3 Exercise Solutions: Chapter 3

1. (a) Suppose $W = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \; x_1 = 0\}$.

   Let $\lambda, \mu \in \mathbb{R}$ and $\boldsymbol{x} = (0, x_2, x_3), \; \boldsymbol{y} = (0, y_2, y_3) \in W$.

   Then
   $$\boldsymbol{z} = \lambda\boldsymbol{x} + \mu\boldsymbol{y} = (0, \lambda x_2 + \mu y_2, \; \lambda x_3 + \mu y_3)$$
   so $z_1 = 0$ and $\boldsymbol{z} \in W$. This holds $\forall \lambda, \mu \in \mathbb{R}$ and $\forall \boldsymbol{x}, \boldsymbol{y} \in W$. Hence W is a subspace of $\mathbb{R}^3$.

   (b) Suppose $U = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_3 = 0\}$.

   Let $\lambda, \mu \in \mathbb{R}$ and $\boldsymbol{x} = (x_1, x_2, -x_1), \; \boldsymbol{y} = (y_1, y_2, -y_1) \in U$.

   Then
   $$\boldsymbol{z} = \lambda\boldsymbol{x} + \mu\boldsymbol{y} = (\lambda x_1 + \mu y_1, \; \lambda x_2 + \mu y_2, -\lambda x_1 - \mu y_1)$$
   so $z_1 + z_3 = 0$ and $\boldsymbol{z} \in U$. This holds $\forall \lambda, \mu \in \mathbb{R}$ and $\forall \boldsymbol{x}, \boldsymbol{y} \in U$. Hence U is a subspace of $\mathbb{R}^3$.

   (c) Suppose $W = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.

   Here W is a subspace of $\mathbb{R}^3$—use same argument as in (a), (b) above. Here typical vector is $\boldsymbol{x} = (x_1, x_2, -(x_1 + x_2)) \in W$.

   (d) Suppose $W = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1\}$

   W is **NOT** a subspace of $\mathbb{R}^3$.

   - For example, it does not contain $\mathbf{0} = (0, 0, 0)$.
   - **Alternatively**, let $\boldsymbol{x} = (0, 0, 1), \; \boldsymbol{y} = (1, 0, 0)$ then $\boldsymbol{x} + \boldsymbol{y} = (1, 0, 1)$
     $\notin W \Rightarrow$ Not closed for addition.
     {Only one of the arguments needed!}

   (e) Suppose $W = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 3x_1 - 2x_2\}$.

   Let $\lambda, \mu \in \mathbb{R}$ and $\boldsymbol{x} = (x_1, x_2, 3x_1 - 2x_2)$, $\boldsymbol{y} = (y_1, y_2, 3y_1 - 2y_2) \in W$. Then
   $$\boldsymbol{z} = \lambda\boldsymbol{x} + \mu\boldsymbol{y} = (\lambda x_1 + \mu y_1, \; \lambda x_2 + \mu y_2, \lambda(3x_1 - 2x_2) + \mu(3y_1 - 2y_2))$$
   $$\Rightarrow \; \boldsymbol{z} = (\lambda x_1 + \mu y_1, \; \lambda x_2 + \mu y_2, \; 3(\lambda x_1 + \mu y_1) - 2(\lambda x_2 + \mu y_2))$$
   so $z_3 = 3z_1 - 2z_2$ and $\boldsymbol{z} \in W$. This holds $\forall \lambda, \mu \in \mathbb{R}$ and $\forall \boldsymbol{x}, \boldsymbol{y} \in W$. Hence W is a subspace of $\mathbb{R}^3$.

   (f) Suppose $U = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > x_2\}$

   U is not a subspace of $\mathbb{R}^3$. It does not contain $\mathbf{0} = (0, 0, 0)$.

(g) Suppose $W = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| < 1, i = 1, 2, 3\}$.
W is NOT a subspace of $\mathbb{R}^3$.
For example, let $\boldsymbol{x} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in W$. Then $4\boldsymbol{x} = (2, 2, 2) \notin W$.
W is not closed under scalar multiplication.

(h) Suppose $V = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 x_2 > 0\}$.
V is not a subspace of $\mathbb{R}^3$.
It does not contain $\mathbf{0}$.
Alternatively, not closed under addition.
Take $\boldsymbol{x} = (-2, -2, 0)$, $\boldsymbol{y} = (4, 1, 0)$ then $\boldsymbol{x} + \boldsymbol{y} \notin V$.

2. (a) W is not a subspace as the 2×2 zero matrix is not in W (as it has zero determinant).
(b) Let $p(x) = \alpha_1 x^2 + \beta_1 x^4$ and $q(x) = \alpha_2 x^2 + \beta_2 x^4$ be two elements of W. Then, for $\lambda, \mu \in \mathbb{R}$, we have

$$r(x) = \lambda p(x) + \mu q(x) = \lambda(\alpha_1 x^2 + \beta_1 x^4) + \mu(\alpha_2 x^2 + \beta_2 x^4) = \gamma_1 x^2 + \gamma_1 x^4$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$, so $r(x) \in W$. As this holds $\forall \lambda, \mu \in \mathbb{R}$ and $\forall p(x), q(x) \in W$, W is a subspace of P^4, so is a real vector space.
(c) W is not a real vector space, as it does not contain the zero polynomial.

3. Let X and Y be matrices in W. Then, for a matrix $Z = \lambda X + \mu Y$ with $\lambda, \mu \in \mathbb{R}$, we have

$$AZ = A(\lambda X + \mu Y) = \lambda AX + \mu AY = \lambda XA + \mu YA = (\lambda X + \mu Y)A = ZA$$

so $Z \in W$. As this holds $\forall \lambda, \mu \in \mathbb{R}$ and $\forall X, Y \in W$, W is a subspace of $\mathbb{R}^{n \times n}$, so is a real vector space.

4. $U = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$. Then

$$\begin{aligned} \boldsymbol{x} \in U \quad &\Leftrightarrow \quad \boldsymbol{x} = (x_1, x_2, 0) \\ &\Leftrightarrow \quad \boldsymbol{x} = x_1(1, 0, 0) + x_2(0, 1, 0) \\ &\Leftrightarrow \quad \boldsymbol{x} \in sp((1, 0, 0), (0, 1, 0)) \end{aligned}$$

Hence $U = sp((1, 0, 0), (0, 1, 0))$.

U contains position vectors of points on the (x_1, x_2) plane.

$V = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 - x_2\}$.

Then

$$\begin{aligned} \boldsymbol{x} \in V \quad &\Leftrightarrow \quad \boldsymbol{x} = (x_1, x_2, x_1 - x_2) \\ &\Leftrightarrow \quad \boldsymbol{x} = x_1(1, 0, 1) + x_2(0, 1, -1) \\ &\Leftrightarrow \quad \boldsymbol{x} \in sp((1, 0, 1), (0, 1, -1)). \end{aligned}$$

Hence $V = sp((1, 0, 1), (0, 1, -1))$.

5. $\boldsymbol{x}_2 \neq \lambda \boldsymbol{x}_1$ for any real λ so retain $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

$$\begin{aligned} \boldsymbol{x}_3 = \lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 \;&\Leftrightarrow\; \lambda_1 + 2\lambda_2 = 0,\ -\lambda_1 + \lambda_2 = -3,\ -2\lambda_2 = 2,\ 2\lambda_1 = 4,\ \lambda_1 = 2 \\ &\Leftrightarrow\; \lambda_1 = 2 \text{ and } \lambda_2 = -1. \text{ Thus } \boldsymbol{x}_3 = 2\boldsymbol{x}_1 - \boldsymbol{x}_2. \end{aligned}$$

$\Rightarrow$ Eliminate $\boldsymbol{x}_3$.

Similarly, $\boldsymbol{x}_4 = -\boldsymbol{x}_1 + 2\boldsymbol{x}_2 \Rightarrow$ eliminate $\boldsymbol{x}_4$.

$\boldsymbol{x}_5 = \lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 \Leftrightarrow \lambda_1 + 2\lambda_2 = 2,\ -\lambda_1 + \lambda_2 = 4,\ -2\lambda_2 = 1,\ 2\lambda_1 = 0,\ \lambda_1 = 1.$

This system has no solution so retain $\boldsymbol{x}_5$.

$$\begin{aligned} \boldsymbol{x}_6 &= \lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 + \lambda_3 \boldsymbol{x}_5 \;\Leftrightarrow \\ &\quad \lambda_1 + 2\lambda_2 + 2\lambda_3 = 5,\ -\lambda_1 + \lambda_2 + 4\lambda_3 = 7,\ -2\lambda_2 + \lambda_3 = -3, \\ &\quad 2\lambda_1 = -2,\ \lambda_1 + \lambda_3 = 0. \end{aligned}$$

This has solution $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 1 \Rightarrow \boldsymbol{x}_6 = -\boldsymbol{x}_1 + 2\boldsymbol{x}_2 + \boldsymbol{x}_5$

$\Rightarrow$ Eliminate $\boldsymbol{x}_6$.

Hence $sp(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5, \boldsymbol{x}_6) = sp(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_5)$.

6. (a) $(3, -6, 9) = 3(1, -2, 3)$
   $\Rightarrow$ set is LD in $\mathbb{R}^3$.

   (b) $\lambda_1(0, 1, -2) + \lambda_2(1, -1, 1) + \lambda_3(1, 2, 1) = \mathbf{0}$
   $\Leftrightarrow \lambda_2 + \lambda_3 = 0,\ \lambda_1 - \lambda_2 + 2\lambda_3 = 0,\ -2\lambda_1 + \lambda_2 + \lambda_3 = 0.$
   The only solution of this system is $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
   Hence the set is LI in $\mathbb{R}^3$.

   (c) $\lambda_1(1, -1, -1) + \lambda_2(2, 3, 1) + \lambda_3(-1, 4, -2) + \lambda_4(3, 10, 8) = \mathbf{0} \;\Leftrightarrow$

$$\begin{array}{rcrcrcrcl} \lambda_1 & + & 2\lambda_2 & - & \lambda_3 & + & 3\lambda_4 & = & 0 \\ -\lambda_1 & + & 3\lambda_2 & + & 4\lambda_3 & + & 10\lambda_4 & = & 0 \\ -\lambda_1 & + & \lambda_2 & - & 2\lambda_3 & + & 8\lambda_4 & = & 0 \end{array} \qquad \text{Reduce matrix to echelon form}$$

$$\begin{array}{rrrr|} 1 & 2 & -1 & 3 \\ -1 & 3 & 4 & 10 \\ -1 & 1 & -2 & 8 \end{array} \sim \begin{array}{rrrr|} 1 & 2 & -1 & 3 \\ 0 & 5 & 3 & 13 \\ 0 & 3 & -3 & 11 \end{array} \sim \begin{array}{rrrr|} 1 & 2 & -1 & 3 \\ 0 & 5 & 3 & 13 \\ 0 & 0 & -24 & 16 \end{array} \sim \begin{array}{rrrr} 1 & 2 & -1 & 3 \\ 0 & 1 & \frac{3}{5} & \frac{13}{5} \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}$$

   Solution is $\lambda_4 = 3\lambda,\ \lambda_3 = 2\lambda,\ \lambda_2 = -9\lambda,\ \lambda_1 = 11\lambda$ for any $\lambda \in \mathbb{R}$.
   If $\lambda \neq 0$ we have a non-trivial solution, so set is LD.
   [See Note 3.19—any set of more than n vectors in $\mathbb{R}^n$ is a LD set].

   (d) $\lambda_1(0, 1, 2, 3, 4, 5) + \lambda_2(0, 2, 3, 4, 5, 6) + \lambda_3(0, 0, 0, 1, 2, 3) = \mathbf{0}$
   $\Leftrightarrow\ \lambda_1 = \lambda_2 = \lambda_3 = 0.$
   Hence set is LI.

7. Only (a), (b), (c) and (e) are subspaces.

   (a) $$\begin{aligned} \boldsymbol{x} \in W &\Leftrightarrow \boldsymbol{x} = (0, x_2, x_3) \\ &\Leftrightarrow \boldsymbol{x} = x_2(0,1,0) + x_3(0,0,1) \\ &\Leftrightarrow \boldsymbol{x} \in sp((0,1,0),(0,0,1)) \end{aligned}$$

   Hence $\{(0,1,0),(0,0,1)\}$ spans W.

   Since $(0,1,0)$ and $(0,0,1)$ are LI (prove this), we conclude that $\{(0,1,0),(0,0,1)\}$ is a basis for W.

   Also $\dim W = 2$.

   (b) $$\begin{aligned} \boldsymbol{x} \in U &\Leftrightarrow \boldsymbol{x} = (x_1, x_2, -x_1) \\ &\Leftrightarrow \boldsymbol{x} = x_1(1,0,-1) + x_2(0,1,0) \\ &\Leftrightarrow \boldsymbol{x} \in sp((1,0,-1),(0,1,0)) \end{aligned}$$

   Hence $U = sp((1,0,-1),(0,1,0))$.

   These vectors are LI so they form a basis for U. $\dim U = 2$.

   (c) $$\begin{aligned} \boldsymbol{x} \in W &\Leftrightarrow \boldsymbol{x} = (x_1, x_2, -x_1 - x_2) \\ &\Leftrightarrow \boldsymbol{x} = x_1(1,0,-1) + x_2(0,1,-1) \\ &\Leftrightarrow \boldsymbol{x} \in sp((1,0,-1),(0,1,-1)) \end{aligned}$$

   Hence $W = sp((1,0,-1),(0,1,-1))$.

   Vectors in this spanning set are LI so they form a basis for W. $\dim W = 2$.

   (e) Use arguments as above to show that $\{(1,0,3),(0,1,-2)\}$ is a basis for W.
   $\dim W = 2$.

8. $\lambda_1(-1,1) + \lambda_2(1,2) = \mathbf{0} \Leftrightarrow \lambda_1 = \lambda_2 = 0$.

   (a) Hence $(-1,1)$ and $(1,2)$ are LI, and it follows that this pair forms a basis for $\mathbb{R}^2$. (any 2 LI vectors in $\mathbb{R}^2$ form a basis for $\mathbb{R}^2$.)

   (b) $\{(-1,3,1),(2,1,4)\}$ is not a basis for $\mathbb{R}^3$—a basis requires 3 vectors (Note 3.19).

   (c) $\lambda_1(-1,3,4) + \lambda_2(1,5,-1) + \lambda_3(1,13,2) = \mathbf{0} \;\Leftrightarrow$

   $$\left.\begin{array}{rcrcrcl} -\lambda_1 & + & \lambda_2 & + & \lambda_3 & = & 0 \\ 3\lambda_1 & + & 5\lambda_2 & + & 13\lambda_3 & = & 0 \\ 4\lambda_1 & - & \lambda_2 & + & 2\lambda_3 & = & 0 \end{array}\right\} \begin{array}{rcl} r_2 & := & r_2 + 3r_1 \\ r_3 & := & r_3 + 4r_1 \end{array} \quad \text{on } A\boldsymbol{\lambda} = \mathbf{0}$$

   $$\begin{array}{rrr|r} -1 & 1 & 1 & 0 \\ 0 & 8 & 16 & 0 \\ 0 & 3 & 6 & 0 \\ \hline \end{array} \sim \begin{array}{rrr|r} -1 & 1 & 1 & 0 \\ 0 & 8 & 16 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array}$$

   Solution is $\lambda_1 = \lambda,\ \lambda_2 = -2\lambda,\ \lambda_3 = \lambda$ for any $\lambda \in \mathbb{R}$.

   Vectors are LD and they do **not** form a basis for $\mathbb{R}^3$.

(d) $\lambda_1(2,1,0,2)+\lambda_2(2,-3,1,0)+\lambda_3(3,2,0,0)+\lambda_4(5,0,0,0)=\mathbf{0}$
$\Leftrightarrow\ \lambda_1=\lambda_2=\lambda_3=\lambda_4=0$ (prove this).
Vectors are LI and so they form a basis for $\mathbb{R}^4$.
(Any 4 LI vectors in $\mathbb{R}^4$ will span $\mathbb{R}^4$ and thus form a basis.)

9. (a) We need to find α, β, γ such that

$$(1,-3,-5)=\alpha(1,0,1)+\beta(-2,2,2)+\gamma(0,1,0),$$

that is, we need $\alpha-2\beta=1$, $2\beta+\gamma=-3$ and $\alpha+2\beta=-5$. Solving these equations gives $\alpha=-2$, $\beta=-3/2$, $\gamma=0$ which are the required coordinates.

(b) We need to find α, β, γ, δ, ϵ, μ such that

$$x+x^3+x^5=\alpha+\beta(1+x)+\gamma(1+x^2)+\delta(1-x^3)+\epsilon(1-x^4)+\mu(1-x^5),$$

that is, we require $\beta=1$, $\delta=-1$, $\mu=-1$, $\gamma=\epsilon=0$, and $\alpha+\beta+\gamma+\delta+\epsilon+\mu=0$. Substituting the given values into this last equation gives $\alpha=1$, so the required coordinates are $(1,1,0,-1,0,-1)$.

(c) We need to find α, β, γ, δ such that

$$\begin{bmatrix}2&3\\-3&0\end{bmatrix}=\alpha\begin{bmatrix}1&0\\0&1\end{bmatrix}+\beta\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\gamma\begin{bmatrix}0&-1\\1&0\end{bmatrix}+\delta\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

that is, we need $\alpha+\beta=2$, $-\gamma+\delta=3$, $\gamma+\delta=-3$ and $\alpha-\beta=0$. Solving these equations gives $\alpha=1$, $\beta=1$, $\gamma=-3$ and $\delta=0$, which are the required coordinates.

10. $V=\{\boldsymbol{x}=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : x_4=ax_1+bx_2+cx_3\}$, where $a,b,c\in\mathbb{R}$.

Let $\lambda,\mu\in\mathbb{R}$ and
$$\begin{aligned}\boldsymbol{x}&=(x_1,x_2,x_3,ax_1+bx_2+cx_3),\\ \boldsymbol{y}&=(y_1,y_2,y_3,ay_1+by_2+cy_3)\in V.\end{aligned}$$

Then

$$\lambda\boldsymbol{x}+\mu\boldsymbol{y}=(\lambda x_1+\mu y_1,\lambda x_2+\mu y_2,\ \lambda x_3+\mu y_3,\ \lambda(ax_1+bx_2+cx_3)+\mu(ay_1+by_2+cy_3))$$

Since $\lambda(ax_1+bx_2+cx_3)+\mu(ay_1+by_2+cy_3)$

$$=a(\lambda x_1+\mu y_1)+b(\lambda x_2+\mu y_2)+c(\lambda x_3+\mu y_3)$$

it follows that $\lambda\boldsymbol{x}+\mu\boldsymbol{y}\in V$.

This holds $\forall\lambda,\mu\in\mathbb{R}$ and $\forall\boldsymbol{x},\boldsymbol{y}\in V$. Hence V is a subspace of $\mathbb{R}^4$.

Also,

$$\begin{aligned}\boldsymbol{x}\in V\ &\Leftrightarrow\ \boldsymbol{x}=(x_1,x_2,x_3,ax_1+bx_2+cx_3)\\ &\Leftrightarrow\ \boldsymbol{x}=x_1(1,0,0,a)+x_2(0,1,0,b)+x_3(0,0,1,c)\\ &\Leftrightarrow\ \boldsymbol{x}\in sp((1,0,0,a),(0,1,0,b),(0,0,1,c))\end{aligned}$$

Hence $V = sp((1,0,0,a),\ (0,1,0,b),\ (0,0,1,c))$.

Vectors in spanning set are LI so they form a basis for V.

Also, $\dim V = 3$.

11. Let $S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$.

    Clearly any $\boldsymbol{x} \in \mathbb{R}^4$ can be written as a LC of the elements in S, so S spans $\mathbb{R}^4$.

    Can we eliminate any of the vectors in S?

    $\boldsymbol{x}_1 \neq \boldsymbol{0}$ so retain $\boldsymbol{x}_1$, and $\boldsymbol{x}_2 \neq \lambda \boldsymbol{x}_1$ for any $\lambda \in \mathbb{R}$ so retain $\boldsymbol{x}_2$.

    $\boldsymbol{e}_1 = (1,0,0,0) \neq \lambda \boldsymbol{x}_1 + \mu \boldsymbol{x}_2$ for any $\lambda, \mu \in \mathbb{R}$, so retain $\boldsymbol{e}_1$.

    Let $\boldsymbol{e}_2 = \lambda \boldsymbol{x}_1 + \mu \boldsymbol{x}_2 + \gamma \boldsymbol{e}_1$.

    This holds

    $$\begin{aligned} &\Leftrightarrow \quad 3\lambda + \gamma = 0,\ -2\lambda + \mu = 1,\ \mu = 0 \\ &\Leftrightarrow \quad \lambda = -\frac{1}{2},\ \mu = 0,\ \gamma = \frac{3}{2} \end{aligned}$$

    $\Rightarrow$ $\boldsymbol{e}_2$ is a LC of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{e}_1 \Rightarrow$ Remove $\boldsymbol{e}_2$.

    Let $\boldsymbol{e}_3 = \lambda \boldsymbol{x}_1 + \mu \boldsymbol{x}_2 + \gamma \boldsymbol{e}_1$.

    It is readily shown that there is no solution for $\lambda, \mu, \gamma \ \Rightarrow\ \boldsymbol{e}_3$ is not a LC of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{e}_1 \ \Rightarrow$ Retain $\boldsymbol{e}_3$.

    Thus $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{e}_1, \boldsymbol{e}_3$ is a LI set in $\mathbb{R}^4$ and since we cannot have more than 4 LI vectors in $\mathbb{R}^4$ we must eliminate $\boldsymbol{e}_4$.

    Hence $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{e}_1, \boldsymbol{e}_3\}$ is a basis for $\mathbb{R}^4$.

12. $V = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \,.\, \boldsymbol{y} = 0 \ \ \forall \boldsymbol{y} \in K\}$.

    Let $\lambda, \mu \in \mathbb{R},\ \boldsymbol{x}, \boldsymbol{z} \in V$.

    $$\begin{aligned} \text{For any } \boldsymbol{y} \in K,\ (\lambda \boldsymbol{x} + \mu \boldsymbol{z}) .\, \boldsymbol{y} &= (\lambda \boldsymbol{x} \,.\, \boldsymbol{y}) + (\mu \boldsymbol{z} \,.\, \boldsymbol{y}) \\ &= \lambda(\boldsymbol{x} \,.\, \boldsymbol{y}) + \mu(\boldsymbol{z} \,.\, \boldsymbol{y}) \\ &= \lambda \,.\, 0 + \mu \,.\, 0, \end{aligned}$$

    since $\boldsymbol{x}, \boldsymbol{z} \in V$.

    $$\Rightarrow \ (\lambda \boldsymbol{x} + \mu \boldsymbol{z}) .\, \boldsymbol{y} = 0 \ \ \forall \boldsymbol{y} \in K.$$

    This holds $\forall \lambda, \mu \in \mathbb{R}$ and $\forall \boldsymbol{x}, \boldsymbol{z} \in V$.

    Hence V is a subspace of $\mathbb{R}^n$.

13. $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$ is an orthonormal basis for $W \subseteq \mathbb{R}^n$.

For any $\boldsymbol{x} \in W$, $\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$, with λ_i uniquely defined. Now

$$\begin{aligned} \boldsymbol{x} \,.\, \boldsymbol{x}_j = \boldsymbol{x}_j \,.\, \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i &= \sum_{i=1}^{k} \lambda_i (\boldsymbol{x}_j \,.\, \boldsymbol{x}_i) \\ &= \sum_{i=1}^{k} \lambda_i \delta_{ji} = \lambda_j \end{aligned}$$

Holds $\forall j = 1, 2, \ldots, k$, so $\boldsymbol{x} = \sum_{j=1}^{k} (\boldsymbol{x} \,.\, \boldsymbol{x}_j) \boldsymbol{x}_j$.

14. It is readily shown that $\boldsymbol{x}_1 \,.\, \boldsymbol{x}_j = 0$ for $j = 2, 3, 4$;
$\boldsymbol{x}_2 \,.\, \boldsymbol{x}_j = 0$ for $j = 3, 4$;
$\boldsymbol{x}_3 \,.\, \boldsymbol{x}_4 = 0$.

{Example: $\boldsymbol{x}_1 \,.\, \boldsymbol{x}_2 = -2 - 2 + 6 - 2 = 0$—check others}.

Hence $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ is an orthogonal set of 4 vectors in $\mathbb{R}^4$. Follows that it is LI.

Since $\dim \mathbb{R}^4 = 4$, **any** 4 LI vectors in $\mathbb{R}^4$ will form a basis for $\mathbb{R}^4$. Thus $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ is an *orthogonal* basis for $\mathbb{R}^4$.

Denote $(1, 1, 1, 1,)$ by $\mathbf{x}$. Thus $\boldsymbol{x} = \sum_{j=1}^{4} \mu_j \boldsymbol{x}_j$, $\mu_i \in \mathbb{R}$.

For $i = 1, 2, 3, 4$, we have
$$\begin{aligned} \boldsymbol{x} \,.\, \boldsymbol{x}_i &= \sum_{j=1}^{4} \mu_j (\boldsymbol{x}_j \,.\, \boldsymbol{x}_i) \\ &= \mu_i (\boldsymbol{x}_i \,.\, \boldsymbol{x}_i) \\ &= \mu_i \|\boldsymbol{x}_i\|^2 \end{aligned}$$

Hence $\mu_i = \dfrac{(\boldsymbol{x} \,.\, \boldsymbol{x}_i)}{\|\boldsymbol{x}_i\|^2}$ for $i = 1, 2, 3, 4$.

The coordinates of $\boldsymbol{x}$ w.r.t. this basis are $(\mu_1, \mu_2, \mu_3, \mu_4)$. Coordinates readily seen to be $\left(\frac{1}{7}, \frac{5}{21}, \frac{1}{3}, 1\right)$.

Aside

Note that $\boldsymbol{x}_j / \|\boldsymbol{x}_j\|$, $j = 1, 2, 3, 4$ is an *orthonormal* basis for $\mathbb{R}^4$. Coordinates w.r.t. this orthonormal basis are $\dfrac{\boldsymbol{x} \,.\, \boldsymbol{x}_j}{\|\boldsymbol{x}_j\|}$, $j = 1, 2, 3, 4$.

These are $\left(\frac{1}{\sqrt{7}}, \frac{5}{\sqrt{21}}, \frac{2}{\sqrt{6}}, \sqrt{2}\right)$.

$$\boldsymbol{x} = \sum_{j=1}^{4} \frac{(\boldsymbol{x} \,.\, \boldsymbol{x}_j)}{\|\boldsymbol{x}_j\|} \frac{\boldsymbol{x}_j}{\|\boldsymbol{x}_j\|}$$

15. Take $\boldsymbol{y}_1 = \boldsymbol{x}_1 = (0, 2, 1, 0)$.

    Let $\boldsymbol{y}_2 = \boldsymbol{x}_2 + \lambda \boldsymbol{y}_1$ with λ chosen s.t. $\boldsymbol{y}_2 \,.\, \boldsymbol{y}_1 = 0$.

    Thus $\lambda = -(\boldsymbol{x}_2, \boldsymbol{y}_1)/(\boldsymbol{y}_1 \,.\, \boldsymbol{y}_1) = +\frac{2}{5}$.

    Thus $\boldsymbol{y}_2 = \boldsymbol{x}_2 + \frac{2}{5}\boldsymbol{y}_1 = \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$.

    For convenience, choose $\boldsymbol{y}_2 = (5, -1, 2, 0)$.

    Let $\boldsymbol{y}_3 = \boldsymbol{x}_3 + \lambda_1 \boldsymbol{y}_1 + \lambda_2 \boldsymbol{y}_2$.

$$\begin{aligned} \boldsymbol{y}_3 \,.\, \boldsymbol{y}_1 &= 4 + 5\lambda_1 = 0 \quad \text{if} \quad \lambda_1 = -4/5 \\ \boldsymbol{y}_3 \,.\, \boldsymbol{y}_2 &= 3 + 30\lambda_2 = 0 \quad \text{if} \quad \lambda_2 = -1/10 \end{aligned}$$

    Thus $\boldsymbol{y}_3 = \left[\frac{1}{2}, \frac{1}{2}, -1, -1\right)$. Select $\boldsymbol{y}_3 = (1, 1, -2, -2)$.

    Let $\boldsymbol{y}_4 = \boldsymbol{x}_4 + \lambda_1 \boldsymbol{y}_1 + \lambda_2 \boldsymbol{y}_2 + \lambda_3 \boldsymbol{y}_3$

$$\begin{aligned} \boldsymbol{y}_4 \,.\, \boldsymbol{y}_1 &= 0 + 5\lambda_1 = 0 \quad \text{if} \quad \lambda_1 = 0;\ \boldsymbol{y}_4 \,.\, \boldsymbol{y}_2 = 5 + 30\lambda_2 = 0 \text{ if } \lambda_2 = -\frac{1}{6} \\ \boldsymbol{y}_4 \,.\, \boldsymbol{y}_3 &= -1 + 10\lambda_3 = 0 \quad \text{if} \quad \lambda_3 = 1/10. \end{aligned}$$

    Thus
$$\boldsymbol{y}_4 = \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{12}{15}\right).$$

    For convenience choose $\boldsymbol{y}_4 = (1, 1, -2, 3)$.

    $\Rightarrow$ Orthogonal basis for $\mathbb{R}^4$ is $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4\}$.

16. (a) Take $\boldsymbol{y}_1 = \boldsymbol{x}_1 = (1, 1)$. Let $\boldsymbol{y}_2 = \boldsymbol{x}_2 + \lambda \boldsymbol{y}_1$
    $\boldsymbol{y}_2 \,.\, \boldsymbol{y}_1 = 3 + 2\lambda = 0$ if
    $\lambda = -\frac{3}{2}$. Hence $\boldsymbol{y}_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$. For convenience take $\boldsymbol{y}_2 = (-1, 1)$. Hence $\{(1, 1), (-1, 1)\}$ is an orthogonal basis and $\left\{\frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(-1, 1)\right\}$ is an orthonormal basis.

    (b) Take $\boldsymbol{y}_1 = \boldsymbol{x}_1 = (2, 1, 1)$. Let $\boldsymbol{y}_2 = \boldsymbol{x}_2 + \lambda \boldsymbol{y}_1$
    $\boldsymbol{y}_2 \,.\, \boldsymbol{y}_1 = 3 + 6\lambda = 0$ if $\lambda = -\frac{1}{2} \Rightarrow \boldsymbol{y}_2 = \left(0, \frac{1}{2}, -\frac{1}{2}\right)$.

    Take $\boldsymbol{y}_2 = (0, 1, -1)$ for convenience.

    Orthogonal basis is $\{(2, 1, 1), (0, 1, -1)\}$.

    Orthonormal basis is $\left\{\frac{1}{\sqrt{6}}(2, 1, 1), \frac{1}{\sqrt{2}}(0, 1, -1)\right\}$.

(c) Take $\boldsymbol{y}_1 = \boldsymbol{x}_1 = (1,1,1)$. Let $\boldsymbol{y}_2 = \boldsymbol{x}_2 + \lambda \boldsymbol{y}_1$

$\boldsymbol{y}_2 \,.\, \boldsymbol{y}_1 = 2 + 3\lambda = 0$ if $\lambda = -\dfrac{2}{3} \Rightarrow \boldsymbol{y}_2 = \left(-\dfrac{2}{3}\,, \dfrac{1}{3}\,, \dfrac{1}{3}\right)$

Take $\boldsymbol{y}_2 = (-2,1,1)$.

Let $\boldsymbol{y}_3 = \boldsymbol{x}_3 + \lambda_1 \boldsymbol{y}_1 + \lambda_2 \boldsymbol{y}_2$: $\boldsymbol{y}_3 \,.\, \boldsymbol{y}_1 = 0$ gives $\lambda_1 = -\dfrac{1}{3}$

$\boldsymbol{y}_3 \,.\, \boldsymbol{y}_2 = 0$ gives $\lambda_2 = -\dfrac{1}{6}\,.$

Thus $\boldsymbol{y}_3 = \left(0, -\dfrac{1}{2}\,, \dfrac{1}{2}\right)$. Take $\boldsymbol{y}_3 = (0,-1,1)$.

Orthogonal basis is $\{(1,1,1),(-2,1,1),(0,-1,1)\}$.

Orthonormal basis is $\left\{\dfrac{1}{\sqrt{3}}\,(1,1,1),\ \dfrac{1}{\sqrt{6}}\,(-2,1,1),\ \dfrac{1}{\sqrt{2}}\,(0,-1,1)\right\}$.