PREFACTORIZATION ALGEBRAS IN QFT

A MINIMALIST THEORY OF OBSERVABLES

(IN 10 MINUTES)

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3. The prefix 'pre-' is not there by accident.

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1. Monoidal categories;
2. Multicategories (AKA colored operads) and their algebras - symmetric version;

Prefactorization Algebras and Disjoint Opens

Definition

[Costello & Gwilliam [CG1] (2016), §3.1.2, Definition 1.2.1] Let $\mathbf{Disj}_M$ denote the following - *symmetric* - multicategory associated to M.

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1. The objects consist of all *connected* open subsets of M;
2. For every (possibly empty) finite collection of open sets $\{U_\alpha\}_{\alpha \in A}$ and open set V, there is a set of maps $\mathbf{Disj}_M \left(\{U\}_{\alpha \in A} | V \right)$.
   If the U_α are pairwise disjoint and all contained in V, then the set of maps is a single point. Otherwise, the set of maps is empty;

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[ibid., §1.2, 40, line 6] A prefactorization algebra is just an algebra over this - *symmetric* - coloured operad $\mathbf{Disj}_M$.

Open connected sets as thin multicategory

Definition

Let $\left(\mathrm{Open}^c_X, \subseteq\right)$ be the ordered set of connected open parts of a topological space X with set-theoretical inclusion as preorder. The associated *symmetric* poset multicategory $\mathbf{Open}^c_X$ consists of the following:

(I) $\left(\mathbf{Open}^c_X\right)_0$ as objects;

(II) For any finite string $\left(U_1, \ldots, U_n\right) \in \prod^n \left(\mathbf{Open}^c_X\right)_0$ an hom-set $\mathbf{Open}^c_X\left(U_1, \ldots, U_n; V\right)$, where:

$$\mathbf{Open}^c_X\left(U_1, \ldots, U_n; V\right) = \begin{cases} \{\emptyset\} & \iff \dot\cup_{i=1}^n U_i \not\subseteq V \\ \{f\} & \iff \dot\cup_{i=1}^n U_i \subseteq V \ \wedge \ U_i \cap U_j = \emptyset \ \ \forall\, i \neq j \end{cases} \tag{1}$$

(III) An operation of composition: $\forall n, k_1, \ldots, k_n \in \mathbb{N}, V, U_i, U_i^j \in \left(\mathbf{Open}^c_X\right)_0$

$$\mathbf{Open}^c_X\left(U_1, \ldots, U_n; V\right) \times \mathbf{Open}^c_X\left(U_1^1, \ldots, U_1^{k_1}; U_1\right) \times \cdots \times \mathbf{Open}^c_X\left(U_1^1, \ldots U_n{}^{k_n}; U_n\right)$$

$$\downarrow$$

$$\mathbf{Open}^c_X\left(U_1^1, \ldots, U_1^{k_1}, \ldots, U_n^1, \ldots, U_n^{k_n}; V\right)$$

$$(f, \ldots, f_n) \mapsto f \circ \left(f_1, \ldots f_n\right) \tag{2}$$

whenever the arrows exist and are sequentially composable.

(IV) An identity arrow: $\forall\, U \in \left(\mathbf{Open}^c_X\right)_0, \exists\, 1_U \in \mathbf{Open}^c_X(U; U)$

satisfying associativity and identity law.

The category of prefactorization algebras 1/2

Definition

Let **C** a symmetric multicategory, a **prefactorisation algebra** with values in **C** is a multifunctor

$$\mathbf{Open}^{c}_{X} \xrightarrow{\mathcal{F}} \mathbf{C} \tag{3}$$

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Let $\mathcal{F} : \mathbf{Open}^c_X \to \mathbf{C}$, $\mathcal{G} : \mathbf{Open}^c_X \to \mathbf{C}$ be two PFAs taking values in the symmetric multicategory $\mathbf{C}$, an **arrow of prefactorisation algebras** is a natural trasformation between them

$$\mathcal{F} \overset{\phi}{\Rightarrow} \mathcal{G} \tag{4}$$

is a family of maps

$$\left\{ \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \right\}_{U \in \left(\mathbf{Open}^c_X\right)_0} \tag{5}$$

such that

$$\phi_V \circ (\mathcal{F}(f)) = \mathcal{G}(f) \circ (\phi_{U_1}, \ldots, \phi_{U_n}) \tag{6}$$

for all $f : \mathcal{F}(U_1), \ldots, \mathcal{F}(U_n) \to \mathcal{F}(V)$.

The category of Prefactorization Algebras 2/2

Definition

Let X be a topological space, **C** be a symmetric multicategory, the **category of prefactorisation algebras over X with values in C** consists of an objects class made out of PFAs and, as morphisms, natural transformation between them. We denote such category by the symbol $\mathbf{PFA}_X(\mathbf{C})$.

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(II) For $\{V_i\}_i$ a Weiss cover of the open $U \subset M$,

$$\bigoplus_{i \neq j} \mathcal{F}(V_i \cap V_j) \to \bigoplus_i \mathcal{F}(V_i) \to \mathcal{F}(U) \to 0$$

is an exact sequence on the right and in the middle.

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References

- [CG1] Costello, K. and Gwilliam, O., Factorization Algebras in Quantum Field Theory, Volume 1 (May 8th, 2016), available online at: https://people.math.umass.edu/ gwilliam/vol1may8.pdf;
- [CG2] Costello K. and Gwilliam O., Factorization Algebras in Quantum Field Theory, Volume 2 (October 2th, 2020), available at: https://people.math.umass.edu/ gwilliam/factorization2.pdf;
- [PS] Pasqualone, F., Prefactorisation Algebras: Introduction and Examples, MSc Thesis, 2023, available at: https://github.com/Federica-Rike