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Geometry and Linear Algebra
Chapter 4: 3D Geometry

One of the principal advantages of treating geometry in the plane algebraically is that we can immediately extend many results to higher dimensions. For example, in n dimensions a straight line can be written $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and an affine map can be represented in the form $A\mathbf{x} + \mathbf{p}$ where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ are vectors of length n and A is an $n \times n$ matrix.

Nevertheless, a number of concepts change when we have more dimensions to work with. In this chapter we briefly consider what the main similarities and main differences are when we move from two to three dimensions. We now require three coordinates to specify a point in the plane: $P(x, y, z)$ and three unit vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Relative to some fixed origin O, it is conventional to let the x, y and z directions represent position in the left-right, forward-backward and up-down senses, respectively.

4.1 The Straight Line

In all dimensions, the shortest path between any two points is a straight line. They remain axiomatic building blocks, but once we move beyond two dimensions they have less of a significant role to play, as the extra dimensions mean that we can work with more elaborate objects. Lines take up very little room in three dimensions, they no longer have one less dimension than space as a whole and so, for example, two straight lines will rarely intersect.

Of the three ways of representing a line in two dimensions, only the vector form extends to higher dimensions. We'll come back to generalisations of the implicit form when we consider planes in three dimensions, but the explicit form is of very limited use.

Given the position vector $\mathbf{a}$ of a point A and a vector $\mathbf{b}$, the vector form of the straight line through A travelling in the direction of $\mathbf{b}$ is still

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}.$$

The only difference is that $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$ each now have three components. Given two points P and Q the unique straight line through these points is

$$\left\{ \mathbf{x} \in \mathbb{R}^3 \,\middle|\, \mathbf{x} = \overrightarrow{OP} + t\overrightarrow{PQ}, t \in \mathbb{R} \right\},$$

although there are many different representations of the same line ($\overrightarrow{OQ} + t\overrightarrow{QP}$ would work).

We can derive a scalar form for a straight line from the vector form as follows. Suppose

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

and that $P(x, y, z)$ lies on $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. Then

$$x = a_1 + tb_1, \;\; y = a_2 + tb_2, \;\; z = a_3 + tb_3,$$

and rearranging gives

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}.$$

Thus our straight line is the set of all points for which these ratios hold. This is known as the **scalar parametric form** of a straight line. This extends in an obvious way to any dimension. In 2D we can rearrange it into the implicit or explicit form of a straight line.

Examples 4.1.1

(i) The line through $(1, -2, -4)$ parallel to $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ can be written in vector form as

$$\mathbf{r} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Its scalar parametric form is

$$\frac{x - 1}{3} = y + 2 = \frac{z + 4}{2}.$$

We can find if it crosses the axes by setting x, y and z to nought in the parametric form. Solving in turn

$$\begin{aligned} -\frac{1}{3} &= y + 2 = \frac{z+4}{2}, \\ 2 &= \frac{x-1}{3} = \frac{z+4}{2}, \\ 2 &= \frac{x-1}{3} = y + 2, \end{aligned}$$

gives these points as $(0, -7/3, -14/3)$ and $(7, 0, 0)$, hence the line crosses the x-axis.

(ii) The line through $(1, 2, 1)$ and $(3, 2, 3)$ can be written in many ways. For example, some of the equivalent forms are

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \text{ or } \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ or } \mathbf{r} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \text{ or } \mathbf{r} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Distance and Section Formula

The formula for the distance between two points extends directly to n dimensions. In particular, the distance between $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ is $d_{PQ} = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}$ or, in terms of position vectors, $d_{PQ} = \|\mathbf{p} - \mathbf{q}\|$.

To find the minimal distance between a point and a line we no longer use a formula. Instead we note that if the nearest point to P on $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ is Q then $\overrightarrow{PQ}$ meets the line at a right angle which can be written algebraically as $\overrightarrow{PQ} \cdot \mathbf{b} = 0$. Using the vector form of the line we can find the value of the parameter t which makes this true and hence find Q and d_{PQ}.

Example 4.1.2

- Find the minimum distance between $P(3, 2, 4)$ and $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

  If Q lies on $\mathbf{r}$ then $\overrightarrow{PQ} = \mathbf{a} - \overrightarrow{OP} + t\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. If $\overrightarrow{PQ}$ meets $\mathbf{r}$ at a right angle then $\overrightarrow{PQ} \cdot \mathbf{b} = 0$. So,

  $$3(-2 + 3t) + 2(2t) - 1(1 - t) = 0,$$

  hence $14t - 7 = 0$, or $t = 1/2$. We can deduce that the coordinates of Q are $(1 + 3t, 2 + 2t, 5 - t) = (5/2, 3, 9/2)$ and

  $$\|\overrightarrow{PQ}\| = \sqrt{\frac{1}{2^2} + 1^2 + \frac{1}{2^2}} = \sqrt{\frac{3}{2}}.$$

Example 4.1.3

The section formula for finding points part way along a line works in any dimension.

Example 4.1.4

- Find the point C on AB which is twice as far from A as it is from B when $A(3, 2, 7)$ and $B(9, 8, -5)$.

  Since $AC : CB = m : n = 2 : 1$, $\mathbf{c} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$ where $\lambda = m/(m + n) = 2/3$. Hence the coordinates of C are $(7, 6, -1)$.

Intersecting Lines

In two dimensions, all pairs of lines intersect at a unique point unless they are parallel. In three dimensions it is much more unusual for lines to intersect. It should be fairly clear to picture this idea geometrically: think of the ways you could position a pair of broom handles so that they don't meet. Algebraically, the intersection problem can be resolved as follows. Suppose that $\mathbf{r}_1 = \mathbf{a} + t\mathbf{b}$ and

$\mathbf{r}_2 = \mathbf{c} + t\mathbf{d}$. If the lines meet then there are parameter values t_1 and t_2 such that $\mathbf{a} + t_1\mathbf{b} = \mathbf{c} + t_2\mathbf{d}$. In other words,

$$\begin{aligned} b_1t_1 - d_1t_2 &= c_1 - a_1, \\ b_2t_1 - d_2t_2 &= c_2 - a_2, \\ b_3t_1 - d_3t_2 &= c_3 - a_3. \end{aligned}$$

With three equations to solve and only two variables, it is highly unlikely that a pair of lines chosen at random will satisfy this **overdetermined system**.

Example 4.1.5

- Determine whether $\mathbf{r}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ intersects $\mathbf{r}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

  Notice that the z-coordinate of points on $\mathbf{r}_2$ always equals zero, so the only possible point of intersection is the point where this is true for $\mathbf{r}_1$, at $(3, 3, 0)$. But when the y-coordinate of $\mathbf{r}_2$ is 3 it is at $(8, 3, 0)$, so the lines do not intersect.

Examples 4.1.6

4.2 Planes

Many of the roles of the straight line in two dimensions are inherited by the plane in three dimensions. While a unique straight line passes through any two distinct points, a plane can be uniquely determined by any **three** points, so long as they form a triangle. The vector form of a straight line can be generalised to a plane by adding a second parametrised direction to give

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} + u\mathbf{c}.$$

If P, Q and R form a triangle then they lie on the plane

$$\mathbf{r} = \overrightarrow{OP} + t\overrightarrow{PQ} + u\overrightarrow{PR}.$$

Different choices of P, Q and R can lead to very different looking vector forms of a plane.

A handy way of representing a plane is via the **Hessian form**,

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| ax + by + cz = d \right\},$$

or simply $ax + by + cz = d$. This gives a plane in three dimensions as we are placing a single constraint on a point, reducing our **degrees of freedom** by one and forcing it to lie in a two dimensional world.

Notice that the Hessian form generalises the implicit form of a straight line in two dimensions. In n dimensions it can be used to describe an $n-1$ dimensional hyperplane.

Observe that the Hessian form can be written $\mathbf{n} \cdot \mathbf{r} = d$, where

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Comparing the vector and Hessian form gives

$$\mathbf{n} \cdot (\mathbf{a} + t\mathbf{b} + u\mathbf{c}) = \mathbf{n} \cdot \mathbf{a} + t\mathbf{n} \cdot \mathbf{b} + u\mathbf{n} \cdot \mathbf{c} = d.$$

This can only be true **for all** values of t and u if $\mathbf{n} \cdot \mathbf{a} = d$ and $\mathbf{n} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{c} = 0$. Thus the vector $\mathbf{n}$ must lie at right angles to the directions defining the plane or, more simply, $\mathbf{n}$ makes a right angle with the plane. Such a vector is known as a **normal** direction and the Hessian form is also known as the normal form.

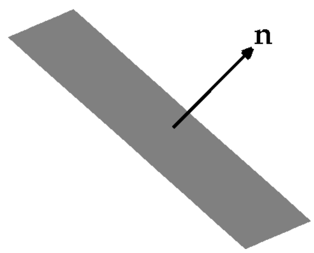


Figure 4.1: The normal to a plane.

Given two vectors in three dimensions we can find a normal vector by computing their **vector product**.

If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ then the vector product, $\mathbf{a} \times \mathbf{b}$, is given by the formula

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Thus given three points on the plane we can find its vector form and then its Hessian form.

Notice that the result of the vector product is a vector (while that of the scalar product is a scalar). Unlike the scalar product, it can only be applied to three dimensional vectors, although normal vectors to $n-1$ dimensional surfaces in n dimensional space can always be found (you should be able to do this in 2 dimensions, where you need a normal vector to a line). In high dimensions, the best way to find normal directions is to solve an appropriate **linear system**.

Vector products have applications in areas such as multivariate calculus and mechanics, so expect to see them in future courses.

Examples 4.2.1

(i) To confirm the normality of $\mathbf{n} = \mathbf{a} \times \mathbf{b}$, note that

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{n} &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\
&= a_1a_2b_3 - a_2a_1b_3 - a_1a_3b_2 + a_3a_1b_2 + a_2a_3b_1 - a_3a_2b_1 = 0, \\
\mathbf{b} \cdot \mathbf{n} &= b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) \\
&= a_2b_1b_3 - b_3a_2a_1 - b_1a_3b_2 + b_2a_3b_1 - b_2a_1b_3 + b_3a_1b_2 = 0.
\end{aligned}$$

(ii) Note that

$$\mathbf{b} \times \mathbf{a} = \begin{bmatrix} b_2a_3 - b_3a_2 \\ b_3a_1 - b_1a_3 \\ b_1a_2 - b_2a_1 \end{bmatrix} = -\mathbf{a} \times \mathbf{b}.$$

The vector product is not commutative, but as we're primarily interested in the direction (which doesn't change) this will not cause us any problems.

Examples 4.2.2

Recall that in two dimensions the distance of the point $P(x, y)$ from the line $ax + by = c$ is given by

$$\frac{|ax + by - c|}{\sqrt{a^2 + b^2}}.$$

In three dimensions, the (perpendicular) distance from (x, y, z) to the plane $ax + by + cz = d$ can be shown to be

$$\frac{|ax + by + cz - d|}{\sqrt{a^2 + b^2 + c^2}}, \tag{4.2.1}$$

and we can extend this in an obvious way to higher dimensions.

We can recognise parallel planes in Hessian form because their normal vectors lie in the same direction. The distance between parallel planes can be calculated using (**??**).

Example 4.2.3

- The distance from $(6, 2, -1)$ to $6x + 2y - z = 4$ is

$$\frac{|36 + 4 + 1 - 4|}{\sqrt{36 + 4 + 1}} = \frac{37}{\sqrt{41}}.$$

- Find the distance between the parallel planes $4x - 2y + 6z = 7$ and $2x - y + 3z = 8$.

  We note that $(4, 0, 0)$ lies on the second plane. Plugging these coordinates and the coefficients of the second plane into the distance formula gives the value

$$\frac{16 - 7}{\sqrt{16 + 4 + 36}} = \frac{9}{\sqrt{56}}.$$

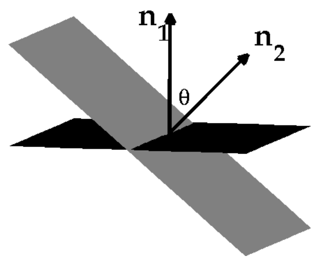


Figure 4.2: Angle between two planes.

Suppose two planes meet at an angle θ (we can always choose θ so that it lies between 0 and $\pi/2$). We can find this angle because it is the same angle at which the normals of the planes meet (see Figure **??**). Thus the problem of finding angles between planes reduces to that of finding the angle between two vectors.

Example 4.2.4

- Find the acute angle, θ, between the planes $x - 2y + 2z = 1$ and $3x + 4y - 12z = 5$.

  The normal directions are $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $\mathbf{m} = \begin{bmatrix} 3 \\ 4 \\ -12 \end{bmatrix}$, so,

$$\cos\theta = \frac{|\mathbf{m}\cdot\mathbf{n}|}{\|\mathbf{m}\|\|\mathbf{n}\|} = \frac{|3-8-24|}{\sqrt{9}\sqrt{169}} = \frac{29}{39},$$

  and $\theta = 0.73$ radians.

Intersections of Lines and Planes

We have already seen that lines rarely cross in three dimensions. Lines do generally intersect planes at a unique point, so long as the line is not parallel to the plane. To find the intersection point we can simply substitute the formula for a general point on the line into the Hessian form and solve the resulting problem.

Two planes meet in a line, unless those two planes are parallel. To find the intersecting line we can exploit the fact the it must be perpendicular to the normal directions of the two planes simultaneously. So we can use our formula for finding the normal direction to two vectors on the normal directions of the two planes. If we can then find a point on this line we have enough information to write down its vector form.

If we take three planes they will generally meet at a unique point. In the exceptional case, the three normal directions lie in the same plane and either the planes do not intersect or they share a common direction (see Figure **??**).

If there is a unique intersection point we can find it by solving a set of simultaneous equations. This can be expressed in the form

$$A\mathbf{x} = \mathbf{b}$$

Figure 4.3: Planes without a unique intersection point.

where A is the matrix whose rows are the normal directions of the three planes and the vector $\mathbf{b}$ contains the right hand side constants in their Hessian forms. This is an obvious generalisation of the intersecting line problem in two dimensions, and can be extended to higher dimensions. Solving **linear systems** of the form $A\mathbf{x} = \mathbf{b}$ is an important practical problem for large values of n. For example, the results provided by search engines such as Google require the solution of linear systems where n is around 10 billion. Luckily this doesn't have to be done for every single search! When solving linear systems with $n \geq 3$ it is best to avoid computing the matrix inverse. You may be familiar with the method of Gaussian elimination.

Examples 4.2.5

(i) Find the point P where the line $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ meets the plane $2x - y + z = 9$.

If P lies on $\mathbf{r}$ then its coordinates are $(2 - t, 1 + 3t, 2 + t)$. Substituting into the equation for the plane gives

$$2(2-t) - (1+3t) + (2+t) = 5 - 4t = 9,$$

hence $t = -1$ and the intersection point is $(3, -2, 1)$.

(ii) Find the line where the planes $2x - 2y + z = 6$ and $x - 4y - 3z = 0$ meet.

The "normal to the normals" is $\begin{bmatrix} 6+4 \\ 1+6 \\ -8+2 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ -6 \end{bmatrix}$. Now suppose this line passes through the plane $z = 0$. At this point,

$$\begin{cases} 2x - 2y = 6, \\ x - 4y = 0, \end{cases}$$

which is true when $x = 4$ and $y = 1$.

Thus the line of intersection passes through $(4, 1, 0)$ and since we know its direction its vector form is

$$\mathbf{r} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 10 \\ 7 \\ -6 \end{bmatrix}.$$

(iii) Show that the planes $2x - y = 3$, $4x + y + z = 2$ and $3x + y + z = 0$ meet at $(2, 1, -7)$.

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

Example 4.2.6

Exercises: Lines and Planes

1. Find the vector form of the lines through the following pairs of points.

   Try and make the constants involved as simple as possible.

   (a) $(0,0,0)$, $(1,1,1)$ (b) $(2,1,2)$, $(3,2,1)$ (c) $(1,5,-3)$, $(2,10,-6)$ (d) $(1,6,3)$, $(0,-2,1)$

2. Find, in vector form, a line which passes through the origin and is perpendicular to the line $\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$.

   How many such lines can you find?

3. Find the minimum distance between the following points and lines.

   (a) $(0,0,1)$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 13 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

   (b) $(-1,3,2)$ and $\mathbf{r} = \begin{bmatrix} -4 \\ 6 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

   (c) $(5,1,0)$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 13 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

4. Find (where possible) the value of α which ensures that

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

   intersects the following lines.

   (a) $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 13 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -10 \\ 4 \end{bmatrix}$

5. Find the Hessian form of the following planes.

   (a) $\mathbf{r} = \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} + u \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ (b) $\mathbf{r} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + u \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$

   (c) $\mathbf{r} = t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

6. Find the Hessian form of the plane that includes the points $(1,2,-1)$, $(3,1,5)$ and $(-1,-1,-1)$.

7. Find a vector form of the following planes. (a) $x - 3y + z = 0$ (b) $2x - 3y + z = 1$ (c) $x - 2y = 10$

8. Find the minimum distance between the following points and lines.

   (a) $(0, 0, 1)$ and $x - 3y + 7z = 1$

   (b) $(-1, 3, 2)$ and $4y - 2z = 1$

   (c) $(5, 1, 0)$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 13 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

9. Find the angle the following planes make with $x = 0$.

   (a) $2x - y + 3z = 0$ (b) $y + z = 9$ (c) $x = 10$

10. Find the points where the following lines meet the plane $x + y + z = 1$.

    (a) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ (c) $t \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

11. Find the lines where the following planes intersect.

    (a) $2x + 3y - z = 9$ and $x + 3y - z = 6$ (b) $3x - 2y + 4z = -1$ and $4x - y = 0$
    (c) $x - 4y + 3z = 7$ and $-3x + 12y - 9z = 0$ (d) $y + 3z = 0$ and $x - 3y = 0$

4.3 Transformations

The concepts behind affine transformations generalise directly from 2 to 3 (or more) dimensions. They are characterised geometrically by the same collinearity and ratio preserving rules introduced in Chapter 3 and algebraically as transformations of the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{p}$, where A is a 3×3 matrix and $\mathbf{p}$ a vector of length three.

When transforming shapes, we have an extra degree of freedom to work with, allowing us to map sets of four points onto any other four points. Recall that in two dimensions, where we had three points to play with, we could map any triangle onto any other triangle (or parallelogram to parallelogram). In three dimensions we can map any **tetrahedron** onto any other tetrahedron (or **parallelepiped** to parallelepiped).

Example 4.3.1

- Find an affine transformation, $f(\mathbf{x})$ to map the unit cube onto the parallelepiped with a vertex $P(2, 1, -3)$ connected to vertices $Q(1, -1, 4)$, $R(0, 1, 2)$ and $S(-2, 3, 1)$.

  Generalising the method of Chapter 3, we let $\mathbf{p} = \overrightarrow{OP}$ and choose as columns of A the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$. So

$$f(\mathbf{x}) = \begin{bmatrix} -1 & -2 & -4 \\ -2 & 0 & 2 \\ 7 & 5 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

Let $O(0, 0, 0)$, $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$. To map vertices P, Q, R and S onto W, X, Y and Z we find transformation f and g that map $OABC$ to $PQRS$ and $WXYZ$, respectively, and the map we want is $g \circ f^{-1}$. Finding inverses of 3×3 matrices is beyond the scope of this course, but you should appreciate how easy it is to generalise results from two dimensions to higher dimensions. A clear picture of what's going on in two dimensions provides insight to high dimensions, which are impossible to picture so explicitly.

Affine transformations in any dimension can be built up from a small set of simple operations such as translations, rotations, reflections and stretches. Let us now look for simple matrix representations of some of these, restricting ourselves to linear transformations.

Reflections

While we can define reflections thorough a line in three dimensions, the most common reflection is through a plane (for example, when we look through a mirror). The reflection through the plane $ax + by + cz = 0$ is given by the Householder matrix generated by its normal vector.

Examples 4.3.2

(i) A normal vector to the plane $3x + 4y - 5z = 0$ is

$$\mathbf{n} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}.$$

Thus vectors can be reflected through this plane using the Householder matrix

$$P = I - \frac{2}{\mathbf{n}^T\mathbf{n}}\mathbf{n}\mathbf{n}^T = I - \frac{2}{50}\begin{bmatrix} 9 & 12 & -15 \\ 12 & 16 & -20 \\ -15 & -20 & 25 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 16 & -12 & 15 \\ -12 & 9 & 20 \\ 15 & 20 & 0 \end{bmatrix}.$$

(ii) To reflect through the plane $3x - y = 0$ we use the Householder matrix generated by

$$\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix},$$

namely,

$$I - \frac{2}{10}\begin{bmatrix} 9 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compare this result with Example 3.2.9.

Reflections can also be made through lines. These can be represented by combining plane reflections whose planes meet at right angles along the desired axis of reflection. This gives a way of finding a matrix to represent the reflection.

Rotations

In three dimensions we rotate around an axis of rotation. Rotations around the coordinate axes are the simplest to picture, and can be described by matrices that look very similar to those we saw in two dimensions. To find the destination of points rotated through an angle θ around the x-axis we can use the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Similarly, to rotate around the y- and z-axes we use

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

All possible rotations can be described by composing rotations around each of the axes in turn through appropriate angles. Aeroplane pilots can describe their turns in terms of rotations in the x-, y- and z-axes. Navigators call these the roll, pitch and yaw.

Example 4.3.3

- We can represent a rotation with roll θ followed by pitch ϕ and then yaw ψ by

$$\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\cos\psi & -\cos\theta\sin\psi - \sin\theta\sin\phi\cos\psi & -\cos\theta\sin\phi\cos\psi + \sin\theta\sin\psi \\ \cos\phi\sin\psi & \cos\theta\cos\psi - \sin\theta\sin\phi\sin\psi & -\cos\theta\sin\phi\sin\psi - \sin\theta\cos\psi \\ \sin\phi & \sin\theta\cos\phi & \cos\theta\cos\phi \end{bmatrix}.$$

Note that while rotations in the plane commute, this is no longer the case when the axes of rotation vary.

Stretches and Shears

By simply expanding the dimensions of the matrices involved, we can generalise stretches and shears to higher dimensions. The diagonal matrix

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

represents a stretch which pulls in the x, y and z directions by factors s_x, s_y and s_z, respectively. Again, negative stretch factors indicate a simultaneous reflection and factors smaller than 1 in size represent shrinkages.

When we shear vectors in three dimensions, one component is kept fixed while the other two are varied in proportion to the fixed component. For example, a shear that fixes the z component of factor k in the x direction and l in the y direction moves the point (x, y, z) to $(x + kz, y + lz, z)$. This can be achieved with the matrix

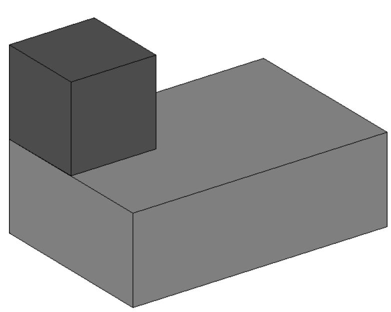
$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix}.$$

Examples 4.3.4

(i) A stretch by factors 2, 3 and -1 in the x, y and z directions respectively can be achieved by applying the matrix

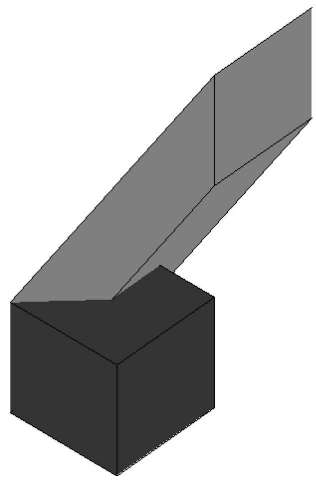
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It maps the unit cube onto a cuboid with vertices at $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, -1)$ all connected to the origin.



(ii) The matrix representation of the shear that maps (x, y, z) to $(x, x+y, 2x+z)$ is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$



4.3.1 Degenerate Transformations

As for two dimensions, degenerate transformations can be represented by singular matrices. Recall that degenerate transformations squash space so that it loses dimensions. The situation is complicated slightly as now a degenerate transformation can lose us one, two or all three dimensions. If a linear transformation maps us into k dimensional space then it is said to have rank k.

If a linear transformation has rank k then we can find the image of the transformation by identifying a k dimensional object that the columns of the associated matrix lie in. The image is called the **range space** of the transformation/matrix.

Examples 4.3.5

(i) The matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 4 \\ -2 & -9 & -15 \end{bmatrix}$$

is singular because the columns are related (the third column can be formed by adding three times the first to the second).

The linear transformation induced by this matrix maps all vectors onto the plane containing the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. To find the Hessian form of this plane we compute the normal vector

$$\mathbf{a}_1 \times \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -9 \end{bmatrix} = \begin{bmatrix} -9+2 \\ -4+9 \\ 1-2 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ -1 \end{bmatrix}.$$

Since the origin lies in this plane, the Hessian form is

$$-7x + 5y - z = 0.$$

For example,

$$A\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ -26 \end{bmatrix}, \quad A\mathbf{e}_2 = \begin{bmatrix} 2 \\ 1 \\ -9 \end{bmatrix},$$

and

$$-7 + 5 + 2 = -56 + 30 + 26 = -14 + 5 + 9 = 0.$$

(ii) All columns of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ 3 & 6 & -6 \end{bmatrix}$$

are multiples of the vector $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$.

The range of the transformation induced by A is the line $\mathbf{r} = t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

The Null Space

Can we account for the dimensions which disappear when we apply a degenerate transformation? In some sense we can. If $f(\mathbf{x}) = A\mathbf{x}$ is **not** degenerate then its action can be reversed by applying the inverse transformation $f^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$. Notice that this means that if $f(\mathbf{x}) = f(\mathbf{y})$ then $A\mathbf{x} = A\mathbf{y}$ and applying the inverse we have $\mathbf{x} = \mathbf{y}$. In other words, an invertible map is one-to-one.

If $f(\mathbf{x})$ is degenerate then we have no reason to expect it to be one-to-one. The matrix behind the transformation is singular, and we can show that there are many vectors which map to the same point. For example, using the matrix from Example 4.3.8(ii) we have

$$A\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Suppose that $B\mathbf{x} = B\mathbf{y}$. Then, using linearity $B(\mathbf{x} - \mathbf{y}) = B\mathbf{z} = \mathbf{0}$. For an invertible transformation we must have $\mathbf{z} = \mathbf{0}$, but for degenerate maps we have a collection of vectors which map to the zero vector. The set of vectors which map to zero under a linear transformation is called the **null space** of the transformation/matrix.

An important theorem which we won't prove (and is true in n dimensions, not just 3) is the following.

Theorem 4.3.1 *If a 3×3 matrix has rank k then its null space is $3 - k$ dimensional.*

Thus the null space accounts for the missing dimensions of a degenerate transformation.

Suppose $\mathbf{n}$ is in the null space of a matrix A and let $\mathbf{x}$ be any vector. Then $\mathbf{x} \cdot (A\mathbf{n}) = 0$ since $A\mathbf{n} = \mathbf{0}$. But notice that

$$0 = \mathbf{x} \cdot (A\mathbf{n}) = (A\mathbf{n}) \cdot \mathbf{x} = (A\mathbf{n})^T \mathbf{x} = \mathbf{n}^T (A^T \mathbf{x}) = \mathbf{n} \cdot (A^T \mathbf{x}).$$

Now as we vary $\mathbf{x}$, $A^T\mathbf{x}$ will pick out the range space of A^T. Since the inner product of $\mathbf{n}$ with these vectors is always zero (but $A^T\mathbf{x}$ is not always zero) we have proved the following theorem

Theorem 4.3.2 *The null space of a matrix A lies at right angles to the range space of A^T.*

We have seen how to compute the range space of a matrix. By computing the range space of A^T we can now deduce the null space of A.

Example 4.3.6

- To find the null space of
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 4 \\ -2 & -9 & -15 \end{bmatrix}$$
we need to find all vectors at right angles to the range space of A^T. We don't need to find the range space of A^T explicitly as we're looking for the space at right angles to it. This means we need to find the normal vector to the plane. That is, the null space of A is the line $\mathbf{r} = t\mathbf{n}$ where
$$\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$
Notice that the vector product is formed from the first two **rows** of A. To confirm,
$$A\mathbf{r} = tA\mathbf{n} = t \begin{bmatrix} 3+2-5 \\ 3+1-4 \\ -6-9+15 \end{bmatrix} = \mathbf{0}.$$

- All rows of the matrix
$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ 3 & 6 & -6 \end{bmatrix}$$
are multiples of the vector $\mathbf{n} = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}$. Thus the null space of A is the plane to which $\mathbf{n}$ acts as a normal, namely
$$x + 2y - 2z = 0.$$
We can find members of this null space by inspection. For example
$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$
all lie on this plane and
$$A\mathbf{u} = A\mathbf{v} = \begin{bmatrix} 2-2 \\ 4-4 \\ 6-6 \end{bmatrix} = A\mathbf{w} = \begin{bmatrix} 4+6-10 \\ 8+12-20 \\ 12+18-30 \end{bmatrix} = \mathbf{0}.$$

Exercises: Transformations

1. Find an affine transformation to map the unit cube onto the parallelepiped with a vertex P connected to vertices Q, R and S with the following coordinates.

   (a) $P(0,1,1)$, $Q(2,1,-1)$, $R(3,2,1)$, $S(3,3,3)$

   (b) $P(2,3,0)$, $Q(1,-1,2)$, $R(0,0,1)$, $S(1,2,1)$

   (c) $P(0,0,0)$, $Q(1,0,-1)$, $R(1,1,1)$, $S(0,1,1)$

   (d) $P(3,2,3)$, $Q(4,2,3)$, $R(3,3,3)$, $S(3,2,4)$

2. Find matrices whose effect is to reflect vectors through the following planes.

   (a) $2x - y = 0$ (b) $x = 0$ (c) $x + y + z = 0$ (d) $x - y - 2z = 0$

3. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ \mu & 1 & 0 \\ \nu & 0 & 1 \end{bmatrix}$.

   Compute AB and show that for an appropriate choice of μ and ν, $B = A^{-1}$.

4. Find the range and null space of the following singular matrices.

   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 7 & 8 & 15 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 5 & -1 & -8 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

4.4 Summary

Here is a list of skills you are expected to have picked up from this chapter.

1. Understand all terms in **bold face** (all sections).
2. Understand all the numbered definitions and statement of all numbered theorems and recognise the relevance of the numbered equations (all sections).
3. Work with lines in three dimensions in vector and scalar parametric form (§4.1).
4. Use the section and distance formula in 3D (§4.1 and §4.2).
5. Use the vector form of a straight line to determine intersection points of lines (if any) in 3D (§4.1).
6. Convert between the vector form and Hessian form of a plane (§4.2).
7. Recognise immediately when a plane or line passes through the origin (§4.1 and §4.2).
8. Use the vector product to find normal vectors to planes (§4.2).
9. Find intersections between planes (§4.2).
10. Find angles between planes (§4.2).
11. Use formulae to generate reflection, rotation and stretch matrices in 3D (§4.3).
12. Find the range space and null space of degenerate transformations (§4.3).