DIVISION ALGEBRAS AND SUPERSYMMETRY

AN INTRODUCTION

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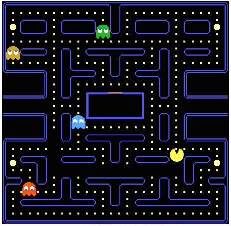
Question #1

What structure we need to define a geometry essentially ?



- A vector space V over a field $\mathbb{K}$;
- A notion of length, i.e. a bilinear form (positive-definite).

Algebras

Definition (Real Algebra)

An algebra A over $\mathbb{R}$ is a real vector space, together with a bilinear form

$$b : A \times A \to A \tag{1}$$

that is distributive, but not necessarily commutative or associative.

Definition (Division Algebra)

An algebra A is called division algebra if it does not contain zero divisors, id est:

$$x \cdot y = 0 \ \Rightarrow x = 0 \ \ or \ \ y = 0 \qquad \forall\ x, y \in A \tag{2}$$

Definition (Normed Algebra)

A normed algebra is a real algebra A with a multiplicative unit and norm $||\cdot||$ such that

$$||x \cdot y|| = ||x|| \cdot ||y|| \qquad \forall\ x, y \in A \tag{3}$$

(!) Every normed algebra is a division algebra.

Hopf spaces

Definition (H-Space)

A space X is an H-space if there is a continuous multiplication map $\mu : X \times X \to X$ and an identity element $e \in X$ such that the two maps $x \to \mu(x, e)$ $x \to \mu(e, x)$ are homotopic to the identity through maps $(X, e) \to (X, e)$. In particular, $\mu(e, e) = e$.

(!) This is weaker than having a topological group structure, as associativity and existence of multiplicative inverses are not required properties.
The topological groups are H-spaces, in fact they are defined as topological spaces with a multiplication and inverse maps both continuous. In addition, associativity holds.

Example

The linear groups $GL_n(\mathbf{.})$ on $\mathbb{R}, \mathbb{C}, \mathbb{H}$, topologised as subspaces of $M_k(\mathbf{.})$, are Lie groups. Since they are opens of Euclidean spaces, they are not compact.
However, they are of the same homotopy type of the compact Lie groups $O(n), U(n), Sp(n)$.

Example

Only S^1 and S^3 are topological groups, since associativity on Cayley octonions $\mathbb{O}$ fails, but S^1, S^3 and S^7 are H-spaces, restricting the ambient space multiplications on the unit spheres.

Parallelizability

Definition (Parallelizable manifold)

A differentiable manifold M is said to be parallelizable iff there exists a linear diffeomorphism $t : M \times \mathbb{R}^n \to TM$ such that $t|_{\{x\}\times\mathbb{R}^n} : \{x\} \times \mathbb{R}^n \to T_x M$, called trivialization. Thus, $TM \cong M \times \mathbb{R}^n$.
Equivalently:
A differentiable manifold M $\subset \mathbb{R}^n$ is said to be parallelizable if there exist a global frame, namely a family $\{\xi_1, ..., \xi_n\}$ of linearly independent vectors being a basis for $T_x M \quad \forall x \in M$.

Proposition

If the oriented vector bundle ξ possesses a nowhere zero cross-section, then the Euler class $e(\xi)$ must be zero.

Corollary

If M compact manifold is parallelizable then $\chi(M) = 0$.

Example

The 2-dimensional sphere S^2 is not parallelizable. Indeed:

$$\chi(S^2) = 2 - 2 \cdot g + 2 \cdot 0 = 2$$

Parallelizability and H-Spaces

Lemma

If $\mathbb{R}^n$ is a division algebra, or if S^{n-1} is parallelizable, then S^{n-1} is an H-space.

Proof.

If $\mathbb{R}^n$ is a division algebra, we obtain on S^{n-1} a well-defined continuous multiplication map

$$\mu(x, y) := \frac{x \cdot y}{||x \cdot y||} \tag{4}$$

with two-sided multiplicative inverse (see Remark 1.1.3) . Thus, S^{n-1} is an H-space, by definition. Suppose S^{n-1} is parallelizable with basis $\{\xi_1, ..., \xi_{n-1}\}$ for $T_x M \quad \forall x \in M$, then the Gram-Schmidt orthonormalisation process on $\{x, \xi_1(x), ..., \xi_{n-1}(x)\}$ gives an orthonormal frame at x. We center this basis at $\hat{e}_1$ and assume it coincides with the canonical basis,$\{\hat{e}_i\}_{i=2,\dots,n}$, up to a change of sign and a local deformation.
Let $\alpha_x \in$ SO(n) the rotation into $\{x, \xi_1(x), ..., \xi_{n-1}(x)\}$ of the standard basis $\{\hat{e}_i\}_{i=1,\dots,n}$, then the map

$$\mu(x, y) := \alpha_x(y) \tag{5}$$

defines an H-space structure on S^{n-1} , with identity element $\hat{e}_1$. □

(!) Take this in mind, we will use it later ...

THE RING STRUCTURE OF K-THEORY

We recall some relevant results already obtained:

THEOREM (REDUCED BOTT PERIODICITY THEOREM)

Given X a compact T_2 space, the homomorphism

$$\beta : \tilde{K}(X) \to \tilde{K}\left(S^2 X\right) \tag{6}$$

$$\beta(\alpha) = (H - 1) * \alpha \tag{7}$$

is an isomorphism.

COROLLARY

The reduced K-theory of spheres is:

$$\tilde{K}\left(S^k\right) = \begin{cases} \mathbb{Z} & k = 2n : n \in \mathbb{Z} \\ \{0\} & k \text{ odd} \end{cases} \tag{8}$$

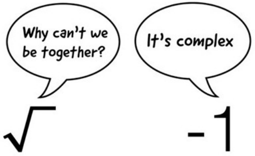
Moreover, the generator for the reduced K-theory of even spheres is given by:

$$(H - 1) * \cdots * (H - 1) \tag{9}$$

(!) H denotes the canonical line bundle over S^2, $\mathbb{CP}^1$ and $S^2 X$ the double suspension of X.

Question #2

How did we obtained the complex numbers starting from the reals (algebraically) ?



Taking an element with no roots inside the reals and going quotient, i.e. imposing a relation ...

$$\phi : \frac{\mathbb{R}[x]}{(x^2+1)} \to \mathbb{C} \tag{10}$$

(!) For this point of view in detail, see Michael Artin, Algebra.

The Fundamental Product Theorem - 1/2

Theorem (The Fundamental Product Theorem)

For every compact, T_2 space X, the homomorphism of rings

$$\mu : K(X) \otimes \frac{\mathbb{Z}[H]}{(H-1)^2} \to K(X) \otimes K\left(S^2\right) \to K\left(X \times S^2\right) \tag{11}$$

is an isomorphism of rings.

Corollary

The following isomorphism of rings holds:

$$\phi : \frac{\mathbb{Z}[H]}{(H-1)^2} \to K\left(S^2\right) \tag{12}$$

Corollary

For every integer k,

$$\tilde{K}\left(S^{2k}\right) \otimes \tilde{K}(X) \cong \tilde{K}\left(S^{2k} \wedge X\right) \tag{13}$$

$$K\left(S^{2k}\right) \otimes K(X) \cong K\left(S^{2k} \times X\right) \tag{14}$$

The Fundamental Product Theorem - 2/2

Remark

In particular, if $X = S^{2l}$, we obtain the following:

$$K\left(S^{2k} \times S^{2l}\right) \cong K\left(S^{2k}\right) \otimes K\left(S^{2l}\right) \tag{15}$$

Therefore

$$K\left(S^{2k} \times S^{2l}\right) \cong \frac{\mathbb{Z}[\alpha]}{(\alpha^2)} \times \frac{\mathbb{Z}[\beta]}{(\beta^2)} \cong \frac{\mathbb{Z}[\alpha, \beta]}{(\alpha^2, \beta^2)} \tag{16}$$

where α and β are the pullbacks of generators of $\tilde{K}\left(S^{2k}\right), \tilde{K}\left(S^{2l}\right)$ under the canonical projections.

Thus, we choose $\{1,\ \alpha,\ \beta,\ \alpha\beta\}$ as basis for $K\left(S^{2k} \times S^{2l}\right)$.

(!) Why? Because every other multiple of these elements gets reduced either to this set or to zero, via the relations we have on the quotient in (16), i.e. going module α^2, β^2.

Adams Operation

We summarize here the main features of the power operation introduced by Adams on the K-theory of a compact, T_2 space. This ring homomorphism is one of the fundamental result in the theory and will allow us to finish the proof of Hurwitz Theorem.

Theorem

If X is a T_2, compact topological space, there exists a ring homomorphism $\psi^k : K(X) \to K(X)$, for all $k \geq 0$, with the following properties:

1. $\phi^k f^* = f^* \psi^k$ *for all morphism* $f : X \to Y$.
2. $\psi^k (L) = L^k$, *if L line bundle.*
3. $\psi^k \circ \psi^l = \psi^{kl}$
4. $\psi^p (\alpha) \overset{(p)}{\equiv} \alpha$, *if p prime.*

Proposition

$\psi^k : K\left(S^{2n}\right) \to K\left(S^{2n}\right)$ *is multiplication by k^n, i.e.*

$$\psi^k (\alpha * \beta) = k^n (\alpha * \beta)$$

(!) Take this in mind, we will use it later ...

Proving Hurwitz Theorem via Adams Theorem

We will prove the classical result by Hurwitz[HW], using the geometrical point of view in Hatcher [H1].

Theorem

The following statements are true only for $n = 1, 2, 4$ and 8:

- $\mathbb{R}^n$ *is a division algebra.*
- S^{n-1} *is parallelizable.*

via

Theorem (Adams Theorem)

There exist a function $f : S^{4n-1} \to S^{2n}$ *such that* $Hp(f) = \pm 1$ *only if* $n \in \{1, 2, 4\}$.

The link is provided by H-Spaces.

Remark

If $\mathbb{R}^n$ *is a division algebra, or if* S^{n-1} *is parallelizable, then* S^{n-1} *is an H-space.*

Proving Hurwitz Theorem via Adams Theorem - Even Spheres

Proposition (Even Spheres)

If $k > 0$, S^{2k} is not an H-space.

Proof.

If $\exists\ \mu : S^{2k} \times S^{2l} \to S^{2k}$, H-space multiplication, $\Rightarrow \mu^* : K\left(S^{2k}\right) \to K\left(S^{2k} \times S^{2l}\right)$ is an homomorphism of K-rings. From remark 1.3.2,

$$\mu^* : \frac{\mathbb{Z}[\gamma]}{(\gamma^2)} \to \frac{\mathbb{Z}[\alpha, \beta]}{(\alpha^2, \beta^2)} \tag{17}$$

Assume it is written in terms of the basis as

$$\mu^*(\gamma) \ = \ \alpha + \beta + m\alpha\beta : m \in \mathbb{Z} \tag{18}$$

The identity on S^{2k} can always be obtained as $S^{2k} \overset{i}{\hookrightarrow} S^{2k} \times S^{2l} \xrightarrow{\mu} S^{2k}$, where i is the inclusion as first or second factor, i.e. as $S^{2k} \times \{e\}$ or $\{e\} \times S^{2k}$, e multiplicative H-identity. Therefore, $i^*(\alpha) = \gamma$ and $i^*(\beta) = 0$ (or viceversa). and similarly for β. $\Rightarrow$ the coefficient of α in (18) is 1 and similarly for β.
This leads however to a contradiction, since $\gamma^2 \equiv 0$, but, as μ^* is a ring morphism:

$$0 = \mu^*\left(\gamma^2\right) \ = (\alpha + \beta + m\alpha\beta)^2 \equiv 2\alpha\beta \neq 0 \qquad \Rightarrow\Leftarrow$$

PROVING HURWITZ TH. VIA ADAMS TH. - ODD SPHERES - 1/2

PROPOSITION (ODD SPHERES)

If n even: $n \notin \{2,4,8\}$, S^{2n-1} is not an H-space.

PROOF.

Consider a function $g : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$ and observe the following:

$$S^{2n-1} \;=\; \partial(D^n \times D^n) = (\partial D^n \times D^n) \cup (D^n \times \partial D^n)$$

We set $\hat{g} : S^{2n-1} \longrightarrow S^n$ to be the continuous, well-defined map

$$\hat{g}\,(x,y) = \begin{cases} ||y|| \cdot g\left(x, \dfrac{y}{||y||}\right) & \text{on } {D^n}_+ \\ ||x|| \cdot g\left(\dfrac{x}{||x||}, y\right) & \text{on } {D^n}_- \end{cases}$$

such that $\hat{g} \equiv g$ on $S^{n-1} \times S^{n-1}$.
For $n = 2k : k \in \mathbb{Z}$,

$$f := \hat{g} : S^{4k-1} \longrightarrow S^{2k} \tag{19}$$

$$C_f := S^{2k} \cup_f \left\{e^{4k}\right\} \;\Rightarrow \frac{C_f}{S^{2k}} \cong S^{4k} \tag{20}$$

As $\tilde{K}^1\left(S^{4k}\right) = \tilde{K}^1\left(S^{2k}\right) = 0$, the S.E.S. of the pair $\left(C_f, S^{2k}\right)$, becomes

Proving Hurwitz Th. via Adams Th. - Odd Spheres - 2/2

$$0 \to \tilde{K}\left(S^{4k}\right) \to \tilde{K}\left(C_f\right) \to \tilde{K}\left(S^{2k}\right) \overset{(16)}{\cong} \frac{\mathbb{Z}\left[\beta\right]}{\left(\beta^2\right)} \to 0 \tag{21}$$

If the generator of $\tilde{K}\left(S^{4k}\right)$ is mapped into α and an element $\beta \in \tilde{K}\left(C_f\right)$ into the generator of $\tilde{K}\left(S^{2k}\right)$.

By exactness and modular arithmetic, we conclude $\beta^2 = h \cdot \alpha$, for some integer h called **Hp(f), Hopf invariant of f**. The Hopf invariant is well-defined and unique. Observe that β is unique module α, and

$$(\beta + m \cdot \alpha)^2 = \alpha^2 + \beta^2 + 2m\alpha\beta \overset{(\alpha^2)}{\equiv} \beta^2 + 2m\alpha\beta \tag{22}$$

We want to show $\alpha\beta = 0$. Since α maps to 0 in $\tilde{K}\left(S^{2k}\right)$, so does the product, i.e.

$\alpha\beta = k \cdot \alpha \ : \ k \in \mathbb{Z} \ \Rightarrow \ k \cdot \alpha\beta = \alpha\beta^2 = \alpha\left(h\alpha\right) = h \cdot \alpha^2 \overset{(\alpha^2)}{\equiv} 0. \ \Rightarrow \ k\alpha\beta = 0 \Rightarrow \alpha\beta = 0$ as the image of $\tilde{K}\left(S^{2k}\right)$ is an infinite cyclic subgroup of $\tilde{K}\left(C_f\right)$

Lemma

If $g : S^{2n-1} \times S^{2n-1} \longrightarrow S^{n-1}$ is an H-space multiplication, the associated map $\hat{g} : S^{4n-1} \longrightarrow S^{2n}$ has Hopf invariant ± 1.

The following Theorem will conclude the proof of the proposition.

Adams Theorem - 1/2

Theorem (Adams Theorem)

There exist a function $f: S^{4n-1} \to S^{2n}$ *such that* $Hp(f) = \pm 1$ *only if* $n \in \{1, 2, 4\}$.

Proof.

Consider α, β in $\tilde{K}(C_f)$ as we have already done introducing $f := \hat{g} : S^{4k-1} \longrightarrow S^{2k}$.
By the prop. in slide 11,

$$\psi^k(\alpha) = k^{2n} \cdot \alpha \tag{23}$$

$$\psi^k(\beta) = k^n \cdot \beta + \mu_k \cdot \alpha \ : \mu_k \text{ in } \mathbb{Z}. \tag{24}$$

as the first refers to the 4n-dimensional sphere.
Thus,

$$\psi^k \psi^l(\beta) = \psi^k(l^n \beta + \mu_l \alpha) = k^n l^n \beta + \left(k^{2n} \mu_l + l^n \mu_k\right) \alpha \tag{25}$$

Observe that, by property 3. of Adams operation,

$$\left(k^{2n} \mu_l + l^n \mu_k\right) \overset{3.}{=} \left(l^{2n} \mu_k + k^n \mu_l\right) \iff \left(k^{2n} - k^n\right) \mu_l = \left(l^{2n} - l^n\right) \mu_k \tag{26}$$

By property 4. [mod a prime number],

$$\psi^2(\beta) \overset{(2)}{\equiv} \beta^2 = h\alpha \overset{eq.(24)}{=} 2^n \beta + \mu_2 \alpha \quad \Rightarrow \mu_2 \overset{(2)}{\equiv} h \quad \text{if } h = \pm 1 \text{ or odd}. \tag{27}$$

Adams Theorem - 2/2

By property 4. [mod a prime number],

$$\psi^2(\beta) \overset{(2)}{\equiv} \beta^2 = h\alpha \overset{eq.(24)}{=} 2^n\beta + \mu_2\alpha \quad \Rightarrow \quad \mu_2 \overset{(2)}{\equiv} h \quad \textit{if } h = \pm 1 \textit{ or odd}. \tag{28}$$

For $l = 3$, $k = 2$, (24) becomes:

$$\left(2^{2n} - 2^n\right)\mu_3 = \left(3^{2n} - 3^n\right)\mu_2 \quad \Longleftrightarrow \quad 2^n(2^n - 1)\mu_3 = 3^n(3^n - 1)\mu_2 \tag{29}$$

Thus $2^n \mid 3^n(3^n - 1)\mu_2$. Since 3^n and μ_2 are odd,

$$2^n \mid (3^n - 1) \tag{30}$$

Lemma

If $n \in \{1, 2, 4\}$, *then* $2^n \mid (3^n - 1)$.

□

Question #3

What is supersymmetry and why is so relevant (ideas)?

Hint ▶

Hint

- Supersymmetry is a principle, not a particular theory;
- A theory is said to be supersymmetric if it has the same number of fermions and bosons;
- To each particle is associated a superpartner, example: electron - "selectron";
- The existence of these supersymmetric particles may solve a lot of open issues in physics (see the links for more).

Link 1 Link 2 Link 3

Simple Yang-Mills Theories

Definition

A Yang-Mills theory is said to be simple if each gauge boson has just one super-partner and there are no additional scalar fields.

In order to find all the possible simple YM theories, we restrict to dimensions d in which the degrees of freedom of a vector and a spinor coincides.
A massless vector field has therefore d - 2 physical modes, whereas for a spinor:

$$d.o.f. = \begin{cases} 2^{\frac{d}{2}-1} & \text{d even} \\ 2^{\frac{d-1}{2}} & \text{d odd} \end{cases}$$

as in the first case we have two inequivalent spinor representations.
Only one half of these components is actually independent, since Majorana reality constraints have to be taken into account as well.

Therefore, the only possible dimensions are d = 3, 4, 6 and 10, for Majorana, Majorana-Dirac, Weyl and Majorana-Weyl spinors respectively, with corresponding physical degrees of freedom n = 1, 2, 4, 8.

Thus, we have equality of fermionic and bosonic d.o.f. for $n = d - 2$.

Existence of a SYMT - 1/2

Theorem

There exists a supersymmetric YM-theory if and only if n = d - 2.

Proof.

Suppose there exists a SYMT [(32), (33), (34)], we want to show that this implies $n = d - 2$. Consider the following Lagrangian

$$L \;=\; -\frac{1}{4} F_{mn} F^{mn} \;+\; \frac{1}{2}\; i\bar{\psi}\gamma^m \nabla_m \psi \tag{31}$$

and assume we are working in a flat space with metric $\eta = diag\,(-1, 1, \ldots, 1)$ and Dirac matrices such that $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$, the superysymmetric transformations are defined as:

$$\delta\,(\epsilon)\,A_m = \frac{1}{2}\;i\left(\bar{\epsilon}\gamma_m\psi - \bar{\psi}\gamma_m\epsilon\right) \tag{32}$$

$$\delta\,(\epsilon)\,\psi = \frac{1}{2}\;F_{mn}\gamma^{mn}\epsilon \tag{33}$$

Thus

$$\delta\,(\epsilon)\,L = -\frac{1}{4}\;\bar{\psi}\gamma^m\left[\bar{\epsilon}\gamma_m\psi \;-\; \bar{\psi}\gamma_m\epsilon, \psi\right] \tag{34}$$

L is invariant when the R.H.S. vanishes for arbitrary anticommuting ψ and ϵ of the type considered above.

Existence of a SYMT - 2/2

There is a unified way to discuss supersymmetric YM theories, making use of a real spinor Ψ and matrices $\Gamma^m{}_{\alpha\beta}$, $\tilde{\Gamma}^{m\alpha\beta}$ such that

$$\Gamma_m \tilde{\Gamma}_n = \Gamma_n \tilde{\Gamma}_m = 2\eta_{mn} \tag{35}$$

in terms of which (31), (32) and (33) become

$$L = -\frac{1}{4} F_{mn} F^{mn} + \frac{1}{2} i \Psi^T \Gamma^m \nabla_m \Psi \tag{36}$$

$$\delta(\epsilon) A_m = i \epsilon^T \Gamma_m \Psi \tag{37}$$

$$\delta(\epsilon) \Psi = \frac{1}{2} F_{mn} \Gamma^{mn} \epsilon \tag{38}$$

where $\Gamma_{mn} = \tilde{\Gamma}_{[m} \Gamma_{n]}$ is the generator of the Lorentz transformation on Ψ.
The condition for invariance is therefore reduced to

$$\Gamma_{m\alpha(\beta} \Gamma^m{}_{\gamma\delta)} = 0 \qquad (!)\ (!)\ (!) \tag{39}$$

Contracting (39) with $\tilde{\Gamma}^{l\gamma\delta}$ and using (35), we obtain:

$$tr\left(\tilde{\Gamma}^l \Gamma^m\right) \Gamma_{m\alpha\beta} + 2 \cdot \left(\Gamma_m \tilde{\Gamma}^l \Gamma^m\right)_{\alpha\beta} = 0 \quad \Longrightarrow \quad n = d - 2 \tag{40}$$

where Ψ has dimension 2n.
This is exactly the condition of equality of bosonic and fermionic degrees of freedom. $\Diamond$

For the other direction, we have to introduce further tools.

SYMMETRIES

Let Q_α denote the generator of the transformations in (3.8), (3.9) , i.e. $\delta(\epsilon) = \epsilon^\alpha Q_\alpha$, the supersymmetry algebra

$$\left\{ Q_\alpha,\ Q_\beta \right\} = 2\Gamma^m{}_{\alpha\beta} P_m \tag{41}$$

can be verified up to field equations and gauge transformations from the invariant condition. It is Lorentz covariant and can have automorphisms of the form

$$\begin{aligned} &Q \to g \cdot Q \quad \text{for some matrix g} \\ &P \to P \end{aligned}$$

i.e. the Lagrangian has global symmetries

$$\begin{aligned} \Psi &\to g^T \cdot \Psi \\ A &\to A \end{aligned}$$

because they both amount to an invariance

$$g\ \Gamma^m g^T = \Gamma^m \tag{42}$$

REMARK

In dimension 4 there is a $U(1)$ symmetry, for $n = 6$ an $SU(2)$ one. These are the only internal symmetries possible in the four cases of interest.

Light-cone decomposition

Consider the light-cone decomposition of a vector $U_\pm = U_0 \pm U_{d-1}$ and the helicity group $SO\,(d-2)$ fixing the $+$ direction, the gauge field and its partner spinor reduce to

$$A_m = (A_+, A_-, A_i) \qquad \Psi = \begin{pmatrix} W^\alpha \\ W^{\dot\alpha} \end{pmatrix} \tag{43}$$

where $i \ \in \ \{1, \ldots, d-2\}$ refers to the vector representation V on SO(d - 2) and $\alpha, \dot\alpha \ \in \ \{1, \ldots, n\}$ to the spinor representations $S_\pm$, defined as eigenspaces of $\Gamma_\pm$, generator of boosts along the $(d-1)$ - axis.
We obtain an equivalent formulation of the invariance in (39) as

$$\gamma_{i\alpha\dot\alpha}\gamma_{i\beta\dot\beta} + \gamma_{i\beta\dot\alpha}\gamma_{i\alpha\dot\beta} = 2\delta_{\alpha\beta}\delta_{\dot\alpha\dot\beta} \quad (!) \tag{44}$$

V and $S_\pm$ carry representations of G = SO(d - 2) $\times$ I, where I denotes the internal symmetries group.

The superalgebra decomposes into:

$$\left\{ Q_\alpha, \ Q_\beta \right\} = 2 \ \delta_{\alpha\beta} P_+ \qquad \left\{ Q_\alpha, \ Q_{\dot\beta} \right\} = 2 \ \gamma_{i\alpha\dot\beta} P_i \qquad \left\{ Q_{\dot\alpha}, \ Q_{\dot\beta} \right\} = 2 \ \delta_{\dot\alpha\dot\beta} P_- \tag{45}$$

Thus, G consists of the linear transformations $Q_\alpha, \ Q_{\dot\alpha}$ and P_i that leave these expressions invariant.

Light-cone Gauge

The problem can be further reduced considering the light-cone gauge $A_+ = 0$ and observing that in the equations of motion the vector A_i and the spinor $W^{\dot{\alpha}}$ can be regarded as the only dynamical degrees of freedom. These physical fields are related, in terms of the generator, by:

$$Q_\alpha A_i = i\gamma_{i\alpha\dot{\alpha}} W_{\dot{\alpha}} \qquad Q_\alpha W_{\dot{\alpha}} = -\,\gamma_{i\alpha\dot{\alpha}} \partial_+ A_i \tag{46}$$

Consider any real commuting spinor ξ_α in S_+ such that $\xi_\alpha \xi_\alpha = 1$.
The matrix $\Xi := \gamma_{i\alpha\dot{\alpha}}\xi_\alpha$ defines a map from V to S_- such that

$$\Xi^T \cdot \Xi = 1 \tag{47}$$

we will have a right inverse as well if the dimensions of the two spaces coincide, i.e.

$$\Xi \cdot \Xi^T = 1 \quad \Longrightarrow \quad \gamma_{i\alpha\dot{\alpha}}\ \gamma_{j\beta\dot{\beta}}\ \xi_\alpha\ \xi_\beta = \delta_{ij} \tag{48}$$

and this is equivalent to (44). Thus:

$$n = d - 2 \quad \Longrightarrow \quad \Gamma_{m\alpha(\beta}\ \Gamma^m{}_{\gamma\delta)} = 0 \tag{49}$$

i.e. we have an associated SYMT.

□

Remark

This always occur when considering spinors of an even orthogonal group, but is rare to have it for both vector and spinor representations.

The conjugation operation

Definition (Conjugation map)

Let A be a real algebra, a map $x \to \bar{x}$ is called conjugation if the following properties are verified:

$$\overline{(\lambda x = \mu y)} = \lambda \bar{x} + \mu \bar{y} \qquad \overline{x \cdot y} = \bar{x} \cdot \bar{y} \qquad \forall\, x, y \in A,\ \forall\, \lambda, \mu \in \mathbb{R} \tag{50}$$

Given an algebra A of dimension n, with conjugation map as above, $A \times A$ is a 2n-dimensional algebra with the following inherited operations: (!) Cayley-Dickson construction

$$(a, b) \cdot (c, d) = \left(ac - d\bar{b}, da + b\bar{c}\right) \qquad \forall\, a, b, c, d \in A \quad \text{(multiplication)} \tag{51}$$

$$\overline{(a, b)} = (\bar{a}, -b) \qquad \forall\, a, b \in A \quad \text{(conjugation)} \tag{52}$$

To obtain a normed division algebra, consider the following inner-product:

$$\frac{1}{2}\left(\bar{x}y + \bar{y}x\right) = \langle x, y \rangle 1 \tag{53}$$

and choose an orthonormal basis $\{e_0, e_1, \dots, e_{n-1}\}$, for n = 1,2,4,8.

Proposition

The norm induced by (53) is multiplicative if the algebra A is associative.

Corollary

The induced norm on $\mathbb{R}, \mathbb{C}, \mathbb{H}$ is multiplicative, i.e. they are normed division algebras.

Quaternions and Octonions

Let us focus a bit more on these two algebras:

- Quaternions $\mathbb{H}$: i, j, k such that

$$i^2 = j^2 = k^2 = -1$$

  with basis $\{(1,0), (0,i), (0,j), (0,k)\}$ and associativity rule as in the diagram below

$$\begin{tikzcd}
& i \arrow[dr] & \\
k \arrow[ur] & & j \arrow[ll]
\end{tikzcd}$$

(!) They are non-commutative obviously.

- Octonions or Cayley numbers $\mathbb{O}$: non-commutative, non-associative, alternative (!) normed division algebra over the reals, with basis:

$$\{(1,0), (0,i), (0,j), (0,k), (0,1), (i,0), (j,0), (k,0)\}$$

Moreover,

A) $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$

B) $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$

C) The items in A) are all and the only normed division algebras over the reals.

(!) Why? see Cayley-Dickson contruction.

(!) Lee-Chevalley for a proof using Clifford Algebras ... ▸ Link

THE SOLUTION: ALTERNATIVITY

DEFINITION (ASSOCIATOR)

Let x, y, z in A, their associator is defined as

$$[x, y, z] = x\,(yz) - (xy)\,z \tag{54}$$

DEFINITION (ALTERNATIVITY)

An algebra A is said to be alternative if the associator of any three elements is totally anti-symmetric.

The issue about the non-associativity of the octonions can be fixed introducing the associator and weakening the requirement for division algebras from associativity to alternativity.
In fact, alternativity ensures:

- the existence of a unique identity;
- the existence of a unique inverse for every non-zero element in the algebra, such that $x^{-1}\,(xy) = y = (yx)\,x^{-1}$.

THEOREM

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ *are the only alternative division algebras over the reals.*

Dualities and Trialities

Definition (Duality)

Given two vector spaces U and V, a bi-linear map $f : U \times V \longrightarrow \mathbb{R}$ defines a duality if $\forall u \in U$ non-zero $\exists\ v \ \in \ V$ such that $f\,(u, v) \neq 0$ and this holds also for V $\times$U.

Definition (Triality)

Given three vector spaces U, V and W, a tri-linear map $f : U \times V \times W \longrightarrow \mathbb{R}$ defines a triality if $\forall u \in U, v \ \in \ V$ non-zero, $\exists\ w \ \in \ W$ such that $f\,(u, v, w) \neq 0$ and this holds also interchanging the role of the factors.

Roughly speaking, fixing a non-zero element in one factor, f defines a duality on the other two.

Definition (Normed Triality)

Given three vector spaces U, V and W, a triality $f : U \times V \times W \longrightarrow \mathbb{R}$ defines a normed triality if each of the factors has a scalar product such that for every triple $(u, v, w) \ \in \ U \times V \times W$

$$|f\,(u, v, w)| \leq ||u|| \cdot ||v|| \cdot ||w|| \tag{55}$$

and for any two fixed elements there exists a non-zero choice of the third so that this bound is attained.

These two conditions determines the multiplication in A completely.

Division Algebras and Trialities

Proposition

Given a normed division algebra A, the map

$$f : A^3 \longrightarrow \mathbb{R} \qquad f(x, y, z) = \langle \bar{z}, xy \rangle \tag{56}$$

defines a triality.

Theorem

There exists a one-to-one correspondence between (normed) trialities and (normed) division algebras.

The group of triples and its automorphisms

Definition (Triple)

Given a normed triality $f : U \times V \times W \longrightarrow \mathbb{R}$, a set of endomorphisms $(\mu,\ \nu\ \rho)$, one on each factor, defines a triple if they preserve scalar products and

$$f\left(\mu\left(u\right),\nu\left(v\right),\rho\left(w\right)\right)=f\left(u,v,w\right)\qquad\forall\ \left(u,v,w\right)\ \in\ U\times V\times W \tag{57}$$

Each element in a triple is therefore an element of $SO(n)$, for some integer n.
Triples form a group T, under the componentwise matrix multiplication.

Proposition

A permutation σ on the entries of a triple defines an internal operation on T.

Proposition

Σ_3 *consists of automorphisms of T.*

Remark

Although we have to take conjugates for those automorphisms which are odd permutations, the representations above are still sent into one another up to similarity transformations.

Application to SYMT

Each of the four possible SYMT are related to matrices $\gamma_{i\alpha\dot{\alpha}}$ which define a map

$$\gamma : V \times S_+ \times S_- \longrightarrow \mathbb{R} \qquad \gamma(v, \xi, \eta) = \gamma_{i\alpha\dot{\alpha}}\ v_i\ \xi_\alpha\ \eta_{\dot{\alpha}} \tag{58}$$

For the matrices $\gamma_{i\alpha\dot{\alpha}}$ the following relations hold:

$$\gamma_{i\alpha\dot{\alpha}}\gamma_{j\beta\dot{\alpha}} + \gamma_{j\alpha\dot{\alpha}}\gamma_{i\beta\dot{\alpha}} = 2\delta_{ij}\delta_{\alpha\beta} \qquad \gamma_{i\alpha\dot{\alpha}}\gamma_{j\alpha\dot{\beta}} + \gamma_{j\alpha\dot{\alpha}}\gamma_{i\alpha\dot{\beta}} = 2\delta_{ij}\delta_{\dot{\alpha}\dot{\beta}} \tag{59}$$

$$\gamma_{i\alpha\dot{\alpha}}\gamma_{i\beta\dot{\beta}} + \gamma_{i\beta\dot{\alpha}}\gamma_{i\alpha\dot{\beta}} = 2\delta_{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}} \tag{60}$$

Proposition

If (59) and (60) are verified, the above-defined map γ defines a normed triality. Viceversa, given any normed triality on that space, its components in a suitable basis must verify (59) and (60).

Hence, we have just established the following 1:1 correspondences:

SYMT d = 3, 4, 6, 10 $\longleftrightarrow$ normed trialities for n = 1, 2, 4, 8 $\longleftrightarrow$ division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Application to SYMT

The group of outer automorphisms considering γ as triality, exactly G = SO(d - 2) $\times$ I.

G is the invariance group of the light-cone algebra, i.e.:
Any $g \in G$ is represented by $\lambda,\ \sigma_+,\ \sigma_- \in M_n(\mathbb{R})$ acting on $P_i \in V,\ Q_\alpha \in S_+$ and $Q_{\dot\alpha} \in S_-$ respectively.
The following possibilities hold:

- g is a rotation in SO(d -2) and the matrices are vector and spinor representations;
- g is an internal symmetry with associated λ= 1 and $\sigma_\pm$ in I;
- g is the product of such things.

Hence λ is orthogonal and the invariance of the first and third parentheses in

$$\left\{Q_\alpha,\ Q_\beta\right\} = 2\ \delta_{\alpha\beta} P_+ \qquad \left\{Q_\alpha,\ Q_{\dot\beta}\right\} = 2\ \gamma_{i\alpha\dot\beta} P_i \qquad \left\{Q_{\dot\alpha},\ Q_{\dot\beta}\right\} = 2\ \delta_{\dot\alpha\dot\beta} P_- \tag{61}$$

implies that also $\sigma_\pm$ are orthogonal.
The second relation above, gives $(\lambda,\ \sigma_+\ \sigma_-)$ is a triple for the triality γ.
Conversely, given any triple, it leaves the superalgebra above unchanged and hence it is an element of G.

Conclusion

When dealing with division algebras we are in fact dealing with the very structure of the Universe; I think it is not a case that K-theory involves techniques from algebra, number theory and representation theory ...
... because maths is not "just maths". Always.

Sir Atiyah, what is a spinor?

"We do not know! [...] I think at the end of my career I'd like to leave some problems for the next generation. So let me know when you've discovered what a spinor is. I'll be listening from above."

- Sir Michael Atiyah, Conférence en l'honneur de Jean-Pierre Bourguignon, IHES, France, 13 September 2013

Link

THANK YOU FOR YOUR ATTENTION !!!

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