CS-E4740 - Federated Learning

FL Design Principle

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Playlist

Glossary

Course Site

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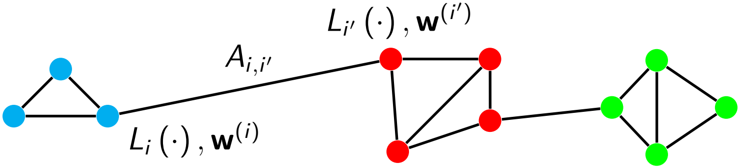
An FL Network



- FL network consisting of devices $i = 1, \ldots, n$.
- Some i, i' connected by an edge with weight $A_{i,i'} > 0$.
- Device i learns model params. $\mathbf{w}^{(i)} \in \mathbb{R}^d$.
- Usefulness of $\mathbf{w}^{(i)}$ measured by some local loss, e.g.,

$$L_i\left(\mathbf{w}^{(i)}\right) := \frac{1}{m_i} \sum_{r=1}^{m_i} \left(y^{(i,r)} - \left(\mathbf{w}^{(i)}\right)^T \mathbf{x}^{(i,r)} \right)^2.$$

FL via Regularization

- Each node carries a linear model $h^{(\mathbf{w}^{(i)})}(\mathbf{x}) := \mathbf{x}^{T}\mathbf{w}^{(i)}$.
- Each node carries m_i labelled data points.
- Node-wise ML fails if $m_i \ll d$ (overfitting).

Idea:

Use the neighbours $\mathcal{N}^{(i)} := \{i' : \{i, i'\} \in \mathcal{E}\}$ to regularize!

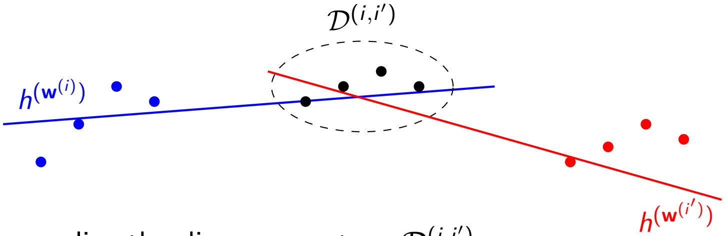
FL via Regularization (ctd.)

As for basic ML, regularization can be done either via

- **Data augmentation** using data from the neighbours.
- **Prune local models** by requiring them to agree across edges.
- **Add a penalty term** to the local loss function.

Building a Penalty Across Edges

- Consider two nodes i, i' with local datasets $\mathcal{D}^{(i)}, \mathcal{D}^{(i')}$.
- Assume there is a non-empty overlap $\mathcal{D}^{(i)} \cap \mathcal{D}^{(i')}$.



We penalize the disagreement on $\mathcal{D}^{(i,i')}$:

$$\sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \left(h^{(\mathbf{w}^{(i)})}(\mathbf{x}) - h^{(\mathbf{w}^{(i')})}(\mathbf{x}) \right)^2 = \sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \left(\mathbf{x}^T \mathbf{w}^{(i)} - \mathbf{x}^T \mathbf{w}^{(i')} \right)^2$$

$$= \left(\mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right)^T \left[\sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \mathbf{x}^T \mathbf{x} \right] \left(\mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right).$$

Generalized TV Minimization (GTVMin)

Learn model params. $\widehat{\mathbf{w}}^{(i)}$ by balancing local loss and GTV

$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)} \in \mathbb{R}^d} \sum_{i=1}^{n} \left[L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} \phi\left(\mathbf{w}^{(i)} - \mathbf{w}^{(i')}\right) \right]$$

- Penalty function $\phi(\mathbf{u})$ is a design choice.
- Previous slide used $\phi(\mathbf{u}) = \mathbf{u}^T \mathbf{Q} \mathbf{u}$ with $\mathbf{Q} := \sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \mathbf{x}\mathbf{x}^T$.
- Our focus is on the choice $\phi(\mathbf{u}) := \|\mathbf{u}\|_2^2$.
- Another popular choice is $\phi(\mathbf{u}) := \|\mathbf{u}\|$.[1]

[1] Y. SarcheshmehPour, et.al, "Clustered Federated Learning via Generalized Total Variation Minimization," in IEEE Trans. Sig. Proc, 2023, doi: 10.1109/TSP.2023.3322848.

Model-Agnostic GTVMin

Replacing $\phi\big(\mathbf{w}^{(i)} - \mathbf{w}^{(i')}\big)$ with the disagreement measure

$$D\big(h^{(i)}, h^{(i')}\big) := \sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \big(h^{(i)}(\mathbf{x}) - h^{(i')}(\mathbf{x})\big)^2$$

yields a model-agnostic generalization of GTVMin

$$\min_{\substack{h^{(i)} \in \mathcal{H}^{(i)} \\ i \in \mathcal{V}}} \sum_{i \in \mathcal{V}} \Bigg[L_i \left(h^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} D\big(h^{(i)}, h^{(i')}\big) \Bigg].$$

This allows for VERY heterogeneous FL networks, e.g.,

$\mathcal{H}^{(1)} =$ lin.model, $\mathcal{H}^{(2)} =$ LLM, $\mathcal{H}^{(3)} =$ decision tree.

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Computational Aspects

$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)}\in\mathbb{R}^d} \sum_{i=1}^{n} \left[L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\}\in\mathcal{E}} A_{i,i'}\phi\left(\mathbf{w}^{(i)} - \mathbf{w}^{(i')}\right) \right]$$

- How can we solve it efficiently over an FL network?
- How much compute/comm. is needed at least?
- What is the effect of different choices for the edges $\mathcal{E}$, loss funcs. $L_i(\cdot)$, and GTV penalty ϕ?

Computational Aspects - Smooth GTVmin

Consider a GTVMin instance

$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)}\in\mathbb{R}^d} \sum_{i=1}^{n} \left[L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\}\in\mathcal{E}} A_{i,i'} \left\|\mathbf{w}^{(i)} - \mathbf{w}^{(i')}\right\|_2^2 \right]$$

with a smooth (differentiable) $L_i(\cdot)$.

If we use (distributed) gradient descent to solve GTVMin:

- How many iterations should we run?
- What is a good choice for the learning rate?
- How to communicate gradients over comm. links?

Characterizing GTVMin Solutions

- Consider GTVMin solution $\widehat{\mathbf{w}}^{(i)} \in \mathbb{R}^d$, for $i = 1, \ldots, n$.
- We stack them into a long vector

$$\widehat{\mathbf{w}} := \left(\widehat{\mathbf{w}}^{(1)}, \ldots, \widehat{\mathbf{w}}^{(n)}\right)^T \in \mathbb{R}^{dn}.$$

- We characterize the solutions as a fixed-point of some $\mathcal{F}$,

$$\boxed{\widehat{\mathbf{w}} \text{ solves GTVMin } \Leftrightarrow \widehat{\mathbf{w}} = \mathcal{F}\widehat{\mathbf{w}}}$$

- The operator $\mathcal{F}$ is not unique (design choice!).

Convex and Smooth GTVMin

Consider GTVMin with a smooth and convex $L_i(\cdot)$,

$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)} \in \mathbb{R}^d} \sum_{i=1}^{n} \left[L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} \left\| \mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right\|_2^2 \right] \quad (1)$$

$$\boxed{\widehat{\mathbf{w}} \text{ solves } (1) \Leftrightarrow \widehat{\mathbf{w}} = \mathcal{F}^{(\eta)} \widehat{\mathbf{w}}}$$

$\mathcal{F}^{(\eta)}$ maps $\mathbf{u} = \left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}\right)^T$ to $\mathbf{v} = \left(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\right)^T$,

$$\mathbf{v}^{(i)} = \mathbf{u}^{(i)} - \eta \left[\nabla L_i\left(\mathbf{u}^{(i)}\right) + 2\alpha \sum_{i' \in \mathcal{N}^{(i)}} A_{i,i'} \left(\mathbf{u}^{(i)} - \mathbf{u}^{(i')}\right) \right].$$

Different choices for "step-size" $\eta > 0$ yield different $\mathcal{F}$.

Fixed-Point Iterations

Q: How to compute a fixed point $\widehat{\mathbf{w}}$ of $\mathcal{F}$?

A: Start with initial guess $\widehat{\mathbf{w}}^{(0)}$ and iterate

$$\widehat{\mathbf{w}}^{(k)} = \mathcal{F}\widehat{\mathbf{w}}^{(k-1)}, \text{ for } k = 1, 2, \ldots .$$

If $\mathcal{F}$ is **firmly non-expansive** $\lim_{k\to\infty} \widehat{\mathbf{w}}^{(k)} = \widehat{\mathbf{w}}$.

If $\mathcal{F}$ is even **contractive** with constant $\kappa < 1$,

$$\left\| \widehat{\mathbf{w}}^{(k)} - \widehat{\mathbf{w}} \right\|_2 \leq \kappa^k \left\| \widehat{\mathbf{w}}^{(0)} - \widehat{\mathbf{w}} \right\|_2 .$$

Gradient Descent as Fixed-Point Iteration

GD for smooth and convex objective function $f(\mathbf{w})$,

$$\mathbf{w}^{(k)} = \mathbf{w}^{(k-1)} - \eta \nabla f\left(\mathbf{w}^{(k-1)}\right)$$

is a fixed-point iteration with $\mathcal{F}^{(\eta)} : \mathbf{w} \mapsto \mathbf{w} - \eta \nabla f(\mathbf{w})$.

- In general, $\mathcal{F}^{(\eta)}$ is neither firmly non-exp. nor contractive.
- Convergence can still be ensured if η is sufficiently small.
- E.g., using learning rate $\eta_k = 1/k$ for smooth $f\left(\mathbf{w}\right)$.

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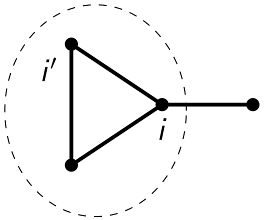
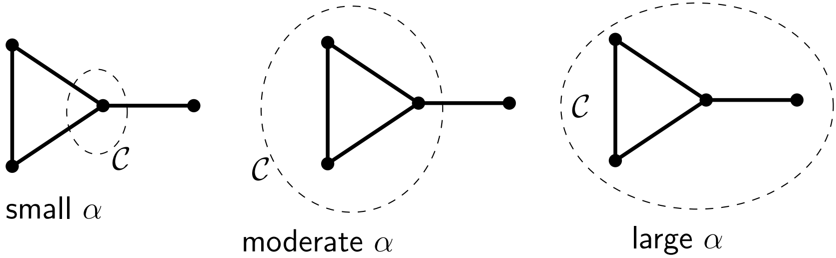
Statistical Aspects

- GTVMin solution yields model params. $\widehat{\mathbf{w}}^{(i)}$, $i = 1, \ldots, n$
- How useful are these model params. ?
- The local loss $L_i\left(\widehat{\mathbf{w}}^{(i)}\right)$ can be misleading (why?)
- Better to use aggregate $\sum_{i \in \mathcal{C}^{(i)}} L_i\left(\widehat{\mathbf{w}}^{(i)}\right)$, with cluster



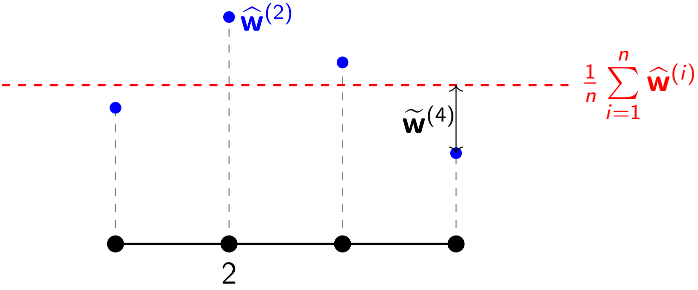
$$\mathcal{C}^{(i)} := \left\{ i' : \widehat{\mathbf{w}}^{(i)} \approx \widehat{\mathbf{w}}^{(i')} \right\}.$$

Clustering of GTVMin



Analysis of Clustering - Assumptions

- Consider a connected FL network $\mathcal{G}$ with $\lambda_2 > 0$.
- Assume loss funcs. satisfy $\min_{\mathbf{v} \in \mathbb{R}^d} \sum_{i=1}^{n} L_i(\mathbf{v}) \leq \varepsilon$
- Use GTVMin to learn local params. $\widehat{\mathbf{w}}^{(i)}$.
- Define the variation $\widetilde{\mathbf{w}}^{(i)} := \widehat{\mathbf{w}}^{(i)} - \frac{1}{n} \sum_{i=1}^{n} \widehat{\mathbf{w}}^{(i)}$.



Analysis of Clustering - Upper Bound

The variation $\widetilde{\mathbf{w}}^{(i)}$ is upper bounded as

$$\sum_{i=1}^{n} \left\| \widetilde{\mathbf{w}}^{(i)} \right\|_2^2 \leq \frac{\varepsilon}{\alpha \lambda_2}.$$

This bound involves the

- connectivity of FL network (via λ_2),
- the properties of local loss functions (via ε), and
- the GTVMin parameter α.

A large $\alpha\lambda_2$ results in nearly identical local params. $\widetilde{\mathbf{w}}^{(i)} \approx \mathbf{0}$.

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Interpretations

We next discuss some interpretations of GTVMin

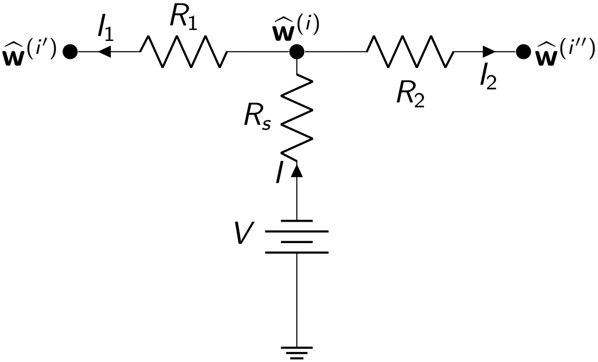
$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)} \in \mathbb{R}^d} \sum_{i=1}^{n} \left[L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} \left\| \mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right\|_2^2 \right]$$

for some FL network with weighted undirected graph $\mathcal{G}$ and smooth and convex loss func. $L_i\left(\mathbf{w}^{(i)}\right)$.

We assume that there exists a solution $\widehat{\mathbf{w}}^{(1)}, \ldots, \widehat{\mathbf{w}}^{(n)}$. (Do we really need to make this assumption?)

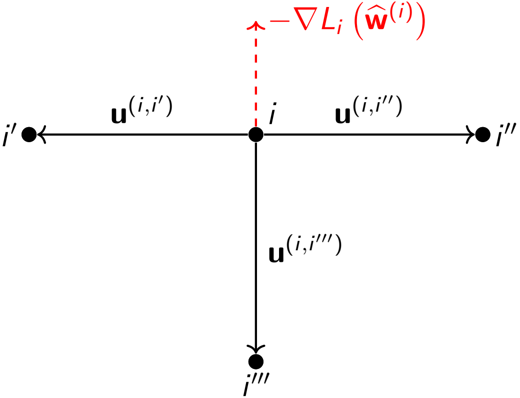
Electronic Circuit

Consider a node i with neighbours $\mathcal{N}^{(i)} = \{i', i''\}$.



$$\underbrace{-\nabla L_i\left(\widehat{\mathbf{w}}^{(i)}\right)}_{I} = \left[\underbrace{A_{i,i'}\left(\widehat{\mathbf{w}}^{(i)} - \widehat{\mathbf{w}}^{(i')}\right)}_{I_1} + \underbrace{A_{i,i''}\left(\widehat{\mathbf{w}}^{(i)} - \widehat{\mathbf{w}}^{(i'')}\right)}_{I_2}\right]$$

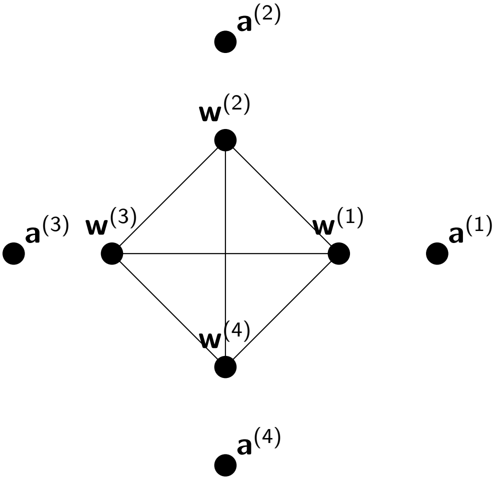
Vector-Valued Flows



Vector-valued flow $\mathbf{u}^{(i,i')} := \nabla\phi\big(\mathbf{u}\big)\Big|_{\mathbf{u}=\widehat{\mathbf{w}}^{(i)}-\widehat{\mathbf{w}}^{(i')}}$.

Locally Weighted Learning

Generalized Convex Clustering



$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)}} \sum_{i=1}^{n} \left\|\mathbf{w}^{(i)} - \mathbf{a}^{(i)}\right\|_2^2 + \alpha \sum_{i,i' \in \mathcal{V}} \left\|\mathbf{w}^{(i)} - \mathbf{w}^{(i')}\right\|_2 .$$

What's Next?

The next module applies optimization methods to solve GTVMin.

We can implement these methods as message passing over the edges of an FL network.

Further Resources

- **YouTube:** @alexjung111
- **LinkedIn:** Alexander Jung
- **GitHub:** alexjungaalto