UNIVERSITY OF STRATHCLYDE
MATHEMATICS & STATISTICS

MM103 Part II: Applications

5. What goes up... must come down

The first main aim of this section is to solve the following practical problem.

Problem: A cannon on level ground fires out a shell at speed U. What is the furthest distance that the shell can travel?

As always, to make progress we need the **relevant equation(s)** and some **data**.

Newton's laws of motion

Isaac Newton (1643–1727) proposed three laws of motion for particles in the late seventeenth century, and these form the basis of "Mechanics":

1. *A body remains in a state of rest or of uniform motion in a straight line unless it is acted on by an external force.*
2. *The rate of change of the momentum of a body is proportional to the force acting upon it.*
3. *To every action there is an equal and opposite reaction.*

We choose units to make the constant of proportionality in the second law above equal to 1. Assuming that the mass of a body stays constant, its rate of change of momentum is equal to its mass $\times$ acceleration, and so **Newton's second law** says that

$$\text{force} \;=\; \text{mass} \;\times\; \text{acceleration}\,.$$

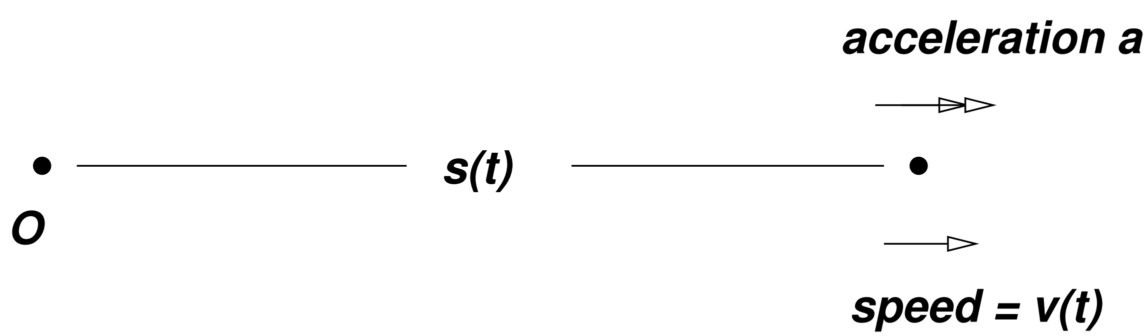
Let us look at the implications of this for a falling object with no external forces acting on it (i.e. we assume that there is no air resistance or wind). In this case, the only force acting on the object is its **weight** acting vertically downwards, and so by Newton's second law, this must equal its mass $\times$ its acceleration. This means that we can regard an object's weight as being its mass multiplied by the **acceleration due to gravity**:

$$\text{weight} \;=\; \text{mass} \;\times\; \text{acceleration due to gravity}\,.$$

The "acceleration due to gravity" acts vertically downwards (towards the centre of the earth) and its size, g, is approximately 9.8 ms^{-2} (it is often taken to be 10 ms^{-2} in numerical examples).

We can use this to derive the relevant equations of motion for the shell problem. Before this we shall look at more general equations for velocity and acceleration

5.1. One dimensional motion



Suppose that a particle is moving in a straight line and at time t it is at a **displacement** $s(t)$ from a fixed origin O, where O is its position at time $t = 0$. The convention is that if the particle is to the right of O in the picture above then $s > 0$, and if it is to the left of O then $s < 0$.

The **velocity** of the particle at time t is

$$v(t) = \frac{ds}{dt} = \dot{s}(t)$$

and its **acceleration** at time t is

$$a(t) = \frac{dv}{dt} = \dot{v}(t)\,,$$

so $a(t) = \ddot{s}(t)$. The **speed** of the particle is the size of its velocity.

Example 5.1 The displacement of a particle from the origin at time t is $s(t) = b \sin \omega t$ where b and ω are positive constants.

(a) Calculate its velocity v and acceleration a at time t.

(b) Calculate s and a at the time at which its speed is first zero.

5.2. One dimensional motion under constant acceleration

Although there are many interesting practical problems which involve non-constant acceleration (i.e. a depends on t, as in the example above), we only need to consider the (easier) case of **constant acceleration** in order to look at the shell (and other projectile problems in which air resistance can be ignored). So between here and the end of Section 5.4 we shall assume that the acceleration a is a constant (it does not change with time). The equation for the velocity is then

$$\frac{dv}{dt} = a = \text{ constant}$$

and integrating with respect to t gives $v = a\,t + C$ where C is a constant of integration. Setting $t = 0$ we see that C is just the velocity at time $t = 0$, so the velocity at time t is

$$v = v_0 + a\,t$$

where v_0 is the velocity at time $t = 0$.

The equation for the displacement s from O (the position at time $t = 0$) is $\dot{s} = v$, i.e.

$$\frac{ds}{dt} = v_0 + a\,t$$

(where a and v_0 are constants) and integrating with respect to t gives

$$s = v_0\,t + \tfrac{1}{2}\,a\,t^2 + C$$

where C is an arbitrary constant of integration. But we know that $s = 0$ at time $t = 0$, and so setting $t = 0$ gives $C = 0$ to give $s = v_0\,t + \frac{1}{2}\,a\,t^2$. These equations are summarised below.

A particle is moving under constant acceleration a, and at time $t = 0$ it is at the point O and is travelling at velocity v_0. The particle's velocity v and displacement s from O at time t are given by

$$\begin{aligned} v &= v_0 + a\,t \\ s &= v_0\,t + \tfrac{1}{2}\,a\,t^2\,. \end{aligned}$$

Example 5.2 A particle is launched from point O at time $t = 0$ at velocity 10 ms^{-1} to the right and is moving under constant deceleration of 2 ms^{-2} in the same direction.

(a) Find the first time t at which its speed is zero. What is its displacement from O at this time?

(b) At what time does it return to the point O?

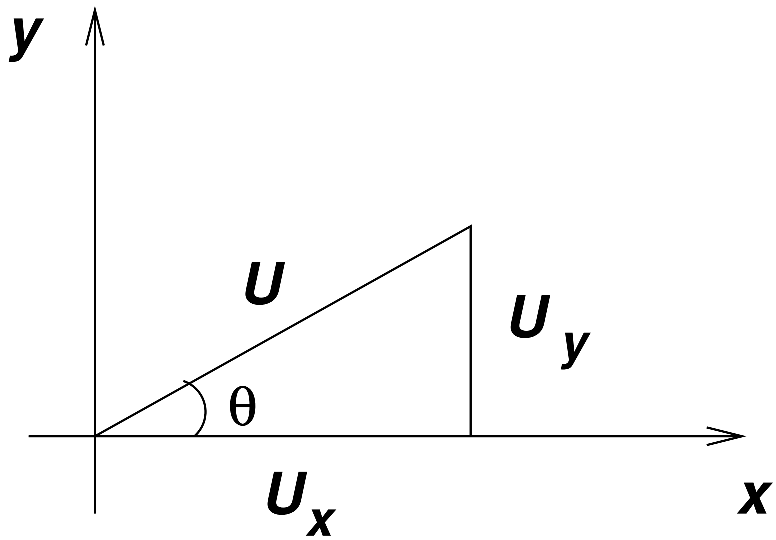
5.3. Components of velocity

Projectile problems (such as the cannon shell) are not one dimensional, although we can look at their horizontal and vertical motion separately. But in order to do this we need to know how to "resolve" the initial velocity into its horizontal and vertical components. This is easily done using a right-angled triangle.

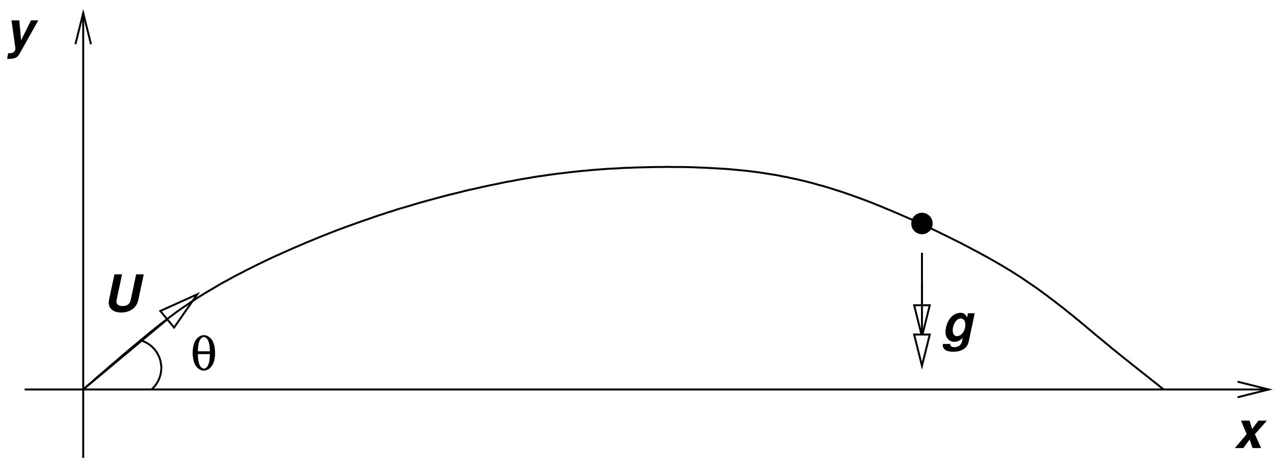
Suppose that the speed is U at an angle θ to the horizontal, as shown below. We draw a right-angled triangle with hypotenuse of length U, then the speed U_x in the horizontal direction is the length of the triangle base and the speed U_y in the vertical direction is the triangle height.

But $\cos\theta = U_x/U$ and $\sin\theta = U_y/U$, and rearranging these gives:

$$\begin{aligned} U_x &= U\,\cos\theta \\ U_y &= U\,\sin\theta\,. \end{aligned}$$



5.4. Projectile problems: motion under gravity



Suppose that a projectile is fired at speed U and angle θ to the horizontal as illustrated above. We have seen that if the initial 1D velocity is v_0 and the (constant) acceleration is a, then the velocity v and displacement s at time t are $v = v_0 + a\,t$ and $s = v_0\,t + \frac{1}{2}\,a\,t^2$. We now use these equations in the horizontal (x) and vertical (y) directions.

$x-$direction:

- initial velocity: $v_0 = U\,\cos\theta$
- acceleration: $a = 0$
- velocity equation $\Rightarrow$ velocity at time t is $v_x = U\,\cos\theta$
- displacement equation $\Rightarrow$ displacement at time t is $x = U\,\cos\theta\,t$

$y-$**direction:**

- initial velocity: $v_0 = U \sin\theta$
- acceleration: $a = -g$
- velocity equation $\Rightarrow$ velocity at time t is $v_y = U \sin\theta - g\,t$
- displacement equation $\Rightarrow$ displacement at time t is $y = U \sin\theta\, t - \frac{1}{2}\, g\, t^2$

Example 5.3 Find the time at which this projectile reaches its maximum height (in terms of U, θ and g). What is its maximum height?

Example 5.4 At what time does the projectile hit the ground, and how far has it travelled along the ground when it does so?

We have seen that the horizontal distance travelled when the particle again hits the ground is

$$x = \frac{U^2 \sin(2\theta)}{g}\ .$$

This means we can now solve our original **cannon problem**.

Example 5.5 What is the furthest distance that the cannon shell can travel, and at what angle does it need to be fired out to achieve this distance?

Example 5.6 A cannon shoots a shell at speed U at an angle of 45^o above the horizontal from level ground below a vertical cliff of height 40 m. The aim is to position the cannon so that the shell will land on the cliff.

(i) Find the smallest value of U for which the cannon shell can land on the cliff (take the acceleration due to gravity to be $g = 10\text{ms}^{-2}$).

(ii) How far away from the foot of the cliff does the cannon need to be for the shell to land on the cliff when U takes this value?

Exercises 5.1

1. A body moves with uniform acceleration for 3 seconds and covers 54 m. It then moves with constant velocity and covers 120 m in the next 5 seconds. Find its initial velocity and acceleration.

2. A train is uniformly accelerated and passes successive kilometre markers at the side of the track with speeds of 10 Km/h and 20 Km/h respectively. Calculate the speed of the train when it passes the next kilometre mark, and the times taken for each of these two intervals of 1Km.

3. A particle is projected from a point on level ground with velocity 60 m/s at an angle of 30^o above the horizontal. Take $g = 10$ m/s^2 and find:
   (a) the time taken for the particle to reach its maximum height;
   (b) the maximum height;
   (c) the time of flight;
   (d) the horizontal range of the particle.

4. A vertical tower on a horizontal plane is 12 m high. A ball is thrown from the top of the tower with velocity of 16 m/s horizontally and 17 m/s vertically upwards. How far will it be from the top of the tower after 1 second? (Take $g = 10$ m/s^2.)

5. How long will it take the ball in the previous example to reach the ground, and how far from the foot of the tower will it be when it does so?

6. A shot is fired from the top of a cliff of height 75 m with a velocity of 20 m/s at an angle of 30^o above the horizontal. Calculate the horizontal distance that the shot has travelled when it hits the ground (take $g = 10$ m/s^2).

7. A particle is projected vertically upwards under gravity with speed U. At time t_0 seconds later another particle is fired upwards from the same point. Find the initial speed of this particle in order that the two particles will collide when the first has reached its highest point.

8. A cannon is pointed up a hill which is at an angle α to the horizontal. The cannon fires a shell at speed U at an angle of θ to the hill.

   (a) Find the time at which the shell hits the ground. (Hint: it hits the ground when $y/x = \tan\alpha$.)

(b) Show that it hits the ground a distance of L away, where

$$L = \frac{U^2}{g \cos^2 \alpha} \left(\sin(2\theta + \alpha) - \sin \alpha\right) .$$

(c) What is the maximum possible value of L, and at what angle θ should the cannon be aimed to achieve this distance?

9. The cannon in the previous question is now pointed down the hill. What is the maximum distance the shell can travel now before hitting the ground, and at what angle should it be aimed to achieve this distance?

10. A body is projected from a horizontal plane at such an angle that the horizontal range is three times the greatest height. Find the angle of projection, and if with this angle the range is 60 m, then find the necessary speed of projection and time of flight (take $g = 10$ m/s^2).

11. A paintball gun can fire a capsule with speed U in any direction. Show that if the capsule is fired out of the gun at an angle of θ to the horizontal then its (x, y) coordinates relative to the gun satisfy

$$g\, x^2 \tan^2 \theta - 2\, U^2\, x \tan \theta + 2\, U^2\, y + g\, x^2 = 0 .$$

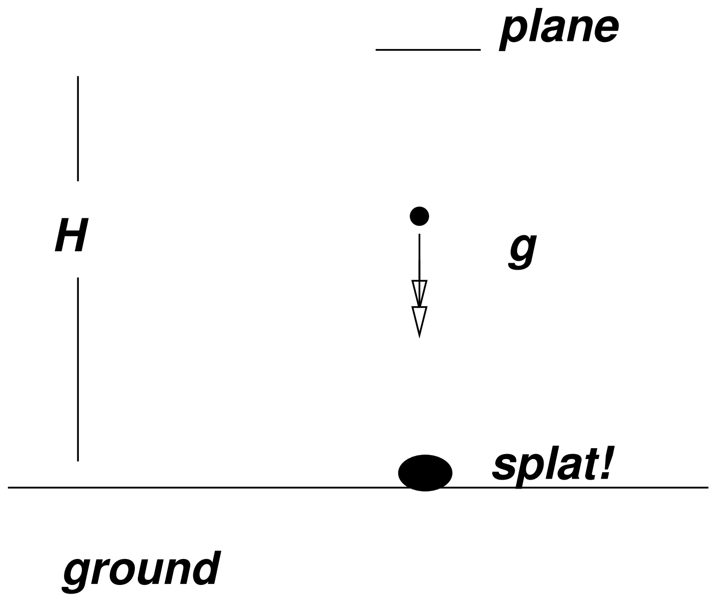
Hence show that the capsule can reach any target within the surface

$$g^2\, x^2 + 2\, g\, U^2\, y = U^4 .$$

5.5. What happens when there is air resistance?

Although air resistance can be neglected when we want to work out the trajectory of a small projectile (or at least one which is not travelling too fast), it is very important to people who jump out of aeroplanes with parachutes. For example, we shall first work out the approximate speed someone would hit the ground at if they stepped out of a plane at height H and fell without a parachute (in this case we can treat the person as a projectile and ignore air resistance).

We measure the displacement s vertically downwards and assume that the vertical speed on leaving the aeroplane is zero, so $v_0 = 0$ and $a = g$. The vertical



speed downwards at time t is then $v = g\,t$, and we can use the displacement equation $s = v_0\,t + \frac{1}{2}\,a\,t^2$ to find the time at which the person hits the ground. This gives $H = \frac{1}{2}\,g\,t^2$, so $t = \sqrt{2\,H/g}$, and the speed at the ground is

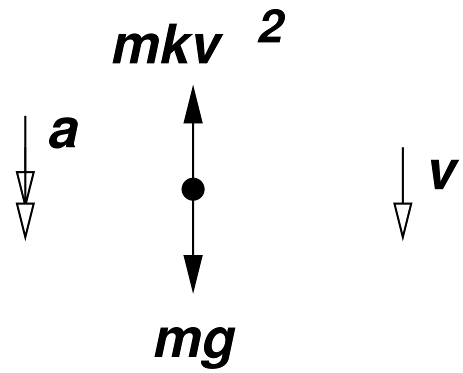
$$V = \sqrt{2\,g\,H}\,.$$

The following table shows how V varies with H, taking $g = 10$ m/s^2.

H (m)	V (m/s)	V (Km/h)
1.8	6.0	21.6
5.0	10.0	36.0
20.0	20.0	72.0
2000.0	200.0	720.0

Even falling from a low height (such as from a ladder, rather than out of a plane) would give an unpleasantly fast landing, and there is no limit to the impact speed, provided H is large enough – the speed grows like $\sqrt{H}$.

Fortunately things are very different with a parachute! This has a large surface area, which impedes or "resists" the motion downwards through the air. The resistance imposed by the air on a falling object is a force which is typically taken to be proportional to the square of the object's speed. If the object is a person without a parachute then the constant of proportionality will be low (and can be neglected), but if the object is a person with a parachute then the constant will be much larger, and air resistance will largely govern the behaviour. For simplicity we shall write the resisting force as $m\,k\,v^2$, where m is the object's mass.



The total force acting vertically downwards on the object is $m\,g - m\,k\,v^2$, and by Newton's second law this is equal to $m\,a$, where a is the downwards acceleration. That is

$$a = g - k\,v^2 \,. \tag{5.1}$$

In order to find out how the speed for this object depends on the distance it has fallen we shall write this equation as an ODE. Letting s denote the vertical displacement and v the vertical speed (both downwards), we know that

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt} \,.$$

But by the **chain rule** for derivatives

$$a = \frac{dv}{dt} = \frac{dv}{ds}\,\frac{ds}{dt} = \frac{dv}{ds}\,v$$

and so we can write

$$a = v\,\frac{dv}{ds}$$

in (5.1) to get the following **separable** ODE

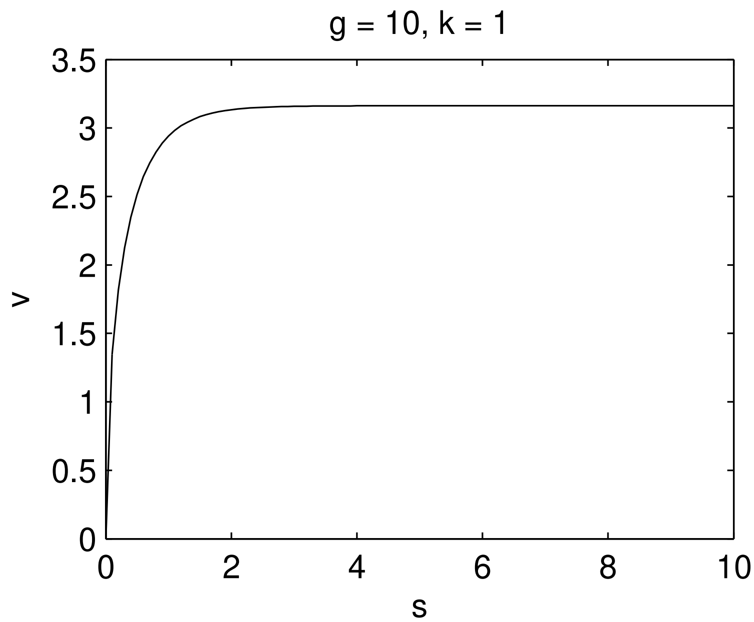
$$v\,\frac{dv}{ds} = g - k\,v^2 \,. \tag{5.2}$$

Example 5.7 Show that the general solution of the ODE (5.2) is

$$g - k\,v^2 = A\,e^{-2\,k\,s}$$

where A is an arbitrary constant of integration. If the speed of descent is zero at $s = 0$, then find the maximum possible speed v_T in terms of g and k.

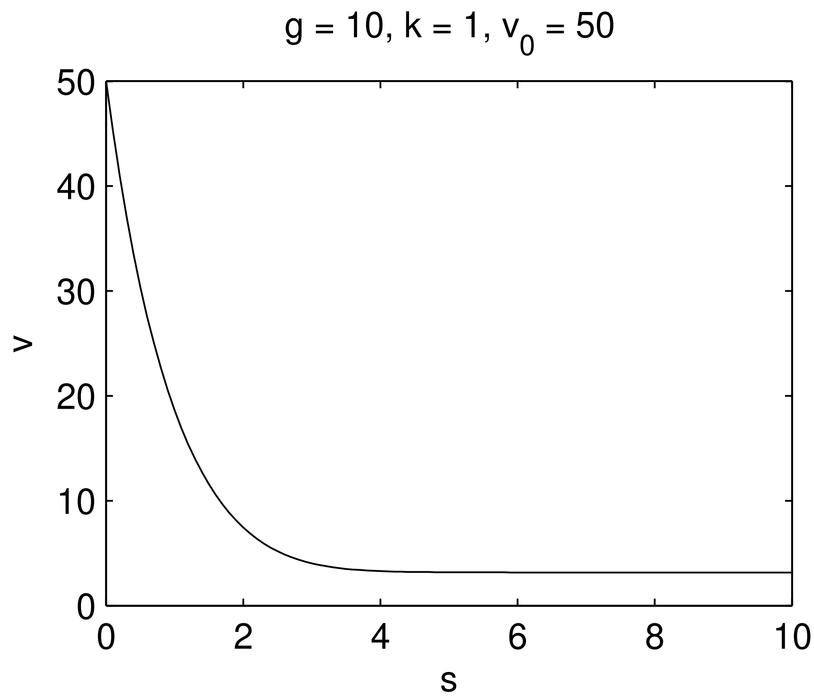
The limiting speed of $\sqrt{g/k}$ is called the **terminal speed**. The aim with a parachute is to design it so that k is large enough for the terminal speed to give a safe landing. The figure below shows a plot of speed against fall distance s when $k = 1$ and g is taken to be 10 m/s^2.



In real life, parachutists free fall for a distance before inflating their parachutes, and the parachute is typically opened at a speed $v_0 \gg v_T$.

Example 5.8 Suppose that the parachute is opened at $s = 0$ at speed v_0. Calculate the speed v at a distance of s m below the parachute opening, and show that the terminal speed is again $v_T = \sqrt{g/k}$.

The figure below shows a plot of v against s with $k = 1$, $v_0 = 50$ m/s and $g = 10$ m/s^2. Notice how quickly the speed drops to v_T.



Exercises 5.2

1. The falling speed v of a parachutist is governed by the ODE

$$v\,\frac{dv}{ds} = g - k\,v^2\,.$$

with $k = 1$, where s is the vertical distance fallen. Assuming that he opens the parachute at $s = 0$ at zero speed, then calculate his speed when he has fallen a distance of s. How far will he have fallen by the time he reaches half his terminal speed?

2. A parachutist drops from an aeroplane and falls under gravity with negligible air resistance for 5 seconds. She then opens her parachute and her

falling speed v is governed by the ODE

$$v\,\frac{dv}{ds} = g - k\,v^2\,.$$

with $k = 1$. Take $g = 10$ m/s^2 and calculate her speed v when she is s m below the point at which she opened her parachute. How far does she need to fall for her speed to be within 1 m/s of her terminal speed?

3. Suppose now that she is using a different parachute for which $k = 0.4$, which she opens when she is travelling at speed v_0. Take $g = 10$ m/s^2 and calculate her speed v a distance s below the place she opened her parachute in terms of v_0. Sketch graphs of v against s in the three cases

   (i) $v_0 > 5$ (ii) $v_0 < 5$ (iii) $v_0 = 5\,.$

   If $v_0 = 5$, then calculate how long it takes her to reach the ground if she opens her parachute 200 m above the ground.

4. Suppose you have two small metallic spheres of the same size, with sphere A being twice as heavy as sphere B, and you drop both of them and a feather at the same time from the top of a tall tower on a still day. What order will the three objects hit the ground in?

5. Your alien friend takes the two spheres and the feather to his tower home on boring planet Zog (it has no atmosphere), and performs the same experiment. What order will the three objects hit the Zog ground in?