\,. \end{aligned}$$

Sketch the feasible region and find the optimal solution to the problem.

Exercises 4

1. If $x + y = 2$, find x and y that minimize: (a) $x^2 + y^2$, (b) $x^2\,y^2$.

2. Find a point on the curve $y^2 - x^2 = 1$ closest to the point $(1, 0)$.

3. A wall $\sqrt{2}$ m high is 0.5 m from the vertical wall of a house. Find the length of the shortest ladder that will reach over the wall to touch the house.

4. Consider the following linear programming problem:

$$\text{maximise } Z = 5x + y$$

subject to

$$\begin{aligned} x + y &\leq 3 \\ y &\leq 1 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

5. Consider the following linear programming problem:

$$\text{maximise } Z = x + 2y$$

subject to

$$\begin{aligned} x + y &\leq 5 \\ x - y &\leq 2 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

6. Consider the following linear programming problem:

$$\text{maximise } Z = x - y$$

subject to

$$\begin{aligned} x + y &\leq 10 \\ 2x - y &\leq 15 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

7. Consider the following linear programming problem:

$$\text{maximise } Z = x - y$$

subject to

$$\begin{aligned} x + y &\leq 10 \\ 2x - y &\geq 15 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

8. Consider the following linear programming problem:

$$\text{maximise } Z = 45x + 80y$$

subject to

$$\begin{aligned} x + 4y &\leq 80 \\ 2x + 3y &\leq 90 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

9. Consider the following linear programming problem:

$$\text{maximise } Z = 2x + 5y$$

subject to

$$\begin{aligned} x + 2y &\leq 7 \\ 2x + y &\leq 12 \\ y &\leq 2 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

10. Consider the following linear programming problem:

$$\text{minimise } Z = y$$

subject to

$$\begin{aligned} 5x + y &\geq 12 \\ -5x + y &\geq 4 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

11. Iron ore is bought from two different sources X and Y, which contain different amounts of metal elements. Ore X has 2 Kg of manganese, 2 Kg of silicon and 3 Kg of copper per tonne, and costs £40 per tonne. Ore Y has 1 Kg of manganese, 4 Kg of silicon and 7 Kg of copper per tonne, and costs £50 per tonne.

    An amount of steel alloy is to be produced from a mixture of these two ores, and it is necessary that the mixture should contain at least 1 Kg of manganese and 3 Kg of silicon, with no more than 6 Kg of copper. The aim is to satisfy these requirements as cheaply as possible, and use at most 1 tonne of ore X.

(a) Suppose that x tonnes of ore X and y tonnes of ore Y are blended together. Write down the objective function (cost) in terms of x and y, and also list all the constraints on x and y.

(b) Sketch the feasible region for the problem and find the optimal solution.

12. A diet of bread and milk is to be prescribed containing an intake per meal of at least 6 units of vitamin A and 10 units of vitamin B. The meal must contain at least one glass of milk. Find the most economical diet possible (i.e. minimise its cost) given the following information:

	bread/slice	milk/glass
Vitamin A	1 unit	2 units
Vitamin B	2 units	3 units
Cost	5p	12p