5 Lecture examples: Chapter 5

Examples 5A

1 Verify that $\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 2$.

$$A\boldsymbol{x} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\boldsymbol{x}.$$

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2 Find the eigenvalues of

$$\text{(a) } A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}, \quad \text{(b) } B = \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix}.$$

(a)

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) + 2 = \lambda^2 - 3\lambda + 2$$

$$\Rightarrow \det(A - \lambda I) = 0 \Leftrightarrow \lambda^2 - 3\lambda + 2 = 0 \Leftrightarrow (\lambda - 2)(\lambda - 1) = 0 \Leftrightarrow \lambda = 1 \text{ } or \text{ } \lambda = 2.$$

Eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

(b)

$$\det(B - \lambda I) = \begin{vmatrix} -3-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} = (-3-\lambda)(3-\lambda) + 10 = \lambda^2 + 1$$

$$\Rightarrow \det(B - \lambda I) = 0 \Leftrightarrow \lambda^2 = -1 \Leftrightarrow \lambda = \pm\, i.$$

Eigenvalues of B are $\lambda = +\, i$ and $\lambda = -i$.

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3 (a) Find the eigenvalues of $A = \begin{bmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{bmatrix}$.

(b) Show that $(1, 0, 0)$ and $(0, 2, 1)$ are eigenvectors of A corresponding to two of the eigenvalues, and find an eigenvector corresponding to the third eigenvalue.

(a)

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 6 & -12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} -13-\lambda & 30 \\ -9 & 20-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\left(\lambda^2 - 7\lambda + 10\right) = -(1+\lambda)(\lambda-5)(\lambda-2).$$

Eigenvalues are therefore -1, 2, 5.

(b)

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow (1,0,0) \text{ is eigenvector for } \lambda = -1.$$

$$A\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Rightarrow (0,2,1) \text{ is eigenvector for } \lambda = 2.$$

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be an eigenvector corresponding to $\lambda = 5$ and solve $(A-5I)\,\boldsymbol{x} = \mathbf{0}$.

Use EROs:

$$\begin{bmatrix} -6 & 6 & -12 \\ 0 & -18 & 30 \\ 0 & -9 & 15 \end{bmatrix} \begin{matrix} r_1' = -r_1/6 \\ \\ r_3' = r_3 - \frac{1}{2}r_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -18 & 30 \\ 0 & 0 & 0 \end{bmatrix} \; r_2' = -r_2/18 \; \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

That is,

$$x_1 - x_2 + 2x_3 = 0, \qquad x_2 - \frac{5}{3}x_3 = 0.$$

Let $x_3 = \mu$. Then $x_2 = \frac{5}{3}x_3 = \frac{5}{3}\mu$ and $x_1 = x_2 - 2x_3 = \frac{5}{3}\mu - 2\mu = -\frac{1}{3}\mu$. Hence $\boldsymbol{x} = \mu\left(-\frac{1}{3}, \frac{5}{3}, 1\right)$ is an eigenvector corresponding to $\lambda = 5$ for any $\mu \neq 0$. Choose $\mu = 3$ to obtain $\boldsymbol{x} = (-1, 5, 3)$.

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4 Show that $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ has only two distinct eigenvalues, and find bases for the two eigenspaces.

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left(\lambda^2 - 3\lambda + 2\right) = -\left(\lambda - 2\right)^2\left(\lambda - 1\right).$$

The eigenvalues are $\lambda = 2$ (with algebraic multiplicity 2) and $\lambda = 1$ (with algebraic multiplicity 1).

Eigenspace for $\lambda = 2$**:** Solve $(A - 2I)\,\boldsymbol{x} = \mathbf{0}$.

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} r_1' = -r_1/2 \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution is $x_3 = s$, $x_2 = t$, $x_1 = -s$ for $s, t \in \mathbb{R}$, that is,

$$\boldsymbol{x} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Basis for eigenspace is $\{(-1,0,1), (0,1,0)\}$ and its dimension is 2. Hence the geometric multiplicity of eigenvalue $\lambda = 2$ is 2.

Eigenspace for $\lambda = 1$**:** Solve $(A - I)\,\boldsymbol{x} = \mathbf{0}$.

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{matrix} r_1' = -r_1 \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{matrix} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution is $x_3 = s$, $x_2 = s$ and $x_1 = -2s$, that is,

$$\boldsymbol{x} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

Basis for the eigenspace is $\{(-2,1,1)\}$ and its dimension is 1. Hence the geometric multiplicity of eigenvalue $\lambda = 1$ is 1.

Examples 5B

1 Given that the matrix

$$A = \begin{bmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{bmatrix}$$

has eigenvalues -1, 2, 5 with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix},$$

find a matrix P that diagonalises A.

$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$ diagonalises A.

[**Check:**

$$AP = \begin{bmatrix} -1 & 0 & -5 \\ 0 & 4 & 25 \\ 0 & 2 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = PD,$$

say, where $D = \mathrm{diag}(-1, 2, 5)$. Hence $P^{-1}AP = D$.]

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2 Given that the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

has eigenvalues 2 (with algebraic multiplicity= geometric multiplicity=2) and 1 (with algebraic multiplicity = geometric multiplicity=1), with corresponding eigenvectors

$$(-1, 0, 1)\,, \qquad (0, 1, 0)\,, \qquad (-2, 1, 1)\,,$$

find a matrix P that diagonalises A.

$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalises A and $P^{-1}AP = \mathrm{diag}\,(2, 2, 1)$.

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3 Show that $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalisable.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^3\,.$$

Hence A has one eigenvalue of $\lambda = 2$ with algebraic multiplicity 3. To find the related eigenspace, solve $(A - 2I)\,\boldsymbol{x} = \mathbf{0}$. This is

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

with solution is $x_2 = 0$, $x_3 = 0$, $x_1 = s$ for any $s \in \mathbb{R}$, i.e. $\boldsymbol{x} = s(1,0,0)$. This eigenspace is spanned by $(1,0,0)$: it has dimension 1 and the geometric multiplicity of $\lambda = 2$ is 1.

Since algebraic multiplicity $\neq$geometric multiplicity, A does not have 3 linearly independent eigenvectors and A is not diagonalisable.

Examples 5C

1 If $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{bmatrix}$, find an orthogonal matrix P such that $P^T AP = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Eigenvalues are given by $\begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix} = 0$, so

$$(1-\lambda)\left[(5-\lambda)(3-\lambda) - 16\right] + 4\left[-4(5-\lambda)\right] = 0$$

$$\Rightarrow \lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0 \Rightarrow (\lambda+3)(\lambda-3)(\lambda-9) = 0.$$

Hence eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 3$, $\lambda_3 = 9$.

Solve $(A+3I)\boldsymbol{x}_1 = \mathbf{0}$, $(A-3I)\boldsymbol{x}_2 = \mathbf{0}$, $(A-9I)\boldsymbol{x}_3 = \mathbf{0}$: corresponding eigenvectors are

$$\boldsymbol{x}_1 = k_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{x}_2 = k_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_3 = k_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

where k_1, k_2, k_3 are non-zero constants.

Choose

$$\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

with corresponding

$$\boldsymbol{z}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{z}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{z}_3 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

(it is readily seen that $\boldsymbol{z}_i \cdot \boldsymbol{z}_j = \delta_{ij}$). So

$$P = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

satisfies

$$P^T P = I, \qquad P^T AP = \operatorname{diag}(-3, 3, 9).$$

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2 If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, find an orthogonal matrix P such that $P^T AP = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Eigenvalues are given by $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$, so

$$-\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = 0 \Rightarrow \lambda^3 - 3\lambda - 2 = 0 \Rightarrow (\lambda - 2)(\lambda + 1)^2 = 0.$$

Hence eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = -1$.

Solve $(A - 2I)\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{matrix} r_1' = r_2 \\ r_2' = r_1 \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{matrix} \\ r_2' = r_2 + 2r_1 \\ r_3' = r_3 - r_1 \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{matrix} \\ \\ r_3' = r_3 + r_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \; r_2' = r_2/(-3) \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let free variable $x_3 = s$ so $x_2 - x_3 = 0 \Rightarrow x_2 = s$, $x_1 - 2x_2 + x_3 = 0 \Rightarrow x_1 = s$ and eigenvector for $\lambda = 2$ is $s(1, 1, 1)$. Choose $s = 1$ to give $\boldsymbol{x}_1 = (1, 1, 1)$.

Solve $(A + I)\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let free variables $x_2 = s$ and $x_3 = t$ so $x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -s - t$ and eigenvector for $\lambda = -1$ is $\boldsymbol{x} = s(-1, 1, 0) + t(-1, 0, 1)$. Eigenspace is $\operatorname{sp}\{(-1, 1, 0), (-1, 0, 1)\}$ so $\lambda = -1$ has geometric multiplicity 2. Choose $s = 1, t = 0$ and $s = 0, t = 1$ to get two eigenvectors $\boldsymbol{x}_2 = (-1, 1, 0)$, $\boldsymbol{x}_3 = (-1, 0, 1)$. So full set of eigenvectors is

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Vectors $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are not orthogonal, so apply Gram-Schmidt:

Step 1: Let $\boldsymbol{y}_1 = (-1, 1, 0)$.

Step 2: Let $\boldsymbol{y}_2 = (-1, 0, 1] + \alpha(-1, 1, 0) = (-1 - \alpha, \alpha, 1)$.

$$\boldsymbol{y}_2 \cdot \boldsymbol{y}_1 = 0 \Rightarrow 1 + 2\alpha = 0 \Rightarrow \alpha = -\frac{1}{2}.$$

So $\boldsymbol{y}_2 = (-1/2, -1/2, 1)$. We will use $\boldsymbol{y}_2 = (-1, -1, 2)$.

The required orthonormal set is therefore

$$\boldsymbol{z}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{z}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{z}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

and

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

satisfies

$$P^T P = I, \qquad P^T A P = \operatorname{diag}(2, -1, -1).$$