2 Exercise Solutions: Chapter 2

1. (i) $\{x \mid x^2 = 1\} = \{-1, 1\}$. (ii) $\{y \mid y^2 = -5\} = \emptyset$.
   (iii) $\{z \mid z \text{ is the square of an integer and } z < 100\} = \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$.

2. $B \subset A$, $C \subset A$, $C \subset D$.

3. (a) $S = \{-1, 1\}$. (b) $T = \{0, 2\}$. (c) $\emptyset$. (d) $\mathbb{Z}$.

4. (i) $2, 4, 6, 162, -10$ (for $k = 1, 2, 3, 81, -5$).
   (ii) $16, 9, 4, 1, 0$ (for $y = -4, -3, -2, -1, 0$).
   (iii) $4, 1, -2, \frac{5}{2}, 3$ (for $z = 0, 1, 2, \frac{1}{2}, \frac{1}{3}$.)
   Note: samples only, any other entries must be of a similar form.

5. (i) Vectors are of the form $(0, x_2, x_3)$ for $x_2, x_3 \in \mathbb{R}$,
   e.g. $(0, 0, 0)$,$(0, 2, 5)$,$(0, -3.2, 17004)$.
   (ii) Vectors are of the form $(4 - 3y_2, y_2, y_3)$ for $y_2, y_3 \in \mathbb{R}$,
   e.g. $(4, 0, 0)$, $(1, 1, 2)$, $(0, 4/3, 17004)$.
   (iii) Vectors are of the form $(z_1, z_2, z_3, -(z_1 + z_2 + z_3))$ for $z_1, z_2, z_3 \in \mathbb{R}$,
   e.g. $(0, 0, 0, 0)$, $(1, 2, 3, -7)$, $(-1, 3, 17004, -17006)$.
   (iv) Vectors are of the form (w_1, w_2, w_3, w_3) for $w_1, w_2, w_3 \in \mathbb{R}$,
   e.g. $(0, 0, 0, 0)$, $(1, 2, 3, 3)$, $(-1, 3, 17004, 17004)$.

6. $\boldsymbol{x} = (1, 0, 2, 2)$, $\boldsymbol{y} = (3, 6, 2, 0)$.

$$\|\boldsymbol{x}\|^2 = 1 + 0 + 4 + 4 = 9 \Rightarrow \|\boldsymbol{x}\| = 3$$

$$\|\boldsymbol{y}\|^2 = 9 + 36 + 4 + 0 = 49 \Rightarrow \|\boldsymbol{y}\| = 7$$

$$\|4\boldsymbol{x}\|^2 = \|(4, 0, 8, 8)\|^2 = 16 + 0 + 64 + 64 = 144 \Rightarrow \|4\boldsymbol{x}\| = 12 = 4\|\boldsymbol{x}\|$$

$$\| - 9\boldsymbol{y}\|^2 = \|(-27, -54, -18, 0)\|^2 = 729 + 2916 + 324 = 3969 \Rightarrow \| - 9\boldsymbol{y}\| = 63 = 9\|\boldsymbol{y}\|$$

$$\boldsymbol{x} + \boldsymbol{y} = (4, 6, 4, 2) \Rightarrow \|\boldsymbol{x} + \boldsymbol{y}\|^2 = 16 + 36 + 16 + 4 = 72 \Rightarrow \|\boldsymbol{x} + \boldsymbol{y}\| = 6\sqrt{2}$$

   But $\|\boldsymbol{x}\| + \|\boldsymbol{y}\| = 3 + 7 = 10$ so $\|\boldsymbol{x} + \boldsymbol{y}\| < \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

$$\cos\theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} = \frac{3 + 0 + 4 + 0}{21} = \frac{7}{21} = \frac{1}{3}$$

   so $\theta = \cos^{-1}(1/3)$.

7. (a) $\boldsymbol{x} \cdot \boldsymbol{y} = 6 + 2 - 16 + 15 = 7$.

   (b) $\boldsymbol{x} \cdot \boldsymbol{y} = 2 - 2 + 0 + 8 + 3 = 11$.

8. (a) $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| \geq 0 \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ by definition of $\| \cdot \|$.

   (b) $d(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \|\boldsymbol{x} - \boldsymbol{y}\| = 0 \Leftrightarrow \boldsymbol{x} - \boldsymbol{y} = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$.

(c) $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| = \|(-1)(\boldsymbol{y} - \boldsymbol{x})\| = |-1|\|\boldsymbol{y} - \boldsymbol{x}\| = \|\boldsymbol{y} - \boldsymbol{x}\| = d(\boldsymbol{y}, \boldsymbol{x})$.

(d) $d(\boldsymbol{x}, \boldsymbol{z}) = \|\boldsymbol{x} - \boldsymbol{z}\| = \|(\boldsymbol{x} - \boldsymbol{y}) + (\boldsymbol{y} - \boldsymbol{z})\| \leq \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y} - \boldsymbol{z}\| = d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$. (by the Triangle Inequality).

Distance between $(-1, 2, 1, 4, 7, -3)$ and $(2, 1, -3, 5, 4, 5)$ in $\mathbb{R}^6$ is

$$\|(-3, 1, 4, -1, 3, -8)\| = \sqrt{9 + 1 + 16 + 1 + 9 + 64} = 10.$$

9. (a) $\|\boldsymbol{x} + \boldsymbol{y}\|^2 = (\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y} + 2(\boldsymbol{x} \cdot \boldsymbol{y}) = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2(\boldsymbol{x} \cdot \boldsymbol{y})$ so $\|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 \Leftrightarrow \boldsymbol{x} \cdot \boldsymbol{y} = 0$, that is, $\boldsymbol{x} \perp \boldsymbol{y}$.

(b) $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2(\boldsymbol{x} \cdot \boldsymbol{y}) = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\|\boldsymbol{x}\|\,\|\boldsymbol{y}\| \cos\theta$ (θ the angle between $\mathbf{x}$ and $\mathbf{y}$)

(c) Using (a) and (b),

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2(\boldsymbol{x} \cdot \boldsymbol{y}) + \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2(\boldsymbol{x} \cdot \boldsymbol{y}) = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2.$$

Theorems of Euclidean geometry for $\mathbb{R}^2$:
(a) is Pythagoras's theorem;
(b) is the cosine rule for triangles;
(c) states that in a parallelogram, the sum of squares on diagonals is twice the sum of squares of lengths of 2 adjacent nodes.