

DIFFERENTIAL CALCULUS ON PFAs

STARTER PACK

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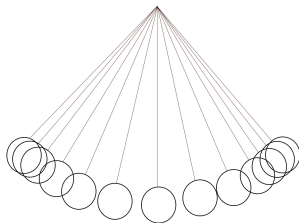


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ONCE UPON A TIME ...

This story starts with a simple innocent object: The pendulum



$$\ddot{\theta} = -\frac{g}{l} \sin\theta$$

$$\text{For } \sin\theta \sim \theta, \ddot{\theta} = -\frac{g}{l} \theta$$

WHY?

1. The last equation is completely equivalent to the one of an harmonic oscillator;
2. If we freeze time, particles form a crystal, whose vibrational modes are exactly described by the same physics of a pendulum;
3. It provides an example, from classical mechanics, of how we study a physical system. In particular, it is clear that if we want to measure some *observable*, such as momentum or velocity, it must be a function over the trajectories cut by the system and not over the whole space-time.

In general, observables are functions over trajectories, i.e. functions over solutions of the Euler-Lagrange equations, a system of pdes. Equivalently, by Hamilton's principle, they can be seen as functions over the stationary points of the action functional S

$$S[q] := \int_{t_1}^{t_2} dt \mathcal{L}(q(t), \dot{q}(t), t)$$

where \mathcal{L} denotes the Lagrangian of the system.

MODELLING OBSERVABLES: WISHLIST

We are studying solutions of pdes, that we know exist at least locally, therefore it is reasonable to ask for the following:

Let $U \subseteq M$ be an open set of a manifold M , that is our space-time,

- 1 An assignment of a 'vector space' of quantum observables to every connected open set

$$U \mapsto \text{Obs}^q(U)$$

2. For U_1, U_2 disjoint open connected sets contained in another open V of M

$$\text{Obs}^q(U_1) \otimes \text{Obs}^q(U_2) \rightarrow \text{Obs}^q(V)$$

i.e. we require the possibility of combining observables defined over disjoint regions of the space-time.

3. Deformation quantization constraint:

$$\lim_{\hbar \rightarrow 0} \frac{1}{\hbar} \text{Obs}^q(U) = \text{Obs}^{cl}(U)$$

(!) SR disclaimer: Observables defined over space-like separated regions are uncorrelated.

PREFACTORIZATION ALGEBRAS AND DISJOINT OPENS

DEFINITION

[Costello & Gwilliam [CG1] (2016), §3.1.2, Definition 1.2.1] Let \mathbf{Disj}_M denote the following - *symmetric* - multicategory associated to M .

1. The objects consist of all *connected* open subsets of M ;
2. For every (possibly empty) finite collection of open sets $\{U_\alpha\}_{\alpha \in A}$ and open set V , there is a set of maps $\mathbf{Disj}_M(\{U_\alpha\}_{\alpha \in A} | V)$.
If the U_α are pairwise disjoint and all contained in V , then the set of maps is a single point. Otherwise, the set of maps is empty;
3. The composition of maps is defined in the obvious way.

DEFINITION

[ibid., §1.2, 40, line 6] A prefactorization algebra is just an algebra over this - *symmetric* - coloured operad \mathbf{Disj}_M .

OPEN CONNECTED SETS AS THIN MULTICATEGORY

DEFINITION

Let $(\text{Open}_X^c, \subseteq)$ be the ordered set of connected open parts of a topological space X with set-theoretical inclusion as preorder. The associated *symmetric* poset multicategory \mathbf{Open}_X^c consists of the following:

1. $(\text{Open}_X^c)_0$ as objects;
2. For any finite string $(U_1, \dots, U_n) \in \prod^n (\text{Open}_X^c)_0$ an hom-set $\mathbf{Open}_X^c(U_1, \dots, U_n; V)$, where:

$$\mathbf{Open}_X^c(U_1, \dots, U_n; V) = \begin{cases} \{\emptyset\} & \iff \bigcup_{i=1}^n U_i \not\subseteq V \\ \{f\} & \iff \bigcup_{i=1}^n U_i \subseteq V \wedge U_i \cap U_j = \emptyset \quad \forall i \neq j \end{cases} \quad (1)$$

3. An operation of composition: $\forall n, k_1, \dots, k_n \in \mathbb{N}, V, U_i, U_i^j \in (\text{Open}_X^c)_0$

$$\mathbf{Open}_X^c(U_1, \dots, U_n; V) \times \mathbf{Open}_X^c(U_1^1, \dots, U_1^{k_1}; U_1) \times \dots \times \mathbf{Open}_X^c(U_1^1, \dots, U_n^{k_n}; U_n)$$

\downarrow

$$\mathbf{Open}_X^c(U_1^1, \dots, U_1^{k_1}, \dots, U_n^1, \dots, U_n^{k_n}; V)$$

$$(f, \dots, f_n) \mapsto f \circ (f_1, \dots, f_n) \quad (2)$$

whenever the arrows exist and are sequentially composable.

- (iv) An identity arrow: $\forall U \in (\text{Open}_X^c)_0, \exists 1_U \in \mathbf{Open}_X^c(U; U)$

satisfying associativity and identity law.

THE CATEGORY OF PREFACTORIZATION ALGEBRAS 1/2

DEFINITION

Let \mathbf{C} a symmetric multicategory, a **prefactorization algebra** with values in \mathbf{C} is a multifunctor

$$\mathbf{Open}_X^c \xrightarrow{\mathcal{F}} \mathbf{C} \quad (3)$$

DEFINITION

Let $\mathcal{F} : \mathbf{Open}_X^c \rightarrow \mathbf{C}$, $\mathcal{G} : \mathbf{Open}_X^c \rightarrow \mathbf{C}$ be two PFAs taking values in the symmetric multicategory \mathbf{C} , an **arrow of prefactorization algebras** is a natural transformation between them

$$\mathcal{F} \xRightarrow{\phi} \mathcal{G} \quad (4)$$

is a family of maps

$$\left\{ \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \right\}_{U \in (\mathbf{Open}_X^c)_0} \quad (5)$$

such that

$$\phi_V \circ (\mathcal{F}(f)) = \mathcal{G}(f) \circ (\phi_{u_1}, \dots, \phi_{u_n}) \quad (6)$$

for all $f : \mathcal{F}(U_1), \dots, \mathcal{F}(U_n) \rightarrow \mathcal{F}(V)$.

THE CATEGORY OF PREFACTORIZATION ALGEBRAS 2/2

DEFINITION

Let X be a topological space, \mathbf{C} be a symmetric multicategory, the **category of prefactorization algebras over X with values in \mathbf{C}** consists of an objects class made out of PFAs and, as morphisms, natural transformation between them. We denote such category by the symbol $\mathbf{PFA}_X(\mathbf{C})$.



FACTORIZATION ALGEBRAS

If we have a global solution of our pdes this descends to local ones, but what about gluing local solutions to a global one? This is exactly what factorization algebras model.

DEFINITION

A factorization algebra is a prefactorization algebra \mathcal{F} satisfying two additional axioms:

1. For $U_i, U_j \subset M$ any two open sets of a manifold M , there exists an isomorphism

$$\mathcal{F}(U_i) \otimes \mathcal{F}(U_j) \xrightarrow{\cong} \mathcal{F}(U_i \dot{\cup} U_j)$$

2. For $\{V_i\}_i$ a Weiss cover of the open $U \subset M$,

$$\bigoplus_{i \neq j} \mathcal{F}(V_i \cap V_j) \rightarrow \bigoplus_i \mathcal{F}(V_i) \rightarrow \mathcal{F}(U) \rightarrow 0$$

is an exact sequence on the right and in the middle.

Given a factorization algebra \mathcal{F} on M , its global sections define the *factorization homology* of \mathcal{F} on M , usually denoted by $\int_M \mathcal{F}$.

OBSERVABLES OF A FREE SCALAR FIELD THEORY

Consider a Riemannian manifold (M, g) , fields are smooth functions $C^\infty(M)$ and the action functional is quadratic in the fields

$$S(\phi) = \frac{1}{2} \int_M \phi (\Delta_g + m^2) \phi \, \text{dvol}_g$$

Correlation functions are of the form

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int_{\phi} d\phi \, \phi(x_1) \cdots \phi(x_n)$$

Therefore, there exists an observable

$$O(x_1, \dots, x_n) : \phi \mapsto \phi(x_1) \cdots \phi(x_n)$$

THE DIVERGENCE OPERATOR - 1/2

Observables are defined in terms of co-kernels of some divergence operator.

For it, let

$$\text{Vect}'(C^\infty(U)) := \text{Sym}(C_c^\infty(U)) \otimes C_c^\infty(U)$$

then an element of this space is a finite sum of monomials $f_1 \cdots f_n \frac{\partial}{\partial \phi}$ for $f_i, \phi \in C_c^\infty(U)$ acting on functions as

$$f_1 \cdots f_n \frac{\partial}{\partial \phi}(g_1 \cdots g_m) = f_1 \cdots f_n \sum_i g_1 \cdots \hat{g}_i \cdots g_m \int_U g_i(x) \phi(x) \text{dvol}_g$$

DEFINITION

The divergence operator associated to the action functional defined before is given by the linear map

$$\text{Div}' : \text{Vect}'_c(C^\infty(U)) \rightarrow \text{Sym}(C_c^\infty(U))$$

$$\text{Div}'\left(f_1 \cdots f_n \frac{\partial}{\partial \phi}\right) = -f_1 \cdots f_n (\Delta + m^2) \phi + \sum_i f_1 \cdots \hat{f}_i \cdots f_n \int_U \phi(x) f_i(x) \text{dvol}_g$$

THE DIVERGENCE OPERATOR - 2/2

(!) Since the compactly supported functions over an open set form a topological vector space, we switch to the completed version of the spaces.

[[CG1], Lemma 2.0.2] The divergence operator Div' extends continuously to a linear map

$$\text{Div} : \bigoplus_{n \geq 0} C_c^\infty(U^{n+1})_{S_n} \rightarrow \bigoplus_{n \geq 0} C_c^\infty(U^n)_{S_n}$$

This holds by virtue of the fact that Vect'_c is a dense subspace of

$$\text{Vect}_c(C_c^\infty(U)) := \bigoplus_{n \geq 0} C_c^\infty(U^{n+1})_{S_n}$$

and, similarly, $\text{Sym}_c^\infty(U)$ is a dense subspace of

$$P(C^\infty(U)) := \bigoplus_{n \geq 0} C_c^\infty(U^n)_{S_n}$$

QUANTUM OBSERVABLES - 1/2

DEFINITION

[[CG1], Definition 2.0.3] The quantum observables of a free field theory are defined as

$$H^0(\text{Obs}^q(U)) = \frac{P(C^\infty(U))}{\text{Im Div}}$$

for $U \subseteq M$ an open set.

(!) This extends to co-chain complexes.

For observables of a free scalar field theory, consider the Gaussian measure

$$\exp\left(-\frac{1}{\hbar} \int_M \phi (\Delta + m^2) \phi\right) d\phi$$

and define the divergence operator as follows

$$\text{Div}_\hbar : \text{Vect}_c(C^\infty(U)) \rightarrow P(C^\infty(U))$$

$$f_1 \cdots f_n \frac{\partial}{\partial \phi} \mapsto -\frac{1}{\hbar} f_1 \cdots f_n (\Delta + m^2) \phi + \sum f_1 \cdots \hat{f}_i \cdots \int_M f \phi$$

QUANTUM OBSERVABLES - 2/2

LEMMA

[[CG1], Lemma 4.0.1] *There exists a prefactorization algebra $H^0 \left(\text{Obs}_\hbar^q (U) \right)$ over $\mathbb{C} [\hbar]$ such that*

$$H^0 \left(\text{Obs}_\hbar^q (U) \right) = \begin{cases} H^0 \left(\text{Obs}^q (U) \right) & \text{for } \hbar = 1 \\ H^0 \left(\text{Obs}^{cl} (U) \right) & \text{for } \hbar = 0 \end{cases}$$

that assigns to each open set U the co-kernel of the map

$$\hbar \text{Div}_\hbar : \text{Vect}_c \left(C^\infty (U) \right) [\hbar] \rightarrow P \left(C^\infty (U) \right) [\hbar]$$

For M a compact Riemannian manifold and $m > 0$, $H^0 \left(\text{Obs}^q (M) \right) \cong \mathbb{R}$, and the correlator coincides with the PFA structure map

$$\langle - \rangle : H^0 \left(\text{Obs}^q (U_1) \right) \otimes \cdots \otimes H^0 \left(\text{Obs}^q (U_n) \right) \rightarrow H^0 \left(\text{Obs}^q (M) \right) \cong \mathbb{R}$$

for $U_i \subseteq M$ connected disjoint opens.

SKETCHES OF AN ELEPHANT - 1/2

For a free scalar field theory with action functional $\int_M \phi \Delta \phi$, the solutions of E-L equations are harmonic functions. Namely, the derived space of solutions is given by

$$\mathcal{E}(U) = \left(C^\infty(U) \xrightarrow{\Delta} C^\infty(U)[-1] \right)$$

$\mathcal{E}(M)$ is a sheaf of vector spaces on the site of smooth manifolds.

As before, we define polynomial functions as

$$P(\mathcal{E}(M)) = \oplus_n P_n(\mathcal{E}(M)) = \oplus_n \text{Hom}_{DVS} \left(\mathcal{E}(M)^{\times n}, \mathbb{R} \right)_{S_n} = \oplus_n \mathcal{D}_c \left(M^n, (E^!)^{\boxtimes n} \right)_{S_n}$$

Define the bundle $E^! := E^\vee \otimes \text{Dens}_M$, then

$$\mathcal{E}^!(U) \cong (C_c^\infty(U)[1] \rightarrow C^\infty(U))$$

The classical observables are given by the following

$$\text{Obs}^{cl}(U) = \text{Sym}(\mathcal{E}^!(U)) = \text{Sym}\left(C_c^\infty(U)[1] \xrightarrow{\Delta} C_c^\infty(U)\right)$$

SKETCHES OF AN ELEPHANT - 2/2

Define the following sheaf

$$\hat{\mathcal{E}}(U) := \mathcal{E}_c(U) \oplus \mathbb{R} \cdot \hbar$$

with Lie bracket given by

$$[\alpha, \beta] = \hbar \langle \alpha, \beta \rangle$$

then the quantum observables are given by the Chevalley-Eilenberg complex

$$\text{Obs}^q(U) = C_\bullet(\hat{\mathcal{E}}(U)) = (\text{Sym}(\hat{\mathcal{E}}(U)[1]), d) = (\text{Obs}^{cl}(U)[\hbar], d)$$

1. As graded vector spaces

$$\text{Obs}^{cl}(U)[\hbar] \cong \text{Obs}^q(U)$$

2. The previous iso does not respect differentials!
3. The quantum observables constitute a PFA valued in BD algebras that quantizes the P_0 -algebras valued PFA of the classical observables. In other words, the Poisson bracket measures the failure for d to be a differential.

REFERENCES



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