University of Strathclyde, Department of Mathematics and Statistics

MM102 Applications of Calculus
Exercises for Week 4
Solutions

Q1. Determine the equations for the tangent and the normal to the graph of the function

$$f(x) = \sin x$$

at the point $x = \dfrac{\pi}{4}$.

Solution:
The point through which tangent and normal go has coordinates

$$a = \frac{\pi}{4}, \qquad b = f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

The derivative of f is

$$f'(x) = \cos x.$$

Hence for the tangent we have

$$m = f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},$$

and the equation of the tangent is

$$\boxed{y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)}$$

For the normal we have

$$m = -\frac{1}{f'\left(\frac{\pi}{4}\right)} = -\sqrt{2}$$

and the equation of the normal is

$$\boxed{y - \frac{1}{\sqrt{2}} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)}$$

Q2. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x and y given that

2(a) $x^3 + y^3 = 1$

Solution:
Differentiate both sides with respect to x:

$$3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Longrightarrow \quad y^2\frac{\mathrm{d}y}{\mathrm{d}x} = -x^2$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2}{y^2}}$$

2(b) $2x^3 \sin y + y^2 - xy^3 = 1$

Solution:
Differentiate both sides with respect to x:

$$\begin{aligned}
6x^2 \sin y + 2x^3 \cos y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^3 - 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x}\Big(2x^3 \cos y + 2y - 3xy^2\Big) &= y^3 - 6x^2 \sin y \\
\implies \quad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3 - 6x^2 \sin y}{2x^3 \cos y + 2y - 3xy^2}}
\end{aligned}$$

2(c) $\sqrt{xy} + \sin x + \cos y = 0$

Solution:
First we write the equation as follows:

$$(xy)^{1/2} + \sin x + \cos y = 0.$$

Now differentiate both sides with respect to x:

$$\begin{aligned}
&\frac{1}{2}(xy)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}(xy) + \cos x - \sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\
\implies \quad &\frac{1}{2\sqrt{xy}}\Big(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\Big) + \cos x - \sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\
\implies \quad &y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\sqrt{xy}\cos x - 2\sqrt{xy}\sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\
\implies \quad &\frac{\mathrm{d}y}{\mathrm{d}x}\big(x - 2\sqrt{xy}\sin y\big) = -y - 2\sqrt{xy}\cos x \\
\implies \quad &\frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{\frac{2\sqrt{xy}\cos x + y}{2\sqrt{xy}\sin y - x}}
\end{aligned}$$

2(d) $\sin(xy) = \cos x \cdot \cos y$

Solution:
Differentiate both sides with respect to x:

$$\begin{aligned}
\cos(xy)\Big(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\Big) &= -\sin x \cdot \cos y + \cos x \cdot \big(-\sin y\big)\frac{\mathrm{d}y}{\mathrm{d}x} \\
\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x}\Big(x\cos(xy) + \cos x \cdot \sin y\Big) &= -y\cos(xy) - \sin x \cdot \cos y \\
\implies \quad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y\cos(xy) - \sin x \cdot \cos y}{x\cos(xy) + \cos x \cdot \sin y}}
\end{aligned}$$

2(e) $\sin(x + y^2) = y$

Solution:
Differentiate both sides with respect to x:

$$\begin{aligned}
&\cos(x + y^2)\left(1 + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}y}{\mathrm{d}x} \\
\implies\ &\frac{\mathrm{d}y}{\mathrm{d}x}\left(2y\cos(x + y^2) - 1\right) = -\cos(x + y^2) \\
\implies\ &\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\cos(x + y^2)}{2y\cos(x + y^2) - 1}}
\end{aligned}$$

2(f) $\sin x + \cos y = 1$

Solution:
Differentiate both sides with respect to x:

$$\cos x - \sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \implies \quad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{\sin y}}$$

2(g) $e^y - xy^2 = 3$

Solution:
Differentiate both sides with respect to x:

$$\begin{aligned}
e^y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 - 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad &\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x}\left(e^y - 2xy\right) = y^2 \\
&\implies \quad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{e^y - 2xy}}
\end{aligned}$$

Q3. Show that the given point lies on the curve. Moreover, find the tangent to the curve at that point.

3(a) $y^2 = 2x^3, \qquad (2, -4)$

Solution:
We plug the coordinates of $(2, -4)$ into the given equation:

$$\text{LHS} = (-4)^2 = 16; \qquad \text{RHS} = 2 \times 2^3 = 16.$$

Hence $(2, -4)$ lies on the curve.
Differentiate both sides with respect to x:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 \quad \implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{y}$$

At the point $(2, -4)$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3 \times 2^2}{-4} = -3.$$

Hence the equation for the tangent is

$$\boxed{y + 4 = -3(x - 2)}$$

3(b) $(x+y)^3 = 2x + y + 3, \qquad (3, -1)$

Solution:
We plug the coordinates of $(3, -1)$ into the given equation:

$$\text{LHS} = (3-1)^3 = 8; \qquad \text{RHS} = 2 \times 3 - 1 + 3 = 8.$$

Hence $(3, -1)$ lies on the curve.
Differentiate both sides with respect to x:

$$
\begin{aligned}
3(x+y)^2\Big(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\Big) &= 2 + \frac{\mathrm{d}y}{\mathrm{d}x} \\
\Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\Big(3(x+y)^2 - 1\Big) &= 2 - 3(x+y)^2 \\
\Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2 - 3(x+y)^2}{3(x+y)^2 - 1}
\end{aligned}
$$

At the point $(3, -1)$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 - 3 \times (3-1)^2}{3 \times (3-1)^2 - 1} = -\frac{10}{11}\,.$$

Hence the equation for the tangent is

$$\boxed{y + 1 = -\frac{10}{11}\big(x - 3\big)}$$

3(c) $xy^3 - x^3y = 30, \qquad (2, 3)$

Solution:
We plug the coordinates of $(2, 3)$ into the given equation:

$$\text{LHS} = 2 \times 3^3 - 2^3 \times 3 = 54 - 24 = 30 = \text{RHS}.$$

Hence $(2, 3)$ lies on the curve.
Differentiate both sides with respect to x:

$$
\begin{aligned}
y^3 + 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2y - x^3\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\big(3xy^2 - x^3\big) &= 3x^2y - y^3 \\
\Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2y - y^3}{3xy^2 - x^3}
\end{aligned}
$$

At the point with $x = 2$, $y = 3$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3 \times 2^2 \times 3 - 3^3}{3 \times 2 \times 3^2 - 2^3} = \frac{9}{46}$$

Hence the equation for the tangent is

$$\boxed{y - 3 = \frac{9}{46}\Big(x - 2\Big)}$$

3(d) $x = y - \cos y, \qquad \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:
We plug the coordinates of $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ into the given equation:

$$\text{LHS} = \frac{\pi}{2}; \qquad \text{RHS} = \frac{\pi}{2} - \underbrace{\cos\frac{\pi}{2}}_{=0} = \frac{\pi}{2}.$$

Hence $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ lies on the curve.
Differentiate both sides w.r.t. x:

$$\begin{aligned} 1 &= \frac{\mathrm{d}y}{\mathrm{d}x} + \sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\ \implies \quad 1 &= \frac{\mathrm{d}y}{\mathrm{d}x}\Big(1 + \sin y\Big) \\ \implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1 + \sin y} \end{aligned}$$

At the point with $x = \frac{\pi}{2}$, $y = \frac{\pi}{2}$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + \sin\frac{\pi}{2}} = \frac{1}{2}$$

Hence the equation for the tangent is

$$\boxed{y - \frac{\pi}{2} = \frac{1}{2}\Big(x - \frac{\pi}{2}\Big)}$$

Q4. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as functions of x and y given that

4(a) $xy^2 + y = 1$

Solution:
Differentiate both sides w.r.t. x:

$$y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{1}$$

$$\begin{aligned} \implies \quad & \frac{\mathrm{d}y}{\mathrm{d}x}\big(2xy + 1\big) = -y^2 \\ \implies \quad & \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{2xy + 1}} \end{aligned}$$

Differentiate this again w.r.t. x:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{y^2}{2xy+1}\right)$$

$$= -\frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}(y^2)\right)(2xy+1) - y^2\frac{\mathrm{d}}{\mathrm{d}x}(2xy+1)}{(2xy+1)^2} = -\frac{2y\frac{\mathrm{d}y}{\mathrm{d}x}(2xy+1) - y^2\left(2y+2x\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(2xy+1)^2}$$

$$= -\frac{2y\frac{-y^2}{2xy+1}(2xy+1) - y^2\left(2y+2x\frac{-y^2}{2xy+1}\right)}{(2xy+1)^2} \quad \left(\text{where the result for } \frac{\mathrm{d}y}{\mathrm{d}x} \text{ was used}\right)$$

$$= -\frac{-2y^3-2y^3+\frac{2xy^4}{2xy+1}}{(2xy+1)^2} = -\frac{-4y^3+\frac{2xy^4}{2xy+1}}{(2xy+1)^2} = -\frac{-4y^3(2xy+1)+2xy^4}{(2xy+1)^3}$$

$$= -\frac{-8xy^4-4y^3+2xy^4}{(2xy+1)^3} = \frac{6xy^4+4y^3}{(2xy+1)^3} = \boxed{\frac{2y^3(3xy+2)}{(2xy+1)^3}}$$

Alternative solution:
Differentiate both sides of (1):

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

$$\implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(2xy+1\right) = -4y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

$$\implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(2xy+1\right) = 4y\frac{y^2}{2xy+1} - 2x\left(-\frac{y^2}{2xy+1}\right)^2$$

$$\implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(2xy+1\right) = \frac{4y^3(2xy+1)-2xy^4}{(2xy+1)^2}$$

$$\implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(2xy+1\right) = \frac{8xy^4+4y^3-2xy^4}{(2xy+1)^2}$$

$$\implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{6xy^4+4y^3}{(2xy+1)^3} = \frac{2y^3(3xy+2)}{(2xy+1)^3}$$

4(b) $y^4 + y = x^3$

Solution:
Differentiate both sides w.r.t. x:

$$4y^3\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \tag{2}$$

$$\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x}\left(4y^3+1\right) = 3x^2$$

$$\implies \quad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{4y^3+1}}$$

Differentiate this again w.r.t. x:

$$\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3x^2}{4y^3+1}\right) \\
&= \frac{6x(4y^3+1) - 3x^2\frac{\mathrm{d}}{\mathrm{d}x}\big(4y^3+1\big)}{(4y^3+1)^2} = \frac{6x(4y^3+1) - 3x^2 \times 12y^2\frac{\mathrm{d}y}{\mathrm{d}x}}{(4y^3+1)^2} \\
&= \frac{6x(4y^3+1) - 36x^2y^2\dfrac{3x^2}{4y^3+1}}{(4y^3+1)^2} \quad \left(\text{where we used the result for } \frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= \boxed{\frac{6x(4y^3+1)^2 - 108x^4y^2}{(4y^3+1)^3}}
\end{aligned}$$

Alternative solution:
Differentiate both sides of (2):

$$\begin{aligned}
& 12y^2\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + 4y^3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x \\
\implies \quad & \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\big(4y^3+1\big) = 6x - 12y^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\implies \quad & \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\big(4y^3+1\big) = 6x - 12y^2\left(\frac{3x^2}{4y^3+1}\right)^2 \\
\implies \quad & \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\big(4y^3+1\big) = \frac{6x(4y^3+1)^2 - 12y^2 \times 9x^4}{(4y^3+1)^2} \\
\implies \quad & \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{6x(4y^3+1)^2 - 108x^4y^2}{(4y^3+1)^3}
\end{aligned}$$

Q5. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of the parameter t when x and y are given by

5(a) $x = 4t^2 - 1, \quad y = 2t + 1$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = 8t, \qquad \dot{y} = 2.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{8t}{2} = \boxed{4t}$$

5(b) $x = 2\sec t, \quad y = \tan t$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = 2\sec t \cdot \tan t, \qquad \dot{y} = \sec^2 t.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{\sec^2 t}{2\sec t \cdot \tan t} = \frac{\sec t}{2\tan t} = \frac{\dfrac{1}{\cos t}}{2\dfrac{\sin t}{\cos t}} = \boxed{\frac{1}{2\sin t}}$$

5(c) $x = \dfrac{1-t^2}{1+t^2}, \quad y = \dfrac{2t}{1+t^2}$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2} = \frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

$$\dot{y} = \frac{2(1+t^2) - 2t \cdot 2t}{(1+t^2)^2} = \frac{2 + 2t^2 - 4t^2}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{\dfrac{2-2t^2}{(1+t^2)^2}}{\dfrac{-4t}{(1+t^2)^2}} = \frac{2-2t^2}{-4t} = \boxed{\frac{t^2-1}{2t}}$$

5(d) $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = 1 - \frac{1}{t^2},$$

$$\dot{y} = 1 + \frac{1}{t^2}.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{1 + \dfrac{1}{t^2}}{1 - \dfrac{1}{t^2}} = \boxed{\frac{t^2+1}{t^2-1}}$$

Q6. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as functions of the parameter t when x and y are given by

6(a) $x = \ln t + 2, \quad y = t^3 + 2t$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = \frac{1}{t},$$
$$\dot{y} = 3t^2 + 2.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{3t^2+2}{1/t} = \boxed{3t^3 + 2t}$$

Let us denote this function by $z(t)$. The derivative of z with respect to t is

$$\dot{z} = 9t^2 + 2$$

Hence

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\dot{z}}{\dot{x}} = \frac{9t^2+2}{1/t} = \boxed{9t^3 + 2t}$$

6(b) $x = \cos t + t, \quad y = \sin t + t^2$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = 1 - \sin t, \qquad \dot{y} = \cos t + 2t.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \boxed{\frac{\cos t + 2t}{1 - \sin t}}$$

Let us denote this function by $z(t)$. The derivative of z with respect to t is

$$\begin{aligned}\dot{z} &= \frac{(1-\sin t)(-\sin t + 2) - (\cos t + 2t)(-\cos t)}{(1 - 2\sin t)^2}\\ &= \frac{-\sin t + \sin^2 t + 2 - 2\sin t + \cos^2 t + 2t\cos t}{(1-\sin t)^2}\\ &= \frac{3 - 3\sin t + 2t\cos t}{(1-\sin t)^2}\end{aligned}$$

Hence

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\dot{z}}{\dot{x}} = \boxed{\frac{3 - 3\sin t + 2t\cos t}{(1-\sin t)^3}}$$

6(c) $x = t^2, \quad y = t^3$

Solution:
The derivatives of x and y with respect to t are

$$\begin{aligned} \dot{x} &= 2t, \\ \dot{y} &= 3t^2. \end{aligned}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{3t^2}{2t} = \boxed{\frac{3}{2}t.}$$

Let us denote this function by $z(t)$. Then

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\dot{z}}{\dot{x}} = \frac{\frac{3}{2}}{2t} = \boxed{\frac{3}{4t}}$$

6(d) $x = t^2 + t, \quad y = 2t^3 + t^2 + 1$

Solution:
The derivatives of x and y with respect to t are

$$\begin{aligned} \dot{x} &= 2t + 1, \\ \dot{y} &= 6t^2 + 2t. \end{aligned}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \boxed{\frac{6t^2 + 2t}{2t + 1}}$$

Let us denote this function by $z(t)$. The derivative of z with respect to t is

$$\begin{aligned} \dot{z} &= \frac{(12t + 2)(2t + 1) - 2(6t^2 + 2t)}{(2t + 1)^2} \\ &= \frac{24t^2 + 4t + 12t + 2 - 12t^2 - 4t}{(2t + 1)^2} \\ &= \frac{12t^2 + 12t + 2}{(2t + 1)^2} \end{aligned}$$

Hence

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\dot{z}}{\dot{x}} = \boxed{\frac{12t^2 + 12t + 2}{(2t + 1)^3}}$$

Q7. Find the equations for the tangent and the normal to the curve given parametrically by

$$x = t^2 + \frac{1}{t}, \qquad y = t^2 - t + 1$$

at the point where $t = 1$.

Solution:
The derivatives of x and y w.r.t. t are

$$\dot{x} = 2t - \frac{1}{t^2}, \qquad \dot{y} = 2t - 1\,.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{2t-1}{2t - \frac{1}{t^2}}$$

For $t = 1$ we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\times 1 - 1}{2\times 1 - \frac{1}{1^2}} = 1.$$

The coordinates of the point for $t = 1$ are

$$x = 2, \qquad y = 1.$$

Hence the equation for the tangent is

$$y - 1 = x - 2 \qquad \text{or} \qquad \boxed{y = x - 1}$$

Q8. Find the length of the given curve:

8(a)

$$x = t - \frac{t^2}{2}, \quad y = \frac{4}{3}t^{\frac{3}{2}} \qquad t \in [0, 1]$$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = 1 - t,$$
$$\dot{y} = \frac{4}{3} \times \frac{3}{2} t^{\frac{1}{2}} = 2t^{\frac{1}{2}}.$$

The arc length is equal to

$$\begin{aligned}
s &= \int_0^1 \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\mathrm{d}t = \int_0^1 \sqrt{(1-t)^2 + 4t}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{1 - 2t + t^2 + 4t}\,\mathrm{d}t = \int_0^1 \sqrt{1 + 2t + t^2}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{(1+t)^2}\,\mathrm{d}t = \int_0^1 (1+t)\mathrm{d}t \\
&= \left[t + \frac{t^2}{2}\right]_0^1 = 1 + \frac{1}{2} - 0 = \boxed{\frac{3}{2}}
\end{aligned}$$

8(b)
$$x = \ln t \qquad t \in [1,2]$$
$$y = \frac{1}{2}\Big(t + \frac{1}{t}\Big)$$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = \frac{1}{t},$$
$$\dot{y} = \frac{1}{2}\Big(1 - \frac{1}{t^2}\Big).$$

Let us consider the expression

$$\begin{aligned}\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2 &= \frac{1}{t^2} + \frac{1}{4}\Big(1 - \frac{1}{t^2}\Big)^2 = \frac{1}{t^2} + \frac{1}{4} - \frac{1}{2t^2} + \frac{1}{4t^4} \\ &= \frac{1}{4} + \frac{1}{2t^2} + \frac{1}{4t^4} = \frac{t^4 + 2t^2 + 1}{4t^4} \\ &= \frac{(t^2+1)^2}{4t^4} = \left(\frac{t^2+1}{2t^2}\right)^2.\end{aligned}$$

The arc length is equal to

$$s = \int_1^2 \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\,\mathrm{d}t = \int_1^2 \frac{t^2+1}{2t^2}\,\mathrm{d}t$$
$$= \frac{1}{2}\int_1^2 \Big(1 + \frac{1}{t^2}\Big)\mathrm{d}t = \frac{1}{2}\Big[t - \frac{1}{t}\Big]_1^2 = \frac{1}{2}\Big(2 - \frac{1}{2} - (1-1)\Big) = \boxed{\frac{3}{4}}$$

8(c)
$$x = 3t^2 \qquad t \in [0,1]$$
$$y = 3t^3 - t$$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = 6t, \qquad \dot{y} = 9t^2 - 1.$$

Let us consider the expression

$$\begin{aligned}\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2 &= (6t)^2 + \big(9t^2 - 1\big)^2 = 36t^2 + 81t^4 - 18t^2 + 1 \\ &= 81t^4 + 18t^2 + 1 = (9t^2+1)^2.\end{aligned}$$

The arc length is equal to

$$s = \int_0^1 \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\,\mathrm{d}t = \int_0^1 (9t^2+1)\mathrm{d}t$$
$$= \Big[3t^3 + t\Big]_0^1 = 3 + 1 - 0 = \boxed{4}$$

8(d)
$$x = 2t^{3/2} + 1 \qquad t \in [0,1]$$
$$y = 4t - 2$$

Solution:
The derivatives of x and y with respect to t are

$$\dot{x} = 3t^{1/2}, \qquad \dot{y} = 4.$$

The arc length is equal to

$$s = \int_0^1 \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\,\mathrm{d}t = \int_0^1 \sqrt{9t+16}\,\mathrm{d}t$$

$$\begin{bmatrix} u = 9t+16, & \mathrm{d}u = 9\,\mathrm{d}t \\ t = 0 \quad \Rightarrow & u = 16 \\ t = 1 \quad \Rightarrow & u = 25 \end{bmatrix}$$

$$= \frac{1}{9}\int_{16}^{25} u^{1/2}\mathrm{d}u = \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_{16}^{25} = \frac{2}{27}\big(125 - 64\big) = \boxed{\frac{122}{27}}$$

Q9. Find the surface area when the following parametric curve is rotated about the x-axis by 360°:

$$x = t - \frac{t^2}{2} \qquad t \in [0,1]$$
$$y = \frac{4}{3}t^{3/2}$$

Solution:
The derivatives of x and y w.r.t. t are

$$\dot{x} = 1 - t,$$
$$\dot{y} = \frac{4}{3} \times \frac{3}{2}t^{1/2} = 2\sqrt{t}.$$

The surface area is equal to

$$S = 2\pi \int_0^1 \frac{4}{3}t^{3/2}\sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\;\mathrm{d}t = \frac{8\pi}{3}\int_0^1 t^{3/2}\sqrt{(1-t)^2 + 4t}\;\mathrm{d}t$$

$$= \frac{8\pi}{3}\int_0^1 t^{3/2}\sqrt{1 - 2t + t^2 + 4t}\;\mathrm{d}t = \frac{8\pi}{3}\int_0^1 t^{3/2}\sqrt{1 + 2t + t^2}\;\mathrm{d}t$$

$$= \frac{8\pi}{3}\int_0^1 t^{3/2}\sqrt{(1+t)^2}\;\mathrm{d}t = \frac{8\pi}{3}\int_0^1 t^{3/2}(1+t)\,\mathrm{d}t$$

$$= \frac{8\pi}{3}\int_0^1 \big(t^{3/2} + t^{5/2}\big)\,\mathrm{d}t$$

$$= \frac{8\pi}{3}\left[\frac{2}{5}t^{5/2} + \frac{2}{7}t^{7/2}\right]_0^1 = \frac{8\pi}{3}\left(\frac{2}{5} + \frac{2}{7} - 0\right) = \boxed{\frac{64\pi}{35}}$$