DEPARTMENT OF MATHEMATICS & STATISTICS
MM102 APPLICATIONS OF CALCULUS
Complex Numbers: Exercise Sheet for Week 5 – Solutions

1. Solve by completing the square or via the Quadratic Formula:

(a) $z^2-4z+5 \;=\; (z-2)^2+1z \;=\; 0 \quad\Longrightarrow\quad (z-2)^2 \;=\; -1 \;=\; i^2 \quad\Longrightarrow\quad z \;=\; 2\pm i.$

(b) $z \;=\; \pm\sqrt{-6} \;=\; \pm\sqrt{6}i.$

(c) $z^2-24z+26 \;=\; (z-12)^2-144+26 \;=\; (z-12)^2-118 \;=\; 0$

$\Longrightarrow\quad z-12 \;=\; \pm\sqrt{118} \quad\Longrightarrow\quad z \;=\; 12\pm\sqrt{118}.$

(d) $z \;=\; \dfrac{-(-1)\pm\sqrt{(-1)^2-4\times3\times3}}{2\times3} \;=\; \dfrac{1\pm\sqrt{-35}}{6} \;=\; \dfrac{1}{6}\pm\dfrac{\sqrt{35}}{6}i.$

2. **(a)** $-4+11i$ **(b)** $-5+12i$ **(c)** $18+i$ **(d)** $-6-17i$ **(e)** $-\dfrac{6}{25}+\dfrac{17}{25}i.$

3. The answers to **(a)**–**(i)** are straightforward by considering real and imaginary parts (after multiplying out the complex numbers if necessary).

(a) $-2+8i$, **(b)** $-2+5i$, **(c)** $3+13i$, **(d)** $1+4i$, **(e)** $14+2i$,

(f) $23-29i$, **(g)** $9-3i$, **(h)** $-46+2i$, **(i)** $-208-52i$,

(j) $\dfrac{3+4i}{4-3i} \;=\; \dfrac{3+4i}{4-3i}\times\dfrac{4+3i}{4+3i} \;=\; \dfrac{12+9i+16i+12i^2}{4^2+3^2} \;=\; \dfrac{25i}{25} \;=\; i,$

(k) $\dfrac{2+3i}{7-i} \;=\; \dfrac{(2+3i)(7+i)}{(7-i)(7+i)} \;=\; \dfrac{14+2i+21i+3i^2}{(7^2+1)} \;=\; \dfrac{11}{50}+\dfrac{23}{50}i$

(l) $\dfrac{(1+i)(2+3i)}{1-i} \;=\; \dfrac{-1+5i}{1-i}\times\dfrac{1+i}{1+i} \;=\; \dfrac{-1-5+5i-i}{1+1} \;=\; \dfrac{-6+4i}{2} \;=\; -3+2i,$

(m) $\dfrac{1}{4-3i}+\dfrac{1}{4+3i} \;=\; \dfrac{(4+3i)+(4-3i)}{4^2+3^2} \;=\; \dfrac{8}{25},$

(n) $\dfrac{1+i}{2+i}+\dfrac{3-i}{1-i} \;=\; \dfrac{(1+i)(1-i)+(3-i)(2+i)}{(2+i)(1-i)} \;=\; \dfrac{9+i}{3-i}\times\dfrac{3+i}{3+i}$

$\;=\; \dfrac{26+12i}{10} \;=\; \dfrac{13}{5}+\dfrac{6}{5}i.$

(o) $\dfrac{10i}{1+3i} \;=\; \dfrac{10i}{1+3i}\times\dfrac{1-3i}{1-3i} \;=\; \dfrac{10i(1-3i)}{1^2+3^2} \;=\; \dfrac{1}{10}\times10(i-3i^2) \;=\; 3+i.$

4. **(a)** $z^2+4z+7=0 \quad\Longrightarrow\quad (z+2)^2-4+7=0 \quad\Longrightarrow\quad (z+2)^2=-3=3i^2.$

Hence $z+2=\pm\sqrt{3}\,i$ and so $z=-2\pm\sqrt{3}\,i.$

The solutions are $z=-2+\sqrt{3}i$ and $z=-2-\sqrt{3}i.$

(b) $z^2+4iz+7=0 \quad\Longrightarrow\quad (z+2i)^2-(2i)^2+7=0 \quad\Longrightarrow\quad (z+2i)^2=-11=11i^2.$

Hence $z+2i=\pm\sqrt{11}\,i$ and so $z=(-2+\sqrt{11})i$ or $z=(-2-\sqrt{11})i.$

5. **(a)** $x + iy - 4i = 3y - 2ix + 9 \implies (x - 3y - 9) + i(y - 4 + 2x) = 0$

$\implies x - 3y - 9 = 0 \quad \& \quad y - 4 + 2x = 0 \implies x = 3,\ y = -2,$

(b) $x + iy - 2 = \dfrac{1}{x - iy}$

$\implies (x - iy)(x + iy) - 2(x - iy) = 1 \implies (x^2 + y^2 - 2x - 1) + 2iy = 0.$

By considering the imaginary part, $y = 0$. The real part now tells us that $x^2 - 2x - 1 = 0$. In other words, $x = 1 \pm \sqrt{2}$.

So the solution is $x = 1 \pm \sqrt{2},\ \ y = 0$.

(c) $\dfrac{1 - x + 2iy}{2x - iy} = 1 - 3i$

$\implies (1 - x) + i(2y) = (1 - 3i)(2x - iy) = (2x - 3y) + i(-6x - y).$

By equation real and imaginary parts: $1 - x = 2x - 3y$ and $2y = -6x - y$.

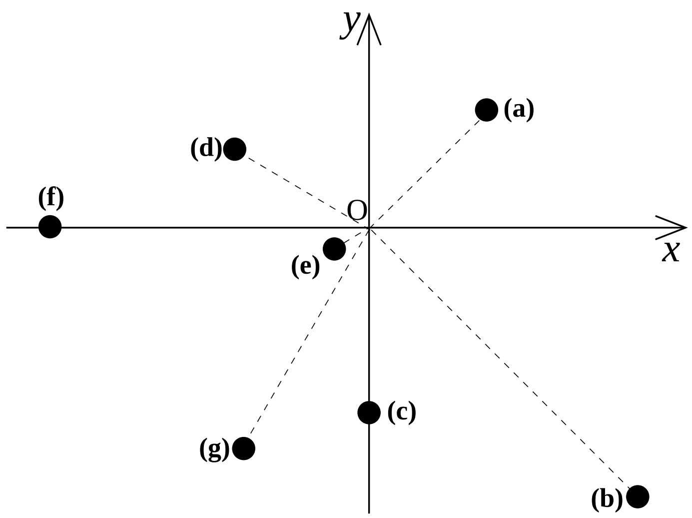
These simultaneous equations have solution $x = \dfrac{1}{9},\ y = -\dfrac{2}{9}$.

6. **(a)** $|z| = \sqrt{8},\ \ \text{Arg } z = \dfrac{\pi}{4},$ **(b)** $|z| = \sqrt{32},\ \ \text{Arg } z = -\dfrac{\pi}{4},$

(c) $|z| = 3,\ \ \text{Arg } z = -\dfrac{\pi}{2},$ **(d)** $|z| = 2,\ \ \text{Arg } z = \dfrac{5\pi}{6},$

(e) $\dfrac{1}{-\sqrt{3} + i} \times \dfrac{-\sqrt{3} - i}{-\sqrt{3} - i} = -\dfrac{\sqrt{3}}{4} - \dfrac{1}{4}i,$ $|z| = \dfrac{1}{2},\ \ \text{Arg } z = -\dfrac{5\pi}{6},$

(f) $|z| = 5,\ \ \text{Arg } z = \pi,$ **(g)** $|z| = 4,\ \ \text{Arg } z = -\dfrac{2\pi}{3}.$



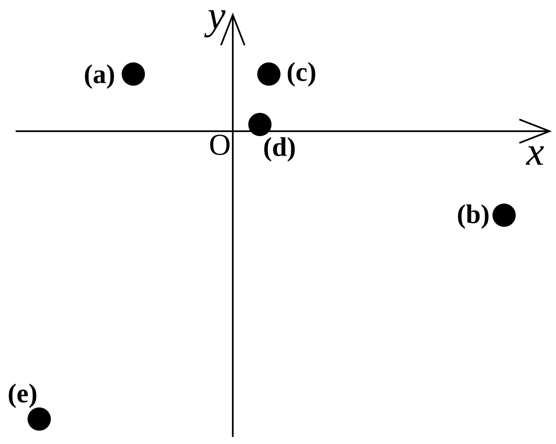
7. **(a)** $-2+i$

(b) $6-3i$

(c) $\dfrac{1}{3}+i$

(d) $0.3+0.1i$

(e) $-4-11i$.



8. **(a)** Circle, radius 1, centred on 0.

(b) A ray (straight line emanating from the origin) making an acute angle of 1 radian (57°) with the positive x-axis.

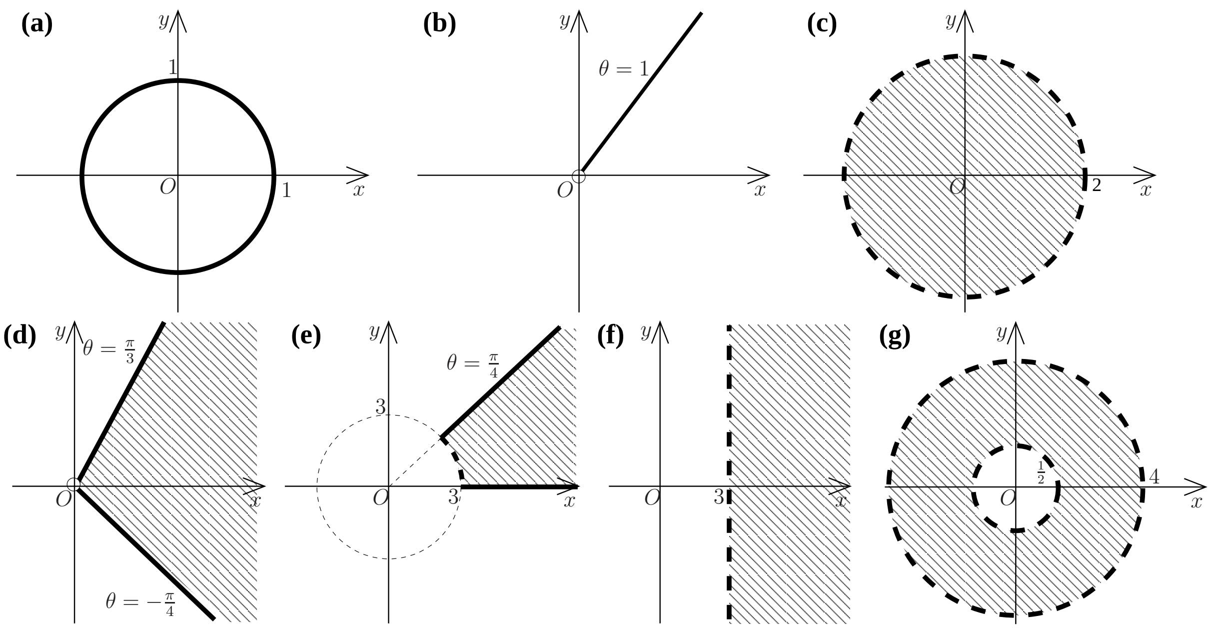
(c) An open disk of radius 2.

(d) An infinite wedge of width 105°.

(e) The intersection of a wedge and the outside of a circle.

(f) Right of the vertical line $x=3$.

(g) An annulus (washer shaped region).



9. For each question, we assume that the given number is represented by z. Therefore, the question is asking us to find the modulus $|z|$ and the principal value of the argument, $\text{Arg}(z)$. In each case, **use the Argand diagram to find the principal value**. Remember, it is not enough to simply consider $\arctan(y/x)$ as this will always lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and will not provide the correct argument for any angle in the 2nd or 3rd quadrants. Use arctan to find an appropriate **acute angle** with respect to the real axis, then use this to find the argument and its principal value.

(a) Modulus $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. The complex number lies in the first quadrant of the Argand diagram, so $\text{Arg}(z) = \arctan(1) = \dfrac{\pi}{4}$.

(b) Modulus $|z| = \sqrt{3+1} = 2$.
z is in the 4th quadrant, so $\text{Arg}(z) = \arctan\left(-\dfrac{1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$.

(c) Modulus $|z| = \sqrt{16+48} = 8$.
z is in the 3rd quadrant, $\tan\big(\text{Arg}(z)\big) = \sqrt{3} \implies \text{Arg}(z) = -\pi + \dfrac{\pi}{3} = -\dfrac{2\pi}{3}$.

(d) Modulus $|z| = 6$. The number -6 lies on the $-$ve real axis (between the 2nd and 3rd quadrants) $\implies \text{Arg}(z) = \pi$.

(e) Modulus $|z| = 2$. Principal value satisfies $\sin\theta = \dfrac{2}{2} = 1, \quad \cos\theta = \dfrac{0}{2} = 0$.
z lies on the +ve imaginary, between the 1st and 2nd quadrants $\implies \text{Arg}(z) = \dfrac{\pi}{2}$.

(f) Modulus $|z| = \sqrt{16+16} = 4\sqrt{2}$.
z lies in 2nd quadrant, $\tan\big(\text{Arg}(z)\big) = -1 \implies \text{Arg}(z) = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$.

(g) Modulus $|z| = 1$, $\text{Arg}(z) = \pi/12$.

(h) Modulus $|z| = 1$, $\text{Arg}(z) = -\pi/4$.

10. In each case, the polar form is given by $z = r\,\text{cis}(\theta)$ where $r = |z|$ and $\theta = \text{Arg}\ z$.
Use the Argand diagram to help find the principal value of the argument in each case, recalling that $\text{Arg}(z)$ is a unique value for each z and always satisfies $-\pi < \text{Arg}(z) \leq \pi$.

(a) $4\,\text{cis}\left(-\dfrac{\pi}{3}\right)$, **(b)** $5\,\text{cis}\,(0.9273)$, **(c)** $12\,\text{cis}\,(\pi)$, **(d)** $128\sqrt{2}\,\text{cis}\left(-\dfrac{3}{4}\pi\right)$,
(e) $\sqrt{72}\,\text{cis}\left(\dfrac{\pi}{4}\right)$, **(f)** $4\,\text{cis}\left(\dfrac{\pi}{2}\right)$, **(g)** $2\,\text{cis}\left(-\dfrac{\pi}{2}\right)$, **(h)** $\sqrt{12}\,\text{cis}\left(\dfrac{5\pi}{6}\right)$.

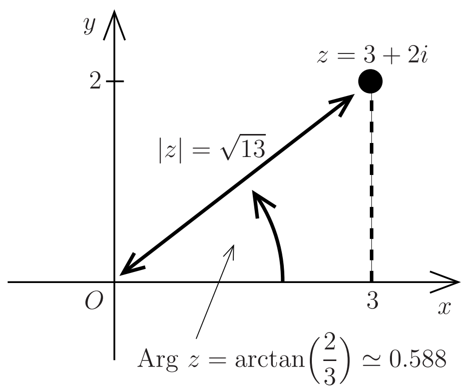
11. $|3+2i| = \sqrt{3^2+2^2} = \sqrt{13}$.

The number $z = 3+2i$ lies in the first quadrant in the Argand diagram.

$$\text{Arg}\,(3+2i) = \arctan\left(\frac{2}{3}\right) \simeq 0.588,$$

$$\arg(3+2i) = \arctan\left(\frac{2}{3}\right) + 2k\pi \quad (k \in \mathbb{Z}).$$

In polar form, $3+2i = \sqrt{13}\,\text{cis}\,(0.588)$.



12. In each case, $z = r\,\mathrm{cis}(\theta) = x + iy$ where $x = r\cos\theta$ and $y = r\sin\theta$.

(a) $\dfrac{3}{\sqrt{2}} - \dfrac{3}{\sqrt{2}}i$, **(b)** $3\sqrt{3} - 3i$, **(c)** -2.

13. Consider the modulus of the three numbers in the expression:

$|6-i| = \sqrt{6^2 + (-1)^2} = \sqrt{37}$, $|12+5i| = \sqrt{12^2 + 5^2} = 13$,

$|-7-24| = \sqrt{(-7)^2 + (-24)^2} = 25$.

Therefore, $\left|\dfrac{(6-i)^2(12+5i)}{-7-24i}\right| = \dfrac{|6-i|^2\ |12+5i|}{|-7-24i|} = \dfrac{37 \times 13}{25} = \dfrac{481}{25}$.