UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS
Lecture Notes for Week 6

4. Complex Numbers – Second Half

§4.8 de Moivre's Theorem and Applications

Recall that

$$|z_1\, z_2| = |z_1|\,|z_2| \qquad \text{and} \qquad \arg(z_1\, z_2) = \arg z_1 + \arg z_2 \tag{1}$$

$$(\text{cis } \theta_1)(\text{cis } \theta_2) = \text{cis}\,(\theta_1 + \theta_2) \tag{2}$$

$$\frac{1}{\text{cis } \theta} = \text{cis}\,(-\theta). \tag{3}$$

Equation (1) with $z_1 = z_2 = z$ gives $|z^2| = |z|^2$ and $\arg(z^2) = 2\arg(z)$.

The more general result is obtained in a similar way for natural number n:

$$|z^n| = |z|^n \qquad \text{and} \qquad \arg(z^n) = n\arg(z).$$

If we multiply together several complex numbers in polar form we have

$$\begin{aligned} z_1\, z_2\, \cdots\, z_n &= (r_1 \text{ cis } \theta_1)\,(r_2 \text{ cis } \theta_2)\, \ldots\, (r_n \text{ cis } \theta_n) \\ &= (r_1\, r_2\, \ldots\, r_n)\,(\text{cis } \theta_1)\,(\text{cis } \theta_2)\, \ldots\, (\text{cis } \theta_n) \\ &= (r_1\, r_2\, \ldots\, r_n) \text{ cis}\,(\theta_1 + \theta_2 + \ldots + \theta_n), \end{aligned}$$

where $z_k = r_k \text{ cis } \theta_k \quad (k = 1, 2, \ldots, n)$ and n is a positive integer.

If we now let $z_1 = z_2 = \ldots = z_n = z$ where $z = r \text{ cis } \theta$, we see that

$$z^n = r^n \text{ cis}\,(n\theta) = r^n \left(\cos(n\theta) + i\sin(n\theta)\right).$$

In the special case where $r = 1$ our expression simplifies to

$$(\text{cis } \theta)^n = \text{cis}\,(n\theta), \tag{4}$$

or, alternatively $\qquad (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$

This is **de Moivre's Theorem**, proved here for positive integers.

In fact, de Moivre's Theorem also holds for negative integers and for fractions. To prove this, if n is a **negative** integer then let $n = -m$, where m is a positive integer. Hence, on using (3) and (4), we have

$$(\text{cis}\ \theta)^n = (\text{cis}\ \theta)^{-m} = \frac{1}{(\text{cis}\ \theta)^m} = \frac{1}{\text{cis}\,(m\theta)} = \text{cis}\,(-m\theta) = \text{cis}\,(n\theta). \qquad (5)$$

Equations (4) and (5) together give de Moivre's Theorem for an integral index, namely

$$\boldsymbol{(\text{cis}\ \theta)^n = \text{cis}\,(n\theta)} \qquad \textbf{where } \boldsymbol{n} \textbf{ is any integer.} \qquad (6)$$

Now let $n = p/q$, where p and $q(\neq 0)$ are integers with no common factor. Then, on using equation (6), we have

$$\left[\text{cis}\left(\frac{\theta}{q}\right)\right]^q = \text{cis}\left(\frac{q\theta}{q}\right) = \text{cis}\ \theta.$$

Hence, taking the q-th root of both sides, $\text{cis}\,(\theta/q)$ is one of the values of $(\text{cis}\ \theta)^{1/q}$. If we now raise both sides to the power of p, it follows that $(\text{cis}\,(\theta/q))^p$ is one of the values of $((\text{cis}\ \theta)^{1/q})^p$. Finally, on using equation (6) again,

$$\boldsymbol{\text{cis}\left(\frac{p}{q}\,\theta\right)} \quad \textbf{is one of the values of} \quad \boldsymbol{(\text{cis}\ \theta)^{p/q}} \qquad (7)$$

where p and q are integers with no common factor.
This is de Moivre's Theorem for a fractional index.

Equation (7) may seem a little strange – why is $\text{cis}\,(p\theta/q)$ only one of the values?

Consider as an example the special case $p = 1$, $q = 2$ and $\theta = \pi$. First of all,

$$\text{cis}\ \theta = \text{cis}\ \pi = (\cos\pi + i\sin\pi) = -1 + 0 = -1 \qquad \text{and}$$

$$\text{cis}\left(\frac{p}{q}\theta\right) = \text{cis}\left(\frac{\theta}{2}\right) = \text{cis}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i.$$

As expected, we have shown that i is one of the values of $(-1)^{1/2}$. (This should be obvious because $i^2 = -1$.) However, it's not the only root – there is another, namely $-i$.

(Check: $(-i)^2 = (-1)^2 \times i^2 = 1 \times (-1) = -1$.)

In fact, we will see later that when we take the n-th root of a complex number, we will find **n distinct roots**.

Example

4.1 Use de Moivre's theorem to simplify (i) $z = (1-i)^8$, (ii) $\dfrac{(-\sqrt{3}-i)^4}{(\sqrt{2}-\sqrt{2}i)^2}$.

(i) Since $1 - i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$, we have,

$$\begin{aligned} z &= (\sqrt{2})^8\left[\cos\left(-8\times\frac{\pi}{4}\right) + i\sin\left(-8\times\frac{\pi}{4}\right)\right] \\ &= 2^4\Big(\cos(-2\pi) + i\sin(-2\pi)\Big) = 16\ \text{cis}\,(0) = 16(1+0i) = 16. \end{aligned}$$

(ii) In polar form, $-\sqrt{3} - i = 2\ \text{cis}\left(-\frac{5\pi}{6}\right)$, $\sqrt{2}-\sqrt{2}i = 2\ \text{cis}\left(-\frac{\pi}{4}\right)$.

$$\begin{aligned} \frac{(-\sqrt{3}-i)^4}{(\sqrt{2}-\sqrt{2}i)^2} &= \frac{\left[2\ \text{cis}\left(-\frac{5\pi}{6}\right)\right]^4}{\left[2\ \text{cis}\left(-\frac{\pi}{4}\right)\right]^2} \\ &= \frac{16\ \text{cis}\left(-\frac{10\pi}{3}\right)}{4\ \text{cis}\left(-\frac{\pi}{2}\right)} \\ &= 4\ \text{cis}\left(-\frac{10\pi}{3}+\frac{\pi}{2}\right) \\ &= 4\ \text{cis}\left(-\frac{17\pi}{6}\right) \\ &= 4\ \text{cis}\left(-\frac{17\pi}{6}+2\pi\right) \\ &= 4\ \text{cis}\left(-\frac{5\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = \underline{\underline{-2\sqrt{3}-2i}}. \end{aligned}$$

Examples covered in video

4.2 Use de Moivre's theorem to simplify

(i) $(1-i)^{10}(-1+\sqrt{3}i)^8$, (ii) $\dfrac{(1-i)^8}{(-1+\sqrt{3}i)^3}$.

§4.9 Application 1: Trigonometric Identities

de Moivre's Theorem with n a positive integer can be used to obtain $\cos(n\theta)$ and $\sin(n\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$. To accomplish this we expand the left hand side of equation (6) using the binomial theorem, namely

$$(a+b)^n \;=\; a^n + na^{n-1}b + \frac{n(n-1)}{2!}\,a^{n-2}b^2 + \ldots + \frac{n(n-1)}{2!}\,a^2b^{n-2} + nab^{n-1} + b^n$$

with $a = \cos\theta$ and $b = i\sin\theta$, and then equate real and imaginary parts. (Note that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.)

Alternatively, it is also possible to obtain $\sin^n\theta$ and $\cos^n\theta$ in terms of sines and cosines of multiple angles. This is very useful for the integration of $\sin^n\theta$ and $\cos^n\theta$.

The trick in this case is to write

$$z \;=\; \cos\theta + i\sin\theta \qquad \text{so that} \qquad z^{-1} \;=\; \frac{1}{z} \;=\; \cos(-\theta) + i\sin(-\theta) \;=\; \cos\theta - i\sin\theta.$$

Thus, by de Moivre's Theorem (6) with m a positive integer,

$$\begin{aligned} z^m &= \cos(m\theta) + i\sin(m\theta), \\ z^{-m} = \frac{1}{z^m} &= \cos(m\theta) - i\sin(m\theta). \end{aligned}$$

Hence, by adding or subtracting the appropriate terms, with $\boldsymbol{z = \text{cis}\,\theta}$:

$$\boldsymbol{\cos\theta \;=\; \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right),} \tag{8}$$

$$\boldsymbol{\cos(m\theta) \;=\; \frac{1}{2}\left(z^m + \frac{1}{z^m}\right), \qquad \sin(m\theta) \;=\; \frac{1}{2i}\left(z^m - \frac{1}{z^m}\right).} \tag{9}$$

From equation (8), $\cos^n\theta$ or $\sin^n\theta$ may be expressed in terms of a binomial expansion of either $(z + z^{-1})^n$ or $(z - z^{-1})^n$, respectively.

Similarly, equation (9) may be used to write expressions involving z^m and z^{-m} $(m = 0,\, 1, \ldots, n)$ in terms of $\cos(m\theta)$ or $\sin(m\theta)$.

Example

4.3 By using $\cos\theta = \frac{1}{2}(z + z^{-1})$ where $z = \text{cis } \theta$, find constants a and b such that $\cos^3\theta = a\cos\theta + b\cos(3\theta)$.

$$\begin{aligned}
\cos^3\theta &= \left[\frac{1}{2}\Big(z + \frac{1}{z}\Big)\right]^3 \\
&= \frac{1}{8}\Big(z + \frac{1}{z}\Big)^3 \\
&= \frac{1}{8}\Big(z^3 + 3z^2 \times \frac{1}{z} + 3z \times \frac{1}{z^2} + \frac{1}{z^3}\Big) \\
&= \frac{1}{8}\Big(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\Big) = \frac{3}{8}\Big(z + \frac{1}{z}\Big) + \frac{1}{8}\Big(z^3 + \frac{1}{z^3}\Big) = \underline{\underline{\frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)}}.
\end{aligned}$$

Examples covered in video

4.4 Express $\cos(3\theta)$ in terms of $\cos\theta$ and $\sin(3\theta)$ in terms of $\sin\theta$.

4.5 Express $\sin^4\theta$ in terms of cosines of integer multiples of θ.

4.6 Express $\sin\theta\,\cos^3\theta$ as a linear combination of sines of multiples of θ. Use this result to evaluate $\displaystyle\int_0^{\pi/4} \sin\theta\,\cos^3\theta\,\mathrm{d}\theta$.

§4.10 Application 2: Roots of Complex Numbers

We know that all positive real numbers have two real square roots, and all real numbers have a single real cube root. More generally, if n is a natural number and x is a real number, then there is one real n-th root if n is odd (denoted $\sqrt[n]{x}$, or $x^{1/n}$) and two real n-th roots, $\sqrt[n]{x}$ and $-\sqrt[n]{x}$, if n is even and $x > 0$.

When we generalise to complex numbers the situation alters significantly.

Example

4.7 Find the cube roots of -1.

The cube roots of -1 will be the zeros of the polynomial $z^3 + 1$. We know that if $z = -1$ then $z^3 + 1 = (-1)^3 + 1 = -1 + 1 = 0$, so $\big(z - (-1)\big) = (z + 1)$ must be a factor of $z^3 + 1$.

In other words, $z^3 + 1 = (z + 1)q(z)$, for some quadratic $q(z)$. We can obtain $q(z)$ by long division (or synthetic division):

$$
\begin{array}{r|l}
 & z^2 - z + 1 \quad \longleftarrow \quad \underline{\underline{q(z)}} \\
\cline{2-2}
z+1 & z^3 \qquad\qquad + 1 \\
 & z^3 + z^2 \\
\cline{2-2}
 & \quad - z^2 \qquad + 1 \\
 & \quad - z^2 - z \\
\cline{2-2}
 & \qquad\qquad z + 1 \\
 & \qquad\qquad z + 1 \\
\cline{2-2}
 & \qquad\qquad\quad \underline{\underline{0}}
\end{array}
$$

The quadratic $q(z) = z^2 - z + 1$ is irreducible and has complex roots

$$z = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

We can check that both of these satisfy $z^3 + 1 = 0$:

$$
\begin{aligned}
\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^3 &= \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2} i\right) + 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} i\right)^2 + \left(\frac{\sqrt{3}}{2} i\right)^3 \\
&= \frac{1}{8} + \frac{3\sqrt{3}}{8} i + \frac{3 \times 3}{8} i^2 + \frac{3\sqrt{3}}{8} i^3 \\
&= \frac{1}{8} - \frac{9}{8} + \frac{3\sqrt{3}}{8} i - \frac{3\sqrt{3}}{8} i = -1.
\end{aligned}
$$

Similarly, we can show that $\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^3 = -1$. So these two numbers are the complex cube roots of -1. Notice that they are complex conjugates.

As with many of the other processes we have seen, finding n-th roots is simplified considerably if we use the polar form of a complex number.

Suppose that $z_0 = r_0 \text{ cis}\,(\theta_0)$ is an n-th root of the complex number $z = r \text{ cis}\,(\theta)$.

Then $(z_0)^n = z$ and so, by de Moivre's theorem,

$$r \text{ cis}\,(\theta) \;=\; \big(r_0 \text{ cis}\,(\theta_0)\big)^n \;=\; (r_0)^n \text{ cis}\,(n\theta_0).$$

This equation can be solved by equating the moduli and the arguments. That is,

$$r_0 = r^{1/n}, \qquad \theta_0 = \frac{\theta}{n}.$$

Since r and r_0 are real and positive (assume $z \neq 0$), we can take r_0 to be the unique real and positive n-th root of r.

This simple equation can be used to find a single n-th root of any complex number, but we can do more as the argument of a complex number is a multi-valued function. Whether we take the argument of the complex number z to be

$$\theta, \quad \theta + 2\pi, \quad \theta - 2\pi \quad \text{or} \quad \theta + 2k\pi \quad (k \in \mathbb{Z})$$

will have no bearing on where the complex number lies in the Argand diagram. However

$$\frac{\theta}{n}, \quad \frac{(\theta + 2\pi)}{n}, \quad \frac{(\theta - 2\pi)}{n} \quad \text{or} \quad \frac{(\theta + 2k\pi)}{n} \quad (k \in \mathbb{Z})$$

can represent very different angles. We use this fact to motivate the following strategy for finding complex n-th roots.

Suppose we are given a complex number z and wish to find the n distinct n-th roots of z, namely the n distinct values of $z^{1/n}$.

- **Write z in polar form $z = r \text{ cis}\,(\theta + 2k\pi)$ where k is any integer.**
- **The n-th roots of z are given by**

$$w_k \;=\; z^{1/n} \;=\; r^{1/n} \text{ cis}\left(\frac{\theta + 2\pi k}{n}\right) \tag{10}$$

  where k is an integer and $r^{1/n}$ is the real and positive n-th root of r.
- **The n distinct roots are found by taking <u>n consecutive values of k</u>, for example, $k = 0, 1, \ldots, n-1$.**

  Each of these roots lies on the circle centred at $z = 0$ with radius $r^{1/n}$, and the roots are regularly spaced at angular intervals of $2\pi/n$.
- **Once we have found the polar form of the roots we can convert them into standard form.**

Examples covered in video

4.8 Find all the square roots of $-1-\sqrt{3}\,i$ (or, equivalently, solve $z^2+1+\sqrt{3}\,i=0$ for z).

4.9 Find all three cube roots of 1.

4.10 Find the fourth roots of $z=-8+8\sqrt{3}i$.

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§4.11 Polynomial Equations

We often require solutions of polynomial equations of the form

$$a_0\,z^n + a_1\,z^{n-1} + a_2\,z^{n-2} + \cdots + a_{n-1}\,z + a_n \;=\; 0 \tag{11}$$

where $a_0 \neq 0$, $a_1, \ldots, a_n$ are given complex numbers and n is a positive integer called the **degree** of the equation. Solutions of polynomial equation (11) are called the **zeros of the polynomial** or the **roots of the equation**.

When finding real solutions to polynomial equations we try to factorise the polynomial, possibly using long division. This method will still hold when complex numbers are involved, and by applying our formulae for complex n-th roots, we can find complex solutions to equations of the form $p(z)=0$ where $p(z)$ is a polynomial.

For example, consider the polynomial equation

$$p(z) \;=\; z^n - z_0 \;=\; 0\,,$$

where $z_0 \in \mathbb{C}$. Since $z^n = z_0$, z is a solution to this equation if and only if z is an n-th root of z_0. We now know that z_0 has n distinct n-th roots. Labelling these roots $w_1, w_2, \ldots, w_n$, we have $p(z)=0$ if and only if $z=w_i$. By the Remainder Theorem, $(z-w_i)$ must be a factor of $p(z)$ for each $i=1,\ldots,n$. Since these n factors are all linear in z, we must be able to rewrite $p(z)$ as

$$p(z) \;=\; z^n - z_0 \;=\; (z-w_1)(z-w_2)\ldots(z-w_n).$$

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§4.12 Factorising Polynomials

The **fundamental theorem of algebra** states that any polynomial, $p_n(z)$, of degree n with real or complex coefficients can be factorised completely into linear factors of the form $(z-z_i)$, where z_i is either real or complex. That is,

$$p_n(z) \;=\; a_0(z-z_1)(z-z_2)\ldots(z-z_n),$$

where a_0 is the coefficient of z^n in $p_n(z)$, and some of the factors may be repeated.

Theorem

Roots of polynomial equations in which the ***coefficients are all real*** *occur in complex conjugate pairs. That is, given a polynomial*

$$P(z) \;=\; a_0\, z^n + a_1\, z^{n-1} + a_2\, z^{n-2} + \ldots + a_{n-1}\, z + a_n$$

in which the coefficients a_0, $a_1, \ldots,$ a_n are all real, if $P(\alpha) = 0$ for some complex number α then $P(\overline{\alpha}) = 0$ as well.

Proof

As a preliminary, note that if n is a positive integer then

$$\overline{z_1 + z_2 + \ldots + z_n} \;=\; \overline{z}_1 + \overline{z}_2 + \ldots + \overline{z}_n \tag{A}$$

$$\overline{z_1\, z_2 \ldots\, z_n} \;=\; \overline{z}_1\, \overline{z}_2 \ldots \overline{z}_n \tag{B}$$

$$\overline{z^n} \;=\; (\overline{z})^n \tag{C}$$

Now suppose α is a root of $P(z) = 0$ so that

$$a_0\, \alpha^n + a_1\, \alpha^{n-1} + \ldots + a_{n-1}\, \alpha + a_n \;=\; 0. \tag{D}$$

On taking the complex conjugate of equation (D) we obtain

$$\overline{a_0\, \alpha^n + a_1\, \alpha^{n-1} + \ldots + a_{n-1}\, \alpha + a_n} \;=\; \overline{0} \;=\; 0.$$

Using equation (A) we have

$$\overline{a_0\, \alpha^n} + \overline{a_1\, \alpha^{n-1}} + \ldots + \overline{a_{n-1}\, \alpha} + \overline{a}_n \;=\; 0. \tag{E}$$

On using equations (B) and (C) we have $\overline{a_0\, \alpha^n} = \overline{a}_0(\overline{\alpha})^n$, etc. Hence equation (E) gives

$$\overline{a}_0(\overline{\alpha})^n + \overline{a}_1(\overline{\alpha})^{n-1} + \ldots + \overline{a}_{n-1}\, \overline{\alpha} + \overline{a}_n \;=\; 0. \tag{F}$$

Thus, if a_0, $a_1, \ldots,$ a_n are all real (so that $\overline{a}_0 = a_0$, $\ldots$, $\overline{a}_n = a_n$), equation (F) becomes

$$a_0(\overline{\alpha})^n + a_1(\overline{\alpha})^{n-1} + \ldots + a_{n-1}(\overline{\alpha}) + a_n = 0,$$

In other words, $P(\overline{\alpha}) = 0$.

The following observation helps us to **factorize polynomials** that have **real** coefficients:

The Remainder Theorem states that $(z-\alpha)$ is a **factor of the polynomial** $P(z)$ if and only if $P(\alpha) = 0$, i.e. if and only if α is a root of $P(z) = 0$. It follows that, if the coefficients in $P(z)$ are real, both $(z-\alpha)$ **and** $(z-\overline{\alpha})$ are factors of $P(z)$. Thus $P(z) \equiv (z-\alpha)(z-\overline{\alpha})\,Q(z)$, where $Q(z)$ is a polynomial of degree 2 less than the degree of $P(z)$. To find $Q(z)$ we need to divide $P(z)$ by $(z-\alpha)(z-\overline{\alpha})$.

Examples covered in video

4.11 Verify that $z = 3$ is a root of the equation

$$P(z) \;=\; z^3 + z^2 - 7z - 15 \;=\; 0.$$

Hence find all roots of this equation. Express $P(z)$ as
(i) the product of linear factors, and
(ii) the product of factors with only real coefficients.

4.12 Verify that $z = -1 + i$ is a root of the equation

$$P(z) \;=\; z^4 - 4z^3 + 8z + 20 \;=\; 0.$$

Hence find all roots of the equation. Express $P(z)$ as
(i) the product of linear factors, and
(ii) the product of quadratic factors with only real coefficients.

4.13 Find all solutions of $(z-3)^4 - 16 = 0$.

§4.13 The Sum of The Roots

If w_1 and w_2 are the two square roots of a number we know that $w_2 = -w_1$, so $w_1 + w_2 = 0$. We can show that this zero sum property generalises to n-th roots of a complex number .

Suppose that the complex number a has distinct n-th roots $w_0, w_1, \ldots, w_{n-1}$. Then using the methods introduced in the previous section, we can factorise the polynomial $z^n - a$ as

$$\begin{aligned} z^n - a &= (z - w_0)(z - w_1)\ldots(z - w_{n-2})(z - w_{n-1}) \\ &\equiv z^n - (w_0 + w_1 + \ldots + w_{n-1})z^{n-1} + (\text{smaller powers of } z) \end{aligned}$$

Now we equate the z^{n-1} terms on either side of this expression:

$$0 = -(w_0 + w_1 + \cdots + w_{n-1}),$$

This confirms that the sum of all the n-th roots of a complex number is zero.

Example

4.14 Confirm that the sum of the roots is zeros in each of the following cases:

(i) the cube roots of 1,

(ii) the fourth roots of $-8 + 8\sqrt{3}i$.

(i) In Example 3.28 we saw that the cube roots of 1 are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$. The sum of the roots is

$$1 - \frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i = 0.$$

(ii) In Example 3.29 we saw that the fourth roots of $-8 + 8\sqrt{3}i$ are $\sqrt{3} + i$, $-1 + \sqrt{3}i$, $-\sqrt{3} - i$ and $1 - \sqrt{3}i$. The sum of the roots is

$$\sqrt{3} + i - 1 + \sqrt{3}i - \sqrt{3} - i + 1 - \sqrt{3}i = 0.$$

§4.14 Introduction to the Complex Exponential and Logarithm

The complex valued functions e^z and $\log(z)$ may be defined rigorously. Here we introduce them in a **non-rigorous** way.

Note: later in the Semester you will see that many functions can be expressed as the sum of a sequence of terms involving integer powers of the x (or the appropriate variable). For example, for any $x \in \mathbb{R}$, the exponential function can be written as

$$\mathrm{e}^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots .$$

For values of x close to zero, if we replace the value of x in the right hand side of the equation and start adding up the terms, very quickly the sum will approach the value of e^x.

In a similar way, the cosine and sine functions can be expressed as

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots ,\\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots .\end{aligned}$$

for any real values of x close to zero, although these definitions also extend to complex numbers. Each expansion is called the **Maclaurin Series** for the function.

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§4.15 The Complex Exponential

The Maclaurin series for e^x, with x real, is

$$\mathrm{e}^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots .$$

If we replace all occurrences of x in this series by $i\theta$, we obtain

$$\begin{aligned}\mathrm{e}^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots .\\ &= \underline{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)} \;+\; i\underline{\underline{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}}\\ &\qquad\quad \uparrow \qquad\qquad\qquad\qquad \nearrow\\ &\equiv \underline{\cos(\theta)} \;+\; i\;\underline{\underline{\sin(\theta)}}\end{aligned}$$

where we have recognised the Maclaurin series for $\cos\theta$ and $\sin\theta$.

Thus we have

$$\boldsymbol{\mathrm{e}^{i\theta} \;=\; \cos\theta + i\sin\theta \;=\; \mathrm{cis}\,(\theta) \qquad\qquad (\theta \in \mathbb{R}).}$$

This is known as Euler's formula and defines $\mathrm{e}^{i\theta}$.

- Note, earlier we introduced the result

$$\text{cis}\,(\theta_1 + \theta_2) \;=\; \text{cis}\;\theta_1\;\text{cis}\;\theta_2$$

and de Moivre's Theorem (4)

$$(\text{cis}\;\theta)^n \;=\; \text{cis}\,(n\theta),$$

where n is an integer. These results may now be interpreted as simple applications of the laws of indices, namely

$$\mathrm{e}^{i(\theta_1+\theta_2)} \;=\; \mathrm{e}^{i\theta_1}\,\mathrm{e}^{i\theta_2} \qquad \text{and} \qquad \left(\,\mathrm{e}^{i\theta}\,\right)^n \;=\; \mathrm{e}^{in\theta},$$

respectively.

- Also note that

$$\mathrm{e}^{i\theta} \;=\; \cos\theta \;+\; i\sin\theta \quad\Longrightarrow\qquad \mathrm{e}^{-i\theta} \;=\; \cos\theta \;-\; i\sin\theta.$$

By adding or subtracting the equations for $\mathrm{e}^{i\theta}$ and $\mathrm{e}^{-i\theta}$, we obtain

$$\boxed{\boldsymbol{\cos\theta \;=\; \frac{\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}}{2} \qquad \textbf{and} \qquad \sin\theta \;=\; \frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{2i}.}}$$

These are equivalent to equations (8) where we used z to represent $\mathrm{e}^{i\theta}$.

- If $z = x + iy$ is any complex number ($x,\, y \in \mathbb{R}$), its <u>**exponential e^z**</u> may be defined as

$$\boxed{\boldsymbol{\mathrm{e}^z \;=\; \mathrm{e}^{x+iy} \;=\; \mathrm{e}^x\,\mathrm{e}^{iy} \;=\; \mathrm{e}^x(\cos y \;+\; i\sin y) \qquad (x,\; y \in \mathbb{R}).}}$$

We may use Euler's formula to establish properties of e^z. For example, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ $(x_1,\, x_2,\, y_1,\, y_2 \in \mathbb{R})$, we have

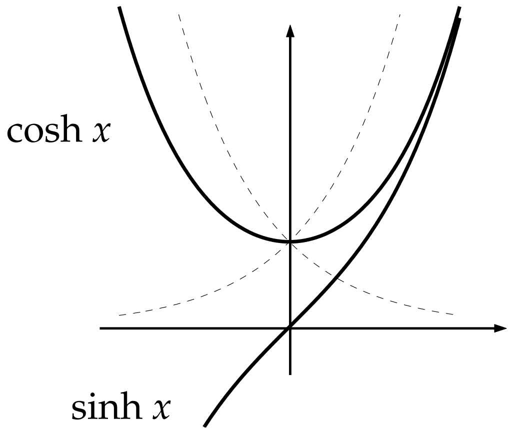
$$\mathrm{e}^{z_1}\mathrm{e}^{z_2} \;=\; \mathrm{e}^{x_1+iy_1}\,\mathrm{e}^{x_2+iy_2} \;=\; \mathrm{e}^{x_1}\;\text{cis}\,(y_1)\;\mathrm{e}^{x_2}\;\text{cis}\,(y_2) \;=\; \mathrm{e}^{x_1+x_2}\;\text{cis}\,(y_1+y_2) \;=\; \mathrm{e}^{z_1+z_2},$$

i.e. the usual rule for indices holds.

§4.16 Hyperbolic Functions

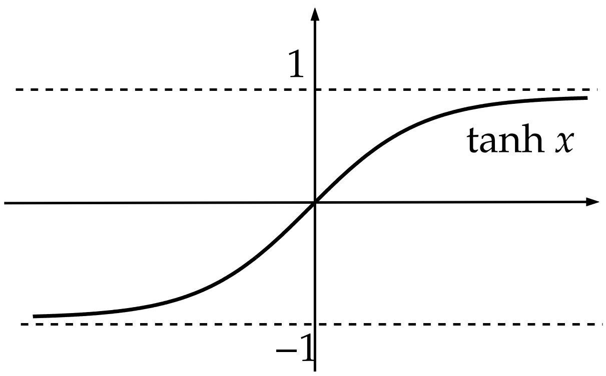
We define the **hyperbolic sine and cosine** functions by

$$\boldsymbol{\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad (x \in \mathbb{R})}$$



As you might expect, the hyperbolic tangent function is defined as

$$\boldsymbol{\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$$



Other hyperbolic functions can be defined in terms of these.

$$\coth x = \frac{1}{\tanh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

The hyperbolic functions defined above are real functions of real numbers, but they are a useful tool in complex number theory. $\sinh x$ is an odd function, $\cosh x$ is an even function, just like their trigonometric counterparts $\sin x$ and $\cos x$. Recall the definition of the trigonometric functions in terms of exponentials,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \textbf{and} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

If we assume that the domain and definition of the hyperbolic functions can be extended from $\mathbb{R}$ to $\mathbb{C}$, then it is clear that hyperbolic and trigonometric functions are related:

$$\begin{aligned}
\cosh(i\theta) &= \frac{e^{i\theta}+e^{-i\theta}}{2} = \cos(\theta) \\
\cos(i\theta) &= \frac{e^{i^2\theta}+e^{-i^2\theta}}{2} = \frac{e^{-\theta}+e^{\theta}}{2} = \cosh(\theta) \\
\sinh(i\theta) &= \frac{e^{i\theta}-e^{-i\theta}}{2} = i\sin(\theta) \\
\sin(i\theta) &= \frac{e^{i^2\theta}-e^{-i^2\theta}}{2i} = \frac{-i(e^{-\theta}-e^{\theta})}{2} = i\sinh(\theta).
\end{aligned}$$

We do not extend this analysis here, but these properties will be useful in classes later in your course.

§4.17 The Complex Logarithm

Consider a complex number in polar form $z = r\,e^{i\theta}$, where $r = |z|$ and $\theta = \arg(z)$. If we assume that the usual laws for the logarithm of a real function hold, we have

$$\log(z) \;=\; \log(re^{i\theta}) \;=\; \log(r) \;+\; \log(e^{i\theta}) \;=\; \log(r) \;+\; i\theta.$$

Since $r = |z|$ is always a real number, we normally write $\log(r)$ as $\ln|z|$. (Recall that $\ln(x)$ the natural log function for a real variable x.)

The logarithm of a complex number is defined as

$$\log(z) \;=\; \ln|z| \;+\; i\arg(z), \qquad (z \in \mathbb{C}).$$

The real part of $\log(z)$ is $\ln|z|$ and its imaginary part is $\arg(z)$.

In §4.6 we examined how the argument of a complex number is multi-valued. We can obtain infinitely many values of the argument by adding integer multiples of 2π. For example, $\arg(1+i) = \dfrac{\pi}{4} + 2k\pi \quad (k \in \mathbb{Z})$.

However, $\log(z)$ is defined in terms of $\arg(z)$, so the function $\log(z)$ **is also multi-valued**. A complex number can have infinitely many values for its logarithm!

In §4.6 we also discussed how we often concentrate on the argument of z that lies in the interval $(-\pi,\, \pi]$ and denote this angle by $\mathrm{Arg}(z)$ – this is the **principal value of the argument** of z.

We can also use the principal value of the argument to define the principal value of the logarithm function:

$$\mathrm{Log}(z) \;=\; \ln|z| \;+\; i\;\mathrm{Arg}(z), \qquad (z \in \mathbb{C}).$$

Since $\mathrm{Arg}(z)$ has a unique value, $\mathrm{Log}(z)$ must also have a unique value for every $z \in \mathbb{C}$.

Examples covered in video

4.15 Evaluate the following:

$$\textbf{(i)} \quad \log(1-\sqrt{3}i), \qquad \textbf{(ii)} \quad \log(4i), \qquad \textbf{(iii)} \quad \log(-1-i).$$

For **(i)**–**(ii)** also derive the principal value of the logarithm function, $\mathrm{Log}(z)$.

4.16 Find $\log(-1)$ and its principal value $\mathrm{Log}(-1)$.

4.17 Solve $\mathrm{e}^z = 1 - i$.

4.18 Find all values of z for which

$$\textbf{(i)} \quad \mathrm{e}^{3z} = 1, \qquad \textbf{(ii)} \quad \mathrm{e}^{4z} = i.$$

§4.18 Regions in the Complex Plane

The following are some examples of sets of complex numbers that can be interpreted geometrically in the Argand diagram.

Examples

4.19 (i) $S = \{ z \in \mathbb{C} \ : \ \mathrm{Re}\,(z) \geq 0 \}$

(ii) $S = \{ z \in \mathbb{C} \ : \ z = \overline{z} \}$

(iii) $S = \{ z \in \mathbb{C} \ : \ z = -\overline{z} \}$

(iv) $S = \{ z \in \mathbb{C} \ : \ |z - 2 + 3i| = 1 \}$

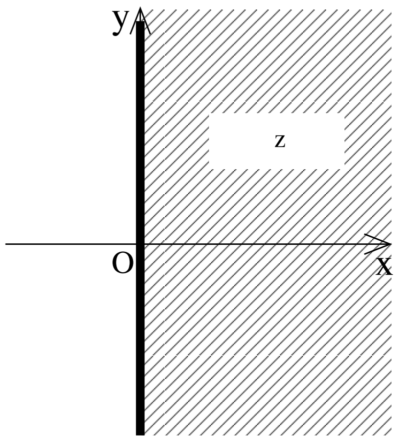
(v) $S = \{ z \in \mathbb{C} \ : \ |z - 2 + i| = |z + 2 - i| \}$

(vi) $S = \left\{ z \in \mathbb{C} \ : \ \mathrm{Arg}\ z = -\dfrac{\pi}{6} \right\}$

(vii) $S = \left\{ z \in \mathbb{C} \ : \ \mathrm{Arg}\,(z - 2 + 4i) = \dfrac{3\pi}{4} \right\}$

(i) $S = \{ z \in \mathbb{C} \ : \ \mathrm{Re}\,(z) \geq 0 \}$

$\mathrm{Re}\,(z) \geq 0 \quad \Longrightarrow \quad x \geq 0$ with no restriction on y. Hence z lies to the right of, or is on, the y-axis.

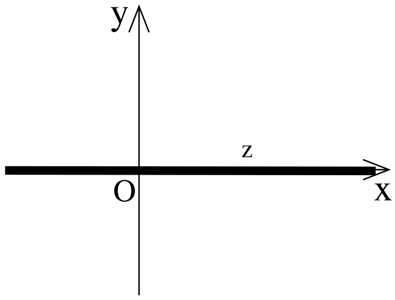


(ii) $S = \{ z \in \mathbb{C} \ : \ z = \overline{z} \}$

$z = \overline{z} \quad \Longleftrightarrow \quad x + iy = x - iy.$

Hence $2iy = 0$ and so $y = 0$.

All the points z lie on the x axis, so S is the entire x-axis.

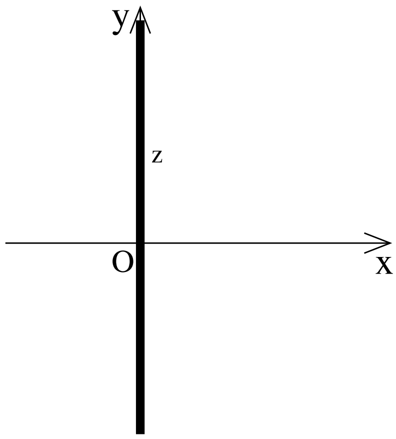


(iii) $S = \{ z \in \mathbb{C} \; : \; z = -\overline{z} \}$

$z = -\overline{z} \quad \Longleftrightarrow \quad x + iy = -x + iy,$ so $x = 0$.

All the points z lie on the y axis.

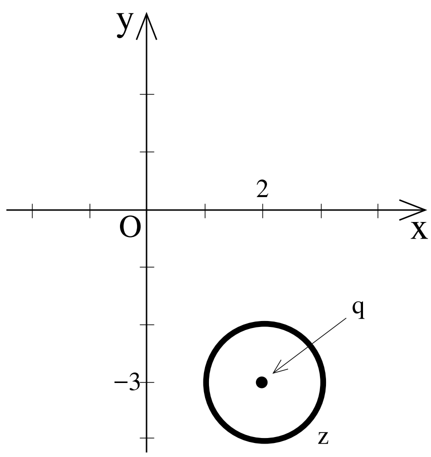
Set S represents the y-axis.



(iv) $S = \{ z \in \mathbb{C} \; : \; |z - 2 + 3i| = 1 \}$

$|z - (2 - 3i)| = 1$ if and only if the distance from the point representing $2 - 3i$ to the point representing z is equal to 1.

Hence S is the circle centred at $2 - 3i$ in the complex plane with radius 1.

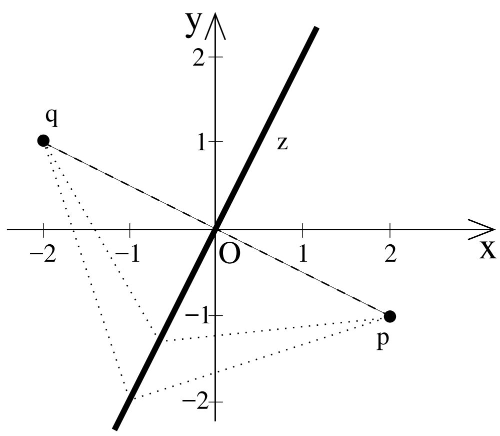


(v) $S = \{ z \in \mathbb{C} \; : \; |z - 2 + i| = |z + 2 - i| \}$

$|z - (2 - i)| = |z - (-2 + i)|$ implies that the point representing z is the same distance from $2 - i$ as from $-2 + i$ in the complex plane.

Therefore z lies on the perpendicular bisector of the line segment joining $2 - i$ to $-2 + i$.

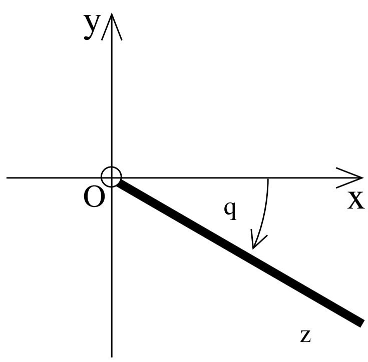
Hence S is a line through the origin with slope 2.



(vi) $S = \left\{ z \in \mathbb{C} \; : \; \text{Arg}\, z = -\dfrac{\pi}{6} \right\}$

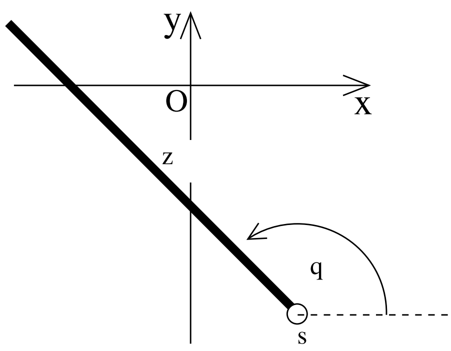
If Arg $z = -\dfrac{\pi}{6}$ then the point representing z lies on the ray directed from the origin that makes an angle $-\dfrac{\pi}{6}$ with the positive x-axis.

This line, with the origin excluded, is S.



(vii) $S = \left\{ z \in \mathbb{C} \ : \ \mathrm{Arg}\,(z - 2 + 4i) = \dfrac{3\pi}{4} \right\}$

$\mathrm{Arg}\,(z - (2 - 4i)) = \dfrac{3\pi}{4}$ means that the line joining $(2,\,-4)$ to $(x,\,y)$ in the complex plane makes angle of $\dfrac{3\pi}{4}$ with the positive x-axis.



Hence S is the ray emanating from $z = 2 - 4i$, with $z = 2 - 4i$ excluded, which makes an angle $\dfrac{3\pi}{4}$ with the positive x-axis direction.

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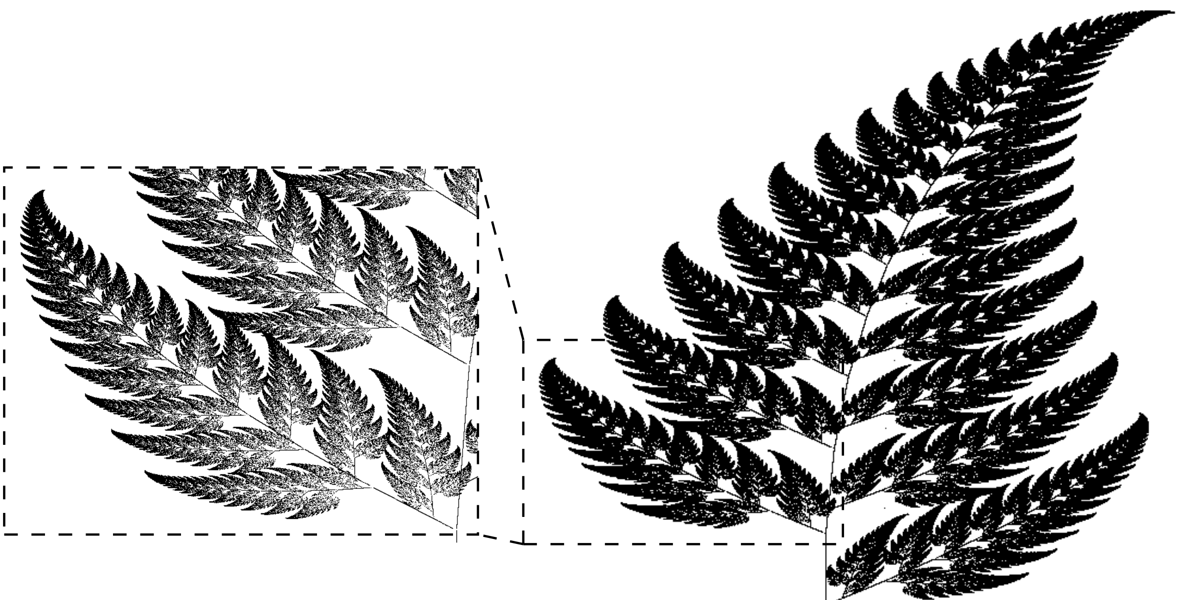
§4.19 Fractals and Complex Numbers: Mandelbrot and Julia Sets

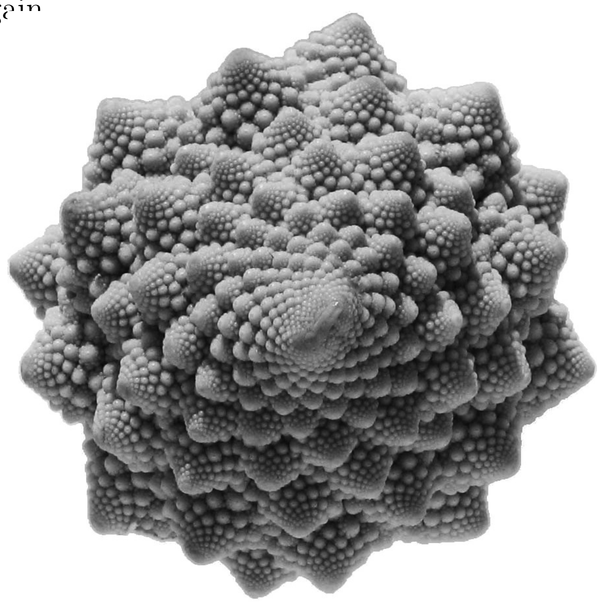
A **fractal** is a rough, geometric shape which can be split into parts, each of which is (approximately) a reduced-size version of the original. Natural objects that are approximated by fractals to a degree include coastlines, snow flakes, clouds, mountain ranges, lightning bolts, ferns, systems of blood vessels and even vegetables such as cauliflowers or broccoli. Fractals have also been popularised through fractal art – algorithmic art created by producing fractal objects mathematically and representing the calculation results as still images or animations. Hundreds of examples of fractals and fractal art can be found via a quick search online.

The reproducing property of a fractal when you zoom in is known as **self-similarity** and it can have important consequences in the structure of coastlines, teletraffic engineering and even stock market transactions. To demonstrate self-similarity, the figure below shows a mathematical shape known

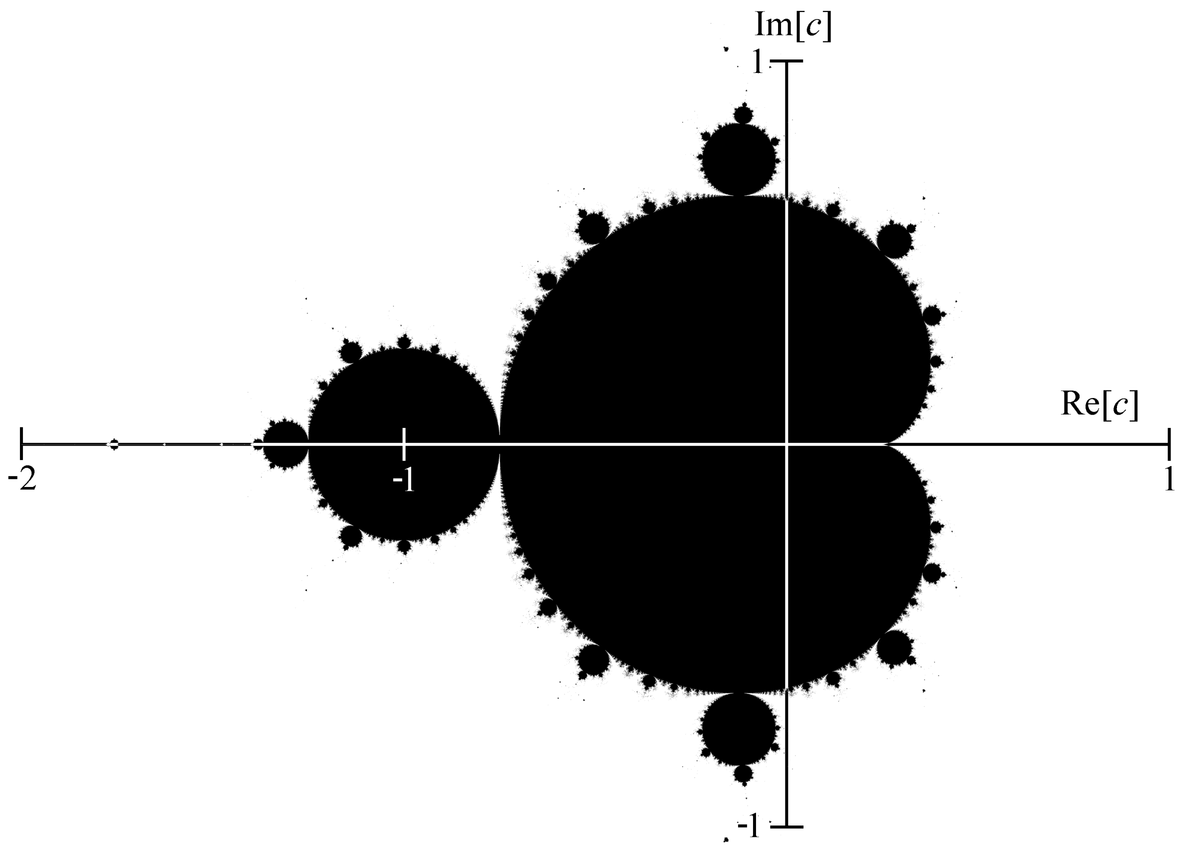


The fern on the right is produced using a mathematical process called **iteration** and is formed from over 2 million points! When we zoom in on one of the fronds (leaves) of the fern, we see the same fern shape as the original larger version, albeit at a lower resolution. If we zoom in even more, we see the fern shape appearing once again.

The figure on the right is another example of self-similarity in nature – but this isn't a mathematical or computer generated image, it's a photograph of a vegetable known as a **romanesco broccoli**! The main stem of the broccoli forms a logarithmic spiral shape. Individual buds are composed of a series of smaller buds also arranged in logarithmic spirals. Mathematically, this same design can be modelled and produced on a computer using a recursive helical arrangement of cones.



The **Mandelbrot Set** is a set of points in the complex plane whose boundary forms a fractal. A given complex number $c \in \mathbb{C}$ either belongs to the Mandelbrot set or it doesn't. A picture of the Mandelbrot Set can be made by colouring all the points in the complex plane which belong to the set as black, while all other points are marked white. This leads to the shape shown in the next figure.



If you zoom in closer and closer on the boundary of the set, you notice small round regions very similar in shape to the round feature in the original. Zoom in even further on the boundary and the pattern repeats once again – the boundary of the Mandelbrot Set is self-similar.

To construct the Mandelbrot set, we form a sequence of complex numbers

$$z_0, \ z_1, \ z_2, \ z_3, \ z_4, \ z_5, \ \ldots$$

where each number is related to the previous one via the formula

$$z_{n+1} \ = \ (z_n)^2 \ + \ c \qquad\qquad (n = 1, 2, 3, \ldots)$$

for some complex number c. To start the sequence, we also assume that the first number $z_0 = 0$. The Mandelbrot Set, M, is then defined as the set of all values of $c \in \mathbb{C}$ for which the sequence **always remains bounded in modulus**. Mathematically this can be written as

$$M \ = \ \{ \ c \in \mathbb{C} \ : \quad \text{sequence} \ \ z_0 = 0, \ \ z_{n+1} = (z_n)^2 + c \ \ \text{remains bounded} \ \}.$$

For example, consider the case $c = 1$. The sequence is then

$$\begin{aligned}
z_0 &= 0, \\
z_1 &= (z_0)^2 + c \ = \ 0^2 + 1 \ = \ 1, \\
z_2 &= (z_1)^2 + c \ = \ 1^2 + 1 \ = \ 2, \\
z_3 &= (z_2)^2 + c \ = \ 2^2 + 1 \ = \ 5, \\
z_4 &= (z_3)^2 + c \ = \ 5^2 + 1 \ = \ 26, \\
z_5 &= (z_4)^2 + c \ = \ 26^2 + 1 \ = \ 677, \ \ldots
\end{aligned}$$

An n increases, the sequence continues to grow (in modulus), therefore it must be unbounded. So the point $c = 1$ does **not** lie in the Mandelbrot Set M. However, if $c = -1$ then

$$\begin{aligned}
z_0 &= 0, \\
z_1 &= (z_0)^2 + c \ = \ 0^2 - 1 \ = \ -1, \\
z_2 &= (z_1)^2 + c \ = \ (-1)^2 - 1 \ = \ 0, \\
z_3 &= (z_2)^2 + c \ = \ 0^2 - 1 \ = \ -1, \\
z_4 &= (z_3)^2 + c \ = \ (-1)^2 - 1 \ = \ 0, \\
z_5 &= (z_4)^2 + c \ = \ 0^2 - 1 \ = \ -1, \ \ldots
\end{aligned}$$

The sequence repeats every two terms, alternating between 0 and -1.
Therefore $|z_n| \ \leq \ |-1| \ = \ 1$ for all $n \ = \ 0, 1, 2, \ldots$, i.e. the sequence remains bounded in modulus and the point $c = -1$ lies in the Mandelbrot Set M.

Now consider the case $c = i$. For this choice of c the sequence is

$$\begin{aligned}
z_0 &= 0, \\
z_1 &= (z_0)^2 + c \;=\; 0^2 + i \;=\; i, \\
z_2 &= (z_1)^2 + c \;=\; i^2 + i \;=\; -1 + i, \\
z_3 &= (z_2)^2 + c \;=\; (-1+i)^2 + i \;=\; -2i + i \;=\; -i, \\
z_4 &= (z_3)^2 + c \;=\; (-i)^2 + i \;=\; -1 + i, \\
z_5 &= (z_4)^2 + c \;=\; (-1+i)^2 + i \;=\; -2i + i \;=\; -i, \;\ldots
\end{aligned}$$

Again the sequence repeats every two terms, alternating between $-i$ and $-1+i$.
Also, $|z_n| \leq |-1+i| = \sqrt{2}$ for all $n = 0, 1, 2, \ldots$, i.e. the sequence remains bounded in modulus and the point $c = i$ belongs to the Mandelbrot Set M.

Another type of set of complex numbers linked to fractals is called a **Julia Set**. The definition of a Julia Set is very similar to the definition of the Mandelbrot Set, with one important difference. When defining the Mandelbrot Set, the start point of the sequence $z_{n+1} = (z_n)^2 + c$ is fixed at $z_0 = 0$. The Mandelbrot Set is comprised of points $c \in \mathbb{C}$ for which the sequence remains bounded. A Julia Set is defined using a similar sequence. However, instead of fixing z_0 to be zero and considering all the different values of c, for the Julia Set we **fix the value of c at the start** and consider different values for the start point z_0. As a result, there are infinitely many Julia Sets, each one corresponding to a different value of c. Mathematically, we can define the Julia Set, J_c, corresponding to a specific number $c \in \mathbb{C}$ as

$$J_c \;=\; \{\; k \in \mathbb{C} \;:\quad \text{sequence} \;\; z_0 = k, \;\; z_{n+1} = (z_n)^2 + c \;\; \text{remains bounded} \;\}.$$

For example, if $c = i$ and $k = 1 + i$ then

$$\begin{aligned}
z_0 &= 1 + i, \\
z_1 &= (z_0)^2 + c \;=\; (1+i)^2 + i \;=\; 2i + i \;=\; 3i, \\
z_2 &= (z_1)^2 + c \;=\; (3i)^2 + i \;=\; -9 + i, \\
z_3 &= (z_2)^2 + c \;=\; (-9+i)^2 + i \;=\; 80 - 18i + i \;=\; 80 - 17i, \\
z_4 &= (z_3)^2 + c \;=\; (80 - 17i)^2 + i \;=\; 6111 - 2720i + i \;=\; 6111 - 2719i, \;\ldots
\end{aligned}$$

The sequence z_n continues to grow in modulus as n increases so it is unbounded. Therefore, the point $k = 1 + i$ does not lie in J_i, the Julia Set corresponding to $c = i$. However, if change

c from i to $-i$ then

$$\begin{aligned} z_0 &= 1+i, \\ z_1 &= (z_0)^2 + c \;=\; (1+i)^2 - i \;=\; 2i - i \;=\; i, \\ z_2 &= (z_1)^2 + c \;=\; i^2 - i \;=\; -1-i, \\ z_3 &= (z_2)^2 + c \;=\; (-1-i)^2 - i \;=\; 2i - i \;=\; i, \\ z_4 &= (z_3)^2 + c \;=\; i^2 - i \;=\; -1-i, \;\ldots \end{aligned}$$

The sequence alternates and each term $|z_n| \le |-1+i| = \sqrt{2}$, so the sequence is bounded. Hence, the point $k = 1+i$ **does** lie in the Julia Set for $c = -i$.

As an illustration, the figure below represents $J_{-0.08+0.8i}$, the Julia Set corresponding to $c = -0.08 + 0.8i$. As with the Mandelbrot Set, the boundary of this Julia Set is self-similar with the same basic shape re-appearing at different length scales.