UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MATHEMATICS AND STATISTICS

MM201 Linear Algebra and Differential Equations

1 Lecture examples: Chapter 1

Examples 1A

1 Find x, y and z by solving the matrix equation

$$\begin{bmatrix} x^2+x & y^2+z \\ y & x^2+x \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 3 & 12 \end{bmatrix}.$$

We have $y=3$, $y^2+z=12$ and $x^2+x=12$ so $y=3$, $z=3$ and $x=3$ or -4.

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2 If $A = \begin{bmatrix} 4 & -2 \\ \frac{1}{2} & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{3}{2} \end{bmatrix}$, evaluate $A+B$ and $A-B$.

$$A+B = \begin{bmatrix} 4+0 & -2+1 \\ \frac{1}{2}-1 & 4-\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}, \quad A-B = \begin{bmatrix} 4-0 & -2-1 \\ \frac{1}{2}+1 & 4+\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ \frac{3}{2} & \frac{11}{2} \end{bmatrix}.$$

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3 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, evaluate AB and BA.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Note: $AB \neq BA$. This is generally the case. If $AB = BA$, we say that they commute under matrix multiplication.

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4 If $C = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$, evaluate CD and DC.

$$CD = \begin{bmatrix} 1 & -1 & 2 \\ -3 & -1 & -4 \end{bmatrix}, \quad DC \text{ cannot be done.}$$

Examples 1B

1 If $A = \begin{bmatrix} 17 & 2 \\ 0 & \frac{1}{2} \end{bmatrix}$, find AI and IA.

$$AI = \begin{bmatrix} 17 & 2 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 2 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Similarly $IA = A$.

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2 Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $A - 3A^T + 2I$.

The identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so

$$\begin{aligned} A - 3A^T + 2I &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 3 \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 9 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ -3 & -6 \end{bmatrix}. \end{aligned}$$

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3 Given $A = \begin{bmatrix} 4 & 0 & -5 \\ -3 & 0 & 4 \\ -18 & 1 & 24 \end{bmatrix}$, verify that $A^{-1} = \begin{bmatrix} 4 & 5 & 0 \\ 0 & -6 & 1 \\ 3 & 4 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 4 & 0 & -5 \\ -3 & 0 & 4 \\ -18 & 1 & 24 \end{bmatrix} \begin{bmatrix} 4 & 5 & 0 \\ 0 & -6 & 1 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

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4 Verify that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{bmatrix}$ is orthogonal.

$$A^T A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ 2 & 1 & -2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I_3.$$

Examples 1C

1 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find A^{-1}.

$$A^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

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2 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, find A^{-1}.

Here the determinant $ad - bc = 1\cdot 6 - 2\cdot 3 = 0$ so the inverse does not exist, i.e. A is singular.

Examples 1D

1 State whether the following matrices are in row echelon form, reduced row echelon form or neither:

$$A = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A,C,D are in reduced row echelon form; B is in row echelon form.

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2 Solve the equations

$$\begin{aligned} x_1 - 3x_2 + 5x_3 - 7x_4 &= -2\ . \\ -2x_1 + 4x_2 - 6x_3 + 8x_4 &= 2 \\ x_1 + x_2 + x_3 + x_4 &= 2 \\ x_1 + 5x_2 + 2x_3 + 5x_4 &= 7 \end{aligned}$$

$$\begin{array}{rrrrcr} 1 & -3 & 5 & -7 & | & -2 \\ -2 & 4 & -6 & 8 & | & 2 \\ 1 & 1 & 1 & 1 & | & 2 \\ 1 & 5 & 2 & 5 & | & 7 \end{array} \begin{array}{l} \\ r_2' = r_2 + 2r_1 \\ r_3' = r_3 - r_1 \\ r_4' = r_4 - r_1 \end{array} \rightarrow \begin{array}{rrrrcr} 1 & -3 & 5 & -7 & | & -2 \\ 0 & -2 & 4 & -6 & | & -2 \\ 0 & 4 & -4 & 8 & | & 4 \\ 0 & 8 & -3 & 12 & | & 9 \end{array} \begin{array}{l} \\ r_2' = -\frac{1}{2}r_2 \\ \\ \\ \end{array}$$

$$\rightarrow \begin{array}{rrrrcr} 1 & -3 & 5 & -7 & | & -2 \\ 0 & 1 & -2 & 3 & | & 1 \\ 0 & 4 & -4 & 8 & | & 4 \\ 0 & 8 & -3 & 12 & | & 9 \end{array} \begin{array}{l} \\ \\ r_3' = r_3 - 4r_2 \\ r_4' = r_4 - 8r_2 \end{array} \rightarrow \begin{array}{rrrrcr} 1 & -3 & 5 & -7 & | & -2 \\ 0 & 1 & -2 & 3 & | & 1 \\ 0 & 0 & 4 & -4 & | & 0 \\ 0 & 0 & 13 & -12 & | & 1 \end{array} \begin{array}{l} \\ \\ r_3' = \frac{1}{4}r_3 \\ \\ \end{array}$$

$$\rightarrow \begin{array}{rrrrcr} 1 & -3 & 5 & -7 & | & -2 \\ 0 & 1 & -2 & 3 & | & 1 \\ 0 & 0 & 1 & -1 & | & 0 \\ 0 & 0 & 13 & -12 & | & 1 \end{array} \begin{array}{l} \\ \\ \\ r_4' = r_4 - 13r_3 \end{array} \rightarrow \begin{array}{rrrrcr} 1 & -3 & 5 & -7 & | & -2 \\ 0 & 1 & -2 & 3 & | & 1 \\ 0 & 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 1 & | & 1 \end{array}$$

so

$$x_4 = 1; \quad x_3 - x_4 = 0 \Rightarrow x_3 = 1; x_2 - 2x_3 + 3x_4 = 1 \Rightarrow x_2 = 0;$$

$$x_1 - 3x_2 + 5x_3 - 7x_4 = -2 \Rightarrow x_1 = 0$$

and the solution is

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Examples 1E

1 Solve the system

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 3x_3 & = & 13 \\ 2x_1 & + & 5x_2 & + & 4x_3 & = & 23 \\ 2x_1 & + & 7x_2 & + & 8x_3 & = & 29. \end{array}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 3 & | & 13 \\ 2 & 5 & 4 & | & 23 \\ 2 & 7 & 8 & | & 29 \end{bmatrix} \begin{array}{l} \\ r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 2r_1 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 3 & | & 13 \\ 0 & -1 & -2 & | & -3 \\ 0 & 1 & 2 & | & 3 \end{bmatrix} \begin{array}{l} \\ r_2' = -r_2 \\ \\ \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 3 & | & 13 \\ 0 & 1 & 2 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{array}{l} \\ \\ r_3' = r_3 - r_2 \end{array}$$

The equations are now

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 3x_3 & = & 13 \\ & & x_2 & + & 2x_3 & = & 3 \end{array}$$

so x_3 is a free variable: put $x_3 = \lambda$, say, to get

$$x_2 = 3 - 2\lambda; \quad x_1 = 13 - 3(3 - 2\lambda) - 3\lambda = 4 + 3\lambda.$$

Solution is

$$\boldsymbol{x} = \begin{bmatrix} 4 + 3\lambda \\ 3 - 2\lambda \\ \lambda \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

2 Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 8 \\ -3 & 1 & 2 & 1 \end{bmatrix}.$$

Use EROs on the augmented matrix $[A|\boldsymbol{b}]$ where $\boldsymbol{b} = [1\,2\,0]^T$ to find the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{x} = [x_1\,x_2\,x_3\,x_4]^T$.

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 4 & 1 \\ 2 & 2 & 2 & 8 & 2 \\ -3 & 1 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} \\ r_2' = r_2 - 2r_1 \\ r_3' = r_3 + 3r_1 \end{array}$$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 13 & 3 \end{array}\right] \begin{array}{l} \\ r_2' = r_3 \\ r_3' = r_2 \end{array}$$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 4 & 1 \\ 0 & 4 & 5 & 13 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ r_2' = r_2/4 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 4 & 1 \\ 0 & 1 & \frac{5}{4} & \frac{13}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1' = r_1 - r_2 \\ \\ \\ \end{array} \qquad \text{echelon form}$$

The variables x_1 and x_2 (associated with the leading 1s) are called **leading** variables: x_3 and x_4 are called **free** variables. From the echelon form, the original system is equivalent to

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & 4x_4 & = & 1 \\ & & x_2 & + & \frac{5}{4}x_3 & + & \frac{13}{4}x_4 & = & \frac{3}{4} \end{array}$$

so let $x_4 = \lambda$, $x_3 = \mu$, say, to obtain

$$x_2 = \frac{3}{4} - \frac{5}{4}\mu - \frac{13}{4}\lambda,$$

$$x_1 = 1 - \left(\frac{3}{4} - \frac{5}{4}\mu - \frac{13}{4}\lambda\right) - \mu - 4\lambda = \frac{1}{4} + \frac{1}{4}\mu - \frac{3}{4}\lambda.$$

The solution is therefore

$$\boldsymbol{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{3}{4} \\ -\frac{13}{4} \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} \frac{1}{4} \\ -\frac{5}{4} \\ 1 \\ 0 \end{bmatrix}.$$

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3 Show that the system

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & x_3 & = & k \\ 4x_1 & + & 11x_2 & - & x_3 & = & -1 \end{array}$$

is inconsistent (i.e. has no solution) if $k \neq 1$. Find the general solution when $k = 1$.

Augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 2 & 3 & 1 & k \\ 4 & 11 & -1 & -1 \end{array}\right] \begin{array}{l} \\ r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 4r_1 \end{array}$$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 0 & 5 & -3 & k-4 \\ 0 & 15 & -9 & -9 \end{array}\right] \begin{array}{l} \\ \\ r_3' = r_3 - 3r_2 \end{array}$$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 0 & 5 & -3 & k-4 \\ 0 & 0 & 0 & 3-3k \end{array}\right] \; r_2' = r_2/5$$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 0 & 1 & -\frac{3}{5} & \frac{k-4}{5} \\ 0 & 0 & 0 & 3(1-k) \end{array}\right]$$

The final equation shows there is only a solution if $k = 1$. When $k = 1$ we have

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 2 \\ & & x_2 & - & \frac{3}{5}x_3 & = & -\frac{3}{5} \end{array}$$

so x_3 is a free variable: put $x_3 = \lambda$, say, to get

$$x_2 = \frac{3}{5}\lambda - \frac{3}{5}; \quad x_1 = 2 - \frac{3}{5} + \frac{3}{5}\lambda - 2\lambda = \frac{7}{5} - \frac{7}{5}\lambda.$$

General solution is

$$\boldsymbol{x} = \begin{bmatrix} \frac{7}{5} \\ -\frac{3}{5} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}.$$

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4 Use EROs to transform the coefficient matrix in

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 9 \\ 2x_1 & + & 4x_2 & - & 3x_3 & = & 1 \\ 3x_1 & + & 6x_2 & - & 5x_3 & = & 0 \end{array}$$

to row echelon form. Hence obtain the solution using back substitution (this process is called **Gaussian Elimination**). Continue EROs to obtain the coefficient matrix in reduced echelon form and hence obtain the solution (this additional work incorporates the back substitution into the matrix transformation, and is called **Gauss-Jordan Elimination**).

Augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \begin{array}{l} \\ r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 3r_1 \end{array}$$

$$\to \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \begin{array}{l} \\ \\ r_3' = r_3 - \frac{3}{2}r_2 \end{array}$$

$$\to \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right] \begin{array}{l} \\ r_2' = r_2/2 \\ r_3' = -2r_3 \end{array}$$

$$\to \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Back substitution:

$$x_3 = 3; \quad x_2 - \frac{7}{2}x_3 = -\frac{17}{2} \Rightarrow x_2 = 2; \quad x_1 + x_2 + 2x_3 = 9 \Rightarrow x_1 = 1$$

so

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Now continue the EROs:

$$\to \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} r_1' = r_1 - 2r_3 \\ r_2' = r_2 + \frac{7}{2}r_3 \\ \\ \end{array}$$

$$\to \left[\begin{array}{rrr|r} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} r_1' = r_1 - r_2 \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

so solution is

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Examples 1F

1 List all permutations of $S = \{1, 2, 3\}$.

If

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ j_1 & j_2 & j_3 \end{pmatrix},$$

then j_1 can take 3 values; for each choice of j_1, j_2 can take 2 values; for each choice of j_1 and j_2, j_3 can take only 1 value.

The 3! permutations are

$$(1, 2, 3), \quad (1, 3, 2), \quad (2, 1, 3), \quad (2, 3, 1), \quad (3, 1, 2), \quad (3, 2, 1).$$

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2 Determine the number of inversions in each of the following permutations:

(a) σ =(3,6,5,1,4,2), (b) σ =(4,2,1,3).

(a) No. of inversions = 2 + 4 + 3 + 0 + 1 = 10.

(b) No. of inversions = 3 + 1 + 0 = 4.

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3 Determine the sign of the following permutations:

(a) σ =(4,1,3,2), (b) σ =(3,6,1,5,4,2).

(a) No. of inversions = 3 + 0 + 1 = 4 so $\text{sgn}(\sigma) = +1$.

(b) No. of inversions = 2+4+0+2+1=9 so $\text{sgn}(\sigma) = -1$.

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4 For $n = 2$, list the set of products $a_{1\sigma(1)}\, a_{2\sigma(2)}$, the associated permutations, signs of permutations and signed products $\operatorname{sgn}(\sigma) a_{1\sigma(1)}\, a_{2\sigma(2)}$. Hence write down $\det(A)$. Repeat for $n = 3$.

$n = 2$:

Product	**Permutation**	**Sign**	**Signed Product**
$a_{11}\, a_{22}$	$(1,\ 2)$	$+1$	$a_{11}\, a_{22}$
$a_{12}\, a_{21}$	$(2,\ 1)$	-1	$-a_{12}\, a_{21}$

$$\Rightarrow\ \det(A) = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$n = 3$:

Product	**Permutation**	**Sign**	**Signed Product**
$a_{11}\, a_{22}\, a_{33}$	$(1, 2, 3)$	$+1$	$a_{11}\, a_{22}\, a_{33}$
$a_{11}\, a_{23}\, a_{32}$	$(1, 3, 2)$	-1	$-a_{11}\, a_{23}\, a_{32}$
$a_{12}\, a_{21}\, a_{33}$	$(2, 1, 3)$	-1	$-a_{12}\, a_{21}\, a_{33}$
$a_{12}\, a_{23}\, a_{31}$	$(2, 3, 1)$	$+1$	$a_{12}\, a_{23}\, a_{31}$
$a_{13}\, a_{21}\, a_{32}$	$(3, 1, 2)$	$+1$	$a_{13}\, a_{21}\, a_{32}$
$a_{13}\, a_{22}\, a_{31}$	$(3, 2, 1)$	-1	$-a_{13}\, a_{22}\, a_{31}$

$$\Rightarrow\ \det(A) = a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{22}\, a_{31}.$$

Examples 1G

1 Identify the minor and cofactor of a_{32} where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 5 \\ 2 & -1 & 5 \end{bmatrix}.$$

$M_{32} = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$ so minor of $a_{32} = \det(M_{32}) = \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = 5 + 3 = 8.$

Cofactor of $a_{32} = (-1)^{3+2} \det(M_{32}) = -8.$

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2 If

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & -3 \\ 5 & 4 & -2 \end{bmatrix},$$

find $\det(A)$ by cofactor expansion along row 2 and also along column 3.

$$\textbf{Row 2}:\ \det(A) = 2\,(-1)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 4\,(+1)\begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix} + (-3)\,(-1)\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= -2\,(-2) + 4\,(-6) + 3\,(12-5) = 4 - 24 + 21 = 1$$

$$\textbf{Col 3}:\ \det(A) = 0\,(+1)\begin{vmatrix} 2 & 4 \\ 5 & 4 \end{vmatrix} + (-3)\,(-1)\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} + (-2)\,(+1)\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= 0 + 3\,(12-5) - 2\,(12-2) = 21 - 20 = 1$$

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3 Find $\det(A)$ if

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Expand along row 1:

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22}a_{33}a_{44}$$

$\Rightarrow \det(A) =$ product of elements on the main diagonal.

Note: This result holds for lower triangular (and upper triangular) matrices of any order. The strategy is to expand along the row (or column) containing at most one non-zero element.

Examples 1H

1 For $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$, verify that

(i) $\det(A+B) \neq \det(A) + \det(B)$ (ii) $\det(AB) = \det(A)\det(B)$.

$$A + B = \begin{bmatrix} 2 & 4 \\ 7 & 9 \end{bmatrix}, \qquad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

so

$$\det(A) = 3 - 2 = 1, \quad \det(B) = -8 - 15 = -23,$$

$$\det(A+B) = 18 - 28 = -10, \quad \det(AB) = 28 - 51 = -23.$$

Hence results.

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2 Find $\det(A)$ for the following matrices (using EROs to simplify if necessary):

$$(a)\, A = \begin{bmatrix} 5 & 1 & 6 \\ 0 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix} \quad (b)\, A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 5 & 8 & 9 \end{bmatrix} \quad (c)\, A = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix}.$$

(a) $\det(A) = -0 \begin{vmatrix} 1 & 6 \\ 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} - 0 \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} = 0.$

(b) $\det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 5 & 8 & 9 \end{vmatrix} \begin{matrix} r_1' = r_1 - r_2 \\ \\ \\ \end{matrix} = \begin{vmatrix} 0 & 0 & 0 \\ 2 & -1 & 3 \\ 5 & 8 & 9 \end{vmatrix} = 0.$

(c) $\det(A) = \begin{vmatrix} 3 & 1 & 5 \\ 1 & 2 & 3 \\ 1 & -1 & 4 \end{vmatrix} \begin{matrix} r_1' = r_3 \\ \\ r_3' = r_1 \end{matrix} = -\begin{vmatrix} 1 & -1 & 4 \\ 1 & 2 & 3 \\ 3 & 1 & 5 \end{vmatrix} \begin{matrix} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - 3r_1 \end{matrix} = -\begin{vmatrix} 1 & -1 & 4 \\ 0 & 3 & -1 \\ 0 & 4 & -7 \end{vmatrix}$

$= -\begin{vmatrix} 3 & -1 \\ 4 & -7 \end{vmatrix} = -(-21 + 4) = 17.$

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3 Use row and column operations to find $\det(A)$ when $A = \begin{bmatrix} \lambda & x & x \\ x & \lambda & x \\ x & x & \lambda \end{bmatrix}$.

$$\det(A) = \begin{vmatrix} \lambda & x & x \\ x & \lambda & x \\ x & x & \lambda \end{vmatrix} \begin{matrix} c_1' = c_1 + c_2 \\ \\ \\ \end{matrix} = \begin{vmatrix} \lambda + x & x & x \\ x + \lambda & \lambda & x \\ 2x & x & \lambda \end{vmatrix} \begin{matrix} c_1' = c_1 + c_3 \\ \\ \\ \end{matrix}$$

$$= \begin{vmatrix} \lambda + 2x & x & x \\ \lambda + 2x & \lambda & x \\ \lambda + 2x & x & \lambda \end{vmatrix} \begin{matrix} c_1' = c_1/(\lambda + 2x) \\ \\ \\ \end{matrix} = (\lambda + 2x) \begin{vmatrix} 1 & x & x \\ 1 & \lambda & x \\ 1 & x & \lambda \end{vmatrix} \begin{matrix} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{matrix}$$

$$= (\lambda + 2x) \begin{vmatrix} 1 & x & x \\ 0 & \lambda - x & 0 \\ 0 & 0 & \lambda - x \end{vmatrix} = (\lambda + 2x)(\lambda - x)^2.$$
