UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS
Lecture Notes for Weeks 7 and 8

5. Ordinary Differential Equations

§5.1 Introduction

Ordinary Differential Equations (or ODEs) are equations involving unknown functions (for example, y), their derivatives (y', y'', $\ldots$) and the variable upon which they depend (x). They play an important role in the application of mathematics to practical problems. For example in biology, chemical reactions, dynamics, economics, electromagnetic waves, finance, fluid flow, particle motion, population dynamics, radio-active decay, temperature changes, etc. Differential equations provide a fundamental way of constructing mathematical models (i.e. equations) of real life phenomena. When such models are being developed, the symbols used for functions and variables are usually chosen to remind us of the quantity being studied, e.g., t for time, v for velocity, T for temperature, C for cost, etc. However in these notes, unless we are considering a particular differential equation modelling some stated phenomena, we will generally use the generic variables x and y, with y being the unknown function of x. We also adopt the standard notation $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, $y'' = y^{(2)} = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $y^{(n)} = \dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$, etc.

Examples of Differential Equations

(A) Newton's law of cooling

Newton's law of cooling states that the rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\kappa(T - T_0),$$

where $T(t)$ is the temperature of the object at time t, T_0 is the ambient temperature and κ is a positive constant.

(B) Bending of a beam

The displacment, $y(x)$, of a loaded beam at a point x units along its length is described by

$$E\,I\,\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = W(x),$$

where E is the Young's modulus of the beam, I is its moment of inertia and $W(x)$ is the load distribution per unit length.

(C) Electrical charge in a condenser

A condenser of capacity C (farads) is charged from a source of electricity of potential E (volts) through a non-inductive resistance R (ohms). The charge $Q(t)$ (coulombs) at time t (seconds) is given by

$$R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = E.$$

(D) Newton's second law of motion

Newton's second law states that the force applied to an object produces a proportional acceleration. Since accelereration is the second derivative of position with respect to time, Newton's second law for motion in one dimenion can be written as

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F,$$

where m is the mass of the object, F is the applied force, x is the position of the object and t is time.

For the special case of motion of an object falling freely under gravity, the force is $F = -mg$ where g is the gravitational acceleration $g = 9.8\mathrm{ms}^{-2}$. In this case Netwon's second law can be written as

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -mg \quad\Longrightarrow\quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -g.$$

(E) Lotka–Volterra (or predator-prey) equations

The predator-prey equations are a pair of first-order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta\, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -y(\gamma - \delta x),$$

where y is the number of some predator (for example, wolves) and x is the number of its prey (rabbits). The derivative terms $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ represent the growths of the two populations at some time t. The constants α, β, γ and δ are parameters representing the interaction between the two species.

§5.2 Definitions and terminology

This section of the class will concentrate on techniques for the solution of ordinary differential equations. It will not examine the construction/origin of the ODE. Many examples are, however, taken from actual applications and may be encountered in other classes.

There is some basic terminology with which you must be familiar:-

- The **order** of an ODE is the highest derivative appearing in the equation.
  - $m\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x, y)$ **2nd order**
  - $\dfrac{\mathrm{d}N}{\mathrm{d}t} = -kN$ **1st order**
  - $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ **2nd order**

- An ODE is **linear** if it is linear in the unknown function and its derivatives, i.e. it takes the form

$$a_n(x)\,\frac{\mathrm{d}^ny}{\mathrm{d}x^n} \;+\; a_{n-1}(x)\,\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} \;+\; \dots \;+\; a_1(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; a_0(x)\,y \;=\; f(x),$$

  for functions $a_r(x)$ $(r = 0, 1, \dots, n)$.

- An equation that is not linear is **nonlinear**.
  - General form for **1st order linear**: $a(x)\,\dfrac{\mathrm{d}y}{\mathrm{d}x} + b(x)\,y = c(x)$
  - General form for **2nd order linear**: $a(x)\,\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b(x)\,\dfrac{\mathrm{d}y}{\mathrm{d}x} + c(x)\,y = d(x)$
  - $m\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t)$ **2nd order linear**
  - $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 = \sin(x)$ **1st order nonlinear** (y^2 is a nonlinear term)
  - $x^3\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \sin(x)\,y = 0$ **2nd order linear**
  - $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} + x\sin(y) = 0$ **4th order nonlinear** ($\sin(y)$ is a nonlinear term)
  - $y''' + yy'' = x^2$ **3rd order nonlinear** (yy'' is a nonlinear term)

§5.3 The Solution of an ODE

In general, ODEs can be written in the form

$$F\Big(x,\, y,\, \frac{\mathrm{d}y}{\mathrm{d}x},\, \frac{\mathrm{d}^2y}{\mathrm{d}x^2},\, \frac{\mathrm{d}^3y}{\mathrm{d}x^3},\, \ldots\Big) \;=\; 0$$

for some function F.

A function $y = k(x)$ is called a **solution** of the ODE if $F(x, y, y', y'', y''', \ldots)$ equals zero when

$$y \text{ is replaced by } k(x), \quad \frac{\mathrm{d}y}{\mathrm{d}x} \text{ is replaced by } k'(x), \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \text{ is replaced by } k''(x), \quad \text{etc.}$$

Given an ODE for y and its derivatives in terms of independent variable x, our aim is to determine the solution $y = k(x)$ that satisfies the ODE. For example, consider the ordinary differential equations introduced at the start of this section.

(A) Newton's law of cooling $\quad \dfrac{\mathrm{d}T}{\mathrm{d}t} = -\kappa(T - T_0)$

The function $T(t) = T_0 + H\mathrm{e}^{-\kappa t}$ is a solution of Example **(A)**, where H is an arbitrary constant. To check this, differentiate $T(t)$ with respect to t:

$$\frac{\mathrm{d}T}{\mathrm{d}t} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(T_0 + H\mathrm{e}^{-\kappa t}\Big) \;=\; -\kappa H\mathrm{e}^{-\kappa t} \;=\; -\kappa(\underline{\underline{T_0 + H\mathrm{e}^{-\kappa t}}} - T_0) \;=\; -\kappa(\underline{\underline{T}} - T_0) \quad \text{as required.}$$

(B) Bending of a beam $\quad E\,I\,\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} = W(x).$

When $W(x) \equiv 0$ then it is easy to show that $y(x) = Ax^3 + Bx^2 + Cx + D$ is a solution of Example **(B)**, where A, B, C and D are arbitrary constants. (It is straightforward to confirm this because the fourth derivative of a cubic polynomial is always zero.)

(C) Electrical charge in a condenser $\quad R\dfrac{\mathrm{d}Q}{\mathrm{d}t} + \dfrac{Q}{C} = E.$

The function $Q(t) = CE(1 - K\mathrm{e}^{-t/RC})$ is a solution of Example **(C)** for arbitrary constant K. To confirm this:

$$\begin{aligned}
R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} \;&=\; R\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(CE(1 - K\mathrm{e}^{-t/RC})\Big) + \frac{1}{C}\Big(CE(1 - K\mathrm{e}^{-t/RC})\Big) \\
&=\; -RCEK\,\mathrm{e}^{-t/RC}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{t}{RC}\Big) + E(1 - K\mathrm{e}^{-t/RC}) \\
&=\; \frac{RCEK}{RC}\,\mathrm{e}^{-t/RC} + E(1 - K\mathrm{e}^{-t/RC}) \\
&=\; EK\mathrm{e}^{-t/RC} + E - EK\mathrm{e}^{-t/RC} \;=\; E \quad \text{as required.}
\end{aligned}$$

If we required the particular solution to **(C)** that exhibits the specific property $Q(0) = 0$ then we would choose $K = 1$.

(D) Motion under gravity $\quad \dfrac{d^2x}{dt^2} = -g.$

The solution to Example **(D)** is $x(t) = -\frac{1}{2}gt^2 + Mt + N$, where M and N are arbitrary constants. In the special case where the initial velocity of the object is 10ms^{-1} and its initial position is 5m from the origin (in other words, $x = 5$ and velocity $= \frac{dx}{dt} = 10$ when $t = 0$), we obtain the particular solution $x(t) = -\frac{1}{2}gt^2 + 10t + 5$.

Note that in all four cases the number of arbitrary constants in the solutions is equal to the order of the original equation, in other words

the solutions of **(A & C)** (first order) have one arbitrary constant each,
the solution of **(D)** (second order) has two arbitrary constants,
while the solution of **(B)** (fourth order) has four arbitrary constants.

- **A solution of the ordinary differential equation that involves arbitrary constants is known as the General Solution (or G.S.).**

- **Conditions may be specified on the solution of the ordinary differential equation, leading to specific values of the arbitrary constants. In this case the solution is known as the Particular Solution (or P.S.)**

- **The additional conditions on a differential equation that lead to a Particular Solution are known as boundary conditions.**

- **In the special case in which all the boundary conditions are given at the same value of the independent variable, the boundary conditions are known as initial conditions.**

  For example, the conditions we introduced above at $t = 0$ in Examples **(C)** and **(D)** are initial conditions on the differential equation.

Example

5.1 Verify that $y = C\mathrm{e}^{3x}$ is the General Solution of the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3y$ for any constant C. What is the Particular Solution with the property that $y = 2$ when $x = 1$?

Substitute $y = C\mathrm{e}^{3x}$ into the LHS of the ODE:

$$\text{LHS} \;=\; \frac{\mathrm{d}}{\mathrm{d}x}\Big(C\mathrm{e}^{3x}\Big) \;=\; 3(\underline{\underline{C\mathrm{e}^{3x}}}) \;=\; \text{RHS (when } y = C\mathrm{e}^{3x}\text{)}.$$

Therefore, $y = C\mathrm{e}^{3x}$ is a solution of the ODE for any constant C, so it must be the General Solution. If we now require that $y = 2$ when $x = 1$, then C must satisfy

$$2 = C\mathrm{e}^{3} \quad\Longrightarrow\quad C = \frac{2}{\mathrm{e}^{3}}.$$

So the Particular Solution is $y \;=\; \dfrac{2}{\mathrm{e}^{3}}\mathrm{e}^{3x} \;\equiv\; 2\mathrm{e}^{3(x-1)}$.

Examples covered in video

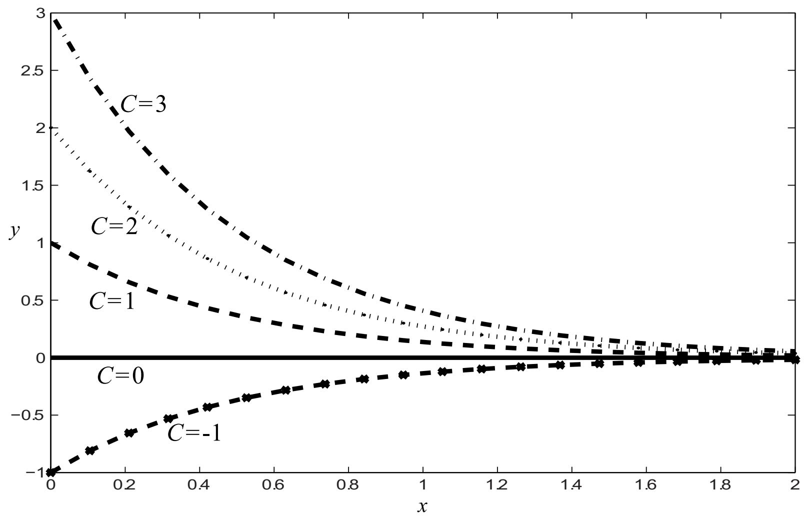
5.2 Verify that $y = \dfrac{C}{x}$ is the General Solution of the ODE $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 0$ for any constant C.

5.3 Verify that $y = A\mathrm{e}^{x} + B\mathrm{e}^{2x}$ is the General Solution of the ODE $y'' - 3y' + 2y = 0$.

§5.4 Solution Curves

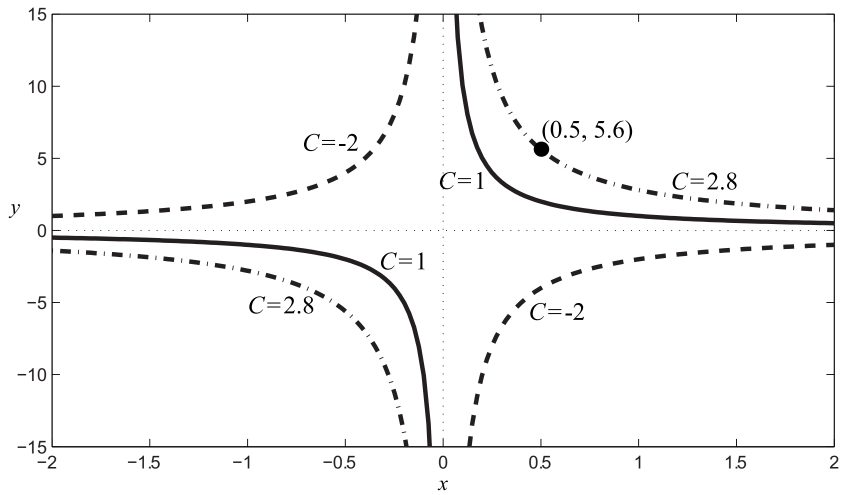
The ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2y$ can be solved to obtain the General Solution $y = C\mathrm{e}^{-2x}$ for any constant C. (In fact, for any given constant λ, the General Solution of $y' = \lambda y$ is $y = C\mathrm{e}^{\lambda x}$ for arbitrary C.)

The General Solution $y = C\mathrm{e}^{-2x}$ represents a **family of solutions**. Each value of C corresponds to a curve in the family of solutions. The figure below shows some of the solutions curves, specifically the cases $C = -1$, 0, 1, 2 and 3, for the interval $x \in [0,\ 2]$. (The case $C = 0$ corresponds to the trivial zero solution $y(x) \equiv 0$ for all values of x.)



As a second example, consider the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{x} = 0$. In Example 4.1 we verified that $y = \dfrac{C}{x}$ is the General Solution. A selection of the family of solutions are shown below for x in the interval $[-2,\ 2] \setminus \{0\}$. (Note, that the ODE is not defined when $x = 0$ as this would require division by zero in the y/x term.)

The solution curves are hyperbolae with two branches, one on either side of the origin. If we are asked to find the Particular Solution with the property that $y = 5.6$ when $x = 0.5$, then the appropriate curve passing through the point (0.5, 5.6) corresponds to $C = 2.8$.



§5.5 First Order ODEs

The general form for a first-order ordinary differential equation can be written as (for some function F):

$$F\Big(\frac{\mathrm{d}y}{\mathrm{d}x},\ x,\ y\Big) \ = \ 0.$$

For example, $y\,y' + y\cos(x) + x^2 + 1 = 0$ is a non-linear, first-order ODE.

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§5.6 Directly Integrable ODEs

The simplest type of first-order ODE occurs when we can re-arrange the differential equation into the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} \ = \ f(x),$$

where the right-hand side (RHS), namely $f(x)$, is a function of the independent variable x only, and we can simply integrate the equation to find y.

Examples covered in video

5.4 Find y as a function of x if $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2$.

5.5 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ to find the solution that satisfies the boundary condition $y = 0$ when $x = 0$.

§5.7 Separable Equations

The general form of a **separable** first-order ODE is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)\,h(y).$$

In other words, the RHS of the differential equation factorises as a product of a function of the independent variable x, and a function of the dependent variable y. We can solve such equations as follows:

- Re-arrange the ODE into the form $\displaystyle \frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)\,.$

- Integrate both sides (with respect to x) to find the General Solution:

$$\int \frac{1}{h(y)}\,\frac{\mathrm{d}y}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int g(x)\,\mathrm{d}x \qquad \Longrightarrow \qquad \int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x\,,$$

  (assuming $h(y) \neq 0$). Essentially all the terms involving y are grouped into one integral, while all terms involving x are grouped into another.

- Note that although there are two separate integrations involved there will only be **one** arbitrary constant. This is because the two constants of integration can be combined into a single constant.

- If $h(y) = 0$ has one or more solutions $y =$ (a constant), then these constant solutions (if not obtainable from the G.S.) are **singular solutions** of the ODE. In most cases the singular solution will not be of any practical significance in a real (modelled) problem.

Note: when working with ODEs, be **very** careful when dealing with integrals of the form

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C \qquad \text{(arbitrary constant } C\text{)}.$$

In some problems, you may know that $f(x) \geq 0$ so you can forget about the modulus signs, for example, $f(x) = 1 + x^2$ or $f(x) = \mathrm{e}^{-x^2}$.

In other examples, you may be given boundary conditions which imply that retaining the modulus signs is unnecessary. You might even be told that $x > 0$ and/or $y > 0$ so that $\ln|x| = \ln x$, $\ln|y| = \ln y$.

However, the rest of the time it may be necessary to retain modulus signs when doing your calculation and deriving the General Solution.

Example

5.6 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy$ (where $y \geq 0$).

Assume that $y \neq 0$ and divide both sides of the ODE by y to obtain $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = x$.

Now separate the variables and integrate both sides:

$$\begin{aligned}
\int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int x\,\mathrm{d}x \\
\Longrightarrow \quad \int \frac{1}{y}\,\mathrm{d}y &= \int x\,\mathrm{d}x \\
\Longrightarrow \quad \ln|y| = \ln y &= \frac{x^2}{2} + C \qquad (C \text{ is an arbitrary constant, } y \geq 0) \\
\Longrightarrow \quad \mathrm{e}^{\ln y} &= \mathrm{e}^{(\frac{1}{2}x^2 + C)} = \mathrm{e}^{\frac{1}{2}x^2}\mathrm{e}^{C} = A\mathrm{e}^{\frac{1}{2}x^2} \\
&\qquad (\text{where we have set constant } A = \mathrm{e}^{C}) \\
\Longrightarrow \quad y &= A\mathrm{e}^{\frac{1}{2}x^2}.
\end{aligned}$$

So the General Solution is $\underline{\underline{y = A\mathrm{e}^{\frac{1}{2}x^2} \equiv A\exp\left(\frac{1}{2}x^2\right)}}$, where A is an arbitrary constant.

Note: we don't need to consider the modulus of y after integrating $\dfrac{1}{y}$ because the question tells us that $y \geq 0$.

You can check the General Solution by substituting into the original ODE:

$$\text{LHS} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(A\mathrm{e}^{\frac{1}{2}x^2}\right) = A\mathrm{e}^{\frac{1}{2}x^2} \times x = xy = \text{RHS}.$$

There is also a singular solution that satisfies the ODE, $y(x) \equiv 0$ for all values of x. (If $y(x) \equiv 0$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 = x \times 0 = xy$.)

Examples covered in video

5.7 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

5.8 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x^2+1)(y^2+1)$.

5.9 Solve $y\dfrac{\mathrm{d}y}{\mathrm{d}x} = x(y^2+1)$ subject to the condition $y(0) = 2$.

5.10 Solve $x(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = 3$ subject to $y(1) = 3$.

§5.8 First-Order Linear ODE

A first order linear ODE is usually written in the 'standard' form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)\,y = q(x).$$

The solution is obtained with the aid of a function known as the **integrating factor (or I.F.)**, defined to be:

$$\boldsymbol{I(x) = \mathrm{e}^{\int p(x)\,\mathrm{d}x} = \exp\Big(\int p(x)\,\mathrm{d}x\Big).}$$

This function has the special property that $\dfrac{\mathrm{d}I}{\mathrm{d}x} = I(x)\,p(x)\,.$

Multiply both sides of the ODE by $I(x)$ to obtain:

$$I(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + \underline{\underline{I(x)\,p(x)}}\,y = I(x)\,q(x)$$

$$\downarrow$$

$$\Longrightarrow \qquad I(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}I}{\underline{\underline{\mathrm{d}x}}}\,y = I(x)\,q(x)$$

$\Big[$Note that by the product rule: $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big(I(x)\,y\Big) \equiv I(x)\,\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{\mathrm{d}I}{\mathrm{d}x}\,y.\Big]$

$$\Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}\Big(I(x)\,y\Big) = I(x)\,q(x) \qquad (1)$$

$$\Longrightarrow \qquad I(x)\,y = \int I(x)\,q(x)\,\mathrm{d}x \qquad (2)$$

$$\Longrightarrow \qquad y = \frac{1}{I(x)}\int I(x)\,q(x)\,\mathrm{d}x \qquad (3)$$

The General Solution of the differential equation can be calculated from one of the equations (1), (2) and (3) – **try to remember one (or all) of them!**

Example

5.11 Solve $\dfrac{dy}{dx} + y = x$ subject to the condition $y(0) = 1$.

This is a linear ODE with $p(x) = 1$, $q(x) = x$.

Integrating Factor: $I(x) = \exp\Big(\displaystyle\int 1\,dx\Big) = e^x$.

(Note: it's not necessary to include an arbitrary constant of integration when calculating the integrating factor.)

Now multiply the ODE by the I.F.:

$$e^x\,y' + e^x\,y \;=\; x\,e^x \quad\Longrightarrow\quad \frac{d}{dx}\Big(e^x\,y\Big) \;=\; x\,e^x$$

Integrate both sides: $e^x\,y \;=\; \displaystyle\int x\,e^x\,dx \;=\; x\,e^x - e^x + C$ (integration by parts)

$\Longrightarrow\quad y \;=\; x - 1 + C\,e^{-x}.$ (G.S., arbitrary C)

When $x = 0$, $y(0) = 1$ $\Longrightarrow$ $1 = -1 + C$, so $C = 2$.

Particular Solution: $\underline{\underline{y \;=\; x - 1 + 2e^{-x}}}$.

Examples covered in video

5.12 Solve $x\dfrac{dy}{dx} - y \;=\; x^2 e^{2x}$ $(x \neq 0)$.

5.13 Solve $x(x-1)y' - y = x - 1$ $(x > 1)$.

§5.9 Homogeneous Equations

If the right-hand side of a first-order ODE can be written so that the two variables x and y always appear in the ratio y/x,

$$\text{i.e.} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right) \qquad \text{for some function } f,$$

then we say that we have a **homogeneous ODE**.

Alternatively, the ODE is homogeneous if the right-hand side $F(x,\,y)$ satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,\,y) = F(kx,\,ky) \quad \text{for all} \quad k \in \mathbb{R}.$$

If we introduce a new dependent variable

$$\boldsymbol{v(x) = \frac{y(x)}{x}}, \qquad \textbf{i.e.} \quad \boldsymbol{y(x) = x\,v(x)}$$

so that $\boldsymbol{\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\,\dfrac{\mathrm{d}v}{\mathrm{d}x} + v}$**,** by the product rule the ODE can be re-written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\,\frac{\mathrm{d}v}{\mathrm{d}x} + v = f(v)\,.$$

Therefore

$$\boldsymbol{\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{f(v) - v}{x}}, \tag{$**$}$$

which is a separable equation for v.

Once $(**)$ is solved for $v(x)$, we re-introduce $y(x)$ via $y(x) = x\,v(x)$.

Example

5.14 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{y} + \dfrac{y}{x} + 1$, assuming $x > 0$ and $y > 0$.

The right-hand side of the ODE can be written as

$$\frac{x}{y} + \frac{y}{x} + 1 = \frac{1}{\left(\frac{y}{x}\right)} + \frac{y}{x} + 1 \equiv f\left(\frac{y}{x}\right), \qquad \text{where} \quad f(v) = \frac{1}{v} + v + 1.$$

Alternatively, the right-hand side could be denoted by $F(x,\,y)$, a function with the property

$$F(kx,\,ky) = \frac{kx}{ky} + \frac{ky}{kx} + 1 = \frac{x}{y} + \frac{y}{x} + 1 = F(x,\,y).$$

Therefore, the ODE must be homogeneous.

Substitute $\quad v(x) = \dfrac{y(x)}{x} \quad$ so that $\; y(x) = x\,v(x) \quad$ and $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = x\,\dfrac{\mathrm{d}v}{\mathrm{d}x} + v.$

The $\frac{y}{x}$ terms in the right-hand side of the ODE can now be replaced by v and the ODE re-written as

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} + v \;=\; \frac{1}{v} + v + 1 \quad\Longrightarrow\quad x\,\frac{\mathrm{d}v}{\mathrm{d}x} \;=\; \frac{1+v}{v} \quad\Longrightarrow\quad x\,\frac{\mathrm{d}v}{\mathrm{d}x} \;=\; \frac{1+v}{x\,v}$$

Note: $\dfrac{1+v}{x\,v} \equiv \dfrac{f(v)-v}{x}$, so this step in the calculation is equivalent to equation (**) introduced on the previous page.

We now have a separable ODE which we can integrate using the method introduced in §5.7.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x} \;&=\; \int \frac{v}{1+v}\,\mathrm{d}v \;=\; \int \frac{(1+v)-1}{1+v}\,\mathrm{d}v \;=\; \int \Big(1 - \frac{1}{1+v}\Big)\,\mathrm{d}v \\
\Longrightarrow \qquad \ln|x| \;&=\; v - \ln(1+v) + C \\
&=\; \frac{y}{x} - \ln\Big(1 + \frac{y}{x}\Big) + C \\
&=\; \frac{y}{x} - \ln\Big(\frac{x+y}{x}\Big) + C \qquad \text{(arbitrary } C\text{)} \\
&=\; \frac{y}{x} - \ln(x+y) + \ln(x) + C.
\end{aligned}$$

Therefore the G.S. is $\dfrac{y}{x} = \ln\big[A(x+y)\big]$ (taking $C = -\ln A$), or $\underline{\underline{\mathrm{e}^{y/x} = A(x+y)}}$.
Note that it's not always possible to write y explicitly as a function of x.

Examples covered in video

5.15 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2+2y^2}{xy}$, subject to the condition $y(1) = 1$.

5.16 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-y}{x+y}$ assuming $y > x > 0$.

§5.10 Direction Fields

Many first order ODES can be reduced to the form

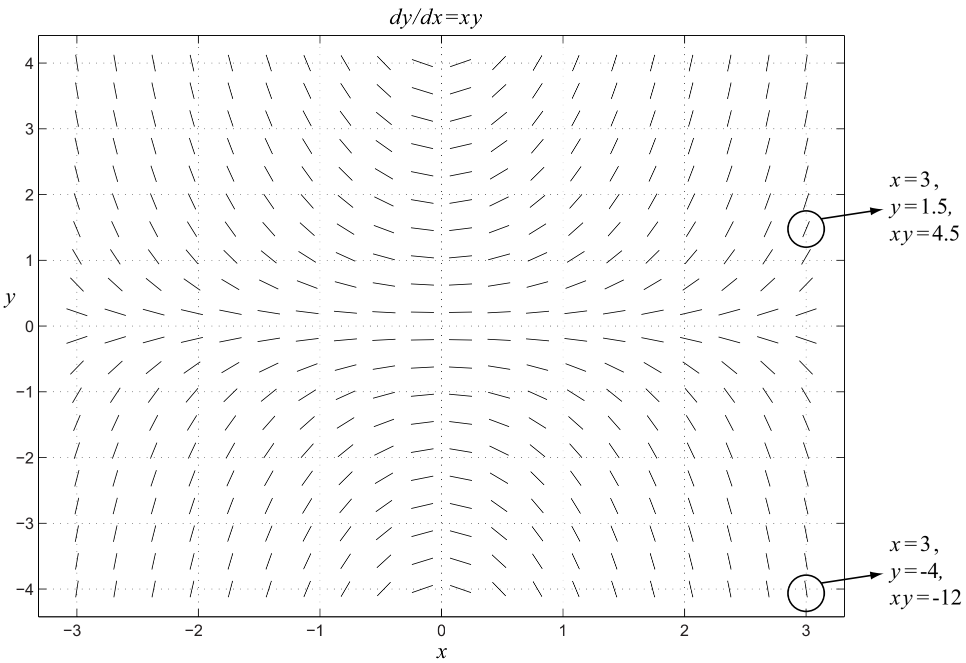
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,\, y)$$

where f can be a function of both x and y. This ODE has a geometric interpretation which we can determine from the nature of the function $f(x,\, y)$.

From calculus we know that the derivative $y'(x)$ represents the slope of the function $y(x)$ at the point $(x,\, y)$. Hence, a solution curve for the ODE $y' = f(x,\, y)$ that passes through a point $(x_0,\, y_0)$ must have a slope $y'(x_0)$ equal to the value of the function f at that point; in other words, for any point $(x_0,\, y_0)$ in the xy-plane we can compute the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ via

$$y'(x_0) = f(x_0,\, y_0).$$

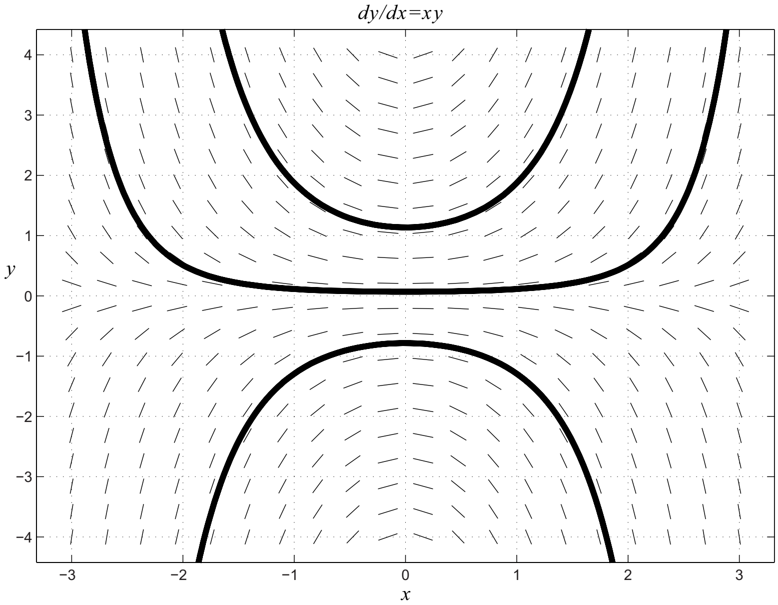
If we do this for a grid of points in the xy-plane, we can draw a picture similar to that below.



The figure above corresponds to the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = x\,y.$

At each point in our xy-grid we have drawn a short line segment with gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$. (For example, at the point (3, 1.5) the gradient is $3 \times 1.5 = 4.5$, while the gradient at $(3, -4)$ is -12.)

This diagram is called the **direction field** for the ODE. A solution curve for the ODE has the property that its gradient is the same as the direction of the direction field at every point on the curve. Therefore, the direction field consists of lines that are **tangential to the solution curves**. By carefully examining the direction field, we are then able to derive the appearance of the solution curves, as shown in the next figure. By continuing this process, we could cover the whole xy-plane with infinitely many different solution curves, each one a Particular Solution of the original ODE.

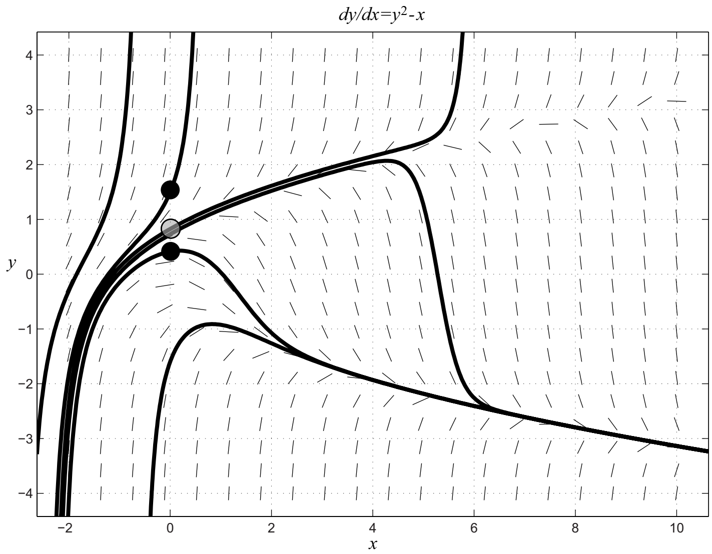


We can compare these curves via the General Solution obtained analytically:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy \quad \Longrightarrow \quad \int \frac{\mathrm{d}y}{y} = \int x\,\mathrm{d}x \quad \Longrightarrow \quad \ln(y) = \frac{1}{2}x^2 + C \quad \Longrightarrow \quad y = \mathrm{e}^{\frac{1}{2}x^2+C} \equiv A\mathrm{e}^{\frac{1}{2}x^2},$$

where C is an arbitrary constant and $A = \mathrm{e}^C$. The three solution curves in the figure above correspond to three different values of the constant A.

As a second example, consider the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x$. This ODE is very difficult to solve analytically. Nonetheless, we can plot solution curves by examining the direction field, as shown below.



In the figure we have drawn six solution curves, each corresponding to a different Particular Solution. The behaviour of a Particular Solution is very sensitive to the initial (or boundary) conditions imposed on the problem. The solution with the property that $y = 1.5$ when $x = 0$ appears to grow very rapidly as x increases, while the solution with the property that $y = 0.4$ when $x = 0$ has a maximum turning point then appears to decrease slowly. Indeed, the solutions are extremely sensitive close to the point $x = 0$, $y = 0.8$. Even a tiny change in the initial value of y when $x = 0$ can lead to two solutions which are very different in behaviour.

§5.11 Autonomous First Order ODEs

A first order ODE (with independent variable t) of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y),$$

for some function $f(y)$, is said to be **autonomous**. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin(y), \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = y(y+2)(y-1)$$

are all autonomous first order ODES, while

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t\,y^2 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos(t)$$

are not.

It may be possible to obtain a solution for the autonomous ODE by separation of variables – this would rely on you being able to integrate $1/f(y)$ as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}t} \;=\; f(y) \qquad \Longrightarrow \qquad \int \frac{\mathrm{d}y}{f(y)} \;=\; \int \mathrm{d}t \;=\; t + C, \quad \text{etc.}$$

However, in some cases it may prove too difficult to integrate $1/f(y)$. Nonetheless, it may be possible to obtain qualitative information about possible solutions. (Here, *qualitative information* means that you may be able to obtain information about how the solution behaves without necessarily calculating the solution itself.)

As an example, consider the following ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} \;=\; Ay - By^2,$$

where A and B are constants. This equation is known as the **logistic equation** and it plays an important role in population dynamics, an area of mathematics that models the evolution of populations of plants, animals or humans over time t.

If $B = 0$ then by separation of variables we can show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} \;=\; Ay \qquad \Longrightarrow \qquad y(t) \;=\; C\mathrm{e}^{At} \quad \text{for arbitrary constant } C.$$

Assuming A is positive, this models exponential growth of the population with nothing to prevent the population increasing over time.

The term $-By^2$ is called the *braking term* that prevents the population from growing without bound. If we re-arrange the ODE as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ay - By^2 = Ay\Big(1 - \frac{B}{A}y\Big),$$

we can see that (for $A > 0$)

- if $y < \dfrac{A}{B}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}t} > 0$, so an initially small population keeps growing for as long as $y < \dfrac{A}{B}$;
- if $y > \dfrac{A}{B}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}t} < 0$ and the population is decreasing for as long as $y > \dfrac{A}{B}$.

Clearly the value of $\dfrac{A}{B}$ is very significant in determining the behaviour of the solution of the ODE. In fact, the *constant function*

$$y(t) = \frac{A}{B} \quad \text{(a constant) for } \underline{\textbf{all}} \text{ times } t$$

is a solution of the original logistic equation. (The derivative of a constant function is zero, while $Ay - By^2$ is also zero when $y \equiv \dfrac{A}{B}$.) This constant solution is an example of an **equilibrium solution**.

Consider the autonomous first order ODE

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{t}} = \boldsymbol{f(y)} \quad \textbf{for some function } \boldsymbol{f(y)}.$$

The ODE has constant solutions, called equilibrium solutions, determined by the zeros of $f(y)$.

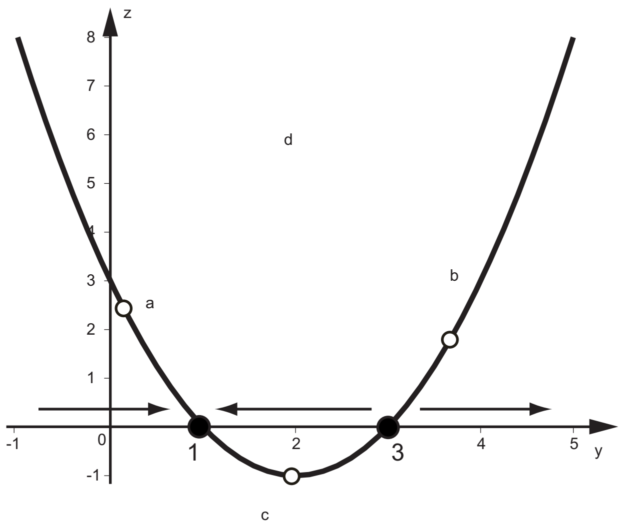
Remember, if $f\big(y(t)\big) = 0$ for all values of t then $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ for all t, so $y(t)$ must be constant.

The zeros of $f(y)$ are called equilibrium or critical points of the ODE.

Consider the autonomous ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (y-1)(y-3).$$

This ODE has two (constant) equilibrium solutions, $y_1(t) = 1$ and $y_2(t) = 3$ for all t. The critical points y_1 and y_2 are the zeros of the quadratic function, as shown in the parabola below.



- Now consider any (non-constant) solution to the ODE that exhibits a value of y smaller than the critical point $y_1 = 1$ at some time t.

  We can see from the parabola that if $y < 1$ then $\dfrac{\mathrm{d}y}{\mathrm{d}t} = f(y) > 0$, so this solution $y(t)$ must be **increasing towards $\boldsymbol{y_1 = 1}$** as t increases.

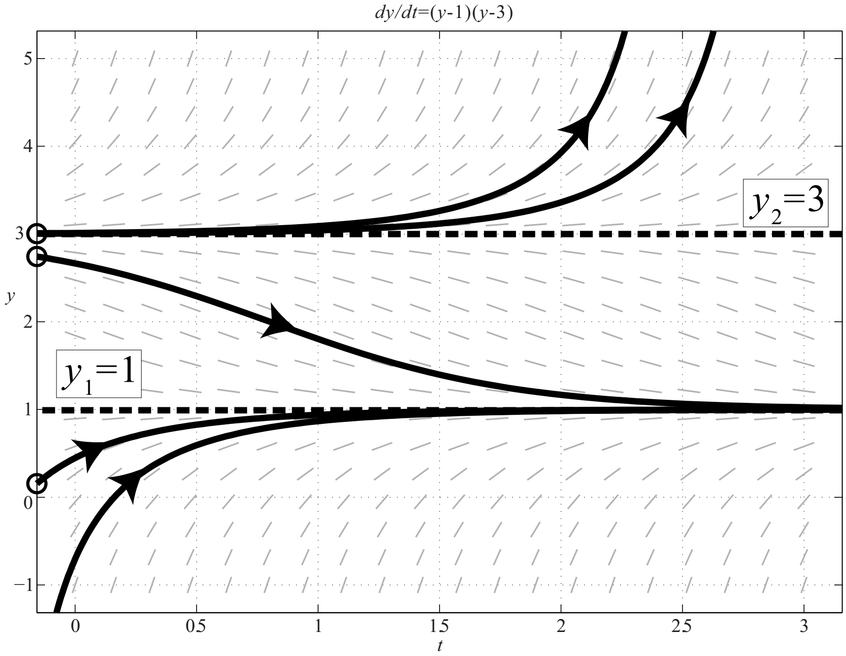
- Similarly, for any solution lying between $y_1 = 1$ and $y_2 = 3$ then $\dfrac{\mathrm{d}y}{\mathrm{d}t} = f(y) < 0$.
  In this case the solution must be **decreasing towards $\boldsymbol{y_1 = 1}$** as t increases.

  Provided a solution is close enough to the equilibrium solution $y_1 = 1$ at some time t, it will remain close to it for all later values of t. The equilibrium solution $y_1 = 1$ is known as a **stable equilibrium solution**.

- Conversely, consider the behaviour of a solution $y(t)$ which, for some time t, lies close to the critical point $y_2 = 3$.

  Regardless of whether $y < 3$ or $y > 3$ at some time t, the solution will **move away from $\boldsymbol{y_2 = 3}$**. Solutions with a value of $y > 3$ will grow as t increases; solutions with $y < 3$ will decrease towards the stable solution at $y_1 = 1$.

  In this case $y_2 = 3$ is called an **unstable equilibrium solution**.



The direction fields in the figure above help visualize stable and unstable solutions. The solutions that pass through the points $(t, y) = (0, 0.2)$ and $(0, 2.7)$ both tend towards the stable equilibrium solution $y_1(t) = 1$ as time t increases. However, the solution passing through $(t, y) = (0, 3.05)$ rapidly moves away from the unstable solution $y_2(t) = 3$.

Example

5.17 Find the critical points of the autonomous ODE

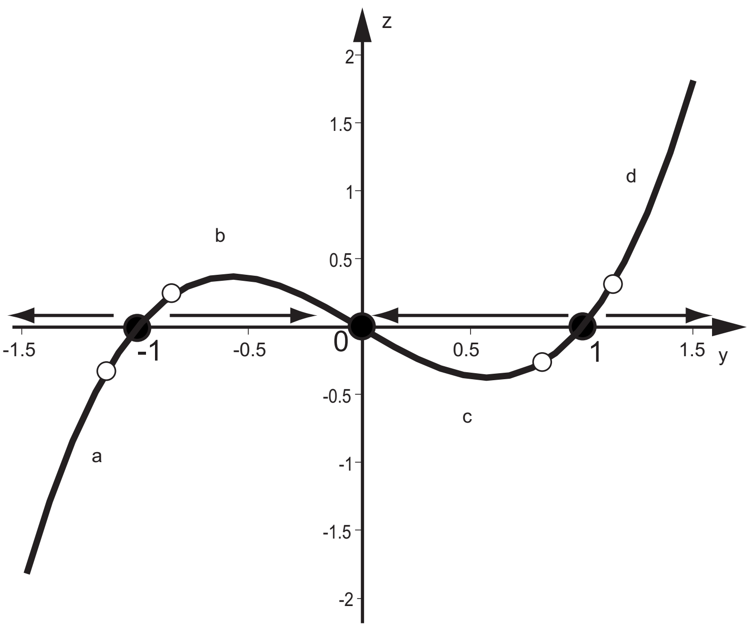
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^3 - y,$$

and determine whether they correspond to stable or unstable equilibrium solutions.

The critical points of the ODE correspond to the zeros for the RHS of the ODE.

$$y^3 - y = y(y^2 - 1) = y(y-1)(y+1) = 0 \qquad \Longleftrightarrow \qquad y = 0,\ 1 \text{ or } -1.$$

So the three critical points are $y = 0$, $y = 1$ and $y = -1$.



To begin, consider a solution $y(t)$ for which $y < -1$ at some time t.

If $y < -1$ then $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y^3 - y < 0$, so the solution must be decreasing away from the critical point $y = -1$ as t increases.

If a solution lies between $y = -1$ and $y = 0$ then $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ is positive and y will increase towards the critical point $y = 0$ as t increases.

Following similar procedures, solutions lying in the interval between $y = 0$ and $y = -1$ will decrease towards $y = 0$, while solutions beyond $y = 1$ will increase away from that critical point.

From the figure we can see that $y(t) = 0$ is a stable equilibrium solution while $y(t) = -1$ and $y(t) = 1$ are unstable equilibrium solutions.

§5.12 Second Order, Linear ODEs with Constant Coefficients

One type of second order differential equation that arises frequently is the linear equation with constant coefficients. These equations are very important in many applications and take the form

$$\boldsymbol{a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; b\,\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; c\,y \;=\; f(x),} \qquad \textbf{(A)}$$

where a, b and c are constants with $a \neq 0$, and $f(x)$ is a given function.

When $f(x)$ isn't the zero function, the General Solution of equation **(A)** is a combination of two terms:

- One is the general solution to the **homogeneous equation** where $f(x) = 0$.

  The General Solution to the homogeneous equation is known as the **Complementary Function (C.F.)** of **(A)**.

- The second is **any** solution to the original non-homogeneous equation **(A)**.

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§5.13 Second Order, Linear Homogenous Equations

A special case of a second order, linear ODE is the homogeneous equation with $f(x) = 0$ on the right hand side:

$$\boldsymbol{a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; b\,\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; c\,y \;=\; 0,} \qquad \textbf{(B)}$$

where a, b and c are constants with $a \neq 0$.

We look for solutions to **(B)** of the form $y(x) = \mathrm{e}^{mx}$, where m is a constant. If $y(x) = \mathrm{e}^{mx}$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = m\mathrm{e}^{mx} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = m^2\mathrm{e}^{mx}.$$

Hence, $y = \mathrm{e}^{mx}$ is a solution of **(B)** if and only if, for all values of x,

$$a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; b\,\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; c\,y \;=\; am^2\mathrm{e}^{mx} + bm\mathrm{e}^{mx} + c\mathrm{e}^{mx} \;=\; (\underline{\underline{am^2 + bm + c}})\,\mathrm{e}^{mx} \;=\; 0.$$

Thus, $y(x) = \mathrm{e}^{mx}$ will be a solution to the homogeneous equation provided

$$\boldsymbol{am^2 \;+\; bm \;+\; c \;=\; 0.} \qquad \textbf{(C)}$$

This quadratic equation is known as the **auxiliary equation (A.E.)** of differential equation **(A)** or **(B)**. There are three possible cases based on the roots of the auxiliary equation.

<u>Case 1.</u> **The auxiliary equation has distinct real roots $m = m_1,\ m_2 \quad (m_1 \neq m_2)$**

In this case $\mathrm{e}^{m_1 x}$ and $\mathrm{e}^{m_2 x}$ are both solutions of equation **(B)**, and so is

$$y(x) \;=\; A\mathrm{e}^{m_1 x} + B\mathrm{e}^{m_2 x}$$

for any constants A and B. To verify this:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(A\mathrm{e}^{m_1 x} + B\mathrm{e}^{m_2 x}\Big) \;=\; Am_1\mathrm{e}^{m_1 x} + Bm_2\mathrm{e}^{m_2 x} \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(A\mathrm{e}^{m_1 x} + B\mathrm{e}^{m_2 x}\Big) \;=\; Am_1^{\,2}\mathrm{e}^{m_1 x} + Bm_2^{\,2}\mathrm{e}^{m_2 x} \\
a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; b\,\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; c\,y &= A\,\underbrace{(am_1^{\,2} + bm_1 + c)}_{0}\,\mathrm{e}^{m_1 x} + B\,\underbrace{(am_2^{\,2} + bm_2 + c)}_{0}\,\mathrm{e}^{m_2 x} \\
&= 0 \;\; \text{because both } m_1 \text{ and } m_2 \text{ are both roots of the A.E.}
\end{aligned}$$

The function $y(x) \;=\; A\mathrm{e}^{m_1 x} + B\mathrm{e}^{m_2 x}$, for arbitrary constants A and B, is the general solution of **(B)**, and hence the Complementary Function (C.F.) of **(A)**.

<u>Case 2.</u> **The auxiliary equation has one repeated root, $m = m_1$**

In this case $\mathrm{e}^{m_1 x}$ is one solution of **(B)**. We can show that $y = x\mathrm{e}^{m_1 x}$ is also a solution of **(B)**:

$$\begin{aligned}
a\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x\mathrm{e}^{m_1 x}) + b\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^{m_1 x}) + cx\mathrm{e}^{m_1 x} &= a(m_1^{\,2}x + 2m_1)x\mathrm{e}^{m_1 x} + b(m_1 x + 1)\mathrm{e}^{m_1 x} + cx\mathrm{e}^{m_1 x} \\
&= (am_1^{\,2} + bm_1 + c)x\mathrm{e}^{m_1 x} + (2am_1 + b)\mathrm{e}^{m_1 x} \\
&= (2am_1 + b)\mathrm{e}^{m_1 x} \qquad \text{since } m_1 \text{ is the root of the A.E.}
\end{aligned}$$

Now, since $m = m_1$ is the repeated root of the A.E., it follows that

$$am^2 + \underline{\underline{b}}\,m + c \;\equiv\; a(m - m_1)^2 \;=\; am^2 + \underline{\underline{(-2am_1)}}\,m + am_1^{\,2}.$$

By comparing the coefficient of m on either side, we see that m_1 must satisfy $-2am_1 = b$, so that $2am_1 + b = 0$. This confirms that $y = x\mathrm{e}^{m_1 x}$ is also a solution of **(B)**, independent of solution $\mathrm{e}^{m_1 x}$ in the sense that neither is a constant multiple of the other.

Combining the two solutions gives a General Solution of **(B)**, which is also the C.F. for **(A)**:

$$y(x) \;=\; A\mathrm{e}^{m_1 x} \;+\; Bx\mathrm{e}^{m_1 x} \;=\; (A + Bx)\mathrm{e}^{m_1 x},$$

for arbitrary constants A and B.

Case 3. The auxiliary equation has complex roots $m = \alpha \pm i\beta$ $(\alpha,\ \beta \in \mathbb{R},\ \beta > 0)$

Since $m = \alpha + i\beta$ is a root of the A.E.,

$$\begin{aligned} a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = 0 \quad &\Longrightarrow \quad a(\alpha^2 - \beta^2) + b\alpha + c + i(2a\alpha + b)\beta = 0 \\ &\Longrightarrow \quad a(\alpha^2 - \beta^2) + b\alpha + c \ \text{ and } \ 2a\alpha + b = 0. \qquad (\Delta) \end{aligned}$$

We can now show that $\mathrm{e}^{\alpha x}\cos(\beta x)$ and $\mathrm{e}^{\alpha x}\sin(\beta x)$ are independent solutions of **(B)**.

$$\begin{aligned} & a\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathrm{e}^{\alpha x}\cos(\beta x)) + b\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\alpha x}\cos(\beta x)) + c\,\mathrm{e}^{\alpha x}\cos(\beta x) \\ = \ & a\{(\alpha^2 - \beta^2)\mathrm{e}^{\alpha x}\cos(\beta x) - 2\alpha\beta\mathrm{e}^{\alpha x}\sin(\beta x)\} \\ & \qquad + b\{\alpha\mathrm{e}^{\alpha x}\cos(\beta x) - \beta\mathrm{e}^{\alpha x}\sin(\beta x)\} + c\,\mathrm{e}^{\alpha x}\cos(\beta x) \\ = \ & \{\underline{\underline{(\alpha^2 - \beta^2) + b\alpha + c}}\}\mathrm{e}^{\alpha x}\cos(\beta x) - \underline{\underline{(2a\alpha + b)}}\mathrm{e}^{\alpha x}\sin(\beta x) \\ = \ & 0 \quad \text{using } (\Delta) \text{ above.} \end{aligned}$$

There is a similar result for $\mathrm{e}^{\alpha x}\sin(\beta x)$. Consequently, for arbitrary constants A and B, the function

$$y(x) = \mathrm{e}^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$$

is the General Solution of **(B)** and, in turn, the complementary function of **(A)**.

Note: this General Solution can also be obtained in a similar manner to Case 1 by considering

$$y(x) = P\mathrm{e}^{(\alpha+i\beta)x} + Q\mathrm{e}^{(\alpha-i\beta)x} = \mathrm{e}^{\alpha x}(P\mathrm{e}^{i\beta x} + Q\mathrm{e}^{-i\beta x}),$$

for constants P and Q. By expressing $\mathrm{e}^{i\beta x} = \cos(\beta x) + i\sin(\beta x)$ and re-arranging, it is possible to re-write $y(x)$ in the form

$$y(x) = P\mathrm{e}^{(\alpha+i\beta)x} + Q\mathrm{e}^{(\alpha-i\beta)x} \equiv \mathrm{e}^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right],$$

where A and B are constants.

Examples covered in video

5.18 Find the General Solution of the second order, linear homogeneous ODE

$$y'' - 3y' + 2y = 0.$$

5.19 Find the General Solution of $y'' + 5y' = 0$.

5.20 Find the General Solution of $y'' + 2y' + y = 0$.

5.21 Find the General Solution of $y'' + 2y' + 5y = 0$.

5.22 Find the General Solution of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = 0$.

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§5.14 Second Order, Linear Non-Homogenous Equations

We now turn our attention to finding solutions to the non-homogenous second order, linear equation introduced earlier:

$$\boldsymbol{a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + c\,y = f(x),} \qquad \mathbf{(A)}$$

where a, b and c are constants with $a \neq 0$, and $f(x)$ is a given non-zero function.

The method of solution is based on

1) obtaining the **Complementary Function, y_{CF}** (i.e. the General Solution of the homogeneous equation as discussed in §5.12),

2) finding a **Particular Integral (P.I.), y_{PI},** for equation **(A)**.

   The Particular Integral is **any** solution of the ODE and is normally determined by the nature of $f(x)$.

The General Solution, y_{GS}, of ordinary differential equation (A) is then the combination of the Complementary Function and the Particular Integral,

$$y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}}.$$

§5.15 Finding Particular Integrals

- In most applications, $f(x)$ is a simple polynomial, an exponential, a sine or cosine, or a combination of these. Generally, the Particular Integral is assumed to be a trial function in the same general form as $f(x)$. Essentially we take an "educated guess" for the Particular Integral based on the general form of the function $f(x)$.

- However, if our guess (or trial function) contains a term, $g(x)$ say, that already appears in the Complementary Function, then the guess will not be appropriate for the Particular Integral.

  Instead we must first multiply the term $g(x)$ by x and try $x\,g(x)$. If this new guess still appears in the C.F. then we multiply it by a further x and try $x^2\,g(x)$. (See the examples below).

- Note that any conditions on y at given values of x must be applied **after** the Particular Integral has been found and we have formed the General Solution of **(A)**.

a) $\boldsymbol{f(x)}$ is a polynomial

Suppose $f(x)$ is a polynomial of degree n,

$$f(x) \;=\; a_0 \;+\; a_1\,x \;+\; \ldots \;+\; a_n\,x^n\,,$$

where a_0, a_1, … a_n are known constants. Consider a, b and c in the ODE **(A)**.

- **if $\boldsymbol{c \neq 0}$** then the P.I. is also a polynomial of degree n,

$$y_{\mathrm{PI}} \;=\; b_0 \;+\; b_1\,x \;+\; \ldots \;+\; b_n\,x^n\,.$$

- **if $\boldsymbol{c = 0,\;\; b \neq 0}$** then the P.I. is a polynomial of the form

$$y_{\mathrm{PI}} \;=\; x\,(b_0 \;+\; b_1\,x \;+\; \ldots \;+\; b_n\,x^n)\,.$$

- **if $\boldsymbol{c = b = 0,\;\; a \neq 0}$** then the P.I. is a polynomial of the form

$$y_{\mathrm{PI}} \;=\; x^2\,(b_0 \;+\; b_1\,x \;+\; \ldots \;+\; b_n\,x^n)\,.$$

- In each case, substitute y_{PI} into the ODE **(A)** and then determine the b_i constants by comparing the coefficients of the corresponding powers of x.

- Note: the polynomial $b_0 \;+\; b_1\,x \;+\; \ldots \;+\; b_n\,x^n$ contains **every** power of x from 0 to n, even if some of the terms are missing from $f(x)$.

b) $f(x)$ is an exponential function

Suppose $f(x)$ is an exponential function

$$f(x) \;=\; k\,\mathrm{e}^{\lambda x}, \qquad (k,\,\lambda \neq 0).$$

The nature of the Particular Integral will depend on how the exponent coefficient λ is related to the root(s) of the Auxiliary Equation.

- **if λ does not equal any of the roots of the A.E.,**

$$y_{\mathrm{PI}} \;=\; P\,\mathrm{e}^{\lambda x} \qquad \text{for some constant } P.$$

- **if λ equals one, but not both, of the roots of the A.E.,**

$$y_{\mathrm{PI}} \;=\; P\,x\,\mathrm{e}^{\lambda x} \qquad \text{for some constant } P.$$

- **if λ equals a (repeated) double root of the A.E.,**

$$y_{\mathrm{PI}} \;=\; P\,x^2\,\mathrm{e}^{\lambda x} \qquad \text{for some constant } P.$$

- In each case, substitute y_{PI} into the ODE **(A)** and then determine the constant P by comparing the coefficients of $\mathrm{e}^{\lambda x}$

c) $f(x)$ is a sine and/or cosine function

Suppose

$$f(x) \;=\; C\,\cos(\mu x) \;+\; D\,\sin(\mu x).$$

Once again the nature of the Particular Integral will depend on how the coefficient μ is related to the root(s) of the Auxiliary Equation.

- **if $\pm i\mu$ are not roots of the A.E.,**

$$y_{\mathrm{PI}} \;=\; P\,\cos(\mu x) \;+\; Q\,\sin(\mu x) \qquad \text{for some constants } P,\, Q.$$

- **if $\pm i\mu$ are roots of the A.E.,**

$$y_{\mathrm{PI}} \;=\; x\Big(P\,\cos(\mu x) \;+\; Q\,\sin(\mu x)\Big) \qquad \text{for some constants } P,\, Q.$$

- In each case, substitute y_{PI} into the ODE **(A)** and then determine the constants P, Q by comparing the coefficients of the $\cos(\mu x)$ and $\sin(\mu x)$ terms.
- Note that even if there is only a sine or a cosine appearing in $f(x)$, we must take both in our trial form for the P.I.

d) **$f(x)$ is a combination of functions from a), b) and c)**

In this case obtain the Particular Integral for each term in $f(x)$ separately and then add them.

e) **$f(x)$ is a product of functions from a), b) and c)**

We can sometimes obtain a Particular Integral by considering the product of functions of the same form as the inhomogeneous term $f(x)$.

Examples covered in video

5.23 Solve $y'' - 2\,y' - 3\,y \;=\; 6x - 2\,.$

5.24 Solve $y'' - 2\,y' - 3\,y \;=\; 3\,\mathrm{e}^{2x}.$

5.25 Solve $y'' - 2\,y' - 3\,y \;=\; 6x - 2 + 3\,\mathrm{e}^{2x}.$

5.26 Solve $y'' - 2\,y' \;=\; 6x - 2\,.$

5.27 Solve $y'' - 2\,y' + y = \mathrm{e}^{x}.$

5.28 Solve $y'' + 3\,y' = 18x^2 + 8.$

5.29 Solve $y'' - 5\,y' + 6\,y = 4\,\mathrm{e}^{2x},$ subject to conditions $y(0) = 0,\quad y'(0) = 1.$

5.30 Solve $y'' + 2\,y' + y = 2\sin(x),$ subject to conditions $y(0) = 0,\quad y'(0) = 1.$

5.31 Solve $y'' + 9\,y = \cos(3x),$ subject to conditions $y(0) = y\left(\dfrac{\pi}{6}\right) = 0.$

5.32 Solve $y'' - 2\,y' - 3\,y = 6x\mathrm{e}^{2x}.$

§5.16 Using Complex Numbers to Find Particular Integrals

In the last section we saw that when the RHS of the ODE

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + c\,y = f(x)$$

is of the form $f(x) = P\cos(\mu x) + Q\sin(\mu x)$, then the choice of particular integral is linked to whether $m = \pm i\mu$ are roots of the Auxilliary Equation. Following the discussion in §5.13 (case 3), this is related to the fact that if the roots of the Auxilliary Equation are $m = \alpha \pm i\beta$, then the Complementary Function may written in an exponential **OR** a trigonometric form:

$$\begin{aligned} y_{\mathrm{CF}} &= A\mathrm{e}^{(\alpha+i\beta)x} + B\mathrm{e}^{(\alpha-i\beta)x} \qquad \text{(for constants } A \text{ and } B\text{)} \\ &= A\mathrm{e}^{\alpha x}\big(\cos(\beta x) + i\sin(\beta x)\big) + B\mathrm{e}^{\alpha x}\big(\cos(\beta x) - i\sin(\beta x)\big) \\ &= (A+B)\mathrm{e}^{\alpha x}\cos(\beta x) + i(A-B)\mathrm{e}^{\alpha x}\sin(\beta x) \\ &\equiv \mathrm{e}^{\alpha x}\big(C\cos(\beta x) + D\sin(\beta x)\big), \end{aligned}$$

where $C = A + B$ and $D = i(A - B)$ are also constants. However, in a problem involving real coefficients where you are asked to find a Particular Solution, it is likely that A and B will be complex numbers whereas C and D will be real numbers.

If $\alpha = 0$ and $\beta = \mu$ then the Complementary Function would have the same general form as $f(x)$ above (the RHS of the ODE), leading to the introduction of an extra x coefficient when we seek the Particular Integral.

Not only do complex numbers play a role in finding the Complementary Function for a second-order linear ODE, they can also be useful when trying to find Particular Integrals. In particular, complex numbers can be employed to find the Particular Integral when the RHS of the ODE takes the form

$$f(x) = P\,\mathrm{e}^{\lambda x}\cos(\mu x) \quad \text{or} \quad f(x) = P\,\mathrm{e}^{\lambda x}\sin(\mu x), \qquad (\dagger)$$

for some (real) constants P, λ and μ.
(Note, that we have already considered the case when $\lambda = 0$ leading to $f(x) = P\cos(\mu x)$ or $f(x) = P\sin(\mu x)$, see §5.13 (case 3).)

To find the Particular Integral appropriate for $f(x)$ given by **(†)**, we make use of the fact that, since x is real,

$$\mathrm{e}^{(\lambda+i\mu)x} = \mathrm{e}^{\lambda x}\,\mathrm{e}^{i\mu x} = \mathrm{e}^{\lambda x}\big(\cos(\mu x) + i\sin(\mu x)\big) = \mathrm{e}^{\lambda x}\cos(\mu x) + i\mathrm{e}^{\lambda x}\sin(\mu x).$$

- $\mathrm{e}^{\lambda x}\cos(\mu x)$ is the **real part** of $\mathrm{e}^{(\lambda+i\mu)x}$, and
- $\mathrm{e}^{\lambda x}\sin(\mu x)$ is the **imaginary part** of $\mathrm{e}^{(\lambda+i\mu)x}$.

When we require a Particular Integral for an ODE of the form

$$a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; b\,\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; c\,y \;=\; P\,\mathrm{e}^{\lambda x}\,\cos(\mu x) \quad \text{or} \quad P\,\mathrm{e}^{\lambda x}\,\sin(\mu x),$$

we start by considering the Particular Integral for the complex version of the ODE,

$$a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; b\,\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; c\,y \;=\; P\mathrm{e}^{(\lambda+i\mu)x}\,. \qquad (\Diamond)$$

- To do this we mimic the approach adopted in §5.13 (case 1) when the RHS of the ODE is $f(x) = k\mathrm{e}^{\lambda x}$ $(k, \lambda \in \mathbb{R})$, namely, look for a P.I. of the form

$$y_{\mathrm{PI}} \;=\; Q\,\mathrm{e}^{(\lambda+i\mu)x} \qquad (Q \in \mathbb{C}).$$

  (If $\lambda + i\mu$ is also a root of the A.E. then we would need to multiply our guess P.I. by x.)

- We now substitute this guess for the P.I. into the ODE $(\Diamond)$ and solve to find an appropriate value for Q.

- Significantly, the derivatives of y_{PI} will involve manipulating complex numbers and the constant Q may also be a **complex number**.

- Recall that $P\mathrm{e}^{\lambda x}\cos(\mu x)$ and $P\mathrm{e}^{\lambda x}\sin(\mu x)$ are, respectively, the real and imaginary parts of $P\mathrm{e}^{(\lambda+i\mu)x}$. It follows that
  - the P.I. for $P\mathrm{e}^{\lambda x}\cos(\mu x)$ is the **real part** of the P.I. for $P\mathrm{e}^{(\lambda+i\mu)x}$
  - the P.I. for $P\mathrm{e}^{\lambda x}\sin(\mu x)$ is the **imaginary part** of the P.I. for $P\mathrm{e}^{(\lambda+i\mu)x}$.

- The Complementary Function and the Particular Integral can now be combined in the usual way to obtain the General Solution of the original second order ODE. Because the real and imaginary parts of the P.I. are *real* numbers, the General Solution will be comprised of **only real numbers.**

Example

5.33 Solve $y'' - 6y' + 13y \;=\; 145\,\mathrm{e}^{x}\sin(3x)$.

The A.E. of the ODE is

$$m^2 - 6m + 13 \;=\; (m-3)^2 + 4 \;=\; 0 \quad \Longleftrightarrow \quad m \;=\; 3 \pm 2i.$$

We will write the C.F. in its trigonometric form

$$y_{\mathrm{CF}} \;=\; \mathrm{e}^{3x}\,(A\cos 2x \;+\; B\sin 2x),$$

for arbitrary constants A and B.

The RHS of the ODE is equivalent to

$$\mathrm{Im}\left(145\mathrm{e}^{x}(\cos 3x \;+\; i\,\sin 3x)\right) \;=\; \mathrm{Im}\left(145\mathrm{e}^{x}\mathrm{e}^{3ix}\right) \;=\; \mathrm{Im}\left(145\mathrm{e}^{(1+3i)x}\right).$$

So our aim now is to find a P.I. appropriate for $145\mathrm{e}^{(1+3i)x}$ on the RHS of the ODE, then take the imaginary part to obtain a P.I. for $145\mathrm{e}^{x}\sin 3x$.

Since $1+3i$ is not a root of the A.E., our guess for a P.I. is

$$\begin{aligned} y &= Q\,\mathrm{e}^{(1+3i)x} \qquad (\text{for } Q \in \mathbb{C}), \qquad \text{so that} \\ y' &= (1+3i)\,Q\,\mathrm{e}^{(1+3i)x}, \\ y'' &= (1+3i)^2\,Q\,\mathrm{e}^{(1+3i)x} \;=\; (-8+6i)\,Q\,\mathrm{e}^{(1+3i)x}. \end{aligned}$$

Now substitute our guess for the P.I. into the complex version of the ODE with $145\,\mathrm{e}^{(1+3i)x}$ on the RHS:

$$Q\,\mathrm{e}^{(1+3i)x}\left\{(-8+6i) - 6(1+3i) + 13\right\} \;=\; 145\,\mathrm{e}^{(1+3i)x}.$$

Comparing coefficients:

$$\begin{aligned} Q(-1-12i) &= 145 \\ \Longrightarrow \qquad Q &= \frac{145}{-1-12i} \\ &= \frac{145}{(-1-12i)}\,\frac{(-1+12i)}{(-1+12i)} \\ &= \frac{145(-1+12i)}{1^2+12^2} \;=\; -1+12i. \end{aligned}$$

So the P.I. for the complex version of the ODE is

$$y \;=\; (-1+12i)\,\mathrm{e}^{(1+3i)x}.$$

Since we are only concerned with the imaginary part of $145\,\mathrm{e}^{(1+3i)x}$ on the RHS of the ODE, we restrict our P.I. to

$$\begin{aligned} y_{\mathrm{PI}} &= \mathrm{Im}\left((-1+12i)\,\mathrm{e}^{(1+3i)x}\right) \\ &= \mathrm{Im}\left((-1+12i)\,\mathrm{e}^{x}\,(\cos 3x + i\sin 3x)\right) \;=\; \mathrm{e}^{x}(12\cos 3x - \sin 3x). \end{aligned}$$

Therefore our G.S. is

$$y_{\mathrm{PS}} \;=\; y_{\mathrm{CF}} + y_{\mathrm{PI}} \;=\; \mathrm{e}^{3x}(A\cos 2x + B\sin 2x) \;+\; \mathrm{e}^{x}(12\cos 3x - \sin 3x).$$

Summary of the method of solution

- Determine the **Complementary Function (C.F.)**, $\boldsymbol{y_{\mathrm{CF}}}$, the general solution to the **homogeneous equation**

$$\boldsymbol{a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\,\frac{\mathrm{d}y}{\mathrm{d}x} + c\,y = 0.} \qquad \text{(B)}$$

- To do this, look for solutions of the form $\boldsymbol{y = \mathrm{e}^{mx}}$, where m satisfies the **Auxiliary Equation (A.E.)**

$$\boldsymbol{a\,m^2 + b\,m + c = 0.}$$

- The roots $m = m_1,\, m_2$ of the quadratic A.E. fall into three categories.
  The nature of the complementary function y_{CF} depends on the type of roots.
  1. **real, distinct roots** $\boldsymbol{m_1 \neq m_2}$
  $$y_{\mathrm{CF}} = A\,\mathrm{e}^{m_1x} + B\,\mathrm{e}^{m_2x} \qquad \text{(arbitrary } A,\, B)$$
  2. **real, equal roots** $\boldsymbol{m_1 = m_2}$
  $$y_{\mathrm{CF}} = (Ax + B)\,\mathrm{e}^{m_1x} \qquad \text{(arbitrary } A,\, B)$$
  3. **complex conjugate roots** $\boldsymbol{m_1 = \alpha + i\beta,\quad m_2 = \alpha - i\beta}$
  $$y_{\mathrm{CF}} = \mathrm{e}^{\alpha x}\,\big[\,A\cos(\beta x) + B\sin(\beta x)\,\big] \qquad \text{(arbitrary } A,\, B)$$

- Now determine the **Particular Integral (or P.I.)**, $\boldsymbol{y_{\mathrm{PI}}}$, which is **any** solution to the original inhomogeneous equation

$$\boldsymbol{a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\,\frac{\mathrm{d}y}{\mathrm{d}x} + c\,y = f(x).} \qquad \text{(A)}$$

- The **General Solution** for the ODE, $\boldsymbol{y_{\mathrm{GS}}}$, is formed by combining the Complementary Function and Particular Integral,

$$\boldsymbol{y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}}\,.}$$