5. **(a)** ODE: $y'' - y' - 6y = 0$;

A.E.: $m^2 - m - 6 = 0$, with solutions $m = -2,\ 3$, so the G.S. is $y(x) = Ae^{-2x} + Be^{3x}$.

(b) ODE: $y'' + 4y' + 3y = 0$;

A.E.: $m^2 + 4m + 3 = 0$, with solutions $m = -1, -3$, so the G.S. is $y(x) = Ae^{-x} + Be^{-3x}$.

(c) ODE: $y'' - 4y' + 4y = 0$;

A.E.: $m^2 - 4m + 4 = 0$, i.e. $(m-2)^2 = 0$, with solution $m = 2$ (twice), so the G.S. is $y(x) = (Ax + B)e^{2x}$.

(d) ODE: $y'' - 2y' + 17y = 0$;

A.E.: $m^2 - 2m + 17 = 0$, with solution
$m = \dfrac{2 \pm \sqrt{4 - 68}}{2} = \dfrac{2 \pm 8i}{2} = 1 \pm 4i$, so the G.S. is $y(x) = e^x(A\cos 4x + B\sin 4x)$.

(e) ODE: $y'' + 8y' + 16y = 0$;

A.E.: $m^2 + 8m + 16 = 0$, i.e. $(m+4)^2 = 0$, with solution $m = -4$ (twice), so the G.S. is $y(x) = (Ax + B)e^{-4x}$.

If $y(0) = 1$ then $(A \times 0 + B) \times 1 = 1 \implies B = 1$.

If $y(1) = 0$ then $(A + B)e^{-4} = 0 \implies A = -1 \implies y(x) = (1 - x)e^{-4x}$.

(f) ODE: $y'' + 2y' = 0$;

A.E.: $m^2 + 2m = 0$, i.e. $m(m+2) = 0$, with solutions $m = 0, -2$, so the G.S. is $y(x) = Ae^{0x} + Be^{-2x} = A + Be^{-2x}$.

Hence, $y'(x) = -2Be^{-2x}$. If $y(0) = 1$ then $A + B = 1$.

If $y'(0) = 2$ then $-2B = 2 \implies B = -1,\ A = 2$. Thus, $y(x) = 2 - e^{-2x}$.

Qu. 5 cont'd next sheet

5. (g) ODE: $y'' + 9y = 0$;

A.E.: $m^2 + 9 = 0$, i.e. $m^2 = -9$, with solutions $m = \pm 3i$, so the G.S. is $y(x) = A\cos(3x) + B\sin(3x)$.

Hence, $y'(x) = -3A\sin(3x) + 3B\cos(3x)$.

If $y(0) = 1$ then $A \times 1 + B \times 0 = 1 \quad \Longrightarrow \quad A = 1$.

If $y'\left(\frac{\pi}{3}\right) = 6$ then $-3A \times 0 + 3B \times (-1) = 6 \quad \Longrightarrow \quad B = -2$.

Thus, $y(x) = \cos(3x) - 2\sin(3x)$.

6. (a) ODE: $y'' - 4y' + 3y = 1$;

A.E.: $m^2 - 4m + 3 = 0$, i.e. $(m-1)(m-3) = 0$

$\Longrightarrow \quad m = 1, 3$, so the C.F. is $y_{\text{CF}}(x) = Ae^x + Be^{3x}$.

RHS of the ODE is a polynomial of degree 0, so as P.I. try $y_{\text{PI}}(x) = P$ (a constant); then $y'_{\text{PI}} = 0$, $y''_{\text{PI}} = 0$.

Substitute in ODE: $0 - 0 + 3P = 1 \quad \Longrightarrow \quad P = \dfrac{1}{3}$.

Therefore G.S. is $y_{\text{GS}}(x) = y_{\text{CF}}(x) + y_{\text{PI}}(x) = Ae^x + Be^{3x} + \dfrac{1}{3}$.

(b) ODE: $y'' + 2y' + y = x^2$;

A.E.: $m^2 + 2m + 1 = 0$, i.e. $(m+1)^2 = 0$, so that $m = -1$ (twice), and the C.F. is $y_{\text{CF}} = (Ax + B)e^{-x}$.

RHS of the ODE is a polynomial of degree 2, so as P.I. try $y_{\text{PI}} = Px^2 + Qx + R$; then $y'_{\text{PI}} = 2Px + \beta$, $y''_{\text{PI}} = 2P$.

Substitute in ODE: $2P + 2(2Px + \beta) + (Px^2 + Qx + R) = x^2$.

Coefficient of x^2 : $P = 1$

Coefficient of x^1 : $4P + Q = 0 \quad \Longrightarrow \quad Q = -4$

Constant term : $2P + 2Q + R = 0 \quad \Longrightarrow \quad R = 6$.

So the P.I. is $y_{\text{PI}} = x^2 - 4x + 6$ and the G.S. is now

$y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = (Ax + B)e^{-x} + x^2 - 4x + 6$.

(c) ODE: $y'' + 6y' + 9y = e^{-x}$;

A.E.: $m^2 + 6m + 9 = 0$, i.e. $(m+3)^2 = 0$ so that $m = -3$ (twice), and the C.F. is $y_{\text{CF}} = (Ax + B)e^{-3x}$.

As P.I. try $y_{\text{PI}} = Pe^{-x}$; then $y'_{\text{PI}} = -Pe^{-x}$, $y''_{\text{PI}} = Pe^{-x}$.

Substitute in ODE:

$Pe^{-x} - 6Pe^{-x} + 9Pe^{-x} = e^{-x} \quad \Longrightarrow \quad 4P = 1 \quad \Longrightarrow \quad P = \dfrac{1}{4} \quad \Longrightarrow \quad y_{\text{PI}} = \dfrac{1}{4}e^{-x}$.

$$\Longrightarrow \quad \text{G.S. is } y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = (Ax + B)e^{-3x} + \frac{1}{4}e^{-x}.$$

Qu. 6 cont'd next sheet

6. (d) ODE: $y'' + 2y' + 2y = 17e^{3x}$;

A.E.: $m^2 + 2m + 2 = 0$, so that $m = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$, so the C.F. is $y_{\mathrm{CF}} = e^{-x}(A\cos x + B\sin x)$.

As P.I. try $y_{\mathrm{PI}} = Pe^{3x}$; then $y'_{\mathrm{PI}} = 3Pe^{3x}$, $y''_{\mathrm{PI}} = 9Pe^{3x}$.

Substitute in ODE: $9Pe^{3x} + 6Pe^{3x} + 2Pe^{3x} = 17e^{3x} \implies 17P = 17 \implies P = 1$.

$$y_{\mathrm{PI}} = e^{3x} \implies \text{G.S. is } y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}} = e^{-x}(A\cos x + B\sin x) + e^{3x}.$$

(e) ODE: $y'' + y' + y = \cos x + \sin x$;

A.E.: $m^2 + m + 1 = 0$, so that $m = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$, and the C.F. is $y_{\mathrm{CF}} = e^{-x/2}\Big[A\cos\Big(\dfrac{\sqrt{3}}{2}x\Big) + B\sin\Big(\dfrac{\sqrt{3}}{2}x\Big)\Big]$.

As P.I. try $y_{\mathrm{PI}} = P\cos x + Q\sin x$; then $y'_{\mathrm{PI}} = -P\sin x + Q\cos x$ and $y''_{\mathrm{PI}} = -P\cos x - Q\sin x$.

Substitute into ODE:

$$(-P\cos x - Q\sin x) + (-P\sin x + Q\cos x) + (P\cos x + Q\sin x) = \cos x + \sin x.$$

Coefficient of $\cos x$: $Q = 1$

Coefficient of $\sin x$: $-P = 1 \implies P = -1$

$\implies y_{\mathrm{PI}} = -\cos x + \sin x$.

Therefore G.S. is $y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}} = e^{-x/2}\Big[A\cos\Big(\dfrac{\sqrt{3}}{2}x\Big) + B\sin\Big(\dfrac{\sqrt{3}}{2}x\Big)\Big] - \cos x + \sin x$.

(f) ODE: $y'' - 2y' + 5y = \sin(2x)$;

A.E.: $m^2 - 2m + 5 = 0$, so that $m = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$, and C.F. is $y_{\mathrm{CF}} = e^{x}[A\cos(2x) + B\sin(2x)]$.

As P.I. try $y_{\mathrm{PI}} = P\cos(2x) + Q\sin(2x)$; then $y'_{\mathrm{PI}} = -2P\sin(2x) + 2Q\cos(2x)$ and $y''_{\mathrm{PI}} = -4P\cos(2x) - 4Q\sin(2x)$. Substitute $y = y_{\mathrm{PI}}$ into ODE:

$$-4P\cos(2x) - 4Q\sin(2x) - 2[-2P\sin(2x) + 2Q\cos(2x)] + 5[P\cos(2x) + Q\sin(2x)] = \sin(2x).$$

Coefficient of $\cos(2x)$: $P - 4Q = 0$

Coefficient of $\sin(2x)$: $4P + Q = 1$

$\implies P = \dfrac{4}{17}$, $Q = \dfrac{1}{17} \implies y_{\mathrm{PI}} = \dfrac{1}{17}[4\cos(2x) + \sin(2x)]$

$$\implies \text{G.S. is } y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}} = e^{x}[A\cos(2x) + B\sin(2x)] + \frac{1}{17}[4\cos(2x) + \sin(2x)].$$

Qu. 6 cont'd next sheet

6. (g) ODE: $y'' - 6y' + 25y = 50x + 13 + 16e^{-x}$;

A.E.: $m^2 - 6m + 25 = 0$, so that $m = \dfrac{6 \pm \sqrt{36-100}}{2} = \dfrac{6 \pm 8i}{2} = 3 \pm 4i$,

and C.F. is $y_{\text{CF}} = e^{3x}[A\cos(4x) + B\sin(4x)]$.

As P.I. try $y_{\text{PI}} = Px + Q + Re^{-x}$; then $y'_{\text{PI}} = P - Re^{-x}$, $y''_{\text{PI}} = Re^{-x}$.

Substitute in ODE: $Re^{-x} - 6(P - Re^{-x}) + 25(Px + Q + Re^{-x}) = 50x + 13 + 16e^{-x}$.

Coefficient of e^{-x} : $32R = 16 \implies R = \dfrac{1}{2}$

Coefficient of x : $25P = 50 \implies P = 2$

constant term: $-6P + 25Q = 13 \implies Q = 1 \implies y_{\text{PI}} = 2x + 1 + \dfrac{1}{2}e^{-x}$.

Therefore G.S. is $y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = e^{3x}[A\cos(4x) + B\sin(4x)] + 2x + 1 + \dfrac{1}{2}e^{-x}$.

(h) ODE: $y'' + 4y' + 13y = 52 + 12\sin x + 4\cos x$;

A.E.: $m^2 + 4m + 13 = 0 \implies m = \dfrac{-4\sqrt{16-52}}{2} = \dfrac{-4 \pm 6i}{2} = -2 \pm 3i$,

so the C.F. is $y_{\text{CF}} = e^{-2x}[A\cos(3x) + B\sin(3x)]$.

As P.I. try $y_{\text{PI}} = P + Q\cos x + R\sin x$; then $y'_{\text{PI}} = -Q\sin x + R\cos x$,

$y''_{\text{PI}} = -Q\cos x - R\sin x$.

Substitute $y = y_{\text{PI}}$ into the ODE: (with $c \equiv \cos x$, $s \equiv \sin x$)

$$(-Qc - Rs) + 4(-Qs + Rc) + B(P + Qc + Rs) = 52 + 12s + 4c \quad (c := \cos x, s := \sin x)$$

Coefficient of c : $12Q + 4R = 4$

Coefficient of s : $-4Q + 12R = 12$

Constant term: $13P = 52 \implies P = 4 \implies Q = 0, \quad R = 1$.

Therefore $y_{\text{PI}} = 4 + \sin x$, and the G.S. is

$$y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = e^{-2x}[A\cos(3x) + B\sin(3x)] + 4 + \sin x.$$

(i) $y'' + 4\,y' + 3\,y = 13\,\cos(2x)$.

A.E.: $m^2 + 4m + 3 = 0 \iff m = -1$ or -3.

So C.F. is $y_{\text{CF}} = A\,e^{-x} + B\,e^{-3x}$ for arbitrary constants A and B.

As $f(x) = 13\,\cos(2x)$, for the P.I. try $y_{\text{PI}} = C\cos(2x) + \underline{\underline{D\sin(2x)}}$ for constants C, D. Substituting the P.I. into the ODE:

$$(-4C + 8D + 3C)\cos(2x) + (-4D - 8C + 3D)\sin(2x) = 13\,\cos(2x).$$

Equating the coefficients of $\cos(2x)$ and $\sin(2x)$:

$$-C + 8D = 13 \qquad \text{and} \qquad -8C - D = 0.$$

Therefore $C = -\dfrac{1}{5}$, $D = \dfrac{8}{5}$.

So the General Solution is $y_{\text{GS}} = A\,e^{-x} + B\,e^{-3x} + \dfrac{1}{5}\left[8\,\sin(2x) - \cos(2x)\right]$.

7. **(a)** ODE: $y'' + y' - 6y = e^{2x}$;

A.E.: $m^2 + m - 6 = 0$, with solutions $m = -3, 2$, so the C.F. is $y_{\text{CF}} = Ae^{-3x} + Be^{2x}$.

P.I. is $y_{\text{PI}} = Pxe^{2x}$ since 2 is a simple (non-repeated) root of A.E.

Then $y'_{\text{PI}} = P(2xe^{2x} + e^{2x})$, $y''_{\text{PI}} = P(4xe^{2x} + 4e^{2x})$.

Substitute into the ODE:

$$(P(4xe^{2x} + 4e^{2x}) + P(2xe^{2x} + e^{2x}) - 6Pxe^{2x} = e^{2x} \quad \Longrightarrow \quad 5Pe^{2x} = e^{2x}$$

$$\Longrightarrow \quad P = 1/5, \qquad \text{and} \quad y_{\text{PI}} = \frac{1}{5}xe^{2x}$$

Therefore G.S. is $y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = Ae^{-3x} + Be^{2x} + \dfrac{1}{5}xe^{2x}$.

(b) ODE: $y'' + 6y' + 9y = 4e^{-3x}$;

A.E.: $m^2 + 6m - 9 = 0$, with solutions $m = -3$ (twice), so the C.F. is $y_{\text{CF}} = (Ax + B)e^{-3x}$.

P.I. is $y_{\text{PI}} = Px^2e^{-3x}$ since 3 is a repeated root of the A.E.

Then $y'_{\text{PI}} = P(-3x^2e^{-3x} + 2xe^{-3x})$, $y''_{\text{PI}} = P(9x^2e^{-3x} - 12xe^{-3x} + 2e^{-3x})$.

Substitute $y = y_{\text{PI}}$ into ODE:

$P(9x^2 - 12x + 2)e^{-3x} + 6P(-3x^2 + 2x)e^{-3x} + 9Px^2e^{-3x} = 4e^{-3x} \quad \Longrightarrow \quad 2Pe^{-3x} = 4e^{-3x}$

$\Longrightarrow \quad P = 2, \qquad$ and $y_{\text{PI}} = 2x^2e^{-3x}$

Therefore G.S. is $y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = (Ax + B)e^{-3x} + 2x^2e^{-3x} = (2x^2 + Ax + B)e^{-3x}$.

(c) ODE: $y'' + 25y = 20\cos(5x)$;

A.E.: $m^2 + 25 = 0$, i.e. $m^2 = -25$ with solutions $m = \pm 5i$, so the C.F. is $y_{\text{CF}} = A\cos(5x) + B\sin(5x)$.

For P.I. try $y_{\text{PI}} = x[P\cos(5x) + \beta\sin(5x)]$; then (with $s = \sin(5x)$ and $c = \cos(5x)$)

$$y'_{\text{PI}} = x(-5Ps + 5Qc) + Pc + Qs, \quad y''_{\text{PI}} = x(-25Pc - 25Qs) - 10Ps + 10Qc,$$

Substitute $y = y_{\text{PI}}$ into the ODE:

$x(-25Pc - 25Qs) - 10Ps + 10Qc + 25x(Pc + Qs) = 20c \quad \Longrightarrow \quad -10Ps + 10Qc = 20c$

Coefficient of s: $-10P = 0$. $\qquad$ Coefficient of c: $10Q = 20 \quad \Longrightarrow \quad Q = 2$.

Therefore $y_{\text{PI}} = 2x\sin(5x)$, and the G.S. is

$$y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = A\cos(5x) + B\sin(5x) + 2x\sin(5x).$$

Qu. 7 cont'd next sheet

7. (d) ODE: $y'' + y' = 1$;

A.E.: $m^2 + m = 0$, i.e. $m(m+1) = 0$, with solutions $m = 0, -1$, so the C.F. is $y_{\mathrm{CF}} = Ae^{\circ} + Be^{-x}$, i.e. $y_{\mathrm{CF}} = A + Be^{-x}$.

P.I. is $y_{\mathrm{PI}} = Px$ (RHS of the ODE is a polynomial of degree 0, but the coefficient of y is 0). Then $y'_{\mathrm{PI}} = P$, $y''_{\mathrm{PI}} = 0$. Substitute $y = y_{\mathrm{PI}}$ into the ODE:

$0 + P = 1 \implies P = 1 \implies y_{\mathrm{PI}} = x$

Therefore the G.S. is $y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}} = A + Be^{-x} + x$.

(e) ODE: $y'' - y = e^x + \frac{1}{2}x$;

A.E.: $m^2 - 1 = 0 \implies m = \pm 1$, so the C.F. is $y_{\mathrm{CF}} = Ae^x + Be^{-x}$.

Since 1 is a simple (non-repeated) root of the

A.E., for the P.I. try $y_{\mathrm{PI}} = Pxe^x + Qx + R$. Then $y'_{\mathrm{PI}} = P(xe^x + e^x) + \beta$, $y''_{\mathrm{PI}} = P(xe^x + 2e^x)$. Substitute into the ODE:

$$P(x+2)e^x - (Pxe^x + Qx + R) = e^x + \frac{1}{2}x \implies 2Pe^x - Qx - R = e^x + \frac{1}{2}x.$$

$$\implies P = \frac{1}{2},\ Q = -\frac{1}{2},\ R = 0, \qquad \text{and} \quad y_{\mathrm{PI}} = \frac{1}{2}xe^x - \frac{1}{2}x.$$

Therefore G.S. is $y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}} = Ae^x + Be^{-x} + \frac{1}{2}x(e^x - 1)$.

$y(0) = 0 \implies A + B = 0$, $\quad y(1) = 0 \implies Ae + B/e + \frac{1}{2}(e-1) = 0$.

Solve to find $A = \dfrac{-e}{2(e+1)}$, $B = \dfrac{e}{2(e+1)}$.

Therefore the Particular solution is

$$y(x) = \frac{-e \times e^x}{2(e+1)} + \frac{e \times e^{-x}}{2(e+1)} + \frac{1}{2}x(e^x - 1) = \frac{-e}{e+1}\sinh x + \frac{1}{2}x(e^x - 1).$$

Qu. 7 cont'd next sheet

7. (f) ODE: $y'' - 9y = 12\cosh(3x) = 6e^{3x} + 6e^{-3x},;$

A.E.: $m^2 - 9 = 0 \quad \Longrightarrow \quad m = \pm 3,$ so the C.F. is $y_{\text{CF}} = Ae^{3x} + Be^{-3x}$.
As P.I. try $y_{\text{PI}} = Pxe^{3x} + Qxe^{-3x};$ then $y'_{\text{PI}} = P(3x+1)e^{3x} + \beta(-3x+1)e^{-3x}$,
$y''_{\text{PI}} = P(9x+6)e^{3x} + Q(9x-6)e^{3x}$. Substitute $y = y_{\text{PI}}$ into ODE:

$$P(9x+6)e^{3x} + Q(9x-6)e^{-3x} - 9Pxe^{3x} - 9Qxe^{-3x} = 6e^{3x} + 6e^{-3x},$$

$$\text{i.e.}\quad 6Pe^{3x} - 6Qe^{-3x} = 6e^{3x} + 6e^{-3x}.$$

$$\Longrightarrow \quad P = 1,\ Q = -1, \qquad \text{so} \quad y_{\text{PI}} = xe^{3x} - xe^{-3x}.$$

Therefore G.S. is
$y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = Ae^{3x} + Be^{-3x} + xe^{3x} - xe^{-3x} = (A+x)e^{3x} + (B-x)^{-3x}$

$$\Longrightarrow \quad y'_{\text{GS}} = (3A + 3x + 1)e^{3x} + (-3B + 3x - 1)e^{-3x}$$

$y(0) = 0 \quad \Longrightarrow \quad A + B = 0, \qquad y'(0) = 0 \quad \Longrightarrow \quad 3A + 3B = 0.$
Solve to find $A = 0,\ B = 0$.
So the Particular solution is $y(x) = xe^{3x} - xe^{-3x} = 2x\sinh(3x)$.

8. A.E.: $m^2 - 2m + 5 = 0 \iff m = \dfrac{2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2} = 1 \pm 2i.$

Therefore C.F. is $y_{\text{CF}} = e^x(A\cos 2x + B\sin 2x)$ for arbitrary constants A and B.

For the complex particular integral try $y = Ce^{(3+i)x}$ for some complex number C.

$$\frac{dy}{dx} = (3+i)C\,e^{(e+i)x}, \qquad \frac{d^2y}{dx^2} = (3+i)^2 C\,e^{(3+i)x} = (8+6i)C\,e^{(3+i)x}.$$

Substitute into the complex version of the ODE and compare coefficients:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = \big((8+6i)C - 2(3+i)C + 5C\big)\,e^{(3+i)x} = 65e^{(3+i)x}$$

$$\implies \quad C(8 + 6i - 6 - 2i + 5) = 65$$

$$\implies \quad C = \frac{65}{7+4i} \times \frac{7-4i}{7-4i} = \frac{65(7-4i}{49+16} = 7 - 4i.$$

Therefore the complex P.I. is $y = (7-4i)e^{(3+i)x}$.

Since $65e^{3x}\cos x$ is the real part of $65e^{(3+i)x}$, the P.I. for the original ODE is

$$\begin{aligned} y_{\text{PI}} &= \text{Re}\big((7-4i)\,e^{3x}\,(\cos x + i\sin x)\big) = \text{Re}\big(e^{3x}(7\cos x + 4\sin x) + ie^{3x}(7\sin x - 4\cos x)\big) \\ &= e^{3x}(7\cos x + 4\sin x). \end{aligned}$$

The General Solution of the original (real) differential equation is

$$y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = e^x(A\cos 2x + B\sin 2x) + e^{3x}(7\cos x + 4\sin x).$$

9. A.E.: $\quad m^2+2m+5 \;=\; 0 \quad\iff\quad m \;=\; \dfrac{-2\pm\sqrt{4-4\times1\times5}}{2} \;=\; -1\pm 2i.$

Therefore C.F. is $\;y_{\text{CF}} \;=\; e^{-x}(A\cos 2x + B\sin 2x)\;$ for arbitrary constants A and B.

For complex particular integral try $\;y \;=\; Ce^{(1-i)x}\;$ for some complex number C.

$$\frac{dy}{dx} \;=\; C(1-i)e^{(1-i)x}, \qquad\qquad \frac{d^2y}{dx^2} \;=\; C(1-i)^2e^{(1-i)x} \;=\; -2iCe^{(1-i)x}.$$

Substitute into the complex version of the ODE and compare coefficients:

$$\begin{aligned}
\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y \;&=\; Ce^{(1-i)x}\,(-2i+2-2i+5) \;=\; 195\,e^{(1-i)x} \\
\Longrightarrow \qquad C(7-4i) \quad &\quad 195 \\
\Longrightarrow \qquad C \;=\; \frac{195}{7-4i}\times\frac{7+4i}{7+4i} \;&=\; \frac{195(7+4i)}{49+16} \;=\; (21+12i).
\end{aligned}$$

Therefore the complex P.I. is $\;y \;=\; (21+12i)e^{(1-i)x}$.

Since $\;-195e^x\sin x\;$ is the imaginary part of $\;195e^{(1-i)x}$, the P.I. for the original ODE is

$$\begin{aligned}
y_{\text{PI}} \;=\; \operatorname{Im}\big((21+12i)e^x(\cos x - i\sin x)\big) \;&=\; \operatorname{Im}\big(e^x(21\cos x+12\sin x)+ie^x(12\cos x-21\sin x)\big) \\
&=\; e^x(12\cos x-21\sin x).
\end{aligned}$$

The General Solution of the original (real) differential equation is

$$y_{\text{GS}} \;=\; y_{\text{CF}}+y_{\text{PI}} \;=\; e^{-x}(A\cos 2x+B\sin 2x) \;+\; e^x(12\cos x-21\sin x).$$

10. ODE: $\;\dfrac{dy}{dx}=\dfrac{y}{2x}-\dfrac{x}{2y}\;$ is homogeneous.

Set $y=vx$: $\quad x\dfrac{dv}{dx}+v=\dfrac{vx}{2x}-\dfrac{x}{2vx}=\dfrac{v}{2}-\dfrac{1}{2v}\,.$

Therefore $\quad x\dfrac{dv}{dx}=\dfrac{v}{2}-\dfrac{1}{2v}-v=-\dfrac{v}{2}-\dfrac{1}{2v}=-\dfrac{(v^2+1)}{2v}\,,\;$ which is separable

$$\Longrightarrow \quad \int\frac{2v}{1+v^2}\,dv=-\int\frac{dx}{x} \quad\Longrightarrow\quad \ln(1+v^2)=-\ln|x|+A.$$

$$\Longrightarrow \quad \ln\!\Big(1+\frac{y^2}{x^2}\Big)=-\ln|x|+A \qquad \text{(arbitrary constant } A\text{)}$$

$y(1)=0 \quad\Longrightarrow\quad \ln 1=-\ln 1+A \quad\Longrightarrow\quad A=0 \quad$ (since $\ln 1=0$).

$$\Longrightarrow \quad \ln\!\Big(1+\frac{y^2}{x^2}\Big)=-\ln x=\ln x^{-1} \quad\Longrightarrow\quad 1+\frac{y^2}{x^2}=\frac{1}{x} \quad\Longrightarrow\quad y^2=x-x^2.$$

11. ODE: $y' + 3x^2 y = e^{-x^3}$ is linear, with I.F. $I(x) = \exp\left(\int 3x^2 dx\right) = \exp(x^3) = e^{x^3}$.

$$e^{x^3}\ y = \int e^{x^3} \times e^{-x^3}\ dx = \int dx = x + A.$$

$$\text{Therefore} \quad y = (x+A)e^{-x^3} \quad \Longrightarrow \quad y' = e^{-x^3} - 3x^2(x+A)e^{-x^3}.$$

$y' = 0$ when $x = 1 \quad \Longrightarrow \quad 0 = e^{-1} - 3(1+A)e^{-1} \quad \Longrightarrow \quad 1 + A = \dfrac{1}{3}, \quad$ i.e. $A = -\dfrac{2}{3}$.

So the required solution is $\quad y(x) = \left(x - \dfrac{2}{3}\right)e^{-x^3}$.

12. ODE: $y'' - 3y' = e^{3x} - 2y$, i.e. $y'' - 3y' + 2y = e^{3x}$, has A.E. $m^2 - 3m + 2 = 0$

$\Longrightarrow \quad m = 1,\ 2,$ so C.F. is $y_{\text{CF}} = Ae^x + Be^{2x}$.

As P.I. try $y_{\text{PI}} = Pe^{3x}$; then $y'_{\text{PI}} = 3Pe^{3x}$, $y''_{\text{PI}} = 9Pe^{3x}$.

Substitute in the ODE: $\quad 9Pe^{3x} - 9Pe^{3x} + 2Pe^{3x} \quad \Longrightarrow \quad P = \dfrac{1}{2}, \quad$ and $\quad y_{\text{PI}} = \dfrac{1}{2}e^{3x}$.

Therefore G.S. is $\quad y = y_{\text{CF}} + y_{\text{PI}} = Ae^x + Be^{2x} + \dfrac{1}{2}e^{3x}$.

$y(0) = 0 \quad \Longrightarrow \quad A + B + \dfrac{1}{2} = 0$

$y(\ln 2) = 0 \quad \Longrightarrow \quad 2A + 4B + 4 = 0.$

So $A = 1,\ B = -\dfrac{3}{2}, \quad$ and the Particular Solution is $\quad y(x) = e^x - \dfrac{3}{2}e^{2x} + \dfrac{1}{2}e^{3x}$.

13. ODE: $x = e^{x+y}y' = e^x \times e^y \times \dfrac{dy}{dx}$, which is is separable.

$$\int xe^{-x}dx = \int e^y dy \quad \Longrightarrow \quad e^y = -(x+1)e^{-x} + A.$$

As $x \to \infty$ we have $y \to 0$, $e^y \to 1$, $(x+1)e^{-x} \to 0$, so that $A = 1$.

Therefore the Particular Solution is $\quad e^y = 1 - (x+1)e^{-x}$.

14. ODE: $y'' + 2y' - 10 = \sin 3x - 5y$ (i.e. $y'' + 2y' + 5y = 10 + \sin 3x$);

A.E.: $m^2 + 2m + 5 = 0 \implies m = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$, so the C.F. is $y_{\text{CF}} = e^{-x}[A\cos(2x) + B\sin(2x)]$.

As P.I. try $y_{\text{PI}} = P + Q\cos(3x) + R\sin(3x)$; then $y'_{\text{PI}} = -3Qs + 3Rc$, $y''_{\text{PI}} = -9Qc - 9Rs$, where $c = \cos(3x)$, $s = \sin(3x)$. Substitute into ODE:

$$-9Qc - 9Rs + 2(-3Qs + 3Rc) + 5(P + Qc + Rs) = 10 + s$$

Coefficient of c: $-4Q + 6R = 0$

Coefficient of s: $-6Q - 4R = 1$,

therefore $B = -\dfrac{3}{26}$, $R = -\dfrac{1}{13}$

The constant term: $5P = 10 \implies P = 2$.

So the G.S. is

$$y_{\text{GS}} = y_{\text{CF}} + y_{\text{PI}} = e^{-x}[A\cos(2x) + B\sin(2x)] + 2 - \frac{1}{26}[3\cos(3x) + 2\sin(3x)].$$

$$y' = e^{-x}[-2A\sin(2x) + 2B\cos(2x)] - e^{-x}[A\cos(2x) + B\sin(2x)] - \frac{1}{26}[-9\sin(3x) + 6\cos(3x)].$$

$$y(0) = 0 \implies A + 2 - \frac{3}{26} = 0 \implies A = -\frac{49}{26}$$

$$y'(0) = 0 \implies 2B - A - \frac{6}{26} = 0 \implies B = -\frac{43}{52}.$$

$$\text{Therefore} \quad y(x) = -\frac{1}{52}e^{-x}[98\cos(2x) + 43\sin(2x)] + 2 - \frac{1}{26}[3\cos(3x) + 2\sin(3x)].$$

15. ODE: $xy' - y = 2x^2\cos^2(3x)$ is equivalent to $y' - \dfrac{1}{x} \times y = 2x\cos^2(2x)$, which is linear, with integrating factor

$$I(x) = \exp\Big(\int -\frac{1}{x}\,dx\Big) = \exp(-\ln x) = \exp(\ln x^{-1}) = x^{-1} = \frac{1}{x}.$$

$$\frac{1}{x}y = \int \frac{1}{x} \times 2x\cos^2(2x)\,dx = \int 2\cos^2(2x)\,dx$$

(substitute $u = 2x$ and use integrate)

$$\implies \frac{1}{x} \times y = \frac{1}{4}\sin(4x) + x + A \implies y = \frac{1}{4}x\sin(4x) + x^2 + Ax.$$

$$y = \frac{\pi^2}{4} \text{ when } x = \frac{\pi}{2} \implies \frac{\pi^2}{4} = \frac{\pi}{8}\sin(2\pi) + \frac{\pi^2}{4} + \frac{A\pi}{2} \implies A = 0.$$

Therefore the Particular Solution is $y(x) = \dfrac{1}{4}x\sin(4x) + x^2$.

16. ODE: $y' + \dfrac{y}{x} = \dfrac{x^2}{y^2}$ is homogeneous, so set $y = vx$:

$$x\,\frac{dv}{dx} + v + \frac{vx}{x} = \frac{x^2}{v^2x^2} \quad\Longrightarrow\quad x\,\frac{dv}{dx} = \frac{1}{v^2} - 2v = \frac{1-2v^3}{v^2}$$
$$\Longrightarrow\quad \int \frac{v^2}{1-2v^3}\,dv = \int \frac{dx}{x}\,.$$

Set $1 = 2v^3 = u$, $\;-6v^2dv = du$, $\;v^2\,dv = -\dfrac{1}{6}\,du$:

$$\int \frac{v^2}{1-2v^3}\,dv = \int -\frac{1}{6}\,\frac{du}{u} = -\frac{1}{6}\,\ln|u| + C = -\frac{1}{6}\,\ln|1-2v^3| + C$$

$$\Longrightarrow\quad -\frac{1}{6}\,\ln|1-2v^3| = \ln|x| + A \quad\Longrightarrow\quad \ln|1-2v^3| + 6\ln|x| = -6A.$$

$$\Longrightarrow\quad \ln|x^6(1-2v^3)| = -6A \quad\Longrightarrow\quad x^6 - 2x^3y^3 = B \qquad (B = \pm e^{-6A} \text{ arbitrary}).$$

$y(1) = 2 \quad\Longrightarrow\quad 1 - 16 = B \quad\Longrightarrow\quad B = -15,$ and now $x^6 - 2x^3y^3 = -15.$

Re-arrange this expression to obtain the soltution $y(x)$:

$$y^3 = \frac{15}{2x^3} + \frac{x^3}{2} \quad\Longrightarrow\quad y = \left(\frac{15}{2x^3} + \frac{x^3}{2}\right)^{1/3}.$$

17. ODE: $y'' + y = x^2 + \sin x$; A.E.: $m^2 + 1 = 0 \;\Longrightarrow\; m = \pm i$, so the C.F. is

$$y_{\mathrm{CF}} = A\cos x + B\sin x.$$

As P.I. try $y_{\mathrm{PI}} = Px^2 + Qx + R + x(S\cos x + T\sin x)$; then, with $c = \cos x$, $s = \sin x$,

$$y'_{\mathrm{PI}} = 2Px + Q + (Sc + Ts) + x(-Ss + Tc), \quad y''_{\mathrm{PI}} = 2P + 2(-Ss + Tc) - x(Sc + Ts).$$

Substitute into ODE: $2P + 2(-Ss+Tc) - x(Sc+Ts) + Px^2 + Qx + R + x(Sc+Ts) = x^2 + s.$

Coefficient of x^2 : $P = 1$

Coefficient of x : $Q = 0$

Constant term : $2P + R = 0 \;\Longrightarrow\; R = -2$

Coefficient of c : $2T = 0 \;\Longrightarrow\; T = 0$

Coefficient of s : $-2S = 1 \;\Longrightarrow\; S = -\dfrac{1}{2}.$

So the P.I. is $y_{\mathrm{PI}} = x^2 - 2 - \dfrac{1}{2}\,x\cos x$ and the G.S. is

$$y_{\mathrm{GS}} = y_{\mathrm{CF}} + y_{\mathrm{PI}} = A\cos x + B\sin x + x^2 - 2 - \frac{1}{2}\,x\cos x.$$

$y(0) = 0 \quad\Longrightarrow\quad A - 2 = 0 \quad\Longrightarrow\quad A = 2.$

$y\left(\dfrac{\pi}{2}\right) = -2 \quad\Longrightarrow\quad B + \dfrac{\pi^2}{4} - 2 + 0 = -2 \quad\Longrightarrow\quad B = -\dfrac{\pi^2}{4}.$

Therefore the Particular Solution is $y(x) = 2\cos x - \dfrac{\pi^2}{4}\,\sin x + x^2 - 2 - \dfrac{1}{2}\,x\cos x.$