UNIVERSITY OF STRATHCLYDE

MATHEMATICS & STATISTICS

MM103 Part II: Applications

0. Intoduction to mathematical modelling

This term describes any attempt to describe and/or simulate economic, physical, chemical, medical, biological etc. phenomena using mathematics. It usually involves three main stages.

- We use what we know about the phenomena to be modelled to write down an equation (or equations) describing its behaviour. These equations and any data form the mathematical model.
- We then 'solve' this equation by using purely mathematical reasoning.
- Next we interpret the solution we get in terms of the phenomena we are trying to model. If it doesn't give good agreement with the real world, then we go back, refine the model and try again.

MATHEMATICAL MODELLING:

$$\boxed{\text{Design model}} \longrightarrow \boxed{\text{Solve equations}}$$

$$\nwarrow \qquad \swarrow$$

$$\boxed{\text{Compare with reality}}$$

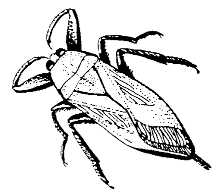
Why model anything mathematically?

Many reasons, for example:

- To simulate things that cannot be tested experimentally. E.g.?
- To simulate things that are dangerous and/or expensive to do in real life experiments. E.g.?
- To gain understanding of how things work and which factors are important. E.g.?
- To help in the design of products and optimize performance of processes. E.g.?
- ...
- ...

In this half of the MM103 class we shall look at a number of different real–world situations that can be modelled by equations that you may already know how to solve. In the second and future years of your degree you will encounter even more complicated and realistic models. We'll begin by looking at predicting insect population growth.

1. How many bugs will there be?



Suppose that we know the number of insects of a certain type that live around a pond. The problem is to predict the size of the insect population at the pond in future years.

1.1. First some biology

We shall simplify things as much as possible, just concentrating on the most important features.

Assumptions:

- no insects can enter or leave the region around the pond (the colony is *isolated*);
- the proportion of males to females stays constant, so we can just look at how many female insects there are;
- there is a single breeding season in the summer of each year. Eggs hatch out in early spring, and the insects become adults in the late summer of that year. They then breed, the females lay their eggs and die in the autumn.

1.2. A mathematical description

- Suppose that at the start there are x_0 female insects at the pond, and let x_1, x_2, x_3,..., x_n be the numbers of females at the start of the breeding season after 1, 2, 3,..., n years.
- On average r of the female eggs laid by each female survive to adulthood and breed the following year.
- Because all the adult insects die every year, the number of females alive in one year will be equal to r times the number of females that were alive the previous year.

> if x_n is the number of females in year n, and
> r is the average number of female offspring of each female, then
>
> $$x_{n+1} = r\,x_n \qquad (1.1)$$

Equation (1.1) is our first mathematical model. We next need to solve it.

1.3. Solving equation (1.1)

We first assume that r is a constant: i.e. it is the same every year, and doesn't depend on the size of the insect population. We want to 'solve' the equation

$$x_{n+1} = r\,x_n, \quad r = \text{constant}$$

to find x_n in terms of n, r, and the initial population x_0.

Example: if $r = 2$ and $x_0 = 3$ million we calculate

$$\begin{array}{lllllllll} n=0: & x_1 & = & r\,x_0 & = & 2\times 3 & = & 6 & \text{million} \\ n=1: & x_2 & = & r\,x_1 & = & 2\times 6 & = & 12 & \text{million} \\ n=2: & x_3 & = & r\,x_2 & = & 2\times 12 & = & 24 & \text{million.} \end{array}$$

We now look for a solution of equation (1.1) which is valid for all r and x_0. Notice that

$$\begin{array}{lllllll} x_1 & = & r\,x_0 & & & & \\ x_2 & = & r\,x_1 & = & r\,(r\,x_0) & = & r^2\,x_0 \\ x_3 & = & r\,x_2 & = & r\,(r^2\,x_0) & = & r^3\,x_0. \end{array}$$

A sensible guess would be

$$x_m = r^m\,x_0, \quad \text{for } m = 1, 2, \ldots$$

We can 'prove' that this is in fact a solution by showing that it fits into our equation $x_{n+1} = r\,x_n$.

If $x_m = r^m\,x_0$, for $m = 1, 2, \ldots$ then for any n we have $x_n = r^n\,x_0$ and $x_{n+1} = r^{n+1}\,x_0$. That is

$$x_{n+1} = r\,(r^n\,x_0) = r\,x_n$$

which is our equation. It also gives the right starting value because $r^0 = 1$.

> The equation $x_{n+1} = r\,x_n$ has the solution $\underline{x_n = r^n\,x_0}$

This formula means that we can now easily calculate the size of the population predicted by the model, provided that we know r and x_0.

Example: If $x_0 = 1.5$ (million) and $r = 2$, then find x_6.

From the formula:

$$x_6 = r^6\, x_0 = 2^6 \times 1.5 = 64 \times 1.5 = 96.$$

What happens to the population size for different values of r?

1. $\boldsymbol{r = 1}$. If $r = 1$ then $r^n = 1$ for any n, and so the population size doesn't change: it is always equal to x_0.
2. $\boldsymbol{r > 1}$. In this case the population size is multiplied by a number bigger than 1 each year, and so the population grows every year.
3. $\boldsymbol{r < 1}$. In this case the population size is multiplied by a number less than 1 every year, and so it shrinks. After long enough it will die out completely.

Exercises 1.1

1. What numbers are suitable choices for r, i.e. can r be negative/ a fraction/ an irrational number (like π)?
2. By looking at the model equation show that the starting population x_0 is not important in predicting what will happen to the population size, but that population trends are completely governed by the size of r.
3. Do you think that the model produces physically realistic answers? Why?

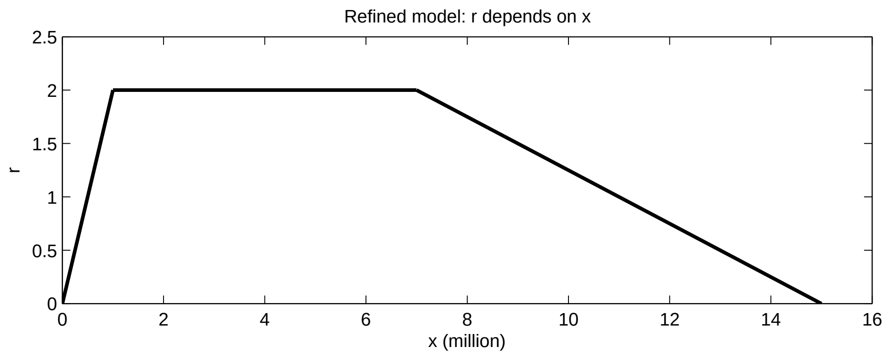
1.4. Refined model: r is not constant

Our first model equation (1.1) relied on the assumption that the average number of surviving female offspring r is the same every year. This is not very realistic, and in general r will depend on a number of things, including the number x of adult females.

- We have seen that if r is constant and less than 1 then the population eventually dies out completely. So if the species is to survive at all there must be some range of values of x for which $r > 1$.
- If the population is very large then we would expect r to be less than 1, as there will be a lot of competition for food and other resources and only those insects which can find enough food will be able to breed successfully.
- We would also expect r to be less than 1 if x is too small, because many females will not be able to find mates.

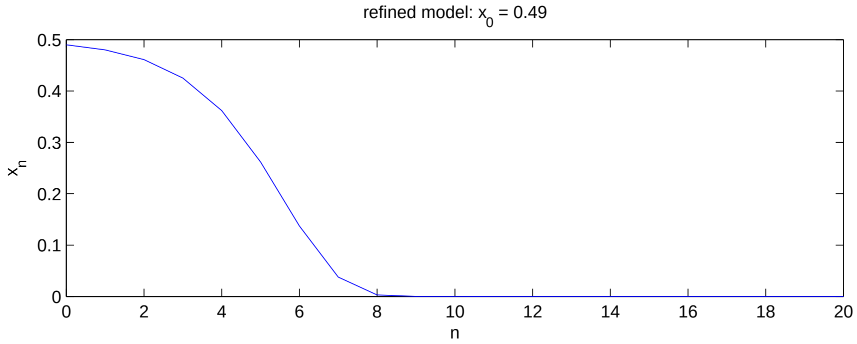
REQUIREMENTS FOR r:
$r > 1$ for some values of x (species survival)
$r < 1$ if x is too big (lack of food etc.)
$r < 1$ if x is too small (few females breed)

The following graph shows a simple model that has all these features. In this, r is constant (equal to 2) when x lies between 1 and 7 million, and decreases to zero if x is less than 1 million or greater than 7 million. If x is bigger than 15 million then r is zero.

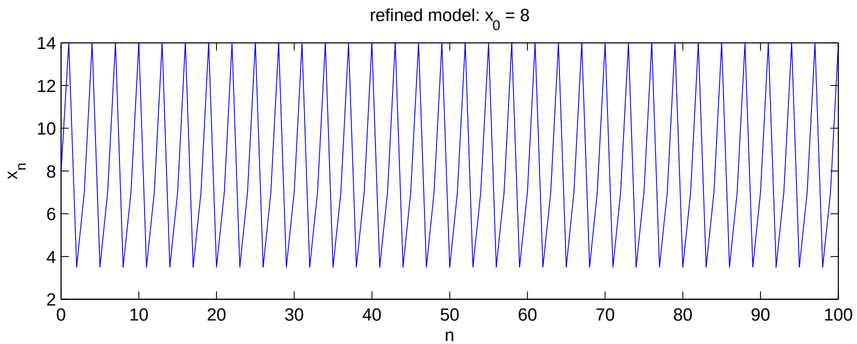


Let's test out the behaviour of this new model for different values of x_0.

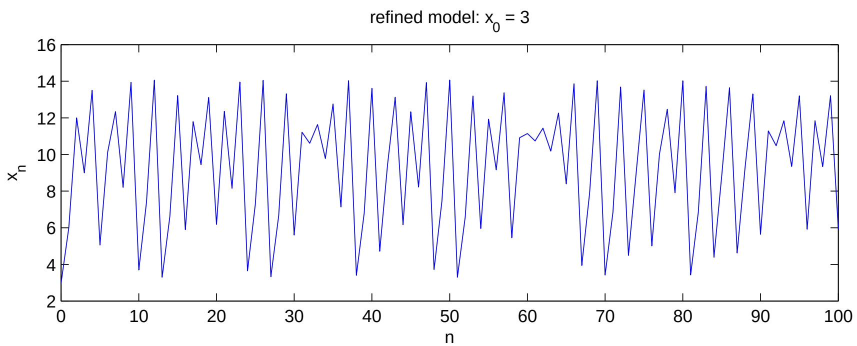
• If $x_0 < 0.5$ then the population dies out quickly:



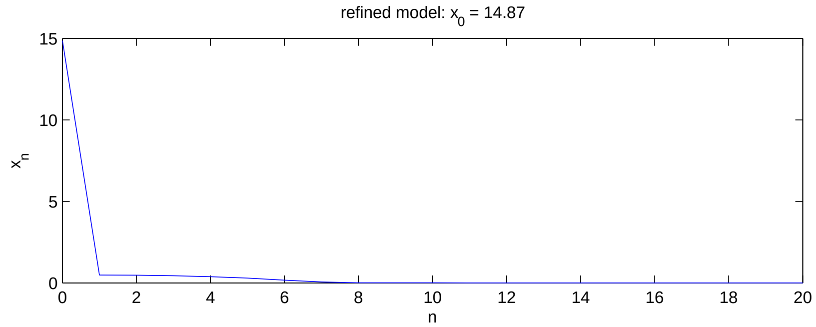
• If $x_0 = 2^p$ for $p = 0 : 3$ then the population oscillates:



• Most other values of x_0 between 0.5 and 14.86 give more complicated behaviour:



• If $x_0 > 14.87$ then the population dies out:



Exercises 1.2

1. Does this model look more or less realistic than the previous one?

2. What other features do you think a good model should take into account?

1.5. Further reading

The book *Mathematical Ideas in Biology* by J. Maynard Smith contains a lot of (more advanced) material on mathematical modelling in biology. We shall look at ODE population models in Section 6.

2. Fibonacci's rabbits and other equations

Leonardo of Pisa, known as Fibonacci, wrote a book *Liber abaci* in 1202. One problem is the growth of a (biologically unrealistic!) population of rabbits:

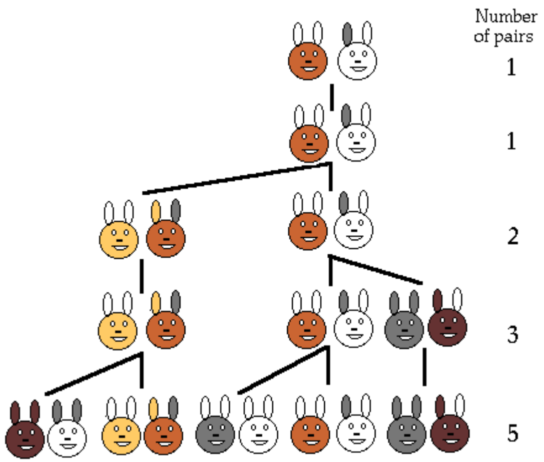
> *A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?*

This problem and its underlying mathematical equation are very common in mathematics articles (e.g. Google lists over three million hits for the word "Fibonacci"). A good reference for Fibonacci (the person) is the St Andrews history of mathematics web site `www-history.mcs.st-and.ac.uk/Biographies/`

`Fibonacci.html` and the following description of the problem is adapted from "Fibonacci numbers and nature" at `www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html`.

What equation governs the number of pairs?

- At time zero, there is one pair of rabbits.
- At the end of the first month the original pair of rabbits mate, but there is still one only 1 pair.
- At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
- At the end of the fourth month, the original female has produced yet another new pair, and the female born two months ago produces her first pair, making 5 pairs in total.



- At the end of the nth month, the number of new pairs of rabbits is equal to the number of pairs in month $n-2$, and the total number of pairs is this plus the number of rabbits alive in month $n-1$. So, if x_n is the number of pairs at the end of month n and $n \geq 2$, then $x_n = x_{n-1} + x_{n-2}$.

It is easier if we look at the equation for x_{n+2} rather than x_n, and this is:

$$x_{n+2} = x_{n+1} + x_n\,, \tag{2.1}$$

where we also know that $x_0 = 1$. In this notation, Fibonacci's original question was: what is x_{12}?

Observation: We don't have enough information to find x_{12}!

In fact, we don't even have enough information to find x_2: if we substitute $n = 0$ into (2.1), then this gives $x_2 = x_1 + x_0 = x_1 + 1$ because $x_0 = 1$. So we cannot find x_2 (or any x_n with $n > 2$) without also specifying x_1.

Why is this? We only needed to specify x_0 in order to solve equation (1.1), $x_{n+1} = r\,x_n$, in the previous section. This is because (1.1) only involves the population size x at two time levels (x_{n+1} is given in terms of **one** past value), but equation (2.1) involves the population size at three time levels (x_{n+2} is given in terms of **two** past values). So for (2.1) we need to specify the first **two** values, i.e. x_0 and x_1, and we saw earlier that both of these are equal to 1.

Definition. The Fibonacci sequence is

$$x_{n+2} = x_{n+1} + x_n \quad \text{for } n \geq 0,$$

with $x_0 = 1$ and $x_1 = 1$.

We can now calculate x_{12}.

- Set $n = 0$ in (2.1) and use $x_0 = 1$ and $x_1 = 1$ to get $x_2 = 1 + 1 = 2$.
- Set $n = 1$ in (2.1) to get $x_3 = x_2 + x_1 = 2 + 1 = 3$.
- Set $n = 2$ in (2.1) to get $x_4 = x_3 + x_2 = 3 + 2 = 5$.
- ...
- Set $n = 10$ in (2.1) to get $x_{12} = x_{11} + x_{10} = 144 + 89 = 233$.

The sequence x_n for $n = 0{:}12$ is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 – notice that each entry is the sum of the previous two. Although we have managed to find x_{12} by direct calculation, it is a fairly tedious way to do it, and would not be a very practical way to find x_{100}, say. Fortunately there is a much better way to solve equations like (2.1), and we shall look at the solution in general before going back to (2.1).

2.1. Solving 3-step difference equations

Equations like (1.1) and (2.1) in which the unknown x is labelled by an integer are called **difference equations**.

Both (1.1) and (2.1) are **linear** equations, because a constant multiple of a solution is also a solution, and the sum of any two solutions is also a solution.

Example 2.1

1. Suppose that the sequence y_n solves (2.1). Show that $c\,y_n$ is also a solution for any constant c.
2. Suppose that the sequences y_n and z_n both solve (2.1). Show that $u_n = y_n + z_n$ is also a solution.

We want to solve equations of the form $a\,x_{n+2} + b\,x_{n+1} + c\,x_n = 0$ (where a, b and c are constants), and we motivate the method by going back to (1.1).

Equation (1.1) is $x_{n+1} = r\,x_n$, and we originally solved it by constructing the solution. We could instead have looked for a solution of the form $x_n = A\,\lambda^n$ by substituting this into (1.1) and looking for a value of λ that "works".

If $x_n = A\,\lambda^n$ for any n, then $x_{n+1} = A\,\lambda^{n+1}$ and (1.1) gives

$$A\,\lambda^{n+1} = r\,A\,\lambda^n$$

and cancelling out A and λ^n from both sides gives $\lambda = r$, so the **general solution (GS)** of (1.1) is $x_n = A\,r^n$, where A is a constant. (It satisfies the equation (1.1) for any value of A.)

The **particular solution (PS)** is the solution of (1.1) that has a particular value when $n = 0$. For example, if we know that $x_0 = 2.3$, then substituting $n = 0$ in the GS gives

$$2.3 = x_0 = A\,r^0 = A\,1 = A\,,$$

so the particular solution in this case is $x_n = 2.3\,r^n$.

We do very much the same thing for a 3-step equation like (2.1), first substituting in $x_n = A\,\lambda^n$ and looking for values of λ that "work", to find the GS. We see below that λ for a 3-step equation is the root of a quadratic equation, and so there are typically two different values of λ that work, and the GS involves both of them, each multiplied by a constant.

We first illustrate the method for an equation for which the calculations are a bit easier than for the Fibonacci equation (2.1).

Consider the 3-step equation

$$x_{n+2} - x_{n+1} - 6\,x_n = 0 \tag{2.2}$$

with

$$x_0 = 3 \quad \text{and} \quad x_1 = -1\,. \tag{2.3}$$

We first find the general solution (GS) of (2.2) and then find the particular solution (PS) which also satisfies (2.3).

Method:

- First set $x_n = A\,\lambda^n$ in (2.2), $x_{n+2} - x_{n+1} - 6\,x_n = 0$:
$$A\,\lambda^{n+2} - A\,\lambda^{n+1} - 6\,A\,\lambda^n = 0\,.$$
- Divide this equation through by A and λ^n to get:
$$\lambda^2 - \lambda - 6 = 0\,.$$
- Solve the above equation for λ. It is a quadratic equation and has **two** roots:
$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = -2\,.$$
- The **general solution** of (2.2) is
$$x_n = A\,\lambda_1^n + B\,\lambda_2^n = A\,3^n + B\,(-2)^n\,, \tag{2.4}$$
where A and B are constants.

- The final stage is to use the values $x_0 = 3$ and $x_1 = -1$ given in (2.3) to find A and B to obtain the particular solution.

  Set $n = 0$ in (2.4) to get $3 = A\,(3)^0 + B\,(-2)^0 = A + B$.

  Set $n = 1$ in (2.4) to get $-1 = A\,(3)^1 + B\,(-2)^1 = 3\,A - 2\,B$.

  The solution to the pair of simultaneous equations

$$\begin{aligned} A + B &= 3 \\ 3\,A - 2\,B &= -1 \end{aligned}$$

  is $A = 1$ and $B = 2$. Hence the PS of (2.2)–(2.3) is

$$x_n = 3^n + 2\,(-2)^n .$$

Note that $x_n \to \infty$ as $n \to \infty$ because the dominant term in the solution is 3^n.

Quadratic roots formula: Recall that the two roots of the quadratic equation $a\,\lambda^2 + b\,\lambda + c = 0$ are

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

Example 2.2

1. Find the general solution of $3\,x_{n+2} - 7\,x_{n+1} + 2\,x_n = 0$. Find the particular solution when:

   (a) $x_0 = -2$ and $x_1 = 1$;

   (b) $x_0 = 3/5$ and $x_1 = 1/5$.

2. Find the general solution of $9\,x_{n+2} - x_n = 0$.

How do these solutions behave for large n?

2.2. Solving the Fibonacci equation (2.1)

We now apply the above method to solve the Fibonacci problem. We first find the GS of (2.1) in terms of two constants. The PS is the solution for which $x_0 = 1$ and $x_1 = 1$.

Solution:

- First set $x_n = A\,\lambda^n$ in (2.1), $x_{n+2} = x_{n+1} + x_n$:

$$A\,\lambda^{n+2} = A\,\lambda^{n+1} + A\,\lambda^n .$$

- Divide this equation through by A and λ^n to get:

$$\lambda^2 - \lambda - 1 = 0 .$$

- Solve the above equation for λ to obtain the two roots:

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$$

- The **general solution** of (2.1) is

$$x_n = A\,\lambda_+^n + B\,\lambda_-^n , \tag{2.5}$$

  where A and B are constants.

- The final stage is to use the values $x_0 = 1$ and $x_1 = 1$ to find A and B to obtain the particular solution.

  Set $n = 0$ in (2.5) to get $1 = x_0 = A\,\lambda_+^0 + B\,\lambda_-^0 = A + B$.

  Set $n = 1$ in (2.5) to get $1 = x_1 = A\,\lambda_+^1 + B\,\lambda_-^1 = A\,\lambda_+ + B\,\lambda_-$.

  That is, A and B solve the pair of simultaneous equations:

$$\begin{aligned} A + B &= 1 \\ A\,\lambda_+ + B\,\lambda_- &= 1 \end{aligned}$$

  The solution is $A = (5 + \sqrt{5})/10$ and $B = (5 - \sqrt{5})/10$.

This calculation shows that the **particular solution** of (2.1) (i.e. the solution x_n of the equation which also satisfies $x_0 = 1$ and $x_1 = 1$) is

$$x_n = \left(\frac{5 + \sqrt{5}}{10}\right) \lambda_+^n + \left(\frac{5 - \sqrt{5}}{10}\right) \lambda_-^n ,$$

where $\lambda_\pm = \left(1 \pm \sqrt{5}\right)/2$.

Verification:

We use this formula to compute x_n for $n = 0 : 20$. The results are output below.

n	x
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233
13	377
14	610
15	987
16	1597
17	2584
18	4181
19	6765
20	10946

Observations

1. The formula looks to be giving the correct answers.

2. The multipliers in equation (2.1) and the two starting conditions x_0 and x_1 are all integers (they're actually all equal to 1), and yet the formula involves $\sqrt{5}$! This is a good illustration that you often need much more sophisticated mathematics to solve a problem than to state it. (Fermat's Last Theorem is an even more striking example of this.)

3. We have $\lambda_+ = \left(1+\sqrt{5}\right)/2 \approx 1.6180$ and $\lambda_- = \left(1-\sqrt{5}\right)/2 \approx -0.6180$. For large n, λ_+^n will grow unbounded, while λ_-^n will go to zero, so the ratio x_{n+1}/x_n will be close to λ_+ when n is large. E.g. to 16 decimal places:

$$\frac{x_{20}}{x_{19}} = \frac{10946}{6765} = 1.618033998521803\,, \quad \lambda_+ = 1.618033988749895\,.$$

2.3. Solving 3-step difference equations with repeated roots

One case we have not yet considered is what happens when there is only one solution λ of the quadratic equation (i.e. the quadratic has repeated roots).

Example: Find the general solution of $x_{n+2} - 4\,x_{n+1} + 4\,x_n = 0$. Find the particular solution when $x_0 = -1$ and $x_1 = 6$.

Proceeding as before, we look for a GS of the form $x_n = A\,\lambda^n$. After cancellation this gives

$$\lambda^2 - 4\,\lambda + 4 = 0\,,$$

and the solution is $\lambda_\pm = 2$, (the quadratic has two equal roots). In this case we cannot write the GS (which must involve two distinct constants) as $x_n = A\,2^n + B\,2^n$, because this can be rewritten as $C\,2^n$ for $C = A + B$, and so it really only involves one constant. Instead the GS is $x_n = A\,2^n + n\,B\,2^n$ – i.e. one of the terms must be multiplied by n. (Exercise: show that this solves the equation.)

For the PS, setting $n = 0$ gives $A = -1$, and $n = 1$ gives $B = 4$, so the PS is $x_n = (4\,n - 1)\,2^n$.

In theory any linear difference equation in which the terms x_m are just multiplied by constants can be solved by first looking for a solution of the form $x_n = A\,\lambda^n$, but if the equation involves x_n and x_{n+p} for $p > 2$, then λ will satisfy a cubic or higher polynomial, and it might not be possible to calculate it exactly.

Exercises 2

1. Find the general solution of $x_{n+1} = 4\,x_n$.
2. Find the general solution of $2\,x_{n+1} + x_n = 0$. Find the particular solution when $x_0 = 3$.
3. Find the particular solution of $2\,x_{n+1} = x_n$ which satisfies $x_0 = -6$.
4. Find the general solution of $x_{n+2} - 3\,x_{n+1} + 2\,x_n = 0$.
5. Find the general solution of $8\,x_{n+2} - 6\,x_{n+1} + x_n = 0$. Find the particular solution when $x_0 = 2$ and $x_1 = 3/4$.
6. Find the particular solution of $x_{n+2} - x_n = 0$ for which $x_0 = 5$ and $x_1 = 1$.
7. Find the general solution of $4\,x_{n+2} - 4\,x_{n+1} + x_n = 0$.
8. Find the general solution of $x_{n+2} - 2\,x_{n+1} + x_n = 0$. Find the particular solution when $x_0 = -2$ and $x_1 = 3$.
9. Find the general solution of $x_{n+3} - 2\,x_{n+2} - x_{n+1} + 2\,x_n = 0$.

In each case describe how the solution x_n behaves when n is large.

3. Some like it hot...

We begin this section with a practical problem.

Problem: You are making a cup of tea to drink in 5 minutes time. Should you add the (refrigerated) milk now or just before you drink the tea, if you want the tea to lose as little heat as possible?

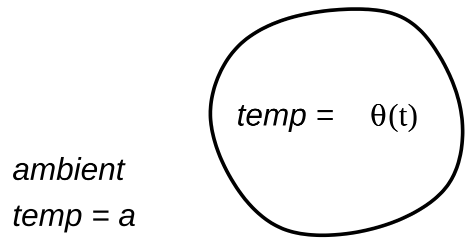
In order to answer this question we need a **relevant equation** and **some data**.

3.1. Newton's law of cooling

This states that *the rate at which the temperature of a hot object decreases with respect to time is proportional to the difference between the object's temperature and that of the surrounding medium.*

How do we write Newton's law of cooling mathematically?

Notation: Let t denote time, and suppose that the object's temperature at time t is $\theta(t)$, and it is in a medium with fixed ambient (background) temperature a, as illustrated below.



The rate of change of the object's temperature with respect to time is $\dfrac{d\theta}{dt}$. Newton's law of cooling says that

$$\frac{d\theta}{dt} = -k\,(\theta - a) \tag{3.1}$$

for some positive constant (of proportionality) k.

Equation (3.1) is a first order separable ordinary differential equation (ODE), and so it can be solved by **separating and integrating**.

$$\int \frac{1}{\theta - a}\,d\theta = -\int k\,dt\,,$$

which gives

$$\ln(\theta - a) = -k\,t + C$$

where C is an arbitrary constant of integration. Taking the exponential of both sides gives

$$\theta - a = \exp(-k\,t + C) = A\,e^{-k\,t},$$

where $A = e^C$ (and so A is also an arbitrary constant, but it must be positive). This gives the temperature $\theta(t)$ as

$$\theta = a + A\,e^{-k\,t},$$

in terms of the arbitrary constant A. We will first work through some examples based on (3.1) before returning to the cup of tea problem. The first two examples ask for the temperature of an object in terms of the Newton's law of cooling constant of proportionality k, when both its initial temperature $\theta(0)$ and a are given.

Example 3.1 A volcano throws out a rock which is initially at 500^{o}C into a wintry landscape at temperature 0^{o}C. Beginning with Newton's law of cooling (3.1), find the temperature of the rock after t minutes in terms of k.

Example 3.2 A frozen pie at temperature -20^oC is placed in an oven at temperature 180^{o}C. Beginning with Newton's law of cooling (3.1), find the temperature of the pie after t minutes in terms of k.

If the temperature of an object is known at two different times (e.g. at time $t = 0$ and one other time), then k can be found, and its temperature can then be calculated at any time t.

Example 3.3 Suppose that the volcanic rock in the example above has temperature 400^{o}C at time $t = 2$ minutes. Use this to find k for this problem, and calculate the rock's temperature at time $t = 8$ minutes.

Example 3.4 Suppose that the frozen pie in the example above has temperature 30^{o}C at time $t = 10$ minutes. Use this to find k for this problem, and

calculate the pie's temperature at time $t = 20$ minutes. If the pie needs to cook at a temperature of at least 80^{o}C for 30 minutes, then how long will it need to stay in the oven (to the nearest minute)?

Now that we have some experience of using Newton's law of cooling (NLC) we can go back to look at the best "milk strategy" for the cup of tea problem. We need one more piece of information first though – if we mix together two liquids at two different temperatures, then what is the temperature of the mixed liquid?

Liquid mixing. Suppose that a volume V_1 of liquid at temperature θ_1 is instantaneously mixed with a volume V_2 of the same type of liquid at temperature θ_2. Then the temperature θ of the resulting mixture is

$$\theta = \frac{V_1\,\theta_1 + V_2\,\theta_2}{V_1 + V_2}\,. \tag{3.2}$$

Example 3.5

1. Suppose that 0.2 litres of water at 100^{o}C is added to 1 litre of water at 25^{o}C. What is the resulting temperature?
2. What volume of water at 85^{o}C needs to be added to 1 litre of water at 10^{o}C to obtain water at 40^{o}C?

Example 3.6 **Cup of tea problem**

Suppose that a cup of tea of volume V_T is made at an initial temperature of θ_T in a room of temperature a, and that a (smaller) volume V_M of milk at temperature θ_M is to be added to the tea before it is drunk, where $\theta_M \le a < \theta_T$. Beginning with Newton's law of cooling

$$\frac{d\theta}{dt} = -k\,(\theta - a)$$

calculate the temperature of the tea/milk mixture at time $t > 0$ in two situations:

1. The tea is allowed to cool and the milk is added at time t. Call this temperature $\theta_1(t)$.
2. The milk is added at time $t = 0$, and the tea/milk mixture then cools. Call this temperature $\theta_2(t)$.

Which of these two temperatures $\theta_1(t)$ and $\theta_2(t)$ is bigger, and how does their difference depend on the sizes of the various temperatures and volumes? (Assume that k has the same value in both cases.)

3.2. Other rate of change problems

Many other physical situations can be modelled as rate of change problems, and the plan of attack is usually the same:

- Convert the word description of the physical rate of change problem into an ODE.
- Separate and integrate to find the general solution of the ODE (the general solution of a first order ODE involves **one** arbitrary constant).
- Use any initial or other conditions to find the specific solution to the problem.

Example 3.7 A factory which manufactures industrial fume cupboards wants to test how efficiently they work. A chemical is released into one of the fume cupboards at an initial concentration of 160 parts per million (ppm), and the time rate of change of its concentration $c(t)$ is measured to be

$$\frac{dc}{dt} = 400 - 10\,c$$

where time is measured in minutes.

(i) Solve the ODE to find the chemical concentration c at time t.

(ii) Find the time at which the chemical concentration is half the initial concentration.

(iii) What is the limiting value of the chemical concentration after a long time?

Exercises 3

1. An iron is left to cool in a room with temperature 20°C. Denote by $\theta(t)$ the temperature (measured in °C) of the iron at time t (measured in min). According to Newton's Law of Cooling

$$\frac{d\theta}{dt} = -k(\theta - 20),$$

where $k > 0$ is a constant.

(a) If $\theta(0) = 100$ show that

$$\theta(t) = 20 + 80\mathrm{e}^{-kt},\ t \geq 0.$$

(b) Find $\lim_{t\to\infty} \theta(t)$ and sketch the function $\theta(t)$.

(c) It takes the iron 5 min to cool from 100°C to 60°C. Find k and hence calculate how long it takes the iron to cool to 25°C.

2. The balance in a bank account is $y(t)$ where time t is measured in years. Starting with £200, no withdrawals or further investments are made. Interest is paid in proportion to the balance, so that the rate of change of y with respect to t is proportional to y.

(a) Write down the ODE for $y(t)$ and solve it to give the balance in the account as a function of t.

(b) Suppose at $t = 10$ the balance reaches $y(10) = 300$. Find the constant of proportionality in the ODE.

(c) Given the numbers from the previous part, find the time taken to double the initial investment.

3. Toricelli's law states that the rate at which the volume V of water in a draining tank decreases with respect to time is proportional to the square root of the depth y of water in the tank, i.e.

$$\frac{dV}{dt} = -k\sqrt{y}\,.$$

(a) Suppose that the tank is a circular cylinder with vertical sides and radius a. Write down a formula for $V(t)$ in terms of $y(t)$ and a, and hence convert the above differential equation to an ODE for y.

(b) Solve this ODE when $y = 4\,a$ at time $t = 0$.

(c) Suppose that the level reaches $y = a$ at time $t = 8$ minutes. Find the constant of proportionality in the ODE.

(d) Find the time at which the level has dropped to $y = a/4$.

4. The time rate of change of the population of a colony of bacteria is proportional to the size P of the population.

(a) Write down the ODE for $P(t)$ and solve it when the population size at time $t = 0$ is 1000.

(b) Suppose that the population has doubled at time $t = 1$ hour. Use this to find the constant of proportionality for the population equation.

(c) Calculate the population size at time $t = 3$ hours.

(d) How long does it take for the population to reach a million?

5. The ratio of numbers of C14 to C16 atoms is more or less fixed in living things (animals, plants etc.), but when they die the number of C16 atoms stays the same and the number of C14 atoms decreases through radioactive decay. Measuring the relative proportions of C14 to C16 can then indirectly give an estimate of the age of the dead material. This is known as carbon dating. On average, a constant fraction of C14 atoms decays each time unit. In mathematical terms

$$\frac{dy}{dt} = -ky$$

where y is the number of C14 atoms, $k > 0$ is the rate of decay and we measure time t in years here.

(a) Find the formula for the number of atoms in our sample given the initial value $y(0) = 10^9$ atoms.

(b) After 5730 years the number of Carbon 14 atoms remaining is 1/2 of the starting value. Find the constant of proportionality k above. (5730 years is the half-life of C14.)

(c) What proportion of atoms decays each time unit (i.e. each year)?

(d) In an archaeolgical sample, 1/4 of the C14 atoms have decayed away. How old is it?

6. Many common drugs are eliminated from the bloodstream at a rate that is proportional to the amount $y(t)$ still present.

(a) Write down a differential equation which describes this situation and show that

$$y(t) = y_0\, e^{-kt}$$

when y_0 milligrams is the amount initially injected and k is a positive constant.

(b) Suppose that to anaesthetise a dog at least 500mg of anaesthetic must be present in the dog's bloodstream. If the half-life of the anaesthetic in the dog's bloodstream (i.e. the time it takes the amount to halve) is 3 hours, estimate the size of a single dose that will anaesthetise the dog for at least 45 mins.

(c) If 700 mg of anaesthetic is injected into the dog's bloodstream, for how long will it remain anaesthetised?

4. Simply the best!

4.1. Optimization

Optimization is the name given to a class of mathematical problems involving finding the "best possible" value of something (e.g. the **lowest** cost, **biggest** profit, **least** toxic waste produced from a manufacturing process, etc.). As such it is a very important part of applied mathematics, and is used throughout industry and business to boost yields and profits.

Simple examples of optimization problems usually involve finding the global maximum or minimum values of a function, and you should have already seen examples of this in MM102 or MM112.

Example 4.1 Find the maximum and minimum values of the function $f(x) = 2\,x^3 - 3\,x^2 - 12\,x + 1$ for $x \in [-2, 4]$.

You should also have seen some more practical problems which involve more than one variable and a **constraint**. The aim with these is to use the constraint to rewrite the problem as an optimization problem in one variable. It is usually helpful to start with a sketch, and then write down mathematical equations for all the pieces of information in the problem.

Example 4.2 The aim is to design a circular cylindrical drink can of volume 375 ml. The metal used to make the top of the can costs twice as much per square centimetre as the metal used to make the sides and bottom. What height and radius should the can have to minimize the costs? (Note 1 ml = 1 cm^3.)

In this example the constraint is an **equation:** $\pi\,r^2\,h = 375$, but in many problems of practical interest the constraint is an **inequality**. For example

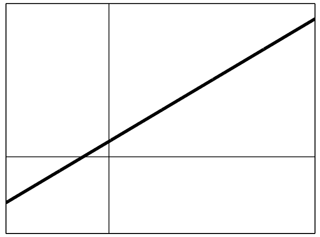
- Maximize the rate of a chemical reaction subject to the concentration of chemical A being non-negative.
- Maximize the profit of company X subject to the factory area being less than the land available.
- Minimize the cost of production of 1 tonne of steel, subject to the finished product having tensile strength greater than some prescribed value.

Linear programming is the name given to solving optimization problems in which the function to be optimized is a **linear** combination of variables, and it is subject to constraints which are all inequalities involving **linear** combinations of variables. Real life linear programming problems typically involve a very large number of variables, but in MM103 we shall just look at problems in two variables, x and y. Before this we shall investigate linear functions of **one** variable.

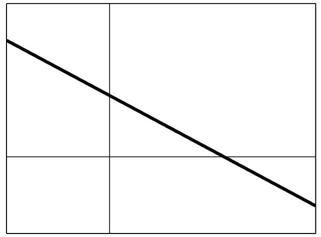
4.2. Linear functions of one variable

If $f(x)$ is a linear function, then its graph is a straight line and it must have the form $f(x) = a\,x + b$ for some constants a and b. The various possibilities are illustrated below.

f(x) = ax+b, a > 0



f(x) = ax+b, a < 0



f(x) = ax+b, a = 0

Result - 1D:

If $f(x)$ is a **linear function** defined for $x \in [x_0, x_1]$ then the maximum value of f occurs at one of the interval endpoints, i.e. it is $f(x_0)$ or $f(x_1)$. The minimum value of f occurs at the other endpoint.

Examples 4.3

(a) Find the maximum and minimum values of $f(x) = 5\,x - 7$ for $x \in [-2, 4]$.

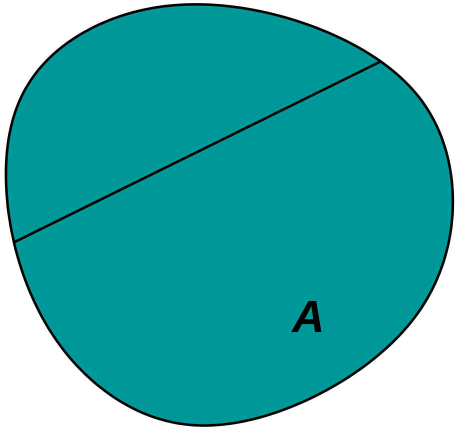
(b) Find the maximum and minimum values of $f(x) = 8 - 3\,x$ for $x \in [1, 5]$.

This simple 1D result is the key to linear programming in two (or more) variables.

4.3. Linear functions of two variables

A linear function in the two variables x and y has the form $f(x, y) = a\,x + b\,y + c$ for some constants a, b, c.

Suppose that the linear function $f(x, y)$ is defined for $(x, y) \in A$ where A is a bounded region in $\mathbb{R}^2$, as shown below.

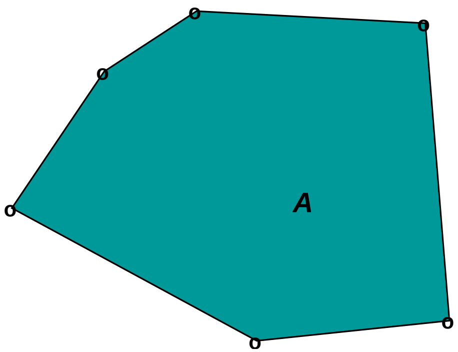


On any straight line through A, $f(x, y)$ is a linear function of one variable, and so by the 1D result, it will attain its maximum and minimum values at the ends of this straight line – i.e. on the boundary of A. This is true for any straight line through the region A, which means that the maximum and minimum values of $f(x, y)$ must occur on the **boundary** of A.

In theory this means that we can find the maximum and minimum values of the 2D linear function f on A just by checking the value of f at all boundary points of A, but in practice this may not be easy. One special case which is easy is when A is a polygon, and we state this as our 2D result.

Result - 2D:

If $f(x, y)$ is a **linear function** defined for $(x, y) \in A$ where A is a bounded **polygon** in $\mathbb{R}^2$, then the maximum value of $f(x, y)$ on A occurs at one of the corner points on the boundary of the region A. The minimum value of $f(x, y)$ on A also occurs at a corner point on the boundary of A.



This result is true because the boundary of A is now formed of straight lines, and the maximum and minimum values of f on a line occurs at the endpoints, i.e. at one of the corners on the boundary of A.

Examples 4.4

Let $f(x, y) = 5\,x - 3\,y + 12$.

(a) Find the maximum and minimum values of $f(x, y)$ on the square with corners at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.

(b) Find the maximum and minimum values of $f(x, y)$ on the triangle with corners at $(-1, 0)$, $(2, 1)$ and $(1, 5)$.

We now use this result to solve linear programming problems in two variables.

4.4. Linear programming in two variables

We are going to look at optimizing (i.e. finding the maximum or minimum values of) a **linear function** $f(x, y)$ which is subject to **linear constraints**. The function f is called the **objective function** and the constraints define the region $A \in \mathbb{R}^2$ over which we want to optimize f. It is called the **feasible region**, and because the constraints are all linear, it will be a polygon. It is easiest to illustrate the method by looking at some concrete examples.

Example: A company makes one product and sells x units of it per day in the UK and y units of it per day in Europe. The profit is £200 per unit sold in the UK and £100 per unit sold in Europe.

Suppose that the total production capacity is at most 100 units per day, and at most 60 units can be sold per day in the UK.

(a) Find the total sales profit f per day, in terms of x and y.

(b) Write down all the constraints on f and draw the feasible region.

(c) Find the values of x and y which maximise f, and calculate the maximum possible daily profit.

Solution

(a) The objective function is the total profit per day: $f(x, y) = 200\,x + 100\,y$.

(b) There are **four** constraints:

$x + y \leq 100$ (production is at most 100 units per day);

$x \leq 60$ (UK sales are at most 60 units per day);

$x \geq 0$ (you can't sell a negative number of units);

$y \geq 0$ (you can't sell a negative number of units).

These four constraints determine the feasible region, A, which is illustrated below. The four corners of A are at: $(0, 0)$, $(60, 0)$, $(60, 40)$ and $(0, 100)$.

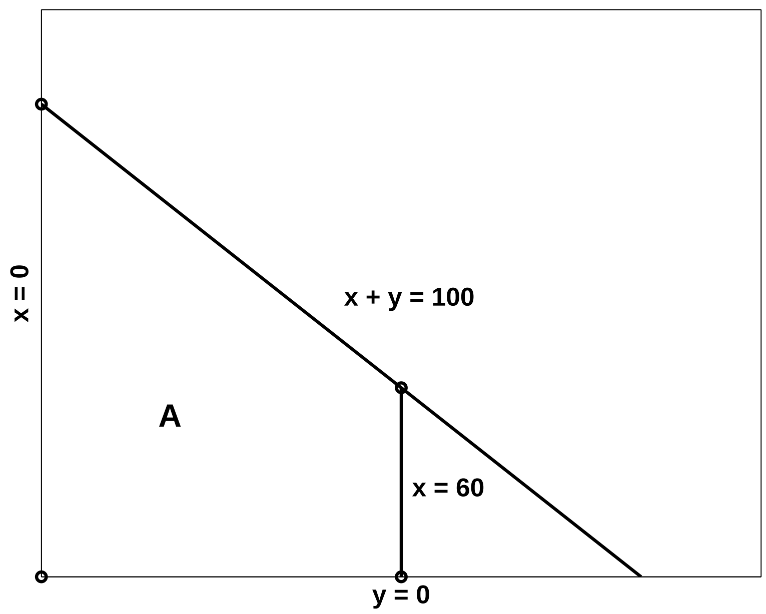
(c) The maximum value of $f(x, y)$ must occur at one of these four corner values. We have $f(x, y) = 200\,x + 100\,y$ and so:

$f(0, 0) = 0$

$f(60, 0) = 12,000$

$f(60, 40) = 16,000$

$f(0, 100) = 10,000$



Hence the maximum daily sales profit is $£16,000$ and occurs when 60 units are sold per day in the UK and 40 units are sold per day in Europe.

In linear programming problems it is usual to use Z rather than f to denote the objective function, and we shall do this from now on.

Example 4.5 Consider the following linear programming problem:

$$\text{maximise } Z = x + y$$

subject to

$$\begin{aligned} 2x + y &\leq 8 \\ 2x + 3y &\leq 12 \\ y &\leq 3 \\ x,\ y &\geq 0\,. \end{aligned}$$

(a) Sketch the feasible region for the problem.

(b) Find the optimal solution to the problem.

Example 4.6 The Belgian chocolate linear programming problem is:

$$\text{maximise } Z = 55M + 89H$$

subject to

$$\begin{aligned} 4M + 18H &\leq 1296 \\ 12M + 6H &\leq 1824 \\ M,\ H &\geq 0\,. \end{aligned}$$

Sketch the feasible region and find the optimal solution to the problem.

Exercises 4

1. If $x + y = 2$, find x and y that minimize: (a) $x^2 + y^2$, (b) $x^2\,y^2$.

2. Find a point on the curve $y^2 - x^2 = 1$ closest to the point $(1, 0)$.

3. A wall $\sqrt{2}$ m high is 0.5 m from the vertical wall of a house. Find the length of the shortest ladder that will reach over the wall to touch the house.

4. Consider the following linear programming problem:

$$\text{maximise } Z = 5x + y$$

subject to

$$\begin{aligned} x + y &\leq 3 \\ y &\leq 1 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

5. Consider the following linear programming problem:

$$\text{maximise } Z = x + 2y$$

subject to

$$\begin{aligned} x + y &\leq 5 \\ x - y &\leq 2 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

6. Consider the following linear programming problem:

$$\text{maximise } Z = x - y$$

subject to

$$\begin{aligned} x + y &\leq 10 \\ 2x - y &\leq 15 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

7. Consider the following linear programming problem:

$$\text{maximise } Z = x - y$$

subject to

$$\begin{aligned} x + y &\leq 10 \\ 2x - y &\geq 15 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

8. Consider the following linear programming problem:

$$\text{maximise } Z = 45x + 80y$$

subject to

$$\begin{aligned} x + 4y &\leq 80 \\ 2x + 3y &\leq 90 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

9. Consider the following linear programming problem:

$$\text{maximise } Z = 2x + 5y$$

subject to

$$\begin{aligned} x + 2y &\leq 7 \\ 2x + y &\leq 12 \\ y &\leq 2 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

10. Consider the following linear programming problem:

$$\text{minimise } Z = y$$

subject to

$$\begin{aligned} 5x + y &\geq 12 \\ -5x + y &\geq 4 \\ x,\ y &\geq 0\,. \end{aligned}$$

Sketch the feasible region for the problem and find the optimal solution.

11. Iron ore is bought from two different sources X and Y, which contain different amounts of metal elements. Ore X has 2 Kg of manganese, 2 Kg of silicon and 3 Kg of copper per tonne, and costs £40 per tonne. Ore Y has 1 Kg of manganese, 4 Kg of silicon and 7 Kg of copper per tonne, and costs £50 per tonne.

    An amount of steel alloy is to be produced from a mixture of these two ores, and it is necessary that the mixture should contain at least 1 Kg of manganese and 3 Kg of silicon, with no more than 6 Kg of copper. The aim is to satisfy these requirements as cheaply as possible, and use at most 1 tonne of ore X.

    (a) Suppose that x tonnes of ore X and y tonnes of ore Y are blended together. Write down the objective function (cost) in terms of x and y, and also list all the constraints on x and y.

    (b) Sketch the feasible region for the problem and find the optimal solution.

12. A diet of bread and milk is to be prescribed containing an intake per meal of at least 6 units of vitamin A and 10 units of vitamin B. The meal must contain at least one glass of milk. Find the most economical diet possible (i.e. minimise its cost) given the following information:

	bread/slice	milk/glass
Vitamin A	1 unit	2 units
Vitamin B	2 units	3 units
Cost	5p	12p

5. What goes up. . . must come down

The first main aim of this section is to solve the following practical problem.

> **Problem:** A cannon on level ground fires out a shell at speed U. What is the furthest distance that the shell can travel?

As always, to make progress we need the **relevant equation(s)** and some **data**.

Newton's laws of motion

Isaac Newton (1643–1727) proposed three laws of motion for particles in the late seventeenth century, and these form the basis of "Mechanics":

1. *A body remains in a state of rest or of uniform motion in a straight line unless it is acted on by an external force.*
2. *The rate of change of the momentum of a body is proportional to the force acting upon it.*
3. *To every action there is an equal and opposite reaction.*

We choose units to make the constant of proportionality in the second law above equal to 1. Assuming that the mass of a body stays constant, its rate of change of momentum is equal to its mass × acceleration, and so **Newton's second law** says that

$$\text{force} = \text{mass} \times \text{acceleration}\,.$$

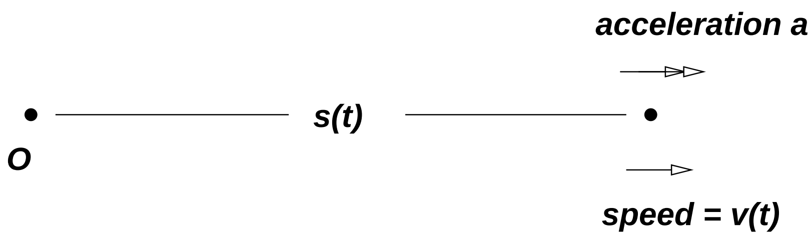
Let us look at the implications of this for a falling object with no external forces acting on it (i.e. we assume that there is no air resistance or wind). In this case, the only force acting on the object is its **weight** acting vertically downwards, and so by Newton's second law, this must equal its mass × its acceleration. This means that we can regard an object's weight as being its mass multiplied by the **acceleration due to gravity**:

$$\text{weight} = \text{mass} \times \text{acceleration due to gravity}\,.$$

The "acceleration due to gravity" acts vertically downwards (towards the centre of the earth) and its size, g, is approximately 9.8 ms^{-2} (it is often taken to be 10 ms^{-2} in numerical examples).

We can use this to derive the relevant equations of motion for the shell problem. Before this we shall look at more general equations for velocity and acceleration

5.1. One dimensional motion



Suppose that a particle is moving in a straight line and at time t it is at a **displacement** $s(t)$ from a fixed origin O, where O is its position at time $t = 0$. The convention is that if the particle is to the right of O in the picture above then $s > 0$, and if it is to the left of O then $s < 0$.

The **velocity** of the particle at time t is

$$v(t) = \frac{ds}{dt} = \dot{s}(t)$$

and its **acceleration** at time t is

$$a(t) = \frac{dv}{dt} = \dot{v}(t)\,,$$

so $a(t) = \ddot{s}(t)$. The **speed** of the particle is the size of its velocity.

Example 5.1 The displacement of a particle from the origin at time t is $s(t) = b\,\sin\omega\,t$ where b and ω are positive constants.

(a) Calculate its velocity v and acceleration a at time t.

(b) Calculate s and a at the time at which its speed is first zero.

5.2. One dimensional motion under constant acceleration

Although there are many interesting practical problems which involve non-constant acceleration (i.e. a depends on t, as in the example above), we only need to consider the (easier) case of **constant acceleration** in order to look at the shell (and other projectile problems in which air resistance can be ignored). **So between here and the end of Section 5.4 we shall assume that the acceleration a is a constant** (it does not change with time). The equation for the velocity is then

$$\frac{dv}{dt} = a = \text{ constant}$$

and integrating with respect to t gives $v = a\,t + C$ where C is a constant of integration. Setting $t = 0$ we see that C is just the velocity at time $t = 0$, so the velocity at time t is

$$v = v_0 + a\,t$$

where v_0 is the velocity at time $t = 0$.

The equation for the displacement s from O (the position at time $t = 0$) is $\dot{s} = v$, i.e.

$$\frac{ds}{dt} = v_0 + a\,t$$

(where a and v_0 are constants) and integrating with respect to t gives

$$s = v_0\,t + \tfrac{1}{2}\,a\,t^2 + C$$

where C is an arbitrary constant of integration. But we know that $s = 0$ at time $t = 0$, and so setting $t = 0$ gives $C = 0$ to give $s = v_0\,t + \frac{1}{2}\,a\,t^2$. These equations are summarised below.

A particle is moving under constant acceleration a, and at time $t = 0$ it is at the point O and is travelling at velocity v_0. The particle's velocity v and displacement s from O at time t are given by

$$\begin{aligned} v &= v_0 + a\,t \\ s &= v_0\,t + \tfrac{1}{2}\,a\,t^2\,. \end{aligned}$$

Example 5.2 A particle is launched from point O at time $t = 0$ at velocity 10 ms^{-1} to the right and is moving under constant deceleration of 2 ms^{-2} in the same direction.

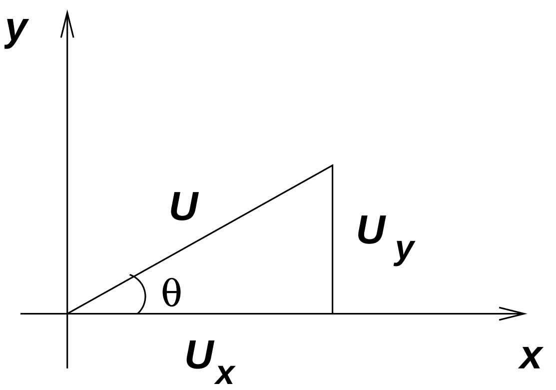
(a) Find the first time t at which its speed is zero. What is its displacement from O at this time?

(b) At what time does it return to the point O?

5.3. Components of velocity

Projectile problems (such as the cannon shell) are not one dimensional, although we can look at their horizontal and vertical motion separately. But in order to do this we need to know how to "resolve" the initial velocity into its horizontal and vertical components. This is easily done using a right-angled triangle.

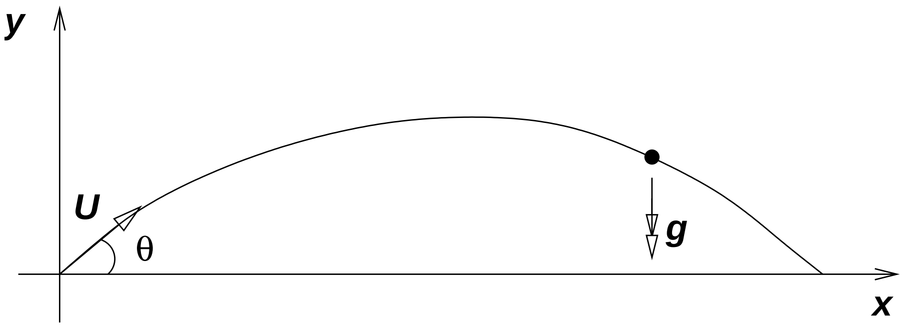
Suppose that the speed is U at an angle θ to the horizontal, as shown below. We draw a right-angled triangle with hypotenuse of length U, then the speed U_x in the horizontal direction is the length of the triangle base and the speed U_y in the vertical direction is the triangle height.



But $\cos\theta = U_x/U$ and $\sin\theta = U_y/U$, and rearranging these gives:

$$U_x = U\cos\theta$$
$$U_y = U\sin\theta\,.$$

5.4. Projectile problems: motion under gravity



Suppose that a projectile is fired at speed U and angle θ to the horizontal as illustrated above. We have seen that if the initial 1D velocity is v_0 and the (constant) acceleration is a, then the velocity v and displacement s at time t are $v = v_0 + a\,t$ and $s = v_0\,t + \frac{1}{2}\,a\,t^2$. We now use these equations in the horizontal (x) and vertical (y) directions.

$x-$direction:

- initial velocity: $v_0 = U\cos\theta$
- acceleration: $a = 0$
- velocity equation $\Rightarrow$ velocity at time t is $v_x = U\cos\theta$
- displacement equation $\Rightarrow$ displacement at time t is $x = U\cos\theta\,t$

$y-$direction:

- initial velocity: $v_0 = U\sin\theta$
- acceleration: $a = -g$
- velocity equation $\Rightarrow$ velocity at time t is $v_y = U\sin\theta - g\,t$
- displacement equation $\Rightarrow$ displacement at time t is $y = U\sin\theta\,t - \frac{1}{2}\,g\,t^2$

Example 5.3 Find the time at which this projectile reaches its maximum height (in terms of U, θ and g). What is its maximum height?

Example 5.4 At what time does the projectile hit the ground, and how far has it travelled along the ground when it does so?

We have seen that the horizontal distance travelled when the particle again hits the ground is

$$x = \frac{U^2 \sin(2\theta)}{g} .$$

This means we can now solve our original **cannon problem**.

Example 5.5 What is the furthest distance that the cannon shell can travel, and at what angle does it need to be fired out to achieve this distance?

Example 5.6 A cannon shoots a shell at speed U at an angle of 45^o above the horizontal from level ground below a vertical cliff of height 40 m. The aim is to position the cannon so that the shell will land on the cliff.

(i) Find the smallest value of U for which the cannon shell can land on the cliff (take the acceleration due to gravity to be $g = 10\text{ms}^{-2}$).

(ii) How far away from the foot of the cliff does the cannon need to be for the shell to land on the cliff when U takes this value?

Exercises 5.1

1. A body moves with uniform acceleration for 3 seconds and covers 54 m. It then moves with constant velocity and covers 120 m in the next 5 seconds. Find its initial velocity and acceleration.

2. A train is uniformly accelerated and passes successive kilometre markers at the side of the track with speeds of 10 Km/h and 20 Km/h respectively. Calculate the speed of the train when it passes the next kilometre mark, and the times taken for each of these two intervals of 1Km.

3. A particle is projected from a point on level ground with velocity 60 m/s at an angle of 30^o above the horizontal. Take $g = 10$ m/s^2 and find:
(a) the time taken for the particle to reach its maximum height;

(b) the maximum height;
(c) the time of flight;
(d) the horizontal range of the particle.

4. A vertical tower on a horizontal plane is 12 m high. A ball is thrown from the top of the tower with velocity of 16 m/s horizontally and 17 m/s vertically upwards. How far will it be from the top of the tower after 1 second? (Take $g = 10$ m/s^2.)

5. How long will it take the ball in the previous example to reach the ground, and how far from the foot of the tower will it be when it does so?

6. A shot is fired from the top of a cliff of height 75 m with a velocity of 20 m/s at an angle of 30^o above the horizontal. Calculate the horizontal distance that the shot has travelled when it hits the ground (take $g = 10$ m/s^2).

7. A particle is projected vertically upwards under gravity with speed U. At time t_0 seconds later another particle is fired upwards from the same point. Find the initial speed of this particle in order that the two particles will collide when the first has reached its highest point.

8. A cannon is pointed up a hill which is at an angle α to the horizontal. The cannon fires a shell at speed U at an angle of θ to the hill.

   (a) Find the time at which the shell hits the ground. (Hint: it hits the ground when $y/x = \tan\alpha$.)

   (b) Show that it hits the ground a distance of L away, where
   $$L = \frac{U^2}{g\cos^2\alpha}\left(\sin(2\theta+\alpha) - \sin\alpha\right) .$$

   (c) What is the maximum possible value of L, and at what angle θ should the cannon be aimed to achieve this distance?

9. The cannon in the previous question is now pointed down the hill. What is the maximum distance the shell can travel now before hitting the ground, and at what angle should it be aimed to achieve this distance?

10. A body is projected from a horizontal plane at such an angle that the horizontal range is three times the greatest height. Find the angle of projection, and if with this angle the range is 60 m, then find the necessary speed of projection and time of flight (take $g = 10$ m/s^2).

11. A paintball gun can fire a capsule with speed U in any direction. Show that if the capsule is fired out of the gun at an angle of θ to the horizontal then its (x, y) coordinates relative to the gun satisfy

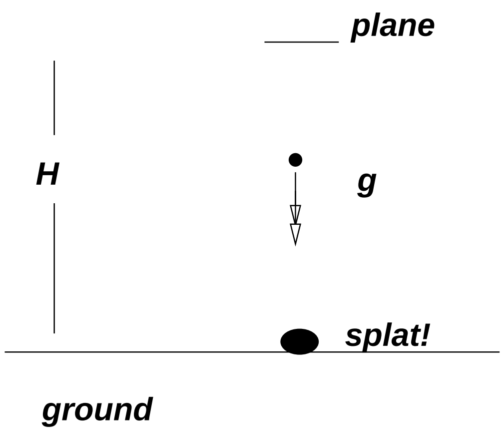
$$g\,x^2\,\tan^2\theta - 2\,U^2\,x\,\tan\theta + 2\,U^2\,y + g\,x^2 = 0\,.$$

Hence show that the capsule can reach any target within the surface

$$g^2\,x^2 + 2\,g\,U^2\,y = U^4\,.$$

5.5. What happens when there is air resistance?

Although air resistance can be neglected when we want to work out the trajectory of a small projectile (or at least one which is not travelling too fast), it is very important to people who jump out of aeroplanes with parachutes. For example, we shall first work out the approximate speed someone would hit the ground at if they stepped out of a plane at height H and fell without a parachute (in this case we can treat the person as a projectile and ignore air resistance).



We measure the displacement s vertically downwards and assume that the vertical speed on leaving the aeroplane is zero, so $v_0 = 0$ and $a = g$. The vertical speed downwards at time t is then $v = g\,t$, and we can use the displacement equation $s = v_0\,t + \frac{1}{2}\,a\,t^2$ to find the time at which the person hits the ground. This gives $H = \frac{1}{2}\,g\,t^2$, so $t = \sqrt{2\,H/g}$, and the speed at the ground is

$$V = \sqrt{2\,g\,H}\,.$$

The following table shows how V varies with H, taking $g = 10$ m/s^2.

H (m)	V (m/s)	V (Km/h)
1.8	6.0	21.6
5.0	10.0	36.0
20.0	20.0	72.0
2000.0	200.0	720.0

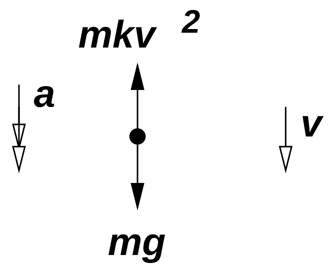
Even falling from a low height (such as from a ladder, rather than out of a plane) would give an unpleasantly fast landing, and there is no limit to the impact speed, provided H is large enough – the speed grows like $\sqrt{H}$.

Fortunately things are very different with a parachute! This has a large surface area, which impedes or "resists" the motion downwards through the air. The resistance imposed by the air on a falling object is a force which is typically taken to be proportional to the square of the object's speed. If the object is a person without a parachute then the constant of proportionality will be low (and can be neglected), but if the object is a person with a parachute then the constant will be much larger, and air resistance will largely govern the behaviour. For simplicity we shall write the resisting force as $m\,k\,v^2$, where m is the object's mass.

The total force acting vertically downwards on the object is $m\,g - m\,k\,v^2$, and by Newton's second law this is equal to $m\,a$, where a is the downwards acceleration. That is

$$a = g - k\,v^2\,. \tag{5.1}$$

In order to find out how the speed for this object depends on the distance it has fallen we shall write this equation as an ODE. Letting s denote the vertical



displacement and v the vertical speed (both downwards), we know that

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt}\,.$$

But by the **chain rule** for derivatives

$$a = \frac{dv}{dt} = \frac{dv}{ds}\,\frac{ds}{dt} = \frac{dv}{ds}\,v$$

and so we can write

$$a = v\,\frac{dv}{ds}$$

in (5.1) to get the following **separable** ODE

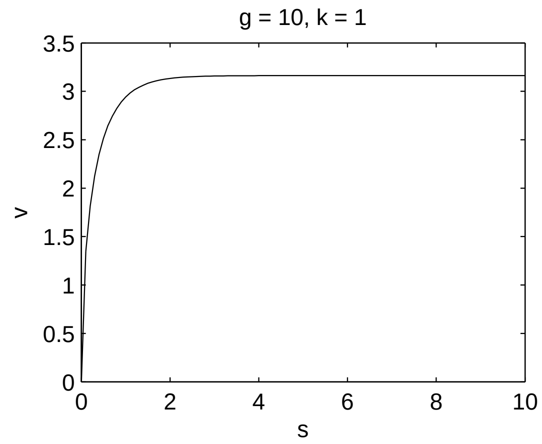
$$v\,\frac{dv}{ds} = g - k\,v^2\,. \qquad (5.2)$$

Example 5.7 Show that the general solution of the ODE (5.2) is

$$g - k\,v^2 = A\,e^{-2\,k\,s}$$

where A is an arbitrary constant of integration. If the speed of descent is zero at $s = 0$, then find the maximum possible speed v_T in terms of g and k.

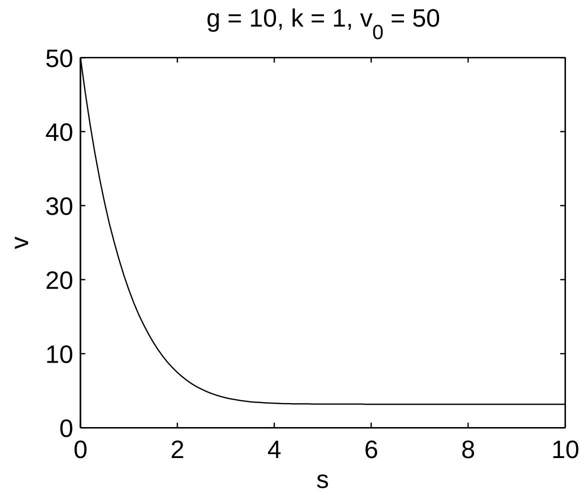
The limiting speed of $\sqrt{g/k}$ is called the **terminal speed**. The aim with a parachute is to design it so that k is large enough for the terminal speed to give a safe landing. The figure below shows a plot of speed against fall distance s when $k = 1$ and g is taken to be 10 m/s^2.



In real life, parachutists free fall for a distance before inflating their parachutes, and the parachute is typically opened at a speed $v_0 \gg v_T$.

Example 5.8 Suppose that the parachute is opened at $s = 0$ at speed v_0. Calculate the speed v at a distance of s m below the parachute opening, and show that the terminal speed is again $v_T = \sqrt{g/k}$.

The figure below shows a plot of v against s with $k = 1$, $v_0 = 50$ m/s and $g = 10$ m/s^2. Notice how quickly the speed drops to v_T.



Exercises 5.2

1. The falling speed v of a parachutist is governed by the ODE
$$v\,\frac{dv}{ds} = g - k\,v^2\,.$$
with $k = 1$, where s is the vertical distance fallen. Assuming that he opens the parachute at $s = 0$ at zero speed, then calculate his speed when he has fallen a distance of s. How far will he have fallen by the time he reaches half his terminal speed?

2. A parachutist drops from an aeroplane and falls under gravity with negligible air resistance for 5 seconds. She then opens her parachute and her falling speed v is governed by the ODE
$$v\,\frac{dv}{ds} = g - k\,v^2\,.$$
with $k = 1$. Take $g = 10$ m/s^2 and calculate her speed v when she is s m below the point at which she opened her parachute. How far does she need to fall for her speed to be within 1 m/s of her terminal speed?

3. Suppose now that she is using a different parachute for which $k = 0.4$, which she opens when she is travelling at speed v_0. Take $g = 10$ m/s^2 and calculate her speed v a distance s below the place she opened her parachute in terms of v_0. Sketch graphs of v against s in the three cases

   (i) $v_0 > 5$ (ii) $v_0 < 5$ (iii) $v_0 = 5\,.$

   If $v_0 = 5$, then calculate how long it takes her to reach the ground if she opens her parachute 200 m above the ground.

4. Suppose you have two small metallic spheres of the same size, with sphere A being twice as heavy as sphere B, and you drop both of them and a feather at the same time from the top of a tall tower on a still day. What order will the three objects hit the ground in?

5. Your alien friend takes the two spheres and the feather to his tower home on boring planet Zog (it has no atmosphere), and performs the same experiment. What order will the three objects hit the Zog ground in?

6. Population growth revisited

We saw in Section 1 how a **difference equation** can be used to model population growth in some circumstances. In other circumstances a **differential equation** (ODE) model can be more realistic. Another advantage of using differential equations is that there are techniques for solving first order nonlinear ODEs (e.g. ones which are separable), while it is usually difficult (or may not be possible) to find the exact solution of a nonlinear difference equation.

6.1. What types of ODEs give realistic results?

We shall (partially) answer this question by solving the following example.

Example 6.1 We know that the human population of the world has been increasing for centuries. Let us start 10,000 years ago and call that $t = 0$, and assume that the population then was 10^6 humans. For comparison, the population now is about 7×10^9 (7 billion) humans. Solve the following ODEs for $y(t)$ (the number of humans alive at time t measured in years) and say which have solutions that could fit these assumptions.

(a) $y' = -2y$ (b) $y' = ky^2$ $(k = 10^{-9})$ (c) $y' = \alpha$
(d) $y' = \beta y$

where α and β are positive constants. In the cases with parameters, find values of the parameters to fit the data given. Are any of these equations good models? Why?

6.2. Designing a model for population growth

Let $y(t)$ be the population of a certain species (e.g. bugs, rabbits, humans, or anything else) at time t. Suppose first of all that the population only changes because of births and deaths, and let

- $\beta(t)$ be the **birth rate** per unit of population at time t; and
- $\delta(t)$ be the **death rate** per unit of population at time t.

Then
$$\frac{dy}{dt} = (\beta(t) - \delta(t))\;y\,.$$

In many populations it is observed that the birth rate β is a **linear** decreasing function of the population size, and the death rate δ is roughly constant – i.e.

$$\beta(t) = \beta_0 - \beta_1 \, y \quad \text{and} \quad \delta(t) = \delta_0 \, .$$

In this case the population growth is governed by the ODE

$$\frac{dy}{dt} = (\beta_0 - \delta_0 - \beta_1 \, y) \; y$$

for suitable values of the constants β_0, β_1 and δ_0. When $\beta_0 - \delta_0 > 0$ and $\beta_1 > 0$ this is called a **logistic** ODE. In this case it is usual to write $\beta_1 = k$ and $m = (\beta_0 - \delta_0)/\beta_1$.

6.3. The logistic ODE

The logistic ODE for $y(t)$ is

$$\frac{dy}{dt} = k \, y \, (m - y) \tag{6.1}$$

where k and m are positive constants. What can we deduce about the behaviour of the solution just from looking at the ODE (6.1)?

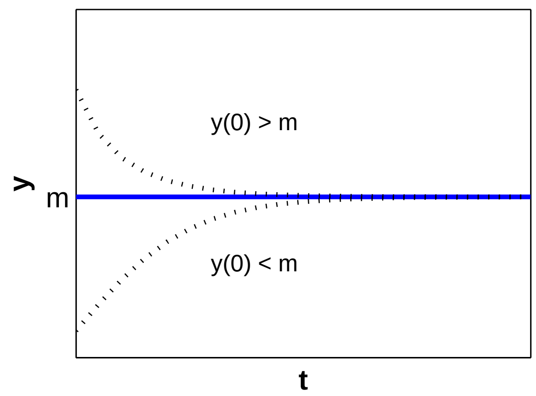
- If $0 < y < m$ then the RHS of (6.1) is positive, and so dy/dt is positive y is **increasing** with time t.
- If $y > m$ then the RHS of (6.1) is negative, and so dy/dt is negative and y is **decreasing** with time t.
- If $y = 0$ or $y = m$ then the RHS of (6.1) is zero, and so $dy/dt = 0$, which means that y **stays fixed**.

These observations mean that the solution shape is governed by $y(0)$, as shown in the figure below.

Example 6.2 Find the solution $y(t)$ of the logistic equation (6.1) which satisfies $y(0) = y_0 > 0$.

The example shows that the solution of (6.1) with initial condition $y(0) = y_0$ is

$$y(t) = \frac{m \, y_0}{y_0 + (m - y_0) \, e^{-mkt}} \, . \tag{6.2}$$



We have assumed that $m > 0$ and $k > 0$, and so as t increases the exponential e^{-mkt} becomes small, and

$$\lim_{t \to \infty} e^{-mkt} = 0 \, .$$

This means that the second term in the denominator of (6.2) goes to zero as t increases, whatever the sign of $m - y_0$, and so $y(t)$ approaches the value m for large t. This is known as the **limiting population** or **carrying capacity** of the population modelled by (6.1). It is the maximum population which that environment can support on a long-term basis.

Example 6.3 Is the ODE (6.1) a feasible model for the population growth described in Example 6.1 for any values of m and k? If so, is it a good model?

6.4. More sophisticated population models

The logistic model (6.1) assumes that the population is isolated – its growth is only determined by the birth and death rates, and members of the species do not move into or out of this closed population. In some cases this will be realistic, but in many others it will not be. For example, there could be movement from one colony to another.

Example 6.4 Suppose that (6.1) is a suitable model for the population size $y(t)$ of an insect colony in isolation, but in addition to this some of the insects leave and others join the colony. Write down a suitable ODE for the population size if the leaving rate per unit of population is ℓ, and the joining rate is p (where both ℓ and p are positive constants).

Of course, some population models are much more sophisticated than this, and take into account spatial movement, specific types of predation or food sources etc. If the population growth depends on its habitat (as would be expected if some areas were safer or had better sources of food), then an appropriate model might be a **partial differential equation**. You will meet partial derivatives in your mathematics classes next year.

Exercises 6

1. Answer Example 6.1 for the following ODEs

   (a) $y' = 5y$ (b) $y' = ky^3$ $(k = 10^{-12})$ (c) $y'' = \alpha$

2. Suppose that the growth of a population is determined by the ODE $y' = ay - by^2$ where a and b are positive constants.

   (a) Find the general solution of the ODE

   (b) Find $\lim_{t\to\infty} y(t)$. How does this value relate to the right hand side of the ODE?

   (c) Set $a = 2$, $b = 9$ and find the particular solutions when $y(0) = 1/9$ and $y(0) = 1$.

   (d) Sketch the solutions from part (c).

3. Suppose that in 1885 the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 million and was then growing at the rate of 1 million per year. Assume that this population satisfies the logistic equation (6.1), and determine both the limiting population m and the population for the year 2020.

4. If their food supply is constant, the number of haddock $y(t)$ at time t (measured in years) in the North Sea is assumed to vary according to the logistic equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{m}\right)$$

where k and m are positive constants.

(a) Solve this equation for $y(t)$ and sketch solutions in the cases $y(0) = m$, $y(0) = m/2$, $y(0) = 2m$. It will help if you calculate $\lim_{t\to\infty} y(t)$.

(b) Now assume that fishing starts and removes a fixed percentage of the fish population every time unit, giving the ODE

$$\frac{dy}{dt} = \underbrace{ky\left(1 - \frac{y}{m}\right)}_{\text{natural population}} - \underbrace{fy}_{\text{caught}}$$

where $f \in (0, k)$ is constant. Solve this ODE for $y(t)$ and find $\lim_{t\to\infty} y(t)$. How does this value relate to the right hand side of the ODE?

(c) An alternative fishing strategy is instead to take a fixed number of fish F per time unit without changing the rest of the circumstances. Write down the ODE modelling haddock numbers in this case. Explain how you could you solve this ODE if you knew the values of k, m and F.

Numerical answers to exercises

Exercises 2

1. GS: $x_n = A\,4^n$
2. GS: $x_n = A\left(-\frac{1}{2}\right)^n$; PS: $x_n = 3\left(-\frac{1}{2}\right)^n$;
3. PS: $x_n = -6\left(\frac{1}{2}\right)^n$
4. GS: $x_n = A + B\,2^n$
5. GS: $x_n = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{4}\right)^n$; PS: $x_n = \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n$
6. PS: $x_n = 3 + 2\,(-1)^n$
7. GS: $x_n = (A + n\,B)\left(\frac{1}{2}\right)^n$
8. GS: $x_n = (A + n\,B)$; PS: $x_n = 5\,n - 2$
9. GS: $x_n = A + B\,(-1)^n + C\,2^n$

Exercises 3

1. (b) $\lim_{t\to\infty}\theta(t) = 20$.
   (c) $k = \frac{1}{5}\ln 2$. It takes 20 min.
2. (a) $y(t) = 200\,e^{kt}$
   (b) $k = 0.1\ln 1.5$
   (c) Takes approx. 17.1 years
3. (b) $y = \left(2\sqrt{a} - \dfrac{kt}{2\pi a^2}\right)^2$
   (c) $k = \pi\,a^{5/2}/4$
   (d) time is $t = 12$ mins
4. (a) $P(t) = 1000\,e^{kt}$
   (b) $k = \ln 2$
   (c) 8000
   (d) Takes approx. 9.97 hours

5. (a) $y(t) = 10^9\,e^{-kt}$
   (b) $k = \ln 2/5730$
   (c) about 0.012%
   (d) about 2378 years old
6. (b) Approx. 594.6 mg
   (c) Approx 87.38 mins (87 mins 23 secs)

Exercises 4

1. (a) $(x,y) = (1,1)$ (b) $(x,y) = (0,2)$ and $(2,0)$
2. $(x,y) = (\frac{1}{2}, \pm\frac{\sqrt{5}}{2})$
3. $3\sqrt{3}/2$ m
4. $Z = 15$ at $(x,y) = (3,0)$
5. $Z = 10$ at $(x,y) = (0,5)$
6. $Z = 15/2$ at $(x,y) = (15/2,0)$
7. $Z = 10$ at $(x,y) = (10,0)$
8. $Z = 2200$ at $(x,y) = (24,14)$
9. $Z = 16$ at $(x,y) = (3,2)$
10. $Z = 8$ at $(x,y) = (4/5,8)$
11. $Z = £40$ at $(x,y) = (1/6,2/3)$
12. 4 slices of bread and 1 glass of milk

Exercises 5.1

1. $v_0 = 12$ ms^{-1}, $a = 4$ ms^{-2}
2. speed $= 10\sqrt{7}$ km/h, times are 1/15 and $(\sqrt{7}-2)/15$ hours
3. (a) 3 secs (b) 45 m (c) 6 secs (d) $180\sqrt{3}$ m
4. 20 m

5. time = 4 secs; distance = 64 m

6. $50\sqrt{3}$ m

7. speed is $\dfrac{U^2+(U-g\,t_0)^2}{2(U-g\,t_0)}$

8. (a) $t=\dfrac{2U\sin\theta}{g\cos\alpha}$ (c) $L=\dfrac{U^2}{g\cos^2\alpha}\,(1-\sin\alpha)$; $\theta=\pi/4-\alpha/2$

9. $L=\dfrac{U^2}{g\cos^2\alpha}\,(1+\sin\alpha)$; $\theta=\pi/4+\alpha/2$

10. angle = $\tan^{-1}(4/3)$; speed = 25 m/s; time = 4 secs

Exercises 5.2

1. speed is $\sqrt{g\,(1-e^{-2s})}$; distance = $\frac{1}{2}\ln(4/3)$

2. speed is $\sqrt{10+2490\,e^{-2s}}$; distance is approx. 2.9 m

3. speed is $\sqrt{25+(v_0^2-25)\,e^{-2ks}}$; time = 40 secs

Exercises 6

2. (a) GS: $y=\dfrac{a}{b+C\,e^{-at}}$, C is an arbitrary constant
   (b) $\lim_{t\to\infty} y(t)=a/b$
   (c) PS: $y=\dfrac{2}{9+9\,e^{-2t}}$; PS: $y=\dfrac{2}{9-7\,e^{-2t}}$

3. $m=200$ million; population in 2020 is approx 166 million

4. (a) solution is $y(t)=\dfrac{m}{1+\left(\frac{m}{y_0}-1\right)\,e^{-kt}}$
   (b) $\lim_{t\to\infty} y(t)=m\,(k-f)/k$