University of Strathclyde, Department of Mathematics and Statistics

MM102 Applications of Calculus
Exercises for Chapter 6 (Week 10)
Solutions

17. For each of the following functions f determine when f is increasing and when it is decreasing.

(a) $f(x) = x^2 - x$

Solution:
The derivative of f is
$$f'(x) = 2x - 1.$$
We have
$$f'(x) > 0 \quad \Longleftrightarrow \quad x > \frac{1}{2}$$
$$f'(x) < 0 \quad \Longleftrightarrow \quad x < \frac{1}{2}$$
Hence f is **increasing** on the interval $(\frac{1}{2}, \infty)$ and **decreasing** on the interval $(-\infty, \frac{1}{2})$.

(b) $f(x) = e^x$

Solution:
The first derivative of f is
$$f'(x) = e^x,$$
which is positive for all x. Hence f is **increasing** on $(-\infty, \infty) = \mathbb{R}$.

(c) $f(x) = x^3 + 3x^2 - 9x - 1$

Solution:
The derivative of f is
$$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x + 3)(x - 1).$$
The first derivative vanishes if $x = 1$ or $x = -3$. In the following table the signs of the factors and f' are summarised.

x		-3		1	
$(x+3)$	$-$	0	$+$	$+$	$+$
$(x-1)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$-$	$\searrow$	$-$	$\nearrow$

The function f is **increasing** on the intervals $\boxed{(-\infty,-3) \text{ and } (1,\infty);}$
it is **decreasing** on the interval $\boxed{(-3,1).}$
(The values at the stationary points are $f(-3) = 26$, $f(1) = -6$.)

(d) $f(x) = x^3 + x$

Solution:
The first derivative of f is

$$f'(x) = 3x^2 + 1.$$

Since $x^2 \geq 0$, we have $f'(x) > 0$ for all $x \in \mathbb{R}$. Hence f is **increasing** on $(-\infty, \infty) = \mathbb{R}$.

(e) $f(x) = \dfrac{1}{1+x^2}$

Solution:
The first derivative of f is

$$f'(x) = -\frac{2x}{(1+x^2)^2}\,.$$

Since the denominator is always positive, we have that $f'(x) > 0$ when $x < 0$ and $f'(x) < 0$ when $x > 0$. Hence f is **increasing** on the interval $(-\infty, 0)$ and **decreasing** on the interval $(0, \infty)$.

18. The function f is given by

$$f(x) = x^3 - 6x^2 - 15x + 75.$$

(i) Find the position and nature of the stationary points. Find when the function f is increasing and when it is decreasing.

(ii) Find the point of inflection. Find when the function is concave up and when it is concave down.

(iii) Examine the behaviour of $f(x)$ as $x \to \pm\infty$.

(iv) Use this information to sketch the graph of f showing all the critical points.

(v) How many zeros does the function f have? Answer this question using the graph and the information obtained above.

Solution:

(i) $$f'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x+1)(x-5)$$

$$f'(x) = 0 \quad \Longleftrightarrow \quad x = -1 \quad \text{or} \quad x = 5 \quad \text{(stationary points)}$$

Sign of f':

x		-1		5	
$(x+1)$	$-$	0	$+$	$+$	$+$
$(x-5)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	—	$\searrow$	—	$\nearrow$
		max		min	

Hence f is **increasing** on the intervals $\boxed{(-\infty,-1) \text{ and } (5,\infty);}$

it is **decreasing** on the interval $\boxed{(-1,5).}$

There is a **local maximum** at $\boxed{x=-1}$ with value $\boxed{f(-1)=83}$ and a **local minimum** at $\boxed{x=5}$ with value $\boxed{f(5)=-25.}$

(ii) $f''(x) = 6x - 12 = 6(x-2)$

$f''(x) = 0 \iff x = 2$

Sign of f'':

x		2	
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$		$\smile$

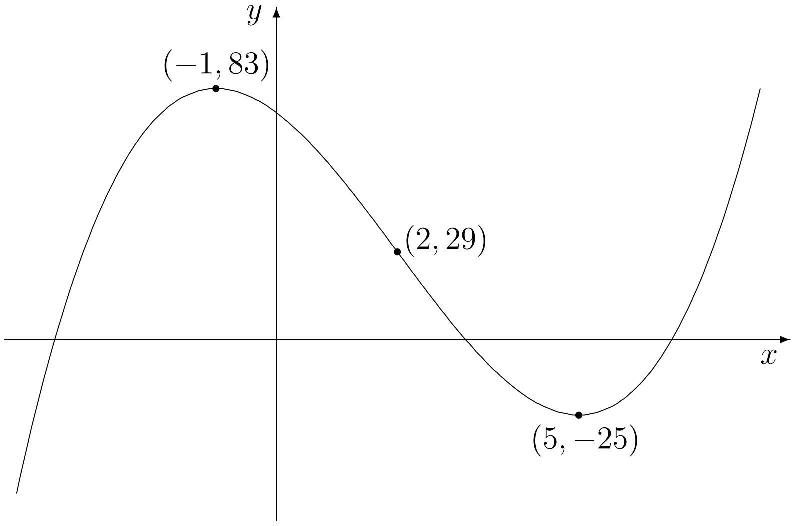
The function f is **concave down** on the interval $\boxed{(-\infty,2)}$ and **concave up** on the interval $\boxed{(2,\infty).}$

Since the concavity changes at $x=2$, there is a **point of inflection** at $\boxed{x=2}$ with value $f(2)=29$.

(iii) As $x \to \infty$ we have $f(x) \sim x^3 \to \infty$.

As $x \to -\infty$ we have $f(x) \sim x^3 \to -\infty$.

(iv)



(v) Since the function is increasing on $(-\infty, -1)$, $f(x) \to -\infty$ as $x \to -\infty$ and $f(-1) > 0$, there must be exactly one zero in the interval $(-\infty, -1)$.

Since the function is decreasing on $(-1, 5)$, $f(-1) > 0$ and $f(5) < 0$, there must be exactly one zero in the interval $(-1, 5)$.

Since the function is increasing on $(5, \infty)$, $f(5) < 0$ and $f(x) \to \infty$ as $x \to \infty$, there must be exactly one zero in the interval $(5, \infty)$.

In total the function f has exactly **3 zeros**.

19. The function f is given by

$$f(x) = x^4 - 6x^3 + 12x^2 - 3.$$

(i) Find the points of inflection. Find when the function is concave up and when it is concave down.

(ii) Show that there is only one stationary point and find its position and nature. Find when the function is increasing and when it is decreasing.

(iii) Examine the behaviour of $f(x)$ as $x \to \pm\infty$.

(iv) Use this information to sketch the graph of f showing all the critical points.

(v) How many zeros does the function f have? Answer this question using the graph and the information obtained above.

Solution:

(i) $f'(x) = 4x^3 - 18x^2 + 24x = 2x(2x^2 - 9x + 12)$

$f''(x) = 12x^2 - 36x + 24 = 12(x^2 - 3x + 2) = 12(x-1)(x-2)$

$$f''(x) = 0 \quad \Longleftrightarrow \quad x = 1 \quad \text{or} \quad x = 2$$

Sign of f'':

x		1		2	
$(x-1)$	$-$	0	$+$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$		$\frown$		$\smile$

The function f is **concave up** on the intervals $\boxed{(-\infty, 1)}$ and $\boxed{(2, \infty);}$ it is **concave down** on the interval $\boxed{(1, 2).}$

Since the concavity changes at $\boxed{x = 1}$ and $\boxed{x = 2,}$ these two points are points of inflection with values $f(1) = 4$ and $f(2) = 13$.

(ii) $f'(x) = 0 \quad \Longleftrightarrow \quad x = 0 \quad \text{or} \quad 2x^2 - 9x + 12 = 0$

The latter equation has no real solution because

$$b^2 - 4ac = (-9)^2 - 4 \times 2 \times 12 = -15 < 0$$

(a, b, c are the coefficients of the quadratic equation and $b^2 - 4ac$ is the expression under the square root in the formula for solutions of a quadratic equation.)

The nature of the stationary point $x = 0$ can be deduced from the second derivative:

$$f''(0) = 24 > 0 \qquad \text{Hence a local minimum.}$$

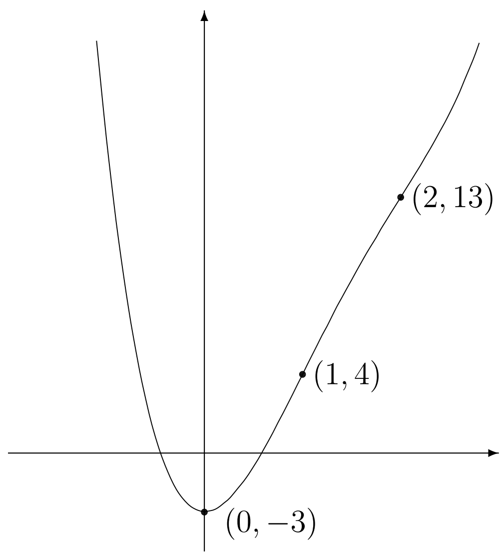
(One could also determine the sign of f' to the left and to the right of $x = 0$.)

The corresponding y-coordinate is $y = f(0) = -3$.

(iii) As $x \to \infty$, we have $f(x) \sim x^4 \to \infty$.

As $x \to -\infty$, we have $f(x) \sim x^4 \to \infty$.

(iv)



(v) Since f is decreasing on $(-\infty, 0)$, $f(x) \to \infty$ as $x \to -\infty$ and $f(0) < 0$, the function f has exactly one zero in the interval $(-\infty, 0)$.

Since f is increasing on $(0, \infty)$, $f(0) < 0$ and $f(x) \to \infty$ as $x \to \infty$, the

function f has exactly one zero in the interval $(0, \infty)$.
Hence f has exactly **2 zeros**.

20. For each of the following functions, f, where

(a) $f(x) = \dfrac{1}{x^2+1}$ (b) $f(x) = \dfrac{x^2-7x+13}{x-2}$

(c) $f(x) = \dfrac{-3x^2+11x-37}{x-2}$ (d) $f(x) = \dfrac{x-3}{x^2-x-2}$

(e) $f(x) = \dfrac{x+2}{x^2+2x-3}$ (f) $f(x) = x\sqrt{x+2}$

(g) $f(x) = x^2 + \dfrac{1}{x^2}$

(i) find the natural domain of f;
(ii) find the asymptotes;
(iii) find the points of intersection with the axes;
(iv) find the position and nature of the stationary points;
(v) determine when the function f is increasing and when it is decreasing;
(vi) use this information to sketch the graph of f showing points of intersection with the axes, stationary points and asymptotes.

Solution:

(a) $f(x) = \dfrac{1}{x^2+1}$

(i) The denominator is always positive; hence the domain of f is

$$\text{dom}(f) = \mathbb{R}.$$

(ii) There are **no vertical asymptotes**.
Since

$$\lim_{x \to \pm\infty} f(x) = 0,$$

the line $\boxed{y = 0}$ is a **horizontal asymptote**.

(iii) Since $f(0) = 1$, the point $\boxed{(0, 1)}$ is a point of intersection of the graph with the y-axis. Since the function has no zeros, the graph has no points of intersection with the x-axis.

(iv) The first derivative of f is

$$f'(x) = -\frac{2x}{(x^2+1)^2}.$$

$$f'(x) = 0 \iff x = 0.$$

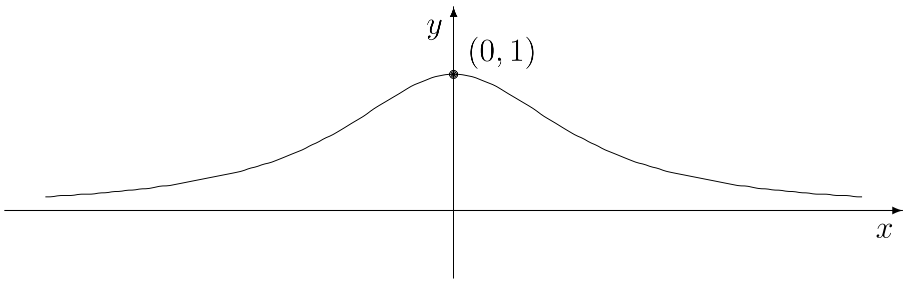
Hence the only stationary point is at $x = 0$. The sign of the first derivative of f is summarised in the following table:

x		0	
x	$-$	0	$+$
$(x^2+1)^2$	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	—	$\searrow$
		max	

Hence there is a **local maximum** at $\boxed{x = 0}$ with $f(0) = 1$.

(v) The function is **increasing** on the interval $\boxed{(-\infty, 0)}$ and **decreasing** on the interval $\boxed{(0, \infty).}$

(vi)



(b) $f(x) = \dfrac{x^2 - 7x + 13}{x - 2}$

(i) The denominator is equal to 0 if and only if $x = 2$. Hence the natural domain of f is equal to

$$\operatorname{dom}(f) = \big\{x \in \mathbb{R} \colon x \neq 2\big\}.$$

(ii) Since the numerator is non-zero at $x = 2$, we have the **vertical asymptote**: $\boxed{x = 2}$

The degree of the numerator is greater than the degree of the denominator by one; hence we expect a slant asymptote. With long division,

$$\begin{array}{r|rrrrr}
 & & & x & - & 5 \\
\hline
x - 2 & x^2 & - & 7x & + & 13 \\
 & x^2 & - & 2x & & \\
\hline
 & & - & 5x & + & 13 \\
 & & - & 5x & + & 10 \\
\hline
 & & & & & 3
\end{array}$$

we can write

$$f(x) = x - 5 + \frac{3}{x - 2}.$$

Since $\dfrac{3}{x - 2} \to 0$ as $x \to \pm\infty$, we have a **slant asymptote**:

$$\boxed{y = x - 5}$$

(iii) Since $f(0) = -\dfrac{13}{2}$, the point $\boxed{\left(0, -\dfrac{13}{2}\right)}$ is a point of intersection of the graph with the y-axis.
With the coefficients $a = 1$, $b = -7$, $c = 13$ of the polynomial in the numerator we have

$$b^2 - 4ac = (-7)^2 - 4 \times 1 \times 13 = 49 - 52 = -3 < 0,$$

which implies that f has no zeros. Hence there are no points of intersection of the graph with the x-axis.

(iv) To find f', one can use either the given form of f or the representation from (ii).

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x - 5 + \frac{3}{x-2}\right) \\ &= 1 - \frac{3}{(x-2)^2} = \frac{(x-2)^2 - 3}{(x-2)^2} \\ &= \frac{x^2 - 4x + 1}{(x-2)^2} \end{aligned}$$

Hence

$$f'(x) = 0 \quad \Longleftrightarrow \quad x^2 - 4x + 1 = 0 \quad \Longleftrightarrow \quad x = 2 \pm \sqrt{3}$$

The stationary points are at

$$\boxed{x = 2 + \sqrt{3}} \qquad \text{and} \qquad \boxed{x = 2 - \sqrt{3}}$$

We can factorise the numerator of the derivative and write

$$f'(x) = \frac{\left(x - 2 - \sqrt{3}\right)\left(x - 2 + \sqrt{3}\right)}{(x-2)^2}$$

For the sign of f' we obtain

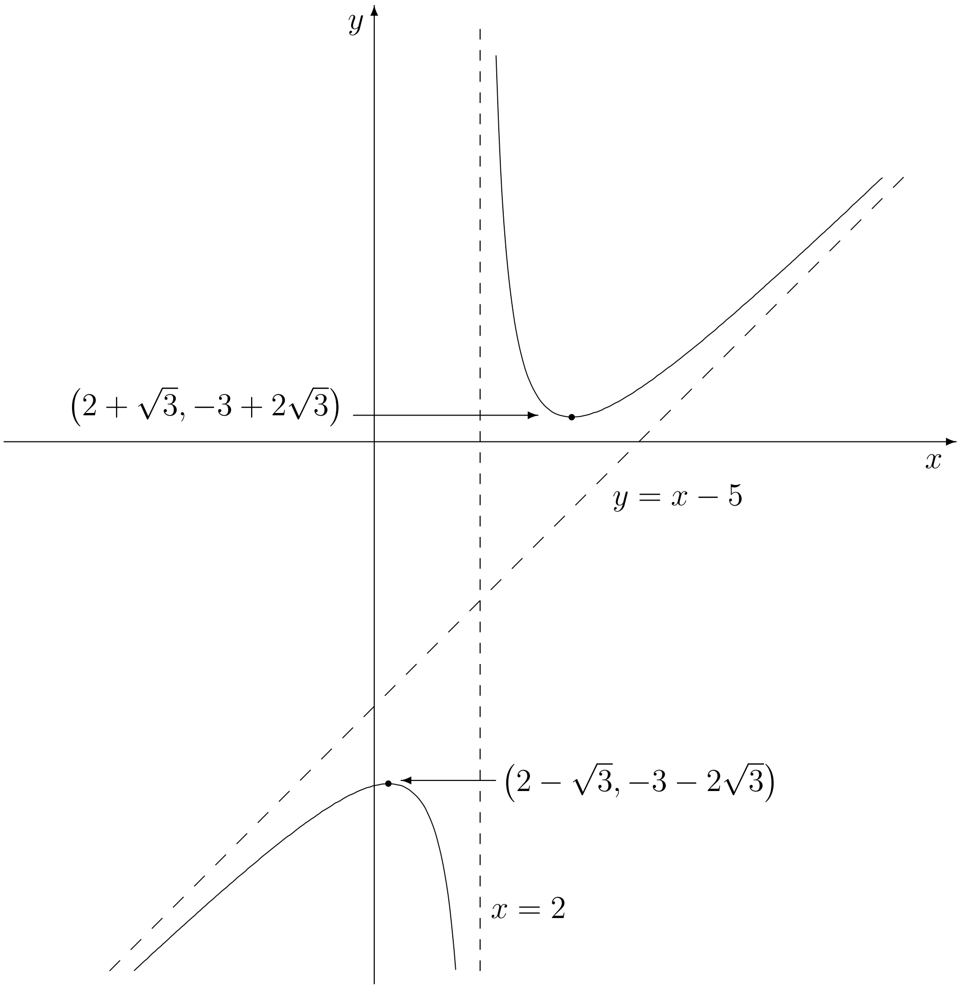
x		$2-\sqrt{3}$		2		$2+\sqrt{3}$	
$(x-2-\sqrt{3})$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x-2+\sqrt{3})$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-2)^2$	$+$	$+$	$+$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	ND	$-$	0	$+$
$f(x)$	$\nearrow$	—	$\searrow$	ND	$\searrow$	—	$\nearrow$
		max				min	

There is a **local maximum** at $\boxed{x = 2 - \sqrt{3}}$ with value $f(2-\sqrt{3}) = -3 - 2\sqrt{3}$ and a **local minimum** at $\boxed{x = 2 + \sqrt{3}}$ with value $f(2+\sqrt{3}) = -3 + 2\sqrt{3}$.

(v) The function f is **increasing** on the intervals $\boxed{(-\infty, 2-\sqrt{3}) \text{ and } (2+\sqrt{3}, \infty);}$

it is **decreasing** on the intervals $\boxed{(2-\sqrt{3}, 2) \text{ and } (2, 2+\sqrt{3}).}$

(vi)



(c) $f(x) = \dfrac{-3x^2 + 11x - 37}{x-2}$

(i) The denominator is equal to 0 if and only if $x = 2$. Hence the natural domain of f is equal to

$$\operatorname{dom}(f) = \left\{x \in \mathbb{R} \colon x \neq 2\right\}.$$

(ii) Since the numerator is non-zero at $x = 2$, we have the **vertical asymptote**: $\boxed{x = 2}$

The degree of the numerator is greater than the degree of the denominator by one; hence we expect a slant asymptote. With long division,

$$\begin{array}{r|rcrcr} & -3x & & +5 & & \\ \cline{2-6} x-2 & -3x^2 & + & 11x & - & 37 \\ & -3x^2 & + & 6x & & \\ \cline{2-5} & & & 5x & - & 37 \\ & & & 5x & - & 10 \\ \cline{3-6} & & & & & -27 \end{array}$$

we can write

$$f(x) = -3x + 5 - \frac{27}{x-2}.$$

Since $\dfrac{27}{x-2} \to 0$ as $x \to \pm\infty$, we have a **slant asymptote**:

$$\boxed{y = -3x + 5}$$

(iii) Since $f(0) = \dfrac{37}{2}$, the point $\boxed{\left(0, \dfrac{37}{2}\right)}$ is a point of intersection of the graph with the y-axis.
With the coefficients $a = -3$, $b = 11$, $c = -37$ of the polynomial in the numerator we have

$$b^2 - 4ac = 11^2 - 4 \times (-3) \times (-37) = 121 - 444 = -323 < 0,$$

which implies that f has no zeros. Hence there are no points of intersection of the graph with the x-axis.

(iv) To find f', one can use either the given form of f or the representation from (ii).

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-3x + 5 - \frac{27}{x-2}\right) \\ &= -3 + \frac{27}{(x-2)^2} = \frac{-3(x-2)^2 + 27}{(x-2)^2} \\ &= \frac{-3(x^2 - 4x + 4) + 27}{(x-2)^2} = \frac{-3x^2 + 12x + 15}{(x-2)^2} \\ &= \frac{-3(x^2 - 4x + 5)}{(x-2)^2} = \frac{-3(x+1)(x-5)}{(x-2)^2} \end{aligned}$$

$$f'(x) = 0 \quad \Longleftrightarrow \quad \boxed{x = -1} \quad \text{or} \quad \boxed{x = 5} \qquad \text{(stationary points)}$$

Sign of f':

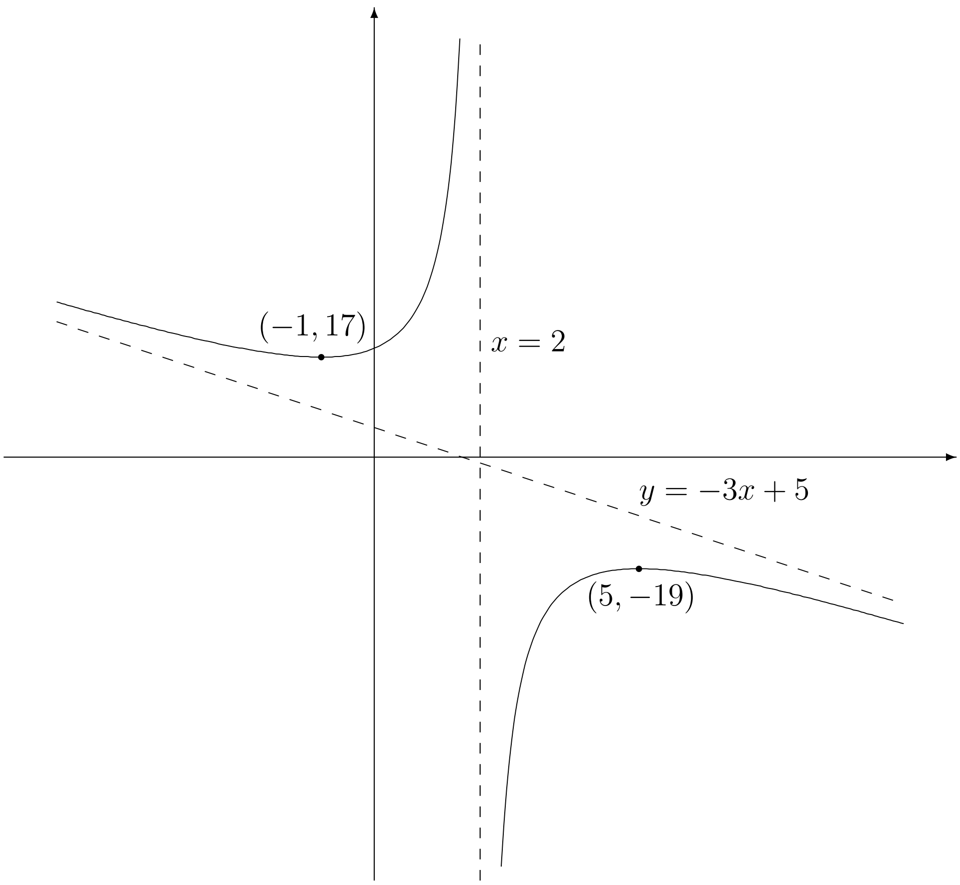
x		-1		2		5	
$(x+1)$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-5)$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x-2)^2$	$+$	$+$	$+$	0	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	ND	$+$	0	$-$
$f(x)$	$\searrow$	—	$\nearrow$	ND	$\nearrow$	—	$\searrow$
		min				max	

There is a **local minimum** at $\boxed{x=-1}$ with $f(-1)=17$ and a **local maximum** at $\boxed{x=5}$ with $f(5)=-19$.

(v) The function f is **increasing** on the intervals $\boxed{(-1,2) \text{ and } (2,5);}$

it is **decreasing** on the intervals $\boxed{(-\infty,-1) \text{ and } (5,\infty).}$

(vi)



(d) $f(x) = \dfrac{x-3}{x^2-x-2}$

(i) The denominator is equal to 0 if and only if

$$x^2 - x - 2 = (x+1)(x-2) = 0$$

i.e. if and only if $\quad x = -1 \quad$ or $\quad x = 2$

Hence the natural domain of f is

$$\mathrm{dom}(f) = \big\{x \in \mathbb{R} \colon x \neq -1 \text{ and } x \neq 2\big\}.$$

(ii) Since the numerator is non-zero at $x = -1$ and $x = 2$, we have the following **vertical asymptotes**:

$$\boxed{x = -1} \qquad \boxed{x = 2}$$

As $x \to \pm\infty$, we have

$$f(x) = \frac{\dfrac{1}{x} - \dfrac{3}{x^2}}{1 - \dfrac{1}{x} - \dfrac{2}{x^2}} \to 0.$$

Hence $\boxed{y = 0}$ is a **horizontal asymptote**.

(iii) Since $f(0) = \dfrac{3}{2}$, the point $\boxed{\left(0, \dfrac{3}{2}\right)}$ is a point of intersection of the graph with the y-axis.
We have $f(x) = 0$ if and only if $x = 3$. Hence $\boxed{(3, 0)}$ is the only point of intersection of the graph with the x-axis.

(iv) The first derivative of f is equal to

$$\begin{aligned} f'(x) &= \frac{x^2 - x - 2 - (2x-1)(x-3)}{(x^2-x-2)^2} \\ &= \frac{x^2 - x - 2 - (2x^2 - 7x + 3)}{(x^2-x-2)^2} = \frac{-x^2 + 6x - 5}{(x^2-x-2)^2} \\ &= -\frac{(x-1)(x-5)}{(x^2-x-2)^2} \end{aligned}$$

$f'(x) = 0 \quad \Longleftrightarrow \quad \boxed{x = 1} \quad$ or $\quad \boxed{x = 5} \qquad$ (stationary points)

Sign of f':

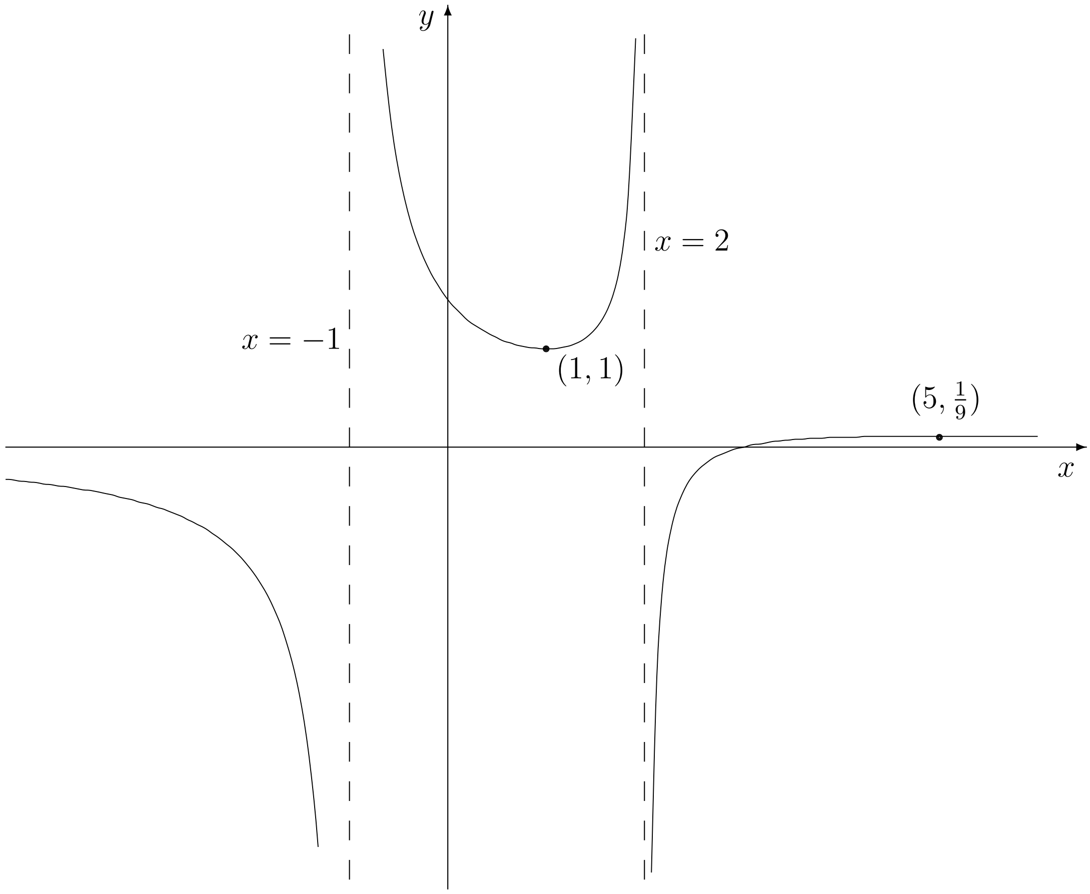
x		-1		1		2		5	
$(x-1)$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-5)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x^2-x-2)^2$	$+$	0	$+$	$+$	$+$	0	$+$	$+$	$+$
$f'(x)$	$-$	ND	$-$	0	$+$	ND	$+$	0	$-$
$f(x)$	$\searrow$	ND	$\searrow$	—	$\nearrow$	ND	$\nearrow$	—	$\searrow$
				min				max	

The function f has a **local minimum** at $\boxed{x = 1}$ with $f(1) = 1$ and a **local maximum** at $\boxed{x = 5}$ with $f(5) = \dfrac{1}{9}$.

(v) The function f is **increasing** on the intervals $\boxed{(1, 2) \text{ and } (2, 5);}$

it is **decreasing** on the intervals $\boxed{(-\infty, -1), (-1, 1) \text{ and } (5, \infty).}$

(vi)



(e) $f(x) = \dfrac{x + 2}{x^2 + 2x - 3}$

(i) The denominator is equal to 0 if and only if $x = 1$ or $x = -3$. Hence

$$\operatorname{dom}(f) = \big\{x \in \mathbb{R} \colon x \neq 1 \text{ and } x \neq -3\big\}.$$

(ii) Since the numerator is non-zero at $x = -3$ and $x = 1$, we have the following **vertical asymptotes**:

$\boxed{x = -3}$ $\boxed{x = 1}$

As $x \to \pm\infty$, we have

$$f(x) = \frac{\dfrac{1}{x} + \dfrac{2}{x^2}}{1 + \dfrac{2}{x} - \dfrac{3}{x^2}} \to 0.$$

Hence $\boxed{y = 0}$ is a **horizontal asymptote**.

(iii) Since $f(0) = -\frac{2}{3}$, the point $\boxed{\left(0, -\frac{2}{3}\right)}$ is a point of intersection of the graph with the y-axis.

We have $f(x) = 0$ if and only if $x = -2$. Hence $\boxed{(-2, 0)}$ is the only point of intersection of the graph with the x-axis.

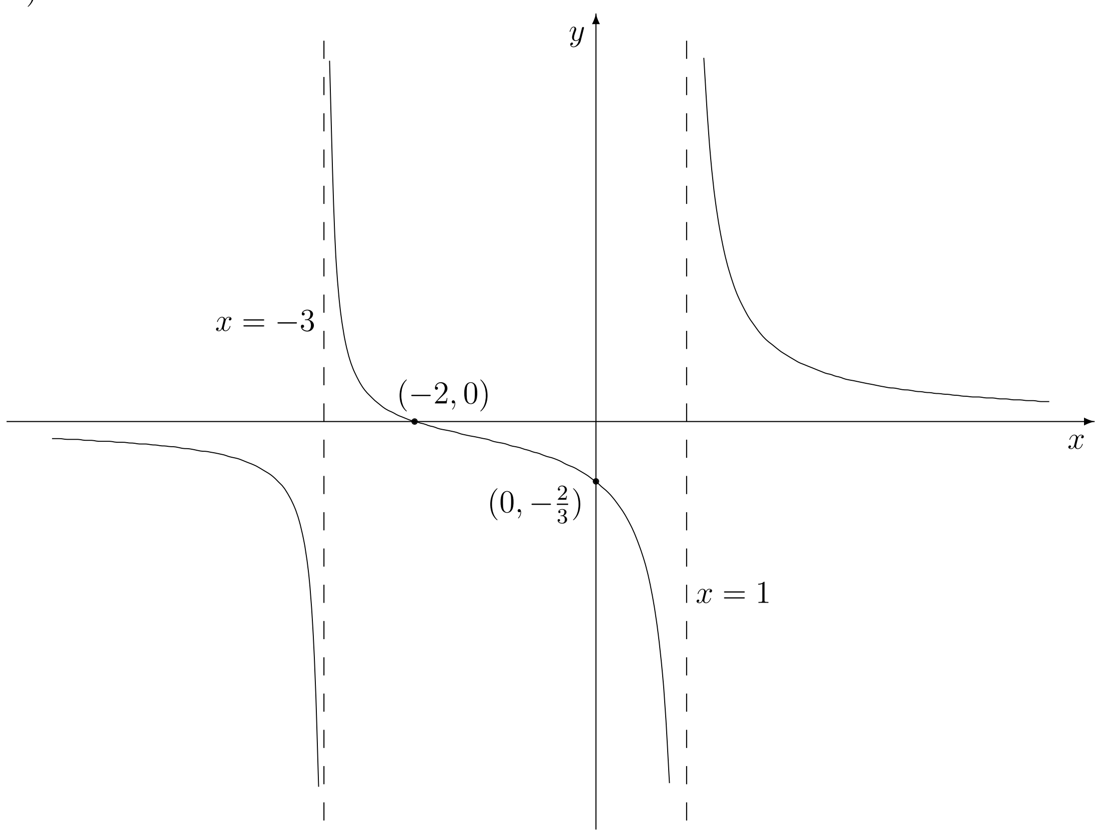
(iv) The first derivative is equal to

$$\begin{aligned} f'(x) &= \frac{x^2 + 2x - 3 - (x+2)(2x+2)}{(x^2 + 2x - 3)^2} \\ &= \frac{x^2 + 2x - 3 - 2x^2 - 6x - 4}{(x^2 + 2x - 3)^2} = \frac{-x^2 - 4x - 7}{(x^2 + 2x - 3)^2} \\ &= -\frac{x^2 + 4x + 7}{(x^2 + 2x - 3)^2} \end{aligned}$$

The numerator of the last fraction is always positive because $x^2+4x+7 = (x+2)^2 + 3$. Hence the function f has no stationary points. Since the function is differentiable on its domain and there are no endpoints of parts of the domain that belong to the domain, the function f has no local maxima or minima.

(v) Since $f'(x) < 0$ on its domain, the function f is **monotonic decreasing** on the intervals $\boxed{(-\infty, -3),\ (-3, 1) \text{ and } (1, \infty).}$

(vi)



(f) $f(x) = x\sqrt{x+2}$

(i) The square root is defined exactly if $x+2 \geq 0$. Hence the natural domain of f is
$$\operatorname{dom}(f) = \{x \in \mathbb{R} : x \geq -2\} = [-2, \infty).$$

(ii) There are no vertical asymptotes since at no point $f(x) \to \infty$ or $f(x) \to -\infty$. There are no horizontal or slant asymptotes since f has no limit as $x \to \infty$ and f does not grow linearly.

(iii) Since $f(0) = 0$, the point $\boxed{(0,0)}$ is a point of intersection of the graph with the y-axis.
Further, $f(x) = 0$ if and only if $x = 0$ or $x = -2$; so $\boxed{(0,0)}$ and $\boxed{(-2,0)}$ are the points of intersection of the graph with the x-axis.

(iv) Using the product rule we can calculate the first derivative of f (for $x > -2$):
$$\begin{aligned} f'(x) &= \sqrt{x+2} + x \cdot \frac{1}{2\sqrt{x+2}} = \frac{2\sqrt{x+2}\sqrt{x+2} + x}{2\sqrt{x+2}} \\ &= \frac{2x+4+x}{2\sqrt{x+2}} = \frac{3x+4}{2\sqrt{x+2}} \end{aligned}$$

We have
$$f'(x) = 0 \iff 3x+4 = 0 \iff \boxed{x = -\frac{4}{3}}$$

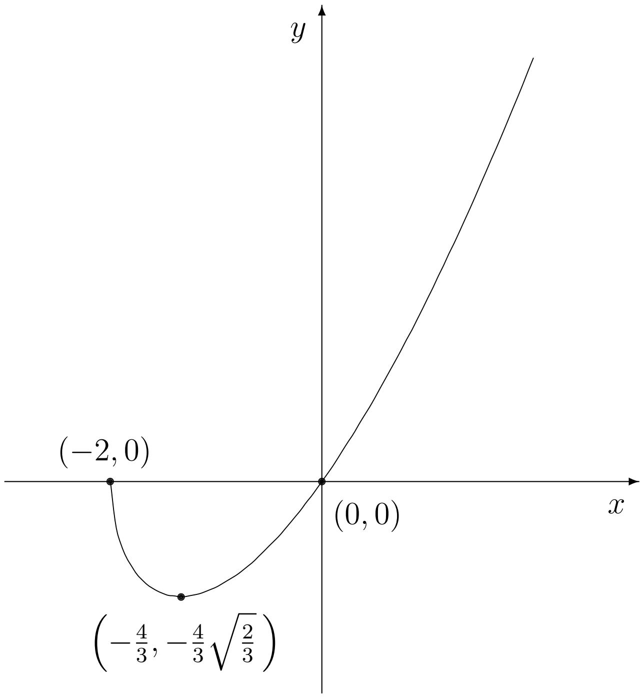
which gives the only stationary point.
For the sign of f' we have

x	-2		$-\frac{4}{3}$	
$3x+4$	$-$	$-$	0	$+$
$2\sqrt{x+2}$	0	$+$	$+$	$+$
$f'(x)$	ND	$-$	0	$+$
$f(x)$	0	$\searrow$	$-$	$\nearrow$
	max		min	

The function f has a **local minimum** at $\boxed{x = -\frac{4}{3}}$ with $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}}$.
We also have to consider the left endpoint of the domain: -2. The function f has a **local maximum** at $\boxed{x = -2}$ with $f(-2) = 0$.

(v) The function f is **monotonic decreasing** on the interval $\boxed{\left(-2, -\frac{4}{3}\right)}$; it is **monotonic increasing** on the interval $\boxed{\left(-\frac{4}{3}, \infty\right)}$.

(vi)



(g) $f(x) = x^2 + \dfrac{1}{x^2}$

(i) The denominator of the fraction vanishes exactly if $x = 0$; hence the natural domain of f is

$$\operatorname{dom}(f) = \{x \in \mathbb{R} \colon x \neq 0\} = (-\infty, 0) \cup (0, \infty).$$

(ii) Since the numerator of the fraction is non-zero at $x = 0$, the line $\boxed{x = 0}$ is a **vertical asymptote**.
For $x \to \pm\infty$ the function f behaves like $f(x) \sim x^2$. Neither does the limit exist nor is the growth linear. Hence there are **no horizontal or slant asymptotes**.

(iii) For $x = 0$ the function is not defined; hence the graph has no point of intersection with the y-axis.
Moreover, $f(x) > 0$ for every $x \in \operatorname{dom}(f)$; hence the graph has no point of intersection with the x-axis.

(iv) The first derivative of f is

$$f'(x) = 2x - \frac{2}{x^3}.$$

$$\begin{aligned} f'(x) = 0 \quad &\Longleftrightarrow \quad 2x = \frac{2}{x^3} \quad \Longleftrightarrow \quad x^4 = 1 \\ &\Longleftrightarrow \quad \boxed{x = 1 \quad \text{or} \quad x = -1} \end{aligned}$$

To determine the sign of $f'(x)$ we can write

$$f'(x) = \frac{2x^4 - 2}{x^3} = \frac{2(x^2+1)(x+1)(x-1)}{x^3}.$$

Since $x^2 + 1 > 0$, we have

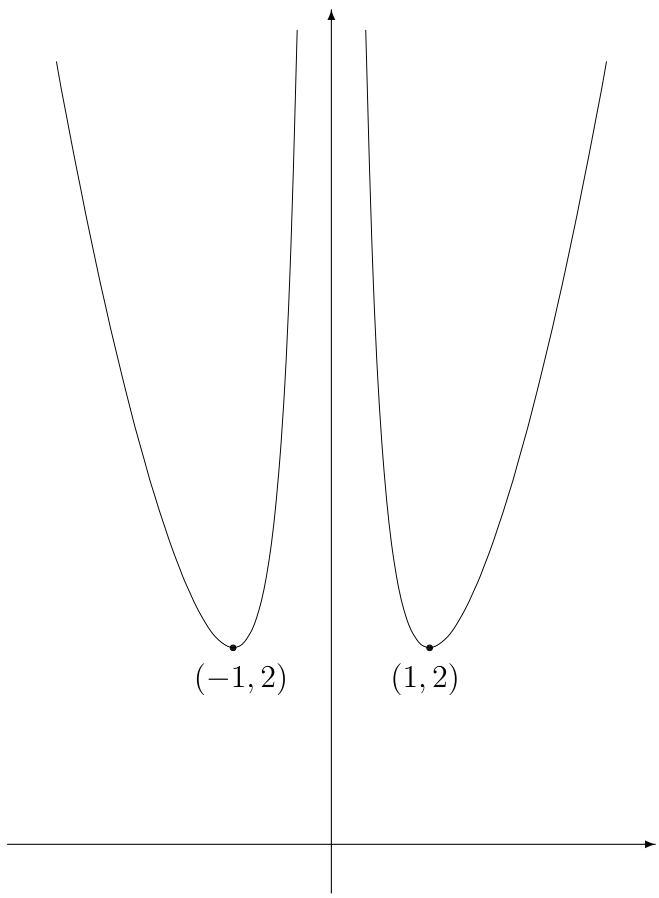
x		-1		0		1	
$(x+1)$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-1)$	$-$	$-$	$-$	$-$	$-$	0	$+$
x^3	$-$	$-$	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	ND	$-$	0	$+$
$f(x)$	$\searrow$	—	$\nearrow$	ND	$\searrow$	—	$\nearrow$
		min				min	

Hence we have two **local minima** at $\boxed{x = -1}$ with $f(-1) = 2$ and $\boxed{x = 1}$ with $f(1) = 2$.

(v) The function is **increasing** on the intervals $\boxed{(-1, 0) \text{ and } (1, \infty);}$

it is **decreasing** on the intervals $\boxed{(-\infty, -1) \text{ and } (0, 1).}$

(vi)



21. Let n be a positive integer and define the function f by

$$f(x) = x^2 e^{-nx}.$$

(i) Determine when the function f is increasing and when it is decreasing. Find the position and nature of the stationary points (they may depend on n).

(ii) Determine when the curve is concave up and when it is concave down. Find the points of inflection.

(iii) Find the position of the critical points relative to each other and use this information to sketch the graph of f qualitatively showing the critical points.

Solution:

(i) $f'(x) = 2xe^{-nx} + x^2(-n)e^{-nx} = (2 - nx)xe^{-nx}$

Since $e^{-nx} > 0$, we have

$$f'(x) = 0 \iff 2 - nx = 0 \quad \text{or} \quad x = 0 \iff \boxed{x = \frac{2}{n}} \quad \text{or} \quad \boxed{x = 0}$$

The values at these stationary points are

$$f(0) = 0; \qquad f\left(\frac{2}{n}\right) = \left(\frac{2}{n}\right)^2 e^{-n\frac{2}{n}} = \frac{4}{n^2}e^{-2} = \frac{4}{e^2 n^2}$$

Sign of f' (The number $x = \dfrac{2}{n}$ is positive since $n > 0$):

x		0		$\frac{2}{n}$	
$(2 - nx)$	$+$	$+$	$+$	0	$-$
x	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	—	$\nearrow$	—	$\searrow$
		min		max	

The function f has a **local minimum** at $\boxed{x = 0}$ with $f(0) = 0$ and a **local maximum** at $\boxed{x = \frac{2}{n}}$ with $f\left(\frac{2}{n}\right) = \frac{4}{e^2 n^2}$.

The function f is **increasing** on the interval $\boxed{\left(0, \frac{2}{n}\right)}$;

it is **decreasing** on the intervals $\boxed{(-\infty, 0)}$ and $\boxed{\left(\frac{2}{n}, \infty\right)}$.

(ii)

$$\begin{aligned}
f''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2xe^{-nx} - nx^2e^{-nx}\right) \\
&= 2e^{-nx} + 2x(-n)e^{-nx} - 2nxe^{-nx} - nx^2(-n)e^{-nx} \\
&= \left(2 - 4nx + n^2x^2\right)e^{-nx}
\end{aligned}$$

$$\begin{aligned}
f''(x) = 0 \quad &\Longleftrightarrow \quad n^2x^2 - 4nx + 2 = 0 \\
&\Longleftrightarrow \quad x = \frac{4n \pm \sqrt{(4n)^2 - 4n^2 \times 2}}{2n^2} = \frac{4n \pm \sqrt{8n^2}}{2n^2} = \frac{2 \pm \sqrt{2}}{n}
\end{aligned}$$

Both numbers are possible points of inflection; they are positive since $2 \pm \sqrt{2} > 0$ and $n > 0$.

f'' can be written as

$$f''(x) = \left(x - \frac{2-\sqrt{2}}{n}\right)\left(x - \frac{2+\sqrt{2}}{n}\right)e^{-nx}$$

Sign of f'':

x		$\frac{2-\sqrt{2}}{n}$		$\frac{2+\sqrt{2}}{n}$	
$\left(x - \frac{2-\sqrt{2}}{n}\right)$	$-$	0	$+$	$+$	$+$
$\left(x - \frac{2+\sqrt{2}}{n}\right)$	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$		$\frown$		$\smile$

The function f is **concave up** on the intervals

$$\boxed{\left(-\infty, \frac{2-\sqrt{2}}{n}\right) \quad \text{and} \quad \left(\frac{2+\sqrt{2}}{n}, \infty\right);}$$

it is **concave down** on the interval $\boxed{\left(\frac{2-\sqrt{2}}{n}, \frac{2+\sqrt{2}}{n}\right).}$

Since the concavity changes, there are **points of inflection** at

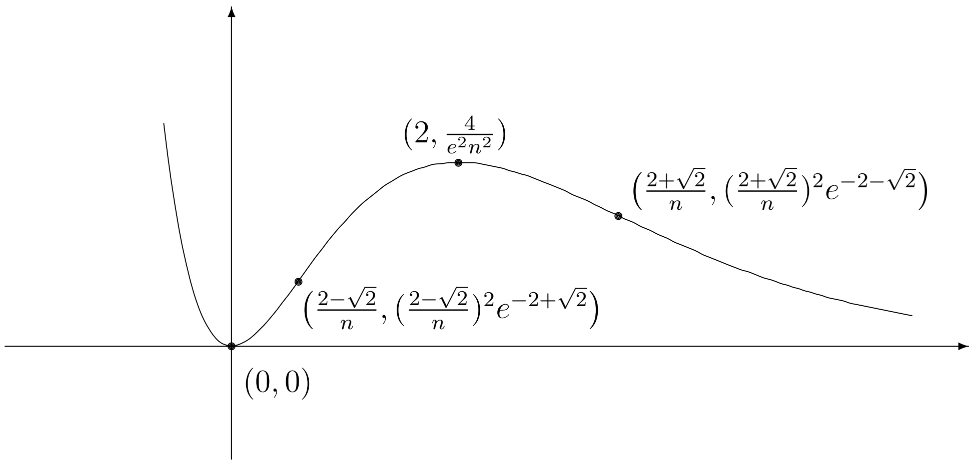
$$x = \boxed{\frac{2-\sqrt{2}}{n}} = \frac{2}{n} - \frac{\sqrt{2}}{n} \quad \text{and} \quad x = \boxed{\frac{2+\sqrt{2}}{n}} = \frac{2}{n} + \frac{\sqrt{2}}{n}$$

with values

$$f\left(\frac{2-\sqrt{2}}{n}\right) = \left(\frac{2-\sqrt{2}}{n}\right)^2 e^{-2+\sqrt{2}}, \qquad f\left(\frac{2+\sqrt{2}}{n}\right) = \left(\frac{2+\sqrt{2}}{n}\right)^2 e^{-2-\sqrt{2}}.$$

(iii) The relative position of the critical points is as follows:

$$0 < \frac{2}{n} - \frac{\sqrt{2}}{n} < \frac{2}{n} < \frac{2}{n} + \frac{\sqrt{2}}{n}$$



22. Let a be a positive constant and define the function

$$f(x) = x\sqrt{x+a}\,.$$

(i) Find the natural domain of f. (It will depend on a.)

(ii) Find the zeros of f. Find when f is positive and when it is negative.

(iii) Find the position and nature of the stationary points. Hence find all local maxima and minima. Determine when the function f is increasing and when it is decreasing.
(These things will depend on a.)

(iv) Sketch the graph of f showing the stationary points.

Solution:

(i) The function is well defined if the expression under the square root is non-negative, i.e. if $x + a \geq 0$. Hence the natural domain is equal to

$$\mathrm{dom}(f) = \big\{x \in \mathbb{R} \colon x \geq -a\big\} = [-a, \infty).$$

(ii) The function f has two zeros: $x = -a$ and $x = 0$.

Since $\sqrt{x+a} \geq 0$, we have $f(x) > 0$ for $x \in (-a, 0)$ and

$f(x) > 0$ for $x \in (0, \infty)$.

(iii) The first derivative is

$$f'(x) = \sqrt{x+a} + \frac{x}{2\sqrt{x+a}} = \frac{2(x+a)+x}{2\sqrt{x+a}} = \frac{3x+2a}{2\sqrt{x+a}}\,.$$

Hence

$$f'(x) = 0 \quad \Longleftrightarrow \quad 3x + 2a = 0 \quad \Longleftrightarrow \quad x = -\frac{2a}{3}.$$

The value at the stationary point $-\dfrac{2a}{3}$ is

$$f\Big(-\frac{2a}{3}\Big) = -\frac{2a}{3}\sqrt{-\frac{2a}{3} + a} = -\frac{2a}{3}\sqrt{\frac{a}{3}}.$$

The sign of f' is summarised in the following table

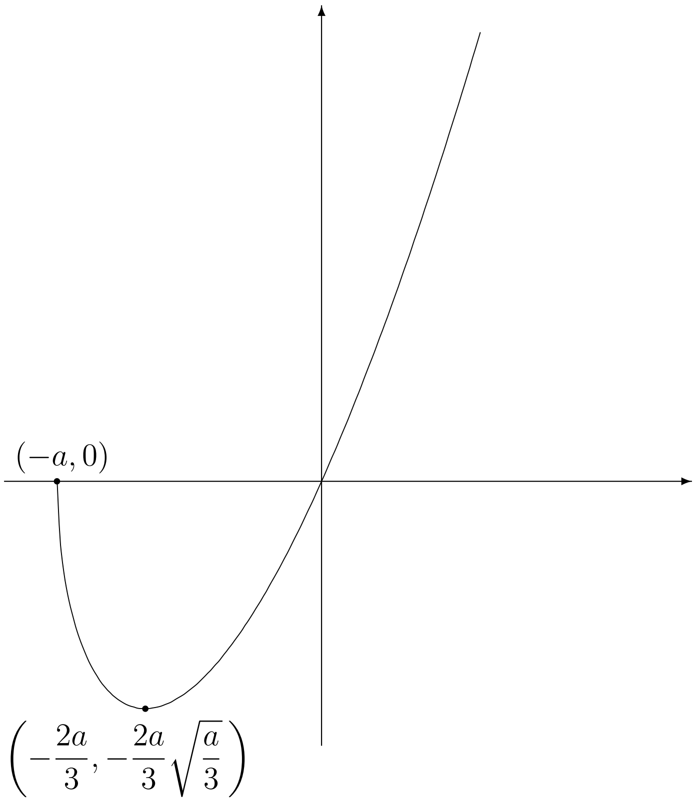
x	$-a$		$-\dfrac{2a}{3}$	
$3x+2a$	$-$	$-$	0	$+$
$\sqrt{x+a}$	0	$+$	$+$	$+$
$f'(x)$	ND	$-$	0	$+$
$f(x)$		$\searrow$	$-$	$\nearrow$
	max		min	

There is a **local minimum** at $\boxed{x = -\dfrac{2a}{3}}$ with $f\Big(-\dfrac{2a}{3}\Big) = -\dfrac{2a}{3}\sqrt{\dfrac{a}{3}}$.

We also have to consider the left endpoint of the domain: $-a$:
the function f has a **local maximum** at $\boxed{x = -a}$ with $f(-a) = 0$.

The function is **decreasing** on $\Big(-a, -\dfrac{2a}{3}\Big)$ and **increasing** on $\Big(-\dfrac{2a}{3}, \infty\Big)$.

(iv)



23. A cylindrical can is to contain $500\,\text{cm}^3$ of a drink. The material used to make the top and bottom of the can costs twice as much per square centimetre as the material used to make the sides. What height and radius should the can have to minimise the costs?

Solution:
The cost is equal to (in appropriate units)

$$\begin{aligned} C &= 2 \times \text{area of bottom} + 2 \times \text{are of top} + \text{area of side} \\ &= 4\pi r^2 + 2\pi r h \end{aligned}$$

Since the volume is $500\,\text{cm}^3$, we have

$$500 = \pi r^2 h,$$

which implies

$$h = \frac{500}{\pi r^2}\,.$$

Hence we can write C as a function of only r:

$$C = C(r) = 4\pi r^2 + 2\pi r \cdot \frac{500}{\pi r^2} = 4\pi r^2 + \frac{1000}{r}$$

The first derivative of C is

$$C'(r) = 8\pi r - \frac{1000}{r^2}\,.$$

Stationary points:

$$C'(r) = 0 \quad \Longleftrightarrow \quad 8\pi r = \frac{1000}{r^2} \quad \Longleftrightarrow \quad r^3 = \frac{125}{\pi}$$
$$\Longleftrightarrow \quad r = \frac{5}{\pi^{1/3}}$$

The second derivative is

$$C''(r) = 8\pi + \frac{2000}{r^3},$$

which is positive for positive r. Hence C has a local minimum at $r = \dfrac{5}{\pi^{1/3}}$.

Since this is the only stationary point, the function has a global minimum at $\boxed{r = \dfrac{5}{\pi^{1/3}}.}$

The corresponding height is

$$h = \frac{500}{\pi\left(\dfrac{5}{\pi^{1/3}}\right)^2} = \frac{500}{\pi \times \pi^{-2/3} \times 25} = \boxed{\frac{20}{\pi^{1/3}}}$$

24. Find the points on the curve $y^2 - x^2 = 1$ which are closest to the point $(2, 0)$.

Solution:

The distance between the points (x, y) and $(2, 0)$ is

$$d = \sqrt{(x-2)^2 + y^2}.$$

If the point (x, y) is on the curve $y^2 - x^2 = 1$, then we can replace y^2 by $x^2 + 1$, which gives

$$d(x) = \sqrt{(x-2)^2 + x^2 + 1} = \sqrt{x^2 - 4x + 4 + x^2 + 1} = \sqrt{2x^2 - 4x + 5}.$$

The first derivative of d with respect to x is

$$d'(x) = \frac{4x - 4}{2\sqrt{2x^2 - 4x + 5}} = \frac{2x - 2}{\sqrt{2x^2 - 4x + 5}}.$$

Stationary points:

$$d'(x) = 0 \quad \Longleftrightarrow \quad x = 1.$$

The derivative d' is negative for $x < 1$; it is positive for $x > 1$.

Hence d has a local minimum at $x = 1$. Since this is the only stationary point, d has a global minimum there. The corresponding y-coordinate is

$$y = \pm\sqrt{1^2 + 1} = \pm\sqrt{2}.$$

Hence there two points where the distance is minimal:

$$\left(1, \sqrt{2}\right) \quad \text{and} \quad \left(1, -\sqrt{2}\right).$$

The minimal distance is

$$d_{\min} = d(1) = \sqrt{2 \times 1^2 - 4 \times 1 + 5} = \sqrt{3}.$$

25. The cost per hour of running a train is proportional to $100+v^2/36$ where v, measured in m.p.h., is the average speed on a trip. Find the speed v that makes the trip Glasgow–London cheapest. (Take the distance Glasgow–London to be 400 miles.)

Solution:
The time for the trip is equal to

$$t = \frac{\text{distance}}{v} = \frac{400}{v}\,.$$

The cost is proportional to the time for the trip times the cost per hour, i.e.

$$C(v) = \frac{400}{v}\cdot\left(100+\frac{v^2}{36}\right) = 400\left(\frac{100}{v}+\frac{v}{36}\right).$$

The derivative with respect to v is

$$C'(v) = 400\left(-\frac{100}{v^2}+\frac{1}{36}\right).$$

Stationary points:

$$\begin{aligned} C'(v)=0 \quad &\iff \quad \frac{100}{v^2}=\frac{1}{36} \quad \iff \quad v^2 = 36\times 100 \\ &\iff \quad v = 60 \qquad (\text{since } v>0). \end{aligned}$$

The second derivative is

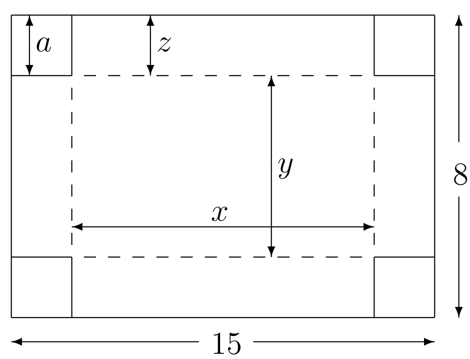
$$C''(v) = 400\times\frac{200}{v^3} > 0.$$

Hence C has a local minimum at $v=60$. Since this is the only stationary point, C has a global minimum at $v = 60\,$m.p.h.

26. From a rectangular piece of cardboard of dimension 8×15, four congruent squares are to be cut out, one at each corner (see diagram). The remaining cross-like piece is then to be folded (along the dashed lines) into an open box. What size of squares should be cut out if the volume of the resulting box is to be a maximum?

Solution:
Let a be the length of a side of the square that is cut out. Then the dimensions of the box are

$$x = 15-2a, \quad y = 8-2a, \quad z = a.$$



Hence the volume is

$$V = V(a) = xyz = (15 - 2a)(8 - 2a)a = 4a^3 - 46a^2 + 120a.$$

Since all lengths x, y, z must be positive, we get the following domain:
$\mathrm{dom}(V) = \{a\colon 0 < a < 4\} = (0, 4)$.
First derivative:

$$\frac{\mathrm{d}V}{\mathrm{d}a} = 12a^2 - 92a + 120 = 4(3a^2 - 23a + 30).$$

Stationary points:

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}a} = 0 \quad &\iff \quad 3a^2 - 23a + 30 = 0 \\ &\iff \quad a = \frac{5}{3} \quad \text{or} \quad a = 6. \end{aligned}$$

6 is not in the domain of V. Hence the only stationary point in the domain is $a = \dfrac{5}{3}$.
The second derivative at this stationary point is

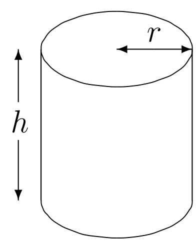
$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}a^2}\right|_{a=\frac{5}{3}} = (24a - 92)\Big|_{a=\frac{5}{3}} = -52 < 0.$$

Hence V has a local maximum at $a = \dfrac{5}{3}$.
The function V has only one stationary point; hence V has a global maximum at $a = \dfrac{5}{3}$.

The size of the square is $\dfrac{5}{3} \times \dfrac{5}{3}$.

27. An open cylindrical can, that is, the can has a bottom but no top, has a surface area of $108\,\pi\,\mathrm{cm}^2$. Find the radius and the height of the can that makes the volume a maximum.

Solution:
Let r and h be the radius and height, respectively.



The volume is

$$V = \pi r^2 h.$$

The surface area is

$$\begin{aligned} &2\pi rh + \pi r^2 = 108\pi \\ \implies \quad &h = \frac{108\pi - \pi r^2}{2\pi r} = \frac{54}{r} - \frac{r}{2} \end{aligned}$$

Hence
$$V = V(r) = \pi r^2\left(\frac{54}{r} - \frac{r}{2}\right) = \pi\left(54r - \frac{r^3}{2}\right).$$
Both r and h must be positive:
$$r > 0 \quad \text{and} \quad \frac{54}{r} - \frac{r}{2} > 0;$$
if r is positive, the latter inequality is equivalent to
$$\frac{54}{r} > \frac{r}{2} \quad \iff \quad r^2 < 108 \quad \iff \quad r < \sqrt{108}.$$
Hence $\operatorname{dom}(V) = (0, \sqrt{108})$.

First derivative:
$$\frac{\mathrm{d}V}{\mathrm{d}r} = \pi\left(54 - \frac{3}{2}r^2\right)$$
Stationary points:
$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}r} = 0 \quad &\iff \quad 54 - \frac{3}{2}r^2 = 0 \\ &\iff \quad r^2 = 36 \\ &\iff \quad r = \pm 6 \end{aligned}$$
But -6 is not in the domain. Hence $r = 6$ is the only stationary point.
The second derivative at the stationary point is
$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}r^2}\right|_{r=6} = -3\pi r\Big|_{r=6} = -18\pi < 0.$$
Hence V has a local maximum at $r = 6$.

The function V has only one stationary point; hence V has a global maximum at $r = 6\,\mathrm{cm}$.
The height is: $h = \dfrac{54}{6} - \dfrac{6}{2} = 6\,\mathrm{cm}$.

28. The pressure P and volume V of a gas satisfy Boyle's law $PV = C$ where C is a constant. If V increases at a rate of $10\,\mathrm{cm}^3$ per second, at what rate is P changing when $V = 2$ litres and $P = 2$ bar.

Solution:
Differentiate both sides of the equation $PV = C$ with respect to time t:
$$\dot{P}V + P\dot{V} = 0$$
since C is a constant. We want to know the rate of change of P; hence we solve for $\dot{P}$:
$$\dot{P} = -\frac{P\dot{V}}{V}.$$
The volume 2 litres is equal to $2000\,\mathrm{cm}^3$. Hence we obtain
$$\dot{P} = -\frac{2 \times 10}{2000} = \boxed{-0.01\,\mathrm{bar\,s^{-1}}}$$

29. Sand falls onto a conical pile at a rate of $0.1\,\mathrm{m^3 s^{-1}}$. The radius of the base of the pile is always equal to half its height. How fast is the height increasing when the pile is
(a) $1\,\mathrm{m}$ high? (b) $2\,\mathrm{m}$ high?

Solution:
The volume of the pile is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}\,.$$

Differentiate both sides of this equation with respect to time t:

$$\dot{V} = \frac{\pi \cdot 3h^2 \dot{h}}{12} = \frac{\pi}{4} h^2 \dot{h}.$$

If we solve this for $\dot{h}$, we obtain

$$\dot{h} = \frac{4\dot{V}}{\pi h^2} = \frac{4 \times 0.1}{\pi h^2} = \frac{0.4}{\pi h^2}\,.$$

(a) For $h = 1\,\mathrm{m}$ we have

$$\dot{h} = \frac{0.4}{\pi} = \boxed{0.127\,\mathrm{m\,s^{-1}}}$$

(b) For $h = 2\,\mathrm{m}$ we have

$$\dot{h} = \frac{0.4}{4\pi} = \boxed{0.0318\,\mathrm{m\,s^{-1}}}$$

30. The centre of a corn field catches fire, and it is observed that the fire spreads circularly. If, when the radius is 15 feet, it is increasing at 3 feet per second, find at what rate is the area increasing.

Solution.
Let r be the radius and A the area of the circular field.
The area is

$$A = \pi r^2.$$

Both A and r are functions of time t. Differentiate both sides of the equation with respect to t (a dot denotes differentiation w.r.t. t):

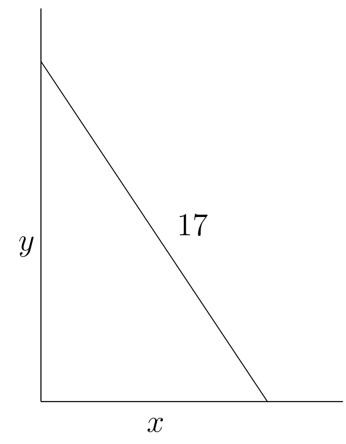
$$\dot{A} = 2\pi r \dot{r}$$

For the given values we obtain:

$$\dot{A} = 2\pi \times 15 \times 3 = 90\pi \approx 282.7\,\mathrm{feet}^2/\mathrm{s}$$

31. A ladder, whose length is 17 feet, stands on horizontal ground against a vertical wall. The ladder starts to slide, and the end in contact with the ground is observed to be moving at 20 feet per second when it is 15 feet from the wall. Find the velocity of the end that is in contact with the wall at this instant.

Solution.
Let x be distance of the bottom of the ladder from the wall and y be the height of the top of the ladder.



By Pythagoras' theorem we have

$$x^2 + y^2 = 17^2.$$

Both x and y are functions of time t. Differentiate both sides of the equation with respect to t:

$$2x\dot{x} + 2y\dot{y} = 0$$

We can solve for $\dot{y}$:

$$\dot{y} = -\frac{x\dot{x}}{y}$$

We know the value of x (namely 15) and $\dot{x}$ (namely 20). We also need the value of y, which we can get from Pythagoras' theorem:

$$y = \sqrt{17^2 - x^2} = \sqrt{17^2 - 15^2} = \sqrt{289 - 225} = \sqrt{64} = 8$$

Hence

$$\dot{y} = -\frac{15 \times 20}{8} = -\frac{75}{2} = -37.5\,\text{feet/s}$$

That $\dot{y}$ is negative shows that the top of the ladder is moving downwards.