UNIVERSITY OF STRATHCLYDE
DEPARTMENT OF MATHEMATICS AND STATISTICS
Lecture Notes for Week 5

4. Complex Numbers

§4.1 A History Lesson . . .

A brief glance at history reveals that as mathematicians have become more ambitious in the problems they wish to solve, so they have had to invent (or discover) new numbers. For example, everyone's earliest experience with mathematics involves the whole numbers, but these would be insufficient to solve an equation as simple as $2x = 3$. This leads us to rational numbers and by the time we encounter trigonometry, say, it becomes necessary to deal with irrational numbers such as $\sqrt{2}$. But how do we deal with numbers like $\sqrt{-2}$, or combinations like $3 - \sqrt{-5}$?

Although there had been references to square roots of negative numbers as early as the 1st century AD, complex numbers did not become prominent in mathematics until the 16th century. In 1545 the Italian Girolamo Cardano published an important book on algebra, *Ars Magna* (latin for *The Great Art*), in which he examined the simultaneous equations

$$x + y = 10, \qquad xy = 40.$$

In other words, find two numbers whose sum is equal to 10 and whose product is equal to 40. Cardano's answer in modern notation was $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$, and he proved this solution was correct on the assumption that the usual algebraic rules apply:

$$x + y = (5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 10; \qquad xy = (5 + \sqrt{-15})(5 - \sqrt{-15}) = 25 - (-15) = 40.$$

Yet he was dismissive because he saw no physical meaning: he wrote that the answer was 'as subtle as it is useless'! In 1572 the Italian mathematician Rafael Bombelli was the first to suggest a way to manipulate 'impossible' roots such as $5 + \sqrt{-15}$ as if they were ordinary numbers. In effect, Bombelli was the first person to present rules for addition, subtraction, multiplication and division of what we now call *complex numbers*.

The concept of a complex number continued to evolve in the centuries that followed Bombelli. In 1637 René Descartes first made the distinction between 'real' and 'imaginary' numbers (although the term imaginary was meant to be derogatory). English mathematician John Wallis in 1673 was the first to represent a complex number geometrically: the real part of the number is measured off along a fixed line (in the direction of its sign), then the imaginary part is measured off at right angles (see §4.5 in the lecture notes). However, these special numbers were still the cause of much confusion. The equation $(\sqrt{-1})^2 = \sqrt{-1}\sqrt{-1} = -1$ seemed to be inconsistent with the identity $\sqrt{a}\sqrt{b} = \sqrt{ab}$ which is valid for positive real numbers a and b, and which was also used in complex number calculations with one of a, b positive and the other negative. This difficulty eventually led to the convention of using the special symbol i in place of $\sqrt{-1}$ to avoid mistakes (see §4.2 in the lecture notes).

In the 18th century complex numbers gained wider use. For example, in 1730 Abraham de Moivre noted that the identities relating trigonometric functions of an integer multiple of an angle to powers of trigonometric functions of that angle could be re-expressed by the formula which now bears his name, *de Moivre's formula* (see §4.8 in next week's lecture notes):

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \qquad (\theta \in \mathbb{R}).$$

By manipulating complex power series, Leonhard Euler extended de Moivre's result in 1748 to derive *Euler's formula* (see §4.15):

$$\cos\theta + i\sin\theta = \mathrm{e}^{i\theta} \qquad (\theta \in \mathbb{R}).$$

In 1797 Caspar Wessel published a paper in Danish describing the representation of a complex number as a point in a plane (the *complex plane*) (see §4.5). Wessel's work went undiscovered until a French translation 100 years later. However, in the meantime Swiss bookkeeper Jean-Robert Argand independently wrote up the same idea in 1806. Today this interpretation of complex numbers in the complex plane is known as an *Argand diagram*. Argand was also the first to define the *modulus* of a complex number (§4.6 in the lecture notes).

In 1837, nearly 300 years after Cardano's impossible roots, Irishman William Rowan Hamilton published the definition of complex numbers as ordered pairs of real numbers subject to certain explicit rules of manipulation (although Carl Friedrich Gauss claims to have had the same idea in 1831!) Finally, complex numbers had been placed on a firm algebraic basis. And despite their theoretical background, complex numbers today have many 'real life' applications, for example, fractals (see §4.18), chaos theory, signal analysis, electrical circuits, fluid mechanics, control theory, quantum mechanics, heat conduction, number theory and structural analysis.

§4.2 Introduction to Complex Numbers

The number system as we know it today has evolved in stages:

natural numbers $\mathbb{N}$ (also called positive integers) 1, 2, 3, 4, ...

$\downarrow$

integers $\mathbb{Z}$ $0, \pm 1, \pm 2, \pm 3, \ldots$

$\downarrow$

rational numbers $\mathbb{Q}$ (or fractions) $\dfrac{m}{n}$ where m, n are integers and $n \neq 0$.

Note: **irrational numbers** such as $\sqrt{2} = 1.41423\ldots$ and $\pi = 3.14159\ldots$ are numbers which are not rational, i.e. they cannot be expressed as m/n where $m, n \in \mathbb{Z}$.

$\downarrow$

real numbers $\mathbb{R}$ the set of all rational and irrational numbers.

A motivation for successive enlargement of the number system comes from trying to solve polynomial equations whose coefficients are drawn from $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. For example,

$x + 1 = 2$	the solution $x = 1$ is a natural number;
$x + 2 = 1$	no natural number solutions, we need to go into the set of integers;
$2x + 4 = 0$	the solution $x = -2$ is an integer;
$4x + 2 = 0$	no integer solutions, we need to go into the set of rational numbers;
$4x^2 - 9 = 0$	the solutions $x = \pm\frac{3}{2}$ are rational numbers;
$4x^2 - 8 = 0$	no rational number solutions, we need to go into the set of real numbers;
$x^2 - 2 = 0$	the solutions $x = \pm\sqrt{2}$ are real numbers;
$x^2 + 1 = 0$	no real number solutions, so what do we do now?

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and other similar equations, we introduce the set of **complex numbers**.

The set of real numbers $\mathbb{R}$ may be extended to include an imaginary number i (sometimes called the imaginary unit) which has the property that

$$\boldsymbol{i^2 = -1}$$

Notice that no real number has this property. Otherwise, i obeys all the usual rules of arithmetic.

The equation $x^2 + 1 = 0$ has solutions $x = \pm i$. Similarly, $x^2 + 2 = 0$ has solutions $x = \pm\sqrt{2}\, i$. More generally, the quadratic equation

$$ax^2 + bx + c = 0$$

can now be solved even when the discriminant $b^2 - 4ac < 0$. (Recall that there are no <u>real</u> solutions of $ax^2 + bx + c = 0$ when $b^2 - 4ac < 0$.)

For example, consider the equation $x^2 + 2x + 2 = 0$ with discriminant $2^2 - (4{\times}1{\times}2) = -4 < 0$.

By completing the square, we can rewrite the equation as

$$(x+1)^2 + 1 = 0, \qquad \text{so that} \quad (x+1)^2 = -1 = i^2.$$

If we now take the square root of both sides then

$$x + 1 = \pm i, \qquad \text{and so} \quad \underline{\underline{x = -1 \pm i}}.$$

The solutions $x = -1 + i$ and $x = -1 - i$ are a combination of a real number and an imaginary number. We could also obtain the solutions using the usual quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4{\times}1{\times}2}}{2{\times}1} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm \sqrt{4}\sqrt{-1}}{2} = \frac{-2 \pm 2i}{2} = \underline{\underline{-1 \pm i}}.$$

§4.3 The Set of Complex Numbers

A <u>complex number</u>, z, may be written in the form

$$\boldsymbol{z = a + ib\,, \quad \text{where} \quad a,\, b \in \mathbb{R} \quad \text{and} \quad i^2 = -1.}$$

The following are all examples of complex numbers:

$$1, \quad i, \quad -i, \quad -4+i, \quad 6-i, \quad \sqrt{3}-i, \quad -2.1+\pi i, \quad \frac{1}{4}-\frac{1}{8}i.$$

- The **<u>complex number system</u>** is denoted by $\mathbb{C}$ and is the set of all numbers of the form $a+ib$, where a, b are real:

  $$\mathbb{C} \;=\; \boldsymbol{\{z : \quad z = a + i\,b\,; \quad a,\, b \in \mathbb{R}\}.}$$

  a is called the **<u>real part of z</u>** and can be written as $\mathrm{Re}(z)$;

  b is called the **<u>imaginary part of z</u>** and can be written as $\mathrm{Im}(z)$.

  Note that both the real and imaginary parts of z are **<u>real numbers</u>**.

- If $b = 0$ then $z = a$ is a real number; the real numbers form a subset of the complex numbers.

- If $a = 0$ then $z = ib$ is said to be (purely) **<u>imaginary</u>**.

- We often use z to denote a complex variable, although other symbols are commonly used, for example, w.

- If $z = a+ib$, then the **<u>complex conjugate of z</u>** is $\boldsymbol{\overline{z} = a - ib.}$

- Having extended the number system $\mathbb{R}$ to $\mathbb{C}$, no further extension is necessary: any polynomial equation

  $$a_n\, z^n + a_{n-1}\, z^{n-1} + \cdots + a_1\, z + a_0 = 0, \quad \text{with} \quad a_0,\, a_1, \ldots, a_n \in \mathbb{C},$$

  has precisely n roots which all lie in $\mathbb{C}$.

  This non-trivial result is called the **<u>Fundamental Theorem of Algebra</u>**.

§4.4 Properties of Complex Numbers

In performing operations with complex numbers we can proceed in the same way as real numbers, replacing i^2 with -1 when it occurs.
Consider two complex numbers $a+ib$ and $c+id$ (where $a,\ b,\ c,\ d \in \mathbb{C}$).

- **Equality:** $\boldsymbol{a+ib \;=\; c+id \quad\Longleftrightarrow\quad a=c \quad \text{and} \quad b=d.}$

  Two complex numbers are equal if and only if their respective real parts are equal AND their imaginary parts are equal.

  Use of this result is known as **equating real and imaginary parts**.

  The result may be proved as follows: $a+ib=c+id \quad\Longrightarrow\quad a-c=i(d-b)$.
  Hence $(a-c)^2 = i^2(d-b)^2$ and so $(a-c)^2 = -(d-b)^2$.
  As the LHS is non-negative and the RHS is non-positive, there is equality if and only if both sides are zero. Hence $a-c=0$ and $b-d=0$, i.e. $a=c$ and $b=d$.

- **Addition and Subtraction**

  $$\boldsymbol{\begin{aligned}(a+ib)\;+\;(c+id) \;&=\; (a+c)\;+\;i(b+d),\\ (a+ib)\;-\;(c+id) \;&=\; (a-c)\;+\;i(b-d).\end{aligned}}$$

  The real part of the sum of two complex numbers is the sum of the individual real parts; similarly, the imaginary part of the sum equals the sum of the individual imaginary parts. Subtraction behaves in the same way as addition.

- **Multiplication:**

  $$\boldsymbol{\begin{aligned}(a+ib)(c+id) \;&=\; a\,(c+id)\;+\;ib\,(c+id)\\ &=\; a\,c\;+\;i\,a\,d\;+\;i\,b\,c\;+\;i^2\,b\,d\\ &=\; a\,c\;+\;i(a\,d+b\,c)\;-\;b\,d\\ &=\; (a\,c-b\,d)\;+\;i(a\,d+c\,b).\end{aligned}}$$

  Note: there is more to multiplying two complex numbers than just multiplying their real and imaginary parts!

  What happens if we multiply a complex number by its own complex conjugate? Using the rule for multiplication:

  $$(a+ib)(a-ib) \;=\; \big(a\times a - b\times(-b)\big) + i\,\big(a\times(-b) + a\times b\big) \;=\; (a^2+b^2) + i\times 0 \;=\; a^2+b^2.$$

  Since a and b are both real, a^2+b^2 must be real. So the product of a complex number and its complex conjugate is a real number.

- **Division:** to divide by a complex number we use the trick of multiplying above and below by the complex conjugate of the denominator. This will convert the denominator into a real number and will allow us to write the result in the usual complex number form:

$$\boldsymbol{\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)\ +\ i(bc-ad)}{c^2+d^2}}$$
$$\boldsymbol{= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right).}$$

<u>Remember the method</u> of multiplying above and below by the complex conjugate of the denominator, not the result.

Examples

4.1 In each of the following examples, we simplify the expression to the form $a+ib$, where a and b are real numbers.

(i) $(2+4i)+(1-3i) \;=\; (2+1)+(4-3)i \;=\; 3+i.$

(ii) $(1-6i)-(-3+4i) \;=\; \big(1-(-3)\big)+(-6-4)i \;=\; 4-10i.$

(iii)
$$\begin{aligned}(3-i)(4+5i) &= (3\times 4) + (3\times 5i) + \big((-i)\times 4\big) + \big((-i)\times 5i\big)\\ &= (12+5) + (15-4)i\\ &= 17+11i.\end{aligned}$$

(iv) $\dfrac{5-i}{2+3i} = \dfrac{5-i}{2+3i}\times\dfrac{2-3i}{2-3i} = \dfrac{10-3+i(-2-15)}{2^2+3^2} = \dfrac{1}{13}(7-17i).$

4.2 Solve $z^2+6z+25=0$ by completing the square.

$$z^2+6z+25 \;=\; (z+3)^2-9+25 \;=\; (z+3)^2+16$$

So if $z^2+6z+25 = (z+3)^2+16=0$ then $(z+3)^2 = -16 = (4i)^2$. Taking the root of both sides gives $z+3=\pm 4i$. The solutions are $\underline{\underline{z=-3+4i}}$ and $\underline{\underline{z=-3-4i}}$.

Note, one solution is the complex conjugate of the other. We could also obtain these solutions by using the quadratic formula (with $a=1,\ b=6,\ c=25$):

$$z = \frac{-b\ \pm\ \sqrt{b^2-4ac}}{2a} = \frac{-6\ \pm\ \sqrt{36-4\times 25}}{2} = \frac{-6\ \pm\ \sqrt{-64}}{2} = \frac{-6\ \pm\ 8i}{2} = \underline{\underline{-3\pm 4i}}.$$

4.3 Find real numbers a and b so that $(a-b)+(3a-2b)i = 2+6i$.

Equate the real and imaginary parts: we must have $a-b=2$ and $3a-2b=6$.

The unique solution of the simultaneous equations is $\underline{\underline{a=2,\ \ b=0}}$.

4.4 Find $x, y \in \mathbb{R}$ if $2x - 3yi = \dfrac{20}{1-3i}$.

$$2x - 3yi \;=\; \frac{20(1+3i)}{(1-3i)(1+3i)} \;=\; \frac{20(1+3i)}{1^2+3^2} \;=\; \frac{20(1+3i)}{10} \;=\; 2+6i.$$

Equating real and imaginary parts, we obtain $2x = 2$ and $-3y = 6$.

Hence $\underline{\underline{x = 1}}$ and $\underline{\underline{y = -2}}$.

4.5 Express i^2, i^3, i^4 and i^5 in their simplest forms. Use the pattern to determine i^{10}, i^{363} and $i^{1234334221}$.

By definition, $i^2 = -1$. Therefore

$$i^3 = i^2 \times i = -i, \qquad i^4 = (i^2)^2 = (-1)^2 = 1 \qquad \text{and} \qquad i^5 = i^4 \times i = i.$$

Notice that the powers of i cycle, so that $i^m = 1$ whenever m is a multiple of 4.

As a result, $i^{10} = i^8 \times i^2 = -1$, $i^{363} = i^3 = -i$ (since $360 = 90 \times 4$) and $i^{1234334221} = i$ as 1234334220 is a multiple of 4 (any whole number that ends in "20" is a multiple of 4).

Examples covered in video

4.6 Simplify the following to the form $a + ib$, where a and b are real:

(i) $(4-i)-(3+2i)$ (ii) $(2+3i)(1+2i)$ (iii) $(6+3i)(6-3i)$

(iv) $(-1+\sqrt{3}i)^3$ (v) $\dfrac{5+4i}{-2+i}$ (vii) $\dfrac{(3+i)(2-4i)}{1+6i}$

4.7 Express $\dfrac{(6+i)(3+4i)}{1+2i} - (7+2i)$ in the form $x + iy$ $(x, y \in \mathbb{R})$.

4.8 Solve the equation $2z^2 + 2z + 1 = 0$.

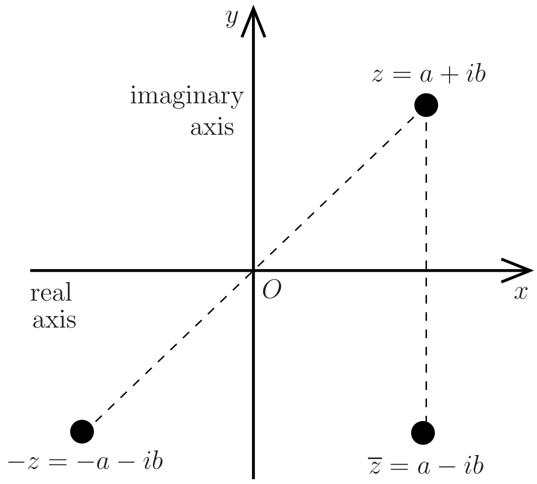
4.9 By equating real and imaginary parts, find real numbers x and y such that

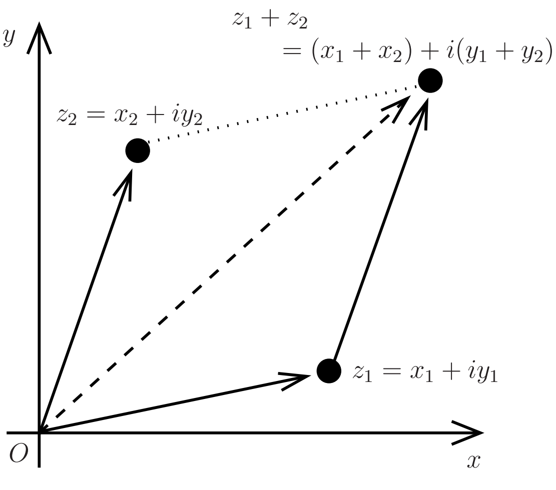
$$\frac{4x+3yi}{2-y+xi} \;=\; 2-5i.$$

§4.5 Graphical Representation: The Argand Diagram

In many applications, it is useful to represent the real numbers with the real line. We can extend this idea to complex numbers and think of $z = a + ib$ as the point in the plane with Cartesian coordinates (a, b). This representation is known as the **Argand diagram** and we often talk about the **complex plane** when discussing complex numbers.

Many properties of complex numbers are easily interpreted geometrically in Argand diagrams. For example, the complex conjugate $\overline{z}$ is the reflection of z in the x-axis and $-z$ is the reflection of z through the origin. Real numbers lie on the x-axis and pure imaginary numbers lie on the y-axis.





If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

which is represented on the Argand diagram by the point $(x_1 + x_2,\ y_1 + y_2)$.

So adding complex numbers can be viewed as geometrically equivalent to adding together the position vectors of two points.

It can be convenient to think of the complex number $z = a + ib$ being represented in the Argand diagram either by the point with coordinates (a, b) or by the position vector of this point.

Examples covered in video

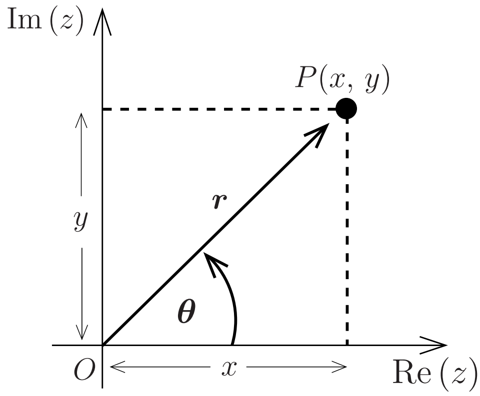
4.10 Let $z_1 = 3 - 2i$ and $z_2 = 4 + i$. Show the numbers z_1, z_2 and $z_1 + z_2$ on the Argand diagram.

4.11 If $z = 2 + 4i$ then plot z, iz and $\dfrac{z}{i}$ on the Argand diagram.

§4.6 Modulus, Argument and Polar Form

Now suppose that $z = x + iy$ is represented by the point $P\,(x,\,y)$ in the complex plane.

Let r denote the distance from the origin to P and θ denotes the angle $\overrightarrow{OP}$ makes with the positive x-axis.



Then by considering the right-angled triangle, we can see that

$$\boldsymbol{x = r\cos\theta, \quad y = r\sin\theta, \quad \text{where} \quad r = |OP| = \sqrt{x^2+y^2}\,.}$$

Thus, $$\boldsymbol{z = x + iy = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta).}$$

This is called the polar form of z and is often abbreviated to

$\boldsymbol{z = r\ \text{cis}\ \theta}$ **or** $\boldsymbol{z = r\angle\theta}$.

- The length $r = \sqrt{x^2+y^2}$ is known as the **modulus z and is denoted by $|z|$**.

  The modulus is always a real number – it represents the length from O to the point P.

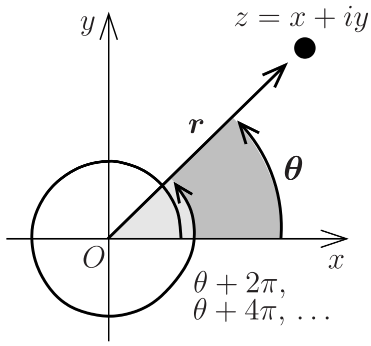
- Recall, from the definition of the complex conjugate,

$$\boldsymbol{z\,\overline{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2.}$$

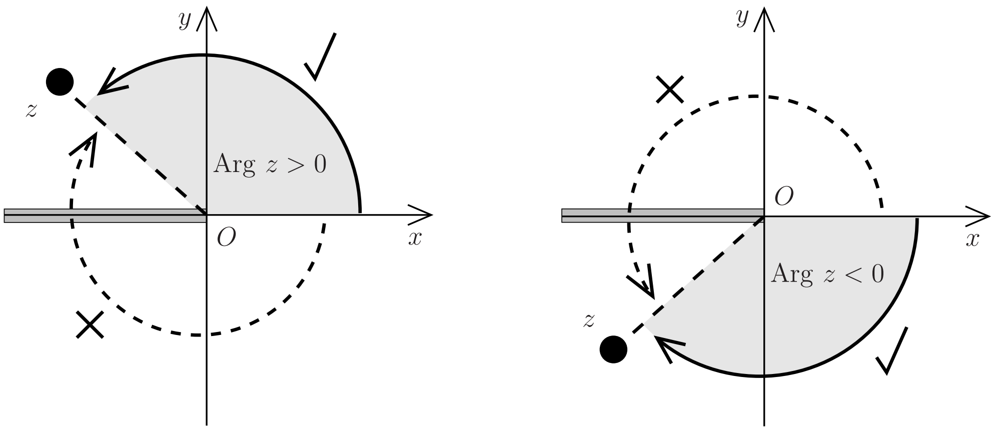
- The angle θ is known as the **argument of z and is written $\arg z$**.

  More formally, the argument of z is defined as the angle between the + ve real axis and the position vector of z measured *anti-clockwise*.

- Note that θ is not unique — the directions defined by angles repeat themselves with period 2π, so the argument is multivalued.
  (Note: we will normally use radians to measure arguments of complex numbers.)



- To assign a unique value, we can use the **principal value of the argument, Arg z**. This is the value that satisfies $-\pi < \text{Arg}\ z \leq \pi$.



Note that $\operatorname{Arg} z$ is positive if z is above the x-axis and negative below.

Since $\theta = \arg(z) = \arg(x+iy)$ satisfies the relation $\tan\theta = \dfrac{y}{x}$, it is tempting to think that $\arg z = \arctan\left(\dfrac{y}{x}\right)$. **However**, the range of the function arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$, which covers only the first and fourth quadrants.

For example, $\tan\left(\dfrac{3\pi}{4}\right) = \tan\left(-\dfrac{\pi}{4}\right) = -1$, **but** $\arctan(-1) = -\dfrac{\pi}{4}$.

So what do we do if we know that z lies in the second or third quadrants? In that case we must add or subtract π to the value of $\arctan\left(\dfrac{y}{x}\right)$ to get the correct argument.

For example, $z = -1 + i$ lies in the second quadrant and

$$\arg(-1+i) = \arctan\left(\frac{1}{-1}\right) + \pi + 2k\pi = -\frac{\pi}{4} + \pi + 2k\pi = \frac{3\pi}{4} + 2k\pi$$

for any integer k. (The argument is a multi-valued function.)

Similarly, $z = -1 - \sqrt{3}i$ lies in the third quadrant and

$$\begin{aligned}\arg\left(-1-\sqrt{3}i\right) &= \arctan\left(\frac{-\sqrt{3}}{-1}\right) - \pi + 2k\pi = \arctan\left(\sqrt{3}\right) - \pi + 2k\pi \\ &= \frac{\pi}{3} - \pi + 2k\pi = -\frac{2\pi}{3} + 2k\pi \quad (k \in \mathbb{Z}).\end{aligned}$$

Note that $\operatorname{Arg}(-1+i) = \dfrac{3\pi}{4}$ and $\operatorname{Arg}\left(-1-\sqrt{3}i\right) = -\dfrac{2\pi}{3}$.

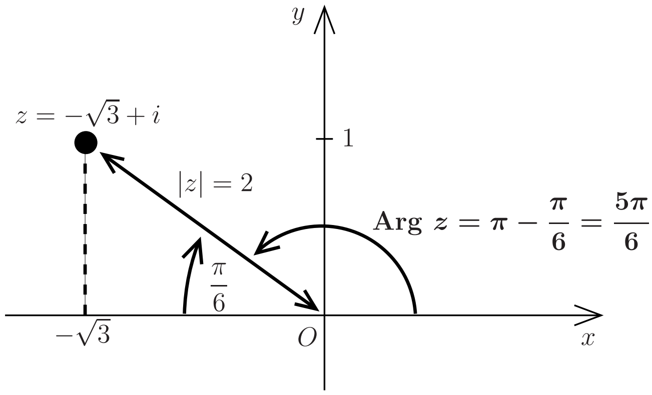
Alternatively, one can compute the argument by first **drawing the Argand diagram** for the complex number, then calculating the **acute angle** with the x-axis. The argument can then be derived from geometrical considerations.

Examples

4.12 Calculate the modulus, argument and principal value of $z = -\sqrt{3} + i$. Also, express z is polar form using the principal value of the argument.

The modulus of z can be calculated via

$$\begin{aligned} |z|^2 &= |-\sqrt{3}+i|^2 \\ &= (-\sqrt{3})^2 + 1^2 \\ &= 3 + 1 = 4 \\ \Longrightarrow \qquad |\boldsymbol{z}| &= \mathbf{2}. \end{aligned}$$



We can see from the figure that the complex number $z = -\sqrt{3} + i$ lies in the second quadrant. However, if $z = -\sqrt{3} + i \equiv x + iy$, then

$$\arctan\left(\frac{y}{x}\right) = \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6},$$

which lies in the fourth quadrant. This illustrates that it is not enough to calculate $\arctan\left(\dfrac{y}{x}\right)$.

Consider the position of $z = -\sqrt{3} + 1$ in the Argand diagram. By considering the right-angled triangle, the point $(-\sqrt{3},\, 1)$ makes an acute angle of $\arctan\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$ with the negative x-axis.

The principal value is, therefore, $\underline{\underline{\theta = \text{Arg}\ z = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}}}.$

The argument of z is multi-valued, with successive values differing by 2π:

$$\begin{array}{lll} \ldots,\ \dfrac{5\pi}{6} - 4\pi = -\dfrac{\mathbf{19}\boldsymbol{\pi}}{\mathbf{6}}, & \dfrac{5\pi}{6} - 2\pi = -\dfrac{\mathbf{7}\boldsymbol{\pi}}{\mathbf{6}}, & \dfrac{\mathbf{5}\boldsymbol{\pi}}{\mathbf{6}}, \\ \dfrac{5\pi}{6} + 2\pi = \dfrac{\mathbf{17}\boldsymbol{\pi}}{\mathbf{6}}, & \dfrac{5\pi}{6} + 4\pi = \dfrac{\mathbf{29}\boldsymbol{\pi}}{\mathbf{6}}, & \dfrac{5\pi}{6} + 6\pi = \dfrac{\mathbf{41}\boldsymbol{\pi}}{\mathbf{6}}, \ \ldots. \end{array}$$

We normally write this in shorthand notation as

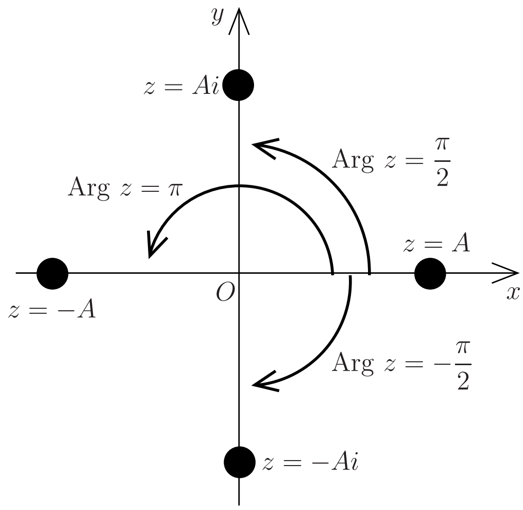
$$\underline{\underline{\arg z = \frac{5\pi}{6} \pmod{2\pi}}} \qquad \text{or} \qquad \underline{\underline{\arg z = \frac{5\pi}{6} + 2k\pi, \quad (k \in \mathbb{Z})}}.$$

In polar form, $\boldsymbol{z} = -\sqrt{\mathbf{3}} + \boldsymbol{i} = \underline{\underline{\mathbf{2}\ \mathbf{cis}\left(\dfrac{\mathbf{5}\boldsymbol{\pi}}{\mathbf{6}}\right) = \mathbf{2}\left(\mathbf{cos}\left(\dfrac{\mathbf{5}\boldsymbol{\pi}}{\mathbf{6}}\right) + \boldsymbol{i}\,\mathbf{sin}\left(\dfrac{\mathbf{5}\boldsymbol{\pi}}{\mathbf{6}}\right)\right)}}.$

4.13 Real and purely imaginary numbers are easy to convert into polar form.

Let A be a real, positive number.

$$\begin{aligned} A &= A\ \mathrm{cis}(0) = A(\cos 0 + i\sin 0), \\ A\,i &= A\ \mathrm{cis}\Big(\frac{\pi}{2}\Big) \\ &= A\left(\cos\Big(\frac{\pi}{2}\Big) + i\sin\Big(\frac{\pi}{2}\Big)\right), \\ -A &= A\ \mathrm{cis}(\pi) = A(\cos\pi + i\sin\pi), \\ -A\,i &= A\ \mathrm{cis}\Big(-\frac{\pi}{2}\Big) \\ &= A\left(\cos\Big(-\frac{\pi}{2}\Big) + i\sin\Big(-\frac{\pi}{2}\Big)\right). \end{aligned}$$



So, for example, in polar form using the principal value,

$$2 = 2\ \mathrm{cis}(0), \qquad 3\,i = 3\ \mathrm{cis}\Big(\frac{\pi}{2}\Big), \qquad -4 = 4\ \mathrm{cis}(\pi), \qquad -5i = 5\ \mathrm{cis}\Big(-\frac{\pi}{2}\Big).$$

4.14 Find $|z|$ and Arg z if $z = 4\ \mathrm{cis}\Big(-\dfrac{27\pi}{2}\Big)$.

The modulus is $\underline{\underline{|z| = 4}}$, and the argument is $\arg z = -\dfrac{27\pi}{2}$.

Since $-\dfrac{27\pi}{2} + (7\times 2\pi) = \dfrac{\pi}{2}$, the principal value of z is $\underline{\underline{\mathrm{Arg}\ z = \dfrac{\pi}{2}}}$.

Examples covered in video

4.15 Find the modulus, argument and principal value of the following complex numbers. Also, express each number in polar form using the principal value.

(i) $-1+i$, (ii) $1-i$, (iii) $-\sqrt{6}-\sqrt{2}i$.

4.16 Find Re(z) and Im(z) if (i) $z = 4\ \mathrm{cis}\Big(-\dfrac{3\pi}{4}\Big)$, (ii) $z = 2\ \mathrm{cis}\Big(\dfrac{5\pi}{3}\Big)$.

§4.7 Multiplication and Division in Polar Form

Complex numbers in polar form can be multiplied and divided very easily. This is because we have the following two useful results.

$$\begin{aligned} \mathbf{(cis\ \boldsymbol{\theta_1})(cis\ \boldsymbol{\theta_2})} &= \mathbf{cis\,(\boldsymbol{\theta_1+\theta_2})} & (1)\\ \frac{\mathbf{1}}{\mathbf{cis\ \boldsymbol{\theta}}} &= \mathbf{cis\,(\boldsymbol{-\theta})} & (2) \end{aligned}$$

(1) and (2) are proved as follows:

$$\begin{aligned} (\text{cis } \theta_1)(\text{cis } \theta_2) &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \;+\; i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\\ &= \cos(\theta_1+\theta_2) \;+\; i\sin(\theta_1+\theta_2), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\text{cis } \theta} = \frac{1}{(\cos\theta + i\sin\theta)} &= \frac{1}{(\cos\theta + i\sin\theta)} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}\\ &= \frac{\cos\theta - i\sin\theta}{(\cos^2\theta + \sin^2\theta)}\\ &= \cos\theta - i\sin\theta \;=\; \cos(-\theta) + i\sin(-\theta). \end{aligned}$$

It follows from (1) that

$$(r_1\ \text{cis}\,\theta_1)(r_2\ \text{cis}\,\theta_2) = r_1 r_2\ \text{cis}(\theta_1+\theta_2). \tag{3}$$

Also, if $r_2 \neq 0$ then using (2) followed by (1):

$$\frac{r_1\ \text{cis}\ \theta_1}{r_2\ \text{cis}\ \theta_2} = \frac{r_1}{r_2}\,[\text{cis}\ \theta_1]\,[\text{cis}\,(-\theta_2)] = \frac{r_1}{r_2}\ \text{cis}\,(\theta_1-\theta_2). \tag{4}$$

Let $z_1 = r_1$ cis θ_1 and $z_2 = r_2$ cis θ_2 where $r_1 = |z_1|$, $r_2 = |z_2|$, $\theta_1 = \arg z_1$ and $\theta_2 = \arg z_2$. Then equation (3) implies that $|z_1\, z_2| = r_1\, r_2$ and $\arg(z_1\, z_2) = \theta_1 + \theta_2$.

Hence

$$\boldsymbol{|z_1\, z_2| \;=\; |z_1|\,|z_2| \qquad \text{and} \qquad \arg(z_1\, z_2) \;=\; \arg z_1 + \arg z_2\,.} \tag{5}$$

In other words,

the modulus of the product $z_1 z_2$ = the product of the moduli of z_1 and z_2 ;

the argument of the product $z_1 z_2$ = the **sum** of the arguments of z_1 and z_2 .

Similarly, equation (4) implies that

$$\boldsymbol{\left|\frac{z_1}{z_2}\right| \;=\; \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad \arg\left(\frac{z_1}{z_2}\right) \;=\; \arg z_1 - \arg z_2\,.} \tag{6}$$

It is possible that when we carry out such calculations, the resulting argument may not lie between $-\pi$ and π. If this is the case, we simply add (or subtract) a suitable multiple of 2π to find the principal argument of our result.

Examples covered in video

4.17 Use the polar form to simplify

$$\text{(i)} \quad u \;=\; \frac{2i(\sqrt{3}+i)}{-3-3\sqrt{3}i}, \qquad \text{(ii)} \quad v \;=\; 2i(\sqrt{3}+i)(-3-3\sqrt{3}i).$$