DEPARTMENT OF MATHEMATICS & STATISTICS
MM102 APPLICATIONS OF CALCULUS
Complex Numbers: Exercise Sheet for Week 6 – Solutions

1. $z_1 = 2 - 2i, \qquad r = \sqrt{2^2 + 2^2} = 2\sqrt{2}; \qquad \arctan\left(\frac{2}{2}\right) = \arctan(1) = \frac{\pi}{4}.$

θ lies in 4th quadrant, therefore $\mathrm{Arg}(z_1) = -\frac{\pi}{4}$ and $z_1 = 2\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$

$z_2 = -1 + \sqrt{3}i, \qquad r = \sqrt{1^2 + 3} = 2; \qquad \arctan(\sqrt{3}) = \frac{\pi}{3}.$

θ lies in 2nd quadrant, therefore $\mathrm{Arg}(z_2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and $z_2 = 2\,\mathrm{cis}\left(\frac{2\pi}{3}\right).$

(a) $z_1\, z_2 \;=\; 2 \times (2\sqrt{2})\,\mathrm{cis}\left(-\frac{\pi}{4} + \frac{2\pi}{3}\right) \;=\; 4\sqrt{2}\,\mathrm{cis}\left(\frac{5\pi}{12}\right).$

(b) $z_1^5 \;=\; (2\sqrt{2})^5\,\mathrm{cis}\left(-\frac{5\pi}{4}\right) \;=\; 2^5(\sqrt{2})^5\,\mathrm{cis}\left(-\frac{5\pi}{4} + 2\pi\right) \;=\; 128\sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right) \;=\; 128.$

(c) $\frac{1}{z_2^3} \;=\; z_2^{-3} \;=\; \frac{1}{8}\left[\,\mathrm{cis}\left(\frac{2\pi}{3}\right)\right]^{-3} \;=\; \frac{1}{8}\,\mathrm{cis}(-2\pi) \;=\; \frac{1}{8}.$

(d)
$$z_1^6\, z_2^4 \;=\; (2\sqrt{2})^6\, 2^4\,\mathrm{cis}\left(-\frac{6\pi}{4} + \frac{8\pi}{3}\right) \;=\; 2^{13}\,\mathrm{cis}\left(\frac{(-18+32)\pi}{12}\right) \;=\; 8192\,\mathrm{cis}\left(\frac{7\pi}{6}\right)$$
$$= 8192\,\mathrm{cis}\left(-\frac{5\pi}{6}\right).$$

(e)
$$\frac{z_1^9}{z_2^7} = \frac{(2\sqrt{2})^9}{2^7}\,\mathrm{cis}\left(-\frac{9\pi}{4} - \frac{14\pi}{3}\right) = 64\sqrt{2}\,\mathrm{cis}\left(\frac{(-27-56)\pi}{12}\right) = 64\sqrt{2}\,\mathrm{cis}\left(-\frac{83\pi}{12}\right)$$
$$= 64\sqrt{2}\,\mathrm{cis}\left(-\frac{11\pi}{12}\right).$$

2. In each example find the polar form of the given complex number, then raise it to the appropriate power using de Moivre's theorem: $|z^n| = |z|^n$, $\arg(z^n) = n\arg(z)$.

The following answers give the principal value of the argument, $\mathrm{Arg}(z)$.

The argument of z can be found via $\arg(z) = \mathrm{Arg}(z) + 2k\pi \quad (k \in \mathbb{Z})$.

(a) $\left|(1-3i)^4\right| = \left|1-3i\right|^4 = (\sqrt{1+9})^4 = (\sqrt{10})^4 = 100$;

$1-3i$ lies on the fourth quadrant, so $\mathrm{Arg}(1-3i) = -\arctan(3) \approx -1.2490$.

$\arg\left[(1-3i)^4\right] = \left[4 \times \mathrm{Arg}(1-3i)\right] + 2k\pi = -4.9962 + 2k\pi \quad (k \in \mathbb{Z})$.

Since -4.9962 does not lie in the interval $(-\pi,\, \pi]$, it cannot be the principal value of the argument. To find the principal value, consider the case $k = 1$:

$\mathrm{Arg}\left[(1-3i)^4\right] \approx -4.9962 + (2 \times \pi) \approx 1.2870$.

(b) In polar form using the principal value, $-1+\sqrt{3}i = 2\mathrm{cis}\left(\frac{2\pi}{3}\right)$.

$\left|(-1+\sqrt{3}i)^5\right| = \left|-1+\sqrt{3}i\right|^5 = 2^5 = 32$;

$\arg\left[(-1+\sqrt{3}i)^5\right] = 5 \times \arg(-1+\sqrt{3}i) = (5 \times \frac{2\pi}{3}) + 2k\pi = \frac{10\pi}{3} + 2k\pi \quad (k \in \mathbb{Z})$;

$\mathrm{Arg}\left[(-1+\sqrt{3}i)^5\right] = \frac{10\pi}{3} - 2\pi = -\frac{2\pi}{3}$.

(c) $\left|(-12-5i)^{-3}\right| = \left|-12-5i\right|^{-3} = 13^{-3} = \frac{1}{2197}$;

$-12-5i$ lies in the third quadrant, so $\mathrm{Arg}(-12-5i) = -\left(\pi - \arctan(\frac{5}{12})\right) \approx -2.7468$.

$\arg\left[(-12-5i)^{-3}\right] = \left[-3 \times \mathrm{Arg}(-12-5i)\right] + 2k\pi = 8.2404 + 2k\pi \quad (k \in \mathbb{Z})$.

Since 8.2404 does not lie in the interval $(-\pi,\, \pi]$, it cannot be be principal value of the argument. To find the principal value consider the case $k = -1$:

$\mathrm{Arg}\left[(-12-5i)^{-3}\right] \approx 8.2404 - (2 \times \pi) \approx 1.9572$.

(d) In polar form using the principal value, $-12-12i = 12\sqrt{2}\mathrm{cis}\left(-\frac{3\pi}{4}\right)$.

$\left|(-12-12i)^5\right| = \left|-12-12i\right|^5 = (12\sqrt{2})^5 = 995328\sqrt{2}$;

$\arg\left[(-12-12i)^5\right] = 5 \times \arg(-12-12i) = (-5 \times \frac{3\pi}{4}) + 2k\pi = -\frac{15\pi}{4} + 2k\pi \quad (k \in \mathbb{Z})$;

$\mathrm{Arg}\left[(-12-12i)^5\right] = -\frac{15\pi}{4} + 4\pi = \frac{\pi}{4}$.

3. Express each complex number in polar form and simplify using de Moivre's theorem:

$$|z^n| = |z|^n, \quad \arg(z^n) = n\arg(z), \quad |z_1 z_2| = |z_1|\,|z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

(a) $$\frac{(1+i)^5}{1-i} = \frac{\left[\sqrt{2}\,\mathrm{cis}(\frac{\pi}{4})\right]^5}{\sqrt{2}\,\mathrm{cis}(-\frac{\pi}{4})} = 2^2\,\mathrm{cis}(5\times\frac{\pi}{4}+\frac{\pi}{4}) = 4\,\mathrm{cis}(\frac{3\pi}{2}) = -4i.$$

(b) $$\frac{(1+\sqrt{3}i)^2}{(1+i)^3} = \frac{\left[2\,\mathrm{cis}(\frac{\pi}{3})\right]^2}{\left[\sqrt{2}\,\mathrm{cis}(\frac{\pi}{4})\right]^3} = \sqrt{2}\,\mathrm{cis}(\frac{2\pi}{3}-\frac{3\pi}{4}) = \sqrt{2}\,\mathrm{cis}(-\frac{\pi}{12}).$$

Note that:

$$\begin{aligned}\mathrm{cis}(-\frac{\pi}{12}) &= \mathrm{cis}(\frac{\pi}{4}-\frac{\pi}{3}) = \mathrm{cis}(\frac{\pi}{4})\,\mathrm{cis}(-\frac{\pi}{3})\\ &= \frac{1}{\sqrt{2}}(1+i)\times\frac{1}{2}(1-\sqrt{3}i) = \frac{1}{2\sqrt{2}}\left[1+\sqrt{3}+i(1-\sqrt{3})\right]\end{aligned}$$

$$\Longrightarrow \quad \frac{(1+\sqrt{3}i)^2}{(1+i)^3} = \frac{1}{2}(1+\sqrt{3})+\frac{1}{2}i(1-\sqrt{3}).$$

(c) $$\begin{aligned}(1+i)^{20}+(1-i)^{20} &= \left[\sqrt{2}\,\mathrm{cis}(\tfrac{\pi}{4})\right]^{20}+\left[\sqrt{2}\,\mathrm{cis}(-\tfrac{\pi}{4})\right]^{20}\\ &= 2^{10}\left[\mathrm{cis}(5\pi)+\mathrm{cis}(-5\pi)\right] = 2^{10}\times 2\cos(5\pi) = 2^{11}\cos(\pi) = -2048.\end{aligned}$$

(d) $$\begin{aligned}\frac{(\sqrt{3}+i)^{10}}{(1-i)^7} &= \frac{\left[2\,\mathrm{cis}(\frac{\pi}{6})\right]^{10}}{\left[\sqrt{2}\,\mathrm{cis}(-\frac{\pi}{4})\right]^7}\\ &= 2^{13/2}\,\mathrm{cis}(\frac{10\pi}{6}+\frac{7\pi}{4}) = 2^{13/2}\,\mathrm{cis}(\frac{41\pi}{12}) = 2^{13/2}\,\mathrm{cis}(-\frac{7\pi}{12}).\end{aligned}$$

Note that:

$$\begin{aligned}\mathrm{cis}(-\frac{7\pi}{12}) &= \mathrm{cis}(-\frac{\pi}{4}-\frac{\pi}{3}) = \mathrm{cis}(-\frac{\pi}{4})\,\mathrm{cis}(-\frac{\pi}{3})\\ &= \frac{1}{\sqrt{2}}(1-i)\times\frac{1}{2}(1-\sqrt{3}i) = \frac{1}{2\sqrt{2}}\left[1-\sqrt{3}-i(1+\sqrt{3})\right]\end{aligned}$$

$$\Longrightarrow \quad \frac{(\sqrt{3}+i)^{10}}{(1-i)^7} = 32(1-\sqrt{3})-32i(1+\sqrt{3}).$$

(e) $$(\sqrt{2}+i\sqrt{2})^{-4} = \left[2\,\mathrm{cis}(\frac{\pi}{4})\right]^{-4} = 2^{-4}\,\mathrm{cis}(-4\times\frac{\pi}{4}) = \frac{1}{16}\,\mathrm{cis}(-\pi) = -\frac{1}{16}.$$

(f) $$(\sqrt{2}+i\sqrt{2})^{8} = \left[2\,\mathrm{cis}(\frac{\pi}{4})\right]^{8} = 2^{8}\,\mathrm{cis}(8\times\frac{\pi}{4}) = 256\,\mathrm{cis}(2\pi) = 256.$$

(g) $$\frac{(\cos\theta+i\sin\theta)^3}{(\sin\theta+i\cos\theta)^2} = \frac{(\cos\theta+i\sin\theta)^3}{\left[i(\cos\theta-i\sin\theta)\right]^2} = \frac{\left[\mathrm{cis}(\theta)\right]^3}{i^2\left[\mathrm{cis}(-\theta)\right]^2} = -\mathrm{cis}(3\theta+2\theta) = -\mathrm{cis}(5\theta).$$

4. $\cos(2\theta)+i\sin(2\theta) = [\,\cos(\theta)+i\sin(\theta)\,]^2 = \cos^2(\theta)-\sin^2(\theta)+2i\cos(\theta)\sin(\theta).$

By equating real and imaginary parts,

$$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta), \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

5. Let $z=\mathrm{cis}\,\theta$, then $\cos\theta = \frac{1}{2}\Big(z+\frac{1}{z}\Big)$ and

$$\cos^2\theta = \left[\frac{1}{2}\Big(z+\frac{1}{z}\Big)\right]^2 = \frac{1}{4}\Big(z^2+\frac{1}{z^2}+2\Big) = \frac{1}{2}\Big(\cos(2\theta)+1\Big).$$

6. Let $z = \operatorname{cis}\theta$, then $\sin\theta = \dfrac{1}{2i}\Big(z - \dfrac{1}{z}\Big)$ and

$$\sin^3\theta = \Big[\frac{1}{2i}\Big(z-\frac{1}{z}\Big)\Big]^3 = -\frac{1}{8i}\Big(z^3 - \frac{1}{z^3} - 3z + \frac{3}{z}\Big)$$

$$= -\frac{1}{4}\sin(3\theta) + \frac{3}{4}\sin\theta. \qquad a = -\frac{1}{4},\ b = \frac{3}{4}$$

$$\int \sin^3\theta\,\mathrm{d}\theta = \int\Big(-\frac{1}{4}\sin(3\theta) + \frac{3}{4}\sin\theta\Big)\,\mathrm{d}\theta = \frac{1}{12}\cos(3\theta) - \frac{3}{4}\cos\theta + c.$$

7.

$$\begin{aligned}
\cos(4\theta) + i\sin(4\theta) &= (\cos\theta + i\sin\theta)^4 \\
&= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta\,\sin^2\theta + \sin^4\theta + i\Big(4\cos^3\theta\,\sin\theta - 4\cos\theta\,\sin^3\theta\Big).
\end{aligned}$$

Equating real parts,

$$\begin{aligned}
\cos(4\theta) &= \cos^4\theta - 6\cos^2\theta\,\sin^2\theta + \sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\
&= 8\cos^4\theta - 8\cos^2\theta + 1. \qquad a = 8,\ b = -8,\ c = 1
\end{aligned}$$

8.

$$\begin{aligned}
\cos^5\theta &= \Big[\frac{1}{2}\Big(z+\frac{1}{z}\Big)\Big]^5 \\
&= \frac{1}{32}\Big(z+\frac{1}{z}\Big)^5 \\
&= \frac{1}{32}\Big[z^5 + 5z^4\Big(\frac{1}{z}\Big) + 10z^3\Big(\frac{1}{z}\Big)^2 + 10z^2\Big(\frac{1}{z}\Big)^3 + 5z\Big(\frac{1}{z}\Big)^4 + \Big(\frac{1}{z}\Big)^5\Big] \\
&= \frac{1}{32}\Big(z^5+\frac{1}{z^5}\Big) + \frac{5}{32}\Big(z^3+\frac{1}{z^3}\Big) + \frac{10}{32}\Big(z+\frac{1}{z}\Big) = \frac{1}{16}\cos(5\theta) + \frac{5}{16}\cos(3\theta) + \frac{5}{8}\cos\theta.
\end{aligned}$$

$$\int\cos^5\theta\ \mathrm{d}\theta = \int\Big(\frac{1}{16}\cos(5\theta)+\frac{5}{16}\cos(3\theta)+\frac{5}{8}\cos\theta\Big)\mathrm{d}\theta = \frac{1}{80}\sin(5\theta)+\frac{5}{48}\sin(3\theta)+\frac{5}{8}\sin\theta+c$$

9. With $c = \cos\theta,\ s = \sin\theta$ (so that $c^2 + s^2 = 1$), de Moivre's theorem gives

$$\begin{aligned}
\cos(5\theta) + i\sin(5\theta) &= (\cos\theta + i\sin\theta)^5 \\
&= (c+is)^5 \\
&= c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5 \\
&= c^5 + 5c^4s\,i + 10c^3s^2i^2 + 10c^2s^3i^3 + 5cs^4i^4 + s^5i^5 \\
&= c^5 + 5c^4s\,i - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i \\
&= (c^5 - 10c^3s^2 + 5cs^4) + (5c^4s - 10c^2s^3 + s^5)i \qquad (**)
\end{aligned}$$

(Recall that $i^2 = -1,\ i^3 = i^2\times i = -i,\ i^4 = i^2\times i^2 = 1,\ i^5 = i^4\times i = i$.)

(Question 9 continued on next page)

9. (cont'd)

(a) Equate real parts in $(**)$ to give

$$\begin{aligned}\cos(5\theta) &= c^5 - 10c^3s^2 + 5cs^4 \\ &= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2 \\ &= c^5 - 10c^3 + 10c^5 + 5c(1-2c^2+c^4) \;=\; c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5,\end{aligned}$$

i.e. $\quad \cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

(b) Imaginary parts of $(**)$ give

$$\begin{aligned}\sin(5\theta) &= 5c^4s - 10c^2s^3 + s^5 \\ &= 5(1-s^2)^2s - 10(1-s^2)s^3 + s^5 \\ &= 5(1-2s^2+s^4)s - 10s^3 + 10s^5 + s^5 \;=\; 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.\end{aligned}$$

(c) Since $\tan\theta$ is well defined for $\theta \neq (2n+1)\dfrac{\pi}{2}$ and $\tan\theta = \dfrac{s}{c}$, equation $(**)$ gives

$$\begin{aligned}\tan(5\theta) = \frac{\sin(5\theta)}{\cos(5\theta)} &= \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4} \\ &= \frac{5\dfrac{s}{c} - 10\left(\dfrac{s}{c}\right)^3 + \left(\dfrac{s}{c}\right)^5}{1 - 10\left(\dfrac{s}{c}\right)^2 + 5\left(\dfrac{s}{c}\right)^4} \quad \text{(dividing numerator \& denominator by } c^5\text{)} \\ &= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}\,.\end{aligned}$$

10. With $c = \cos\theta,\ s = \sin\theta$, we obtain

$$\begin{aligned}
\cos(6\theta) + i\sin(6\theta) &= (\cos\theta + i\sin\theta)^6 \\
&= (c+is)^6 \\
&= c^6 + 6c^5(is) + 15c^4(is)^2 + 20c^3(is)^3 + 15c^2(is)^4 + 6c(is)^5 + (is)^6 \\
&= c^6 + 6c^5 s\, i - 15c^4s^2 - 20c^3s^3 i + 15c^2s^4 + 6cs^5 i - s^6 \\
&= c^6 - 15c^4s^2 + 15c^2s^4 - s^6 + (6c^5s - 20c^3s^3 + 6cs^5)i \qquad (\triangle)
\end{aligned}$$

Real parts of $(\triangle)$ give

$$\begin{aligned}
\cos(6\theta) &= c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^2 \qquad (\text{since } \ s^2 = 1-c^2) \\
&= c^6 - 15c^4 + 15c^6 + 15c^2(1-2c^2+c^4)(1-3c^2+3c^4-c^6) \\
&= c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6 - 1 + 3c^2 - 3c^4 + c^6 \\
&= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.
\end{aligned}$$

Similarly, imaginary parts of $(\triangle)$ give

$$\begin{aligned}
\sin(6\theta) &= 6c^5s - 20c^3s^3 + 6cs^5 \\
&= s\Big(6c^5 - 20c^3(1-c^2) + 6c(1-c^2)^2\Big) \\
&= s\Big(6c^5 - 20c^3 + 20c^5 + 6c(1-2c^2+c^4)\Big) \\
&= s(6c^5 - 20c^3 + 20c^5 + 6c - 12c^3 + 6c^5) \\
&= \sin\theta(32\cos^5\theta - 32\cos^3\theta + 6\cos\theta). \qquad a = 32, \quad b = -32, \quad c = 6.
\end{aligned}$$

11. With $z = \cos\theta + i\sin\theta$, we have $2\cos(n\theta) = z^n + \dfrac{1}{z^n}$, $2i\sin(n\theta) = z^n - \dfrac{1}{z^n}$.

(a)

$$\begin{aligned}
& (2\cos\theta)^4 &=& \left(z+\frac{1}{z}\right)^4 \\
\Longrightarrow\quad & 16\cos^4\theta &=& z^4 + 4z^3\times\frac{1}{z} + 6z^2\times\frac{1}{z^3} + 4z\times\frac{1}{z^3} + \frac{1}{z^4} \\
\Longrightarrow\quad & 16\cos^4\theta &=& \left(z^4+\frac{1}{z^4}\right) + 4\left(z^2+\frac{1}{z^2}\right) + 6 = 2\cos(4\theta) + 4\times 2\cos(2\theta) + 6 \\
\Longrightarrow\quad & \cos^4\theta &=& \frac{1}{8}\Big(\cos(4\theta) + 4\cos(2\theta) + 3\Big).
\end{aligned}$$

(b)

$$\begin{aligned}
& (2\cos\theta)^2(2i\sin\theta)^4 &=& \left(z+\frac{1}{z}\right)^2\left(z-\frac{1}{z}\right)^4 = \left(z+\frac{1}{z}\right)^2\left(z-\frac{1}{z}\right)^2\left(z-\frac{1}{z}\right)^2 \\
\Longrightarrow\quad & 2^2\cos^2\theta\times 2^4i^4\sin^4\theta &=& \left[\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)\right]^2\left(z-\frac{1}{z}\right)^2 = \left(z^2-\frac{1}{z^2}\right)^2\left(z^2-2+\frac{1}{z^2}\right) \\
\Longrightarrow\quad & 2^6\cos^2\theta\sin^4\theta &=& \left(z^4-2+\frac{1}{z^4}\right)\left(z^2-2+\frac{1}{z^2}\right) \\
&&=& z^6 - 2z^4 + z^2 - 2z^2 + 4 - \frac{2}{z^2} + \frac{1}{z^2} - \frac{2}{z^4} + \frac{1}{z^6} \\
&&=& \left(z^6+\frac{1}{z^6}\right) - 2\left(z^4+\frac{1}{z^4}\right) - \left(z^2+\frac{1}{z^2}\right) + 4 \\
&&=& 2\cos 6\theta - 2(2\cos 4\theta) - 2\cos 2\theta + 4 \\
\Longrightarrow\quad & \cos^2\theta\sin^4\theta &=& \frac{1}{32}\Big(\cos(6\theta) - 2\cos(4\theta) - \cos(2\theta) + 2\Big).
\end{aligned}$$

12.(a) With the notation of the previous question,

$$\begin{aligned}
& (2i\sin\theta)^5 &=& \left(z-\frac{1}{z}\right)^5 \\
\Longrightarrow\quad & 2^5i^5\sin^5\theta &=& z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5 \\
\Longrightarrow\quad & 32i\sin^5\theta &=& \left(z^5-\frac{1}{z^5}\right) - 5\left(z^3-\frac{1}{z^3}\right) + 10\left(z-\frac{1}{z}\right) \\
&&=& 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta) \\
\Longrightarrow\quad & \sin^5\theta &=& \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).
\end{aligned}$$

(b)

$$\begin{aligned}
& (2i\sin\theta)^3(2\cos\theta)^3 &=& \left(z-\frac{1}{z}\right)^3\left(z+\frac{1}{z}\right)^3 = \left[\left(z-\frac{1}{z}\right)\left(z+\frac{1}{z}\right)\right]^3 = \left(z^2-\frac{1}{z^2}\right)^3 \\
\Longrightarrow\quad & 2^3i^3\sin^3\theta\times 2^3\cos^3\theta &=& z^6 + 3z^4\left(-\frac{1}{z^2}\right) + 3z^2\left(-\frac{1}{z^2}\right)^2 + \left(-\frac{1}{z^2}\right)^3 \\
&&& = z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6} \\
\Longrightarrow\quad & -2^6i\sin^3\theta\cos^3\theta &=& \left(z^6-\frac{1}{z^6}\right) - 3\left(z^2-\frac{1}{z^2}\right) = 2i\sin(6\theta) - 3\left(2i\sin(2\theta)\right) \\
\Longrightarrow\quad & \sin^3\theta\cos^3\theta &=& -\frac{1}{32}\left(\sin(6\theta) - 3\sin(2\theta)\right) = \frac{1}{32}\left(3\sin(2\theta) - \sin(6\theta)\right).
\end{aligned}$$

$$\textbf{13.(a)}\quad \int_0^{\pi/4} \cos^4\theta \,\mathrm{d}\theta = \int_0^{\pi/4} \frac{1}{8}\Big(\cos(4\theta) + 4\cos(2\theta) + 3\Big)\mathrm{d}\theta$$

$$= \left[\frac{1}{8}\Big(\frac{1}{4}\sin(4\theta) + 4\times\frac{1}{2}\sin(2\theta) + 3\theta\Big)\right]_0^{\pi/4}$$

$$= \frac{1}{8}\left[\Big(0 + 2 + \frac{3\pi}{4}\Big) - (0+0+0)\right] = \frac{1}{4} + \frac{3\pi}{32}.$$

$$\textbf{(b)}\quad \int_{\pi/2}^{\pi} \sin^3\theta\cos^3\theta \,\mathrm{d}\theta = \frac{1}{32}\int_{\pi/2}^{\pi}\Big(3\sin(2\theta) - \sin(6\theta)\Big)\mathrm{d}\theta$$

$$= \frac{1}{32}\left[-\frac{3}{2}\cos(2\theta) + \frac{1}{6}\cos(6\theta)\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{32}\left[\Big(-\frac{3}{2}+\frac{1}{6}\Big) - \Big(\frac{3}{2}-\frac{1}{6}\Big)\right] = \frac{1}{32}\Big(-\frac{8}{3}\Big) = -\frac{1}{2}.$$

14. Clearly 1 itself is a sixth root of 1 because $1^6 = 1$.

So all the sixth roots of 1 must have modulus $|1| = 1$.

All six roots are, therefore, equally spaced on a unit circle centred at the origin at angle $\dfrac{2\pi}{6} = \dfrac{\pi}{3}$.

15.(a) In polar form,

$$i = 1\,\text{cis}\Big(\frac{\pi}{2} + 2k\pi\Big) \quad (k \in \mathbb{Z}) \quad \Longrightarrow \quad w_k \;=\; i^{1/2} = \text{cis}\Big(\frac{\pi}{4} + k\pi\Big) \qquad (k = 0,\, 1).$$

$\underline{k=0}$ gives
$$w_0 \;=\; \text{cis}\Big(\frac{\pi}{4}\Big) \;=\; \cos\Big(\frac{\pi}{4}\Big) + i\sin\Big(\frac{\pi}{4}\Big) \;=\; \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,,$$

$\underline{k=1}$ gives
$$w_1 \;=\; \text{cis}\Big(\frac{5\pi}{4}\Big) \;=\; \cos\Big(\frac{5\pi}{4}\Big) + i\sin\Big(\frac{5\pi}{4}\Big) \;=\; -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i.$$

(b) In polar form,

$$1+\sqrt{3}i = 2\,\text{cis}\Big(\frac{\pi}{3}+2k\pi\Big) \quad (k \in \mathbb{Z}) \quad \Longrightarrow \quad w_k = (1+\sqrt{3}i)^{1/2} = \sqrt{2}\,\text{cis}\Big(\frac{\pi}{6}+k\pi\Big) \quad (k = 0,\, 1).$$

$\underline{k=0}$ gives
$$w_0 \;=\; \sqrt{2}\,\text{cis}\Big(\frac{\pi}{6}\Big) \;=\; \sqrt{2}\Big[\cos\Big(\frac{\pi}{6}\Big) + i\sin\Big(\frac{\pi}{6}\Big)\Big] \;=\; \frac{1}{\sqrt{2}}(\sqrt{3}+i)\,,$$

$\underline{k=1}$ gives
$$w_1 \;=\; \sqrt{2}\,\text{cis}\Big(\frac{7\pi}{6}\Big) \;=\; \sqrt{2}\Big[\cos\Big(\frac{7\pi}{6}\Big) + i\sin\Big(\frac{7\pi}{6}\Big)\Big] \;=\; -\frac{1}{\sqrt{2}}(\sqrt{3}+i).$$

(c) In polar form,

$$-8 = 8\,\text{cis}(\pi+2k\pi) \qquad (k \in \mathbb{Z}) \qquad \Longrightarrow \qquad w_k = (-8)^{1/3} = 2\,\text{cis}\Big(\frac{(2k+1)\pi}{3}\Big) \quad (k = 0,\, 1,\, 2).$$

$\underline{k=0}$
$$w_0 \;=\; 2\,\text{cis}\Big(\frac{\pi}{3}\Big) \;=\; 1+\sqrt{3}i\,,$$

$\underline{k=1}$
$$w_1 \;=\; 2\,\text{cis}(\pi) \;=\; -2\,,$$

$\underline{k=2}$
$$w_2 \;=\; 2\,\text{cis}\Big(\frac{5\pi}{3}\Big) \;=\; 1-\sqrt{3}i.$$

(For the third value we could alternatively use $k = -1$.)

(d) In polar form, $\quad 27i = 27\,\text{cis}\Big(\dfrac{\pi}{2} + 2k\pi\Big) \qquad (k \in \mathbb{Z})$

$$\Longrightarrow \qquad w_k = (27i)^{1/3} = 3\,\text{cis}\Big(\frac{(4k+1)\pi}{6}\Big) \qquad (k = 0,\, 1,\, 2).$$

$\underline{k=0}$
$$w_0 \;=\; 3\,\text{cis}\Big(\frac{\pi}{6}\Big) \;=\; \frac{3}{2}(\sqrt{3}+i)\,,$$

$\underline{k=1}$
$$w_1 \;=\; 3\,\text{cis}\Big(\frac{5\pi}{6}\Big) \;=\; \frac{3}{2}(-\sqrt{3}+i)\,,$$

$\underline{k=2}$
$$w_2 \;=\; 3\,\text{cis}\Big(\frac{9\pi}{6}\Big) \;=\; -3i.$$

Qu. 15 cont'd next sheet

15.(e) In polar form, $\quad -8-8\sqrt{3}i = 16\,\mathrm{cis}\Big(\frac{4\pi}{3}+2k\pi\Big) \quad (k\in\mathbb{Z})$

$$\Longrightarrow \quad w_k = (-8-8\sqrt{3}i)^{1/4} = 2\,\mathrm{cis}\Big(\frac{\pi}{3}+\frac{k\pi}{2}\Big) \qquad (k=0,\,1,\,2,\,3).$$

$\underline{k=0}$ $\quad w_0 = 2\,\mathrm{cis}\Big(\frac{\pi}{3}\Big) = 1+\sqrt{3}i\,,$

$\underline{k=1}$ $\quad w_1 = 2\,\mathrm{cis}\Big(\frac{5\pi}{6}\Big) = -\sqrt{3}+i\,,$

$\underline{k=2}$ $\quad w_2 = 2\,\mathrm{cis}\Big(\frac{4\pi}{3}\Big) = -1-\sqrt{3}i\,,$

$\underline{k=3}$ $\quad w_3 = 2\,\mathrm{cis}\Big(\frac{11\pi}{6}\Big) = \sqrt{3}-i.$

(f) In polar form, $\quad -64 = 64\,\mathrm{cis}(\pi+2k\pi) \qquad (k\in\mathbb{Z})$

$$\Longrightarrow \quad w_k = (-64)^{1/6} = 2\,\mathrm{cis}\Big(\frac{(2k+1)\pi}{6}\Big) \qquad (k=-3,\,-2,\,-1,\,0,\,1,\,2).$$

$\underline{k=-3}$ $\quad w_{-3} = 2\,\mathrm{cis}\Big(-\frac{5\pi}{6}\Big) = -\sqrt{3}-i\,,$

$\underline{k=-2}$ $\quad w_{-2} = 2\,\mathrm{cis}\Big(-\frac{3\pi}{6}\Big) = -2i\,,$

$\underline{k=-1}$ $\quad w_{-1} = 2\,\mathrm{cis}\Big(-\frac{\pi}{6}\Big) = \sqrt{3}-i\,,$

$\underline{k=0}$ $\quad w_0 = 2\,\mathrm{cis}\Big(\frac{\pi}{6}\Big) = \sqrt{3}+i\,,$

$\underline{k=1}$ $\quad w_1 = 2\,\mathrm{cis}\Big(\frac{3\pi}{6}\Big) = 2i\,,$

$\underline{k=2}$ $\quad w_2 = 2\,\mathrm{cis}\Big(\frac{5\pi}{6}\Big) = -\sqrt{3}+i.$

16.(a)
$$\begin{aligned} z^4+81 = 0 \quad \Longrightarrow \quad z^4 &= -81 = 81\,\mathrm{cis}(\pi) \\ \Longrightarrow \quad w_k = z &= 81^{1/4}\,\mathrm{cis}\Big(\frac{1}{4}(\pi+2k\pi)\Big) \\ &= 3\,\mathrm{cis}\Big(\frac{\pi}{4}+\frac{k\pi}{2}\Big) \qquad (k=0,\ 1,\ 2,\ 3). \end{aligned}$$

Therefore, taking principal values of the arguments, the roots are

$$w_0 = 3\,\mathrm{cis}\Big(\frac{\pi}{4}\Big) = \frac{3}{\sqrt{2}}(1+i), \qquad w_1 = 3\,\mathrm{cis}\Big(\frac{3\pi}{4}\Big) = \frac{3}{\sqrt{2}}(-1+i),$$

$$w_2 = 3\,\mathrm{cis}\Big(\frac{5\pi}{4}\Big) = 3\,\mathrm{cis}\Big(-\frac{3\pi}{4}\Big) = -\frac{3}{\sqrt{2}}(1+i),$$

$$w_3 = 3\,\mathrm{cis}\Big(\frac{7\pi}{4}\Big) = 3\,\mathrm{cis}\Big(-\frac{\pi}{4}\Big) = \frac{3}{\sqrt{2}}(1-i)\,.$$

(b)
$$\begin{aligned} z^6+1 = \sqrt{3}i \quad \Longrightarrow \quad z^6 &= -1+\sqrt{3}i = 2\,\mathrm{cis}\Big(\frac{2\pi}{3}\Big) \\ \Longrightarrow \quad w_k = z &= 2^{1/6}\,\mathrm{cis}\Big(\frac{1}{6}(\frac{2\pi}{3}+2k\pi)\Big) \\ &= 2^{1/6}\,\mathrm{cis}\Big(\frac{\pi}{9}+\frac{k\pi}{3}\Big) \qquad (k=0,\ 1,\ 2,\ 3,\ 4,\ 5). \end{aligned}$$

Therefore, taking principal values of the arguments in degrees, the roots are

$$w_0 = 2^{1/6}\,\mathrm{cis}(20^\circ), \qquad w_1 = 2^{1/6}\,\mathrm{cis}(80^\circ),$$

$$w_2 = 2^{1/6}\,\mathrm{cis}(140^\circ), \qquad w_3 = 2^{1/6}\,\mathrm{cis}(200^\circ) = 2^{1/6}\mathrm{cis}(-160^\circ),$$

$$w_4 = 2^{1/6}\,\mathrm{cis}(260^\circ) = 2^{1/6}\mathrm{cis}(-100^\circ), \qquad w_5 = 2^{1/6}\,\mathrm{cis}(320^\circ) = 2^{1/6}\mathrm{cis}(-40^\circ)\,.$$

17. $|2-2\sqrt{3}i| \;=\; 2\sqrt{1^2+(\sqrt{3})^2} \;=\; 4, \qquad \operatorname{Arg}(2-2\sqrt{3}i) \;=\; -\dfrac{\pi}{3}$

In polar form $\qquad 2-2\sqrt{3}i \;=\; 4\operatorname{cis}\Big(-\dfrac{\pi}{3}\Big)$

$$\Longrightarrow \quad w_k \;=\; (2-2\sqrt{3}i)^{1/3} \;=\; 4^{1/3}\operatorname{cis}\left(\frac{-\pi/3+2k\pi}{3}\right),$$

where $k=0,\,1,\,2$ for distinct roots.

Roots are:

$$(k=0) \qquad w_0 \;=\; 4^{1/3}\operatorname{cis}\left(\frac{-\pi/3}{3}\right) \;=\; 4^{1/3}\operatorname{cis}\Big(-\frac{\pi}{9}\Big),$$

$$(k=1) \qquad w_1 \;=\; 4^{1/3}\operatorname{cis}\left(\frac{-\pi/3+2\pi}{3}\right) \;=\; 4^{1/3}\operatorname{cis}\left(\frac{5\pi}{9}\right),$$

$$(k=2) \qquad w_2 \;=\; 4^{1/3}\operatorname{cis}\left(\frac{-\pi/3+4\pi}{3}\right) \;=\; 4^{1/3}\operatorname{cis}\left(\frac{11\pi}{9}\right) \;\equiv\; 4^{1/3}\operatorname{cis}\left(-\frac{7\pi}{9}\right),$$

or equivalently,

$$w_0 \;=\; 4^{1/3}\left(\cos\Big(-\frac{\pi}{9}\Big)+i\sin\Big(-\frac{\pi}{9}\Big)\right) = 1.5874\left(\cos\frac{\pi}{9}-i\sin\frac{\pi}{9}\right) \;=\; 1.492-0.543i,$$

$$w_1 \;=\; 1.5874\left(\cos\frac{5\pi}{9}+i\sin\frac{5\pi}{9}\right) = -0.276+1.563i,$$

$$w_2 \;=\; 1.5874\left(\cos\frac{7\pi}{9}-i\sin\frac{7\pi}{9}\right) = -1.216-1.020i.$$

18. $|-2-2\sqrt{3}i| \;=\; 2\sqrt{(-1)^2+(\sqrt{3})^2} \;=\; 4, \qquad \operatorname{Arg}(-2-2\sqrt{3}i) \;=\; -\dfrac{2\pi}{3}$

In polar form $\qquad -2-2\sqrt{3}i \;=\; 4\operatorname{cis}\Big(-\dfrac{2\pi}{3}\Big)$

$$\Longrightarrow \quad w_k \;=\; (-2-2\sqrt{3}i)^{1/4} \;=\; 4^{1/4}\operatorname{cis}\left(\frac{-2\pi/3+2k\pi}{4}\right) \;=\; \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}+\frac{k\pi}{2}\right),$$

where $k=0,\,1,\,2,\,3$ for distinct roots.

Roots are:

$$(k=0) \qquad w_0 \;=\; 4^{1/4}\operatorname{cis}\left(\frac{-2\pi/3}{4}\right) \;=\; \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}\right) \;=\; \frac{1}{\sqrt{2}}(\sqrt{3}-i),$$

$$(k=1) \qquad w_1 \;=\; 4^{1/4}\operatorname{cis}\left(\frac{-2\pi/3+2\pi}{4}\right) \;=\; \sqrt{2}\operatorname{cis}\left(\frac{\pi}{3}\right) \;=\; \frac{1}{\sqrt{2}}(1+\sqrt{3}i),$$

$$(k=2) \qquad w_2 \;=\; 4^{1/4}\operatorname{cis}\left(\frac{-2\pi/3+4\pi}{4}\right) \;=\; \sqrt{2}\operatorname{cis}\left(\frac{5\pi}{6}\right) \;=\; -\frac{1}{\sqrt{2}}(\sqrt{3}-i),$$

$$(k=3) \qquad w_3 \;=\; 4^{1/4}\operatorname{cis}\left(\frac{-2\pi/3+6\pi}{4}\right)$$

$$=\; \sqrt{2}\operatorname{cis}\left(\frac{4\pi}{3}\right) \;\equiv\; \sqrt{2}\operatorname{cis}\left(-\frac{2\pi}{3}\right) \;=\; -\frac{1}{\sqrt{2}}(1+\sqrt{3}i).$$

19. Follow the method adopted in the previous questions using the polar form:

$$4 + 4\sqrt{3}i \;=\; 8\,\mathrm{cis}\Big(\frac{\pi}{3} + 2k\pi\Big) \quad (k \in \mathbb{Z})$$

$$w_k \;=\; (4 + 4\sqrt{3}i)^{1/3} \;=\; 2\,\mathrm{cis}\Big[\frac{1}{3}(\frac{\pi}{3} + 2k\pi)\Big] \quad (k = 0,\ 1,\ 2).$$

$$w_0 = 2\,\mathrm{cis}\Big(\frac{\pi}{9}\Big), \quad w_1 = 2\,\mathrm{cis}\Big(\frac{7\pi}{9}\Big) \quad \text{and} \quad w_2 = 2\,\mathrm{cis}\Big(\frac{13\pi}{9}\Big).$$

20. In polar form, $-1 = 1\mathrm{cis}(\pi + 2k\pi) \quad (k \in \mathbb{Z})$. So the fifth roots of -1 are

$$w_k \;=\; 1^{1/5}\mathrm{cis}\Big(\frac{\pi}{5}(1+2k)\Big) \;=\; \mathrm{cis}\Big(\frac{\pi}{5}(1+2k)\Big) \qquad k = 0,\ 1,\ 2,\ 3,\ 4.$$

(The arguments are odd-multiples of $\frac{\pi}{5}$. Case $k = 2$ corresponds to $w_2 = \mathrm{cis}(\pi) = -1$.)

$w_0 = 0.8090 - 0.5878i, \qquad w_1 = 0.8090 + 0.5878i, \qquad w_2 = -0.3090 + 0.9511i,$

$w_3 = -1, \qquad w_4 = -0.3090 - 0.9511i. \qquad$ Check: $w_0 + w_1 + w_2 + w_3 + w_4 = 0.$

21. In polar form, $-16 \;=\; 16\,\mathrm{cis}\,(\pi + 2\pi k)$, where k is any integer.

Taking the fourth root,

$$w_k \;=\; (-16)^{1/4} \;=\; 16^{1/4}\,\mathrm{cis}\,\Big(\frac{1}{4}(\pi + 2\pi k)\Big) \;=\; 2\,\mathrm{cis}\,\Big(\frac{\pi}{4} + \frac{k\pi}{2}\Big) \quad (k \in \mathbb{Z}).$$

To find the four roots, we consider four successive values of k, for example, $k = 0,\ 1,\ 2,\ 3$:

$\underline{k=0} \qquad w_0 \;=\; 2\,\mathrm{cis}\,\Big(\frac{\pi}{4}\Big) \;=\; 2\Big(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\Big) \;=\; \sqrt{2}(1+i),$

$\underline{k=1} \qquad w_1 \;=\; 2\,\mathrm{cis}\,\Big(\frac{\pi}{4} + \frac{\pi}{2}\Big) \;=\; 2\,\mathrm{cis}\,\Big(\frac{3\pi}{4}\Big) \;=\; 2\Big(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\Big) \;=\; \sqrt{2}(1-i),$

$\underline{k=2} \qquad w_2 \;=\; 2\,\mathrm{cis}\,\Big(\frac{\pi}{4} + \pi\Big) \;=\; 2\,\mathrm{cis}\,\Big(\frac{5\pi}{4}\Big) \;=\; 2\Big(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\Big) \;=\; -\sqrt{2}(1+i),$

$\underline{k=3} \qquad w_3 \;=\; 2\,\mathrm{cis}\,\Big(\frac{\pi}{4} + \frac{3\pi}{2}\Big) \;=\; 2\,\mathrm{cis}\,\Big(\frac{7\pi}{4}\Big) \;=\; 2\Big(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\Big) \;=\; -\sqrt{2}(1-i).$

PSfrag replacements

x

y

O

$w_3 = -\sqrt{2}(1-i)$

$w_1 = \sqrt{2}(1-i)$

$\frac{\pi}{2}$

$w_2 = -\sqrt{2}(1+i)$

$w_0 = \sqrt{2}(1+i)$

$\sqrt{2}$

All four roots lie on a circle centred at $z = 0$ with radius $\sqrt{2}$, and are separated by an angular interval of $\frac{\pi}{2}$.

22. All four roots have the same modulus as $0.8 + 0.6i$, namely $\sqrt{0.8^2 + 0.6^2} = \sqrt{1} = 1$. So all the roots lie on the circle of radius 1 centred at the origin. There are four roots, so they are separated on the circle by angles $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$, i.e. they are separated by right angles.

PSfrag replacements

x

y

O

$-0.6 + 0.8i$

$0.6 - 0.8i$

$-0.8 - 0.6i$

$0.8 + 0.6i$

1

Once $0.8 + 0.6i$ is plotted on the Argand diagram, it is straightforward to sketch the other three roots.

23. In polar form, $8 = 8\text{cis}(0 + 2k\pi)$ $(k \in \mathbb{Z})$. So the cube roots of 8 are

$$w_k = 8^{1/3}\,\text{cis}\Big(\frac{2k\pi}{3}\Big) = 2\,\text{cis}\Big(\frac{2k\pi}{3}\Big), \qquad k = 0,\,1,\,2.$$

$$w_0 = 2\,\text{cis}(0) = 2, \qquad w_1 = 2\,\text{cis}\Big(\frac{2\pi}{3}\Big) = 2\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big) = -1 + \sqrt{3}i,$$

$$w_2 = 2\,\text{cis}\Big(\frac{4\pi}{3}\Big) \equiv 2\,\text{cis}\Big(-\frac{2\pi}{3}\Big) = 2\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big) = -1 - \sqrt{3}i.$$

If $(w - 3)^3 = 8$, then $w - 3 = 8^{1/3}$. In other words, $w = 3 + 8^{1/3}$. There are three roots, corresponding to the three cube roots of 8:

$w = 3 + 2 = 5$, $\quad w = 3 - 1 + \sqrt{3}i = 2 + \sqrt{3}i \quad$ and $\quad w = 3 - 1 - \sqrt{3}i = 2 - \sqrt{3}i$.

24. It is easy to confirm that $z = -2$ is a solution of the equation. So $(z+2)$ must be a linear factor of $z^3 + 6z + 20$.

$$\begin{array}{r|rrrr}
 & z^2 & -\ 2z & +\ 10 & \\
\hline
z+2 & z^3 & & +\ 6z & +\ 20 \\
 & z^3 & +\ 2z^2 & & \\
\hline
 & & -\ 2z^2 & +\ 6z & +\ 20 \\
 & & -\ 2z^2 & -\ 4z & \\
\hline
 & & & 10z & +\ 20 \\
 & & & 10z & +\ 20 \\
\hline
 & & & & 0
\end{array}$$

So $z^3 + 6z + 20 = (z+2)(z^2 - 2z + 10) = (z+2)\big((z-1)^2 + 9\big) = 0$ when $z = -2$, $z = 1 + 3i$ or $z = 1 - 3i$.

25.(a) $(1+iz)^3 = 8 = 2^3\,\mathrm{cis}\,(0+2k\pi) \quad\Longrightarrow\quad 1+iz = 2\,\mathrm{cis}\left(\dfrac{2k\pi}{3}\right) \quad (k=0,\,1,\,2).$

$$\underline{k=0} \qquad 1+iz \;=\; 2 \;\Longrightarrow\; iz=1 \;\Longrightarrow\; z \;=\; \frac{1}{i} = \frac{i}{i^2} = -i\,,$$

$$\underline{k=1} \qquad 1+iz \;=\; 2\mathrm{cis}\left(\frac{2\pi}{3}\right) \;=\; -1+i\sqrt{3} \;\Longrightarrow\; iz=-2+i\sqrt{3} \;\Longrightarrow\; z=2i+\sqrt{3}\,,$$

$$\underline{k=2} \qquad 1+iz \;=\; 2\mathrm{cis}\left(\frac{4\pi}{3}\right) \;=\; -1-i\sqrt{3} \;\Longrightarrow\; iz=-2-i\sqrt{3} \;\Longrightarrow\; z=2i-\sqrt{3}\,.$$

Hence the roots of the equation are $\sqrt{3}+2i$, $-\sqrt{3}+2i$ and $-i$.

(b) $z^4+13z^2+36=0 \quad\Longrightarrow\quad (z^2)^2+13z^2+36 \quad\Longrightarrow\quad (z^2+4)(z^2+9)=0$
(obtained by introducing $w=z^2$ and factorising the quadratic $w^2+13w+36$).

Hence either $z^2+4=0$ or $z^2+9=0$.
For $z^2+4=0$ either use the factorisation $z^2+4=z^2-4i^2=(z+2i)(z-2i)$ or find the two square roots of -4. Both methods produce $z=\pm 2i$.
Similarly, $z^2+9=0 \quad\Longrightarrow\quad z=\pm 3i.$ So the roots are $\pm 2i$, $\pm 3i$.

26. Since $z=0$ is not a solution, the equation is equivalent to

$$\left(\frac{z+1}{z}\right)^4 = 1 \quad\Longrightarrow\quad 1+\frac{1}{z} = 1,\; i,\; -1,\; -i \quad \text{(the fourth roots of unity).}$$

However there is no complex number z such that $1+\dfrac{1}{z}=1$. That leaves 3 cases.

$$1+\frac{1}{z} \;=\; -i \;\Longrightarrow\; \frac{1}{z} \;=\; -1+i \;\Longrightarrow\; z \;=\; \frac{1}{-1+i} \;=\; \frac{-1-i}{(-1+i)(-1-i)} \;=\; \frac{-1-i}{2}\,.$$

$$]1+\frac{1}{z} \;=\; -1 \;\Longrightarrow\; \frac{1}{z} \;=\; -2 \;\Longrightarrow\; z \;=\; -\frac{1}{2}\,.$$

$$1+\frac{1}{z} \;=\; -i \;\Longrightarrow\; \frac{1}{z} \;=\; -1-i \;\Longrightarrow\; z \;=\; \frac{1}{-1-i} \;=\; \frac{-1+i}{(-1-i)(-1+i)} \;=\; \frac{-1+i}{2}\,.$$

Hence we obtain the three solutions: $z=\dfrac{1}{2}(-1-i)$, $\dfrac{1}{2}(-1+i)$ and $-\dfrac{1}{2}$.

There are only three roots since <u>the equation is a disguised cubic</u>. It is straightforward to show that the quartic terms on either side of the equation cancel out and that

$$(z+1)^4 = z^4 \quad\Longleftrightarrow\quad 4z^3+6z^2+4z+1=0.$$

The left-hand side will factorise as $(2z+1)(2z^2+2z+1)$, from which the roots can be found. However, the first method is probably as quick and involves no guessing of factors.

27.(a) (i) $z^3 - 1 = 0 \implies z^3 = 1 = 1\,\mathrm{cis}(0 + 2k\pi) \implies z = \mathrm{cis}\Big(\dfrac{2k\pi}{3}\Big) \quad (k \in \mathbb{Z}).$

Taking $k = 0,\ 1,\ 2$ (say) gives the values $\quad z = 1,\ -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i,\ -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\,i.$

Hence $z^3 - 1 = (z-1)\Big(z + \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\,i\Big)\Big(z + \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i\Big).$

(ii) The second and third factors combine to give

$$\Big(z + \frac{1}{2}\Big)^2 - \Big(\frac{\sqrt{3}}{2}\,i\Big)^2 = z^2 + z + \frac{1}{4} - \frac{3}{4}\,i^2 = z^2 + z + 1.$$

Hence $z^3 - 1 = (z-1)(z^2+z+1)$, which is the common factorisation of $z^3 - 1$.

(b) (i) $z^4 + 1 = 0 \implies z^4 = -1 = 1\,\mathrm{cis}(\pi + 2k\pi) \implies z = \mathrm{cis}\dfrac{(2k+1)\pi}{4} \quad (k \in \mathbb{Z}).$

The values $k = -2,\ -1,\ 0,\ 1$ give, respectively,

$$\mathrm{cis}\Big(-\frac{3\pi}{4}\Big) = -\frac{1}{\sqrt{2}}(1+i)\,, \qquad \mathrm{cis}\Big(-\frac{\pi}{4}\Big) = \frac{1}{\sqrt{2}}(1-i)\,,$$
$$\mathrm{cis}\Big(\frac{\pi}{4}\Big) = \frac{1}{\sqrt{2}}(1+i)\,, \qquad \mathrm{cis}\Big(\frac{3\pi}{4}\Big) = \frac{1}{\sqrt{2}}(-1+i)\,.$$

Thus,

$$z^4 + 1 = \Big(z + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\Big)\Big(z - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\Big)\Big(z - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\Big)\Big(z + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\Big).$$

(ii) The linear factors pair off, first with last, second with third (corresponding to conjugate roots). We therefore obtain

$$\begin{aligned} z^4 + 1 &= \Big[\Big(z + \frac{1}{\sqrt{2}}\Big)^2 - \Big(\frac{1}{\sqrt{2}}\,i\Big)^2\Big]\Big[\Big(z - \frac{1}{\sqrt{2}}\Big)^2 - \Big(\frac{1}{\sqrt{2}}\,i\Big)^2\Big] \\ &= \Big[z^2 + \frac{2}{\sqrt{2}}\,z + \frac{1}{2} - \frac{1}{2}\,i^2\Big]\Big[z^2 - \frac{2}{\sqrt{2}}\,z + \frac{1}{2} - \frac{1}{2}\,i^2\Big] \\ &= (z^2 + \sqrt{2}z + 1)(z^2 - \sqrt{2}z + 1). \end{aligned}$$

(c) (i) $z^6 + 1 = 0 \implies z^6 = -1 = \mathrm{cis}(2k+1)\pi \implies z = \mathrm{cis}\dfrac{(2k+1)\pi}{6} \quad (k \in \mathbb{Z}).$

From calculations similar to Qu.**28(f)** we obtain the conjugate pairs of roots

$$i,\ -i,\ \frac{1}{2}(\sqrt{3}+i),\ \frac{1}{2}(\sqrt{3}-i),\ \frac{1}{2}(-\sqrt{3}+i),\ \frac{1}{2}(-\sqrt{3}-i).$$

Hence,

$$z^6 + 1 = (z-i)(z+i)\Big(z - \frac{\sqrt{3}}{2} - \frac{1}{2}\,i\Big)\Big(z - \frac{\sqrt{3}}{2} + \frac{1}{2}\,i\Big)\Big(z + \frac{\sqrt{3}}{2} - \frac{1}{2}\,i\Big)\Big(z + \frac{\sqrt{3}}{2} + \frac{1}{2}\,i\Big).$$

(ii) Combining consecutive pairs of linear factors (corresponding to conjugate pairs) gives

$$\begin{aligned} z^6 + 1 &= (z^2 - i^2)\Big[\Big(z - \frac{\sqrt{3}}{2}\Big)^2 - \Big(\frac{1}{2}\,i\Big)^2\Big]\Big[\Big(z + \frac{\sqrt{3}}{2}\Big)^2 - \Big(\frac{1}{2}\,i\Big)^2\Big] \\ &= (z^2+1)\Big[z^2 - \sqrt{3}z + \frac{3}{4} - \frac{1}{4}\,i^2\Big]\Big[z^2 + \sqrt{3}z + \frac{3}{4} - \frac{1}{4}\,i^2\Big] \\ &= (z^2+1)(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1). \end{aligned}$$

Qu. 27 cont'd next sheet

27.(d) (i) $z^5 - 1 = 0 \iff z^5 = 1 = 1\,\mathrm{cis}(0 + 2k\pi) \iff z = \mathrm{cis}\left(\frac{2k\pi}{5}\right) \quad (k \in \mathbb{Z})$.

Take $k = -2, -1, 0, 1, 2$ to give (including two conjugate pairs)

$$\cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right), \qquad \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right), \qquad 1,$$
$$\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right), \qquad \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right).$$

Thus,

$$\begin{aligned} z^5 - 1 &= (z-1)\left[\left(z - \cos\left(\frac{2\pi}{5}\right)\right) - i\sin\left(\frac{2\pi}{5}\right)\right]\left[\left(z - \cos\left(\frac{2\pi}{5}\right)\right) + i\sin\left(\frac{2\pi}{5}\right)\right] \\ &\qquad \times \left[\left(z - \cos\left(\frac{4\pi}{5}\right)\right) - i\sin\left(\frac{4\pi}{5}\right)\right]\left[\left(z - \cos\left(\frac{4\pi}{5}\right)\right) + i\sin\left(\frac{4\pi}{5}\right)\right]. \end{aligned}$$

(ii) The second and third linear factors combine to give

$$\begin{aligned} &\left[z - \cos\left(\frac{2\pi}{5}\right)\right]^2 - i^2\sin^2\left(\frac{2\pi}{5}\right) \\ &= z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) + \sin^2\left(\frac{2\pi}{5}\right) \\ &= z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1. \end{aligned}$$

The fourth and fifth factors combine similarly to give $z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + 1$. Hence

$$z^5 - 1 = (z-1)\left[z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1\right]\left[z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + 1\right].$$

28. If $z = 3i$ then $z^2 = -9$, $z^3 = -27i$, $z^4 = 81$ and $z^5 = 243i$. Hence

$$P(3i) = 243i + 9(-27i) + 8(-9) + 72 = (243 - 243)i - 72 + 72 = 0i + 0 = 0,$$

so $3i$ is a root of equation $P(z) = 0$.

Since $P(z)$ has real coefficients, $\overline{3i} = -3i$ is also a zero. Thus $P(z)$ has linear factors $(z - 3i)$ and $(z - (-3i))$, i.e. $P(z) \equiv (z - 3i)(z + 3i)Q(z) \equiv (z^2 + 9)Q(z)$, where $Q(z)$ is a polynomial of degree 3 given by $Q(z) = P(z)/(z^2 + 9)$. The division is as follows:

$$\begin{array}{r|llllll} \multicolumn{1}{r}{} & z^3 & & & & + 8 & \longleftarrow \underline{\underline{Q(z)}} \\ \cline{2-6} z^2 + 9 & z^5 + 0z^4 & + 9z^3 & + 8z^2 & + 0z & + 72 & \\ & \underline{z^5 \qquad\quad} & \underline{+ 9z^3} & & & & \\ & & & 8z^2 & & + 72 & \\ & & & \underline{8z^2} & & \underline{+ 72} & \\ & & & & & \underline{\underline{0}} & \end{array}$$

Therefore $Q(z) = z^3 + 8$.

Qu. 28 cont'd next sheet

28. (cont'd)

Note that $Q(-2) = 0$, so z^3+8 has a factor $(z-(-2)) = z+2$. Hence $z^3+8 \equiv (z+2)R(z)$, where the polynomial $R(z) = (z^3+8)/(z+2)$ can be found by long division as follows:

$$
\begin{array}{r|l}
 & z^2 \;-\; 2z \;+\; 4 \quad \longleftarrow \quad \underline{\underline{R(z)}} \\
\hline
z+2 & z^3 \;+\; 0z^2 \;+\; 0z \;+\; 8 \\
 & \underline{z^3 \;+\; 2z^2} \\
 & \quad -\; 2z^2 \;+\; 0z \;+\; 8 \\
 & \quad \underline{-\; 2z^2 \;-\; 4z} \\
 & \qquad\qquad 4z \;+\; 8 \\
 & \qquad\qquad \underline{4z \;+\; 8} \\
 & \qquad\qquad\qquad \underline{\underline{0}}
\end{array}
$$

Finally, the roots of $R(z) = z^2 - 2z + 4 = 0$ are given by

$$
\begin{aligned}
z^2 - 2z + 4 = 0 \quad &\Longrightarrow \quad (z-1)^2 - 1 + 4 \;=\; 0 \\
&\Longrightarrow \quad (z-1)^2 \;=\; -3 \;=\; -3i^2 \\
&\Longrightarrow \quad z - 1 \;=\; \pm\sqrt{3}\, i \\
&\Longrightarrow \quad z \;=\; 1 \pm \sqrt{3}\, i.
\end{aligned}
$$

Hence the roots of $P(z) = 0$ are $\pm 3i$, -2 and $1 \pm \sqrt{3}\, i$.

(i) $P(z)$ may be expressed as a product of linear factors as follows:

$$P(z) \;\equiv\; (z-3i)(z+3i)(z+2)(z-1+\sqrt{3}\, i)(z-1-\sqrt{3}\, i).$$

(ii) Alternatively, in terms of quadratic and linear factors involving only real coefficients:

$$P(z) \;\equiv\; (z^2+9)(z+2)(z^2-2z+4).$$

[Note that we could have found the zeros (and hence factors) of $Q(z)$ by solving $z^3+8 = 0$:

$$z^3 = -8 = 8\,\mathrm{cis}\,(\pi + 2\pi k) \qquad \Longrightarrow \qquad z = 2\,\mathrm{cis}\left((2k+1)\frac{\pi}{3}\right), \qquad k = 0,\ 1,\ 2.$$

Hence the zeros of $Q(z) = 0$ are

$$z = 2\,\mathrm{cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}\, i \qquad \text{or} \qquad 2\,\mathrm{cis}\,(\pi) = -2 \qquad \text{or} \qquad 2\,\mathrm{cis}\left(\frac{5\pi}{3}\right) = 1 - \sqrt{3}\, i.]$$

29. Let $w = z^2$ (this is a trick we can always use when a polynomial only involves even powers). Then we can rewrite our equations as $w^2 + 2w + 4 = 0$, a quadratic for w.

We can find solutions for w by completing the square, then use the formula for square roots to compute z:

$$w^2{+}2w{+}4 \;=\; (w{+}1)^2{+}3 \;=\; 0 \quad\Longrightarrow\quad (w{+}1)^2 \;=\; -3 \;=\; 3i^2 \quad\Longrightarrow\quad z^2 \;=\; w \;=\; -1{\pm}\sqrt{3}\,i.$$

In polar form, $-1{+}\sqrt{3}\,i = 2\,\mathrm{cis}\left(\dfrac{2\pi}{3}{+}2k\pi\right)$, $\quad -1{-}\sqrt{3}\,i = 2\,\mathrm{cis}\left(\dfrac{-2\pi}{3}{+}2k\pi\right) \quad (k \in \mathbb{Z})$.

Therefore, the two square roots of $-1 + \sqrt{3}\,i$ are

$$\begin{aligned}
\left(-1 + \sqrt{3}\,i\right)^{1/2} \;&=\; \left[\,2\,\mathrm{cis}\left(\frac{2\pi}{3} + 2k\pi\right)\right]^{1/2} \;=\; \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{3} + k\pi\right) \quad (k = 0,\,1)\\
&=\; \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{3}\right) \quad\text{and}\quad \sqrt{2}\,\mathrm{cis}\left(\frac{4\pi}{3}\right)\\
&=\; \frac{\sqrt{2}}{2}(1+3i) \quad\text{and}\quad -\frac{\sqrt{2}}{2}(1+3i)\\
&=\; \frac{1}{\sqrt{2}}(1+3i) \quad\text{and}\quad -\frac{1}{\sqrt{2}}(1+3i).
\end{aligned}$$

In a similar way, we can show that the two square roots of $-1 - \sqrt{3}\,i$ are

$$\left(-1 - \sqrt{3}\,i\right)^{1/2} \;=\; \frac{1}{\sqrt{2}}(1-3i) \quad\text{and}\quad -\frac{1}{\sqrt{2}}(1-3i).$$

So the four solutions of the quartic equation can be written (in a concise form) as:

$$z \;=\; \pm\frac{1}{\sqrt{2}}(1 \pm 3i)\,.$$

30. $z = 1 + i \quad\Longrightarrow\quad z^2 = 1 + 2i + i^2 = 2i$

$$\Longrightarrow\quad z^3 = z^2 \times z = 2i + 2i^2 = -2 + 2i, \quad z^4 = z^2 \times z^2 = -4.$$

Therefore, when $z = 1 + i$,

$$z^4 - 6z^3 + 23z^2 - 34z + 26 \;=\; -4 + 12 - 12i + 46i - 34 - 34i + 26 \;=\; 0.$$

So $z = 1 + i$ is a root. It follows that $z = \overline{1+i} = 1 - i$ is also a root because the polynomial has real coefficients.

By combining the two linear factors we obtain the real quadratic factor

$$(z - 1 - i)(z - 1 + i) \;=\; (z-1)^2 - i^2 \;=\; z^2 - 2z + 1 - i^2 \;=\; z^2 - 2z + 2.$$

Long division now allows us to factorise the quartic polynomial as

$$z^4 - 6z^3 + 23z^2 - 34z + 26 \;=\; (z^2 - 2z + 2)(z^2 - 4z + 13).$$

The other roots of the original equation satisfy

$$z^2 - 4z + 13 = 0 \quad\Longrightarrow\quad (z-2)^2 = -9 \quad\Longrightarrow\quad z - 2 = \pm 3i \quad\Longrightarrow\quad z = 2 \pm 3i.$$

Hence the four roots are $1 + i$, $1 - i$, $2 + 3i$ and $2 - 3i$.

31.(a) In polar form, $\sqrt{3} - i \;=\; 2\operatorname{cis}\left(-\frac{\pi}{6} + 2k\pi\right) \;=\; 2e^{i(-\frac{\pi}{6}+2k\pi)}$

$$\implies \quad \log(\sqrt{3} - i) \;=\; \ln(2) + i\left(-\frac{\pi}{6} + 2k\pi\right) \qquad (k \in \mathbb{Z}).$$

(b) In polar form, $2 + 2i \;=\; \sqrt{8}\operatorname{cis}\left(\frac{\pi}{4} + 2k\pi\right) \;=\; 2^{\frac{3}{2}}\, e^{i(\frac{\pi}{4}+2k\pi)}$

$$\implies \quad \log(2 + 2i) \;=\; \ln(\sqrt{8}) + i\left(\frac{\pi}{4} + 2k\pi\right) \;=\; \frac{3}{2}\ln 2 + i\left(\frac{\pi}{4} + 2k\pi\right) \qquad (k \in \mathbb{Z}).$$

(c) In polar form, $-i \;=\; 1\operatorname{cis}\left(-\frac{\pi}{2} + 2k\pi\right) \;=\; 1e^{i(-\frac{\pi}{2}+2k\pi)}$

$$\implies \quad \log(-i) \;=\; \ln 1 + i\left(-\frac{\pi}{2} + 2k\pi\right) \;=\; i\left(-\frac{\pi}{2} + 2k\pi\right) \qquad (k \in \mathbb{Z}).$$

(d) $e^{3-4i} \;=\; e^3\, e^{-4i} \;=\; e^3\big(\cos(4) - i\sin(4)\big) \;=\; e^3\cos(4) - i\, e^3\sin(4).$

Real part is $e^3\cos(4)$, imaginary part is $-e^3\sin(4)$.