4 Lecture examples: Chapter 4

Examples 4A

1 If

$$A = \begin{bmatrix} 1 & 1 & -1 & 3 \\ 2 & 2 & -2 & 6 \\ -3 & 1 & 2 & 1 \end{bmatrix},$$

find $r(A)$ and $n(A)$.

Row space $= sp\{(1,1,-1,3],(2,2,-2,6],(-3,1,2,1]\}$. Now $(2,2,-2,6] = 2(1,1,-1,3]$, so remove $(2,2,-2,6]$, giving row space Row space $= sp\{(1,1,-1,3],(-3,1,2,1]\}$. These vectors are linearly independent, so $\rho(A) = r(A) = 2$.

(**Note:** Column space$= sp\{(1,2,-3),(1,2,1),(-1,-2,2),(3,6,1)\}$. But

$$(-1,-2,2) = -\frac{3}{4}(1,2,-3) - \frac{1}{4}(1,2,1), \qquad (3,6,1) = \frac{1}{2}(1,2,-3) + \frac{5}{2}(1,2,1)$$

so Column space$= sp\{(1,2,-3),(1,2,1)\}$ and $c(A) = r(A) = 2$.)

Nullspace:

$$A\boldsymbol{x} = \mathbf{0} \Leftrightarrow \left\{ \begin{array}{rcl} x_1 + x_2 - x_3 + 3x_4 & = & 0 \\ 2x_1 + 2x_2 - 2x_3 + 6x_4 & = & 0 \\ -3x_1 + x_2 + 2x_3 + x_4 & = & 0 \end{array} \right.$$

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 2 & 2 & -2 & 6 \\ -3 & 1 & 2 & 1 \end{bmatrix} \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 + 3r_1 \end{array} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -1 & 10 \end{bmatrix} \begin{array}{l} r_2' = r_3 \\ r_3' = r_2 \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & -1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; r_2' = r_2/4 \; \rightarrow \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -1/4 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $x_3 = \lambda$, $x_4 = \mu$ to find

$$x_2 = \frac{1}{4}\lambda - \frac{5}{2}\mu, \qquad x_1 = -\left[\frac{1}{4}\lambda - \frac{5}{2}\mu\right] + \lambda - 3\mu = \frac{3}{4}\lambda - \frac{1}{2}\mu$$

so

$$\begin{aligned} \boldsymbol{x} &= \lambda\left(\frac{3}{4},\frac{1}{4},1,0\right) + \mu\left(-\frac{1}{2},-\frac{5}{2},0,1\right) \\ &= \frac{\lambda}{4}(3,1,4,0) + \frac{\mu}{2}(-1,-5,0,2) \end{aligned}$$

Hence

$$N(A) = sp\{(3,1,4,0),(-1,-5,0,2)\}$$

so dimension of nullspace $n(A) = 2$.

---

2 Verify the Dimension Theorem for

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 8 \\ -3 & 1 & 2 & 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 8 \\ -3 & 1 & 2 & 1 \end{bmatrix} \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 + 3r_1 \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 13 \end{bmatrix} \begin{array}{l} r_2' = r_3 \\ r_3' = r_2 \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 4 & 5 & 13 \\ 0 & 0 & 0 & 0 \end{bmatrix} r_2' = r_2/4 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 5/4 & 13/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

leading variables: x_1, x_2 so $r(A) = 2$ (= no. of nonzeros rows in B)
free variables: x_3, x_4 so $n(A) = 2$

so

$$r(A) + n(A) = 4 = n \qquad \text{(no. of columns in } A\text{)}$$

Examples 4B

1 For

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 8 \\ -3 & 1 & 2 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

state why the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent and find its general solution.

$$\begin{bmatrix} 1 & 1 & 1 & 4 & | & 1 \\ 2 & 2 & 2 & 8 & | & 2 \\ -3 & 1 & 2 & 1 & | & 0 \end{bmatrix} \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 + 3r_1 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 4 & 5 & 13 & | & 3 \end{bmatrix} \begin{array}{l} r_2' = r_3 \\ r_3' = r_2 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 & | & 1 \\ 0 & 4 & 5 & 13 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} r_2' = r_2/4$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 1 \\ 0 & 1 & \frac{5}{4} & \frac{13}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1' = r_1 - r_2 \\ \\ \\ \end{array} \qquad \text{echelon form}$$

We have $r(A) = r(A|\boldsymbol{b}) = 2$ so system is consistent and the number of parameters in the general solution is $4 - 2 = 2$. Free variables are x_3, x_4, so set $x_4 = \lambda$, $x_3 = \mu$, say. Then

$$x_2 = \frac{3}{4} - \frac{5}{4}\mu - \frac{13}{4}\lambda, \qquad x_1 = 1 - \left(\frac{3}{4} - \frac{5}{4}\mu - \frac{13}{4}\lambda\right) - \mu - 4\lambda = \frac{1}{4} + \frac{1}{4}\mu - \frac{3}{4}\lambda$$

so general solution is

$$\boldsymbol{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} \frac{1}{4} \\ \frac{-5}{4} \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} \frac{-3}{4} \\ \frac{-13}{4} \\ 0 \\ 1 \end{bmatrix}.$$

Note: We could equally well write for example

$$\boldsymbol{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -5 \\ 4 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ -13 \\ 0 \\ 4 \end{bmatrix}.$$

---

2 Apply EROs to reduce

$$[A|\boldsymbol{b}] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & k & 1 & -1 \\ 3 & 2 & k^2-1 & k^2-k & k \end{array}\right]$$

to echelon form. Determine values of k for which system is consistent. Give the number of parameters in the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ when the system is consistent.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & k & 1 & -1 \\ 3 & 2 & k^2-1 & k^2-k & k \end{array}\right] \begin{array}{l} \\ r_2' = r_2 - 2r_1 \rightarrow \\ r_3' = r_3 - 3r_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -1 & k-2 & -1 & -7 \\ 0 & -1 & k^2-4 & k^2-k-3 & k-9 \end{array}\right] r_2' = -r_2 \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2-k & 1 & 7 \\ 0 & -1 & k^2-4 & k^2-k-3 & k-9 \end{array}\right] \begin{array}{l} \\ \qquad\qquad \rightarrow \\ r_3' = r_3 + r_2 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & | & 3 \\ 0 & 1 & 2-k & 1 & | & 7 \\ 0 & 0 & k^2-k-2 & k^2-k-2 & | & k-2 \end{array}\right].$$

Noting that $p(k) = k^2 - k - 2 = (k-2)(k+1)$, we have three separate cases:

1. Suppose $k^2 - k - 2 \neq 0$, i.e. $k \neq 2$, $k \neq -1$. Then we have

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & | & 3 \\ 0 & 1 & 2-k & 1 & | & 7 \\ 0 & 0 & 1 & 1 & | & \frac{1}{k+1} \end{array}\right] \Rightarrow \left\{ \begin{array}{c} r(A) = 3 \\ r(A|\mathbf{b}) = 3 \end{array} \right.$$

   System is therefore consistent with $n - r = 4 - 3 = 1$ parameter (free variable x_4).

2. Suppose $k = 2$. Then we have

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & | & 3 \\ 0 & 1 & 0 & 1 & | & 7 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right] \Rightarrow \left\{ \begin{array}{c} r(A) = 2 \\ r(A|\mathbf{b}) = 2 \end{array} \right.$$

   System is therefore consistent with $n - r = 4 - 2 = 2$ parameters (free variables x_3 and x_4).

3. Suppose $k = -1$. Then we have

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & | & 3 \\ 0 & 1 & 3 & 1 & | & 7 \\ 0 & 0 & 0 & 0 & | & -3 \end{array}\right] \Rightarrow \left\{ \begin{array}{c} r(A) = 2 \\ r(A|\mathbf{b}) = 3 \end{array} \right.$$

   System is therefore inconsistent and has no solutions.

Examples 4C

1 Find the values of k for which

$$A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$$

is invertible. Solve the system $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{b} = (1, 1)$ when $k = 1$.

$$\det(A) = (k-3)(k-2) - 4 = k^2 - 5k + 2$$

so

$$\det(A) = 0 \Leftrightarrow \quad k = \frac{5 \pm \sqrt{17}}{2}.$$

When $k = 1$,

$$A = \begin{bmatrix} -2 & -2 \\ -2 & -1 \end{bmatrix} \quad \Rightarrow A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}$$

so

$$\boldsymbol{x} = \frac{1}{-2}\begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}.$$

---

2 For which value(s) of k is the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$$

singular?

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix} \quad \begin{array}{l} \\ r_2' = r_2 - 3r_1 \\ r_3' = r_3 - kr_1 \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -6 \\ 0 & 3-2k & 2-4k \end{bmatrix}$$

so

$$\det(A) = -5(2-4k) + 6(3-2k) = -10 + 20k + 18 - 12k = 8 + 8k = 8(k+1)$$

so A is singular when $k = -1$.