DEPARTMENT OF MATHEMATICS & STATISTICS
MM102 APPLICATIONS OF CALCULUS
Complex Numbers: Exercise Sheet for Week 6

1. Express $z_1 = 2-2i$ and $z_2 = -1+\sqrt{3}\,i$ in polar form. Hence evaluate the following in polar form using the principal value of the argument in each case:

(a) $z_1\, z_2$, **(b)** z_1^5, **(c)** $\dfrac{1}{z_2^3}$, **(d)** $z_1^6\, z_2^4$, **(e)** $\dfrac{z_1^9}{z_2^7}$.

2. Find the modulus and argument of the following complex numbers, and state the principal value of the argument.

(a) $(1-3i)^4$, **(b)** $(-1+\sqrt{3}i)^5$, **(c)** $(-12-5i)^{-3}$, **(d)** $(-12-12i)^5$.

3. Use the polar form and de Moivre's theorem to simplify the following. (Give your answers in the form $x+iy$ where $x,\, y \in \mathbb{R}$):

(a) $\dfrac{(1+i)^5}{1-i}$ **(b)** $\dfrac{(1+\sqrt{3}i)^2}{(1+i)^3}$ **(c)** $(1+i)^{20} + (1-i)^{20}$

(d) $\dfrac{(\sqrt{3}+i)^{10}}{(1-i)^7}$ **(e)** $(\sqrt{2}+i\sqrt{2})^{-4}$ **(f)** $(\sqrt{2}+i\sqrt{2})^8$ **(g)** $\dfrac{(\cos\theta + i\sin\theta)^3}{(\sin\theta + i\cos\theta)^2}$

4. Use de Moivre's theorem to express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

5. Use de Moivre's theorem to show that $\cos^2\theta = \dfrac{1}{2}\Big(\cos(2\theta)+1\Big)$.

6. Find constants a and b such that $\sin^3\theta = a\sin(3\theta) + b\sin\theta$.
Hence, calculate $\displaystyle\int \sin^3\theta\, \mathrm{d}\theta$.

7. Find constants a, b and c such that $\cos(4\theta) = a\cos^4\theta + b\cos^2\theta + c$.

8. Express $\cos^5\theta$ in terms of cosines of integer multiples of θ.
Hence, calculate $\displaystyle\int \cos^5\theta\ \mathrm{d}\theta$.

9. Use de Moivre's theorem to express

(a) $\cos(5\theta)$ in terms of $\cos\theta$, **(b)** $\sin 5\theta$ in terms of $\sin\theta$

(c) $\tan(5\theta)$ in terms of $\tan\theta$ (where $\theta \neq (2n+1)\dfrac{\pi}{2}$ for any integer n).

10. Express $\cos(6\theta)$ in terms of $\cos\theta$ and find real constants a, b and c such that

$$\sin(6\theta) = \sin\theta\,(a\cos^5\theta\ +\ b\cos^3\theta\ +\ c\cos\theta)$$

for all angles θ.

11. Express the following as linear combinations of cosines of multiples of θ:

(a) $\cos^4\theta$, **(b)** $\cos^2\theta\sin^4\theta$.

12. Express the following as linear combinations of sines of multiples of θ:

(a) $\sin^5\theta$, **(b)** $\sin^3\theta\cos^3\theta$.

13. Use the results of the previous two questions to evaluate

(a) $\displaystyle\int_0^{\pi/4}\cos^4\theta\,\mathrm{d}\theta$, **(b)** $\displaystyle\int_{\pi/2}^{\pi}\sin^3\theta\cos^3\theta\,\mathrm{d}\theta$.

14. Without any calculation, sketch all the sixth roots of 1 on an Argand diagram.

15. Find the following in the form $x+iy$ where $x,\ y\in\mathbb{R}$.

(a) the square roots of i, **(b)** the square roots of $1+\sqrt{3}i$,

(c) the cube roots of -8, **(d)** the cube roots of $27i$,

(e) the fourth roots of $-8-8\sqrt{3}i$, **(f)** the sixth roots of -64.

16. Solve the equations: **(a)** $z^4+81=0$, **(b)** $z^6+1=\sqrt{3}i$.

17. Find all distinct values of $(2-2\sqrt{3}i)^{1/3}$

18. Find the fourth roots of $-2-2\sqrt{3}i$.

19. Determine the roots of the equation $z^3=4+4\sqrt{3}i$, leaving your answers in polar form.

20. Determine the five roots of the equation $z^5=-1$, giving your answers in the form $a+ib$ to four decimal places in real numbers a, b. Verify that the sum of the roots is zero to this accuracy.

21. Find all the fourth roots of -16. [Equivalently, solve $z^4+16=0$ for $z\in\mathbb{C}$.]

22. Given that one of the fourth roots of $z=-0.8432+0.5376\,i$ is $0.8+0.6i$, sketch all the fourth roots of z on an Argand diagram.

23. Determine, in the form $a+ib$ ($a,\ b\in\mathbb{R}$), the roots of the equation $z^3=8$. Hence find the roots of the equation $(w-3)^3=8$.

24. Find all solutions to the equation $z^3+6z+20=0$ for $z\in\mathbb{C}$.

25. Solve the equations

(a) $(1+iz)^3=8$, **(b)** $z^4+13z^2+36=0$.

26. Solve the equation $(z+1)^4 = z^4$. Explain why there are only three solutions.

27. Express each of the following as a product of (i) linear factors and (ii) linear and quadratic factors with only real coefficients:

(a) $z^3 - 1$, **(b)** $z^4 + 1$, **(c)** $z^6 + 1$, **(d)** $z^5 - 1$.

28. Verify that $z = 3i$ is a root of the equation

$$P(z) = z^5 + 9z^3 + 8z^2 + 72 = 0.$$

Hence find all roots of this equation. Express $P(z)$ as

(i) the product of linear factors, and

(ii) the product of linear and quadratic factors with only real coefficients.

29. Find all solutions of $z^4 + 2z^2 + 4 = 0$ where $z \in \mathbb{C}$. (Hint: set $w = z^2$ and solve the quadratic equation for w.)

30. Verify that $z = 1 + i$ is a root of the equation

$$z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$$

and hence find all four roots of the equation.

31. Express the following in the form $a + ib$ $(a, b \in \mathbb{R})$:

(a) $\log(\sqrt{3} - i)$, **(b)** $\log(2 + 2i)$, **(c)** $\log(-i)$, **(d)** e^{3-4i}.