Dangerous Ideas

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Group 4: Applications of Skew Monoidal Categories with Niccolo' Veltri

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Talking can be dangerous ...

Two main ideas I had by looking at the papers:

- Profunctors;
- Duality;

and indeed ...

Long story short

Left-skew monoidal and left-skew closed categories can be derived from a left-skew promonoidal category, i.e. a category $\mathbb{C}$ equipped with the following structure:

$$P : \mathbb{C}^{\text{op}} \times \mathbb{C}^{\text{op}} \times \mathbb{C} \to \mathbf{Set}$$

$$J : \mathbb{C} \to \mathbf{Set}$$

$$\lambda_{A,B} : \mathbb{C}(A,B) \to \int^{X} J(X) \times P(X,A;B)$$

$$\rho_{A,B} : \int^{X} P(A,X;B) \times J(X) \to \mathbb{C}(A,B)$$

$$\alpha_{A,B,C,D} : \int^{X} P(A,X;D) \times P(B,C;X) \to \int^{X} P(X,C;D) \times P(A,B;X)$$

satisfying five axioms:

The Five Axioms 1/3

(!) Dropping commas, $\times$ and with "Einstein summation convention"!

$$
\begin{tikzcd}
(AXE)(BYX)(CDY) \arrow[r, "\alpha\times 1"] \arrow[d, "1\times\alpha"] & (XYA)(ABX)(CDY) \arrow[d, "\cong"] \\
(AYE)(XDY)(BCX) \arrow[d, "\alpha\times 1"] & (XYA)(CDY)(ABX) \arrow[d, "\alpha\times 1"] \\
(YDE)(AXY)(BCX) \arrow[r, "1\times\alpha"] & (YDE)(XCY)(ABX)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(AXD)(BCX)\,JB \arrow[r, "\alpha\times 1"] & (XCD)(ABX)\,JB \arrow[d, "1\times\rho"] \\
(AXD)\,JB\,(BCX) \arrow[u, "\cong"] & (XCD)\,\mathbb{C}(A,X) \arrow[d, "\cong"] \\
(AXD)\,\mathbb{C}(C,X) \arrow[u, "1\times\lambda"] \arrow[r, "1\times\alpha"] & (ACD)
\end{tikzcd}
$$

The Five Axioms 2/3

$$
\begin{tikzcd}
JA\,(AXD)\,(BCX) \arrow[r, "1\times\alpha"] & JA\,(XCD)\,(ABX) \arrow[r, "\cong"] & (XCD)\,JA\,(ABX) \\
(AXD)\,JB\,(BCX) \arrow[u, "\cong"] & & (XCD)\,\mathbb{C}\,(A,X) \arrow[u, "1\times\lambda"'] \\
\mathbb{C}\,(X,D)\,(BCX) \arrow[u, "\lambda\times 1"] \arrow[rr, "\cong"] & & (BCD) \arrow[u, "\cong"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(AXD)\,(BCX)\,JC \arrow[r, "\alpha\times 1"] \arrow[d, "1\times\rho"] & (XCD)\,(ABX)\,JC \arrow[d, "\cong"] \\
(AXD)\,\mathbb{C}\,(B,X) \arrow[d, "\cong"] & (XCD)\,JC\,(ABX) \arrow[d, "\rho\times 1"] \\
(ABD) \arrow[r, "\cong"] & \mathbb{C}\,(X,D)\,(ABX)
\end{tikzcd}
$$

The Five Axioms 3/3

$$
\begin{tikzcd}
JA\,(AXB)\,JX \arrow[r, "1\times\rho"] & JA\mathbb{C}\,(A,B) \arrow[d, "\cong"] \\
\mathbb{C}\,(X,B)\,JX \arrow[u, "\lambda\times 1"] \arrow[r, "\cong"] & JB
\end{tikzcd}
$$

Left-skew monoidal and left-skew closed cats

A left-skew monoidal category $(\mathbb{C}, I, \otimes, \lambda, \rho, \alpha)$ is a category $\mathbb{C}$ with a distinguished object I equipped with a monoidal product and three natural transformations

$$\begin{aligned} \lambda_A &: I \otimes A \to A \\ \rho_A &: A \to A \otimes I \\ \alpha_{A,B.C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \end{aligned}$$

satisfying five axioms.

A left-skew closed category is defined by $\left(\mathbb{C}, I, \multimap : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}, j_A, i_A, L^A_{B,C}\right)$, where j_A, i_A and $L^A_{B,C}$ are the following natural transformations:

$$\begin{aligned} j_A &: I \to A \multimap A \\ i_A &: I \multimap A \to A \\ L^A_{B,C} &: B \multimap C \to (A \multimap B) \multimap (A \multimap C) \end{aligned}$$

satisfying five axioms.

(!) No one said they should be isos.

Of left-skew promonoidal, monoidal and closed cats

A left-skew monoidal category induces a left-skew promonoidal structure where

$$JA = \mathbb{C}\,(I, A)$$
$$P\,(A, B; C) = \mathbb{C}\,(A \otimes B, C)$$

A left-skew closed category induces a left-skew promonoidal structure where

$$JA = \mathbb{C}\,(I, A)$$
$$P\,(A, B; C) = \mathbb{C}\,(A, B \multimap C)$$

Viceversa, from a left-skew promonoidal structure is possible to recover either a left-skew monoidal category or a left-skew closed one by imposing the right representability conditions [[ST13], Proposition 22].

(!) It looks like an adjunction, and indeed ... there is an adjunction!

The Eilenberg-Kelly Theorem

Let $\mathbb{C}$ be a category equipped with both a monoidal and a closed structure, i.e. $(\mathbb{C}, I, \otimes, \multimap)$ such that

$$B \multimap - \vdash - \otimes B$$

natural in B, then

$$(\mathbb{C}, I, \otimes) \text{ is monoidal } \iff (\mathbb{C}, I, \multimap) \text{ is closed.}$$

More is actually true ... [[ST13], Proposition 18]:

$$(\mathbb{C}, I, \otimes) \text{ left-skew monoidal } \iff (\mathbb{C}, I, \multimap) \text{ left-skew closed.}$$

Thus, we call a category $(\mathbb{C}, I, \otimes, \multimap)$ left-skew closed monoidal.

On embeddings

What about embedding skew structures into presheaves?

Brillant results from old papers:

- (Day, 1974) An embedding theorem for closed categories;
- (Laplaza, 1977) Embedding of closed categories into monoidal closed categories;
- (Day and Laplaza, 1978) On embedding closed categories.

[[DL78], Theorem 2.4]: Each (small) symmetric closed category $\mathcal{V}$ admits a symmetric closed full embedding into a symmetric monoidal closed category $[\mathcal{A}, \mathbf{Ens}]$.

References

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