DEPARTMENT OF MATHEMATICS & STATISTICS
MM102 APPLICATIONS OF CALCULUS
Ordinary Differential Equations: Exercise Sheet

In all questions solve the given differential equation subject to any given initial or boundary conditions.

First order equations

1. Separable Equations

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x(y+1)^2$ **(b)** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^3 \cos^2 y$ **(c)** $x^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} = y - y^2$

(d) $x^2(y+1) + y^2(x-1)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ **(e)** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{1+y}$

(f) $x \tan y + (x^2+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

(g) $xy^3 \dfrac{\mathrm{d}y}{\mathrm{d}x} = (1+x^2)(1+y^2)$, if $y = 1$ when $x = 1$

2. First order linear equations

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{2}{x}y = 8x$ **(b)** $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 4x$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y \cot x + \sin 2x = 0$

(d) $x(x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2x$ **(e)** $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = \sin x$

(f) $(1-x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} - xy = 3$, if $y = 1$ when $x = 0$

(g) $x\,y' + 2y = x^2$, if $y(1) = 0$

(h) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin x - 2y$, if $y = 0$ when $x = \dfrac{\pi}{2}$

(i) $x(x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2$, if $y \to 1$ when $x \to \infty$.

3. Homogeneous equations

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$ **(b)** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+y}{x-y}$ **(c)** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y(x+y)}{x(x-y)}$

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2y}{x} + \dfrac{x}{y}$, if $y = 2$ when $x = 1$.

4. Autonomous equations

(a) Draw a graph of the function $f(y) = y(1-y)(y-2)$.

(b) From this graph determine the equilibrium points of the autonomous ODE $\dfrac{\mathrm{d}y}{\mathrm{d}t} = f(y)$.

(c) Determine whether the equilibrium solutions are stable or unstable.

Second order equations

5. Second order linear with constant coefficients: homogeneous

(a) $y'' - y' - 6y = 0$

(b) $y'' + 4y' + 3y = 0$

(c) $y'' - 4y' + 4y = 0$

(d) $y'' - 2y' + 17y = 0$

(e) $y'' + 8y' + 16y = 0, \quad \text{if } y = 1 \text{ at } x = 0 \text{ and } y = 0 \text{ at } x = 1$

(f) $y'' + 2y' = 0, \quad \text{if } y' = 2,\ y = 1 \text{ at } x = 0$

(g) $y'' + 9y = 0, \quad \text{if } y = 1 \text{ at } x = 0 \text{ and } y' = 6 \text{ at } x = \dfrac{\pi}{3}.$

6. Second order linear with constant coefficients: inhomogeneous

(a) $y'' - 4y' + 3y = 1$

(b) $y'' + 2y' + y = x^2$

(c) $y'' + 6y' + 9y = \mathrm{e}^{-x}$

(d) $y'' + 2y' + 2y = 17\mathrm{e}^{3x}$

(e) $y'' + y' + y = \cos x + \sin x$

(f) $y'' - 2y' + 5y = \sin(2x)$

(g) $y'' - 6y' + 25y = 50x + 13 + 16\mathrm{e}^{-x}$

(h) $y'' + 4y' + 13y = 52 + 12\sin x + 4\cos x$

(i) $y'' + 4\,y' + 3\,y = 13\,\cos(2x)\,.$

7. **(a)** $y'' + y' - 6y = \mathrm{e}^{2x}$

(b) $y'' + 6y' + 9y = 4\mathrm{e}^{-3x}$

(c) $y'' + 25y = 20\cos(5x)$

(d) $y'' + y' = 1$

(e) $y'' - y = \mathrm{e}^{x} + \dfrac{x}{2}, \quad \text{if } y = 0 \text{ when } x = 0 \text{ and when } x = 1$

(f) $y'' - 9y = 12\cosh(3x), \quad \text{if } y = 0 \text{ and } y' = 0 \text{ when } x = 0.$

8. Determine the Complementary Function of the second-order linear ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y \;=\; 65\mathrm{e}^{3x}\cos x. \qquad (*)$$

Find a complex Particular Integral for the ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y \;=\; 65\mathrm{e}^{(3+i)x},$$

where x is a real variable. Hence, determine the (real) General Solution of (*).

9. Determine the Complementary Function of the second-order linear ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y \;=\; -195\mathrm{e}^x \sin x. \qquad (**)$$

Find a complex Particular Integral for the ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y \;=\; 195\mathrm{e}^{(1-i)x},$$

where x is a real variable. Hence, determine the (real) General Solution of (**).

<u>Miscellaneous</u>

In the following you must first identify the type of ODE, then adopt an appropriate method in order to obtain the General and Particular Solutions.

10. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{2x} - \dfrac{x}{2y}$, if $y = 1$ when $x = 1$.

11. $y' + 3x^2y = \exp(-x^3)$, if $y'(1) = 0$.

12. $y'' - 3y' = \mathrm{e}^{3x} - 2y$, if $y = 0$ when $x = 0$ and when $x = \ln 2$.

13. $x = \mathrm{e}^{x+y}y'$, if $y \to 0$ as $x \to \infty$.

14. $y'' + 2y' - 10 = \sin(3x) - 5y$, if $y = y' = 0$ when $x = 0$.

15. $xy' - y = 2x^2\cos^2(2x)$, if $y = \dfrac{\pi^2}{4}$ when $x = \dfrac{\pi}{2}$.

16. $y' + \dfrac{y}{x} = \dfrac{x^2}{y^2}$, if $y = 2$ when $x = 1$.

17. $y'' + y = x^2 + \sin x$, if $y = 0$ when $x = 0$ and $y = -2$ when $x = \dfrac{\pi}{2}$.