6 Applications of Differentiation

6.1 Small Variations and Error Analysis

In this section we want to study how much the value of $f(x)$ changes if x changes slightly. This can be used to estimate the error of one quantity if the error of another quantity is known.

Recall the definition of the derivative of a function f at a point x_0:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} .$$

This means that if h is small, then

$$\frac{f(x_0 + h) - f(x_0)}{h} \approx f'(x_0).$$

If we multiply both sides by h and set

$$\Delta x := h, \qquad x := x_0 + \Delta x,$$

then we obtain

$$f(x) - f(x_0) \approx f'(x_0)\Delta x.$$

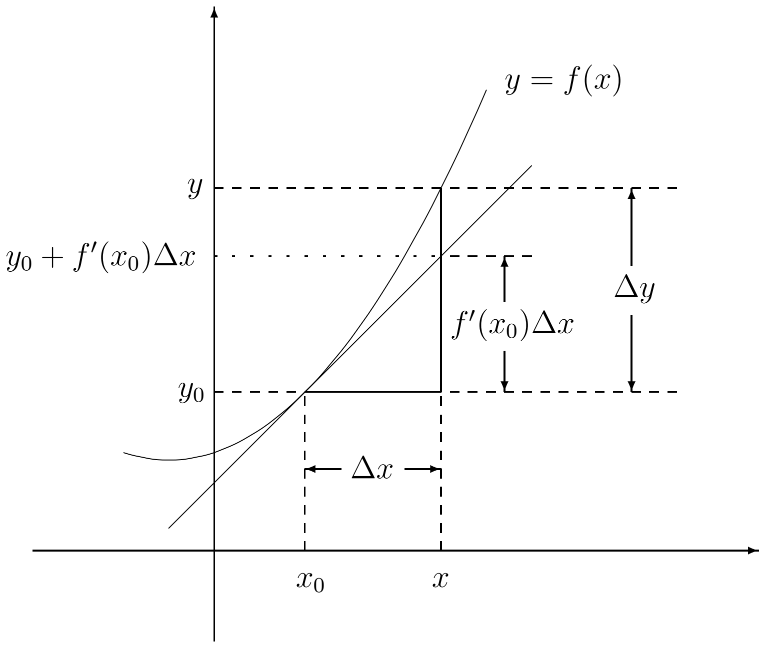
Let us further set

$$y := f(x), \qquad y_0 := f(x_0),$$
$$\Delta y := y - y_0 = f(x) - f(x_0),$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_0} = f'(x_0).$$

Then

$$\boxed{\Delta y \approx \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_0} \Delta x \qquad \text{if } \Delta x \text{ is small.}} \tag{6.1}$$

This formula can be used to estimate the error of $f(x)$ if x is known only with a certain error. The numbers Δx and Δy are the increments of x and y, respectively.



From the picture one can see that the estimate for Δy is obtained by the use of the tangent instead of the curve $y = f(x)$.

Example 6.1.
Let $y = x^{1/4}$. Find the approximate change in y when

(a) x is increased from 16 to 17;

(b) x is decreased from 16 to 14.

Solution in video

Example 6.2.
The pressure p of a gas is related to its volume V by the formula

$$p = \frac{1000}{V}.$$

If V is measured as 4.56 with a maximum error of ± 0.02, estimate the maximum error in p.
Set $V_0 = 4.56$. We know that the error in V is at most 0.02, i.e.

$$|\Delta V| \leq 0.02.$$

The derivative of p with respect to V is

$$\frac{\mathrm{d}p}{\mathrm{d}V} = -\frac{1000}{V^2}.$$

With

$$\left.\frac{\mathrm{d}p}{\mathrm{d}V}\right|_{V_0} = -\frac{1000}{4.56^2} = -48.092$$

we obtain

$$\Delta p \approx \left.\frac{\mathrm{d}p}{\mathrm{d}V}\right|_{V_0} \Delta V = -48.092 \times \Delta V$$

and hence

$$|\Delta p| \approx \left|-48.092 \times \Delta V\right| = 48.092 \times |\Delta V| \leq 48.092 \times 0.02 = 0.962.$$

Often not the **absolute change** but a **relative change** is known. For a quantity x we define the relative change to be

$$\frac{\Delta x}{x_0}$$

if $\Delta x = x - x_0$. The **percentage change** is

$$100 \times \frac{\Delta x}{x_0}.$$

Examples 6.3.

(i) Find the approximate percentage change in the volume of a sphere when the radius is increased by 2%.

Solution in video

(ii) What is the approximate maximal percentage change of the radius of a disc if the area of the disc is decreased by at most 3%?

Solution in video

6.2 Maxima and Minima

In this section we study local and global maxima and minima and their connection with stationary points. Let us start with the definitions of a global maximum and a global minimum. These are the largest and smallest values a function attains.

Definition 6.1. Let f be a function with domain $\mathrm{dom}(f)$.
We say that f has a **global maximum** at a point $x_0 \in \mathrm{dom}(f)$ if

$$f(x_0) \geq f(x) \qquad \text{for all } x \in \mathrm{dom}(f);$$

f has a **global minimum** at a point $x_0 \in \mathrm{dom}\, f$ if

$$f(x_0) \leq f(x) \qquad \text{for all } x \in \mathrm{dom}(f).$$

Example 6.4.
Consider the functions

$$f(x) = x^2, \qquad \mathrm{dom}(f) = (-1,1);$$
$$g(x) = x^2, \qquad \mathrm{dom}(g) = [-1,1].$$

Do the functions f and g have global maxima and/or minima?

Solution in video

There is a situation where we can guarantee that a function f has a global maximum and a global minimum.

Theorem 6.2. *If f is a continuous function defined on a closed, bounded interval $[a,b]$ with $a, b \in \mathbb{R}$, $a < b$, then f has a global maximum and a global minimum on $[a,b]$.*

An interval is called **bounded** if both end-points are numbers in $\mathbb{R}$, i.e. not $+\infty$ or $-\infty$. For the proof of the theorem see, e.g. [Spivak, Theorem 7-3].

The assumption that the domain is a *closed* interval is essential as the example

$$f(x) = \frac{1}{x}, \qquad \mathrm{dom}(f) = (0,1),$$

shows. This function has neither a maximum nor a minimum.

To find maxima and minima of a function, it is often easier to start with local maxima and minima since we can use differentiation to find them. Here we compare the value of the function at a point x only with the values of the function in a neighbourhood of x.

Definition 6.3. Let f be a function with domain $\mathrm{dom}(f)$.
We say that f has a **local maximum** at a point $x_0 \in \mathrm{dom}(f)$ if there exists a $\delta > 0$ so that

$$f(x_0) \geq f(x) \qquad \text{for all } x \in (x_0 - \delta, x_0 + \delta) \cap \mathrm{dom}(f);$$

f has a **local minimum** at a point $x_0 \in \mathrm{dom}(f)$ if there exists a $\delta > 0$ so that

$$f(x_0) \leq f(x) \qquad \text{for all } x \in (x_0 - \delta, x_0 + \delta) \cap \mathrm{dom}(f).$$

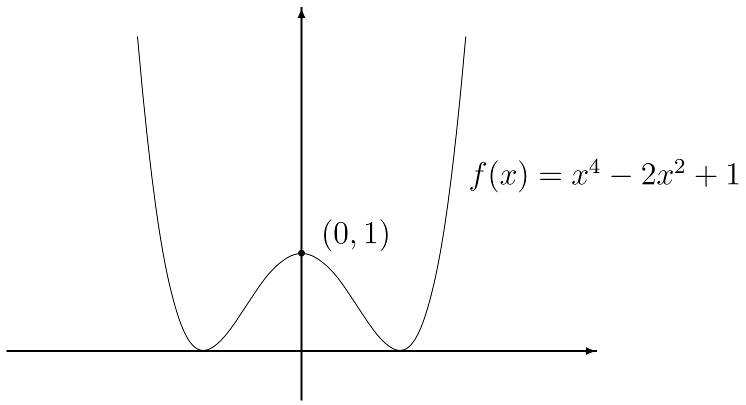
This means that there is a local maximum at x_0 if for all x that are close to x_0, we have $f(x_0) \geq f(x)$.

Every global maximum is of course also a local maximum. Hence, if we know that a function has a global maximum, it is sufficient to find all local maxima and compare the values at these local maxima.

The following example shows that it can happen that a function has a local maximum but no global maximum:

$$f(x) = x^4 - 2x^2 + 1, \qquad \mathrm{dom}(f) = \mathbb{R}.$$

It has a local maximum at $x = 0$ with value $f(1) = 1$. However, $f(x) \to \infty$ as $x \to \pm\infty$, so f cannot have a global maximum.



In the next theorem we see that in many cases the derivative of f must be 0 at a local maximum or minimum.

Theorem 6.4. *Let f be a function defined on* $\mathrm{dom}(f)$ *and $x_0 \in \mathrm{dom}(f)$ a point such that f is differentiable at x_0 and $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \mathrm{dom}(f)$ for some $\varepsilon > 0$. If f has a local maximum or a local minimum at x, then*

$$f'(x_0) = 0.$$

(The condition $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \mathrm{dom}(f)$ means basically that x_0 is not an end-point of the domain.)

Proof. We consider only the case when f has a local maximum at x_0. The case when f has a local minimum is analogous.

Since f has a local maximum at x_0, there exists a $\delta > 0$ such that the condition in the definition of a maximum is satisfied. We can choose $\delta \leq \varepsilon$. Then

$$f(x_0) \geq f(x) \qquad \text{for all } x \in (x_0 - \delta, x_0 + \delta).$$

If $0 < h < \delta$, then

$$f(x_0) \geq f(x_0 + h)$$

and hence

$$f(x_0 + h) - f(x_0) \leq 0$$

and since $h > 0$,

$$\frac{f(x_0+h)-f(x_0)}{h} \le 0.$$

Since f is differentiable at x_0, the limit of the difference quotient exists. By the previous inequality we must have

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \le 0. \tag{$*$}$$

Now choose $-\delta < h < 0$. Since f has a local maximum at x_0 we have

$$\begin{aligned}
&f(x_0) \ge f(x_0+h) \\
\implies \quad & f(x_0+h) - f(x_0) \le 0 \\
\implies \quad & \frac{f(x_0+h)-f(x_0)}{h} \ge 0 \qquad (h \text{ is negative!}) \\
\implies \quad & f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x)}{h} \ge 0 \qquad (**)
\end{aligned}$$

Inequalities $(*)$ and $(**)$ together show that $f'(x_0) = 0$. □

Definition 6.5. A point $x_0 \in \mathrm{dom}(f)$ is called **stationary point** of f if $f'(x_0) = 0$.

Theorem 6.4 says that that if f has a local maximum or minimum at a point x which is not an end-point of the domain and where f is differentiable, then x is a stationary point.

At a stationary point there need not be a local maximum or minimum as the example $f(x) = x^3$ shows. For this function, $f'(x) = 3x^2$ and hence $f'(0) = 0$, but $f(x) > f(0)$ for $x > 0$ and $f(x) < f(0)$ for $x < 0$.

We shall discuss the question when a stationary point is a maximum or minimum later.

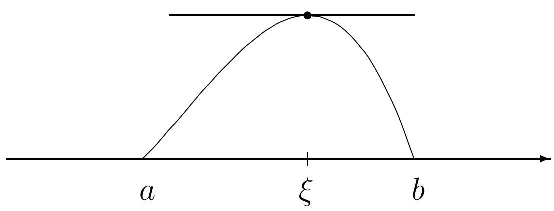
6.3 The Mean Value Theorem

In this section we study a relation between the difference quotient and the derivative, which will be useful later on.

Theorem 6.6. (Rolle's Theorem) *Let f be a differentiable function defined on the closed, bounded interval* $\mathrm{dom}(f) = [a, b]$. *If $f(a) = 0$ and $f(b) = 0$, then there exists a number $\xi \in (a, b)$ such that*

$$f'(\xi) = 0.$$

Proof. If $f(x) = 0$ for all $x \in [a, b]$, then $f'(x) = 0$ for all $x \in [a, b]$, and for ξ we can take any number in (a, b).
Assume now that $f(x_0) \neq 0$ for some $x_0 \in [a, b]$. We consider the case when $f(x_0) > 0$; the case when $f(x_0) < 0$ is analogous. Since f is a continuous function on the closed, bounded interval $[a, b]$, f has a global maximum at a point ξ (see Theorem 6.2). The global maximum cannot occur at an end-point a or b since $f(a) = 0$, $f(b) = 0$ and $f(x_0) > 0$. Hence $\xi \in (a, b)$. Since f is differentiable at ξ, we have $f'(\xi) = 0$ by Theorem 6.4.

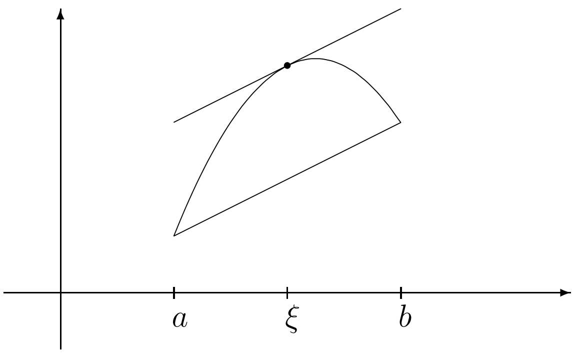


□

Theorem 6.7. (Mean Value Theorem) *If f is a differentiable function defined on the closed, bounded interval* $\mathrm{dom}(f) = [a, b]$, *then there exists a number $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

The theorem tells us that there exists a point strictly between a and b where the tangent is parallel to the line connecting the two points $(a, f(a))$ and $(b, f(b))$.



Proof. We apply Rolle's Theorem to the following function

$$g(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

At the end-points, g vanishes:

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0,$$

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - \Big(f(b) - f(a)\Big) = 0.$$

According to Rolle's Theorem there exists a $\xi \in (a,b)$ so that $g'(\xi) = 0$.
Let us calculate g':

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}\,.$$

Hence

$$0 = g'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a}\,,$$

which implies

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}\,.$$

This finishes the proof. □

We shall use the Mean Value Theorem later to study the behaviour of functions like monotonicity.

Example 6.5.
Use the Mean Value Theorem to show that

$$|\sin x| \leq x \qquad \text{for } \; x > 0.$$

Solution in video

Example 6.6.
Use the Mean Value Theorem to show the following statement.
If f is a differentiable function defined on the interval (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

Solution in video

6.4 Taylor Polynomials

In Section 6.1 we obtained the approximation

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0)$$

if $\Delta x = x - x_0$ is small. We can also write this as

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

if x is close to x_0. In a neighbourhood of x_0, the function f is approximated by the right-hand side,

$$p_{1,x_0}(x) := f(x_0) + f'(x_0)(x - x_0),$$

which is a linear function or a polynomial of degree at most 1. We see that f and p_{1,x_0} have the same value at x_0:

$$p_{1,x_0}(x_0) = f(x_0) + f'(x_0)(x_0 - x_0) = f(x_0).$$

The first derivative of p_{1,x_0} is

$$p'_{1,x_0}(x) = f'(x_0) \qquad \text{for all } x \in \mathbb{R}.$$

In particular, the first derivatives of f and p_{1,x_0} coincide at x_0:

$$p'_{1,x_0}(x_0) = f'(x_0).$$

Since $p''_{1,x_0}(x) = 0$, the second derivatives of f and p_1 will not coincide at x_0 in general.

To obtain better approximations, we can try to use polynomials of higher degree. For simplicity let us first assume that $x_0 = 0$. Let us start with a quadratic polynomial

$$p_{2,0}(x) = c_0 + c_1 x + c_2 x^2.$$

We want to determine the coefficients c_0, c_1, c_2 such that

$$p_{2,0}(0) = f(0), \qquad p'_{2,0}(0) = f'(0), \qquad p''_{2,0}(0) = f''(0). \tag{6.2}$$

The first two derivatives of p_2 are

$$\begin{aligned} p'_{2,0}(x) &= c_1 + 2c_2 x, \\ p''_{2,0}(x) &= 2c_2, \end{aligned}$$

and the equalities in (6.2) yield

$$\begin{aligned} c_0 &= f(0), \\ c_1 &= f'(0), \\ 2c_2 &= f''(0). \end{aligned}$$

From this we can easily determine the coefficients c_j, and $p_{2,0}$ is given by

$$p_{2,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

If we try a polynomial of degree 3:

$$p_{3,0}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

and want that

$$p_{3,0}(0) = f(0), \qquad p'_{3,0}(0) = f'(0), \qquad p''_{3,0}(0) = f''(0), \qquad p'''_{3,0}(0) = f'''(0),$$

then we obtain

$$c_0 = f(0), \qquad c_1 = f'(0), \qquad c_2 = \frac{f''(0)}{2}, \qquad c_3 = \frac{f'''(0)}{2 \times 3} = \frac{f'''(0)}{3!}.$$

Hence

$$p_{3,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

This suggests that we set

$$\boxed{p_{n,0}(x) := f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n.} \tag{6.3}$$

The polynomial $p_{n,0}$ in (6.3) is called **Taylor polynomial of degree n about $x = 0$**.

With induction one can show the following theorem.

Theorem 6.8. *If f is a function that is n times differentiable about 0 and $p_{n,0}$ is the Taylor polynomial of degree n about $x = 0$, then the values and the first n derivatives of $p_{n,0}$ and f coincide at 0:*

$$p_{n,0}(0) = f(0), \qquad p'_{n,0}(0) = f'(0), \qquad \ldots \qquad p^{(n)}_{n,0}(0) = f^{(n)}(0).$$

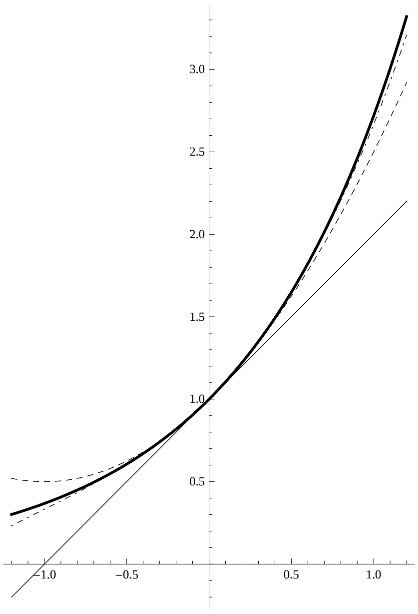
Let us determine the Taylor polynomials for some functions.

Example 6.7.
Find $p_{3,0}$, the Taylor polynomial of degree 3 about $x = 0$, for

$$f(x) = \ln(1 + x).$$

Solution in video

In general the Taylor polynomials approximate the function f better and better if n increases. See the graphs of the first three Taylor polynomials of the function $f(x) = e^x$. The thick line is the graph of f, the thin line the graph of $p_{1,0}$, the dashed line the graph of $p_{2,0}$ and the dot and dash line is the graph of $p_{3,0}$.



We shall come to the estimation of the error a bit later.

Example 6.8.
Find $p_{5,0}$, the Taylor polynomial of degree 5 about $x = 0$, for

$$f(x) = \sin x.$$

Solution in video

Now let us consider an arbitrary point a instead of 0 about which we want to determine the Taylor polynomial. The following analogue of Theorem 6.8 answers this question. We have to replace $f^{(k)}(0)$ by $f^{(k)}(a)$ and x^k by $(x-a)^k$.

Theorem 6.9. *Let f be a function that is differentiable n times at a. The polynomial*

$$\boxed{p_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

has the property that the values and the first n derivatives of $p_{n,a}$ and f coincide at a, i.e.

$$p_{n,a}(a) = f(a), \qquad p'_{n,a}(a) = f'(a), \qquad \ldots \qquad p^{(n)}_{n,a}(a) = f^{(n)}(a).$$

Definition 6.10. The polynomial $p_{n,a}$ in Theorem 6.9 is called the **Taylor polynomial of degree n about $x = a$**.

Example 6.9.
Find $p_{4,\pi}$, the Taylor polynomial of degree 4 about $x = \pi$, for

$$f(x) = \cos x.$$

The first four derivatives of f and their values at $x = \pi$ are

$$\begin{aligned}
f(x) &= \cos x &&\implies & f(\pi) &= -1 \\
f'(x) &= -\sin x &&\implies & f'(\pi) &= 0 \\
f''(x) &= -\cos x &&\implies & f''(\pi) &= 1 \\
f'''(x) &= \sin x &&\implies & f'''(\pi) &= 0 \\
f^{(4)}(x) &= \cos x &&\implies & f^{(4)}(\pi) &= -1.
\end{aligned}$$

Hence

$$\begin{aligned}
p_{4,\pi}(x) &= f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f^{(3)}(\pi)}{3!}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x-\pi)^4 \\
&= -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4.
\end{aligned}$$

6.5 The Remainder Term for Taylor Polynomials

In this section we want to estimate the error we make when we approximate a function f by its Taylor polynomial of degree n about a point a. The next theorem will be proved in the class MM203.

Theorem 6.11. (Taylor's Theorem)
Let f be $n+1$ times continuously differentiable on an interval I. Further, let $a \in I$ and $p_{n,a}$ bet the Taylor polynomial of degree n about $x = a$. Then for $x \in I$,

$$f(x) = p_{n,a}(x) + R_{n,a}(x), \tag{6.4}$$

where $R_{n,a}$ is the (Cauchy form of the) remainder:

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \tag{6.5}$$

for some ξ between a and x, i.e.

$$\begin{aligned}
&\xi \in (a, x) \quad \text{if } a < x \\
&\xi \in (x, a) \quad \text{if } x < a.
\end{aligned}$$

Remark. The number ξ depends on x.

Example 6.10.
Determine the remainder term for Example 6.8 ($p_{5,0}$ for $f(x) = \sin x$ about $x = 0$) and estimate its modulus from above.

Solution in video

Example 6.11.

(a) Find the remainder term $R_{3,0}(x)$ for the Taylor polynomial $p_{3,0}$ of degree 3 about $x = 0$ for the function
$$f(x) = \ln(1 + x).$$

(b) Use $p_{3,0}$ to obtain an approximate value for $f(0.1) = \ln 1.1$.

(c) Find an upper bound for the modulus of $R_{3,0}(0.1)$ that does not depend on ξ.

(a) We already found that $f'(x) = \dfrac{1}{1+x}$, $f''(x) = -\dfrac{1}{(1+x)^2}$, $f'''(x) = \dfrac{2}{(1+x)^3}$ and that the Taylor polynomial is equal to

$$p_{3,0}(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3.$$

The fourth derivative of f is

$$f^{(4)}(x) = -\frac{6}{(1+x)^4}.$$

Hence the remainder term $R_{3,0}(x)$ is

$$R_{3,0}(x) = \frac{f^{(4)}(\xi)}{4!}x^4 = \frac{1}{24}\left(-\frac{6}{(1+\xi)^4}\right)x^4 = -\frac{1}{4(1+\xi)^4}x^4$$

where ξ is between 0 and x.
(b) An approximate value for $f(0.1) = \ln 1.1$ is

$$f(0.1) \approx p_3(0.1) = 0.1 - \frac{1}{2} \times 0.1^2 + \frac{1}{3} \times 0.1^3 = 0.095\,333.$$

(c) The error we make is given by the modulus of the remainder term

$$\begin{aligned} \left|f(0.1) - p_{3,0}(0.1)\right| = \left|R_{3,0}(0.1)\right| &= \left|-\frac{1}{4(1+\xi)^4} \times 0.1^4\right| \\ &= 0.000\,025 \times \frac{1}{(1+\xi)^4} \end{aligned}$$

where $0 < \xi < 0.1$. To estimate the last fraction, we observe that

$$\begin{aligned} 0 < \xi < 0.1 \quad &\Rightarrow \quad 1 < 1 + \xi < 1.1 \\ &\Rightarrow \quad 1 < (1+\xi)^4 < 1.1^4 \\ &\Rightarrow \quad 1 > \frac{1}{(1+\xi)^4} > \frac{1}{1.1^4}. \end{aligned}$$

In particular, $\dfrac{1}{(1+\xi)^4} < 1$ and hence

$$\left|R_{3,0}(0.1)\right| < 0.000\,025.$$

The actual error is $\left|f(0.1) - p_{3,0}(0.1)\right| = 0.000\,023\,15$, which is indeed less than our upper bound.

Example 6.12.

(a) Find $p_{2,4}$, the Taylor polynomial of degree 2 about $x = 4$, for

$$f(x) = \sqrt{x}.$$

(b) Use $p_{2,4}$ to obtain an approximate value for $\sqrt{3.7}$.

(c) Use the remainder term $R_{2,4}$ to estimate the maximum absolute error in this result.

Solution in video

6.6 Taylor Series

In many cases the Taylor polynomials $p_{n,a}$ approximate a function f better and better as n tends to infinity. If

$$R_{n,a}(x) \to 0 \qquad \text{as } n \to \infty,$$

then $p_{n,a}(x)$ converges to $f(x)$, or, equivalently, $f(x)$ is represented by an infinite series:

$$\boxed{\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \\ &= \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!}(x-a)^r. \end{aligned}} \tag{6.6}$$

This series is called **Taylor series** of f about $x = a$. When the Taylor series is about the point $a = 0$, it is also called **Maclaurin series**. Note that there exist functions for which the Taylor series does not converge to $f(x)$ for any $x \neq a$.

There will be more about convergence of Taylor series in the class MM203 (Applicable Analysis 1) in second year.

Examples 6.13.

(i) Find the Maclaurin series for

$$f(x) = e^x.$$

Solution in video

(ii) Find the Maclaurin series for

$$f(x) = \sin x.$$

Solution in video

Example 6.14.
For a positive integer n the expression $(1+x)^n$ can be expanded as a polynomial of degree n by means of the Binomial Theorem:

$$\begin{aligned}(1+x)^n &= \sum_{r=0}^{n} \binom{n}{r} x^r \\ &= 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n.\end{aligned}$$

If n is replaced by an arbitrary real number, this expansion is no longer valid. However, we can find the Maclaurin series, which has a similar form but is an *infinite* series.

Find the Maclaurin series for the function

$$f(x) = (1+x)^\alpha, \qquad x \in (-1, 1),$$

where $\alpha \in \mathbb{R}$ is some constant.
Solution in video

The most frequently occurring Taylor series are summarised below; they converge for the range of x shown.

$$
\begin{array}{rl}
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots & x \in \mathbb{R} \\
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots & x \in \mathbb{R} \\
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots & x \in \mathbb{R} \\
\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots & x \in \mathbb{R} \\
\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots & x \in \mathbb{R} \\
\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots & -1 < x \leq 1 \\
(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots & -1 < x < 1 \\
\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots & -1 \leq x \leq 1
\end{array}
$$

6.7 Monotonicity

In this section we study how the derivative of a function can help us to decide whether the function is increasing or decreasing.

Definition 6.12.
A function f is **(strictly) increasing** on an interval I if for all $a, b \in I$ with $a < b$ we have

$$f(a) < f(b).$$

It is **(strictly) decreasing** on an interval I if for all $a, b \in I$ with $a < b$ we have

$$f(a) > f(b).$$

Theorem 6.13. *Let f be a differentiable function on an open interval I.*

- *If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.*
- *If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.*

Proof. We consider only the first case. Let $a, b \in I$ with $a < b$. By the Mean Value Theorem there exists a number $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Since $f'(\xi) > 0$ and $a - b > 0$, we obtain $f(b) - f(a) > 0$, which proves the statement. $\square$

Example 6.15.
Determine when

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 2$$

is increasing and when it is decreasing.

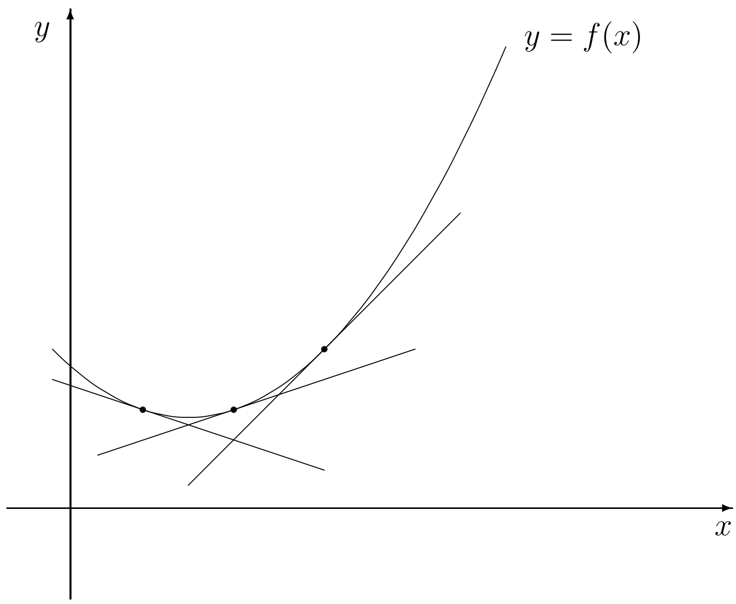
Solution in video

6.8 The Sign of the Second Derivative

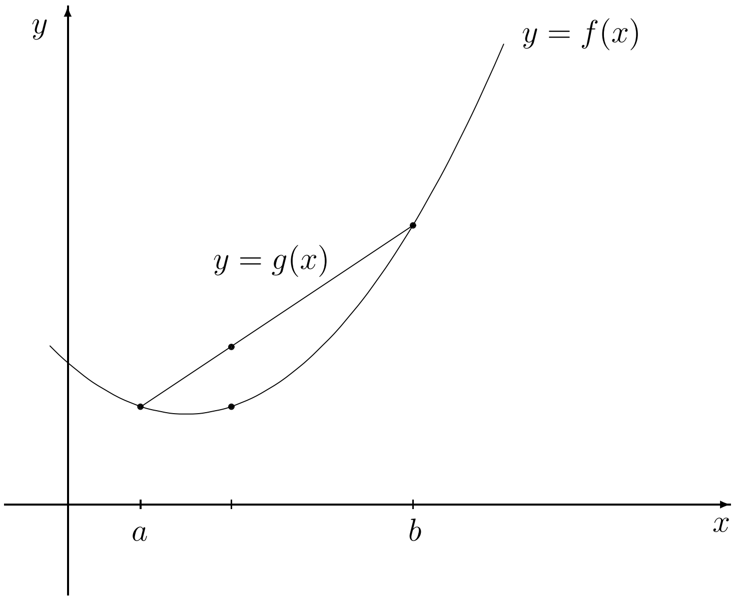
Let us assume that a function f has a positive second derivative on some interval I:

$$f''(x) > 0 \qquad \text{for } x \in I.$$

From the previous subsection we know that f' is increasing on I since f'' is the derivative of f'. This means that the slopes of the tangents at x increase when x increases.



In the next theorem we see that the straight line that connects two points on the graph lies above the graph of the function.



Theorem 6.14. *Assume that $f''(x) > 0$ on some interval I and let $a, b \in I$ with $a < b$. Let*

$$g(x) := \frac{(b-x)f(a) + (x-a)f(b)}{b-a}$$

be the linear function whose graph connects the points $(a, f(a))$ and $(b, f(b))$. Then

$$f(x) < g(x) \qquad \text{for all } x \in (a, b).$$

Proof. First note that g is a linear function and that $g(a) = f(a)$ and $g(b) = f(b)$. Now let $x \in (a, b)$. By Taylor's Theorem (about the point x) we have

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(\xi_1)}{2}(a-x)^2,$$

$$f(b) = f(x) + f'(x)(b-x) + \frac{f''(\xi_2)}{2}(b-x)^2$$

with $\xi_1 \in (a, x)$ and $\xi_2 \in (x, b)$. Then

$$\begin{aligned} g(x) &= \frac{1}{b-a}\Big[(b-x)\Big(f(x) + f'(x)(a-x) + \frac{f''(\xi_1)}{2}(a-x)^2\Big) \\ &\quad + (x-a)\Big(f(x) + f'(x)(b-x) + \frac{f''(\xi_2)}{2}(b-x)^2\Big)\Big] \\ &= \frac{1}{b-a}\Big[(b-a)f(x) + \underbrace{\Big((b-x)(a-x) + (x-a)(b-x)\Big)}_{=0} f'(x) \\ &\quad + \frac{1}{2}\underbrace{(b-x)(a-x)^2}_{>0}\underbrace{f''(\xi_1)}_{>0} + \frac{1}{2}\underbrace{(x-a)(b-x)^2}_{>0}\underbrace{f''(\xi_2)}_{>0}\Big] > f(x), \end{aligned}$$

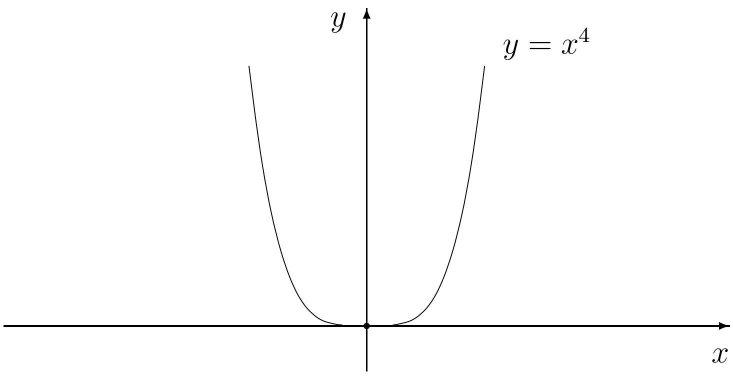
which proves the assertion. $\square$

Definition 6.15.
A function f for which $f''(x) > 0$ on some interval I is called **concave up** on the interval I.
A function f for which $f''(x) < 0$ on some interval I is called **concave down** on the interval I.
A point where the concavity changes is called a **point of inflection**.

Remark. In some textbooks concave up is called *convex* and concave down just *concave*.

For a function that is concave down one can show in a similar way that the graph of the function is always below a tangent.

If f is twice continuously differentiable and x is a point of inflection, then $f''(x) = 0$. This gives us a possibility to find points of inflection. However, not every point x with $f''(x) = 0$ is a point of inflection as the example $f(x) = x^4$ shows, for which $f''(x) = 12x^2$. Hence $f''(0) = 0$ and $f''(x) > 0$ for $x > 0$ and $x < 0$. So 0 is *not* a point of inflection.



Example 6.16.
Determine when the function

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 2$$

from Example 6.15 is concave up and when it is concave down. Find the point of inflection.

Solution in video

6.9 Classification of Stationary Points

In Theorem 6.4 we saw that a point x_0 where a function f has a local maximum or minimum is a stationary point, i.e. $f'(x_0) = 0$ (if x_0 is not an end-point of the domain of f). We also saw that not every stationary point is a maximum or minimum. Now we want to discuss sufficient conditions so that f has a maximum or minimum at a point x_0. There are two alternative methods.

Method 1.
If f' has different sign to the left and to the right of a stationary point x_0, then f has a maximum or minimum:

- If $f'(x) > 0$ for $x \in (a, x_0)$ and $f'(x) < 0$ for $x \in (x_0, b)$ for some $a < x_0$ and $b > x_0$, then f has a **local maximum** at x_0.
- If $f'(x) < 0$ for $x \in (a, x_0)$ and $f'(x) > 0$ for $x \in (x_0, b)$ for some $a < x_0$ and $b > x_0$, then f has a **local minimum** at x_0.

In the first case, for example, the conditions imply that f is increasing in a small interval to the left of x_0 and decreasing in a small interval to the right of x_0; hence f has a local maximum at x_0.

If the sign of f' is the same on both sides of x_0, then x_0 is a point of inflection:

- If $f'(x) > 0$ for $x \in (a, x_0)$ and $f'(x) > 0$ for $x \in (x_0, b)$ for some $a < x_0$ and $b > x_0$, then x_0 is a **point of inflection**.
- If $f'(x) < 0$ for $x \in (a, x_0)$ and $f'(x) < 0$ for $x \in (x_0, b)$ for some $a < x_0$ and $b > x_0$, then x_0 is a **point of inflection**.

Method 2.
If $f''(x_0)$ is positive or negative, f has a minimum or maximum (for this we need that the second derivative exists and is continuous in a neighbourhood of x_0):

- If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a **local maximum** at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a **local minimum** at x_0.

Let us prove the assertion in the second case, i.e. when $f''(x_0) > 0$. Since f'' is continuous, we have $f''(t) > 0$ for $t \in (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$. Let $x \in (x_0 - \delta, x_0 + \delta)$. Then Taylor's Theorem (Theorem 6.11) implies that

$$f(x) = f(x_0) + \underbrace{f'(x_0)}_{=0}(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2$$

for some ξ between x and x_0. In particular, $\xi \in (x_0 - \delta, x_0 + \delta)$ and hence $f''(t) > 0$. Therefore $f(x) \geq f(x_0)$.

Remark. Note that in the case when $f'(x_0) = 0$ and $f''(x_0) = 0$ we cannot decide whether at x_0 there is a maximum, a minimum or a point of inflection. For the functions

$$f_1(x) = x^3, \quad f_2(x) = x^4, \quad f_3(x) = -x^4$$

we have a stationary point at $x_0 = 0$ with $f_j''(0) = 0$; at 0 there is a point of inflection for

f_1, a minimum for f_2 and a maximum for f_3. However, one can use Method 1 to study the nature of the point x_0.

Example 6.17.
For the function

$$f(x) = -x^3 + 6x^2 + 15x - 54$$

(i) find the position and nature of the stationary points;

(ii) find where the function is increasing and where it is decreasing;

(iii) find the points of inflection;

(iv) find where the function is concave up and where it is concave down;

(v) sketch the graph of f showing all these points.

The first two derivatives of f are

$$f'(x) = -3x^2 + 12x + 15 = -3(x^2 - 4x - 5) = -3(x+1)(x-5),$$
$$f''(x) = -6x + 12 = -6(x-2).$$

(i) and (ii)
The stationary points are $x = -1$ and $x = 5$. For the nature of these points and the monotonicity we determine the sign of the factors of $f'(x)$:

x		-1		5	
$(x+1)$	$-$	0	$+$	$+$	$+$
$(x-5)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	—	$\nearrow$	—	$\searrow$
		min		max	

Hence we have a local minimum at $x = -1$, a local maximum at $x = 5$. The function is increasing on the interval $(-1, 5)$ and decreasing on the intervals $(-\infty, -1)$ and $(5, \infty)$.

At the stationary points the values of f are: $f(-1) = -62$, $f(5) = 46$.

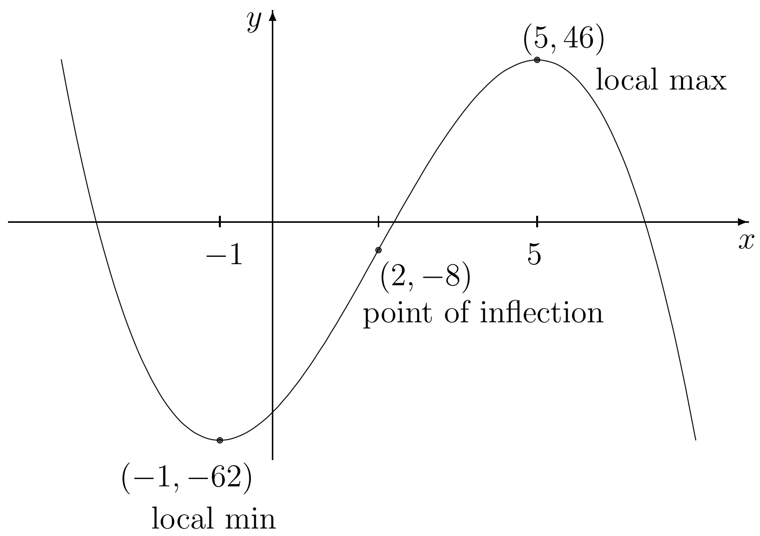
(iii) and (iv)
For the sign of the second derivative we obtain

x		2	
$f''(x)$	$+$	0	$-$
$f(x)$	$\smile$		$\frown$

The function is concave up on the interval $(-\infty, 2)$ and concave down on the interval $(2, \infty)$. Since the concavity changes at $x = 2$, this point is a point of inflection.

The value of f at the point of inflection is $f(2) = -8$.

(v)



6.10 Behaviour at Infinity and Asymptotes

In this subsection we want to discuss the behaviour of a function at infinity. Recall from MM101 (Section 9.6) that

$$\lim_{x\to\infty} f(x) = l$$

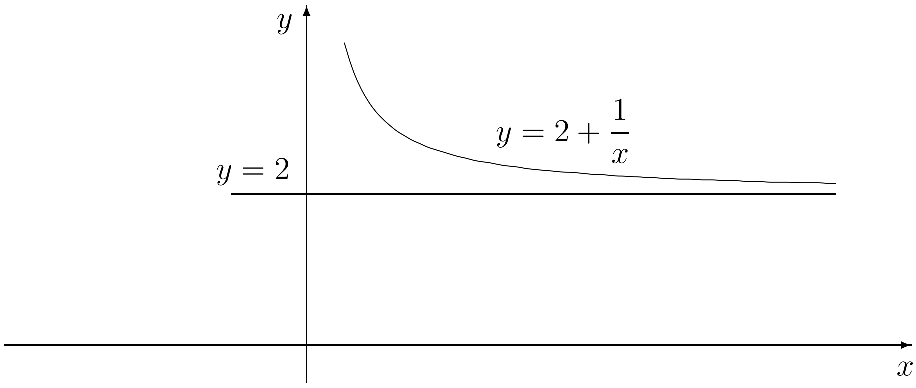
if for every $\varepsilon > 0$ there is a number N such that $|f(x) - l| < \varepsilon$ for all $x > N$. If this is the case, then the graph of the function approaches the horizontal line $y = l$. We call such a horizontal line a **horizontal asymptote**. For instance, the function

$$f(x) = 2 + \frac{1}{x}$$

has the limit

$$\lim_{x\to\infty} f(x) = 2,$$

and $y = 2$ is a horizontal asymptote.



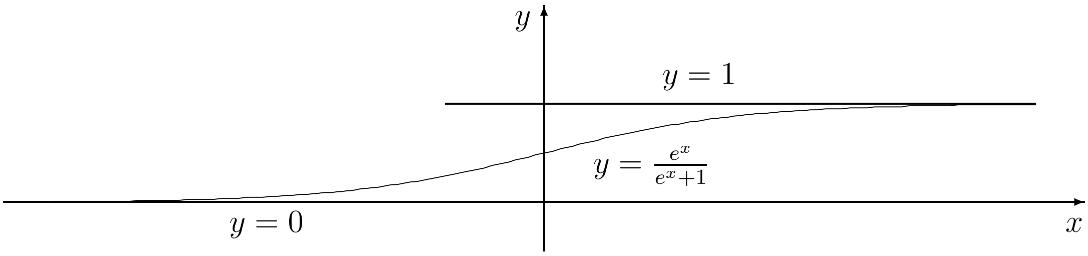
If the limit exists for $x \to -\infty$, then we also have a horizontal asymptote. Note that the limits for $x \to \infty$ and $x \to -\infty$ may be different. For example, the function

$$f(x) = \frac{e^x}{e^x + 1}$$

has the limits

$$\lim_{x \to \infty} f(x) = 1, \qquad \lim_{x \to -\infty} f(x) = 0.$$

Both lines $y = 1$ and $y = 0$ are asymptotes for f.



Often the limit does not exist as $x \to \infty$ or $x \to -\infty$ but $f(x)$ tends to ∞ or $-\infty$. For example, $f(x)$ tends to ∞ as $x \to \infty$ if for every number M there is a number N such that $f(x) > M$ for all $x > N$. This means that $f(x)$ will be greater than every number if x is big enough.

For integer powers of x we have the following behaviour.

$$\begin{array}{ll}
x,\ x^2,\ x^3,\ \ldots \to \infty & \text{as } x \to \infty \\[4pt]
x,\ x^3,\ x^5,\ \ldots \to -\infty & \text{as } x \to -\infty \\
x^2,\ x^4,\ x^6,\ \ldots \to \infty & \text{as } x \to -\infty \\[4pt]
\dfrac{1}{x},\ \dfrac{1}{x^2},\ \dfrac{1}{x^3},\ \ldots \to 0 & \text{as } x \to \pm\infty
\end{array}$$

For polynomials the behaviour is governed by the highest power. For example, if

$$f(x) = x^3 + 3x + 7$$

then $f(x) \sim x^3$ as $x \to \pm\infty$ (the symbol $\sim$ means that the quotient of $f(x)/x^3$ converges to 1 as $x \to \pm\infty$). Hence

$$f(x) \to \infty \quad \text{as } x \to \infty \qquad \text{and} \qquad f(x) \to -\infty \quad \text{as } x \to -\infty.$$

For rational functions we can again consider the highest powers; for example, if

$$f(x) = \frac{x^3 + 2x + 9}{x + 3}$$

then $f(x) \sim \dfrac{x^3}{x} = x^2$, and hence $f(x) \to \infty$ as $x \to \pm\infty$.

Slant Asymptotes.

If the function $f(x)$ grows linearly, then it may approach a line as $x \to \infty$ or $x \to -\infty$. This happens if, e.g. f is a rational function, $f(x) = \dfrac{p(x)}{q(x)}$, where the degree of the polynomial p is **greater** than the degree of q **by one**, i.e.

$$\deg(p) = \deg(q) + 1.$$

Using long division, we can write f as a linear polynomial $mx + c$ plus a proper fraction (degree of numerator is less than degree of denominator). The graph of the function f then approaches the line

$$y = mx + c$$

as $x \to \pm\infty$ since the proper fraction converges to 0.

This line is called **slant asymptote** (or **inclined asymptote** or **oblique asymptote**).

Example 6.18.
Find the slant asymptote for the function

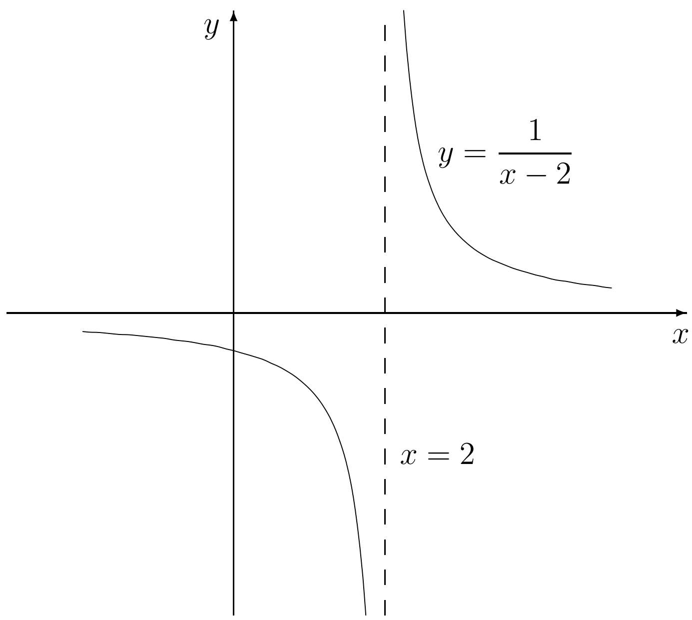
$$f(x) = \frac{2x^2}{x + 1}$$

Solution in video

Vertical Asymptotes.

Consider the graph of the function

$$f(x) = \frac{1}{x - 2}$$



As x approaches 2 from the right, $f(x)$ tends to ∞; as x approaches 2 from the left, $f(x)$ tends to $-\infty$. This implies that the graph of f approaches the vertical line $x = 2$ as $x \to 2^+$ (from the right) or $x \to 2^-$ (from the left). Such a line is called a **vertical asymptote**.

In general, if for some number a, we have $f(x) \to \infty$ as $x \to a^+$ or $x \to a^-$ or if we have $f(x) \to -\infty$ as $x \to a^+$ or $x \to a^-$, then the line $x = a$ is a vertical asymptote.

We can use monotonicity to decide from which side the graph approaches a vertical asymptote. If, e.g. f has a vertical asymptote $x = a$ and f is increasing to the right of a, then $f(x) \to -\infty$ as $x \to a^+$; if f is decreasing to the right of a, then $f(x) \to \infty$ as $x \to a^+$. To the left of a it is the other way round: if f is increasing to the left of a, then $f(x) \to \infty$ as $x \to a^-$; if f is decreasing to the left of a, then $f(x) \to -\infty$ as $x \to a^-$.

If f is a rational function, $f(x) = \dfrac{p(x)}{q(x)}$, we must look for zeros of the denominator q. The line $x = a$ is a vertical asymptote if

$$q(a) = 0 \quad \text{and} \quad p(a) \neq 0.$$

If both $q(a) = 0$ and $p(a) = 0$, then p and q have a common factor $(x - a)$, which can be cancelled. After this factor has been cancelled, check again whether $q(a) = 0$ and $p(a) \neq 0$. The function

$$f(x) = \frac{x^2 - 3x + 2}{x - 1}$$

has a zero in the denominator at $x = 1$. However, also the numerator vanishes at $x = 1$. The fraction can be simplified:

$$f(x) = \frac{(x-1)(x-2)}{x-1} = x - 2, \qquad \text{for } x \neq 1.$$

For $x \neq 1$ the function f coincides with the linear function that maps x onto $x-2$. Therefore there is no vertical asymptote at $x=1$.

Summary for asymptotes of rational functions.

$$f(x) = \frac{p(x)}{q(x)}$$

Vertical asymptotes:	if $q(a)=0$ and $p(a)\neq 0$, then $x=a$ is VA if $q(a)=0$ and $p(a)=0$, cancel $(x-a)$ and check again
Horizontal asymptotes:	if $\deg(p) \leq \deg(q)$, then $b := \lim_{x\to\pm\infty} f(x)$ exists; $y=b$ is HA
Slant asymptote:	if $\deg(p) = \deg(q)+1$, then write (using long division) $f(x) = mx + c +$ proper fraction $y = mx + c$ is a slant asymptote

6.11 Curve Sketching

Using information regarding stationary points, monotonicity, points of inflection, concavity, asymptotes and points of intersection with the axes, we can sketch the graph of a function. Below there is a summary of how one can proceed.

1. Find the natural domain of the function.
2. Find position and nature of stationary points.
3. Find when the function is increasing and when it is decreasing.
4. Find the points of inflection.
5. Find when the function is concave up and when it is concave down.
6. Find vertical, horizontal and slant asymptotes.
7. Examine $f(x)$ as $x \to \pm\infty$.
8. Find all points of intersection with the x-axis and the y-axis.
9. Locate all the above and sketch the graph of f.

Usually, in an assignment or exam question you will be told which of the items above you have to do. In practical problems you have to decide what is needed in order to sketch the graph and answer a given question about f.

One can also use a computer with a mathematical software package to plot the graph of a function. However, one might miss important features of the function if the plot range is not well chosen. Moreover, if a parameter appears in the function, then one cannot plot the graph for all parameters, but with the calculations mentioned above one can find the graph at least qualitatively and how it changes when the parameter changes.

Sometimes information about, say, stationary points can be used to obtain information about zeros of f. See, e.g. Example 6.17 where one can deduce that there must be exactly one zero between the minimum and the maximum.

Example 6.19.
The function f is given by

$$f(x) = \frac{x^2-7}{x^2-4}\,.$$

(i) Find the natural domain of f.

(ii) Find the asymptotes.

(iii) Find the position and nature of the stationary points.

(iv) Find when f is increasing and when it is decreasing.

(v) Find the points of intersection with the x-axis and the y-axis.

(vi) Sketch the graph of the function f showing all critical points and asymptotes.

Solution:

(i) The denominator is 0 if and only if $x^2 = 4$, i.e. when $x = 2$ or $x = -2$. Hence the natural domain is equal to

$$\mathrm{dom}(f) = \big\{x \in \mathbb{R}\colon x \neq 2 \text{ and } x \neq -2\big\}.$$

(ii) Since the numerator is non-zero for $x = 2$ and $x = -2$, the vertical asymptotes are

$$x = 2 \qquad \text{and} \qquad x = -2.$$

The degrees of the numerator and the denominator are the same, so we expect to have a horizontal asymptote. The limit of $f(x)$ as $x \to \pm\infty$ can be calculated by dividing numerator and denominator by x^2:

$$\lim_{x\to\pm\infty} \frac{x^2-7}{x^2-4} = \lim_{x\to\pm\infty} \frac{1-\dfrac{7}{x^2}}{1-\dfrac{4}{x^2}} = \frac{1}{1} = 1.$$

Hence the line $y = 1$ is a horizontal asymptote for both $x \to \infty$ and $x \to -\infty$.

(iii) The first derivative of f is

$$f'(x) = \frac{2x(x^2-4)-(x^2-7)(2x)}{(x^2-4)^2} = \frac{2x^3-8x-2x^3+14x}{(x^2-4)^2} = \frac{6x}{(x^2-4)^2}$$

$f'(x) = 0$ when $6x = 0$, i.e. $x = 0$. For the sign of $f'(x)$ we obtain (we consider also the points where the denominator of $f'(x)$ is 0)

x		-2		0		2	
$6x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x^2-4)^2$	$+$	0	$+$	$+$	$+$	0	$+$
$f'(x)$	$-$	ND	$-$	0	$+$	ND	$+$
$f(x)$	$\searrow$	ND	$\searrow$	—	$\nearrow$	ND	$\nearrow$

where ND means "not defined". It follows that there is a <u>local minimum</u> at $x = 0$. At this point the value of f is $f(0) = \dfrac{7}{4}$.

(iv) The function is <u>decreasing</u> on the intervals $(-\infty, -2)$, $(-2, 0)$ and <u>increasing</u> on the intervals $(0, 2)$, $(2, \infty)$.

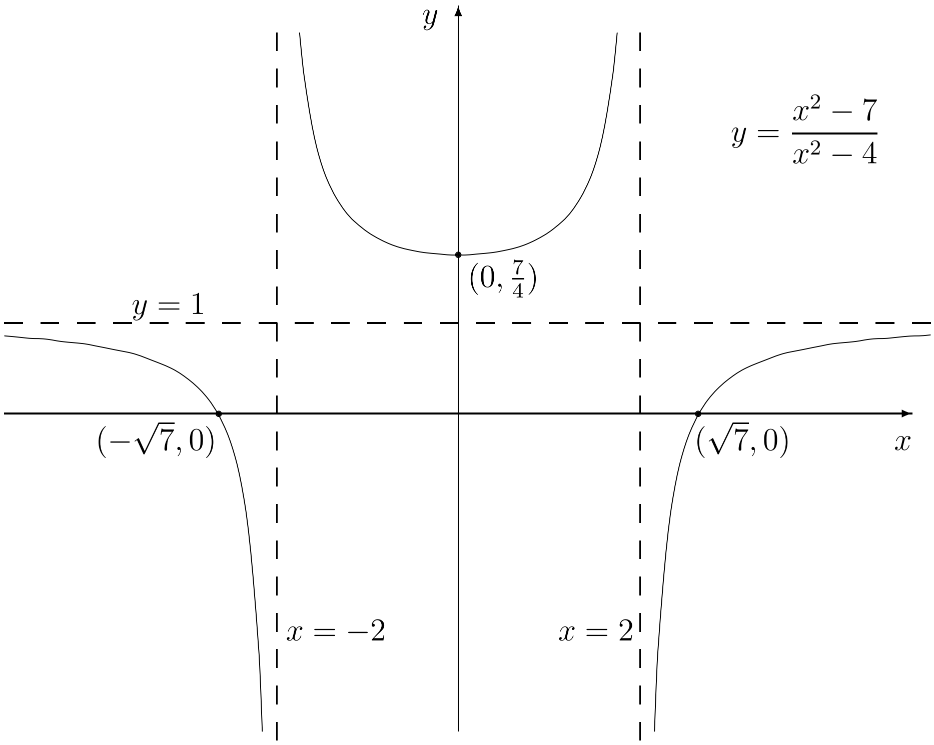
From the monotonicity we can deduce how the graph of f approaches the vertical asymptotes:

$$\begin{aligned} f(x) &\to -\infty \quad \text{as } x \to -2^-, \\ f(x) &\to \infty \quad \text{as } x \to -2^+, \\ f(x) &\to \infty \quad \text{as } x \to 2^-, \\ f(x) &\to -\infty \quad \text{as } x \to 2^+. \end{aligned}$$

(v) The graph of the function intersects the y-axis at $y = f(0) = \dfrac{7}{4}$.
It intersects the x-axis where $f(x) = 0$, i.e. when

$$x^2 - 7 = 0 \qquad \Longleftrightarrow \qquad x = \pm\sqrt{7}.$$

(vi) We can put all this together to obtain a sketch of the graph of f.



Example 6.20.
The function f is given by

$$f(x) = \frac{x^2}{x-1}.$$

(i) Find the natural domain of f.

(ii) Find the asymptotes.

(iii) Find the position and nature of the stationary points.
Find when the function is increasing and when it is decreasing.

(iv) Show that there are no points of inflection. Find when the function is concave up and when it is concave down.

(v) Find the points of intersection with the x-axis and the y-axis.

(vi) Sketch the graph of the function f showing all critical points and asymptotes.

Solution in video

The next two examples involve a parameter. In these examples one has to find the properties of the function and its graph in dependence of the parameter.

Example 6.21.
Let $a > 0$ and consider the function

$$f(x) = x^3 - 3a^2x + 1, \qquad x \in \mathbb{R}.$$

(i) Find the position and nature of the stationary points. Find where the function is increasing and where it is decreasing.

(ii) Sketch the graph.

(iii) How many zeros does the function have?

Solution:

(i) The first derivative is equal to

$$f'(x) = 3x^2 - 3a^2 = 3(x^2 - a^2) = 3(x + a)(x - a).$$

For the stationary points. we have

$$f'(x) = 0 \quad \Longleftrightarrow \quad (x + a)(x - a) = 0 \quad \Longleftrightarrow \quad x = a \quad \text{or} \quad x = -a.$$

The sign of $f'(x)$ is considered in the following table:

x		$-a$		a	
$(x+a)$	$-$	0	$+$	$+$	$+$
$(x-a)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	—	$\searrow$	—	$\nearrow$
		max		min	

Hence f has a local maximum at $x = -a$ and a local minimum at $x = a$. The values of f at these points are:

$$\begin{aligned} f(-a) &= (-a)^3 - 3a^2(-a) + 1 = -a^3 + 3a^3 + 1 = 2a^3 + 1, \\ f(a) &= a^3 - 3a^2a + 1 = -2a^3 + 1. \end{aligned}$$

Since $a > 0$, we have

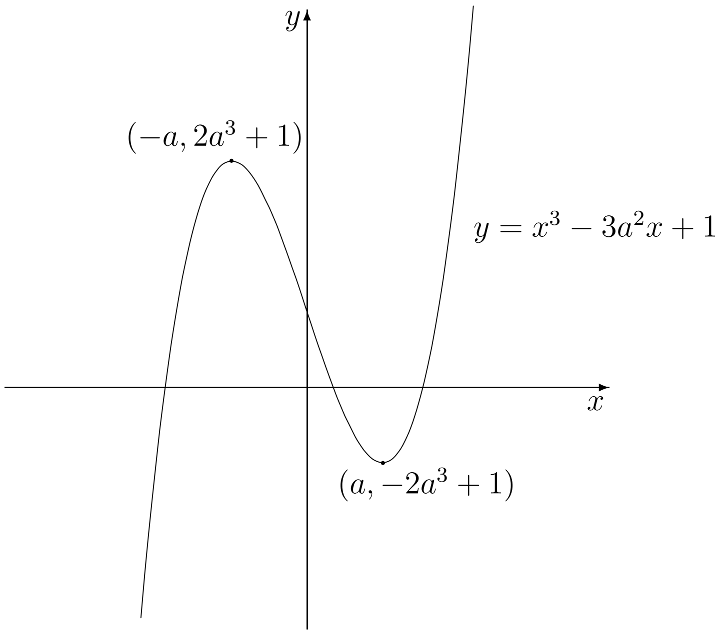
$$f(-a) > 0.$$

The sign of $f(a)$ depends on a:

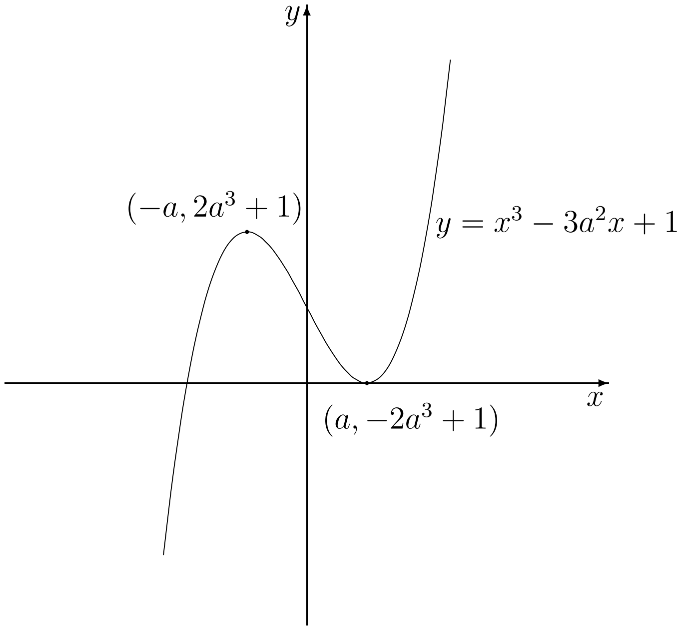
$$\begin{aligned} f(a) > 0 \quad &\Longleftrightarrow \quad a < \frac{1}{\sqrt[3]{2}}, \\ f(a) = 0 \quad &\Longleftrightarrow \quad a = \frac{1}{\sqrt[3]{2}}, \\ f(a) < 0 \quad &\Longleftrightarrow \quad a > \frac{1}{\sqrt[3]{2}}. \end{aligned}$$

Further, the function is increasing on the intervals $(-\infty, -a)$ and (a, ∞); it is decreasing on the interval $(-a, a)$.

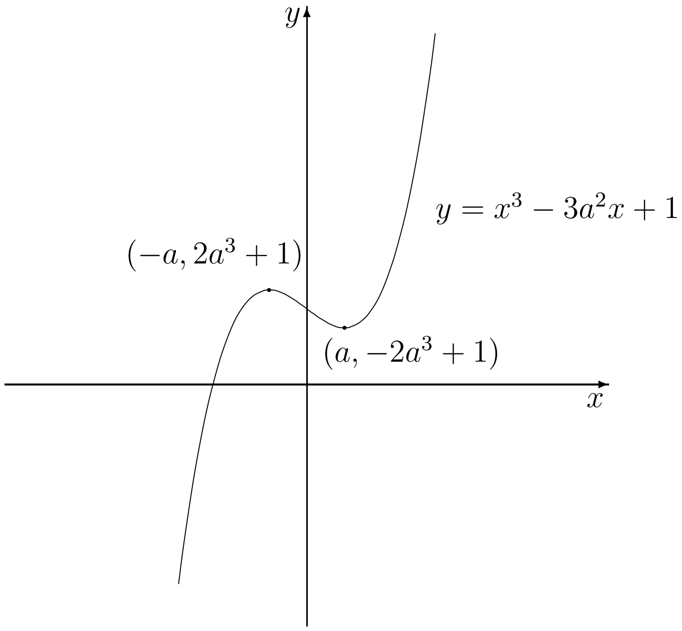
(ii) When $a > \dfrac{1}{\sqrt[3]{2}}$, the graph looks as follows:



When $a = \dfrac{1}{\sqrt[3]{2}}$, the graph looks as follows:



When $a < \dfrac{1}{\sqrt[3]{2}}$, the graph looks as follows:



(iii) When $a > \dfrac{1}{\sqrt[3]{2}}$, the function has 3 zeros. When $a = \dfrac{1}{\sqrt[3]{2}}$, the function has 2 zeros. When $a < \dfrac{1}{\sqrt[3]{2}}$, the function has 1 zero.

Example 6.22.
Let a be a positive constant and define the function f by

$$f(x) = xe^{ax}.$$

(i) Find the position and nature of the stationary points. (These will depend on a.) Find when the function is increasing and when it is decreasing.
(Note that $e^t > 0$ for every $t \in \mathbb{R}$.)

(ii) Find the point of inflection. Find when the function is concave up and when it is concave down.

(iii) Find the points of intersection with the x-axis and the y-axis.

(iv) Find the position of the critical points relative to each other and use this information to sketch the graph of f qualitatively showing the critical points.

Solution in video

Example 6.23.
The function f is given by

$$f(x) = \frac{x-2}{x^2}.$$

(i) Find the natural domain of f.

(ii) Find the asymptotes.

(iii) Find the position and nature of the stationary points.
Find when f is increasing and when it is decreasing.

(iv) Find the points of inflection.
Find when f is concave up and when it is concave down.

(v) Find the points of intersection with the x-axis and the y-axis.

(vi) Sketch the graph of the function f showing all critical points and asymptotes.

Solution.

(i) The denominator vanishes exactly when $x = 0$. Hence the natural domain of the function is

$$\mathrm{dom}(f) = \big\{x \in \mathbb{R} \colon x \neq 0\big\}.$$

(ii) Since the numerator is non-zero for $x = 0$, we have the vertical asymptote $x = 0$.

The degree of the numerator is less than the degree of the denominator, so we expect that the limit of $f(x)$ exists as $x \to \pm\infty$. Indeed,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{1}{x} - \frac{2}{x^2}\right) = 0.$$

Hence the line $y = 0$ is a horizontal asymptote.

(iii) The first two derivatives are

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x} - \frac{2}{x^2}\right) = -\frac{1}{x^2} + \frac{4}{x^3} = \frac{-x+4}{x^3}\,;$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{x^2} + \frac{4}{x^3}\right) = \frac{2}{x^3} - \frac{12}{x^4} = \frac{2(x-6)}{x^4}\,.$$

$f'(x) = 0 \quad \Longleftrightarrow \quad x = 4 \quad$ (stationary point)

We consider the sign of f' in the following table

x		0		4	
$(-x+4)$	$+$	$+$	$+$	0	$-$
x^3	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	ND	$+$	0	$-$
$f(x)$	$\searrow$	ND	$\nearrow$	—	$\searrow$
				max	

The function f is increasing on the interval $(0, 4)$ and decreasing on the intervals $(-\infty, 0)$ and $(4, \infty)$.

There is a local maximum at $x = 4$ with value $f(4) = \dfrac{1}{8}\,.$

(iv) $f''(x) = 0 \quad \Longleftrightarrow \quad x = 6 \quad$ (possible point of inflection)

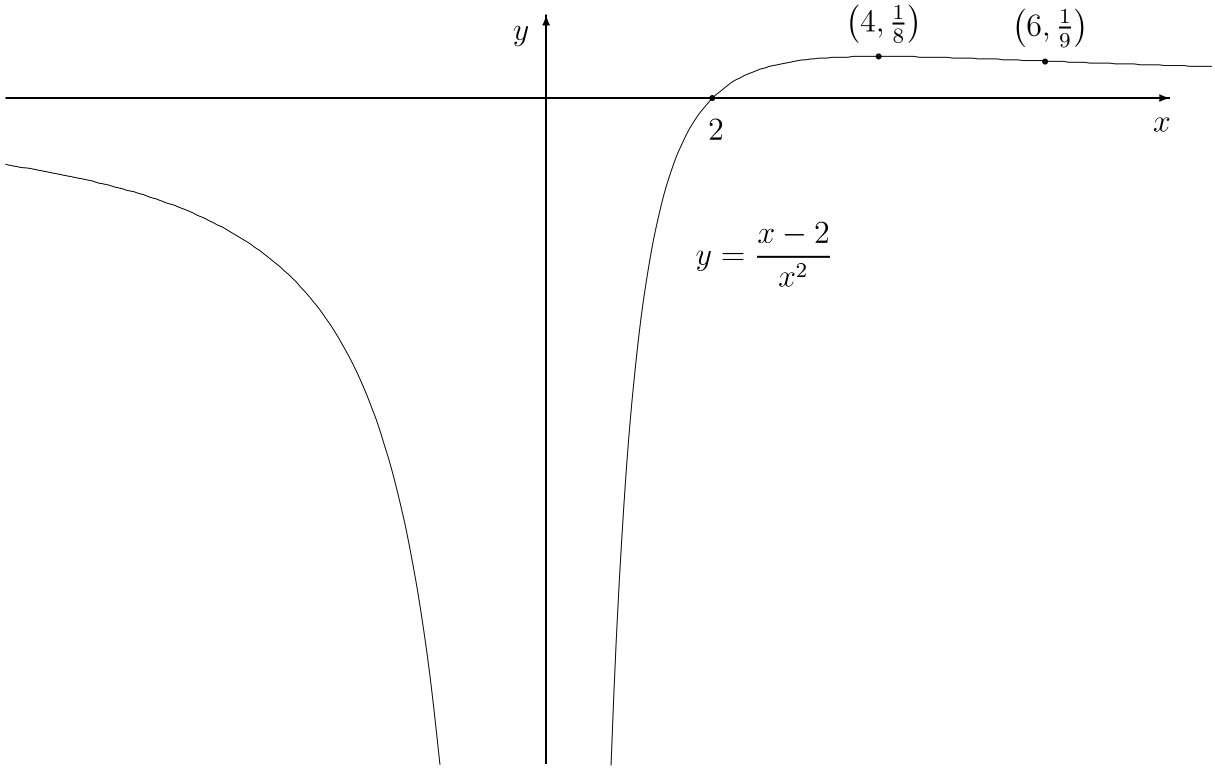
We consider the sign of f'' in the following table

x		0		6	
$(x-6)$	$-$	$-$	$-$	0	$+$
x^4	$+$	0	$+$	$+$	$+$
$f''(x)$	$-$	ND	$-$	0	$+$
$f(x)$	$\frown$	ND	$\frown$		$\smile$

The function is concave up on the interval $(6, \infty)$ and concave down on the intervals $(-\infty, 0)$ and $(0, 6)$. Since concavity changes at $x = 6$, we have a point of inflection there with $f(6) = \dfrac{1}{9}\,.$

(v) The function f is not defined at $x = 0$, so there is no point of intersection with the y-axis.
The point of intersection with the x-axis is where $f(x) = 0$, i.e. at $x = 2$.

(vi) We can sketch the graph of f.



6.12 Practical Optimisation Problems

In Section 6.2 we discussed the connection between maxima/minima of a function f and stationary points. In particular, we have seen that global maxima/minima can occur at stationary points, end-points of the domain and points where f is not differentiable.

If f is continuously differentiable and has exactly one stationary point x_0 where f has a local maximum or minimum, then f has a global maximum/minimum, respectively. This follows from the fact that, e.g. in the case of a maximum, $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$.

In the following we summarise how practical optimisation problems can be solved.

Strategy for solving maximum/minimum problems

1. Identify and name variables. If the problem involves some geometry, make a diagram.
2. Decide which variable is to be optimised (e.g. y) and find a formula for y in terms of the other variables.
3. Write down all other information as mathematical equations.
4. Use these equations to eliminate all variables except one, e.g. x. Now y should be a function of x only, say $y = f(x)$. Determine the domain of the function f.
5. Consider critical points:
   - stationary points in $\mathrm{dom}(f)$;
   - end-points of the domain;
   - points where the function is not differentiable.
6. If the function f is continuous on a closed, bounded interval, then identify the maximum or minimum by evaluating the function at all critical points and comparing the values.

   Otherwise, determine also the behaviour of $f(x)$ as x tends to the end-points of the domain. It may happen that no maximum/minimum exists.
7. If $\mathrm{dom}(f)$ is an interval, f is continuously differentiable on $\mathrm{dom}(f)$ and has exactly one stationary point where f has a local maximum or minimum, then it is not necessary to investigate the behaviour at the end-points of the domain.

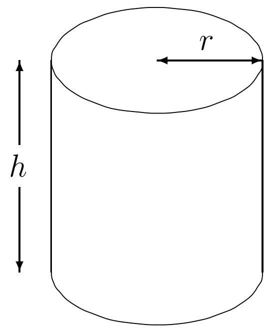
Example 6.24.
A wire of 48 cm is bent to form a rectangle. What are the dimensions of the rectangle for which the area is a maximum.

Solution in video

Example 6.25.
A soft drink manufacturer wants to fabricate a cylindrical can for its product. The can is to have a volume of $330\,\mathrm{cm}^3$. Find the dimensions of the can that will require the least amount of material (i.e. the surface area should become a minimum).

Introduce the following variables:

S ... surface area
r ... radius of the discs at top and bottom
h ... height of the can



The surface area is equal to

$$\begin{aligned} S &= \text{area of side} + \text{top} + \text{bottom} \\ &= 2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2. \end{aligned}$$

We also know that the volume is equal to $330\,\mathrm{cm}^3$. With the formula for the volume in terms of r and h:

$$\text{volume} = \text{area of bottom} \times \text{height} = \pi r^2 h$$

we obtain the relation

$$\pi r^2 h = 330.$$

We can solve this equation for h and eliminate h in the formula for S:

$$\begin{aligned} & h = \frac{330}{\pi r^2} \\ \Longrightarrow \quad & S = S(r) = 2\pi r \frac{330}{\pi r^2} + 2\pi r^2 = \frac{660}{r} + 2\pi r^2, \end{aligned}$$

which is now only a function of r. Every positive number r is allowed, so the domain of the function S is the interval $(0, \infty)$.

To find the stationary points, let us differentiate S with respect to r:

$$\frac{\mathrm{d}S}{\mathrm{d}r} = -\frac{660}{r^2} + 4\pi r.$$

Then

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}r} = 0 \quad & \Longleftrightarrow \quad \frac{660}{r^2} = 4\pi r \\ & \Longleftrightarrow \quad r^3 = \frac{660}{4\pi} = \frac{165}{\pi} \\ & \Longleftrightarrow \quad r = \left(\frac{165}{\pi}\right)^{1/3}. \end{aligned}$$

To determine the nature of this stationary point, we calculate the second derivative:

$$\frac{\mathrm{d}^2 S}{\mathrm{d}r^2} = \frac{2 \times 660}{r^3} + 4\pi,$$

which is positive for every positive r, in particular, also at the stationary point. Hence we have a local minimum.

The function S has only one stationary point, and S is continuously differentiable on the interval $(0, \infty)$. Hence S has a global minimum at

$$r = \left(\frac{165}{\pi}\right)^{1/3} \approx 3.74\,\mathrm{cm}.$$

The corresponding values for h and S are

$$\begin{aligned} h &= \frac{330}{\pi} \times \left(\frac{165}{\pi}\right)^{-2/3} = \frac{2 \times 165^{1/3}}{\pi^{1/3}} \approx 7.49\,\mathrm{cm} \\ S &= 660 \times \left(\frac{165}{\pi}\right)^{-1/3} + 2\pi\left(\frac{165}{\pi}\right)^{2/3} = 4 \times 165^{2/3}\pi^{1/3} + 2 \times 165^{2/3}\pi^{1/3} \\ &= 6 \times 165^{2/3}\pi^{1/3} \approx 264.36\,\mathrm{cm}^2. \end{aligned}$$

Note that $h = 2r$ at the minimum.

Example 6.26.
A ship is sailing due south at 20 mph and a second ship is sailing due east at 15 mph. At a certain time the second ship is 100 miles south of the first ship. When is the distance between the ships a minimum and what is this distance?

Solution in video

6.13 Related Rates

If y is a function of time t, then the **rate of change** of y is $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

In some applications, two variables, say x and y, are functions of time, one rate of change is known and one wants to know the rate of change of the other variable. This task can be achieved by the following strategy if the two variables themselves are related by some equation.

Strategy for related rates problems

1. Identify and name the two variables, say x and y, that depend on time. A diagram may help.
2. Find an equality relating the two variables x, y.
3. Differentiate both sides of this equality with respect to t and treat x and y as functions of t.
4. If $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ is the rate of change you want to know, solve the equation you obtained in the previous step for $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.
5. Evaluate $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ by substituting for x, y and $\dfrac{\mathrm{d}x}{\mathrm{d}t}$.

Example 6.27.
Air is blown into a spherical balloon at a rate of $10\,\mathrm{cm}^3$ per second. How fast is its radius changing with time when the radius is $4\,\mathrm{cm}$?

Solution in video

Example 6.28.
A point moves on the ellipse $4x^2 + 5y^2 = 9$. As it passes $(-1, 1)$, the y-coordinate is increasing at $2\,\mathrm{cm/s}$. How is the x-coordinate changing?

Both x and y depend on t. If we differentiate both sides of the relation

$$4x^2 + 5y^2 = 9$$

with respect to t, we obtain

$$8x\dot{x} + 10y\dot{y} = 0.$$

Since we want to know the rate of change of x, we solve for $\dot{x}$:

$$\dot{x} = -\frac{5y\dot{y}}{4x}.$$

At the point $(-1, 1)$ and with $\dot{y} = 2$ we obtain

$$\dot{x} = -\frac{5 \times 1 \times 2}{4 \times (-1)} = \frac{5}{2}\,\mathrm{cm/s}.$$

Example 6.29.
An inverted cone of height $20\,\mathrm{cm}$ and radius $6\,\mathrm{cm}$ at the top is filled with water at a rate of $12\,\mathrm{cm}^3/\mathrm{s}$. When the height of water is $15\,\mathrm{cm}$, at what rate is the height increasing?

Solution in video