UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MM201 Linear Algebra and Differential Equations

1 Exercise Solutions: Chapter 1

1.

(a)	Not defined.	(b)	Defined: 4×2	(c)	Not defined.	(d)	Not defined.
(e)	Defined: 5×5	(f)	Defined: 5×2	(g)	Not defined.	(h)	Defined: 5×2.

2. (a)

$$4E - 2D = \begin{bmatrix} 24 & 4 & 12 \\ -4 & 4 & 8 \\ 16 & 4 & 12 \end{bmatrix} - \begin{bmatrix} 2 & 10 & 4 \\ -2 & 0 & 2 \\ 6 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$$

(b) $2B - C$ is not defined.

(c)

$$AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

(d) BA is not defined.

(e)

$$(3E)D = 3(ED) = 3 \begin{bmatrix} 14 & 36 & 25 \\ 4 & -1 & 7 \\ 12 & 26 & 21 \end{bmatrix} = \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$$

(f)

$$(AB)C = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(g)

$$A(BC) = (AB)C \quad \text{(same as (f))}$$

(h)

$$DA = \begin{bmatrix} 0 & 12 \\ -2 & 1 \\ 11 & 8 \end{bmatrix} \Rightarrow (DA)^T = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$$

(i)

$$A^T D^T = (DA)^T \quad \text{(same as (h))}$$

(j)

$$4E^T - D = 4\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 23 & -9 & 14 \\ 5 & 4 & 3 \\ 9 & 6 & 8 \end{bmatrix}$$

$$\Rightarrow \mathrm{tr(4E^T - D)} = 23 + 4 + 8 = 35$$

(k)

$$CC^T = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix} \Rightarrow \mathrm{tr(CC^T)} = 21 + 35 = 56$$

3. Equations are

$$\begin{array}{rcrcrcrcl} a & - & b & & & & & = & 8 \\ & & b & + & c & & & = & 1 \\ & & & & c & + & 3d & = & 7 \\ 2a & & & & & - & 4d & = & 6 \end{array}$$

so augmented matrix is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & | & 8 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & 3 & | & 7 \\ 2 & 0 & 0 & -4 & | & 6 \end{bmatrix} \; r_4' = r_4 - 2r_1 \quad \to \begin{bmatrix} 1 & -1 & 0 & 0 & | & 8 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & 3 & | & 7 \\ 0 & 2 & 0 & -4 & | & -10 \end{bmatrix} \; r_4' = r_4 - 2r_2$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 0 & | & 8 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & 3 & | & 7 \\ 0 & 0 & -2 & -4 & | & -12 \end{bmatrix} \; r_4' = r_4 + 2r_3 \quad \to \begin{bmatrix} 1 & -1 & 0 & 0 & | & 8 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & 3 & | & 7 \\ 0 & 0 & 0 & 2 & | & 2 \end{bmatrix}$$

Back substitute:

$$d = 1, \quad c + 3d = 7 \Rightarrow c = 4, \quad b + c = 1 \Rightarrow b = -3, \quad a - b = 8 \Rightarrow a = 5.$$

4.

$$A^{-1} = \frac{1}{6-5}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{8+12}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$AB = \begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix} \Rightarrow (AB)^{-1} = \frac{1}{-70+90} \begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix}$$

$$ABC = (AB)C = \begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 20 & -15 \\ 36 & -21 \end{bmatrix} \Rightarrow (ABC)^{-1} = \frac{1}{120} \begin{bmatrix} -21 & 15 \\ -36 & 20 \end{bmatrix}$$

$$C^{-1}B^{-1}A^{-1} = C^{-1}(B^{-1}A^{-1}) = \frac{1}{20} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix} = \frac{1}{120} \begin{bmatrix} -21 & 15 \\ -36 & 20 \end{bmatrix}$$

5.

$$(7D)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} \Rightarrow 7D = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}^{-1} = \frac{1}{6-7} \begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$$

so

$$D = \frac{1}{7} \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}.$$

6. $\mathrm{tr}(0_{n\times n}) = 0, \qquad \mathrm{tr}(I_n) = n$

$$\text{(a)} \qquad \mathrm{tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{tr}(A) + \mathrm{tr}(B).$$

(b) Write $C = AB$ and $D = BA$. Then $C = [c_{ij}]_{n\times n}$ with $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ and $D = [d_{ij}]_{n\times n}$ with $d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$.

$$\mathrm{tr}(AB) = \mathrm{tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n} d_{kk} = \mathrm{tr}(D) = \mathrm{tr}(BA).$$

$$\text{(c)} \qquad \mathrm{tr}(\lambda A) = \sum_{i=1}^{n}(\lambda a_{ii}) = \lambda \sum_{i=1}^{n} a_{ii} = \lambda \mathrm{tr}(A).$$

7. (a) Both. (b) Neither (leading 1 in r_2 to the right of leading 1 in r_3).
(c) Both. (d) Echelon (NOT reduced echelon: nonzero entry above leading 1 in c_2).
(e) Neither (leading 1 in r_2 in same column as leading 1 in r_1). (f) Both.

8. (a)

$$\begin{array}{ccc|cccl} 1&0&0&1&1&1& \\ 0&1&0&3&1&2& r_2'=r_2-3r_1 \\ 0&0&1&5&1&3& r_3'=r_3-5r_1 \end{array} \rightarrow \begin{array}{ccc|cccl} 1&0&0&1&1&1& \\ -3&1&0&0&-2&-1& \\ -5&0&1&0&-4&-2& r_3'=r_3-2r_2 \end{array}$$

$$\rightarrow \begin{array}{ccc|cccl} 1&0&0&1&1&1& \\ -3&1&0&0&-2&-1& r_2'=-\frac{1}{2}r_2 \\ 1&-2&1&0&0&0& \end{array} \rightarrow \begin{array}{ccc|cccl} 1&0&0&1&1&1& r_1'=r_1-r_2 \\ \frac{3}{2}&-\frac{1}{2}&0&0&1&\frac{1}{2}& \\ 1&-2&1&0&0&0& \end{array}$$

$$\rightarrow \begin{array}{ccc|ccc} -\frac{1}{2}&\frac{1}{2}&0&1&0&\frac{1}{2} \\ \frac{3}{2}&-\frac{1}{2}&0&0&1&\frac{1}{2} \\ 1&-2&1&0&0&0 \end{array}$$

Reduced echelon matrix with

$$P = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

(b)

$$\begin{array}{ccc|ccccl} 1&0&0&1&3&4&3& \\ 0&1&0&1&2&2&2& r_2'=r_2-r_1 \\ 0&0&1&-1&6&4&1& r_3'=r_3+r_1 \end{array} \rightarrow \begin{array}{ccc|ccccl} 1&0&0&1&3&4&3& \\ -1&1&0&0&-1&-2&-1& \\ 1&0&1&0&9&8&4& r_3'=r_3+9r_2 \end{array}$$

$$\rightarrow \begin{array}{ccc|ccccl} 1&0&0&1&3&4&3& \\ -1&1&0&0&-1&-2&-1& r_2'=-r_2 \\ -8&9&1&0&0&-10&-5& r_3'=-\frac{1}{10}r_3 \end{array} \rightarrow \begin{array}{ccc|ccccl} 1&0&0&1&3&4&3& r_1'=r_1-4r_3 \\ 1&-1&0&0&1&2&1& r_2'=r_2-2r_3 \\ \frac{4}{5}&-\frac{9}{10}&-\frac{1}{10}&0&0&1&\frac{1}{2}& \end{array}$$

$$\rightarrow \begin{array}{ccc|ccccl} -\frac{11}{5}&\frac{18}{5}&\frac{2}{5}&1&3&0&1& r_1'=r_1-3r_2 \\ -\frac{3}{5}&\frac{4}{5}&\frac{1}{5}&0&1&0&0& \\ \frac{4}{5}&-\frac{9}{10}&-\frac{1}{10}&0&0&1&\frac{1}{2}& \end{array} \rightarrow \begin{array}{ccc|cccc} -\frac{2}{5}&\frac{6}{5}&-\frac{1}{5}&1&0&0&1 \\ -\frac{3}{5}&\frac{4}{5}&\frac{1}{5}&0&1&0&0 \\ \frac{4}{5}&-\frac{9}{10}&-\frac{1}{10}&0&0&1&\frac{1}{2} \end{array}$$

Reduced echelon form with

$$P = \frac{1}{10}\begin{bmatrix} -4 & 12 & -2 \\ -6 & 8 & 2 \\ 8 & -9 & -1 \end{bmatrix}.$$

(c)

$$\begin{array}{cccc|cccl} 1&0&0&0&1&1&1& \\ 0&1&0&0&1&2&4& r_2'=r_2-r_1 \\ 0&0&1&0&1&3&9& r_3'=r_3-r_1 \\ 0&0&0&1&1&4&16& r_4'=r_4-r_1 \end{array} \rightarrow \begin{array}{cccc|cccl} 1&0&0&0&1&1&1& \\ -1&1&0&0&0&1&3& \\ -1&0&1&0&0&2&8& r_3'=r_3-2r_2 \\ -1&0&0&1&0&3&15& r_4'=r_4-3r_2 \end{array}$$

$$\rightarrow \begin{array}{rrrr|rrr} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 3 \\ 1 & -2 & 1 & 0 & 0 & 0 & 2 \\ 2 & -3 & 0 & 1 & 0 & 0 & 6 \end{array} \begin{array}{l} \\ \\ \\ r_4' = r_4 - 3r_3 \end{array} \quad \rightarrow \begin{array}{rrrr|rrr} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 3 \\ 1 & -2 & 1 & 0 & 0 & 0 & 2 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 \end{array} \begin{array}{l} \\ \\ r_3' = \frac{1}{2}r_3 \\ \\ \end{array}$$

$$\rightarrow \begin{array}{rrrr|rrr} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 3 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 \end{array} \begin{array}{l} r_1' = r_1 - r_3 \\ r_2' = r_2 - 3r_1 \\ \\ \\ \end{array} \rightarrow \begin{array}{rrrr|rrr} \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 1 & 1 & 0 \\ -\frac{5}{2} & 4 & -\frac{3}{2} & 0 & 0 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 \end{array} \begin{array}{l} r_1' = r_1 - r_2 \\ \\ \\ \\ \end{array}$$

$$\rightarrow \begin{array}{rrrr|rrr} 3 & -3 & 1 & 0 & 1 & 0 & 0 \\ -\frac{5}{2} & 4 & -\frac{3}{2} & 0 & 0 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 \end{array}$$

Reduced echelon matrix with

$$P = \frac{1}{2} \begin{bmatrix} 6 & -6 & 2 & 0 \\ -5 & 8 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 6 & -6 & 2 \end{bmatrix}.$$

(d)

$$\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 & 4 & 3 & 4 \\ 0 & 0 & 1 & 0 & 1 & 3 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 & 1 & 3 \end{array} \begin{array}{l} \\ r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \\ \\ \end{array} \rightarrow \begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 & 1 & 3 \end{array} \begin{array}{l} \\ r_2' = r_3 \\ r_3' = r_2 \\ \\ \end{array}$$

$$\rightarrow \begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 1 & 3 \end{array} \begin{array}{l} \\ \\ \\ r_4' = r_4 - 3r_2 \end{array} \rightarrow \begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & -3 & 1 & 0 & 0 & -2 & 0 \end{array} \begin{array}{l} \\ \\ \\ r_4' = r_4 + 2r_3 \end{array}$$

$$\rightarrow \begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & -3 & 1 & 0 & 0 & 0 & 0 \end{array} \begin{array}{l} r_1' = r_1 - r_3 \\ r_2' = r_2 - r_3 \\ \\ \\ \end{array} \rightarrow \begin{array}{rrrr|rrrr} 3 & -1 & 0 & 0 & 1 & 2 & 0 & 2 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & -3 & 1 & 0 & 0 & 0 & 0 \end{array} \begin{array}{l} r_1' = r_1 - 2r_2 \\ \\ \\ \\ \end{array}$$

$$\rightarrow \begin{array}{rrrr|rrrr} 1 & 1 & -2 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & -3 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Reduced echelon matrix with

$$P = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 2 & -3 & 1 \end{bmatrix}.$$

(e)

$$\begin{array}{rrrcrrrl} 1 & 0 & 0 & | & 2 & 2 & 3 & r_1' = r_2 \\ 0 & 1 & 0 & | & 1 & -1 & 0 & r_2' = r_1 \\ 0 & 0 & 1 & | & -1 & 2 & 1 & \end{array} \rightarrow \begin{array}{rrrcrrrl} 0 & 1 & 0 & | & 1 & -1 & 0 & \\ 1 & 0 & 0 & | & 2 & 2 & 3 & r_2' = r_2 - 2r_1 \\ 0 & 0 & 1 & | & -1 & 2 & 1 & r_3' = r_3 + r_1 \end{array}$$

$$\rightarrow \begin{array}{rrrcrrrl} 0 & 1 & 0 & | & 1 & -1 & 0 & \\ 1 & -2 & 0 & | & 0 & 4 & 3 & r_2' = \frac{1}{4}r_2 \\ 0 & 1 & 1 & | & 0 & 1 & 1 & \end{array} \rightarrow \begin{array}{rrrcrrrl} 0 & 1 & 0 & | & 1 & -1 & 0 & \\ 1 & -2 & 0 & | & 0 & 4 & 3 & r_2' = \frac{1}{4}r_2 \\ 0 & 1 & 1 & | & 0 & 1 & 1 & \end{array}$$

$$\rightarrow \begin{array}{rrrcrrrl} 0 & 1 & 0 & | & 1 & -1 & 0 & \\ \frac{1}{4} & -\frac{1}{2} & 0 & | & 0 & 1 & \frac{3}{4} & \\ 0 & 1 & 1 & | & 0 & 1 & 1 & r_3' = r_3 - r_2 \end{array} \rightarrow \begin{array}{rrrcrrrl} 0 & 1 & 0 & | & 1 & -1 & 0 & \\ \frac{1}{4} & -\frac{1}{2} & 0 & | & 0 & 1 & \frac{3}{4} & r_2' = r_2 - 3r_3 \\ -\frac{1}{4} & \frac{3}{2} & 1 & | & 0 & 0 & \frac{1}{4} & \end{array}$$

$$\rightarrow \begin{array}{rrrcrrrl} 0 & 1 & 0 & | & 1 & -1 & 0 & r_1' = r_1 + r_2 \\ 1 & -5 & -3 & | & 0 & 1 & 0 & \\ -\frac{1}{4} & \frac{3}{2} & 1 & | & 0 & 0 & \frac{1}{4} & \end{array} \rightarrow \begin{array}{rrrcrrrl} 1 & -4 & -3 & | & 1 & 0 & 0 & \\ 1 & -5 & -3 & | & 0 & 1 & 0 & \\ -\frac{1}{4} & \frac{3}{2} & 1 & | & 0 & 0 & \frac{1}{4} & r_3' = 4r_3 \end{array}$$

$$\rightarrow \begin{array}{rrrcrrr} 1 & -4 & -3 & | & 1 & 0 & 0 \\ 1 & -5 & -3 & | & 0 & 1 & 0 \\ -1 & 6 & 4 & | & 0 & 0 & 1 \end{array}$$

Reduced echelon matrix with

$$P = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}. \quad \text{(Here } P \text{ is the inverse of the given matrix. Why?)}$$

9. (a)

$$x_3 = 5; \quad x_2 + 2x_3 = 2 \Rightarrow x_2 = -8; \quad x_1 - 3x_2 + 4x_3 = 7 \Rightarrow x_1 = -37$$

so

$$\boldsymbol{x} = \begin{bmatrix} -37 \\ -8 \\ 5 \end{bmatrix}.$$

(b) Set $x_4 = \lambda$. Then

$$x_3 + x_4 = 2 \Rightarrow x_3 = 2 - \lambda; \quad x_2 + 4x_3 - 9x_4 = 3 \Rightarrow x_2 = -5 + 13\lambda;$$

$$x_1 + 8x_3 - 5x_4 = 6 \Rightarrow x_1 = -10 + 13\lambda$$

so

$$\boldsymbol{x} = \begin{bmatrix} -10 \\ -5 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 13 \\ 13 \\ -1 \\ 1 \end{bmatrix}.$$

(c) Set $x_5 = \lambda$. Then

$$x_4 + 3x_5 = 9 \Rightarrow x_4 = 9 - 3\lambda; \quad x_3 + x_4 + 6x_5 = 5 \Rightarrow x_3 = -4 - 3\lambda.$$

Now set $x_2 = \mu$ to get

$$x_1 + 7x_2 - 2x_3 - 8x_5 = -3 \Rightarrow x_1 = -11 - 7\mu + 2\lambda,$$

so

$$\boldsymbol{x} = \begin{bmatrix} -11 \\ 0 \\ -4 \\ 9 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 0 \\ -3 \\ -3 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(d) Inconsistent: no solution.

10. (a)

$$\begin{array}{rrrcl} 2 & 1 & 3 & | & 0 \quad r_1' = r_3 \\ 3 & -2 & 1 & | & 0 \\ 1 & -3 & -2 & | & 0 \quad r_3' = r_1 \end{array} \rightarrow \begin{array}{rrrcl} 1 & -3 & -2 & | & 0 \\ 3 & -2 & 1 & | & 0 \quad r_2' = r_2 - 3r_1 \\ 2 & 1 & 3 & | & 0 \quad r_3' = r_3 - 2r_1 \end{array}$$

$$\rightarrow \begin{array}{rrrcl} 1 & -3 & -2 & | & 0 \\ 0 & 7 & 7 & | & 0 \\ 0 & 7 & 7 & | & 0 \quad r_3' = r_3 - r_2 \end{array} \rightarrow \begin{array}{rrrcl} 1 & -3 & -2 & | & 0 \\ 0 & 7 & 7 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{array}$$

Let $x_3 = \lambda$. Then

$$x_2 = -\lambda; \quad x_1 - 3x_2 - 2x_3 = 0 \Rightarrow x_1 = -\lambda$$

so solution is

$$\boldsymbol{x} = \lambda \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

(b)

$$\begin{array}{rrrcl} 2 & 3 & 4 & | & 1 \quad r_1' = \frac{1}{2}r_1 \\ 5 & 6 & 7 & | & 2 \\ 8 & 9 & 10 & | & 4 \end{array} \rightarrow \begin{array}{rrrcl} 1 & \frac{3}{2} & 2 & | & \frac{1}{2} \\ 5 & 6 & 7 & | & 2 \quad r_2' = r_2 - 5r_1 \\ 8 & 9 & 10 & | & 4 \quad r_3' = r_3 - 8r_1 \end{array}$$

$$\rightarrow \begin{array}{rrrcl} 1 & \frac{3}{2} & 2 & | & \frac{1}{2} \\ 0 & -\frac{3}{2} & -3 & | & -\frac{1}{2} \\ 0 & -3 & -6 & | & 0 \quad r_3' = r_3 - 2r_2 \end{array} \rightarrow \begin{array}{rrrcl} 1 & \frac{3}{2} & 2 & | & \frac{1}{2} \\ 0 & -\frac{3}{2} & -3 & | & -\frac{1}{2} \\ 0 & 0 & 0 & | & 1 \end{array}$$

Equations are inconsistent, and have no solution.

(c)

$$\begin{array}{rrrrcrl} 1 & -3 & 5 & -7 & | & -2 & \\ -2 & 4 & -6 & 8 & | & 2 & r_2' = r_2 + 2r_1 \\ 1 & 1 & 1 & 1 & | & 2 & r_3' = r_3 - r_1 \\ 1 & 5 & 2 & 5 & | & 7 & r_4' = r_4 - r_1 \end{array} \rightarrow \begin{array}{rrrrcrl} 1 & -3 & 5 & -7 & | & -2 & \\ 0 & -2 & 4 & -6 & | & -2 & r_2' = -\frac{1}{2}r_2 \\ 0 & 4 & -4 & 8 & | & 4 & \\ 0 & 8 & -3 & 12 & | & 9 & \end{array}$$

$$\rightarrow \begin{array}{rrrrcrl} 1 & -3 & 5 & -7 & | & -2 & \\ 0 & 1 & -2 & 3 & | & 1 & \\ 0 & 4 & -4 & 8 & | & 4 & r_3' = r_3 - 4r_2 \\ 0 & 8 & -3 & 12 & | & 9 & r_4' = r_4 - 8r_2 \end{array} \rightarrow \begin{array}{rrrrcrl} 1 & -3 & 5 & -7 & | & -2 & \\ 0 & 1 & -2 & 3 & | & 1 & \\ 0 & 0 & 4 & -4 & | & 0 & r_3' = \frac{1}{4}r_3 \\ 0 & 0 & 13 & -12 & | & 1 & \end{array}$$

$$\rightarrow \begin{array}{rrrrcrl} 1 & -3 & 5 & -7 & | & -2 & \\ 0 & 1 & -2 & 3 & | & 1 & \\ 0 & 0 & 1 & -1 & | & 0 & \\ 0 & 0 & 13 & -12 & | & 1 & r_4' = r_4 - 13r_3 \end{array} \rightarrow \begin{array}{rrrrcr} 1 & -3 & 5 & -7 & | & -2 \\ 0 & 1 & -2 & 3 & | & 1 \\ 0 & 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 1 & | & 1 \end{array}$$

so

$$x_4 = 1; \quad x_3 - x_4 = 0 \Rightarrow x_3 = 1; x_2 - 2x_3 + 3x_4 = 1 \Rightarrow x_2 = 0;$$

$$x_1 - 3x_2 + 5x_3 - 7x_4 = -2 \Rightarrow x_1 = 0$$

and solution is

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(d)

$$\begin{array}{rrrrcrl} 3 & 1 & 2 & 4 & | & 3 & r_1' = \frac{1}{3}r_1 \\ 5 & 2 & 3 & 6 & | & 5 & \\ 4 & 1 & 3 & 6 & | & 4 & \\ 5 & 1 & 4 & 8 & | & 5 & \end{array} \rightarrow \begin{array}{rrrrcrl} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & | & 1 & \\ 5 & 2 & 3 & 6 & | & 5 & r_2' = r_2 - 5r_1 \\ 4 & 1 & 3 & 6 & | & 4 & r_3' = r_3 - 4r_1 \\ 5 & 1 & 4 & 8 & | & 5 & r_4' = r_4 - 5r_1 \end{array}$$

$$\rightarrow \begin{array}{rrrrcrl} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & | & 1 & \\ 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & | & 0 & \\ 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & | & 0 & r_3' = r_3 + r_2 \\ 0 & -\frac{2}{3} & \frac{2}{3} & \frac{4}{3} & | & 0 & r_4' = r_4 + 2r_2 \end{array} \rightarrow \begin{array}{rrrrcrl} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & | & 1 & \\ 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & | & 0 & r_2' = 3r_2 \\ 0 & 0 & 0 & 0 & | & 0 & \\ 0 & 0 & 0 & 0 & | & 0 & \end{array}$$

$$\rightarrow \begin{array}{rrrrcr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & | & 1 \\ 0 & 1 & -1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}$$

Let $x_3 = \mu$, $x_4 = \lambda$ to get

$$x_2 - x_3 - 2x_4 = 0 \Rightarrow x_2 = \mu + 2\lambda; \quad x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3 + \frac{4}{3}x_4 \Rightarrow x_1 = 1 - \mu - 2\lambda$$

so the solution is

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(e)

$$\begin{matrix} 2 & 1 & 3 & 5 & | & 6 \\ 3 & 2 & 4 & 6 & | & 8 \\ -1 & 3 & 2 & 7 & | & -3 \end{matrix} \begin{matrix} r_1' = r_3 \\ r_3' = r_1 \end{matrix} \rightarrow \begin{matrix} -1 & 3 & 2 & 7 & | & -3 \\ 3 & 2 & 4 & 6 & | & 8 \\ 2 & 1 & 3 & 5 & | & 6 \end{matrix} \quad r_2' = r_2 + 3r_1 r_3' = r_3 + 2r_1$$

$$\rightarrow \begin{matrix} -1 & 3 & 2 & 7 & | & -3 \\ 0 & 11 & 10 & 27 & | & -1 \\ 0 & 7 & 7 & 19 & | & 0 \end{matrix} \quad r_2' = \tfrac{1}{11} r_2 \quad \rightarrow \begin{matrix} -1 & 3 & 2 & 7 & | & -3 \\ 0 & 1 & \frac{10}{11} & \frac{27}{11} & | & -\frac{1}{11} \\ 0 & 7 & 7 & 19 & | & 0 \end{matrix} \quad r_3' = r_3 - 7r_2$$

$$\rightarrow \begin{matrix} -1 & 3 & 2 & 7 & | & -3 \\ 0 & 1 & \frac{10}{11} & \frac{27}{11} & | & -\frac{1}{11} \\ 0 & 0 & \frac{7}{11} & \frac{20}{11} & | & \frac{7}{11} \end{matrix}$$

Let $x_4 = \lambda$ so

$$\frac{7}{11}x_3 + \frac{20}{11}x_4 = \frac{7}{11} \Rightarrow x_3 = 1 - \frac{20}{7}\lambda; \quad x_2 + \frac{10}{11}x_3 + \frac{27}{11}x_4 = -\frac{1}{11} \Rightarrow x_2 = -1 + \frac{1}{7}\lambda;$$

$$-x_1 + 3x_2 + 2x_3 + 7x_4 = -3 \Rightarrow x_1 = 2 + \frac{12}{7}\lambda$$

so solution is

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{\lambda}{7} \begin{bmatrix} 12 \\ 1 \\ -20 \\ 7 \end{bmatrix}.$$

11. We have $A(\boldsymbol{x}_1 + \alpha \boldsymbol{x}_2) = A\boldsymbol{x}_1 + A(\alpha \boldsymbol{x}_2) = A\boldsymbol{x}_1 + \alpha A \boldsymbol{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$ so $\boldsymbol{x}_1 + \alpha \boldsymbol{x}_2$ is also a solution of the system.

12. One possible system is $x + y = 0$, $z - y = 3$.

13. We assume $|A| \neq 0$ and consider two separate cases. Let us first suppose that $a \neq 0$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \; r_1' = \tfrac{1}{a} r_1 \rightarrow \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \; r_2' = r_2 - c r_1 \rightarrow \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{bmatrix} \; r_2' = \tfrac{a}{ad - bc} r_2$$

$$\rightarrow \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} \; r_1' = r_1 - \tfrac{b}{a} r_2 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Secondly, if $a = 0$, then $ad - bc = -bc \neq 0 \Rightarrow b \neq 0$ and $c \neq 0$, so we have

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \begin{matrix} r_1' = r_2 \\ r_2' = r_1 \end{matrix} \rightarrow \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \begin{matrix} r_1' = \frac{1}{c} r_1 \\ r_2' = \frac{1}{b} r_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \; r_1' = r_1 - \tfrac{d}{c} r_2 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

14. Terms in $\det(A)$ are of the form $sgn(\sigma)\ a_{1\sigma(1)}\ a_{2\sigma(2)}\ a_{3\sigma(3)}\ a_{4\sigma(4)}$, where σ is a permutation of $\{1, 2, 3, 4\}$.

    $a_{13}\ a_{24}\ a_{41}\ a_{32} = a_{13}\ a_{24}\ a_{32}\ a_{41}$ corresponds to $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$, which has sign $= (-1)^5 = -1$.

    Hence $-a_{13}\ a_{24}\ a_{41}\ a_{32}$ occurs in $\det(A)$.

    $a_{31}\ a_{42}\ a_{14}\ a_{33}$ contains no term $a_{2\sigma(2)}$ and hence it does **not** occur in $\det(A)$.

    $a_{12}\ a_{22}\ a_{31}\ a_{44}$ does not correspond to $a_{1\sigma(1)}\ a_{2\sigma(2)}\ a_{3\sigma(3)}\ a_{4\sigma(4)}$ since a permutation is one-to-one: $\sigma(1)$ cannot have same value as $\sigma(2)$. Hence this term is **not** in $\det(A)$.

15. (a) $I_n = [\delta_{ij}]_{n\times n}$, where δ_{ij} is the Kronecker delta. Hence $\det(I_n) = \sum_{\sigma\in S_n} sgn(\sigma)\delta_{1\sigma(1)}$ $\delta_{2\sigma(2)} \dots \delta_{n\sigma(n)}$.

    The only non-zero term occurs when $\sigma(1) = 1$, $\sigma(2) = 2, \dots, \sigma(n) = n$, and this is the identity permutation, with sign $= (-1)^0 = 1$.

    Hence $\det(I_n) = \delta_{11}\,\delta_{22} \dots \delta_{nn} = 1$.

    $0_{n\times n} = [\theta_{ij}]_{n\times n}$ where $\theta_{ij} = 0 \ \ \forall i, j = 1, 2, \dots, n$.

$$\begin{aligned} \Rightarrow\ \det(0_{n\times n}) &= \sum_{\sigma\in S_n} sgn(\sigma)\theta_{1\sigma(1)}\ \theta_{2\sigma(2)} \dots \theta_{n\sigma(n)} \\ &= \sum_{\sigma\in S_n} sgn(\sigma) 0\,.\,0 \dots 0 = 0. \end{aligned}$$

    (b) $A = [a_{ij}]_{n\times n}$. Let $a_{lj} = 0$ for $j = 1, 2, \dots, n$—row l of A is zero.

$$\begin{aligned} \det(A) &= \sum_{\sigma\in S_n} sgn(\sigma) a_{1\sigma(1)}\ a_{2\sigma(2)} \dots a_{l\sigma(l)} \dots a_{n\sigma(n)} \\ &= 0 \ \text{ since } \ a_{l\sigma(l)=0}. \end{aligned}$$

    Proof for a **zero column** is similar.

    (c) *Prove by induction.* Result holds for $n = 2$ since

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\ a_{22} - a_{12}\ a_{21} = \det\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}.$$

    Assume result to be true for any $n \times n$ matrix.

    Consider $A = [a_{ij}]_{(n+1)\times(n+1)}$

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n+1} a_{1j}\ A_{1j} \quad \text{(expand by row 1)} \\ &= \sum_{j=1}^{n+1} a_{1j}(-1)^{1+j} \det(M_{1j}) \quad (\det(M_{1j}) \ \text{ is minor of } a_{1j}) \end{aligned}$$

$$= \sum_{j=1}^{n+1} a_{1j}(-1)^{1+j}\det(M_{j1}^T) \quad \text{(by assumption)}$$

{Here M_{j1}^T is submatrix of A^T obtained by deleting element in row j and column 1 from A^T.

$$= \sum_{j=1}^{n+1} a_{j1}^T\ A_{j1}^T,$$

where $a_{j1}^T = a_{1j} =$ element in row j and column 1 of A^T, and $A_{j1}^T =$ cofactor of a_{j1}^T in A^T.

$$= \det(A^T). \text{ Result follows by induction.}$$

(d) *Prove by induction.* Result holds for $n = 2$.

If $A = [a_{ij}]_{2\times 2}$, then $\det(A) = a_{11}\ a_{22} - a_{12}\ a_{21}$. Now interchange rows 1 and 2 to obtain $E_{12}\,A = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$ which has determinant $a_{21}\ a_{12} - a_{11}\ a_{22} = -\det(A)$.

Assume result is true for any $n\times n$ matrix and consider $A = [a_{ij}]_{(n+1)\times(n+1)}$. Interchanging rows r and s produces $B = E_{rs}\,A$.

Let M_{ij} denote the $n\times n$ matrix obtained by deleting row i and column j of A and let M_{ij}^B be the analogous $n\times n$ matrix obtained by deleting row i and column j of matrix B. If $i \neq r, s$ then M_{ij}^B is derived from M_{ij} by interchanging rows in M_{ij} corresponding to rows r and s of A.

Now

$$\begin{aligned} \det(B) &= \sum_{j=1}^{n+1} b_{ij}\ B_{ij} = \sum_{j=1}^{n+1} a_{ij}\ B_{ij} \quad (i \neq r, s) \\ &= -\sum_{j} a_{ij}\ A_{ij} \end{aligned}$$

(from assumption, noting that $B_{ij} = (-1)^{i+j}\det(M_{ij}^B) = -(-1)^{i+j}\det(M_{ij}) = -A_{ij}$).

Hence $\det(B) = -\det(A)$. Result follows by induction.

(e) If σ is not the identity permutation, then for at least one $i, i < \sigma(i)$, in which case $a_{i\sigma(i)} = 0$ since A is lower triangular. Hence the only non-zero term in $\det(A)$ is that corresponding to identity perm.

Hence $\det(A) = a_{11}\ a_{22}\ a_{33}\ldots a_{nn}$.

16.
$$\begin{aligned}
\det(C) &= \sum_{\sigma\in S_n}(sgn\ \sigma)c_{1\sigma(1)}\ c_{2\sigma(2)}\dots c_{n\sigma(n)}\\
&= \sum_{\sigma}(sgn\ \sigma)(a_{1\sigma(1)}+b_{1\sigma 1})c_{2\sigma(2)}\ c_{3\sigma(3)}\dots c_{n\sigma(n)}\\
&= \sum_{\sigma}(sgn\ \sigma)a_{1\sigma(1)}\ c_{2\sigma(2)}\dots c_{n\sigma(n)}+\sum_{\sigma}(sgn\ \sigma)b_{1\sigma(1)}\ c_{2\sigma(2)}\dots c_{n\sigma(n)}\\
&= \sum_{\sigma\in S_n}(sgn\ \sigma)a_{1\sigma(1)}\ a_{2\sigma(2)}\dots a_{n\sigma(n}+\sum_{\sigma\in S_n}(sgn\ \sigma)b_{1\sigma(1)}\ b_{2\sigma(2)}\dots b_{n\sigma(n)}\\
&= \det(A)+\det(B).
\end{aligned}$$

17. (a)
$$\begin{vmatrix} 2 & 1 & 2 & -1\\ 3 & -1 & 0 & 2\\ 4 & 2 & 3 & 1\\ -2 & 1 & 0 & 2\end{vmatrix} = 2\begin{vmatrix} -1 & 0 & 2\\ 2 & 3 & 1\\ 1 & 0 & 2\end{vmatrix} - \begin{vmatrix} 3 & 0 & 2\\ 4 & 3 & 1\\ -2 & 0 & 2\end{vmatrix}$$
$$+\,2\begin{vmatrix} 3 & -1 & 2\\ 4 & 2 & 1\\ -2 & 1 & 2\end{vmatrix} + \begin{vmatrix} 3 & -1 & 0\\ 4 & 2 & 3\\ -2 & 1 & 0\end{vmatrix}$$

$$= 2[-1(6)+2(-3)]-[3(6)+2(6)]+2[3(3)+10+2(8)]+[3(-3)+6]$$

$$= -24-30+70-3=13.$$

(b)
$$\det = 2\underset{\text{(see above!)}}{\begin{vmatrix} 3 & -1 & 2\\ 4 & 2 & 1\\ -2 & 1 & 2\end{vmatrix}} + 3\begin{vmatrix} 2 & 1 & -1\\ 3 & -1 & 2\\ -2 & 1 & 2\end{vmatrix} \begin{aligned} &= 70+3(-8-10-1)\\ &= 70-57=13\end{aligned}$$

(c)
$$\begin{vmatrix} 2 & 1 & 2 & -1\\ 3 & -1 & 0 & 2\\ 4 & 2 & 3 & 1\\ -2 & 1 & 0 & 2\end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & -1\\ 5 & 0 & 2 & 1\\ 0 & 0 & -1 & 3\\ -4 & 0 & -2 & 3\end{vmatrix} = -\begin{vmatrix} 5 & 2 & 1\\ 0 & -1 & 3\\ -4 & -2 & 3\end{vmatrix}$$

$$\left\{\begin{aligned} r_2 &:= r_2+r_1\\ r_3 &:= r_3-2r_1\\ r_4 &:= r_4-r_1\end{aligned}\right\} \qquad c_3 := c_3+3c_2$$

$$= -\begin{vmatrix} 5 & 2 & 7\\ 0 & -1 & 0\\ -4 & -2 & -3\end{vmatrix} = +\begin{vmatrix} 5 & 7\\ -4 & -3\end{vmatrix} = -15+28=13$$

18. (a)
$$\underset{c_3:=c_3-3c_1}{\begin{vmatrix} 17 & 25 & 13\\ 9 & 11 & 8\\ 19 & -58 & 57\end{vmatrix}} = \begin{vmatrix} 17 & 25 & -38\\ 9 & 11 & -19\\ 19 & -58 & 0\end{vmatrix} = -19\underset{r_1:=r_1-2r_2}{\begin{vmatrix} 17 & 25 & 2\\ 9 & 11 & 1\\ 19 & -58 & 0\end{vmatrix}}$$

$$= -19\begin{vmatrix} -1 & 3 & 0 \\ 9 & 11 & 1 \\ 19 & -58 & 0 \end{vmatrix} = +19\begin{vmatrix} -1 & 3 \\ 19 & -58 \end{vmatrix} = 19.$$

(b) $$\begin{vmatrix} 299 & 199 & 99 \\ 597 & 397 & 197 \\ 999 & 599 & 299 \end{vmatrix} = \begin{vmatrix} 100 & 100 & 99 \\ 200 & 200 & 197 \\ 400 & 300 & 299 \end{vmatrix} = \begin{vmatrix} 0 & 100 & 99 \\ 0 & 200 & 197 \\ 100 & 300 & 299 \end{vmatrix}$$

$\{c_1 := c_1 - c_2; c_2 := c_2 - c_3\}$ $\{c_1 := c_1 - c_2\}$

$$= 10^4\begin{vmatrix} 1 & 99 \\ 2 & 197 \end{vmatrix} = 10^4(197 - 198) = -10^4$$

(c) $$\begin{vmatrix} 11 & 3 & -2 & 6 \\ 20 & 0 & 36 & 17 \\ 38 & 9 & 3 & 22 \\ 30 & 6 & 11 & 19 \end{vmatrix} = \begin{vmatrix} 11 & 3 & -2 & 6 \\ 20 & 0 & 36 & 17 \\ 5 & 0 & 9 & 4 \\ 8 & 0 & 15 & 7 \end{vmatrix}$$

$\{r_3 := r_3 - 3r_1; r_4 := r_4 - 2r_1\}$

$$= -3\underbrace{\begin{vmatrix} 20 & 36 & 17 \\ 5 & 9 & 4 \\ 8 & 15 & 7 \end{vmatrix}}_{r_1 := r_1 - 4r_2} = -3\begin{vmatrix} 0 & 0 & 1 \\ 5 & 9 & 4 \\ 8 & 15 & 7 \end{vmatrix} \begin{aligned} &= -3(75-72) \\ &= -9 \end{aligned}$$

19. (a) $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$$

$$= (y-x)(z-x)(z-y)$$

$$\{= (x-y)(y-z)(z-x)\}$$

(b) $$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix} = (y-x)(z-x)\underbrace{\begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+xz+x^2 \end{vmatrix}}_{r_2 := r_2 - xr_1}$$

$$= (y-x)(z-x)\underbrace{\begin{vmatrix} y+x & z+x \\ y^2 & z^2 \end{vmatrix}}_{c_1 := c_1 - c_2} = (y-x)(z-x)\begin{vmatrix} y-z & z+x \\ y^2-z^2 & z^2 \end{vmatrix}$$

$$= (y-x)(z-x)(y-z)\begin{vmatrix} 1 & z+x \\ y+z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

(c) $$\begin{vmatrix} 1 & 1 & 1 \\ x+y & x+z & y+z \\ xy & xz & yz \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x+y & z-y & z-x \\ xy & x(z-y) & y(z-x) \end{vmatrix}$$

$$= (z-y)(z-x)\begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix} = (z-y)(z-x)(y-x) = (x-y)(y-z)(z-x)$$

(d) $$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} = (x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & (x-a) & 0 & 0 \\ 0 & 0 & (x-a) & 0 \\ 0 & 0 & 0 & (x-a) \end{vmatrix}$$

$\{c_1 := c_1 + c_2 + c_3 + c_4\}\{r_2 := r_2 - r_1; r_3 := r_3 - r_1; r_4 := r_4 - r_1\}$

$= (x+3a)(x-a)^3.$

(e) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ w & x-w & y-w & z-w \\ w^2 & x^2-w^2 & y^2-w^2 & z^2-w^2 \\ w^3 & x^3-w^3 & y^3-w^3 & z^3-w^3 \end{vmatrix}$$

$$= (x-w)(y-w)(z-w)\begin{vmatrix} 1 & 1 & 1 \\ x+w & y+w & z+w \\ x^2+xw+w^2 & y^2+yw+w^2 & z^2+zw+w^2 \end{vmatrix}$$

$$= (x-w)(y-w)(z-w)\begin{vmatrix} 1 & 0 & 0 \\ x+w & y-x & z-x \\ x^2+xw+w^2 & (y-x)(x+y+w) & (z-x)(z+x+w) \end{vmatrix}$$

$$= (x-w)(y-w)(z-w)(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ x+y+w & z+x+w \end{vmatrix}$$

$$= (x-w)(y-w)(z-w)(y-x)(z-x)(z-y)$$

20. $$\begin{vmatrix} 1+x_1 & x_2 & \cdots & x_n \\ x_1 & 1+x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & 1+x_n \end{vmatrix} = \begin{vmatrix} 1+\sum_{i=1}^n x_i & x_2 & \cdots & x_n \\ 1+\sum_{i=1}^n x_i & 1+x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ 1+\sum_{i=1}^n x_i & x_2 & \cdots & 1+x_n \end{vmatrix}$$

$$= \begin{vmatrix} 1+\sum_{i=1}^n x_i & x_2 & x_3 & \cdots & x_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = \left(1+\sum_{i=1}^n x_i\right)\det(I_{n-1}) = 1+\sum_{i=1}^n x_i$$

21. (a) $|A_1| = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, where $\boldsymbol{b} \in \mathbb{R}^3$.

If $A\boldsymbol{x} = \boldsymbol{b}$, then

$$\begin{aligned} |A_1| &= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{12} & a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{22} & a_{23} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & a_{32} & a_{33} \end{vmatrix} \\ &\qquad c_1 := c_1 - x_2c_2 - x_3c_3 \\ &= \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} \\ a_{21}x_1 & a_{22} & a_{23} \\ a_{31}x_1 & a_{32} & a_{33} \end{vmatrix} = x_1\,|A| \end{aligned}$$

Thus, $x_1 = |A_1|/|A|$ since $|A| \neq 0$.

Similar results for x_2 and x_3.

Hence, if $A\boldsymbol{x} = \boldsymbol{b}$ then $x_1 = |A_1|/|A|$, $x_2 = |A_2|/|A|$, $x_3 = |A_3|/|A|$.

(b) Here,

$$A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 0 & 2 \\ -3 & 5 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} -6 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{aligned} |A| &= (-10) - (6) - 3(-5) = -1 \\ |A_1| &= \begin{vmatrix} -6 & 1 & -3 \\ 5 & 0 & 2 \\ 7 & 5 & 0 \end{vmatrix} = -1; \quad |A_2| = \begin{vmatrix} 1 & -6 & -3 \\ -1 & 5 & 2 \\ -3 & 7 & 0 \end{vmatrix} = -2 \\ |A_3| &= \begin{vmatrix} 1 & 1 & -6 \\ -1 & 0 & 5 \\ -3 & 5 & 7 \end{vmatrix} = -3 \quad \text{Hence } x_1 = 1,\ x_2 = 2,\ x_3 = 3. \end{aligned}$$