Degenerate Transformations and the Null Space

Consider the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{x} \mapsto A\mathbf{x}$, where A is the following 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 4 \\ -2 & -9 & -15 \end{bmatrix}.$$

Observe that the third column of this matrix is a linear combination of the other two:

$$3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -9 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -15 \end{bmatrix}. \tag{1}$$

This means A is a singular matrix and hence A^{-1} does not exist. (This result will be proved in MM201 Linear Algebra.) Therefore, the transformation f is **degenerate**.

We wish to understand the range space and null space of this transformation. Let us first define these terms.

The **range space** is the set of all vectors of $\mathbb{R}^3$ that are equal to $A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^3$. In other words, it is the image of the transformation:

$$\text{range}(f) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^3\}.$$

The **null space** is the set of all vectors of $\mathbb{R}^3$ that are mapped to the zero vector:

$$\text{null}(f) = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}.$$

It accounts for the dimensions we "lose" when we apply a degenerate transformation. Let us now compute the range and null space of our transformation.

The range space is the set of all vectors $A\mathbf{x}$, where $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right]$. If $\mathbf{r}$ belongs to the range space, then:

$$\begin{aligned} \mathbf{r} &= \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 4 \\ -2 & -9 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 5x_3 \\ x_1 + x_2 + 4x_3 \\ -2x_1 - 9x_2 - 15x_3 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -9 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 4 \\ -15 \end{bmatrix}. \end{aligned}$$

Technically, this means the range space is *spanned* by the columns of A and this is why the range space is often called the column space. You will learn about spanning sets and the column space in MM201. We can now use (1) to write $\mathbf{r}$ in terms of the first two columns of A:

$$\begin{aligned}\mathbf{r} &= x_1\begin{bmatrix}1\\1\\-2\end{bmatrix} + x_2\begin{bmatrix}2\\1\\-9\end{bmatrix} + x_3\begin{bmatrix}5\\4\\-15\end{bmatrix}\\ &= x_1\begin{bmatrix}1\\1\\-2\end{bmatrix} + x_2\begin{bmatrix}2\\1\\-9\end{bmatrix} + x_3\left(3\begin{bmatrix}1\\1\\-2\end{bmatrix} + \begin{bmatrix}2\\1\\-9\end{bmatrix}\right)\\ &= (x_1+3x_3)\begin{bmatrix}1\\1\\-2\end{bmatrix} + (x_2+x_3)\begin{bmatrix}2\\1\\-9\end{bmatrix}.\end{aligned}$$

Thus, we have shown that the range space is the set of all vectors of the form

$$\mathbf{r} = t\begin{bmatrix}1\\1\\-2\end{bmatrix} + u\begin{bmatrix}2\\1\\-9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\1\\-2\end{bmatrix} + u\begin{bmatrix}2\\1\\-9\end{bmatrix}$$

for all $t, u \in \mathbb{R}$. In other words, the range space is the plane in $\mathbb{R}^3$ that contains the origin and whose normal is equal to the vector product of the first two columns of A. This is a 2-dimensional object, which means that we lose one dimension when we apply the degenerate transformation f.

This lost dimension is the null space and so we should be able to show that the null space is a straight line in $\mathbb{R}^3$. Let $\mathbf{x}$ belong to the null space. Then $A\mathbf{x} = \mathbf{0}$, which is equivalent to the following set of equations:

$$\begin{aligned}x_1 + 2x_2 + 5x_3 &= 0\\ x_1 + x_2 + 4x_3 &= 0\\ -2x_1 - 9x_2 - 15x_3 &= 0.\end{aligned}$$

The solution of this system is $x_1 = 3t$, $x_2 = t$, $x_3 = -t$, where t is any real number. Therefore, the null space is the set of all vectors

$$\left\{ t\begin{bmatrix}3\\1\\-1\end{bmatrix} : t \in \mathbb{R}\right\}, \qquad (2)$$

which is indeed a straight line.

It turns out that the null space of A is at right angles to the range space of A^T. To see this, we can use the same method used to calculate the range space of A to show that the range space of A^T is the plane that contains the origin and whose normal vector is

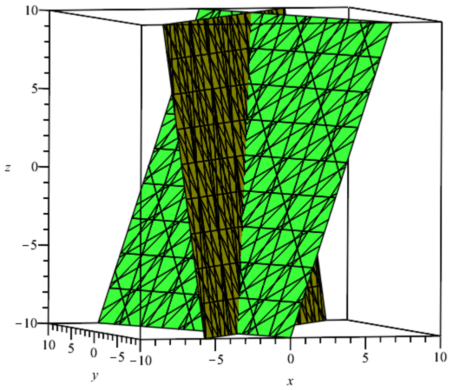

Figure 1: The range space of A is the yellow plane and the range space of A^T is the green plane.

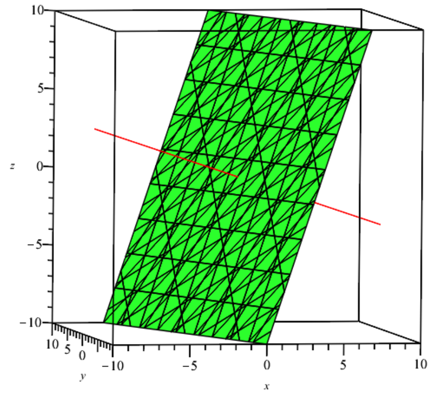

Figure 2: The null space is the red straight line and it is perpendicular to the range space of A^T.

$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. This normal vector is parallel to the straight line (2) and hence this straight line is indeed perpendicular to the plane that corresponds to the range space of A^T.