3 Lecture examples: Chapter 3

Examples 3A

1 Show that $W = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ is a subspace of $\mathbb{R}^3$.

Let $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} = (0, 0, x_3), \boldsymbol{y} = (0, 0, y_3) \in W$. Then

$$\boldsymbol{z} = \lambda\boldsymbol{x} + \mu\boldsymbol{y} = (0, 0, \lambda x_3 + \mu y_3)$$

so $\mathbf{z} = (0, 0, z_3) \in W$, and as this holds $\forall \lambda, \mu \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in W$, W is a subspace of $\mathbb{R}^3$.

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2 Show that $U = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

Let $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} = (x_1, x_2, -(x_1 + x_2)), \boldsymbol{y} = (y_1, y_2, -(y_1 + y_2)) \in U$. Then

$$\begin{aligned}\mathbf{z} = \lambda\boldsymbol{x} + \mu\boldsymbol{y} &= (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, -\lambda(x_1 + x_2) - \lambda(y_1 + y_2)) \\ &= (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, -(\lambda x_1 + \mu y_1) - (\lambda x_2 + \mu y_2))\end{aligned}$$

so $z_1 + z_2 + z_3 = 0$ and $\mathbf{z} \in U$. As this holds $\forall \lambda, \mu \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in U$, U is a subspace of $\mathbb{R}^3$.

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3 State why $V = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 1\}$ is not a subspace of $\mathbb{R}^4$.

$\mathbf{0} \notin V$ so V is not a subspace of $\mathbb{R}^4$.

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4 State why $W = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}$ is not a subspace of $\mathbb{R}^4$.

W is not closed under addition: for example, if $\boldsymbol{x} = (1, 0, 0, 0)$ and $\boldsymbol{y} = (0, 0, 1, 0) \in W$ then $\boldsymbol{x} + \boldsymbol{y} = (1, 0, 1, 0) \notin W$.

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5 Let A be a real $m \times n$ matrix. Show that $U = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Let $\lambda, \mu \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in U$ with $\mathbf{z} = \lambda \boldsymbol{x} + \mu \boldsymbol{y}$. Then

$$A\mathbf{z} = A(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = A(\lambda \boldsymbol{x}) + A(\mu \boldsymbol{y}) = \lambda \mathbf{0} + \mu \mathbf{0} = \mathbf{0}.$$

This holds $\forall \lambda, \mu \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \mathbf{0}$, $A\boldsymbol{y} = \mathbf{0}$. Hence U is a subspace of $\mathbb{R}^n$. (This subspace is called the **nullspace** of A).

Examples 3B

1 State the span of the vectors $(1, 0, 1)$ and $(2, 1, 0)$.

$sp((1, 0, 1), (2, 1, 0))$
$= \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} = \alpha_1(1, 0, 1) + \alpha_2(2, 1, 0)\}$
$= \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} = (\alpha_1 + 2\alpha_2, \alpha_2, \alpha_1), \alpha_1, \alpha_2 \in \mathbb{R}\}$

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2 Verify that $\mathbb{R}^n = sp(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$ where $\boldsymbol{e}_i$ is the ith column of I_n, $i = 1, 2, \ldots, n$.

$$\begin{aligned} \boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n &\Leftrightarrow \boldsymbol{x} = x_1\,\boldsymbol{e}_1 + x_2\,\boldsymbol{e}_2 + \cdots + x_n\,\boldsymbol{e}_n \quad \text{so } \mathbb{R}^n = sp(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n). \\ &\Leftrightarrow \boldsymbol{x} \in sp(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n) \end{aligned}$$

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3 Verify that $U = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\} = sp((1, 1, 0), (0, 1, 1))$.

$$\begin{aligned} \boldsymbol{x} \in U \quad &\Leftrightarrow \quad \boldsymbol{x} = (x_1, x_1 + x_3, x_3) \\ &\Leftrightarrow \quad \boldsymbol{x} = x_1(1, 1, 0) + x_3(0, 1, 1) \\ &\Leftrightarrow \quad \boldsymbol{x} \in sp((1, 1, 0), (0, 1, 1)). \end{aligned}$$

Hence $U = sp((1, 1, 0), (0, 1, 1))$.

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4 Determine whether or not the vectors $\boldsymbol{x}_1 = (1, -1)$ and $\boldsymbol{x}_2 = (1, 1)$ span $\mathbb{R}^2$.

We must determine if an arbitrary vector $\boldsymbol{a} = (a_1, a_2)$ in $\mathbb{R}^2$ can be expressed as a linear combination $\boldsymbol{a} = \lambda_1\,\boldsymbol{x}_1 + \lambda_2\,\boldsymbol{x}_2$ of the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. In component form, we have $(a_1, a_2) = \lambda_1(1, -1) + \lambda_2(1, 1)$, which holds if and only if $\lambda_1 + \lambda_2 = a_1$, $-\lambda_1 + \lambda_2 = a_2$. This system is consistent for all values of a_1 and a_2, with solution $\lambda_1 = \frac{1}{2}(a_1 - a_2)$ and $\lambda_2 = \frac{1}{2}(a_1 + a_2)$, so any $\boldsymbol{a} \in \mathbb{R}^2$ can be expressed as a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as

$$\boldsymbol{a} = \frac{1}{2}(a_1 - a_2)\boldsymbol{x}_1 + \frac{1}{2}(a_1 + a_2)\boldsymbol{x}_2.$$

Hence $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ span $\mathbb{R}^2$.

(Check: e.g. $\mathbf{a} = (2, 3)$:

$$\frac{1}{2}(a_1 - a_2) = -\frac{1}{2}, \quad \frac{1}{2}(a_1 + a_2) = \frac{5}{2} \Rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which is correct.)

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5 Suppose $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are vectors in $\mathbb{R}^3$ chosen such that the corresponding position vectors are not collinear. What is the geometrical interpretation of $sp(\boldsymbol{x}_1, \boldsymbol{x}_2)$?

The space $sp(\boldsymbol{x}_1, \boldsymbol{x}_2)$ consists of all vectors of the form $\lambda\boldsymbol{x}_1 + \mu\boldsymbol{x}_2$, with $\lambda, \mu \in \mathbb{R}$. This is the plane in $\mathbb{R}^3$ determined by the position vectors corresponding to $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Any vector of the form $\lambda\boldsymbol{x}_1 + \mu\boldsymbol{x}_2$ is in this plane.
(Note: if $\boldsymbol{x}_1$ is a nonzero vector in $\mathbb{R}^2$ then $sp(\boldsymbol{x}_1)$ is the straight line through the origin determined by $\boldsymbol{x}_1$.)

Examples 3C

1 Find a smaller spanning set for $U = sp(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)$ where $\boldsymbol{x}_1 = (1, 0, 1)$, $\boldsymbol{x}_2 = (0, 0, 0)$, $\boldsymbol{x}_3 = (0, 1, 1)$, $\boldsymbol{x}_4 = (1, 1, 2) \in \mathbb{R}^3$.

Elimination check:

$$\begin{aligned}
&\boldsymbol{x}_1 \neq \mathbf{0} \;\Rightarrow\; \text{retain } \boldsymbol{x}_1 \\
&\boldsymbol{x}_2 = \mathbf{0} \;\Rightarrow\; \text{eliminate } \boldsymbol{x}_2 \\
&\boldsymbol{x}_3 = \lambda\boldsymbol{x}_1 \;\Leftrightarrow\; (0, 1, 1) = (\lambda, 0, \lambda), \quad \text{which is not possible} \;\Rightarrow\; \text{retain } \boldsymbol{x}_3 \\
&\boldsymbol{x}_4 = \lambda\boldsymbol{x}_1 + \mu\boldsymbol{x}_3 \;\Leftrightarrow\; (1, 1, 2) = (\lambda, 0, \lambda) + (0, \mu, \mu) \\
&\qquad\qquad\qquad\quad\;\Leftrightarrow\; \lambda = \mu = 1 \;\Rightarrow\; \text{eliminate } \boldsymbol{x}_4.
\end{aligned}$$

Hence a smaller spanning set is $\{\boldsymbol{x}_1, \boldsymbol{x}_3\}$.

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2 Show that $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are linearly independent vectors in $\mathbb{R}^n$.

$$\alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 + \ldots + \alpha_n\boldsymbol{e}_n = \mathbf{0} \Leftrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0)$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

Hence $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are linearly independent vectors in $\mathbb{R}^n$.

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3 Show that $\{(1,1,0),(0,1,1)\}$ is a linearly independent set in $\mathbb{R}^3$.

$$\alpha_1(1,1,0)+\alpha_2(0,1,1)=\mathbf{0} \Leftrightarrow (\alpha_1,\alpha_1+\alpha_2,\alpha_2)=(0,0,0) \Leftrightarrow \alpha_1=\alpha_2=0.$$

Hence $\{(1,1,0),(0,1,1)\}$ is a linearly independent set in $\mathbb{R}^3$.

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4 Show that $\{(1,1,0),(0,1,1),(1,2,1)\}$ is a linearly dependent set in $\mathbb{R}^3$.

$$\alpha_1(1,1,0)+\alpha_2(0,1,1)+\alpha_3(1,2,1)=\mathbf{0} \Leftrightarrow (\alpha_1+\alpha_3,\alpha_1+\alpha_2+2\alpha_3,\alpha_2+\alpha_3)=(0,0,0)$$

$$\Leftrightarrow \begin{array}{ccccccc} \alpha_1 & & & + & \alpha_3 & = & 0 \\ \alpha_1 & + & \alpha_2 & + & 2\alpha_3 & = & 0 \\ & & \alpha_2 & + & \alpha_3 & = & 0 \end{array} \Leftrightarrow \begin{array}{ccccc} \alpha_1 & + & \alpha_3 & = & 0 \\ \alpha_2 & + & \alpha_3 & = & 0 \\ \alpha_2 & + & \alpha_3 & = & 0 \end{array} \Leftrightarrow \begin{array}{ccc} \alpha_1 & = & -\lambda \\ \alpha_2 & = & -\lambda \\ \alpha_3 & = & \lambda \end{array}$$

for any $\lambda \in \mathbb{R}$. Hence $\{(1,1,0),(0,1,1),(1,2,1)\}$ is a linearly dependent set. (Note that $(1,2,1)=(1,1,0)+(0,1,1)$.)

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5 Determine whether the vectors $\boldsymbol{v}_1=(1,-1,2)$, $\boldsymbol{v}_2=(3,2,-1)$ and $\boldsymbol{v}_3=(5,0,3)$ form a linearly independent or dependent set in $\mathbb{R}^3$.

$$\alpha_1\boldsymbol{v}_1+\alpha_2\boldsymbol{v}_2+\alpha_3\boldsymbol{v}_3=\mathbf{0} \Leftrightarrow (\alpha_1+3\alpha_2+5\alpha_3,-\alpha_1+2\alpha_2,2\alpha_1-\alpha_2+3\alpha_3)=\mathbf{0}$$

$$\Leftrightarrow \begin{array}{ccccccc} \alpha_1 & + & 3\alpha_2 & + & 5\alpha_3 & = & 0 \\ -\alpha_1 & + & 2\alpha_2 & & & = & 0 \\ 2\alpha_1 & - & \alpha_2 & + & 3\alpha_3 & = & 0 \end{array} \Leftrightarrow A\boldsymbol{a}=\boldsymbol{b}.$$

Solve via EROs:

$$\begin{pmatrix} 1 & 3 & 5 \\ -1 & 2 & 0 \\ 2 & -1 & 3 \end{pmatrix} \begin{array}{c} r_2'=r_2+r_1 \\ r_3'=r_3-2r_1 \end{array} \rightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & 5 & 5 \\ 0 & -7 & -7 \end{pmatrix} \begin{array}{c} r_2'=\frac{1}{5}r_2 \\ r_3'=r_3+\frac{7}{5}r_2 \end{array} \rightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Set $\alpha_3=\lambda$, so $\alpha_2=-\lambda$ and $\alpha_1=-2\lambda$. As this is a non-trivial solution, the vectors are linearly dependent.

(Example: set $\lambda=1$ to obtain $\alpha_3=1$, $\alpha_2=-1$ and $\alpha_1=-2$ so

$$-2\boldsymbol{v}_1-\boldsymbol{v}_2+\boldsymbol{v}_3=\mathbf{0} \Rightarrow \boldsymbol{v}_1=-\frac{1}{2}\boldsymbol{v}_2+\frac{1}{2}\boldsymbol{v}_3 \quad \text{or} \quad \boldsymbol{v}_2=-2\boldsymbol{v}_1+\boldsymbol{v}_3 \quad \text{or} \quad \boldsymbol{v}_3=2\boldsymbol{v}_1+\boldsymbol{v}_2.)$$

Examples 3D

1 Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a basis for $\mathbb{R}^n$.

This is a linearly independent set that spans $\mathbb{R}^n$ (see Examples 3C.2 and 3D.2), so forms a a basis. (Note: this is the standard basis for $\mathbb{R}^n$).

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2 Show that $\{(1,1,0),(0,1,1)\}$ is a basis for $U = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 - x_2 + x_3 = 0\}$.

We have seen that this is a linearly independent set that spans U (see Examples 3C.3, 3D.3).

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3 Show that $B = \{(-3,7),(5,5)\}$ is a basis for $\mathbb{R}^2$.

Note that the two vectors are not parallel so they are linearly independent. To determine whether B spans $\mathbb{R}^2$ we must find out whether any arbitrary vector $\boldsymbol{a} = (a_1, a_2)$ in $\mathbb{R}^2$ can be expressed as a linear combination $\boldsymbol{a} = \alpha_1(-3,7) + \alpha_2(5,5)$. Using components, we see that this holds if and only if $-3\alpha_1 + 5\alpha_2 = a_1, \ \ 7\alpha_1 + 5\alpha_2 = a_2$. This system is consistent for all values of a_1 and a_2 with solution $\alpha_1 = (a_2 - a_1)/10$,
$\alpha_2 = (7a_1 + 3a_2)/50$. Hence B spans $\mathbb{R}^2$, so it is a basis for $\mathbb{R}^2$.

Examples 3E

1 Show that $S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}$ is a basis for $U = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$ and state the dimension of U.

Note first that $S \subset U$. Now

$$
\begin{aligned}
\boldsymbol{x} \in U &\Leftrightarrow \boldsymbol{x} = (x_1, x_2, x_3, -x_1 - x_2 - x_3) \\
&\Leftrightarrow \boldsymbol{x} = x_1(1,0,0,-1) + x_2(0,1,0,-1) + x_3(0,0,1,-1) \\
&\Leftrightarrow \boldsymbol{x} \in sp(S) \\
&\Leftrightarrow U \text{ is spanned by the set } S.
\end{aligned}
$$

We now have to show that S is a linearly independent set in U:

$$\alpha_1(1,0,0,-1) + \alpha_2(0,1,0,-1) + \alpha_3(0,0,1,-1) \ = \ \mathbf{0}$$

$$\begin{aligned} \Leftrightarrow (\alpha_1, \alpha_2, \alpha_3, -\alpha_1 - \alpha_2 - \alpha_3) &= (0,0,0,0) \\ \Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 &= 0 \\ \Leftrightarrow S \text{ is a linearly independent set in } U & \end{aligned}$$

Hence S is a basis for U. Also, $\dim U = 3$ (from Defn 3.9).

Examples 3F

1 Show that $\boldsymbol{v}_1 = (1,1,0)$, $\boldsymbol{v}_2 = (0,1,0)$ and $\boldsymbol{v}_3 = (0,1,1)$ form a basis for $\mathbb{R}^3$.

Since there are 3 vectors in this set and $\dim \mathbb{R}^3 = 3$, we need only show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a linearly independent set.

$$\begin{aligned} \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 &= \mathbf{0} \\ \Leftrightarrow (\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3) &= (0,0,0) \\ \Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 &= 0 \\ \Leftrightarrow \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \text{ is a linearly independent set of vectors in } \mathbb{R}^3. & \end{aligned}$$

Hence this set froms a basis for $\mathbb{R}^3$.

Examples 3G

1 Let $\boldsymbol{b}_1 = (0,1,0)$, $\boldsymbol{b}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$, and $\boldsymbol{b}_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$. Show that the set $S = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is an orthonormal basis for $\mathbb{R}^3$. Express the vector $\boldsymbol{u} = (1,1,1)$ as a linear combination of the vectors in S, and find the coordinates of $\boldsymbol{u}$ relative to S.

We have $\boldsymbol{b}_i \cdot \boldsymbol{b}_j = \delta_{ij}$, $i, j = 1, 2, 3$ so the vectors form an orthonormal set. As the dimension of $\mathbb{R}^3$ is 3, then the fact that these vectors are linearly independent means that they form a basis for $\mathbb{R}^3$. So, for any $\boldsymbol{x} \in \mathbb{R}^3$ we have

$$\boldsymbol{x} = \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + \alpha_3 \boldsymbol{b}_3 = (\boldsymbol{x} \cdot \boldsymbol{b}_1)\boldsymbol{b}_1 + (\boldsymbol{x} \cdot \boldsymbol{b}_2)\boldsymbol{b}_2 + (\boldsymbol{x} \cdot \boldsymbol{b}_3)\boldsymbol{b}_3$$

(as $\|\boldsymbol{b}_i\|^2 = 1$ for $i = 1, 2, 3$). Here $\boldsymbol{u} \cdot \boldsymbol{b}_1 = 1$, $\boldsymbol{u} \cdot \boldsymbol{b}_2 = -\frac{1}{5}$ and $\boldsymbol{u} \cdot \boldsymbol{b}_3 = \frac{7}{5}$, so

$$\boldsymbol{u} = \boldsymbol{b}_1 - \frac{1}{5}\boldsymbol{b}_2 + \frac{7}{5}\boldsymbol{b}_3.$$

$$\text{Check:} \quad (1,1,1) = (0,1,0) - \frac{1}{5}\left(-\frac{4}{5}, 0, \frac{3}{5}\right) + \frac{7}{5}\left(\frac{3}{5}, 0, \frac{4}{5}\right).$$

That is, the coordinates of $\boldsymbol{u}$ relative to S are $\left(1, -\frac{1}{5}, \frac{7}{5}\right)$.

Examples 3H

1 The vectors $\boldsymbol{b}_1 = (1,1,1)$, $\boldsymbol{b}_2 = (0,1,1)$, $\boldsymbol{b}_3 = (0,0,1)$ form a basis for $\mathbb{R}^3$. Use Gram-Schmidt orthogonalisation to transform this into an orthogonal basis $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ for $\mathbb{R}^3$.

Step 1: $\boldsymbol{y}_1 = \mathbf{b}_1 = (1,1,1)$.

Step 2: $\boldsymbol{y}_2 = \mathbf{b}_2 + \alpha_1\boldsymbol{y}_1$. Find α_1:

$$\boldsymbol{y}_2 \cdot \boldsymbol{y}_1 = 0 \Rightarrow (\mathbf{b}_2 + \alpha_1\boldsymbol{y}_1)\cdot\boldsymbol{y}_1 = 0 \Rightarrow \mathbf{b}_2\cdot\boldsymbol{y}_1 + \alpha_1\|\boldsymbol{y}_1\|^2 = 0 \Rightarrow \alpha_1 = -\frac{\mathbf{b}_2\cdot\boldsymbol{y}_1}{\|\boldsymbol{y}_1\|^2}$$

so

$$\mathbf{b}_2\cdot\boldsymbol{y}_1 = 2, \quad \|\boldsymbol{y}_1\|^2 = 3 \Rightarrow \alpha_1 = -\frac{2}{3}$$

and

$$\boldsymbol{y}_2 = (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Step 3: $\boldsymbol{y}_3 = \mathbf{b}_3 + \alpha_1\boldsymbol{y}_1 + \alpha_2\boldsymbol{y}_2$. Find α_1 and α_2:

$$\boldsymbol{y}_3\cdot\boldsymbol{y}_1 = 0 \Rightarrow (\mathbf{b}_3+\alpha_1\boldsymbol{y}_1+\alpha_2\boldsymbol{y}_2)\cdot\boldsymbol{y}_1 = 0 \Rightarrow \mathbf{b}_3\cdot\boldsymbol{y}_1+\alpha_1\|\boldsymbol{y}_1\|^2 = 0 \Rightarrow \alpha_1 = -\frac{\mathbf{b}_3\cdot\boldsymbol{y}_1}{\|\boldsymbol{y}_1\|^2} = -\frac{1}{3}.$$

$$\boldsymbol{y}_3\cdot\boldsymbol{y}_2 = 0 \Rightarrow (\mathbf{b}_3+\alpha_1\boldsymbol{y}_1+\alpha_2\boldsymbol{y}_2)\cdot\boldsymbol{y}_2 = 0 \Rightarrow \mathbf{b}_3\cdot\boldsymbol{y}_2+\alpha_2\|\boldsymbol{y}_2\|^2 = 0 \Rightarrow \alpha_2 = -\frac{\mathbf{b}_3\cdot\boldsymbol{y}_2}{\|\boldsymbol{y}_2\|^2} = -\frac{1}{2}.$$

So

$$\boldsymbol{y}_3 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(0, -\frac{1}{2}, \frac{1}{2}\right).$$

So $\{(1,1,1), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, -\frac{1}{2}, \frac{1}{2}\right)\}$ is an orthogonal basis for $\mathbb{R}^3$.

NOTE: $\boldsymbol{y}_i \cdot \boldsymbol{y}_j = 0$ is not changed by multiplying either vector by a constant. We can sometimes use this to make the arithmetic easier.

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2 The set $S = \{(1,0,0,1), (0,1,0,1), (0,0,1,1)\}$ is a basis for $W = \{\boldsymbol{x} = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_4 = x_1+x_2+x_3\}$. Apply Gram-Schmidt orthogonalisation to produce an orthonormal basis for W. Find coordinates of $\boldsymbol{x} = (1,-1,0,0) \in W$ with respect to this orthonormal basis.

Step 1: $\boldsymbol{y}_1 = (1,0,0,1)$.

Step 2: $\boldsymbol{y}_2 = (0,1,0,1) + \alpha_1(1,0,0,1) = (\alpha_1, 1, 0, 1+\alpha_1)$. Now

$$\boldsymbol{y}_2 \cdot \boldsymbol{y}_1 = 1 + 2\alpha_1 = 0 \Leftrightarrow \alpha_1 = -\frac{1}{2}$$

so

$$\boldsymbol{y}_2 = (0,1,0,1) - \frac{1}{2}(1,0,0,1) = \left(-\frac{1}{2}, 1, 0, \frac{1}{2}\right).$$

For easier arithmetic, we will multiply by 2 and use $\boldsymbol{y}_2 = (-1,2,0,1)$ (this is still orthogonal to $\boldsymbol{y}_1$).

Step3:

$$\begin{aligned}
\boldsymbol{y}_3 &= (0,0,1,1) + \alpha_1(1,0,0,1) + \alpha_2(-1,2,0,1) = (\alpha_1 - \alpha_2, 2\alpha_2, 1, 1+\alpha_1+\alpha_2) \\
\boldsymbol{y}_3 \cdot \boldsymbol{y}_1 &= 1 + 2\alpha_1 = 0 \Leftrightarrow \alpha_1 = -\frac{1}{2} \\
\boldsymbol{y}_3 \cdot \boldsymbol{y}_2 &= 1 + 6\alpha_2 = 0 \Leftrightarrow \alpha_2 = -\frac{1}{6}
\end{aligned}$$

so

$$\boldsymbol{y}_3 = \left(-\frac{1}{3}, -\frac{1}{3}, 1, \frac{1}{3}\right).$$

Choose $\boldsymbol{y}_3 = (-1,-1,3,1)$, so orthogonal basis is

$$\{(1,0,0,1), (-1,2,0,1), (-1,-1,3,1)\}.$$

Orthonormal basis $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3\}$ is

$$\left\{\frac{1}{\sqrt{2}}(1,0,0,1), \frac{1}{\sqrt{6}}(-1,2,0,1), \frac{1}{\sqrt{12}}(-1,-1,3,1)\right\}.$$

Coordinates of $\boldsymbol{x} = (1,-1,0,0) \in W$ with respect to this orthonormal basis are

$$\boldsymbol{x} \cdot \boldsymbol{z}_1 = \frac{1}{\sqrt{2}}; \qquad \boldsymbol{x} \cdot \boldsymbol{z}_2 = -\frac{3}{\sqrt{6}}; \qquad \boldsymbol{x} \cdot \boldsymbol{z}_3 = 0,$$

that is,

$$\boldsymbol{x} = (1,-1,0,0) = \frac{1}{\sqrt{2}}\boldsymbol{z}_1 - \frac{3}{\sqrt{6}}\boldsymbol{z}_2 + 0\boldsymbol{z}_3.$$