University of Strathclyde, Department of Mathematics and Statistics

MM102 Applications of Calculus
Exercises for Chapter 6 (Week 9)
Solutions

1. Let $y = x^2 + 3x + 4$. Find the approximate change in y if x is increased from 2 to 2.08.

   Solution:
   Set $x_0 = 2$ and $x = 2.08$. The first derivative of y is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x + 3$.
   With $\Delta x = x - x_0 = 0.08$ we obtain

$$\Delta y \approx \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} \Delta x = 7 \times 0.08 = \boxed{0.56}$$

2. Let $y = \sqrt{x}$. Find the approximate change in y if x is increased from 4 to 4.01.

   Solution:
   Set $x_0 = 4$ and $x = 4.01$. The first derivative of y with respect to x is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\,.$$

   With $\Delta x = x - x_0 = 0.01$ we obtain

$$\Delta y \approx \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=4} \Delta x = \frac{1}{2\sqrt{4}} \times 0.01 = \boxed{0.0025}$$

   The value of y changes from 2 to $2 + \Delta y \approx 2.0025$.

3. Newton's law of gravitation states that the force F of attraction between two particles having masses m_1 and m_2 is given by $F = Gm_1m_2/s^2$ where G is a constant and s is the distance between the particles. Find the approximate percentage change in F if s is increased by 0.5%.

   Solution:
   The derivative of F with respect to s is

$$\frac{\mathrm{d}F}{\mathrm{d}s} = Gm_1m_2 \times \left(-\frac{2}{s^3}\right) = -\frac{2Gm_1m_2}{s^3}\,.$$

   The absolute change of F is

$$\Delta F \approx \left.\frac{\mathrm{d}F}{\mathrm{d}s}\right|_{s=s_0} \Delta s = -\frac{2Gm_1m_2}{s_0^3}\Delta s;$$

   the relative change is

$$\frac{\Delta F}{F_0} \approx -\frac{2Gm_1m_2}{s_0^3}\Delta s \div \frac{Gm_1m_2}{s_0^2} = -2\frac{\Delta s}{s_0}\,.$$

Hence, if s is increased by 0.5%, then

$$\frac{\Delta F}{F_0} \approx -2 \times 0.5\% = -1\%,$$

i.e. F is decreased by approximately $\boxed{1\%.}$

4. Consider a sphere with radius r. What is the approximate percentage change of the surface area S of the sphere if the radius is decreased by 1.5%?

   Solution:
   The surface area of a sphere is $A = 4\pi r^2$. The first derivative is $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$.
   Hence the absolute change is

   $$\Delta A \approx \left.\frac{\mathrm{d}A}{\mathrm{d}r}\right|_{r=r_0} \Delta r = 8\pi r_0 \Delta r.$$

   If we divide by A_0, we obtain

   $$\frac{\Delta A}{A_0} \approx \frac{8\pi r_0 \Delta r}{4\pi r_0^2} = 2\frac{\Delta r}{r_0} = 2 \times (-1.5\%) = \boxed{-3\%}$$

   Hence the area is decreased by approximately 3%.

5. The period T of a pendulum in seconds is given by

   $$T = 2\pi\sqrt{\frac{L}{g}}\,,$$

   where L is the length of the pendulum in metres and $g = 9.81\,\mathrm{ms}^{-2}$. Find the approximate percentage change in T if the pendulum is lengthened by 1%.

   Solution:
   First we rewrite T:

   $$T = \frac{2\pi}{\sqrt{g}}\sqrt{L}.$$

   The derivative of T with respect to L is

   $$\frac{\mathrm{d}T}{\mathrm{d}L} = \frac{2\pi}{\sqrt{g}}\frac{\mathrm{d}}{\mathrm{d}L}\big(\sqrt{L}\big) = \frac{2\pi}{\sqrt{g}} \times \frac{1}{2\sqrt{L}}\,.$$

   The absolute change of T is

   $$\Delta T \approx \left.\frac{\mathrm{d}T}{\mathrm{d}L}\right|_{L=L_0} \Delta L = \frac{2\pi}{\sqrt{g}} \times \frac{1}{2\sqrt{L_0}}\Delta L;$$

   the relative change is

   $$\frac{\Delta T}{T_0} \approx \frac{2\pi}{\sqrt{g}} \times \frac{1}{2\sqrt{L_0}}\Delta L \div \left(\frac{2\pi}{\sqrt{g}}\sqrt{L_0}\right) = \frac{\Delta L}{2L_0}\,.$$

   If the pendulum is lengthened by 1%, then

   $$\frac{\Delta T}{T_0} \approx \frac{1}{2} \times 1\% = 0.5\%,$$

   i.e. the period increases by approximately $\boxed{0.5\%.}$

6. The volume V of a sphere is measured by estimating its radius r. What is the maximum percentage error in the radius if the maximum error in V that is allowed is 1%?

Solution:
The formula for the volume is

$$V = \frac{4\pi r^3}{3}.$$

Its derivative with respect to r is equal to

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2.$$

The absolute change is approximately

$$\Delta V \approx \left.\frac{\mathrm{d}V}{\mathrm{d}r}\right|_{r_0} \Delta r = 4\pi r_0^2 \Delta r,$$

and the relative change

$$\frac{\Delta V}{V_0} \approx \frac{4\pi r_0^2 \Delta r}{4\pi r_0^3/3} = 3\frac{\Delta r}{r_0}.$$

If $\left|\frac{\Delta V}{V_0}\right| \leq 1\%$, then

$$\left|\frac{\Delta r}{r_0}\right| \approx \frac{1}{3}\left|\frac{\Delta V}{V_0}\right| \leq \frac{1}{3}\%.$$

Hence the percentage error in the radius should be at most $\boxed{\frac{1}{3}\%.}$

7. Find $p_{2,0}$, the Taylor polynomial of degree 2 about $x = 0$, for the function

$$f(x) = \sqrt{1+x}.$$

Solution:
The first two derivatives of f are

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big((1+x)^{1/2}\Big) = \frac{1}{2}(1+x)^{-1/2}$$
$$f''(x) = \frac{1}{2} \times \Big(-\frac{1}{2}\Big)(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2}$$

Evaluating these and f itself at $x = 0$ we obtain

$$f(0) = 1, \qquad f'(0) = \frac{1}{2}, \qquad f''(0) = -\frac{1}{4}.$$

Hence

$$p_{2,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x + \frac{-1/4}{2}x^2$$
$$= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2}$$

8. Find $p_{3,\frac{\pi}{2}}$, the Taylor polynomial of degree $x = 3$ about $\frac{\pi}{2}$, for the function
$$f(x) = x \sin x.$$

Solution:
The first three derivatives of f are

$$f'(x) = \sin x + x \cos x$$
$$f''(x) = \cos x + \cos x - x \sin x = 2\cos x - x \sin x$$
$$f'''(x) = -2\sin x - \sin x - x\cos x = -3\sin x - x\cos x$$

Evaluating these at $x = \frac{\pi}{2}$ we obtain

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$
$$f'\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2} = 1$$
$$f''\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2} = -\frac{\pi}{2}$$
$$f'''\left(\frac{\pi}{2}\right) = -3\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2} = -3$$

Hence the Taylor polynomial of degree 3 about $x = \frac{\pi}{2}$ is

$$p_{3,\frac{\pi}{2}}(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{1}{2!}f''\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{3!}f'''\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)^3$$
$$= \boxed{\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right) - \frac{\pi}{4}\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^3}$$

9. Find $p_{2,1}$, the Taylor polynomial of degree 2 about $x = 1$, for the function
$$f(x) = e^{x^2}.$$

Solution:
The first two derivatives of f are

$$f'(x) = e^{x^2} \times 2x = 2xe^{x^2}$$
$$f''(x) = 2e^{x^2} + 2xe^{x^2} \times 2x = 2e^{x^2} + 4x^2e^{x^2}$$

Evaluating these derivatives and f at $x = 1$ we obtain

$$f(1) = e^1 = e, \qquad f'(1) = 2e^1 = 2e, \qquad f''(1) = 2e^1 + 4e^1 = 6e.$$

Hence

$$p_{2,1}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$
$$= \boxed{e + 2e(x-1) + 3e(x-1)^2}$$

10. Find $p_{2,1}$, the Taylor polynomial of degree 2 about $x = 1$, for the function

$$f(x) = \arctan x.$$

Solution:
We need the first three derivatives of f and their values at $x = 0$:

$$\begin{aligned} f(x) &= \arctan x & \implies \quad & f(1) = \arctan 1 = \frac{\pi}{4} \\ f'(x) &= \frac{1}{1+x^2} & \implies \quad & f'(1) = \frac{1}{2} \\ f''(x) &= -\frac{2x}{(1+x^2)^2} & \implies \quad & f''(1) = -\frac{1}{2}. \end{aligned}$$

Hence

$$\begin{aligned} p_{2,1}(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= \boxed{\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2} \end{aligned}$$

11. For the function $f(x) = \sqrt{1+x}$ in Question 7, determine the remainder term $R_{2,0}$.

Solution:
The third derivative of f is

$$f'''(x) = -\frac{1}{4} \times \left(-\frac{3}{2}\right)(1+x)^{-5/2} = \frac{3}{8}(1+x)^{-5/2}.$$

Hence

$$R_{2,0} = \frac{f'''(\xi)}{3!}x^3 = \frac{1}{6} \times \frac{3}{8}(1+\xi)^{-5/2}x^3 = \boxed{\frac{1}{16}(1+\xi)^{-5/2}x^3}$$

where ξ is between x and 0.

12. For the function $f(x) = x \sin x$ in Question 8, determine the remainder term $R_{3,\frac{\pi}{2}}$.

Solution:
The fourth derivative of f is

$$f^{(4)}(x) = -3\cos x - \Big(\cos x + x(-\sin x)\Big) = -4\cos x + x \sin x.$$

Hence

$$R_{3,\frac{\pi}{2}} = \frac{f^{(4)}(\xi)}{4!}\left(x - \frac{\pi}{2}\right)^4 = \boxed{\frac{1}{24}\Big(-4\cos\xi + \xi\sin\xi\Big)\left(x - \frac{\pi}{2}\right)^4}$$

where ξ is between x and $\frac{\pi}{2}$.

13. (a) Determine $p_{4,0}$, the Taylor polynomial of degree 4 about $x = 0$, for $f(x) = \cos(2x)$.

    (b) Use $p_{4,0}$ to obtain an approximate value for $f(0.2)$.

(c) Determine the remainder term $R_{4,0}(0.2)$.

(d) Use the inequality $|\sin t| \leq 1 \quad (t \in \mathbb{R})$ to obtain an upper bound of the remainder term $R_{4,0}(x)$. Hence find an upper bound for the error that is made when $\cos(0.4)$ is approximated by the Taylor polynomial as in (b).

Solution:

(a) The first five derivatives of f are

$$f'(x) = -2\sin(2x), \qquad f''(x) = -4\cos(2x),$$
$$f'''(x) = 8\sin(2x), \qquad f^{(4)}(x) = 16\cos(2x),$$
$$f^{(5)}(x) = -32\sin(2x).$$

The values of $f, \ldots, f^{(4)}$ at $x = 0$ are:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -4, \quad f'''(0) = 0, \quad f^{(4)}(0) = 16.$$

Hence the Taylor polynomial of degree 4 about 0 is

$$\begin{aligned} p_{4,0}(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 \\ &= \boxed{1 - 2x^2 + \frac{2}{3}x^4} \end{aligned}$$

(b) $f(0.2) \approx p_{4,0}(0.2) = 1 - 2 \times (0.2)^2 + \dfrac{2}{3} \times (0.2)^4 = \boxed{0.921\,067}$

Note that this value is an approximate value for $f(0.2) = \cos(0.4)$.

(c) The Cauchy remainder term is

$$R_{4,0}(x) = \frac{f^{(5)}(\xi)}{5!}x^5 = -\frac{32\sin(2\xi)}{120}x^5 = -\frac{4\sin(2\xi)}{15}x^5$$

where ξ is between 0 and x. For $x = 0.2$ we obtain

$$R_{4,0}(0.2) = -\frac{4\sin(2\xi)}{15} \times (0.2)^5$$

where $0 < \xi < 0.2$.

(d) The modulus of $R_{4,0}(0.2)$ can be estimated as follows

$$|R_{4,0}(0.2)| = \frac{4|\sin(2\xi)|}{15} \times (0.2)^5 \leq \frac{4}{15} \times (0.2)^5 = \boxed{0.000\,085\,33}$$

(The actual error is $|f(0.2) - p_4(0.2)| = 0.000\,005\,67$.)

14. (a) Determine $p_{3,1}$, the Taylor polynomial of degree $x = 3$ about 1, for $f(x) = \ln(x)$.
(b) Use $p_{3,1}$ to obtain an approximate value for $f(1.2)$.
(c) Use the remainder term to estimate the maximum absolute error in this result.

Solution:

(a) The first four derivatives of f are

$$f'(x) = \frac{1}{x}, \qquad f''(x) = -\frac{1}{x^2},$$

$$f'''(x) = \frac{2}{x^3}, \qquad f^{(4)}(x) = -\frac{6}{x^4}.$$

The values of $f, \ldots, f'''$ at $x = 1$ are

$$f(1) = \ln 1 = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f'''(1) = 2.$$

Hence the Taylor polynomial of degree 3 about $x = 1$ is

$$\begin{aligned} p_{3,1}(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 \\ &= \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3} \end{aligned}$$

(b) $\ln 1.2 = f(1.2) \approx p_{3,1}(1.2) = 0.2 - \frac{1}{2} \times (0.2)^2 + \frac{1}{3} \times (0.2)^3 = \boxed{0.182\,667}$

(c) The remainder term is

$$R_{3,1}(x) = \frac{f^{(4)}(\xi)}{4!}(x-1)^4 = \frac{1}{24}\left(-\frac{6}{\xi^4}\right)(x-1)^4 = -\frac{(x-1)^4}{4\xi^4}$$

where ξ is between 1 and x. For $x = 1.2$ we obtain

$$R_{3,1}(1.2) = -\frac{(0.2)^4}{4} \times \frac{1}{\xi^4}$$

where $1 < \xi < 1.2$. The error we made in (b) can be estimated as follows

$$|\ln(1.2) - p_{3,1}(1.2)| = |f(1.2) - p_{3,1}(1.2)| = |R_{3,1}(1.2)| = \frac{(0.2)^4}{4} \times \frac{1}{\xi^4}$$

From $1 < \xi < 1.2$ it follows that

$$\begin{aligned} & 1 < \xi^4 < (1.2)^4 \\ \Longrightarrow \quad & 1 > \frac{1}{\xi^4} > \frac{1}{(1.2)^4} \end{aligned}$$

Hence

$$|\ln(1.2) - p_{3,1}(1.2)| < \frac{(0.2)^4}{4} = \boxed{0.0004}$$

(The actual error is $|\ln(1.2) - p_{3,1}(1.2)| = 0.000\,345$.)

15. (a) Determine $p_{2,9}$, the Taylor polynomial of degree 2 about $x = 9$, for $f(x) = \dfrac{1}{\sqrt{x}}$.

(b) Use $p_{2,9}$ to obtain an approximate value for $\dfrac{1}{\sqrt{9.1}}$.

(c) Use the remainder term to estimate the maximum absolute error in this result.

Solution:

(a) The first three derivatives of f are

$$
\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}x^{-1/2} = -\frac{1}{2}x^{-3/2} \\
f''(x) &= -\frac{1}{2} \times \left(-\frac{3}{2}\right)x^{-5/2} = \frac{3}{4}x^{-5/2} \\
f'''(x) &= \frac{3}{4} \times \left(-\frac{5}{2}\right)x^{-7/2} = -\frac{15}{8}x^{-7/2}
\end{aligned}
$$

The values of f, f' and f'' at $x = 9$ are

$$
\begin{aligned}
f(9) &= \frac{1}{\sqrt{9}} = \frac{1}{3} \\
f'(9) &= -\frac{1}{2} \times 9^{-3/2} = -\frac{1}{2} \times \frac{1}{\left(\sqrt{9}\right)^3} = -\frac{1}{2} \times \frac{1}{27} = -\frac{1}{54} \\
f''(x) &= \frac{3}{4} \times \frac{1}{\left(\sqrt{9}\right)^5} = \frac{3}{4} \times \frac{1}{243} = \frac{1}{324}
\end{aligned}
$$

Hence

$$
\begin{aligned}
p_{2,9}(x) &= f(9) + f'(9)(x-9) + \frac{f''(9)}{2!}(x-9)^2 \\
&= \boxed{\frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{1}{\sqrt{9.1}} &= f(9.1) \approx p_{2,9}(9.1) = \frac{1}{3} - \frac{1}{54}(9.1-9) + \frac{1}{648}(9.1-9)^2 \\
&= \frac{1}{3} - \frac{1}{54} \times 0.1 + \frac{1}{648} \times 0.1^2 = \boxed{0.331\,496\,914}
\end{aligned}
$$

(The actual value is $\dfrac{1}{\sqrt{9.1}} = 0.331\,496\,773$.)

(c) The remainder term is

$$
R_{2,9}(x) = \frac{f'''(\xi)}{3!}(x-9)^3 = \frac{1}{6} \times \left(-\frac{15}{8}\right)\xi^{-7/2}(x-9)^3 = -\frac{5}{16}\xi^{-7/2}(x-9)^3
$$

where ξ is between x and 9. For $x = 9.1$ we obtain

$$
R_{2,9}(9.1) = -\frac{5}{16}\xi^{-7/2}(9.1-9)^3 = -\frac{5}{16}\xi^{-7/2} \times 0.1^3
$$

where $9 < \xi < 9.1$. The error made in (b) can be estimated as follows

$$\begin{aligned}\left|\frac{1}{\sqrt{9.1}} - p_{2,9}(9.1)\right| &= \left|R_{2,9}(9.1)\right| = \frac{5}{16}\xi^{-7/2} \times 0.1^3 \\ &= \frac{5}{16}\xi^{-7/2} \times \frac{1}{1000} = \frac{1}{3200}\xi^{-7/2}\end{aligned}$$

Since $\xi > 9$ we have $\sqrt{\xi} > 3$, which implies

$$\xi^{-7/2} = \frac{1}{\left(\sqrt{\xi}\right)^7} < \frac{1}{3^7} = \frac{1}{2187}\,.$$

Hence

$$\left|\frac{1}{\sqrt{9.1}} - p_{2,9}(9.1)\right| < \frac{1}{3200} \times \frac{1}{2187} = \frac{1}{6\,998\,400} = 0.000\,000\,142\,889$$

(The actual error is $f(9.1) - p_{2,9}(9.1) = 0.000\,000\,141\,514$.)

16. Let

$$f(x) = \frac{1}{1-x}\,.$$

Use induction to show that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \qquad n = 0, 1, \ldots.$$

Hence show that the Maclaurin series of f is

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n.$$

(You do **not** have to show that the remainder converges to 0 if $|x| < 1$.)
Note that this series is a geometric series.

Solution:
We have to show that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \tag{$*$}$$

for $n = 0, 1, \ldots$.
For $n = 0$ the assertion $(*)$ is true.
Let us assume that it is true for $n = k$, i.e.

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}\,.$$

When we differentiate this, we obtain

$$\begin{aligned}f^{(k+1)}(x) &= k!\frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-k-1} = k!(-k-1)(1-x)^{-k-2} \times (-1) \\ &= \frac{(k+1)!}{(1-x)^{k+2}},\end{aligned}$$

which is $(*)$ for $n = k + 1$.

If we evaluate this at $x = 0$, we obtain $f^{(n)}(0) = n!$. Hence the Taylor series of f is equal to

$$\begin{aligned}
&f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} + \dots \\
&= 1 + x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \dots \\
&= 1 + x + x^2 + x^3 + \dots
\end{aligned}$$

or with sigma notation

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n.$$