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Scuola di Ingegneria Industriale e dell'Informazione
Corso di Laurea Magistrale in Mathematical Engineering – Quantitative Finance

Computational Finance: Hedging Competition

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1 Bootstrap of discount factors and forward prices

We compute discount factors and forward prices on 8 December 2017 directly from option quotes. The MATLAB function `bootstrap` takes as input call and put prices, their strikes, and their expiration dates. First, we collect all distinct maturities from call and put options and sort them. For each maturity T, we then:

1. Select all calls and puts with that expiration date.

2. Keep only strikes where both a call and a put are quoted (common strikes).

3. For these common strikes K, compute

$$G(K,T) = C(K,T) - P(K,T).$$

4. Using put–call parity,

$$C(K,T) - P(K,T) = B(0,T)\big(F(0,T) - K\big),$$

we regress $G(K,T)$ on K,

$$G(K,T) = a + bK.$$

The discount factor is given by

$$B(0,T) = -b,$$

and the forward price by

$$F(0,T) = \frac{a}{B(0,T)}.$$

2 Model Calibration

We calibrate the equity volatility surface of 8 December 2017 by fitting a Variance Gamma (VG) Lévy model to S&P 500 option prices. Since the quality of a calibration depends critically on the reliability of the input quotes, the first step consists in constructing a clean and stable cross-section of options.

2.1 Option filtering

The raw dataset contains a heterogeneous mix of maturities, strikes, and liquidity profiles. A selective filtering procedure is therefore required to remove quotes that would introduce instability or bias into the calibration.

Log-moneyness. For each maturity we first compute the forward price F_t and define log-moneyness as

$$m = \log\left(\frac{F_t}{K}\right).$$

For calls we retain options with

$$-0.5 < m < 0,$$

that is, strikes slightly above the forward ($F_t < K$), corresponding to moderately out-of-the-money calls. For puts we retain options with

$$0 < m < 0.5,$$

i.e. strikes slightly below the forward ($F_t > K$), corresponding to moderately out-of-the-money puts.

This choice focuses the calibration on options close to the forward level on each side of the smile, where liquidity is higher and bid–ask spreads are tighter, while excluding very deep ITM/OTM contracts that are more affected by microstructure noise.

Maturity selection. Options expiring in calendar year 2017 are removed. Since the valuation date is 8 December 2017, these contracts have only days (or at most a few weeks) to expiry, making their prices extremely sensitive to short-term effects such as discrete dividends, bid–ask asymmetry and rounding. Such very short-dated instruments carry little information about the intermediate-term dynamics we aim to capture and would disproportionately destabilise the calibration. All maturities from 2018 onward are retained.

Random subsampling. After the above filters, a substantial number of strikes and maturities remain. Calibrating a Lévy model like VG requires repeated evaluation of Fourier-based pricing formulas, and the cost grows linearly with the dataset size. To keep the optimisation tractable while preserving the statistical representativeness of the sample, we randomly retain 30% of the calls and 30% of the puts.

This purely random selection avoids introducing structural biases (for example, favouring specific moneyness regions or maturities) and significantly reduces computation time, while maintaining sufficient coverage of the volatility surface for a stable parameter estimation.

The resulting final dataset offers a balanced compromise between numerical efficiency and informational richness, and forms the basis for the VG model calibration described below.

2.2 Chosen model: Variance Gamma (VG)

We model log-returns under the risk–neutral measure using the Variance Gamma (VG) process, corresponding to the Normal Tempered Stable class with stability parameter $\alpha = 0$, as discussed in the lecture notes.

The characteristic function of the VG process is

$$\hat{\mu}_{Y_t}(u) = \left(1 + \kappa\left(-iu\,\mu + \frac{u^2\sigma^2}{2}\right)\right)^{-t/\kappa}, \qquad \text{where} \quad \mu = -\sigma^2(\eta + 1/2).$$

where σ controls volatility, η controls skewness, and κ governs kurtosis and jump activity.

Option prices under the VG model are computed using the Lewis formula, which evaluates European payoffs via Fourier inversion of the characteristic function.

2.3 Calibration methodology

Let C_i^{mkt} denote the mid-market price of option i in the filtered dataset, and $C_i^{\mathrm{VG}}(\sigma, \kappa, \eta)$ the corresponding model price. The calibration problem consists in solving

$$(\hat{\sigma}, \hat{\kappa}, \hat{\eta}) = \underset{\sigma,\kappa,\eta}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} \left(C_i^{\mathrm{VG}}(\sigma, \kappa, \eta) - C_i^{\mathrm{mkt}}\right)^2.$$

The optimisation is performed in MATLAB using `fmincon`, subject to $\sigma > 0$ and $\kappa > 0$. The parameter α is fixed to zero, as required by the VG specification.

2.4 Calibration results

The Variance Gamma parameters obtained from the optimisation are reported in Table 1. These values define the risk–neutral distribution used in the subsequent pricing and hedging tasks.

Parameter	**Estimated value**
$\hat{\sigma}$	0.1263
$\hat{\kappa}$	2.5310
$\hat{\eta}$	5.0039

Table 1: Calibrated VG parameters as of 8 December 2017.

To assess the quality of the calibration, we compute the Root Mean Squared Error (RMSE) and the Mean Absolute Percentage Error (MAPE):

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(C_i^{\text{VG}} - C_i^{\text{mkt}}\right)^2}, \qquad \text{MAPE} = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{C_i^{\text{VG}} - C_i^{\text{mkt}}}{C_i^{\text{mkt}}}\right|.$$

For our dataset, the calibration achieves:

$$\text{RMSE} = 4.243, \qquad \text{MAPE} = 0.245.$$

These error levels indicate that the VG model provides a satisfactory fit to the central portion of the volatility surface, where option liquidity is highest and mispricing effects are least pronounced.

3 Pricing of the Certificate

The certificate is valued on 8 December 2017 using Monte Carlo simulation under the calibrated NTS (VG) model. For each time interval between observation dates, we simulate log–returns by numerically inverting the model's characteristic function through an FFT-based implementation of the Lewis formula. This provides simulated forward paths, which are converted into spot paths using the discount factors.

For each simulated path, the payoff is computed following the product rules: autocall redemption if the trigger is reached, otherwise final payoff determined by the terminal underlying level and the barrier condition. All payoffs are discounted back to the valuation date.

The resulting certificate price is:

$$\boxed{\text{Price} = 15.3 \text{ million}}$$

4 Hedging Strategy

As the market maker is short the certificate, our goal is to construct a hedging portfolio that neutralises the main risk factors driving the price—the skew rotation sensitivity (η–risk), vega exposure, and delta exposure. The strategy is implemented on 8 December 2017 and rebalanced on subsequent dates.

4.1 Hedging the skew rotation (η–hedge)

The first component of the hedge targets the sensitivity of the certificate to a *smile rotation*s. We model the rotation by shifting the implied volatilities by:

$$\text{OTM calls: } +1\%, \qquad \text{OTM puts: } -1\%.$$

This perturbation captures the asymmetry of the volatility surface (right-tail steepening and left-tail flattening), which is one of the main sources of mis-hedging for path-dependent payoff structures.

We compute the certificate's sensitivity to this asymmetric bump and compare it with the corresponding sensitivity of each call in the dataset. All sensitivities are evaluated in absolute price terms, that is, as the change in the option or certificate price induced by the +1% / −1% volatility perturbation. This ensures a consistent comparison across instruments, independently of their scale or moneyness.

We decided to hedge this metric using a call option. Among all calls available after the filtering step, we select the one that:

1. has maturity strictly beyond the certificate's maturity (to avoid rollover risk), and
2. minimises the bid–ask spread per unit of η–sensitivity.

Let w_η denote its optimal weight. The call position is sized so that the net η–exposure of the portfolio becomes zero.

4.2 Vega hedge

After completing the η–hedge, the portfolio still carries exposure to parallel movements of the implied volatility surface. The goal is therefore to neutralise the portfolio vega while preserving the η–neutrality already achieved.

Definition of vega. Following the lecture notes, the portfolio vega is defined as

$$V_t = \frac{\partial \hat{P}(S_t, \hat{\theta}(S_t, \Sigma_t))}{\partial \Sigma_t},$$

and is computed numerically using a symmetric parallel shift of the volatility surface:

$$\widehat{V}_t = \frac{\hat{P}(S_t, \hat{\theta}(S_t, \Sigma_t + \epsilon)) - \hat{P}(S_t, \hat{\theta}(S_t, \Sigma_t - \epsilon))}{2\epsilon},$$

with $\epsilon \approx 1\%$. All sensitivities are measured in absolute price terms, indeed we didnt divide by the quantity 2ϵ.

Choosing a call–put pair that preserves η neutrality. Both calls and puts have strictly positive vega. By combining one call and one put (each with maturity beyond the certificate's maturity), it is always possible to find a pair of quantities $(w_{\mathrm{C}}, w_{\mathrm{P}})$ such that:

(i) the portfolio vega becomes zero and (ii) the overall η–exposure remains unchanged.

Selecting the optimal instruments. Among all call–put pairs satisfying the maturity constraint, we choose the one with the smallest total bid–ask cost. Once the instruments are chosen, the weights $(w_{\mathrm{C}}, w_{\mathrm{P}})$ are obtained by solving the two linear conditions:

$$\text{vega neutrality}, \qquad \text{preservation of the } \eta\text{–hedge.}$$

This produces a minimal-cost vega hedge that keeps the portfolio invariant to both parallel volatility shifts and skew–rotation movements.

4.3 Delta hedge

Once the η– and vega–hedges have been implemented, the portfolio still carries directional exposure to movements of the underlying. The final step is therefore to neutralise the delta.

Definition of delta. Following the lecture notes, the delta of a derivative is defined as

$$\Delta_t = \frac{\partial \hat{P}(S_t, \hat{\theta}(S_t, \Sigma_t))}{\partial S_t},$$

and is computed numerically by bumping the underlying price:

$$\widehat{\Delta}_t = \frac{\hat{P}(S_t + \epsilon, \hat{\theta}(S_t, \Sigma_t)) - \hat{P}(S_t - \epsilon, \hat{\theta}(S_t, \Sigma_t))}{2\epsilon},$$

where ϵ is a small absolute bump of the spot (e.g., one index point). As in the previous greeks, the sensitivity is measured in absolute price terms.

Choice of the hedging instrument. Delta is hedged using a synthetic forward contract constructed via put–call parity. Forward contracts have zero vega and negligible sensitivity to skew–rotation bumps, so they do not interfere with the hedges already in place.

Among all maturities and strikes producing a valid synthetic forward, we select the one with minimal effective bid–ask cost, using the construction:

$$F^{ASK} = \min_K \left(\frac{C^{ASK}(K) - P^{BID}(K)}{B(0,T)} + K \right), \qquad F^{BID} = \max_K \left(\frac{C^{BID}(K) - P^{ASK}(K)}{B(0,T)} + K \right).$$

Delta neutralisation. Let w_{cert} denote the (negative) position in the certificate, and w_η, w_{C}, w_{P} the quantities of the η–hedge call, the vega–hedge call and the vega–hedge put respectively. Let w_F be the position in the synthetic forward.

Delta neutrality requires the weighted sum of deltas to vanish:

$$w_{\mathrm{cert}}\, \Delta_{\mathrm{cert}} + w_\eta\, \Delta_{\eta\text{-call}} + w_{\mathrm{C}}\, \Delta_{\mathrm{C}} + w_{\mathrm{P}}\, \Delta_{\mathrm{P}} + w_F = 0.$$

Solving for w_F yields the forward position that removes the remaining directional exposure while preserving both the η– and vega–hedges.

4.4 Resulting hedged portfolio

At the end of the calibration and hedging procedure, the portfolio held on 8 December 2017 is:

- short certificate,
- long one call used for the η–hedge,
- long one call and one put used for the vega hedge,
- long or short synthetic forward for delta neutrality,
- a residual cash position reflecting the funding of the above trades.

This portfolio has, by construction, neutral exposure to skew rotation, vega and delta, while minimising hedging costs through bid–ask–efficient instrument selection.

4.5 Portfolio valuation and daily P&L

At each rebalancing date (8, 11, 12 and 13 December 2017) we compute the value of the hedged portfolio using market mid prices for all quoted instruments and the model price for the certificate recalibrated to the current datas.

Portfolio valuation. Let $w_{\text{cert}}, w_\eta, w_{\text{C}}, w_{\text{P}}, w_F$ denote, respectively, the positions in the certificate, the η–hedge call, the vega–hedge call, the vega–hedge put and the synthetic forward. The marked–to–market value of the portfolio at time t is

$$V_t = w_{\text{cert}}\,\hat{P}_{\text{cert}}(t) + w_\eta\,M_\eta(t) + w_{\text{C}}\,M_{\text{C}}(t) + w_{\text{P}}\,M_{\text{P}}(t) + w_F\,F_t + \text{Cash}_t,$$

where:

- $\hat{P}_{\text{cert}}(t)$ is the model price of the certificate using the parameters calibrated on date t,
- $M_\eta(t), M_{\text{C}}(t), M_{\text{P}}(t)$ are the mid prices of the selected options,
- F_t is the mid price of the synthetic forward,
- Cash_t is the liquidity account updated with the funding rate.

All instruments are revalued using *market mid prices*, ensuring that the P&L reflects the performance of the hedge rather than transaction costs.

Daily P&L. The profit and loss between two consecutive dates is computed as the absolute difference in portfolio value:

$$\text{P\&L}_t = V_t - V_{t-1}.$$

This captures both:

- changes in market prices (options, forward, and spot),
- changes in the certificate value caused by the recalibration.

Rebalancing and liquidity update. After computing V_t, the hedge is recomputed from scratch using the model calibrated on date t and the new target positions $(w_\eta, w_\mathrm{C}, w_\mathrm{P}, w_F)$ are obtained. The trades required are determined by comparing the new positions with those held at $t-1$.

When a position needs to be *increased*, the transaction is executed at the ask price; when it needs to be *reduced*, it is executed at the bid price. The resulting cash flows are accumulated in a liquidity account.

The liquidity account is rolled forward using the discount factor between the two dates rather than a continuous funding rate. Denoting by $B(t-1,t)$ the discount factor from $t-1$ to t, the update rule is

$$\mathrm{Cash}_t = \frac{\mathrm{Cash}_{t-1}}{B(t-1,t)} \; - \; \text{Cost of purchases} \; + \; \text{Proceeds from sales.}$$

that the cash component of the portfolio is valued under the same discounting convention as the derivative positions.

This procedure produces a fully self–financing hedging strategy whose P&L can be analysed across the three testing dates.

Discussion of results. The hedging performance we obtain does not meet our expectations. Given the scale of the certificate's price (of order 10^7), we would have expected the daily P&L variability after hedging to be significantly reduced, ideally in the range of 10^2–10^3. However, the realised squared P&L values are substantially larger, at times reaching magnitudes between 10^8 and 10^{10}. This indicates that the hedge is not sufficiently effective in mitigating the risks embedded in the certificate. However, we remain confident in the soundness of the proposed hedging strategy and are convinced that the issue lies in a minor implementation error in the P&L calculation rather than in a conceptual flaw.