Skewing the Sequent Calculus:

Skew Monoidal Categories $\wedge$ Skew Multicategories

Federica Pasqualone, Wilf Offord

March 28, 2024

Outline

Definition

A **skew-monoidal category** is a category $\mathcal{C}$ equipped with:

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- an object $I \in \mathcal{C}_0$
- three natural transformations, called associator, left and right unitor respectively. Component-wise written as:

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$

$$\lambda_x : I \otimes x \to x$$

$$\rho_x : x \to x \otimes I$$

natural in x, y, z. ! They are not required to be natural isos !

such that the following diagrams commute:

Definition

$$\begin{tikzcd}
& & ((x \otimes y) \otimes z) \otimes w \arrow[dll, "\alpha_{x \otimes y, z, w}"'] \arrow[drr, "\alpha_{x,y,z} \otimes w"] & & \\
(x \otimes y) \otimes (z \otimes w) \arrow[dr, "\alpha_{x,y,z \otimes w}"'] & & & & (x \otimes (y \otimes z)) \otimes w \arrow[dl, "\alpha_{x, y \otimes z, w}"] \\
& x \otimes (y \otimes (z \otimes w)) & & x \otimes ((y \otimes z) \otimes w) \arrow[ll, "x \otimes \alpha_{y,z,w}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes y \arrow[r, equal] \arrow[d, "\rho_x \otimes y"'] & x \otimes y \\
(x \otimes I) \otimes y \arrow[r, "\alpha_{x,I,y}"'] & x \otimes (I \otimes y) \arrow[u, "x \otimes \lambda_y"']
\end{tikzcd}$$

$$\begin{tikzcd}
(I \otimes x) \otimes y \arrow[rr, "\alpha_{I,x,y}"] \arrow[dr, "\lambda_x \otimes y"'] & & I \otimes (x \otimes y) \arrow[dl, "\lambda_{x \otimes y}"] \\
& x \otimes y &
\end{tikzcd}
\quad
\begin{tikzcd}
(x \otimes y) \otimes I \arrow[rr, "\alpha_{x,y,I}"] & & x \otimes (y \otimes I) \\
& x \otimes y \arrow[ul, "\rho_{x \otimes y}"] \arrow[ur, "x \otimes \rho_y"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
I \arrow[rr, equal] \arrow[dr, "\rho_I"'] & & I \\
& I \otimes I \arrow[ur, "\lambda_I"'] &
\end{tikzcd}$$

! In the case of monoidal categories, the final three equations follow from the first two.

Examples

- Trivially, any monoidal category is a skew-monoidal category.
- Take $\mathcal{C}$ to be **Ptd**, the category of pointed sets, with $(X, x_0) \otimes (Y, y_0) := (X + Y, \mathsf{inl} x_0)$ and $I := (1, *)$. Then α, λ, ρ can be easily defined (see next slide).
- A skew-monoidal structure can be put on the poset $\mathbb{N}$, as we will see shortly.
- Let $\mathcal{D}$ be cocomplete so that all left Kan extensions exist, and fix a functor $J : \mathcal{C} \to \mathcal{D}$. We define a tensor product on $[\mathcal{C}, \mathcal{D}]$ by $F \otimes G := \mathsf{Lan}_J F \circ G$. The monoidal unit is J. The universal property of Kan extensions gives maps α, λ, ρ pointing in the right direction, but not necessarily invertible.

Ptd as Skew Monoidal Category 1/3

Consider $I := (1, *), (X, x_0), (Y, y_0) \in \mathbf{Ptd}$ pointed sets (or pointed topological spaces !):
with tensor product

$$(X, x_0) \otimes (Y, y_0) := (X + Y, \mathsf{inl}\, x_0)$$

Define associator, left and right unitors as follows:

$$\begin{array}{c} ((X, x_0) \otimes (Y, y_0)) \otimes (Z, z_0) = ((X + Y) + Z, \mathsf{inl}\,(\mathsf{inl}\, x_0)) \\ \Big\downarrow \alpha_{X,Y,Z} \\ (X, x_0) \otimes ((Y, y_0) \otimes (Z, z_0)) = (X + (Y + Z), \mathsf{inl}\, x_0) \end{array}$$

Ptd as Skew Monoidal Category 2/3

$$\begin{aligned}\lambda_X : (1,*) \otimes (X, x_0) = (1 + X, \mathsf{inl}*) &\to (X, x_0)\\ \mathsf{inl}* &\mapsto x_0\\ \mathsf{inr}x &\mapsto x\end{aligned}$$

$$\begin{aligned}\rho_X : (X, x_0) &\to (1 + X, \mathsf{inl}*)\\ x &\mapsto \mathsf{inl}x\end{aligned}$$

! The left unitor is not monic, the right not epic - but the associator is iso !

Ptd as Skew Monoidal Category 3/3

What we have just defined look likes a co-product in **Ptd**, but it is not ! - it is indeed the tensor product.

Here it is the actual co-product:

$$(X, x_0) + (Y, y_0) = ((X + Y) \,/ \sim, [\mathsf{inl} x_0])$$

with equivalence relation induced by gluing $\mathsf{inl} x_0$ and $\mathsf{inr} y_0$.

$\mathbb{N}$ as Posetal SMC

Consider the natural numbers with their standard order as posetal category, meaning

$$n \leq m \iff n \to m$$

for $n, m \in \mathbb{N}$.

For a fixed number $\mathbf{n} \in \mathbb{N}$, that we set as unit of the monoidal structure, define the monoidal product truncating as follows:

$$h \otimes r := (h - \mathbf{n}) + r$$

The left, right unitors and associator reads:

$$\lambda_h : (\mathbf{n} - \mathbf{n}) + h = 0 + h = h \qquad \rho_h : h \leq h \max \mathbf{n} = (h - \mathbf{n}) + \mathbf{n}$$

$$\alpha_{h,r,t} : (((h - \mathbf{n}) + r) - \mathbf{n}) + t \leq (h - \mathbf{n}) + (r - \mathbf{n}) + t$$

Coherence

- When α, ρ, and λ are isomorphisms, we have a **monoidal category**, and the morphisms in these are easily characterised by MacLane's *coherence theorem*.
- This can be stated as follows: the free monoidal category on any set of objects is a preorder.
- In the skew case, this is no longer true. For instance, it is not the case that $\rho_I \circ \lambda_I = \mathrm{id}_{I \otimes I}$. It would be nice therefore to have some other way to characterise the morphisms in free skew-monoidal categories.
- The paper solves the issue by characterizing morphisms as equivalence classes of deduction trees for a sequent calculus, as we will see at the very end.

Multicategories

- The sequent calculus presented is best understood as a calculus for *left representable skew multicategories*. We give first some background on multicategories and their relation to monoidal categories.
- A **multicategory** $\mathcal{C}$ is like a category, except morphisms can have "multiple inputs": for each list $\overline{A}$ of objects of C, and for each object B, we have a Hom-set $\mathrm{Hom}(\overline{A}; B)$. Elements of these homsets are often drawn as follows:

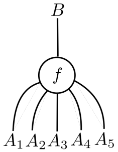
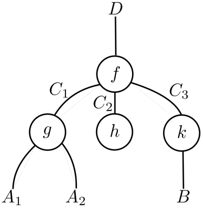


Figure: A multimorphism $f \in \mathrm{Hom}(A_1, A_2, A_3, A_4, A_5; B)$

Multicategories

- There is a multi-ary composition operation, and identity morphisms, depicted as:



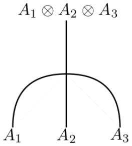
(a) Graphical representation of $f(g, h, k)$. Here $f \in \mathrm{Hom}(C_1, C_2, C_3; D)$, $g \in \mathrm{Hom}(A_1, A_2; C_1)$, $h \in \mathrm{Hom}(; C_2)$, $k \in \mathrm{Hom}(B; C_3)$

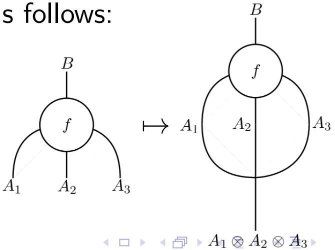
(b) Graphical representation of id_A

- These are subject to obvious multi-ary associativity and unitality constraints, which follow automatically from the graphical representation.

Representable Multicategories

- We say a list $\overline{A}$ of objects is *representable* if there exists an object $m(\overline{A})$ and a morphism $\theta(\overline{A}) : \overline{A} \to m(\overline{A})$ such that precomposing with $\theta(\overline{A})$ gives a natural isomorphism between $\mathrm{Hom}(m(\overline{A}), B)$ and $\mathrm{Hom}(\overline{A}, B)$, for any B - that is, the functor $\mathrm{Hom}(\overline{A}, -)$ is representable.
- We also write $m(A_1, \ldots, A_n)$ as $(A_1 \otimes \cdots \otimes A_n)$.
- We draw θ and the inverse to precomposition, a map $\mathrm{Hom}(\overline{A}; B) \to \mathrm{Hom}(m(\overline{A}); B)$, as follows:

$\theta(A_1, A_2, A_3) =$ 

,


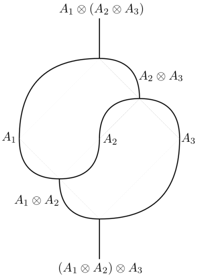
Representable Multicategories

- A multicategory is **weakly representable** if all lists of objects are, and **representable** if additionally composition with $(\mathrm{id}_{A_1}, \ldots, \mathrm{id}_{A_j}, \theta(\overline{B}), \mathrm{id}_{C_1}, \ldots, \mathrm{id}_{C_k})$ induces a bijection between $\mathrm{Hom}(\overline{A}, m(\overline{B}), \overline{C}; D)$ and $\mathrm{Hom}(\overline{A}, \overline{B}, \overline{C}; D)$ for all $\overline{A}, \overline{B}, \overline{C}, D$.
- Representable multicategories are important because they are in equivalence with monoidal categories.

Representable Multicategories

- We can reconstruct the coherences of a monoidal category from the representable multicategory structure, e.g.:

$\alpha_{A_1,A_2,A_3} =$



- We will now define *skew* multicategories, and a *left representability* condition picking out a subcategory of skew multicategories that are in equivalence with skew-monoidal categories.

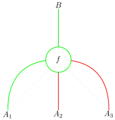
Skew Multicategories

A **skew multicategory** has two types of multimorphism: **tight** and **loose**. Concretely, a skew multicategory is:

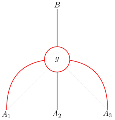
- A class of objects.
- For each object B, a class $\mathrm{Hom}_l(;B)$ of **loose nullary maps** into B.
- For each $n > 0$, and objects $A_1, \ldots, A_n, B$, classes $\mathrm{Hom}_t(A_1, \ldots, A_n; B)$ and $\mathrm{Hom}_l(A_1, \ldots, A_n; B)$ of **tight** and **loose** morphisms, and a function $\lambda_{A_1,\ldots,A_n;B} : \mathrm{Hom}_t(A_1, \ldots, A_n; B) \to \mathrm{Hom}_l(A_1, \ldots, A_n; B)$ allowing us to view tight maps as loose.
- A chosen element $\mathrm{id}_A : \mathrm{Hom}_t(A; A)$
- A composition operation on both tight and loose morphisms, such that $g \circ (f_1, \ldots, f_n)$ is tight exactly when g and f_1 are.

Skew Multicategories

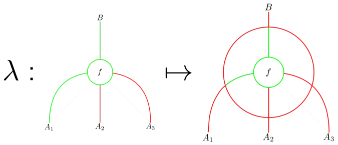
We can graphically represent this structure as follows:



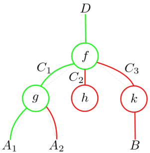
(a) A map in $\mathrm{Hom}_t(A_1, A_2, A_3; B)$



(b) A map in $\mathrm{Hom}_l(A_1, A_2, A_3; B)$



(c) Action of λ



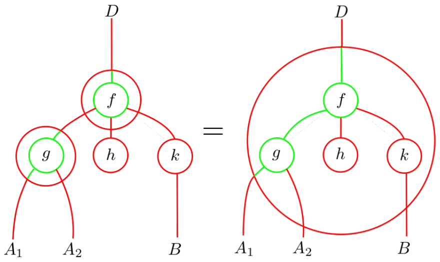
(d) The composite $f(g, h, k)$. Here f and g are tight, and thus the composite is.

Skew Multicategories

subject to:

- Analogues of the unitality and associativity laws.
- Composition commutes with λ, i.e. $(\lambda g) \circ (\lambda f_1, \ldots, f_n) = \lambda(g \circ (f_1, \ldots, f_n))$.

The associativity and unitality are automatic from the graphical representation. The other condition is:



Representable Skew Multicategories

A skew multicategory is **left representable** if for all lists $\overline{A}$ of objects, there are objects $m_t(\overline{A})$ and $m_l(\overline{A})$, and multimorphisms $\theta_t(\overline{A}) \in \mathrm{Hom}_t(\overline{A}; m_t(\overline{A}))$ and $\theta_l(\overline{A}) \in \mathrm{Hom}_l(\overline{A}; m_l(\overline{A}))$ which induce bijections:

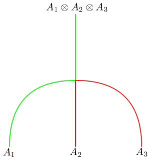
$$\mathrm{Hom}_t(m_t(\overline{A}), \overline{C}; B) \to \mathrm{Hom}_t(\overline{A}, \overline{C}; B)$$

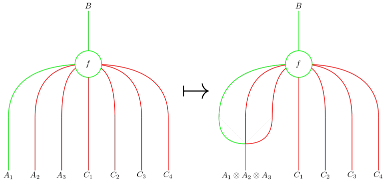
$$\mathrm{Hom}_t(m_l(\overline{A}), \overline{C}; B) \to \mathrm{Hom}_l(\overline{A}, \overline{C}; B)$$

We also write $(A_1 \otimes \cdots \otimes A_n)$ for $m_t(A_1, \ldots, A_n)$ and I for $m_l()$.

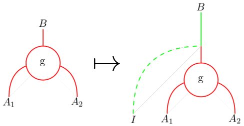
Representable Skew Multicategories

We graphically represent θ_t and the inverse to precomposition with θ_t, a map $\mathrm{Hom}_t(\overline{A}, \overline{C}; B) \to \mathrm{Hom}_t(m_t(\overline{A}), \overline{C}; B)$, as:



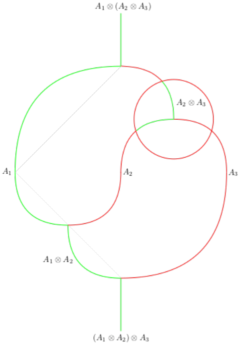


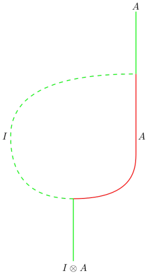
We represent $\theta_I()$ and the inverse to precomposition with $\theta_I()$, a map $\mathrm{Hom}_I(\overline{A}; B) \to \mathrm{Hom}_t(I, \overline{A}; B)$, as:

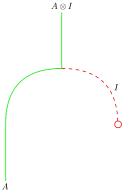


Representable Skew Multicategories

Left representable skew multicategories are in equivalence with skew-monoidal categories. For instance, we can build the associator and unitors as tight morphisms using the above structure:

$\alpha_{A_1,A_2,A_3} =$ 

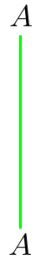
$\lambda_A =$ 

$\rho_A =$ 

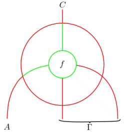
Sequent Calculus

From the definition of left representable skew multicategories, we can derive a sequent calculus for the coherences in a skew-monoidal category. We present these one by one with justification and the corresponding graphical representation. Sequents are of the form $S \mid \Gamma \to A$ where S is the "stoup", either empty or a single object, Γ is a list of objects, and A is an object.

$$\frac{}{A \mid \ \to A}\ \text{id}$$



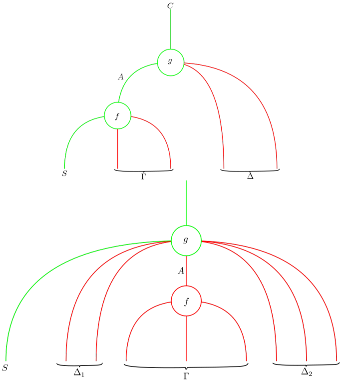
$$\frac{A \mid \Gamma \xrightarrow{f} C}{- \mid A, \Gamma \to C}\ \text{shift}$$



Sequent Calculus

$$\frac{S \mid \Gamma \xrightarrow{f} A \qquad A \mid \Delta \xrightarrow{g} C}{S \mid \Gamma, \Delta \to C}\ \text{scut}$$

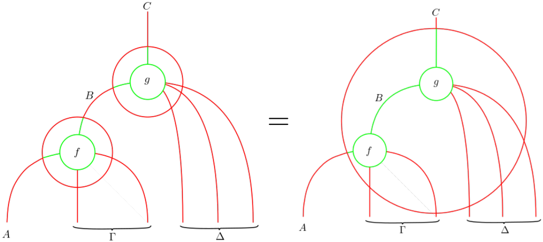
$$\frac{- \mid \Gamma \xrightarrow{f} A \qquad S \mid \Delta_1, A, \Delta_2 \xrightarrow{g} C}{S \mid \Delta_1, \Gamma, \Delta_2 \to C}\ \text{ccut}$$



Sequent Calculus

We impose an equational theory on derivations enforcing the axioms of a skew multicategory, for instance:

$$\dfrac{\dfrac{A \mid \Gamma \xrightarrow{f} B}{- \mid A, \Gamma \to B}\,\text{shift} \qquad B \mid \Delta \xrightarrow{g} C}{- \mid A, \Gamma, \Delta \to C}\,\text{scut} = \dfrac{\dfrac{A \mid \Gamma \xrightarrow{f} B \qquad B \mid \Delta \xrightarrow{g} C}{A \mid \Gamma, \Delta \to C}\,\text{scut}}{- \mid A, \Gamma, \Delta \to C}\,\text{shift}$$

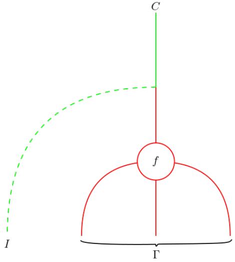


Sequent Calculus

To capture left representable skew multicategories, we add four additional rules.

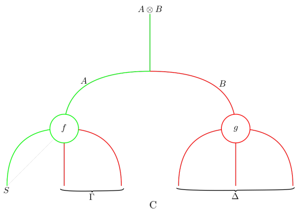
$$\frac{}{- \mid \ \to I}\ \mathsf{IR}$$

$$\frac{- \mid \Gamma \xrightarrow{f} C}{I \mid \Gamma \to C}\ \mathsf{IL}$$

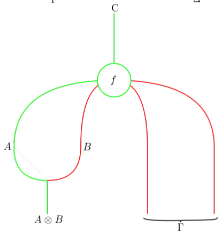


Sequent Calculus

$$\frac{S \mid \Gamma \xrightarrow{f} A \qquad - \mid \Delta \xrightarrow{g} B}{S \mid \Gamma, \Delta \to A \otimes B} \otimes R$$



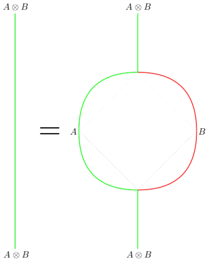
$$\frac{A \mid B, \Gamma \xrightarrow{f} C}{A \otimes B \mid \Gamma \to C} \otimes L$$



Sequent Calculus

These are also subject to equations, like:

$$\dfrac{}{A \otimes B \mid \;\to A \otimes B}\,\mathrm{id} = \dfrac{\dfrac{\dfrac{}{A \mid \;\to A}\,\mathrm{id} \qquad \dfrac{\dfrac{}{B \mid \;\to B}\,\mathrm{id}}{- \mid B \to B}\,\mathrm{shift}}{A \mid B \to A \otimes B}\,\otimes R}{A \otimes B \mid \;\to A \otimes B}\,\otimes L$$



Main Theorem

Theorem

Morphisms from A to B in a free skew-monoidal category on a set of objects are in bijection with equivalence classes of derivations of $A \mid \ \to B$ in the above sequent calculus.

References

[1] John Bourke and Stephen Lack. "Skew monoidal categories and skew multicategories". In: *Journal of Algebra* 506 (July 2018), pp. 237–266. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2018.02.039. URL: http://dx.doi.org/10.1016/j.jalgebra.2018.02.039.

[2] Tarmo Uustalu, Niccolò Veltri, and Noam Zeilberger. *The Sequent Calculus of Skew Monoidal Categories*. 2020. arXiv: 2003.05213 [cs.LO].

[3] Paul Wilson, Dan Ghica, and Fabio Zanasi. *String diagrams for non-strict monoidal categories*. 2022. arXiv: 2201.11738 [math.CT].