# DEPARTMENT OF MATHEMATICS & STATISTICS
# MM102 APPLICATIONS OF CALCULUS
## Ordinary Differential Equations: Exercise Sheet Solutions

**1.** In each question, the ODE can be written as: $\dfrac{dy}{dx} = f(x)g(y)$.

Re-arrange and integrate: $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$. In each solution, $A$ is an arbitrary constant.

**(a)** $\dfrac{dy}{dx} = x(y+1)^2 \quad\Longrightarrow\quad \displaystyle\int \frac{dy}{(y+1)^2} = \int x\,dx \quad\Longrightarrow\quad -\frac{1}{y+1} = \frac{1}{2}x^2 + A.$

$$\text{Therefore}\quad y+1 = -\frac{1}{\frac{1}{2}x^2 + A}\,, \quad\text{or}\quad y = \frac{-2}{x^2+2A} - 1.$$

**(b)** $\dfrac{dy}{dx} = x^3\cos^2 y \quad\Longrightarrow\quad \displaystyle\int \frac{dy}{\cos^2 y} = \int x^3\,dx \quad\Longrightarrow\quad \int \sec^2 y\,dy = \frac{x^4}{4} + A,$

where $A$ is an arbitrary constant. Therefore,

$$\tan y = \frac{x^4}{4} + A, \quad\text{or}\quad y = \tan^{-1}\Big(\frac{x^4}{4} + A\Big) \quad (\text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}).$$

**(c)** $x^2\dfrac{dy}{dx} = y - y^2 \quad\Longrightarrow\quad \displaystyle\int \frac{dy}{y(1-y)} = \int \frac{dx}{x^2}\,.$

$$\ln\Big|\frac{y}{1-y}\Big| = -\frac{1}{x} + A \quad\text{for arbitrary constant } A.$$

**(d)** $x^2(y+1) + y^2(x-1)\dfrac{dy}{dx} = 0 \quad\Longrightarrow\quad y^2(x-1)\dfrac{dy}{dx} = -x^2(y+1)$

$$\Longrightarrow\quad \int \frac{y^2}{y+1}\,dy = -\int \frac{x^2}{x-1}\,dx.$$

$$\int \frac{y^2}{y+1}\,dy = \int \Big(y - 1 + \frac{1}{y+1}\Big)\,dy = \frac{y^2}{2} - y + \ln|y+1| + C_1,$$

$$\int \frac{x^2}{x-1}\,dx = \int \Big(x + 1 + \frac{1}{x-1}\Big)\,dx = \frac{x^2}{2} + x + \ln|x-1| + C_2.$$

Therefore $\dfrac{y^2}{2} - y + \ln|y+1| = -\Big(\dfrac{x^2}{2} + x + \ln|x-1|\Big) + A$ where $A = C_2 - C_1$ is an arbitrary constant. Thus

$$\frac{1}{2}(x^2+y^2) + x - y + \ln|(x-1)(y+1)| = A.$$

**Qu. 1 cont'd next sheet**

**1. (e)** There is no $x$ term (other than $dx$) but we may separate variables as before:

$$\int dx \;\equiv\; \int 1\,dx \;=\; \int \frac{1+y}{y}\,dy \;=\; \int \Big(\frac{1}{y}+1\Big)\,dy$$

$$\Longrightarrow \qquad x \;=\; \ln|y| \,+\, y \,+\, C \qquad \text{(arbitrary constant } C\text{)}.$$

The General Solution is $x \;=\; \ln|y| \,+\, y \,+\, C$.

Note that it is not possible to express the General Solution explicitly as $y = f(x)$ .

**(f)**

$$x\tan y + (x^2+1)\,\frac{dy}{dx} = 0$$

$$\Longrightarrow \quad (x^2+1)\frac{dy}{dx} = -x\tan y$$

$$\Longrightarrow \quad \int \frac{dy}{\tan y} = -\int \frac{x}{x^2+1}\,dx$$

$$\Longrightarrow \quad \ln|\sin y| = -\frac{1}{2}\,\ln(x^2+1) + A = \ln(x^2+1)^{-1/2} + A.$$

$$\Longrightarrow \quad |\sin y| = e^{\ln(x^2+1)^{-1/2}+A} = e^{\ln(x^2+1)^{-1/2}} \times e^A = e^A\,(x^2+1)^{-1/2}$$

$$\Longrightarrow \quad \sin y = B(x^2+1)^{-1/2} \qquad \text{(where } B = \pm\, e^A \text{ is arbitrary).}$$

**(g)**

$$xy^3\,\frac{dy}{dx} = (1+x^2)(1+y^2)$$

$$\Longrightarrow \quad \int \frac{y^3}{1+y^2}\,dy = \int \frac{1+x^2}{x}\,dx$$

$$\Longrightarrow \quad \int \Big(y - \frac{y}{1+y^2}\Big)\,dy = \int \Big(\frac{1}{x}+x\Big)\,dx$$

$$\Longrightarrow \quad \frac{y^2}{2} - \frac{1}{2}\,\ln(1+y^2) = \ln|x| + \frac{x^2}{2} + C$$

$$\Longrightarrow \quad y^2 - x^2 = 2\ln|x| + \ln(1+y^2) + K \qquad \text{(where } 2C \text{ is arbitrary).}$$

If $y=1$ when $x=1$ then $\quad 0 = \ln 2 + K \quad \Longrightarrow \quad K = -\ln 2 = \ln\frac{1}{2}$ and the Particular Solution is $\quad y^2 - x^2 = \ln\Big(\frac{x^2}{2}\,(1+y^2)\Big).$

[Note: $\quad m\ln z = \ln(z^m), \qquad \ln u + \ln v = \ln(uv)$.]

**2.** In all questions, the ODE can be re-arranged into the form $\dfrac{dy}{dx} + P(x)\,y = Q(x)$.

The integrating factor I.F. is $I(x) = \exp\Big(\displaystyle\int P(x)\,dx\Big)$.

This leads to $I(x)y = \displaystyle\int I(x)Q(x)\,dx$. In each solution, $A$ is an arbitrary constant.

**(a)** $\dfrac{dy}{dx} + \dfrac{2}{x}\,y = 8x \quad\Longrightarrow\quad P(x) = \dfrac{2}{x},\;\; Q(x) = 8x.$

$I(x) = \exp\Big(\displaystyle\int \frac{2}{x}\,dx\Big) = \exp(2\ln x) = \exp(\ln x^2) = x^2$

$$x^2y = \int (x^2 \times 8x)\,dx = 8\int x^3\,dx = 2x^4 + A \quad\Longrightarrow\quad y = \frac{2x^4 + A}{x^2} \quad\text{or}\quad y = 2x^2 + \frac{A}{x^2}.$$

**(b)** $\dfrac{dy}{dx} + 2xy = 4x \quad\Longrightarrow\quad P(x) = 2x,\;\; Q(x) = 4x.$

$I(x) = \exp\Big(\displaystyle\int 2x\,dx\Big) = \exp(x^2) = e^{x^2}.$

Therefore $e^{x^2}y = \displaystyle\int 4x\,e^{x^2}\,dx.$ Put $x^2 = u,\;\; 2x\,dx = du$ :

$$\int 4x\,e^{x^2}dx = \int 2e^u\,du = 2e^u + A = 2e^{x^2} + A \quad\Longrightarrow\quad e^{x^2}y = 2e^{x^2} + A \quad\Longrightarrow\quad y = 2 + Ae^{-x^2}.$$

**(c)** $\dfrac{dy}{dx} + 2y\cot x + \sin 2x = 0 \quad\Longrightarrow\quad \dfrac{dy}{dx} + (2\cot x)y = -\sin 2x$

$\Longrightarrow\quad P(x) = 2\cot x = \dfrac{2\cos x}{\sin x},\;\; Q(x) = -\sin 2x.$

$I(x) = \exp\Big(2\displaystyle\int \frac{\cos x}{\sin x}\,dx\Big) = \exp(2\ln|\sin x|) = \exp(\ln \sin^2 x) = \sin^2 x.$

Therefore $(\sin^2 x)y = -\displaystyle\int (\sin^2 x \times \sin 2x)\,dx = -2\int \sin^3 x\cos x\,dx$

(since $\sin 2x = 2\sin x\cos x$). Put $\sin x = u,\;\; \cos x\,dx = du$ to give:

$$\int \sin^3 x\cos x\,dx = \int u^3\,du = \frac{1}{4}\,u^4 + C = \frac{1}{4}\,\sin^4 x + C$$

$$\Longrightarrow\quad (\sin^2 x)y = -2 \times \frac{1}{4}\,\sin^4 x + A \qquad (A = -2C)$$

$$\Longrightarrow\quad y = -\frac{1}{2}\,\sin^2 x + \frac{A}{\sin^2 x} = A\,\mathrm{cosec}^2 x - \frac{1}{2}\,\sin^2 x.$$

**(d)** $x(x+1)\dfrac{dy}{dx} + y = 2x \quad\Longrightarrow\quad \dfrac{dy}{dx} + \dfrac{1}{x(x+1)}\,y = \dfrac{2}{x+1}$

$\Longrightarrow\quad P(x) = \dfrac{1}{x(x+1)},\quad Q(x) = \dfrac{2}{x+1}.$

$$I(x) = \exp\Big\{\int \frac{dx}{x(x+1)}\Big\} = \exp\Big\{\int\Big(\frac{1}{x} - \frac{1}{1+x}\Big)dx\Big\} = \exp\Big\{\ln\Big(\frac{x}{x+1}\Big)\Big\} = \frac{x}{x+1}$$

$$\Longrightarrow\quad \frac{x}{x+1}\,y = \int \frac{x}{x+1} \times \frac{2}{x+1}\,dx = 2\int \frac{x}{(x+1)^2}\,dx. \qquad (**)$$

**Qu. 2 cont'd next sheet**

**2. (d) cont'd** Put $x+1=u,\ x=u-1,\ dx=du$:

$$\int \frac{x}{(x+1)^2}\,dx = \int \frac{u-1}{u^2}\,du = \int \Big(\frac{1}{u}-\frac{1}{u^2}\Big)\,du = \ln|u| + \frac{1}{u} + C = \ln|x+1| + \frac{1}{x+1} + C.$$

Therefore from (∗∗) (with $A=2C$):

$$\Big(\frac{x}{x+1}\Big)y = 2\ln|x+1| + \frac{2}{x+1} + A \quad\Longrightarrow\quad y = \Big(\ln(x+1)^2 + \frac{2}{x+1} + A\Big)\frac{(x+1)}{x}.$$

**(e)** The ODE is $x\dfrac{dy}{dx} + y \;=\; \sin x.$

Divide by $x$ to obtain the standard form,

$$\frac{dy}{dx} + \frac{1}{x}\,y \;=\; \frac{1}{x}\sin x,$$

so $p(x) = \dfrac{1}{x}$ and $q(x) = \dfrac{1}{x}\sin x$. Integrating factor:

$$I(x) \;=\; \exp\Big(\int p(x)\,dx\Big) \;=\; \exp\Big(\int \frac{1}{x}\,dx\Big) \;=\; e^{\ln x} \;=\; x.$$

Multiply the ODE by this I.F. and re-write as

$$\begin{aligned} I(x)\,y \;=\; \int I(x)\,q(x)\,dx \quad &\Longrightarrow\quad yx \;=\; \int \sin x\,dx \\ &\Longrightarrow\quad yx \;=\; -\cos x + C \qquad \text{(arbitrary constant } C\text{)} \\ &\Longrightarrow\quad y \;=\; -\frac{\cos x}{x} + \frac{C}{x}. \qquad \text{(G.S.)} \end{aligned}$$

**(f)** $(1-x^2)\dfrac{dy}{dx} - xy = 3 \quad\Longrightarrow\quad \dfrac{dy}{dx} - \dfrac{x}{1-x^2}\,y = \dfrac{3}{1-x^2}$

$\Longrightarrow\quad P(x) = -\dfrac{x}{1-x^2},\quad Q(x) = \dfrac{3}{1-x^2}\,.$

$I(x) = \exp\Big(-\displaystyle\int \frac{x}{1-x^2}\,dx\Big).$ Put $1-x^2=u,\quad -2x\,dx = du$:

$$-\int \frac{x}{1-x^2}\,dx = -\int -\frac{1}{2}\frac{du}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(1-x^2) = \ln(1-x^2)^{1/2}$$

$$\begin{aligned} &\Longrightarrow\quad I(x) = \exp\Big[\ln(1-x^2)^{1/2}\Big] = (1-x^2)^{1/2} \\ &\Longrightarrow\quad (1-x^2)^{1/2}y = \int (1-x^2)^{1/2}\times\frac{3}{1-x^2}\,dx = 3\int\frac{dx}{(1-x^2)^{1/2}} = 3\sin^{-1}x + A \\ &\Longrightarrow\quad y = (3\sin^{-1}x + A)(1-x^2)^{-1/2}. \end{aligned}$$

If $y=1$ when $x=0$ then $1=(0+A) \Longrightarrow A=1$. So the Particular Solution is $y = (3\sin^{-1}x+1)(1-x^2)^{-1/2}$.

**Qu. 2 cont'd next sheet**

**2. (g)** In standard form, the ODE is $\quad y' + \dfrac{2}{x}\,y = x\,; \qquad p(x) = \dfrac{2}{x}, \quad q(x) = x.$

$$\text{I.F.:} \quad I(x) = \exp\Big(\int \frac{2}{x}\,dx\Big) = \exp\Big(2\ln|x|\Big) = x^2.$$

$$\begin{aligned} I(x)\,y \;=\; \int I(x)\,q(x)\,dx \quad &\Longrightarrow \quad x^2 y \;=\; \int x^3\,dx \;=\; \frac{1}{4}x^4 + C \\ &\Longrightarrow \quad y \;=\; \frac{1}{4}x^2 + \frac{C}{x^2} \qquad \text{(G.S., arbitrary } C\text{)}. \\ y = 0 \ \text{ at } \ x = 1 \quad &\Longrightarrow \quad 0 \;=\; \frac{1}{4} + C \quad \Longrightarrow \quad C = -\frac{1}{4} \\ &\Longrightarrow \quad y \;=\; \frac{1}{4}x^2 - \frac{1}{4x^2} \;=\; \frac{x^4 - 1}{4x^2} \qquad \text{(P.S.)} \end{aligned}$$

**(h)** $x\,\dfrac{dy}{dx} = \sin x - 2y \quad \Longrightarrow \quad \dfrac{dy}{dx} + \dfrac{2}{x}\,y = \dfrac{\sin x}{x} \quad \Longrightarrow \quad P(x) = \dfrac{2}{x}, \quad Q(x) = \dfrac{\sin x}{x}.$

$$I(x) = \exp\Big(\int \frac{2}{x}\,dx\Big) = \exp(2\ln|x|) = \exp(\ln x^2) = x^2$$

$$\Longrightarrow \quad x^2 y = \int x^2 \times \frac{\sin x}{x}\,dx = \int x\sin x\,dx = -\cos x + \sin x + A$$

$$\Longrightarrow \quad y = \frac{1}{x^2}\,(\sin x - x\cos x + A).$$

If $y = 0$ when $x = \dfrac{\pi}{2}$ then $\quad 0 = \dfrac{4}{\pi^2}\,(1 - 0 + A) \quad \Longrightarrow \quad A = -1.$

Particular solution is $\quad y = \dfrac{1}{x^2}\,(\sin x - x\cos x - 1).$

**(i)** $x(x+1)\dfrac{dy}{dx} + y = 2 \quad \Longrightarrow \quad \dfrac{dy}{dx} + \dfrac{1}{x(x+1)}\,y = \dfrac{2}{x(x+1)}$

$\Longrightarrow \quad P(x) = \dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{1+x}, \qquad Q(x) = \dfrac{2}{x(x+1)}.$

$$I(x) = \exp\Big[\int \Big(\frac{1}{x} - \frac{1}{1+x}\Big)\,dx\Big] = \exp\Big[\ln\Big(\frac{x}{x+1}\Big)\Big] = \frac{x}{x+1}$$

$$\Longrightarrow \quad \frac{x}{x+1}\,y = \int \frac{x}{x+1} \times \frac{2}{x(x+1)}\,dx = 2\int \frac{dx}{(x+1)^2} = -\frac{2}{x+1} + A$$

$$\Longrightarrow \quad y = \frac{x+1}{x}\Big(-\frac{2}{x+1} + A\Big) = -\frac{2}{x} + \frac{A(x+1)}{x} = -\frac{2}{x} + A + \frac{A}{x}.$$

In the General Solution above, $y \to A$ as $x \to \infty$. Since we require $y \to 1$ as $x \to \infty$, we must have $A = 1$, so that the Particular Solution is

$$y = -\frac{2}{x} + \frac{x+1}{x} = -\frac{2}{x} + \Big(1 + \frac{1}{x}\Big) = 1 - \frac{1}{x}.$$

**3.** In each question, the ODE can be re-written as $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$.
Although the details have been omitted, in each solution **it is assumed** that we have introduced the new variable $v(x) = \dfrac{y(x)}{x}$, and replaced the derivative in the ODE via:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = f(v) \quad \Longrightarrow \quad x\frac{dv}{dx} = f(v) - v \quad \Longrightarrow \quad \int \frac{dv}{f(v) - v} = \int \frac{dx}{x}.$$

We then intregrate both sides of this equation, before replacing $v$ with $y/x$ and re-writing in terms of $y$. In each solution, $A$ is an arbitrary constant.

**(a)** RHS of the ODE is $\dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right) = v + \tan v = f(v) \quad \Longrightarrow \quad f(v) - v = \tan v.$
Following the process described at the start of the solutions for **Qu.3**:

$$\begin{aligned}
\int \frac{dv}{\tan v} = \int \frac{dx}{x} \quad &\Longrightarrow \quad \ln|\sin v| = \ln|x| + A \\
&\Longrightarrow \quad |\sin v| = \exp(\ln|x| + A) = e^{\ln|x|}\, e^A = |x|\, e^A \\
&\Longrightarrow \quad \sin v = Bx \qquad (B = \pm e^A \text{ is arbitrary}) \\
&\Longrightarrow \quad \sin\left(\frac{y}{x}\right) = Bx \quad \Longrightarrow \quad y = x\sin^{-1}(Bx) \qquad \left(-\frac{\pi}{2} < \frac{y}{x} < \frac{\pi}{2}\right).
\end{aligned}$$

**(b)** $\dfrac{x+y}{x-y} = \dfrac{1+\frac{y}{x}}{1-\frac{y}{x}} = \dfrac{1+v}{1-v} = f(v) \quad \Longrightarrow \quad f(v) - v = \dfrac{(1+v) - v(1-v)}{1-v} = \dfrac{1+v^2}{1-v}.$
Following the process described at the start of the solutions for **Qu.3**:

$$\int \frac{(1-v)}{1+v^2}\, dv = \int \frac{dx}{x}.$$

Now $\displaystyle\int \frac{1-v}{1+v^2}\, dv = \int \left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right) dv = \tan^{-1} v - \frac{1}{2}\ln(1+v^2)$

$$\begin{aligned}
&\Longrightarrow \quad \tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln|x| + A \\
&\Longrightarrow \quad \tan^{-1}\frac{y}{x} = \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) + \frac{1}{2}\ln x^2 + A = \frac{1}{2}\ln(x^2+y^2) + A.
\end{aligned}$$

[Note: $\ln u + \ln v = \ln(uv)$.]

**(c)** $\dfrac{y(x+y)}{x(x-y)} = \dfrac{\frac{y}{x}(1+\frac{y}{x})}{1-\frac{y}{x}} = \dfrac{v(1+v)}{1-v} = f(v)$

$\Longrightarrow \quad f(v) - v = \dfrac{v(1+v)}{1-v} - v = \dfrac{v + v^2 - v(1-v)}{1-v} = \dfrac{2v^2}{1-v}.$
Following the process described at the start of the solutions for **Qu.3**:

$$\begin{aligned}
\int \frac{1-v}{v^2}\, dv = \int \frac{2}{x}\, dx \quad &\Longrightarrow \quad \int \left(\frac{1}{v^2} - \frac{1}{v}\right) dv = 2\ln|x| + A \\
&\Longrightarrow \quad -\frac{1}{v} - \ln|v| = \ln(x^2) + A \\
&\Longrightarrow \quad -\frac{x}{y} = \ln\left|\frac{y}{x}\right| + \ln(x^2) + A = \ln|xy| + A \quad \Longrightarrow \quad \frac{y}{x} + \ln|xy| + A = 0.
\end{aligned}$$

**Qu. 3 cont'd next sheet**

**3. (d)** $\dfrac{2y}{x}+\dfrac{x}{y}=2v+\dfrac{1}{v}=f(v) \quad\Longrightarrow\quad f(v)-v=v+\dfrac{1}{v}=\dfrac{v^2+1}{v}.$

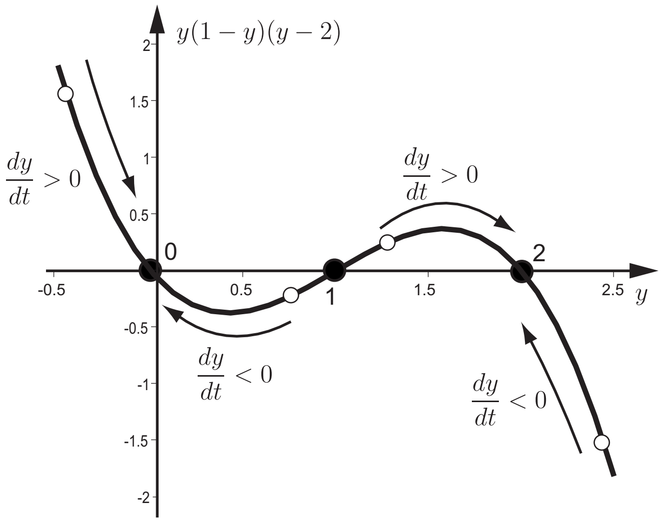
Following the process described at the start of the solutions for **Qu.3**:

$$\begin{aligned}
\int \frac{v}{v^2+1}\,dv = \int \frac{dx}{x} \quad &\Longrightarrow \quad \frac{1}{2}\ln(v^2+1)=\ln|x|+A \\
&\Longrightarrow \quad \ln(v^2+1)=\ln(x^2)+2A \\
&\Longrightarrow \quad v^2+1=e^{\ln(x^2)+2A}=e^{\ln(x^2)}\,e^{2A}=x^2\,e^{2A}=Bx^2 \qquad (B=e^{2A}) \\
&\Longrightarrow \quad \frac{y^2}{x^2}+1=Bx^2 \\
&\Longrightarrow \quad y^2=Bx^4-x^2.
\end{aligned}$$

If $y=2$ when $x=1$ then $\quad 4=B-1 \quad\Longrightarrow\quad B=5.$ So the Particular Solution is

$$\Longrightarrow \quad y^2=5x^4-x^2 \quad\Longrightarrow\quad y=x\sqrt{5x^2-1}.$$

**4.**




**(a,b)** The graph of $f(y)=y(1-y)(y-2)$ is a cubic function with zeros at $y=0$,1 and 2. These zeros are the critical (or equilibrium) points of the autonomous ODE $\dfrac{dy}{dt}=f(y)$.

**(c)** Consider a solution $y(t)$ for which $y<0$ at some time $t$.

If $y<0$ then $\dfrac{dy}{dt}>0$, so the solution must be increasing towards the critical point $y=0$ as $t$ increases.

If a solution lies between $y=0$ and $y=1$ then $\dfrac{dy}{dt}$ is negative and $y$ will decrease towards the critical point $y=0$ as $t$ increases.

Following similar procedures, solutions lying in the interval between $y=1$ and $y=2$ will increase towards $y=2$, while solutions beyond $y=2$ will decrease towards the critical point $y=2$.

From the figure we can see that $y(t)=0$ and $y(t)=2$ are stable equilibrium solutions while $y(t)=1$ is an unstable equilibrium solution.