UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MATHEMATICS & STATISTICS

MM103 Geometry and Algebra

Chapter 4: Lines and Planes

Q1. (a) $\mathbf{r} = t\begin{bmatrix}1\\1\\1\end{bmatrix}$, (b) $\mathbf{r} = \begin{bmatrix}2\\1\\2\end{bmatrix} + t\begin{bmatrix}1\\1\\-1\end{bmatrix}$, (c) $\mathbf{r} = t\begin{bmatrix}1\\5\\-3\end{bmatrix}$, (d) $\mathbf{r} = \begin{bmatrix}1\\0\\6\end{bmatrix} + t\begin{bmatrix}0\\-2\\1\end{bmatrix}$.

Q2. Any such line must be of the form $\mathbf{r} = \mathbf{p} + t\mathbf{u}$, where the vector $\mathbf{u}$ is perpendicular to $\begin{bmatrix}-2 & 3 & -1\end{bmatrix}^T$. Note that $\mathbf{p}$ can be taken to equal $\mathbf{0}$ because the required line passes through the origin. Thus, $\mathbf{r} = t\begin{bmatrix}3 & 2 & 0\end{bmatrix}^T$ is one such line. This line lies on the plane $-2x + 3y - z = 0$ and any such line will be perpendicular to the given line in the question.

Q3. For a given a line $\mathbf{r} = \mathbf{p} + t\mathbf{u}$ and point $A(a_1, a_2, a_3)$, the point B on $\mathbf{r}$ that is closest to A will satisfy

$$\overrightarrow{AB} \cdot \mathbf{u} = 0.$$

Now, B will have coordinates $(p_1 + tu_1, p_2 + tu_2, p_3 + tu_3)$, so we can solve the above equation for t to determine the closest point B. The distance can then be calculated using the standard Euclidean distance formula.

(a) $B = (28/15, 161/15, -11/3)$ and the minimum distance is $\dfrac{\sqrt{31605}}{15}$.

(b) The given point lies on the given line, so the minimum distance is 0.

(c) $B = (-25/29, 97/29, 0)$ and the minimum distance is $\dfrac{\sqrt{33524}}{29}$.

Q4.

(a) The equations to solve are:

$$2 + t = \lambda$$
$$1 - 5t = -2\lambda$$
$$4 + 2t = \alpha + 3\lambda$$

This system has a solution with $\alpha = -\dfrac{11}{3}$.

(b) The equations to solve are:

$$\begin{aligned} -3+t &= \lambda \\ -t &= -2\lambda \\ 2+t &= \alpha+3\lambda \end{aligned}$$

This system has a solution with $\alpha = -1$.

(c) The equations to solve are:

$$\begin{aligned} 3+2t &= \lambda \\ 13-10t &= -2\lambda \\ 4t &= \alpha+3\lambda \end{aligned}$$

This system has a solution with $\alpha = -\dfrac{46}{3}$.

Q5. If the given plane is $\mathbf{r} = \mathbf{a} + t\mathbf{b} + u\mathbf{c}$, then the Hessian form will be given by $\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{a}$, where the normal vector $\mathbf{n}$ equals $\mathbf{b}\times\mathbf{c}$.

(a) $-11x - 8y + 5z = -56$

(b) $3x + 2y - 3z = 17$

(c) $x = 0$

Q6. Label the points as $A(1,2,-1)$, $B(3,1,5)$ and $C(-1,-1,-1)$. Then the plane has equation $\mathbf{r} = \overrightarrow{OA} + t\overrightarrow{AB} + u\overrightarrow{AC}$, i.e.,

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + u\begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}.$$

The Hessian form can now be found in the same way as in Q5. It is $9x - 6y - 4z = 1$.

Q7. From the given equation, find three (non-collinear) points A, B, C that lie on the plane. The vector form can then be given by $\mathbf{r} = \overrightarrow{OA} + t\overrightarrow{AB} + u\overrightarrow{AC}$.

(a) $\mathbf{r} = t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

(c) $\mathbf{r} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Q8. The shortest distance from the point (α, β, γ) to the plane $ax + by + cz = d$ is equal to

$$\frac{|a\alpha + b\beta + c\gamma|}{\sqrt{a^2 + b^2 + c^2}}.$$

(a) $\frac{6}{\sqrt{59}}$, (b) $\frac{7}{\sqrt{20}}$, (c) $\frac{34}{\sqrt{54}}$.

Q9. The angle between two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is equal to the angle between the normal vectors. The normal vector of the plane $x = 0$ is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

(a) $\cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$

(b) $\frac{\pi}{2}$

(c) The planes are parallel, so the angle between them is zero.

Q10. Substitute the equation of the line into the equation of the plane and then solve for t. Then, use this value of t to find the point of intersection.

(a) $(-2/3, 1, 2/3)$

(b) $(-1, -2, 4)$

(c) $(1/3, 0, 2/3)$

Q11. In order to find the line of intersection of two planes, one must find a point P that lies on both planes and a vector $\mathbf{u}$ that is mutually perpendicular to the normal vectors of each plane. The equation of the line is then $\mathbf{r} = \overrightarrow{OP} + t\mathbf{u}$.

(a) $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and P has coordinates $(3, 1, 0)$. The line is then

$$\mathbf{r} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

(b) $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ and P has coordinates $(0, 0, -1/4)$. The line is then

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \end{bmatrix} + t \begin{bmatrix} 4 \\ 16 \\ 5 \end{bmatrix}.$$

(c) The planes are parallel and do not intersect.

(d) $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and P has coordinates $(0, 0, 0)$. The line is then

$$\mathbf{r} = t \begin{bmatrix} -9 \\ 3 \\ -1 \end{bmatrix}.$$

Chapter 4: Transformations

Q1. If the vertices of the unit cube, $(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1)$, are mapped onto P,Q,R,S, respectively, then the required transformation is

$$f(\mathbf{x}) = \begin{bmatrix} \overrightarrow{PQ} & \overrightarrow{PR} & \overrightarrow{PS} \end{bmatrix} \mathbf{x} + \overrightarrow{OP}.$$

(a) $f(\mathbf{x}) = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 1 & 2 \\ -2 & 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $f(\mathbf{x}) = \begin{bmatrix} -1 & -2 & -1 \\ -4 & -3 & -1 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

(c) $f(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \mathbf{x}$

(d) $f(\mathbf{x}) = \mathbf{x} + \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$

Q2. If $\mathbf{n}$ is the normal vector to the given plane of reflection, then the required transformation is given by the Householder matrix $P = I - \dfrac{2}{\mathbf{n}^T\mathbf{n}}\mathbf{n}\mathbf{n}^T$.

(a) $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $P = \begin{bmatrix} -\frac{3}{4} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ and $P = \dfrac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$

Q3. $AB = \begin{bmatrix} 1 & 0 & 0 \\ m+\mu & 1 & 0 \\ n+\nu & 0 & 1 \end{bmatrix}$ and so $AB = I$ if $\mu = -m$ and $\nu = -n$. In this case, $B = A^{-1}$.

Q4

(a) The range space consists of all vectors of the form

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{bmatrix}.$$

We can write these vectors in the form

$$x \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + z \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

and since

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix},$$

we see that the range space consists of all vectors of the form

$$(x-z) \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + (y+2z) \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}.$$

This is a plane that contains the vectors $\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}^T$ and $\begin{bmatrix} 2 & 5 & 8 \end{bmatrix}^T$, and the origin. The normal vector to the plane equals

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \times \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

and so the Hessian form of the plane is given by $-3x+6y-3z=0$, i.e., $x-2y+z=0$.

The null space is at a right angle to the range space of A^T, which is found to be the same plane as above, $x-2y+z=0$. Moreover, the dimension of the null space equals $3-2=1$. Therefore, the null space is a straight line that passes through the origin and is parallel to $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T$ (the normal vector to the plane of the range space). Thus, the null space is given by the line

$$\mathbf{r} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

(b) Range space: $x-2y+z=0$, Null space: $\mathbf{r} = t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(c) Range space: $-2x + y + z = 0$, Null space: $\mathbf{r} = t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

(d) Range space: $\mathbf{r} = t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, Null space: $x + 2y + 3z = 0$.