2 Lecture examples: Chapter 2

Examples 2A

1 List three members of the following sets:
(i) $\{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : \; x_2 = 1\}$.
(ii) $\{\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : \; y_1 - y_2 = 0\}$.
(iii) $\{\boldsymbol{z} = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : \; z_1 + z_2 + z_3 + z_4 > 0\}$.

(i) Vectors are of the form $(x_1, 1)$ for $x_1 \in \mathbb{R}$,
e.g. $(0, 1)$,$(1, 1)$,$(-7, 1)$.
(ii) Vectors are of the form (y_2, y_2, y_3) for $y_2, y_3 \in \mathbb{R}$,
e.g. $(0, 0, 0)$, $(1, 1, 23)$, $(-1, -1, 17004)$.
(iii) Not easy to write a general representation here, but we have e.g. $(1, 1, 1, 1)$, $(1, 2, 3, 4)$, $(10, -1, -2, -3)$.

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2 Given that $\boldsymbol{x} + \boldsymbol{u} = \boldsymbol{v}$, show that $\boldsymbol{x} = \boldsymbol{v} - \boldsymbol{u}$.

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{u} = \boldsymbol{v} \quad &\Rightarrow \quad (\boldsymbol{x} + \boldsymbol{u}) + (-\boldsymbol{u}) = \boldsymbol{v} + (-\boldsymbol{u}) && \\
&\Rightarrow \quad \boldsymbol{x} + (\boldsymbol{u} + (-\boldsymbol{u})) = \boldsymbol{v} - \boldsymbol{u} && \text{(use A2)} \\
&\Rightarrow \quad \boldsymbol{x} + \boldsymbol{0} = \boldsymbol{v} - \boldsymbol{u} && \text{(use A4)} \\
&\Rightarrow \quad \boldsymbol{x} = \boldsymbol{v} - \boldsymbol{u} && \text{(use A3)}
\end{aligned}$$

Examples 2B

1 Simplify $(3\boldsymbol{x} + 2\boldsymbol{y}) \cdot (4\boldsymbol{x} + \boldsymbol{y})$.

$$\begin{aligned}
(3\boldsymbol{x} + 2\boldsymbol{y}) \cdot (4\boldsymbol{x} + \boldsymbol{y}) &= 3\boldsymbol{x} \cdot (4\boldsymbol{x} + \boldsymbol{y}) + 2\boldsymbol{y} \cdot (4\boldsymbol{x} + \boldsymbol{y}) \\
&= 12(\boldsymbol{x} \cdot \boldsymbol{x}) + 3(\boldsymbol{x} \cdot \boldsymbol{y}) + 8(\boldsymbol{y} \cdot \boldsymbol{x}) + 2(\boldsymbol{y} \cdot \boldsymbol{y}) \\
&= 12\|\boldsymbol{x}\|^2 + 11(\boldsymbol{x} \cdot \boldsymbol{y}) + 2\|\boldsymbol{y}\|^2
\end{aligned}$$

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2 Find the angle θ between $\boldsymbol{x} = (1, 0, 2, -1, 3)$ and $\boldsymbol{y} = (0, 1, -1, -2, 1)$.

$$\|\boldsymbol{x}\| = \sqrt{1+0+4+1+9} = \sqrt{15}; \quad \|\boldsymbol{y}\| = \sqrt{0+1+1+4+1} = \sqrt{7}$$

and

$$\boldsymbol{x} \,.\, \boldsymbol{y} = 0 + 0 - 2 + 2 + 3 = 3$$

so

$$\cos\theta = \frac{\boldsymbol{x} \,.\, \boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} = \frac{3}{\sqrt{15}\sqrt{7}} \simeq 0.293$$

and $\theta \simeq 1.27$ radians.

Examples 2C

1 Show that $\boldsymbol{x} = (4, 1, -2, 1)$ and $\boldsymbol{y} = (3, -4, 2, -4)$ are orthogonal vectors in $\mathbb{R}^4$, and use them to construct a pair of orthonormal vectors.

$$\boldsymbol{x} \cdot \boldsymbol{y} = 12 - 4 - 4 - 4 = 0 \Rightarrow \boldsymbol{x} \perp \boldsymbol{y}.$$

Also,

$$\|\boldsymbol{x}\| = \sqrt{(16+1+4+1)} = \sqrt{22}; \quad \|\boldsymbol{y}\| = \sqrt{(9+16+4+16)} = \sqrt{45}.$$

Let

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{1}{\sqrt{22}}(4, 1, -2, 1), \quad \hat{\boldsymbol{y}} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{1}{\sqrt{45}}(3, -4, 2, -4),$$

then $\|\hat{\boldsymbol{x}}\| = 1$ and $\|\hat{\boldsymbol{y}}\| = 1$ (i.e. $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are unit vectors) and $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = 0$ so $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are orthonormal.

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2 Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is an orthonormal set of vectors in $\mathbb{R}^n$, where

$$\boldsymbol{e}_1 = (1, 0, 0, \ldots, 0), \quad \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots \quad \boldsymbol{e}_n = (0, 0, 0, \ldots, 1),$$

(each with n elements).

$$\|\boldsymbol{e}_i\| = \sqrt{0 + 0 + \ldots + 1 + \ldots + 0 + 0} = 1 \Rightarrow \|\boldsymbol{e}_i\| = 1 \quad \text{for } i = 1, 2, \ldots, n.$$

Also

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = \delta_{ij},$$

so $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is an orthonormal set in $\mathbb{R}^n$.