Quantum Categories and Bialgebroids

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1 Introduction

Skew monoidal categories are particularly useful to describe complex objects like bialgebroids, bialgebras over a non-commutative ring. Introducing them, Szlachányi proved a useful duality theorem [**SZ**] between certain skew monoidal structures and bialgebroids that substantially simplify the treatment of the related theory. The reasearch in this area started in the late 90's with the investigation of braided monoidal categories by Joyal, Street, Freyd, Yetter, Drinfel'd, Turaev et al. [**JS**] and still is an hot topic as skew objects are of great interests in low dimensional topology, knot theory and theoretical physics, among other fields. The curious reader may want to have a look at some good Poisson geometry ... or deformation quantization. :)

Skew structures were introduced to study bialgebroids, therefore let us have a first glimpse of this mathematical object before diving into details.
In short, a bialgebroid H over a non-commutative ring R is a monoid in the category ${}_{R^e}\mathbf{Ab}_{R^e}$ of R^e-bimodules and a comonoid in the one of R-bimodules, where $R^e := R^o \otimes R$. Therefore, it comes with two main maps, called multiplication and comultiplication:

$$H \otimes_{R^e} H \xrightarrow{m} H \quad \text{in } {}_{R^e}\mathbf{Ab}_{R^e}$$
$$H \xrightarrow{\Delta} H \otimes H \quad \text{in } {}_{R}\mathbf{Ab}_{R}$$

with fairly complex compatibility conditions between the R-coring and R^e-ring structures. The symbol $R^o = (R, c\Delta, \epsilon)$ denotes the right bidual of R, equipped with the comultiplication map Δ of R post-composed with the braiding c.

To establish the "Szlachányi duality", we need to faimiliarize with the notion of right-monoidal category and closedness.

Definition 1.1 *A right-monoidal category (skew) consists of a category M, equipped with a skew product $* : M \times M \to M$ and three natural transformations*

(skew associator, left and right unitors)

$$L * (M * N) \xrightarrow{\gamma} (L * M) * N$$
$$M \xrightarrow{\eta} R * M$$
$$M * R \xrightarrow{\epsilon} M$$

*that are not required to be invertible and that satisfy analogues of the five coherence equations for monoidal categories, defined in [**CWM**].*

Definition 1.2 *A skew-monoidal category $\langle M, *, R, \gamma, \eta, \epsilon \rangle$ is called left(right) closed iff the endofunctor $-*N$ $(N * -)$ has a right adjoint $hom^l(N, -)$ $(hom^r(N, -))$ $\forall N \in M_0$. It is closed if it is both left and right closed.*

For a right bialgebroid H, the above M coincides with $\mathbf{Ab}_R$, the unit is the ring R, the associator corresponds to the Galois map, and the left and right unitors are the counit and the source map of H. More details are given later.

The "Duality Theorem" states that closed right-monoidal structures over $\mathbf{Ab}_R$ with skew-monoidal unit R are in correspondence with right bialgebroids over R. Similarly, closed left-monoidal structures on ${}_R\mathbf{Ab}$ with skew-monoidal unit R are left R-bialgebroids. More precisely:

Theorem 1.1 *[**SZ**] Let R be a ring. Then closed right-monoidal structures $\langle \boldsymbol{Ab}_R, *, R, \gamma, \eta, \epsilon \rangle$ on the category of right R-modules, with unit object being the right-regular R-module, are precisely the right bialgebroids over R.*

This yields a useful result - for the advanced reader:

Corollary 1.1 *A right monoidal category $\langle M, *, R, \gamma, \eta, \epsilon \rangle$ is equivalent to the right-monoidal category of a right-bialgebroid iff*

(i) M is cocomplete Abelian;

(ii) $*$ preserves colimits in both arguments;

(iii) R is a small projective generator.

A natural question to ask at this point is: Do we have a duality between the "standard" monoidal structure (whose product we denote by $\otimes$) and the skew-monoidal one with product $*$? yes!
Consider a monoidal category M equipped with the both structures, then

$$M * N \cong M \otimes TN$$

for T a bimonad, if there exists a tetrahedal isomorphism

$$\phi : L \otimes (M * N) \to (L \otimes M) * N$$

For further details see [SZ, §8].

2 Quantum Categories

Bialgebroids are not the ultimate skew structure that we can describe purely in categorical term. A further enlargement allows to incorporate bialgebroids and, in particular, Hopf bialgebroids as a particular instance: Quantum categories.

In order to describe a quantum category, according to [**DS**] and [**LS**], we need to remind ourselves what a braiding is.

Definition 2.1 *A braided monoidal category $\mathcal{V}$ is a monoidal category $(\mathcal{V}, \otimes, 1)$ with an additional natural swapping isomorphism $c_{A,B} : A \otimes B \to B \otimes A$, called braiding, subject to coherence constraints.*

Remark 2.1 *Recall that a braided monoidal category is called symmetric when $c_{B,A} \circ c_{A,B} = id_{A,B}$. This condition does not hold for braided monoidal categories in general and, further, we have more than one possible canonical automorphism of $A \otimes B$ [*__CWM__*], namely:*

$$id, c^2, c^4, \ldots, c^{-2}, c^{-4}, \cdots : A \otimes B \to A \otimes B$$

At this point you may start picturing images of braids in your brain, and indeed it is exactly the good way to think about braided categories (and friends) .. and, for the ones familiar with physics, yes - as Mac Lane said - " The realization of a braid by twisted strings directly suggests the use of braided categories for string theory in theoretical physics." [**CWM**]

Let $(\mathcal{V}, \otimes, I, c)$ be a braided monoidal category. Assume further that tensoring with an object $V \in \mathcal{V}_0$

$$V \otimes - : \mathcal{V} \to \mathcal{V} \tag{1}$$

preserves equalizers. In other words, $\mathcal{V}$ has equalizers of coreflexive pairs.

The bicategory of comodules over $\mathcal{V}$, denoted by **Comod** $(\mathcal{V})$ has objects comonoids C in $\mathcal{V}$, i.e. objects equipped with co-multiplication map $\delta : C \to C \otimes C$ and co-unit $\epsilon : C \to I$ satisfying the usual axioms.

The 1-cells, denoted by the arrows $\nrightarrow$ are comodules $M : C \nrightarrow D$ from C to D, i.e. they are objects $M \in \mathcal{V}_0$ with coaction $\delta : M \to C \otimes M \otimes D$. It is also possible to see more of the bimodular structure, by composing the coaction with the unit and splitting it in left and right ones, namely:

$$\delta^l : M \to C \otimes M$$
$$\delta^r : M \to M \otimes D$$

The 2-cells

$$F : M \Rightarrow M' : C \nrightarrow D \tag{2}$$

are given by morphisms $F : M \to M'$ respecting the coactions.

The composition operation for comodules $M : C \nrightarrow D$ and $N : D \nrightarrow E$ is the coreflexive equalizer

$$N^o M = M \otimes_D N \longrightarrow M \otimes N \underset{\delta^r \otimes 1}{\overset{1 \otimes \delta^l}{\rightrightarrows}} M \otimes D \otimes N$$

Remark 2.2 *Any comonoid C is a $C \nrightarrow C$ comodule with coaction*

$$C \overset{\delta_3}{\to} C \otimes C \otimes C$$

The identity comodule $1_C : C \nrightarrow C$ is C.

Not it should be clear why we abuse notation: the same notation δ is used for both comultiplication and coaction.

Reminder for the reader: A bicategory is a category weakly enriched over **Cat**.

To see the dual structure of an algebra [**LS**], define the right bidual C^o of C, $C^o := (C, c\delta, \epsilon)$, with unit $e : C \otimes C^o \to I$ and unit $n : I \to C^o \otimes C$. There is a pseudomonoid, or canonical monoidale, structure on $C^e := C^o \otimes C$ with multiplication and unit, respectively:

$$p = 1 \otimes e \otimes 1 : C^e \otimes C^e \to C^e$$
$$j = n : I \to C^e$$

Definition 2.2 *A quantum category (C, A) of $\mathcal{V}$ is a monoidal comonad A on the canonical monoidale (pseudomonoidal)*

$$C^e = C^o \otimes C \tag{3}$$

This yields the existence in **Comod** $(\mathcal{V})$ of an arrow $A : C^e \to C^e$ with 2-cells

$$\epsilon : A \Rightarrow 1_{C^e} \tag{4}$$
$$\delta : A \Rightarrow A \otimes_{C^e} A \tag{5}$$

under some coherence conditions [**LS**] - three comonad axioms plus seven others for the 2-cells

$$\phi_2 : A \otimes A \Rightarrow A$$

$$\begin{tikzcd}
C^e \otimes C^e \arrow[r, "p"] \arrow[d, "A\otimes A"'] & C^e \arrow[d, "A"] \\
C^e \otimes C^e \arrow[r, "p"'] & C^e
\end{tikzcd}$$

and $\phi_0 : j \Rightarrow A$

$$\begin{tikzcd}
I \arrow[rr, "p"] \arrow[dr, "j"'] & & C^e \arrow[dl, "A"] \\
& C^e &
\end{tikzcd}$$

The equivalence between left-skew monoidal structures and the one of quantum categories is stated in the following theorem.

Theorem 2.1 *[**LS**] There is a bijection between quantum category structures in $\mathcal{V}$ on (C, A) and left-skew monoidal structures on $\mathcal{V}$, with tensor product morphism $\bar{A}$ and unit morphism ϵ^* in Comod$(\mathcal{V})$ using the inverse braiding.*

For details, [**LS**].

There is a further angle on the theory of quantum categories, namely to see them as a 2-functor from the 2-category of braided monoidal categories (satisfying certain conditions) and braided strong monoidal functors to the 2-category of categories. This is a perspective investigated in [**CK**].
We have already seen that quantum categories are defined within a monoidal structure. If we take the opposite category of modules over a ring $\mathbf{Mod}_R$, the associated quantum category is indeed a bialgebroid.

Reminder: Let $(\mathcal{V}, \otimes, I, \gamma)$ be a braided monoidal category with finite colimits such that tensoring preserves equalizers of coreflexive pairs. The monoidal bicategory $\mathcal{C}$ has comonoids as objects

$$C = (C, \delta : C \to C \otimes C, \epsilon : C \to I)$$

and morphisms $\mathcal{C}\,(C, D)$ the Eilenberg-Moore coalgebras for the comonad

$$C \otimes - \otimes D : \mathcal{V} \to \mathcal{V} \tag{6}$$

The 1-cells are $C \twoheadrightarrow D$, a comodule from C to D, i.e. an object M and a coaction map $\delta : M \to C \otimes M \otimes D$ with axioms.

To build quantum categories as functor, observe that: [**CK**] A coreflexive-equalizer-preserving braided strong-monoidal functor $\mathcal{V} \to \mathcal{W}$ induces a 2-functor

$$\mathbf{qCat}\,(\mathcal{V}) \to \mathbf{qCat}\,(\mathcal{W}) \tag{7}$$

Moreover, it is a finite limit preserving functor. Therefore:

$$\mathbf{qCat}\,(\mathcal{V} \times \mathcal{W}) \cong \mathbf{qCat}\,(\mathcal{V}) \times \mathbf{qCat}\,(\mathcal{W}) \tag{8}$$

$$\mathbf{qCat}\,(1) \cong 1 \tag{9}$$

To fix the ideas, we discuss some relevant examples.

Example 2.1 *[**CK**] For $(\mathcal{V}, \otimes, I)$, the operation of tensoring and the unit are symmetric monoidal functors*

$$- \otimes - : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \qquad\qquad I : 1 \to \mathcal{V} \tag{10}$$

that induce functors

$$- \otimes - : \mathbf{qCat}\,(\mathcal{V}) \times \mathbf{qCat}\,(\mathcal{V}) \to \mathbf{qCat}\,(\mathcal{V}) \qquad\qquad I : 1 \to \mathbf{qCat}\,(\mathcal{V}) \tag{11}$$

defining a monoidal category on the category of quantum categories over $\mathcal{V}$.

Example 2.2 *[**CK**] Consider the functor*

$$\phi : \mathbf{Set} \to \mathcal{V} \tag{12}$$

$$S \mapsto S \cdot I \tag{13}$$

taking copowers of the unit, if they exist. It is strong monoidal, whenever the distributivity law is satisfied. Any coreflexive equalizer -not involving the empty set - in **Set** *is split, therefore preserved by any functor. We have thus an induced functor*

$$\phi^* : \mathbf{qCat}\,(\mathbf{Set}) = \mathbf{Cat} \to \mathbf{qCat}\,(\mathcal{V}) \tag{14}$$

Example 2.3 *[**CK**] Consider* $\mathbf{Fam}\,(\mathcal{V})$ *with objects pairs of a set and a family f objects of $\mathcal{V}$ indexed by the set, i.e. $(S, \{A_s\})$ and morphisms*

$$(f, \{\phi_s\}) : (S, \{A_s\}) \to (S', \{A'_s\})$$

where $f : S \to S'$ is a map and $\forall s \in S$ a $\mathcal{V}$ - morphism $\phi_s : A_s \to A_{f(s)}$.
If $\mathcal{V}$ has a monoidal structure, then also the category of families has and the functor

$$\psi : \mathbf{Fam}\,(\mathcal{V}) \to \mathcal{V}$$

$$(S, \{A_s\}) \mapsto \coprod_s A_s$$

is monoidal and preserves coreflexive equalizers. It results in an induced functor

$$\mathbf{qCat}\,(\boldsymbol{Fam}\,(\mathcal{V})) \to \mathbf{qCat}\,(\mathcal{V}) \tag{15}$$

In particular, by taking the indexing set $\boldsymbol{Fam}\,(\mathcal{V}) \to \mathbf{Set}$, we have an induced functor

$$\mathbf{qCat}\,(\mathbf{Fam}\,(\mathcal{V})) \to \mathbf{Cat} \tag{16}$$

to the category of all small categories.

3 Bialgebroids Reloaded

The previous example is of great interest, as leads to bialgebroids. For it, consider $G \in \mathbf{Grp}_0$ a group, the data of Hopf G-coalgebra are a family of algebras indexed by the group $\{A_g\}$, a family of linear maps $\{A_{gh} \to A_g \otimes A_h\}$ with an antipode map satisfying axioms.

Excluding the antipode map, the Hopf structure just described is an object $\mathbf{qCat}\left(\mathbf{Fam}\left(\mathbf{Vect}\right)\right)^{\mathrm{op}}$ with the object of objects $(1, \{I\})$ and the one of morphisms $(G, \{A_g\})$. The underlying category is G.

By equation (16), from a Hopf group coalgebra we obtain a bialgebroid, i.e. a quantum category in $\mathbf{Vect}^{\mathrm{op}}$. If it is equipped with an antipode map, we have back the Hopf algebroid.

We can analyze now more in detail the structure of a bialgebroid mentioned at the very beginning of this post [SZ, §3]:
Consider a right R-bialgebroid with R^e-ring and R-coring structure

$$\begin{aligned} t^H \otimes s^H &: R^e \to H \\ \Delta^H &: H \to H \otimes_{R_1} H \end{aligned}$$

with unit 1^H and counit ϵ^H. The symbol $\otimes_{R_i}$ denotes the tensoring over R with respect to the left action λ_i defined below.
On H we have four actions of R, namely:

$$\begin{aligned} \lambda_1\left(r\right)\left(h\right) &:= ht^H\left(r\right) & \qquad \rho_1\left(r\right)\left(h\right) &:= t^H\left(r\right)h \\ \lambda_2\left(r\right)\left(h\right) &:= s^H\left(r\right)h & \qquad \rho_2\left(r\right)\left(h\right) &:= hs^H\left(r\right) \end{aligned}$$

for $r \in R$ and $h \in H$.

Define the skew-product as follows:

$$M * N := M \otimes_{R_1} \left(N \otimes_{R_2} H\right)$$

where $M, N \in \mathbf{Ab}_R$ right R-modules. Notice that $M * N \in \mathbf{Ab}_R$ with right-action ρ_2 of R over H and its elements are denoted by square brackets

$$[m, n, h] = m \otimes (n \otimes h)$$

Compatibility with the actions is given by

$$\begin{aligned} [m \cdot r, n, h] &= \left[m, n, ht^H\left(r\right)\right] \\ [m, n \cdot r, h] &= \left[m, n, s^H\left(r\right)h\right] \\ [m, n, h] \cdot r &= \left[m, n, hs^H\left(r\right)\right] \end{aligned}$$

yielding the three skew-monoidal natural transformations

$$\begin{array}{lll} \eta_M : M \to R * M & \epsilon_M : M * R \to M & \gamma_{L,M,N} : L * (M * N) \to (L * M) * N \\ m \mapsto \left[1^R, m, 1^H\right] & [m, r, h] \mapsto m \cdot \epsilon^H \left(s^H (r) h\right) & [l, [m, n, g], h] \mapsto \left[\left[l, m, h^{(1)}\right], n, gh^{(2)}\right] \end{array}$$

Therefore, $\langle \mathbf{Ab}_R, *, R_R, \gamma, \eta, \epsilon \rangle$ is a right-monoidal category and the skew-associator on R, i.e. $\gamma_{R,R,R}$ is the Galois map

$$H \otimes_{R_2} H \to H \otimes_{R_1} H$$
$$g \otimes h \mapsto h^{(1)} \otimes gh^{(2)}$$

of H as left H-comodule algebra.
The bialgebroid structure just presented describes an Hopf algebroid when γ is invertible.

To sum up: Quantum categories are skew-monoidal objects of a monoidal bicategory and Hopf skew-monoidal ones are quantum groupoids.

Definition 3.1 *[**DS**] A quantum category over $\mathcal{V}$ is basic data $C, h : C^o \otimes C \to A$ in $\mathbf{Comod}(\mathcal{V})^{co}$. A quantum groupoid over $\mathcal{V}$ is Hopf basic data in $\mathbf{Comod}(\mathcal{V})^{co}$.*

where $\mathbf{Comod}(\mathcal{V})^{co}$ is the bicategory $\mathbf{Comod}(\mathcal{V})$ with 2-cells reversed.

Thankks! I hope you enjoyed reading! :)

References

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