5 Exercise Solutions: Chapter 5

1. (a) $\lambda = 0$ is an eigenvalue of A

   $\Leftrightarrow\ \det(A - 0I) = 0$

   $\Leftrightarrow\ \det(A) = 0\ \Leftrightarrow\ A$ is singular.

   (b) Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}\ \ (\boldsymbol{x} \neq \mathbf{0})$.

   Then $A^2\boldsymbol{x} = A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda(A\boldsymbol{x}) = \lambda(\lambda\boldsymbol{x}) = \lambda^2\boldsymbol{x}$.

   Thus, λ^2 is an eigenvalue of A^2 (with eigenvector $\boldsymbol{x}$) whenever λ is an eigenvalue of A (with eigenvector $\mathbf{x}$).

   Now use induction.

   (c) Let $V_\lambda = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \lambda\boldsymbol{x}\}\quad (\lambda \in \mathbb{R}$ fixed) and let $\boldsymbol{x}_1,\ \boldsymbol{x}_2 \in V_\lambda;\ \alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} A(\alpha\,\boldsymbol{x}_1 + \beta\,\boldsymbol{x}_2) &= \alpha\,A\boldsymbol{x}_1 + \beta\,A\boldsymbol{x}_2 \\ &= \alpha\,\lambda\boldsymbol{x}_1 + \beta\,\lambda\boldsymbol{x}_2 \quad (\boldsymbol{x}_1, \boldsymbol{x}_2 \in V_\lambda) \\ &= \lambda(\alpha\,\boldsymbol{x}_1 + \beta\,\boldsymbol{x}_2) \end{aligned}$$

   $\Rightarrow\ \alpha\,\boldsymbol{x}_1 + \beta\,\boldsymbol{x}_2 \in V_\lambda$, and hence V_λ is a subspace of $\mathbb{R}^n$.

   {Note: For V_λ to be a subspace we must include $\mathbf{0}$ which is **not** regarded as an eigenvector.}

   (d) Let A be nilpotent. Then $\exists\, r \in \mathbb{N}$ such that $A^r = O_{n\times n}$. By part (b) above, if λ is an eigenvalue of A then λ^r is an eigenvalue of $A^r = O_{n\times n}$.

   Hence $\lambda^r = 0$ (only eigenvalue of $O_{n\times n}$ is 0) and it follows that $\lambda = 0$.

2. Let $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ $(a, b \in \mathbb{R})$.

   Then $\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ -b & a-\lambda \end{bmatrix} = (a-\lambda)^2 + b^2$

   $\Rightarrow\ \det(A - \lambda I) = 0\ \Leftrightarrow\ \lambda = a \pm ib\ \ (i^2 = -1)$.

   Eigenvectors

   $\boldsymbol{\lambda = a + ib}$. Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$.

   This is

$$\begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   Solution is

$$\boldsymbol{x} = k \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \left(k = 1 \text{ gives } \begin{bmatrix} 1 \\ i \end{bmatrix}\right).$$

   $\boldsymbol{\lambda = a - ib}$ Solve $\begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} \boldsymbol{x} = \mathbf{0}$

Solution is $\boldsymbol{x} = k \begin{bmatrix} 1 \\ -i \end{bmatrix}$, and $k = 1$ gives $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

3. (a) $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$ when $\lambda = 1$ (three times)

$\Rightarrow$ Algebraic multiplicity $= 3$.

Geometric multiplicity is dimension of $N(A - \lambda I)$—dimension of eigenspace of A corresponding to eigenvalue λ.

Solve

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \mathbf{0}$$

and obtain

$$\boldsymbol{x} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Three linearly independent eigenvectors, so geometric multiplicity $= 3$.

(b) $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$ when $\lambda = 1$ (three times)

$\Rightarrow$ Algebraic multiplicity $= 3$.

For geometric multiplicity we solve

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \mathbf{0}.$$

This gives $x_3 = 0$ and x_1, x_2 arbitrary.

Hence

$$\boldsymbol{x} = \mu \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\exists$ 2 linearly independent eigenvectors, so geometric multiplicity $= 2$.

(c) $\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$ when $\lambda = 1$ (three times)

$\Rightarrow$ Algebraic multiplicity $= 3$.

For geometric multiplicity we solve

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \mathbf{0}.$$

This gives $x_2 = x_3 = 0$, and x_1 arbitrary.

Hence

$$\boldsymbol{x} = \mu \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Geometric multiplicity $= 1$.

4. The characteristic equation of A is

$$f(\lambda) = \lambda^3 - 2\lambda^2 - 2\lambda - 1 = 0.$$

Hence by the Cayley-Hamilton Theorem

$$A^3 - 2A^2 - 2A - I_3 = O_{3\times 3}.$$

In other words,

$$I_3 = A^3 - 2A^2 - 2A.$$

Hence $A^{-1} = A^2 - 2A - 2I_3$. Since

$$A^2 = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 5 \\ 2 & 1 & 5 \end{bmatrix},$$

this gives

$$A^{-1} = \begin{bmatrix} -3 & 2 & 2 \\ -4 & 2 & 3 \\ 2 & -1 & -1 \end{bmatrix}.$$

5. (a)

$$\begin{aligned} \det(A - \lambda I)] = \begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} &= (3-\lambda)[(1-\lambda)(2-\lambda) - 6] \\ &= (3-\lambda)(\lambda^2 - 3\lambda - 4) \\ &= (3-\lambda)(\lambda-4)(\lambda+1) \\ &= 0 \text{ when } \lambda = -1, 3, 4 \end{aligned}$$

Hence eigenvalues are $-1, 3, 4$.

Eigenvectors

$\boldsymbol{\lambda = -1}$: $\quad A - \lambda I = \begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$

Hence $(A-\lambda I)\boldsymbol{x}=\mathbf{0} \;\Rightarrow\; \boldsymbol{x}=\alpha\begin{bmatrix}-3\\1\\0\end{bmatrix}$ for any $\alpha\in\mathbb{R}$.

This is an eigenvector corresponding to $\lambda=-1$ for any $\alpha\neq 0$.

$$\boldsymbol{\lambda=3}: \qquad A-\lambda I=\begin{bmatrix}-2&6&1\\1&-1&0\\0&0&0\end{bmatrix}\sim\begin{bmatrix}-2&6&1\\0&4&1\\0&0&0\end{bmatrix}$$

Hence $(A-\lambda I)\boldsymbol{x}=\mathbf{0} \;\Rightarrow\; \boldsymbol{x}=\beta\begin{bmatrix}-1\\-1\\4\end{bmatrix}$ for any $\beta\in\mathbb{R}$.

This is the eigenvector for any $\beta\neq 0$.

$$\boldsymbol{\lambda=4}: \qquad A-\lambda I=\begin{bmatrix}-3&6&1\\1&-2&0\\0&0&-1\end{bmatrix}\sim\begin{bmatrix}-3&6&1\\0&0&+1\\0&0&-1\end{bmatrix}.$$

Hence $(A-\lambda I)\boldsymbol{x}=\mathbf{0} \;\Rightarrow\; \boldsymbol{x}=\gamma\begin{bmatrix}2\\1\\0\end{bmatrix}$ for any $\gamma\in\mathbb{R}$.

This is the eigenvector for any $\gamma\neq 0$.

(b)
$$\begin{aligned}
\det(A-\lambda I) &= \underset{r_1:=r_1+r_3}{\begin{vmatrix}-2-\lambda&-1&-5\\1&2-\lambda&1\\3&1&6-\lambda\end{vmatrix}} = \begin{vmatrix}1-\lambda&0&1-\lambda\\1&2-\lambda&1\\3&1&6-\lambda\end{vmatrix}\\
&= (1-\lambda)\underset{r_2:=r_2-r_1}{\begin{vmatrix}1&0&1\\1&2-\lambda&1\\3&1&6-\lambda\end{vmatrix}} = (1-\lambda)\begin{vmatrix}1&0&1\\0&2-\lambda&0\\3&1&6-\lambda\end{vmatrix}\\
&= (1-\lambda)[(2-\lambda)(6-\lambda)-3(2-\lambda)]\\
&= (1-\lambda)(2-\lambda)(3-\lambda)
\end{aligned}$$

$\Rightarrow\;\det(A-\lambda I)=0$ when $\lambda=1,2,3$.

Hence eigenvalues of A are 1, 2, 3.

Eigenvectors

$$\boldsymbol{\lambda=1}: \qquad A-\lambda I=\underset{\substack{r_2:=r_2+r_1\\ r_3:=r_3+r_1}}{\begin{bmatrix}-3&-1&-5\\1&1&1\\3&1&5\end{bmatrix}}\sim\begin{bmatrix}-3&-1&-5\\-2&0&-4\\0&0&0\end{bmatrix}$$

Solution for $\boldsymbol{x}=(x_1,x_2,x_3)$ is given by $x_3=\alpha$, $x_2=\alpha$, $x_1=-2\alpha$.

Hence eigenvector is $\boldsymbol{x} = \alpha(-2, 1, 1)$ for any $\alpha \neq 0$.

$$\boldsymbol{\lambda = 2}: \quad A - \lambda I = \begin{bmatrix} -4 & -1 & -5 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} -4 & -1 & -5 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -4 & -1 & -5 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Eigenvector is $\boldsymbol{x} = \beta(-1, -1, 1)$ for any $\beta \neq 0$.

$$\boldsymbol{\lambda = 3}: \quad A - \lambda I = \begin{bmatrix} -5 & -1 & -5 \\ 1 & -1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} -5 & -1 & -5 \\ 6 & 0 & 6 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} -5 & -1 & -5 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvector is $\boldsymbol{x} = \gamma(1, 0, -1)$ for any $\gamma \neq 0$.

$$\begin{aligned}
\text{(c)} \quad \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 2 & 2 & -4 \\ 2 & 4-\lambda & 2 & -1 \\ 1 & 1 & 3-\lambda & -1 \\ 2 & 2 & 2 & -\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & 2 & 2 & -4 \\ 0 & 2-\lambda & 2\lambda-4 & 1 \\ 1 & 1 & 3-\lambda & -1 \\ 0 & 0 & 2\lambda-4 & 2-\lambda \end{vmatrix} \\
& \quad\; \begin{matrix} r_2 := r_2 - 2r_3 \\ r_4 := r_4 - 2r_3 \end{matrix} \\
&= \begin{vmatrix} 4-\lambda & 2 & -6 & -4 \\ 0 & 2-\lambda & 2\lambda-2 & 1 \\ 1 & 1 & 1-\lambda & -1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 4-\lambda & 2 & -6 \\ 0 & 2-\lambda & 2\lambda-2 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\
& \qquad\qquad\qquad\qquad\qquad\qquad r_1 := r_1 + r_2 \\
&= (2-\lambda)(4-\lambda) \begin{vmatrix} 1 & 1 & -2 \\ 0 & 2-\lambda & 2\lambda-2 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\
& \qquad\qquad r_3 := r_3 - r_1 \\
&= (2-\lambda)(4-\lambda) \begin{vmatrix} 1 & 1 & -2 \\ 0 & 2-\lambda & 2\lambda-2 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\
&= (2-\lambda)^2(3-\lambda)(4-\lambda).
\end{aligned}$$

Hence eigenvalues are 2 (with algebraic multiplicity 2), 3, 4.

Eigenvectors

$$\boldsymbol{\lambda = 2}: \quad A - \lambda I = \begin{bmatrix} 2 & 2 & 2 & -4 \\ 2 & 2 & 2 & -1 \\ 1 & 1 & 1 & -1 \\ 2 & 2 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution for $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ is given by $x_4 = 0$,

$$x_3 = \alpha,\ x_2 = \beta,\ x_1 = -\alpha - \beta$$

Hence

$$\boldsymbol{x} = \alpha(-1, 0, 1, 0) + \beta(-1, 1, 0, 0)$$

The geometric multiplicity is 2 (2 linearly independent eigenvectors).

This is the dimension of $N(A - 2I)$.

$\boldsymbol{\lambda = 3}$ gives $\boldsymbol{x} = \gamma(-2, 4, 1, 2)$ and $\boldsymbol{\lambda = 4}$ gives $\boldsymbol{x} = \delta(0, 3, 1, 2)$.

6. The eigenvalues of A are 5 and 4 and the corresponding eigenvectors are $\boldsymbol{u}_1 = (1, 1)$ and $\boldsymbol{u}_2 = (1, 2)$. Hence $D = P^{-1}AP$, where

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},\ D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore $A = PDP^{-1}$ and $A^k = PD^kP^{-1}$. Now,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}.$$

Hence

$$A^k = PD^kP^{-1} = \begin{bmatrix} 2 \cdot 5^k - 4^k & 4^k - 5^k \\ 2(5^k - 4^k) & 2 \cdot 4^k - 5^k \end{bmatrix}.$$

7. The characteristic equation of A is

$$f(\lambda) = (\lambda - 2)^2(\lambda - 4) = 0.$$

Hence it has an eigenvalue $\lambda = 2$ of algebraic multiplicity 2. However, there is only one linearly independent eigenvector, $\boldsymbol{u} = (1, 0, 0)$ corresponding to this eigenvalue, so one cannot construct the 3×3 matrix of eigenvectors and the matrix A cannot therefore be diagonalised.

8. The eigenvalues of A are 4 and 1. Corresponding to to the eigenvalue $\lambda = 1$ we have two linearly independent eigenvectors $\boldsymbol{u}_1 = (1, 1, 0)$ and $\boldsymbol{u}_2 = (-1, 0, 1)$ and corresponding to $\lambda = 4$, we have an eigenvector $\boldsymbol{u}_3 = (1, 1, 1)$. Hence A is diagonalisable and the matrix P is given by

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Hence $A = PDP^{-1}$, where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

(Note: you can use the idea of Q4 to compute P^{-1}!) Therefore if $B^2 = A$, we should have $B = PXP^{-1}$ where $X^2 = D$. There are many (see below) choices of X, e.g.

$$X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } X_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

In the first case we have

$$B = PX_1P^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$

and in the second case

$$C = PX_2P^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & -3 & 2 \end{bmatrix}.$$

Please check that $B^2 = A$ and $C^2 = A$.

As there are 3 (positive) eigenvalues of A, there are $2^3 = 8$ choices of the matrices X_i and hence of square roots of the matrix A.