University of Strathclyde, Department of Mathematics and Statistics

MM102 Applications of Calculus
Exercises for Chapter 6

1. Let $y = x^2 + 3x + 4$. Find the approximate change in y if x is increased from 2 to 2.08.

2. Let $y = \sqrt{x}$. Find the approximate change in y if x is increased from 4 to 4.01.

3. Newton's law of gravitation states that the force F of attraction between two particles having masses m_1 and m_2 is given by $F = Gm_1m_2/s^2$ where G is a constant and s is the distance between the particles. Find the approximate percentage change in F if s is increased by 0.5%.

4. Consider a sphere with radius r. What is the approximate percentage change of the surface area S of the sphere if the radius is decreased by 1.5%?

5. The period T of a pendulum in seconds is given by
$$T = 2\pi\sqrt{\frac{L}{g}}$$
where L is the length of the pendulum in metres and $g = 9.81\,\mathrm{ms}^{-2}$. Find the approximate percentage change in T if the pendulum is lengthened by 1%.

6. The volume V of a sphere is measured by estimating its radius r. What is the maximum percentage error in the radius if the maximum error in V that is allowed is 1%?

7. Find $p_{2,0}$, the Taylor polynomial of degree 2 about $x = 0$, for the function
$$f(x) = \sqrt{1+x}.$$

8. Find $p_{3,\frac{\pi}{2}}$, the Taylor polynomial of degree $x = 3$ about $\dfrac{\pi}{2}$, for the function
$$f(x) = x\sin x.$$

9. Find $p_{2,1}$, the Taylor polynomial of degree 2 about $x = 1$, for the function
$$f(x) = e^{x^2}.$$

10. Find $p_{2,1}$, the Taylor polynomial of degree 2 about $x = 1$, for the function
$$f(x) = \arctan x.$$

11. For the function $f(x) = \sqrt{1+x}$ in Question 7, determine the remainder term $R_{2,0}$.

12. For the function $f(x) = x \sin x$ in Question 8, determine the remainder term $R_{3,\frac{\pi}{2}}$.

13. (a) Determine $p_{4,0}$, the Taylor polynomial of degree 4 about $x = 0$, for $f(x) = \cos(2x)$.
    (b) Use $p_{4,0}$ to obtain an approximate value for $\cos(0.4)$.
    (c) Determine the remainder term $R_{4,0}$.
    (d) Use the inequality $|\sin t| \leq 1 \ (t \in \mathbb{R})$ to obtain an upper bound of the remainder term $R_{4,0}(x)$. Hence find an upper bound for the error that is made when $\cos(0.4)$ is approximated by the Taylor polynomial as in (b).

14. (a) Determine $p_{3,1}$, the Taylor polynomial of degree 3 about $x = 1$, for $f(x) = \ln(x)$.
    (b) Use $p_{3,1}$ to obtain an approximate value for $f(1.2)$.
    (c) Use the remainder term to estimate the maximum absolute error in this result.

15. (a) Determine $p_{2,9}$, the Taylor polynomial of degree 2 about $x = 9$, for $f(x) = \dfrac{1}{\sqrt{x}}$.
    (b) Use $p_{2,9}$ to obtain an approximate value for $\dfrac{1}{\sqrt{9.1}}$.
    (c) Use the remainder term to estimate the maximum absolute error in this result.

16. Let

$$f(x) = \frac{1}{1-x}.$$

Use induction to show that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \qquad n = 0, 1, \ldots .$$

Hence show that the Maclaurin series of f is

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n.$$

(You do **not** have to show that the remainder converges to 0 if $|x| < 1$.)
Note that this series is a geometric series.

17. For each of the following functions, f, determine when f is increasing and when it is decreasing.
    (a) $f(x) = x^2 - x$
    (b) $f(x) = e^x$
    (c) $f(x) = x^3 + 3x^2 - 9x - 1$
    (d) $f(x) = x^3 + x$
    (e) $f(x) = \dfrac{1}{1+x^2}$

18. The function f is given by

$$f(x) = x^3 - 6x^2 - 15x + 75.$$

(i) Find the position and nature of the stationary points. Find when the function f is increasing and when it is decreasing.

(ii) Find the point of inflection. Find when the function is concave up and when it is concave down.

(iii) Examine the behaviour of $f(x)$ as $x \to \pm\infty$.

(iv) Use this information to sketch the graph of f showing all the critical points.

(v) How many zeros does the function f have? Answer this question using the graph and the information obtained above.

19. The function f is given by

$$f(x) = x^4 - 6x^3 + 12x^2 - 3.$$

(i) Find the points of inflection. Find when the function is concave up and when it is concave down.

(ii) Show that there is only one stationary point and find its position and nature. Find when the function is increasing and when it is decreasing.

(iii) Examine the behaviour of $f(x)$ as $x \to \pm\infty$.

(iv) Use this information to sketch the graph of f showing all the critical points.

(v) How many zeros does the function f have? Answer this question using the graph and the information obtained above.

20. For each of the following functions, f, where

(a) $f(x) = \dfrac{1}{x^2+1}$

(b) $f(x) = \dfrac{x^2 - 7x + 13}{x-2}$

(c) $f(x) = \dfrac{-3x^2 + 11x - 37}{x-2}$

(d) $f(x) = \dfrac{x-3}{x^2 - x - 2}$

(e) $f(x) = \dfrac{x+2}{x^2 + 2x - 3}$

(f) $f(x) = x\sqrt{x+2}$

(g) $f(x) = x^2 + \dfrac{1}{x^2}$

(i) find the natural domain of f;

(ii) find the asymptotes;

(iii) find the points of intersection with the axes;

(iv) find the position and nature of the stationary points;

(v) determine when the function f is increasing and when it is decreasing;

(vi) use this information to sketch the graph of f showing points of intersection with the axes, stationary points and asymptotes.

21. Let n be a positive integer and define the function f by

$$f(x) = x^2 e^{-nx}.$$

(i) Determine when the function f is increasing and when it is decreasing. Find the position and nature of the stationary points (they may depend on n).

(ii) Determine when the curve is concave up and when it is concave down. Find the points of inflection.

(iii) Find the position of the critical points relative to each other and use this information to sketch the graph of f qualitatively showing the critical points.

22. Let a be a positive constant and define the function

$$f(x) = x\sqrt{x+a}\,.$$

(i) Find the natural domain of f. (It will depend on a.)

(ii) Find the zeros of f. Find when f is positive and when it is negative.

(iii) Find the position and nature of the stationary points. Determine when the function f is increasing and when it is decreasing.
(These things will depend on a.)

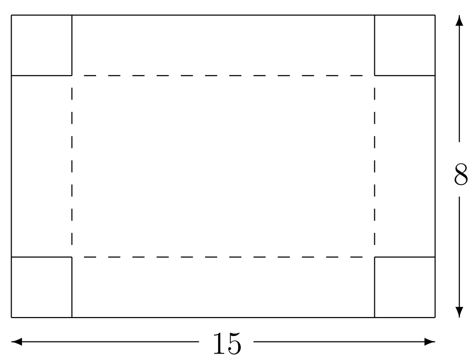
(iv) Sketch the graph of f showing the stationary points.

23. A cylindrical can is to contain $500\,\mathrm{cm}^3$ of a drink. The material used to make the top and bottom of the can costs twice as much per square centimetre as the material used to make the sides. What height and radius should the can have to minimise the costs?

24. Find the points on the curve $y^2 - x^2 = 1$ which are closest to the point $(2, 0)$.

25. The cost per hour of running a train is proportional to $100 + v^2/36$ where v, measured in m.p.h., is the average speed on a trip. Find the speed v that makes the trip Glasgow–London cheapest. (Take the distance Glasgow–London to be 400 miles.)

26. From a rectangular piece of cardboard of dimension 8×15, four congruent squares are to be cut out, one at each corner (see diagram). The remaining cross-like piece is then to be folded (along the dashed lines) into an open box. What size of squares should be cut out if the volume of the resulting box is to be a maximum?



27. An open cylindrical can, that is, the can has a bottom but no top, has a surface area of $108\,\pi\,\mathrm{cm}^2$. Find the radius and the height of the can that makes the volume a maximum.

28. The pressure P and volume V of a gas satisfy Boyle's law $PV = C$ where C is a constant. If V increases at a rate of $10\,\mathrm{cm}^3$ per minute, at what rate is P changing when $V = 2$ litres and $P = 2$ bar.

29. Sand falls onto a conical pile at a rate of $0.1\,\mathrm{m}^3\mathrm{s}^{-1}$. The radius of the base of the pile is always equal to half its height. How fast is the height increasing when the pile is
    (a) 1 m high? (b) 2 m high?

30. The centre of a corn field catches fire, and it is observed that the fire spreads circularly. If, when the radius is 15 feet, it is increasing at 3 feet per second, find at what rate is the area increasing.

31. A ladder, whose length is 17 feet, stands on horizontal ground against a vertical wall. The ladder starts to slide, and the end in contact with the ground is observed to be moving at 20 feet per second when it is 15 feet from the wall. Find the velocity of the end that is in contact with the wall at this instant.