UNIVERSITY OF STRATHCLYDE
MATHEMATICS & STATISTICS

MM103 Part II: Applications

3. Some like it hot. . .

We begin this section with a practical problem.

Problem: You are making a cup of tea to drink in 5 minutes time. Should you add the (refrigerated) milk now or just before you drink the tea, if you want the tea to lose as little heat as possible?

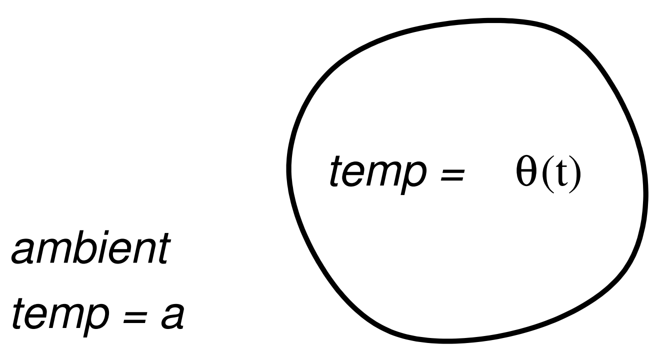
In order to answer this question we need a **relevant equation** and **some data**.

3.1. Newton's law of cooling

This states that *the rate at which the temperature of a hot object decreases with respect to time is proportional to the difference between the object's temperature and that of the surrounding medium.*

How do we write Newton's law of cooling mathematically?

Notation: Let t denote time, and suppose that the object's temperature at time t is $\theta(t)$, and it is in a medium with fixed ambient (background) temperature a, as illustrated below.



The rate of change of the object's temperature with respect to time is $\dfrac{d\theta}{dt}$. Newton's law of cooling says that

$$\frac{d\theta}{dt} = -k\,(\theta - a) \tag{3.1}$$

for some positive constant (of proportionality) k.

Equation (3.1) is a first order separable ordinary differential equation (ODE), and so it can be solved by **separating and integrating**.

$$\int \frac{1}{\theta - a}\,d\theta = -\int k\,dt\,,$$

which gives

$$\ln(\theta - a) = -k\,t + C$$

where C is an arbitrary constant of integration. Taking the exponential of both sides gives

$$\theta - a = \exp(-k\,t + C) = A\,e^{-k\,t}\,,$$

where $A = e^C$ (and so A is also an arbitrary constant, but it must be positive). This gives the temperature $\theta(t)$ as

$$\theta = a + A\,e^{-k\,t}\,,$$

in terms of the arbitrary constant A. We will first work through some examples based on (3.1) before returning to the cup of tea problem. The first two examples ask for the temperature of an object in terms of the Newton's law of cooling constant of proportionality k, when both its initial temperature $\theta(0)$ and a are given.

Example 3.1 A volcano throws out a rock which is initially at 500^oC into a wintry landscape at temperature 0^oC. Beginning with Newton's law of cooling (3.1), find the temperature of the rock after t minutes in terms of k.

Example 3.2 A frozen pie at temperature -20^oC is placed in an oven at temperature 180^oC. Beginning with Newton's law of cooling (3.1), find the temperature of the pie after t minutes in terms of k.

If the temperature of an object is known at two different times (e.g. at time $t = 0$ and one other time), then k can be found, and its temperature can then be calculated at any time t.

Example 3.3 Suppose that the volcanic rock in the example above has temperature 400^oC at time $t = 2$ minutes. Use this to find k for this problem, and calculate the rock's temperature at time $t = 8$ minutes.

Example 3.4 Suppose that the frozen pie in the example above has temperature 30^oC at time $t = 10$ minutes. Use this to find k for this problem, and calculate the pie's temperature at time $t = 20$ minutes. If the pie needs to cook at a temperature of at least 80^oC for 30 minutes, then how long will it need to stay in the oven (to the nearest minute)?

Now that we have some experience of using Newton's law of cooling (NLC) we can go back to look at the best "milk strategy" for the cup of tea problem. We need one more piece of information first though – if we mix together two liquids at two different temperatures, then what is the temperature of the mixed liquid?

Liquid mixing. Suppose that a volume V_1 of liquid at temperature θ_1 is instantaneously mixed with a volume V_2 of liquid at temperature θ_2. Then (assuming no chemical reactions) the temperature θ of the mixed liquid is

$$\theta = \frac{V_1\,\theta_1 + V_2\,\theta_2}{V_1 + V_2}\ . \tag{3.2}$$

Example 3.5

1. Suppose that 0.2 litres of water at 100^oC is added to 1 litre of water at 25^oC. What is the resulting temperature?

2. What volume of water at 85^oC needs to be added to 1 litre of water at 10^oC to obtain water at 40^oC?

Example 3.6 **Cup of tea problem**

Suppose that a cup of tea of volume V_T is made at an initial temperature of θ_T in a room of temperature a, and that a (smaller) volume V_M of milk at temperature θ_M is to be added to the tea before it is drunk, where $\theta_M \leq a < \theta_T$. Beginning with Newton's law of cooling

$$\frac{d\theta}{dt} = -k\,(\theta - a)$$

calculate the temperature of the tea/milk mixture at time $t > 0$ in two situations:

1. The tea is allowed to cool and the milk is added at time t. Call this temperature $\theta_1(t)$.
2. The milk is added at time $t = 0$, and the tea/milk mixture then cools. Call this temperature $\theta_2(t)$.

Which of these two temperatures $\theta_1(t)$ and $\theta_2(t)$ is bigger, and how does their difference depend on the sizes of the various temperatures and volumes? (Assume that k has the same value in both cases.)

3.2. Other rate of change problems

Many other physical situations can be modelled as rate of change problems, and the plan of attack is usually the same:

- Convert the word description of the physical rate of change problem into an ODE.
- Separate and integrate to find the general solution of the ODE (the general solution of a first order ODE involves **one** arbitrary constant).
- Use any initial or other conditions to find the specific solution to the problem.

Example 3.7 A factory which manufactures industrial fume cupboards wants to test how efficiently they work. A chemical is released into one of the fume cupboards at an initial concentration of 160 parts per million (ppm), and the time rate of change of its concentration $c(t)$ is measured to be

$$\frac{dc}{dt} = 400 - 10\,c$$

where time is measured in minutes.

(i) Solve the ODE to find the chemical concentration c at time t.

(ii) Find the time at which the chemical concentration is half the initial concentration.

(iii) What is the limiting value of the chemical concentration after a long time?

Exercises 3

1. An iron is left to cool in a room with temperature 20°C. Denote by $\theta(t)$ the temperature (measured in °C) of the iron at time t (measured in min). According to Newton's Law of Cooling

$$\frac{d\theta}{dt} = -k(\theta - 20),$$

where $k > 0$ is a constant.

(a) If $\theta(0) = 100$ show that

$$\theta(t) = 20 + 80\mathrm{e}^{-kt},\ t \geq 0.$$

(b) Find $\lim_{t\to\infty} \theta(t)$ and sketch the function $\theta(t)$.

(c) It takes the iron 5 min to cool from 100°C to 60°C. Find k and hence calculate how long it takes the iron to cool to 25°C.

2. The balance in a bank account is $y(t)$ where time t is measured in years. Starting with £200, no withdrawals or further investments are made. Interest is paid in proportion to the balance, so that the rate of change of y with respect to t is proportional to y.

(a) Write down the ODE for $y(t)$ and solve it to give the balance in the account as a function of t.

(b) Suppose at $t = 10$ the balance reaches $y(10) = 300$. Find the constant of proportionality in the ODE.

(c) Given the numbers from the previous part, find the time taken to double the initial investment.

3. Toricelli's law states that the rate at which the volume V of water in a draining tank decreases with respect to time is proportional to the square root of the depth y of water in the tank, i.e.

$$\frac{dV}{dt} = -k\sqrt{y}\,.$$

(a) Suppose that the tank is a circular cylinder with vertical sides and radius a. Write down a formula for $V(t)$ in terms of $y(t)$ and a, and hence convert the above differential equation to an ODE for y.

(b) Solve this ODE when $y = 4\,a$ at time $t = 0$.

(c) Suppose that the level reaches $y = a$ at time $t = 8$ minutes. Find the constant of proportionality in the ODE.

(d) Find the time at which the level has dropped to $y = a/4$.

4. The time rate of change of the population of a colony of bacteria is proportional to the size P of the population.

(a) Write down the ODE for $P(t)$ and solve it when the population size at time $t = 0$ is 1000.

(b) Suppose that the population has doubled at time $t = 1$ hour. Use this to find the constant of proportionality for the population equation.

(c) Calculate the population size at time $t = 3$ hours.

(d) How long does it take for the population to reach a million?

5. The ratio of numbers of C14 to C16 atoms is more or less fixed in living things (animals, plants etc.), but when they die the number of C16 atoms stays the same and the number of C14 atoms decreases through radioactive decay. Measuring the relative proportions of C14 to C16 can then indirectly give an estimate of the age of the dead material. This is known

as carbon dating. On average, a constant fraction of C14 atoms decays each time unit. In mathematical terms

$$\frac{dy}{dt} = -ky$$

where y is the number of C14 atoms, $k > 0$ is the rate of decay and we measure time t in years here.

(a) Find the formula for the number of atoms in our sample given the initial value $y(0) = 10^9$ atoms.

(b) After 5730 years the number of Carbon 14 atoms remaining is 1/2 of the starting value. Find the constant of proportionality k above. (5730 years is the half-life of C14.)

(c) What proportion of atoms decays each time unit (i.e. each year)?

(d) In an archaeolgical sample, 1/4 of the C14 atoms have decayed away. How old is it?

6. Many common drugs are eliminated from the bloodstream at a rate that is proportional to the amount $y(t)$ still present.

   (a) Write down a differential equation which describes this situation and show that

$$y(t) = y_0\, e^{-kt}$$

   when y_0 milligrams is the amount initially injected and k is a positive constant.

   (b) Suppose that to anaesthetise a dog at least 500mg of anaesthetic must be present in the dog's bloodstream. If the half-life of the anaesthetic in the dog's bloodstream (i.e. the time it takes the amount to halve) is 3 hours, estimate the size of a single dose that will anaesthetise the dog for at least 45 mins.

   (c) If 700 mg of anaesthetic is injected into the dog's bloodstream, for how long will it remain anaesthetised?