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GEORG-AUGUST-UNIVERSITÄT
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SEMINAR IN K-THEORY

DIVISION ALGEBRAS AND SUPERSYMMETRY

AN INTRODUCTION

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Contents

Chapter 1

Prerequisites and basic results

1.1 Division Algebras

Definition 1.1.1 (Real Algebra) An algebra A over $\mathbb{R}$ is a real vector space, together with a bilinear form

$$b : A \times A \to A \tag{1.1}$$

that is distributive, but not necessarily commutative or associative.

Definition 1.1.2 (Division Algebra) An algebra A is called division algebra if it does not contain zero divisors, id est:

$$x \cdot y = 0 \;\Rightarrow x = 0 \;\; or \;\; y = 0 \qquad \forall \; x, y \in A \tag{1.2}$$

Remark 1.1.1 Notice that no notion of norm is required here.

Remark 1.1.2 Any division algebra is in particular a domain. If it is also commutative, an integral domain.

If the bilinear form is also positive-definite, b gives a scalar product, denoted by $\langle \cdot, \cdot \rangle$. Thus, we have naturally defined on A a normed space structure, in the usual way:

$$||x|| = \sqrt{\langle x, x \rangle} \qquad \forall \; x \in A. \tag{1.3}$$

Definition 1.1.3 (Normed Algebra) A normed algebra is a real algebra A with a multiplicative unit and norm $||\cdot||$ such that

$$||x\cdot y|| = ||x||\cdot||y|| \qquad \forall\ x,y\in A \tag{1.4}$$

Proposition 1.1.1 Every normed algebra is a division algebra.

Proof

$0 = ||x\cdot y|| = ||x||\cdot||y|| \Rightarrow ||x|| = 0\ or\ ||y|| = 0 \Rightarrow x = 0\ or\ y = 0$, as $||\cdot||$ is a norm. The statement follows by arbitrariness of x and y in A.

$\square$

Remark 1.1.3 We can always have a multiplicative inverse for a real algebra.

Consider a unit vector e and an invertible linear map T on $\mathbb{R}^n$ such that $T(e^2) = e$, then we have:

$$T\circ(e\cdot e) = e\cdot T(e) = T(e)\cdot e = T(e^2) = e$$

Therefore, we may assume $e^2 = e$. Observe that the left and right multiplication maps induced by b are linear and invertible for every non-zero element, i.e. $x \to a\cdot x$ and $x \to x\cdot a$ are surjective $\forall\ a\ \in\ \mathbb{R}^n\setminus\{0\}$.

This two maps, restricted to the versor e, are used to build up a new product with e as left and right inverse. For it, if $\alpha(x) := x\cdot e, \beta(x) := e\cdot x$, then

$$\mu(x,y) = \alpha(x)^{-1}\cdot\beta(y)^{-1}$$

is such that $\mu(x,e) = \alpha(x)^{-1}\cdot\beta(e)^{-1} = \alpha(x)^{-1}\cdot e = x$ and $\mu(e,y) = y$.

Thus, $a\cdot x = e$ and $x\cdot a = e$ are solvable for x, whenever a is nonzero.

Our aim is to prove a classical result by Hurwitz:

There are only four normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$ with dimension 1, 2, 4, and 8, respectively.

1.2 H-Spaces and Parallelizability

Firstly, we investigate the correlations between the existence of a division algebra structure on $\mathbb{R}^n$, parallelizability of the embedded unit sphere $\mathbb{S}^{n-1}$ and H-spaces.

1.2.1 Hopf spaces

Definition 1.2.1 A space X is an H-space if there is a continuous multiplication map $\mu : X \times X \to X$ and an identity element $e \in X$ such that the two maps $x \to \mu(x,e) \quad x \to \mu(e,x)$ are homotopic to the identity through maps $(X,e) \to (X,e)$. In particular, $\mu(e,e) = e$.

Remark 1.2.1

- This definition can be modified requiring e strict identity or homotopies not fixing the base-point.
- If a space Y is homotopic to an H-space X, then X is an H-space.

Remark 1.2.2 This is weaker than having a topological group structure, as associativity and existence of multiplicative inverses are not required properties.

Example 1.2.1 The topological groups are H-spaces, in fact they are defined as topological spaces with a multiplication and inverse maps both continuous. In addition, associativity holds.

Only S^1 and S^3 are topological groups, since associativity on Cayley octonions $\mathbb{O}$ fails. However, S^1, S^3 and S^7 are H-spaces, restricting the ambient space multiplications on the unit spheres. See next chapter.

Example 1.2.2 The linear groups $GL_n(\boldsymbol{.})$ on $\mathbb{R}, \mathbb{C}, \mathbb{H}$, topologised as subspaces of $M_k(\boldsymbol{.})$, are Lie groups. Since they are opens of Euclidean spaces, they are not compact.
However, they are of the same homotopy type of the compact Lie groups $O(n), U(n), Sp(n)$.

For more see [KS].

1.2.2 Parallelizability

Definition 1.2.2 A differentiable manifold M is said to be parallelizable iff there exists a linear diffeomorphism $t : M \times \mathbb{R}^n \to TM$ such that $t|_{\{x\}\times\mathbb{R}^n} : \{x\} \times \mathbb{R}^n \to T_xM$, called trivialization. Thus, $TM \cong M \times \mathbb{R}^n$.

Definition 1.2.3 (Equivalent to 1.2.2) A differentiable manifold $M \subset \mathbb{R}^n$ is said to be parallelizable if there exist a global frame, namely a family $\{\xi_1, ..., \xi_n\}$ of linearly independent vectors being a basis for $T_xM \quad \forall x \in M$.

Proposition 1.2.1 If the oriented vector bundle ξ possesses a nowhere zero cross-section, then the Euler class $e(\xi)$ must be zero.

Proof

see Milnor, Stasheff [MS], Chapter 9. ♠

Corollary 1.2.1 If M compact manifold is parallelizable then $\chi(M) = 0$.

Proof

Follows directly from the previous result.

Example 1.2.3 The 2-dimensional sphere S^2 is not parallelizable. Indeed:

$$\chi(S^2) = 2 - 2 \cdot g + 2 \cdot 0 = 2$$

Example 1.2.4 The odd spheres $\left\{S^1,\ S^3,\ S^7\right\}$ are all parallelizable and on $\mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ we have bi-linear forms giving a domain structure.
More explicitly, in the case of S^3, taking $\xi(x) = (-x_2, x_1, -x_4, x_3),\ x \in S^3$, it's easy to show that $\langle \xi(x), x \rangle = 0$.

Lemma 1.2.1 If $\mathbb{R}^n$ is a division algebra, or if S^{n-1} is parallelizable, then S^{n-1} is an H-space.

Proof

If $\mathbb{R}^n$ is a division algebra, we obtain on S^{n-1} a well-defined continuous multiplication map

$$\mu\left(x, y\right) := \frac{x \cdot y}{||x \cdot y||} \tag{1.5}$$

with two-sided multiplicative inverse (see Remark 1.1.3) . Thus, S^{n-1} is an H-space, by definition.
Suppose S^{n-1} is parallelizable with basis $\{\xi_1, ..., \xi_{n-1}\}$ for $T_x M \quad \forall x \in M$, then the Gram-Schmidt orthonormalisation process on $\{x, \xi_1\left(x\right), ..., \xi_{n-1}\left(x\right)\}$ gives an orthonormal frame at x. We center this basis at $\hat{e}_1$ and assume it coincides with the canonical basis,$\{\hat{e}_i\}_{i=2,\ldots,n}$, up to a change of sign and a local deformation.
Let $\alpha_x \in$ SO(n) the rotation into $\{x, \xi_1\left(x\right), ..., \xi_{n-1}\left(x\right)\}$ of the standard basis $\{\hat{e}_i\}_{i=1,\ldots,n}$, then the map

$$\mu\left(x, y\right) := \alpha_x\left(y\right) \tag{1.6}$$

defines an H-space structure on S^{n-1} , with identity element $\hat{e}_1$.

□

1.3 The ring structure of K-theory

We recall here some relevant useful results in view of the next section.

1.3.1 Bott Periodicity Theorem

Theorem 1.3.1 (Reduced Bott Periodicity theorem) Given X a compact T_2 space, the homomorphism

$$\beta : \tilde{K}(X) \to \tilde{K}\left(S^2 X\right) \tag{1.7}$$

$$\beta(\alpha) \ = (H \ - \ 1) * \alpha \tag{1.8}$$

is an isomorphism.

Remark 1.3.1 (Notation) H denotes the canonical line bundle over S^2, $\mathbb{CP}^1$.
$S^2 X$ the 2-fold suspension of X.

Proof

see Hatcher[H1], Theorem 2.11. ♠

Corollary 1.3.1 The reduced K-theory of spheres is:

$$\tilde{K}\left(S^k\right) = \begin{cases} \mathbb{Z} & \text{k} \ = \ \text{2n} \ : \ \text{n} \in \ \mathbb{Z} \\ \{0\} & \text{k odd} \end{cases} \tag{1.9}$$

Moreover, the generator for the reduced K-theory of even spheres is given by:

$$(H-1) * \cdots * (H-1) \tag{1.10}$$

Proof

see Hatcher[H1], Corollary 2.12. ♠

1.3.2 The Fundamental Product Theorem

The next theorem and its corollary will create an isomorphism of rings analogous to the one we have between complex numbers and real polynomials in one variables mod $\left(x^2+1\right)$.

Theorem 1.3.2 (The Fundamental Product Theorem) For every compact, T_2 space X, the homomorphism of rings

$$\mu : K(X) \otimes \frac{\mathbb{Z}[H]}{(H-1)^2} \to K(X) \otimes K\left(S^2\right) \to K\left(X \times S^2\right) \tag{1.11}$$

is an isomorphism of rings.

Proof

see Hatcher[H1], Theorem 2.2. ♠

Corollary 1.3.2 The following isomorphism of rings holds:

$$\phi : \frac{\mathbb{Z}[H]}{(H-1)^2} \to K\left(S^2\right) \tag{1.12}$$

Proof

It is obtained from the previous theorem, just considering X to be a point.

□

Corollary 1.3.3 For every integer k,

$$\tilde{K}\left(S^{2k}\right) \otimes \tilde{K}(X) \cong \tilde{K}\left(S^{2k} \wedge X\right) \tag{1.13}$$

$$K\left(S^{2k}\right) \otimes K(X) \cong K\left(S^{2k} \times X\right) \tag{1.14}$$

Proof

It follows iterating the Bott Periodicity isomorphism, as composition is an internal operation.

□

Remark 1.3.2 In particular, if $X = S^{2l}$, we obtain the following:

$$K\left(S^{2k} \times S^{2l}\right) \cong K\left(S^{2k}\right) \otimes K\left(S^{2l}\right) \tag{1.15}$$

Therefore

$$K\left(S^{2k} \times S^{2l}\right) \cong \frac{\mathbb{Z}[\alpha]}{(\alpha^2)} \times \frac{\mathbb{Z}[\beta]}{(\beta^2)} \cong \frac{\mathbb{Z}[\alpha, \beta]}{(\alpha^2, \beta^2)} \tag{1.16}$$

where α and β are the pullbacks of generators of $\tilde{K}\left(S^{2k}\right), \tilde{K}\left(S^{2l}\right)$ under the canonical projections. Thus, we choose $\{1,\ \alpha,\ \beta,\ \alpha\beta\}$ as basis for $K\left(S^{2k} \times S^{2l}\right)$.

For the relation between reduced and unreduced K-Theory, see [H1] page 58.

Chapter 2

Hurwitz Theorem

In this chapter we will prove the classical result by Hurwitz[HW], using the geometrical point of view in Hatcher [H1].

2.1 Adams Operation

We summarize here the main features of the power operation introduced by Adams on the K-theory of a compact, T_2 space. This ring homomorphism is one of the fundamental result in the theory and will allow us to finish the proof of Hurwitz Theorem.

Theorem 2.1.1 If X is a T_2, compact topological space, there exists a ring homomorphism $\psi^k : K(X) \to K(X)$, for all k ≥ 0, with the following properties:

1. $\phi^k f^* = f^* \psi^k$ for all morphism $f : X \to Y$.

2. $\psi^k(L) = L^k$, if L line bundle.

3. $\psi^k \circ \psi^l = \psi^{kl}$

4. $\psi^p(\alpha) \overset{(p)}{\equiv} \alpha$, if p prime.

Proof

see Hatcher[H1], Theorem 2.20. ♠

Proposition 2.1.1 $\psi^k : K\left(S^{2n}\right) \to K\left(S^{2n}\right)$ is multiplication by k^n, i.e.

$$\psi^k(\alpha * \beta) = k^n(\alpha * \beta)$$

Proof

see Hatcher[H1], Proposition 2.21. ♠

Splitting Principle:

Given a vector bundle $E \to X$, with X compact, T_2 topological space, there is a compact Hausdorff space F(E) and a map p:F(E)→ X such that the induced map $p^* : K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$splits as a sum of line bundles.

For the proof see [H1], page 66 et seq.

2.2 Proving Hurwitz Theorem via Adams Theorem

Theorem 2.2.1 The following statements are true only for n = 1, 2, 4 and 8:

- $\mathbb{R}^n$ is a division algebra.
- S^{n-1} is parallelizable.

Remark 2.2.1 We have already proven lemma 1.2.1:

If $\mathbb{R}^n$ is a division algebra, or if S^{n-1} is parallelizable, then S^{n-1} is an H-space.

Proposition 2.2.1 (Even Spheres) If $k > 0$, S^{2k} is not an H-space.

Proof

$If\ \exists\ \mu : S^{2k} \times S^{2l} \to S^{2k}$, H-space multiplication, $\Rightarrow \mu^* : K\left(S^{2k}\right) \to K\left(S^{2k} \times S^{2l}\right)$ is an homomorphism of K-rings. From remark 1.3.2,

$$\mu^* : \frac{\mathbb{Z}[\gamma]}{(\gamma^2)} \to \frac{\mathbb{Z}[\alpha, \beta]}{(\alpha^2, \beta^2)} \tag{2.1}$$

Assume it is written in terms of the basis as

$$\mu^*(\gamma) \ = \ \alpha + \beta + m\alpha\beta : m \in \mathbb{Z} \tag{2.2}$$

The identity on S^{2k} can always be obtained as $S^{2k} \overset{i}{\hookrightarrow} S^{2k} \times S^{2l} \overset{\mu}{\to} S^{2k}$, where i is the inclusion as first or second factor, i.e. as $S^{2k} \times \{e\}$ or $\{e\} \times S^{2k}$, e multiplicative H-identity. Therefore, $i^*(\alpha) = \gamma$ and $i^*(\beta) = 0$ (or viceversa). and similarly for β. $\Rightarrow$ the coefficient of α in (2.2) is 1 and similarly for β.

This leads however to a contradiction, since $\gamma^2 \equiv 0$, but, as μ^* is a ring morphism:

$$0 = \mu^* \left(\gamma^2\right) \ = \left(\alpha + \beta + m\alpha\beta\right)^2 \equiv 2\alpha\beta \neq 0$$

□

Proposition 2.2.2 (Odd spheres) If n even: n $\notin \{2,4,8\}$, S^{2n-1} is not an H-space.

Proof

Consider a function $g : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$ and observe the following:

$$S^{2n-1} \ = \ \partial(D^n \times D^n) = (\partial D^n \times D^n) \cup (D^n \times \partial D^n)$$

We set $\hat{g} : S^{2n-1} \longrightarrow S^n$ to be the continuous, well-defined map

$$\hat{g}\left(x,y\right) = \begin{cases} ||y|| \cdot g\left(x, \dfrac{y}{||y||}\right) & \text{on } {D^n}_+ \\ ||x|| \cdot g\left(\dfrac{x}{||x||}, y\right) & \text{on } {D^n}_- \end{cases}$$

such that $\hat{g} \equiv g$ on $S^{n-1} \times S^{n-1}$.

For $n = 2k : k \in \mathbb{Z}$,

$$f := \hat{g} : S^{4k-1} \longrightarrow S^{2k} \tag{2.3}$$

$$C_f := S^{2k} \cup_f \left\{e^{4k}\right\} \ \Rightarrow \frac{C_f}{S^{2k}} \cong S^{4k} \tag{2.4}$$

As $\tilde{K}^1\left(S^{4k}\right) = \tilde{K}^1\left(S^{2k}\right) = 0$, the S.E.S. of the pair $\left(C_f, S^{2k}\right)$, becomes

$$0 \to \tilde{K}\left(S^{4k}\right) \to \tilde{K}\left(C_f\right) \to \tilde{K}\left(S^{2k}\right) \overset{(1.16)}{\cong} \frac{\mathbb{Z}\left[\beta\right]}{\left(\beta^2\right)} \to 0 \tag{2.5}$$

If the generator of $\tilde{K}\left(S^{4k}\right)$ is mapped into α and an element $\beta \ \in \ \tilde{K}\left(C_f\right)$ into the generator of $\tilde{K}\left(S^{2k}\right)$. By exactness and modular arithmetic, we conclude $\beta^2 = h \cdot \alpha$, for some integer h called **Hp(f), Hopf invariant of f**. The Hopf invariant is well-defined and unique. Observe that β is unique module α, and

$$\left(\beta + m \cdot \alpha\right)^2 = \alpha^2 + \beta^2 + 2m\alpha\beta \overset{(\alpha^2)}{\equiv} \beta^2 + 2m\alpha\beta \tag{2.6}$$

We want to show $\alpha\beta = 0$. Since α maps to 0 in $\tilde{K}\left(S^{2k}\right)$, so does the product, i.e. $\alpha\beta = k \cdot \alpha \ : \ k \in \mathbb{Z} \ \Rightarrow \ k \cdot \alpha\beta = \alpha\beta^2 = \alpha\left(h\alpha\right) = h \cdot \alpha^2 \overset{(\alpha^2)}{\equiv} 0. \ \Rightarrow \ k\alpha\beta = 0 \Rightarrow \alpha\beta = 0$ as the image of $\tilde{K}\left(S^{2k}\right)$ is an infinite cyclic subgroup of $\tilde{K}\left(C_f\right)$.

Lemma 2.2.1 If $g : S^{2n-1} \times S^{2n-1} \longrightarrow S^{n-1}$ is an H-space multiplication, the associated map $\hat{g} : S^{4n-1} \longrightarrow S^{2n}$ has Hopf invariant ± 1.

Proof

If $f := \hat{g}$, the attaching map defined before. The characteristic map of e^4 in C_f is defined as

$$\Phi : \left(D^{2n} \times D^{2n},\ \partial\left(D^{2n} \times D^{2n}\right)\right) \longrightarrow \left(C_f, S^{2k}\right)$$

Let $e \in S^{2n-1}$ be the H-space multiplicative identity, we have the following commutative diagram:

$$\begin{tikzcd}
\tilde{K}\left(C_f\right) \otimes \tilde{K}\left(C_f\right) \arrow[r, "{\mu_1}^*"] & \tilde{K}\left(C_f\right) \\
\tilde{K}\left(C_f, D^{2n}_-\right) \otimes \tilde{K}\left(C_f, D^{2n}_+\right) \arrow[u, "\cong"] \arrow[r, "{\mu_2}^*"] \arrow[d, "\Phi^*\otimes\Phi^*"] & \tilde{K}\left(C_f, S^{2n}\right) \arrow[u] \arrow[d, "\Phi^*\ \cong"] \\
\tilde{K}\left(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}\right) \otimes \tilde{K}\left(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n}\right) \arrow[r, "{\mu_3}^*"] \arrow[d, "\cong"] & \underbrace{\tilde{K}\left(D^{2n} \times D^{2n}, \partial\left(D^{2n} \times D^{2n}\right)\right)}_{\cong\ \tilde{K}(S^{4n})} \\
\underbrace{\tilde{K}\left(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}\right) \otimes \tilde{K}\left(\{e\} \times D^{2n}, \{e\} \times \partial D^{2n}\right)}_{\cong\ \tilde{K}(S^{2n}) \otimes \tilde{K}(S^{2n})} \arrow[ur, "\cong\ *"] \arrow[uu, bend left, "\phi_|"] \arrow[uu, bend right, "\phi_|"'] &
\end{tikzcd}$$

The diagonal map $*$ denotes the external product. By definition of H-space and f, Φ restricts to an homeomorphism onto the positive and negative part of the 2n-dimensional disk [red arrows above]. The element $\beta \otimes \beta$ in $\tilde{K}\left(C_f\right) \otimes \tilde{K}\left(C_f\right)$ is mapped to a generator of $\tilde{K}\left(S^{2n}\right) \otimes \tilde{K}\left(S^{2n}\right)$. By commutativity,

$$\mu_1^*\left(\beta \otimes \beta\right) = \pm\alpha\ \Rightarrow\ \beta^2 = \pm\alpha\ \Rightarrow\ Hp\left(f\right) = \pm 1$$

where α was defined above as the image of a generator of $\tilde{K}\left(C_f, S^{2n}\right)$.

□

Proposition 2.2.2 is an immediate consequence of the following result:

Theorem 2.2.2 (Adams Theorem) There exist a function $f : S^{4n-1} \to S^{2n}$ such that $Hp(f) = \pm 1$ only if n $\in \{1, 2, 4\}$.

Proof

Consider α, β in $\tilde{K}(C_f)$ as we have already done introducing $f := \hat{g} : S^{4k-1} \longrightarrow S^{2k}$. By Proposition 2.1.1,

$$\psi^k(\alpha) = k^{2n} \cdot \alpha \tag{2.7}$$

$$\psi^k(\beta) = k^n \cdot \beta + \mu_k \cdot \alpha \ : \mu_k \ in \ \mathbb{Z}. \tag{2.8}$$

as the first refers to the 4n-dimensional sphere.

Thus,

$$\psi^k \psi^l(\beta) = \psi^k(l^n \beta + \mu_l \alpha) = k^n l^n \beta + \left(k^{2n}\mu_l + l^n \mu_k\right)\alpha \tag{2.9}$$

Observe that, by property 3. of Adams operation,

$$\left(k^{2n}\mu_l + l^n\mu_k\right) \overset{3.}{=} \left(l^{2n}\mu_k + k^n\mu_l\right) \iff \left(k^{2n} - k^n\right)\mu_l = \left(l^{2n} - l^n\right)\mu_k \tag{2.10}$$

By property 4. [mod a prime number],

$$\psi^2(\beta) \overset{(2)}{\equiv} \beta^2 = h\alpha \overset{eq.(2.8)}{=} 2^n\beta + \mu_2\alpha \quad \Rightarrow \quad \mu_2 \overset{(2)}{\equiv} h \quad if \ h = \pm 1 \ or \ odd. \tag{2.11}$$

For $l = 3,\ k = 2$, (2.10) becomes:

$$\left(2^{2n} - 2^n\right)\mu_3 = \left(3^{2n} - 3^n\right)\mu_2 \iff 2^n\left(2^n - 1\right)\mu_3 = 3^n\left(3^n - 1\right)\mu_2 \tag{2.12}$$

Thus $2^n \mid 3^n(3^n - 1)\mu_2$. Since 3^n and μ_2 are odd,

$$2^n \mid (3^n - 1) \tag{2.13}$$

The following elementary number theory result will conclude the proof.

Lemma 2.2.2 If n $\in \{1, 2, 4\}$, then $2^n \mid (3^n - 1)$.

Proof

see Hatcher[H1], Lemma 2.22. ♠

There are few other ways to prove the result on parallelizability of spheres, see [AH] for further details.

Chapter 3

Supersymmetric Yang-Mills Theories and Division Algebras

In this chapter we will discuss the interplay between division algebras and supersymmetry. Being here an application of the results we have just collected, the physical treatment of the subject is not self-contained. Thus, refer to [IZ] and [LO] for QFT and spinors.
The exposition will follow the article of Evans [ES].

3.1 Simple Yang-Mills Theories

Simple supersymmetric YM-theories and classical superstrings (N=1,2) in the Green-Schwarz covariant formulation appear only in dimension 3, 4, 6, 10.
Roughly speaking, the necessary and sufficient condition for this to happen, is that there exists an identity γ- matrix.
Our aim is to show the following 1:1 correspondences hold:

- division algebras $\longleftrightarrow$ trialities;
- $\exists$ a decomposition of the γ-matrices to give a triality $\longleftrightarrow$ supersymmetry holds;

Definition 3.1.1 A Yang-Mills theory is said to be simple if each gauge boson has just one super-partner and there are no additional scalar fields.

In order to find all the possible simple YM theories, we restrict to dimensions d in which the degrees of freedom of a vector and a spinor coincides. A massless vector field has therefore d - 2 physical modes, whereas for a spinor:

$$d.o.f. = \begin{cases} 2^{\frac{d}{2}-1} & \text{d even} \\ 2^{\frac{d-1}{2}} & \text{d odd} \end{cases}$$

as in the first case we have two inequivalent spinor representations.

Only one half of these components is actually independent, since Majorana reality constraints have to be taken into account as well.

Therefore, the only possible dimensions are d = 3, 4, 6 and 10, for Majorana, Majorana-Dirac, Weyl and Majorana-Weyl spinors respectively, with corresponding physical degrees of freedom n = 1, 2, 4, 8.

Theorem 3.1.1 There exists a supersymmetric YM-theory if and only if n = d - 2.

Proof

($\Rightarrow$)

Consider the following Lagrangian

$$L \;=\; -\frac{1}{4}F_{mn}F^{mn} \;+\; \frac{1}{2}\; i\bar{\psi}\gamma^m\nabla_m\psi \tag{3.1}$$

where ψ is the Dirac spinor, minimally coupled to the gauge field A_m with values in the Lie algebra of some gauge group. The covariant derivative and field strength are, as usual:

$$\nabla_m = \partial_m + A_m \qquad\qquad F_{mn} = [\nabla_m, \nabla_n] \tag{3.2}$$

Assume we are working in a flat space with metric $\eta = diag\,(-1, 1, \ldots, 1)$ and Dirac matrices such that $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$, the superysymmetric transformations are defined as:

$$\delta\,(\epsilon)\,A_m = \frac{1}{2}\; i\left(\bar{\epsilon}\gamma_m\psi - \bar{\psi}\gamma_m\epsilon\right) \tag{3.3}$$

$$\delta\,(\epsilon)\,\psi = \frac{1}{2}\; F_{mn}\gamma^{mn}\epsilon \tag{3.4}$$

Thus

$$\delta\,(\epsilon)\,L = -\frac{1}{4}\;\bar{\psi}\gamma^m\left[\bar{\epsilon}\gamma_m\psi \;-\; \bar{\psi}\gamma_m\epsilon, \psi\right] \tag{3.5}$$

L is invariant when the R.H.S. vanishes for arbitrary anticommuting ψ and ϵ of the type considered above.

There is a unified way to discuss supersymmetric YM theories, making use of a real spinor Ψ and matrices $\Gamma^m{}_{\alpha\beta},\ \tilde{\Gamma}^{m\alpha\beta}$ such that

$$\Gamma_m \tilde{\Gamma}_n = \Gamma_n \tilde{\Gamma}_m = 2\eta_{mn} \tag{3.6}$$

in terms of which (3.3), (3.4) and (3.5) become

$$L \;=\; -\frac{1}{4} F_{mn} F^{mn} \;+\; \frac{1}{2}\, i\Psi^T \Gamma^m \nabla_m \Psi \tag{3.7}$$

$$\delta\left(\epsilon\right) A_m = \; i\epsilon^T \Gamma_m \Psi \tag{3.8}$$

$$\delta\left(\epsilon\right) \Psi = \frac{1}{2}\; F_{mn} \Gamma^{mn} \epsilon \tag{3.9}$$

where $\Gamma_{mn} = \tilde{\Gamma}_{[m} \Gamma_{n]}$ is the generator of the Lorentz transformation on Ψ.

The condition for invariance is therefore reduced to

$$\Gamma_{m\alpha(\beta}\; \Gamma^m{}_{\gamma\delta)} = 0 \tag{3.10}$$

Contracting (3.10) with $\tilde{\Gamma}^{l\gamma\delta}$ and using (3.6), we obtain:

$$tr\left(\tilde{\Gamma}^l\; \Gamma^m\right) \Gamma_{m\alpha\beta} + 2 \cdot \left(\Gamma_m \tilde{\Gamma}^l \Gamma^m\right)_{\alpha\beta} = 0 \qquad \Longrightarrow \qquad n = d-2 \tag{3.11}$$

where Ψ has dimension 2n.

This is exactly the condition of equality of bosonic and fermionic degrees of freedom.

$\square$

For the other direction, we have to introduce further tools.

3.1.1 SYM and the Light-Cone

Let Q_α denote the generator of the transformations in (3.8), (3.9) , i.e. $\delta\left(\epsilon\right) = \epsilon^\alpha Q_\alpha$, the supersymmetry algebra

$$\{Q_\alpha,\ Q_\beta\} = 2\Gamma^m{}_{\alpha\beta} P_m \tag{3.12}$$

can be verified up to field equations and gauge transformations from the invariant condition in (3.10).

It is Lorentz covariant and can have automorphisms of the form

$$
\begin{aligned}
&Q \to g \cdot Q \quad \text{for some matrix g} \\
&P \to P
\end{aligned}
$$

i.e. the Lagrangian has global symmetries

$$
\begin{aligned}
&\Psi \to g^T \cdot \Psi \\
&A \to A
\end{aligned}
$$

because they both amount to an invariance

$$g\ \Gamma^m g^T = \Gamma^m \tag{3.13}$$

Remark 3.1.1 In dimension 4 there is a $U(1)$ symmetry, for n = 6 an $SU(2)$ one. These are the only internal symmetries possible in the four cases of interest.

Consider the light-cone decomposition of a vector $U_{\pm} = U_0 \pm U_{d-1}$ and the helicity group $SO(d-2)$ fixing the + direction, the gauge field and its partner spinor reduce to

$$A_m = (A_+, A_-, A_i) \qquad \Psi = \begin{pmatrix} W^{\alpha} \\ W^{\dot{\alpha}} \end{pmatrix} \tag{3.14}$$

where $i \in \{1, \ldots, d-2\}$ refers to the vector representation V on SO(d - 2) and $\alpha, \dot{\alpha} \in \{1, \ldots, n\}$ to the spinor representations $S_{\pm}$, defined as eigenspaces of $\Gamma_{\pm}$, generator of boosts along the $(d-1)$- axis.

Remark 3.1.2 For n = 6, 10, Ψ is a Weyl spinor and $\alpha, \dot{\alpha}$ are of opposite chirality. For n = 3,4, Ψ is a Dirac spinor, therefore $\alpha, \dot{\alpha}$ are different copies of the same representation.

With an opportune choice of the six Γ-matrices, $\Gamma_{\pm}$ diagonalises and in this basis $\delta_{\alpha\beta}, \delta_{\dot{\alpha}\dot{\beta}}$ define inner products on $S_{\pm}$ invariant under the transverse group. We obtain an equivalent formulation of the invariance in (3.10) as

$$\gamma_{i\alpha\dot{\alpha}}\gamma_{i\beta\dot{\beta}} + \gamma_{i\beta\dot{\alpha}}\gamma_{i\alpha\dot{\beta}} = 2\delta_{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}} \tag{3.15}$$

The internal symmetries I and their representations remain completely unaffected by this light-cone decomposition because they commute with the Lorentz group. Consequently, V and $S_{\pm}$ carry representations of G = SO(d - 2) $\times$ I.

The superalgebra decomposes into:

$$\Big\{Q_{\alpha},\ Q_{\beta}\Big\} = 2\ \delta_{\alpha\beta}P_{+} \qquad \Big\{Q_{\alpha},\ Q_{\dot\beta}\Big\} = 2\ \gamma_{i\alpha\dot\beta}P_{i} \qquad \Big\{Q_{\dot\alpha},\ Q_{\dot\beta}\Big\} = 2\ \delta_{\dot\alpha\dot\beta}P_{-} \tag{3.16}$$

Thus, G consists of the linear transformations $Q_{\alpha},\ Q_{\dot\alpha}$ and P_i that leave these expressions invariant.

The problem can be further reduced considering the light-cone gauge $A_{+} = 0$ and observing that in the equations of motion the vector A_i and the spinor $W^{\dot\alpha}$ can be regarded as the only dynamical degrees of freedom. These physical fields are related, in terms of the generator, by:

$$Q_{\alpha}A_i = i\gamma_{i\alpha\dot\alpha}W_{\dot\alpha} \qquad Q_{\alpha}W_{\dot\alpha} = -\ \gamma_{i\alpha\dot\alpha}\partial_{+}A_i \tag{3.17}$$

Remark 3.1.3 Consider any real commuting spinor ξ_α in S_+ such that $\xi_\alpha\xi_\alpha = 1$.

The matrix $\Xi := \gamma_{i\alpha\dot\alpha}\xi_\alpha$ defines a map from V to S_- such that

$$\Xi^T \cdot \Xi = 1 \tag{3.18}$$

we will have a right inverse as well if the dimensions of the two spaces coincide, i.e.

$$\Xi \cdot \Xi^T = 1 \implies \gamma_{i\alpha\dot\alpha}\ \gamma_{j\beta\dot\beta}\ \xi_\alpha\ \xi_\beta = \delta_{ij} \tag{3.19}$$

and this is equivalent to (3.15).

Thus, (3.10) holds precisely when n = d - 2, i.e. we have equal numbers of bosons and fermions. This always occur when considering spinors of an even orthogonal group, but is rare to have it for both vector and spinor representations.

3.2 Relation with division algebras

The aim of this section is to demonstrate the equivalence between (normed) division algebras and (normed)trialities and show its relevance in SYM-theories, as an application.

3.2.1 Alternativity, Trialities and Outer Automorphisms

Definition 3.2.1 (Conjugation map) Let A be an algebra, a map $x \to \bar{x}$ is called conjugation if the following properties are verified:

$$\overline{(\lambda x = \mu y)} = \lambda \bar{x} + \mu \bar{y} \qquad \overline{x \cdot y} = \bar{x} \cdot \bar{y} \qquad \forall\ x, y\ \in\ A,\ \forall\ \lambda, \mu\ \in\ \mathbb{K} \tag{3.20}$$

Given an algebra A of dimension n, with conjugation map as above, $A \times A$ is a 2n-dimensional algebra with the following inherited operations:

$$(a, b) \cdot (c, d) = \left(ac - d\bar{b}, da + b\bar{c}\right) \qquad \forall\ a, b, c, d\ \in\ A \quad \text{(multiplication)} \tag{3.21}$$

$$\overline{(a, b)} = (\bar{a}, -b) \qquad \forall\ a, b\ \in\ A \quad \text{(conjugation)} \tag{3.22}$$

With this procedure, we can therefore successively construct algebras of dimension 2, 4 and 8, starting from $\mathbb{R}$ with trivial conjugation and usual multiplication (for more, see Cayley-Dickson construction).

To obtain a normed division algebra, consider the following inner-product:

$$\frac{1}{2}\left(\bar{x}y + \bar{y}x\right) = \langle x, y \rangle 1 \tag{3.23}$$

(It is easy to show this is indeed a positive-definite bilinear form) and choose an orthonormal basis $\{e_0, e_1, \ldots, e_{n-1}\}$, for n = 1,2,4,8, where $e_0 = 1$ and all the other elements are pure imaginary.

Proposition 3.2.1 The norm induced by (3.23) is multiplicative if the algebra A is associative.

Proof

It is left to the reader. ♠

Corollary 3.2.1 The induced norm on $\mathbb{R}, \mathbb{C}, \mathbb{H}$ is multiplicative, i.e. they are normed division algebras.

Definition 3.2.2 (Associator) Let x, y, z in A, their associator is defined as

$$[x, y, z] = x\,(yz) - (xy)\,z \tag{3.24}$$

Definition 3.2.3 (Alternativity) An algebra A is said to be alternative if the associator of any three elements is totally anti-symmetric.

The issue about the non-associativity of the octonions can be fixed introducing the associator and weakening the requirement for division algebras from associativity to alternativity. In fact, alternativity ensures the existence of a unique identity and the existence of a unique inverse for every non-zero element in the algebra, such that $x^{-1}(xy) = y = (yx)x^{-1}$.

Theorem 3.2.1 $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only alternative division algebras over the reals.

Proof

It is left to the reader. ♠

Definition 3.2.4 (Duality) Given two vector spaces U and V, a bi-linear map $f : U \times V \longrightarrow \mathbb{R}$ defines a duality if $\forall u \in U$ non-zero $\exists\ v \ \in \ V$ such that $f(u, v) \neq 0$ and this holds also for V $\times$U.

Remark 3.2.1 There exists, therefore, a natural identification $U \equiv V^*$, each time we have a well-defined duality on the direct product of the two vector spaces.

Definition 3.2.5 (Triality) Given three vector spaces U, V and W, a tri-linear map $f : U \times V \times W \longrightarrow \mathbb{R}$ defines a triality if $\forall u \in U, v \ \in \ V$ non-zero, $\exists\ w \ \in \ W$ such that $f(u, v, w) \neq 0$ and this holds also interchanging the role of the factors.

Roughly speaking, fixing a non-zero element in one factor, f defines a duality on the other two.

Definition 3.2.6 (Normed Triality) Given three vector spaces U, V and W, a triality $f : U \times V \times W \longrightarrow \mathbb{R}$ defines a normed triality if each of the factors has a scalar product such that for every triple $(u, v, w) \ \in \ U \times V \times W$

$$|f(u, v, w)| \leq ||u|| \cdot ||v|| \cdot ||w|| \tag{3.25}$$

and for any two fixed elements there exists a non-zero choice of the third so that this bound is attained.

These two conditions determines the multiplication in A completely.

Proposition 3.2.2 Given a normed division algebra A, the map

$$f : A^3 \longrightarrow \mathbb{R} \qquad f(x, y, z) = \langle \bar{z}, xy \rangle \tag{3.26}$$

defines a triality.

Proof

It is left to the reader. ♠

Theorem 3.2.2 There exists a one-to-one correspondence between (normed) trialities and (normed) division algebras.

Proof

It is left to the reader to complete [ES] pages 99, 100. ♠

Definition 3.2.7 (Triple) Given a normed triality $f : U \times V \times W \longrightarrow \mathbb{R}$, a set of endomorphisms $(\mu,\ \nu\ \rho)$, one on each factor, defines a triple if they preserve scalar products and

$$f(\mu(u), \nu(v), \rho(w)) = f(u, v, w) \qquad \forall\ (u, v, w)\ \in\ U \times V \times W \tag{3.27}$$

Each element in a triple is therefore an element of $SO(n)$, for some integer n. Triples form a group T, under the componentwise matrix multiplication.

In terms of the corresponding division algebra A (as in proposition (3.2.2)), (3.27) is equivalent to

$$\langle \overline{\rho(z)}, \mu(x)\nu(y) \rangle = \langle \bar{z}, xy \rangle \tag{3.28}$$

Given an endomorphism α on A, we define $\bar{\alpha}$ as:

$$\bar{\alpha}(z) := \overline{\bar{\alpha}(z)} \qquad \forall\ z\ \in A \tag{3.29}$$

or equivalently, regarding the maps as matrices in the standard basis and considering the metric $\mathcal{X} := diag(1, -1, \ldots, -1)$,

$$\bar{\alpha} = \mathcal{X}\alpha\mathcal{X}^{-1} \tag{3.30}$$

Since ρ is orthogonal, (3.28) can be seen as

$$\bar{\rho}(xy) = \mu(x) \cdot \nu(y) \tag{3.31}$$

We will obtain the automorphisms of T in terms of the simmetric group Σ_3.

Proposition 3.2.3 A permutation σ on the entries of a triple defines an internal operation on T.

Proof

If σ is even, this follows from the cyclic symmetry

$$\langle \bar{z}, xy \rangle = \langle \bar{y}, zx \rangle = \langle \bar{x}, yz \rangle \tag{3.32}$$

For σ odd, (3.28) implies

$$\langle \overline{\bar{\rho}\,(\bar{z})}, \bar{\nu}\,(\bar{y})\,\mu\,(\bar{x}) \rangle = \langle z, \bar{y}\bar{x} \rangle \tag{3.33}$$

The statement follows directly, as (3.27) is equivalent to (3.28).

□

Proposition 3.2.4 Σ_3 consists of automorphisms of T.

Proof

Left to the reader.

Hint: For σ even, this is trivial. For σ odd, observe that: if α and β are endomorphisms on A, $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$ and, since T is defined as a subgroup of $SO(n)^3$, Σ_3 permutes the three representations $(\mu,\ \nu\ \rho) \to \mu$, $(\mu,\ \nu\ \rho) \to \nu$ and $(\mu,\ \nu\ \rho) \to \rho$.

Furthermore, $(\mu,\ \nu\ \rho) \to \mu$ and $(\mu,\ \nu\ \rho) \to \bar{\mu}$ are equivalent because of (3.29).

Hence, although we have to take conjugates for those automorphisms which are odd permutations, the representations above are still sent into one another up to similarity transformations. ♠

3.2.2 SYM and the Algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

From subsection 3.1.1, each of the four possible SYMT are related to matrices $\gamma_{i\alpha\dot{\alpha}}$ which define a map

$$\gamma : V \times S_+ \times S_- \longrightarrow \mathbb{R} \qquad \gamma\,(v, \xi, \eta) = \gamma_{i\alpha\dot{\alpha}}\ v_i\ \xi_\alpha\ \eta_{\dot{\alpha}} \tag{3.34}$$

As a reminder, recall that for the matrices $\gamma_{i\alpha\dot{\alpha}}$ the following relations hold:

$$\gamma_{i\alpha\dot{\alpha}}\gamma_{j\beta\dot{\alpha}} + \gamma_{j\alpha\dot{\alpha}}\gamma_{i\beta\dot{\alpha}} = 2\delta_{ij}\delta_{\alpha\beta} \qquad \gamma_{i\alpha\dot{\alpha}}\gamma_{j\alpha\dot{\beta}} + \gamma_{j\alpha\dot{\alpha}}\gamma_{i\alpha\dot{\beta}} = 2\delta_{ij}\delta_{\dot{\alpha}\dot{\beta}} \tag{3.35}$$

$$\gamma_{i\alpha\dot{\alpha}}\gamma_{i\beta\dot{\beta}} + \gamma_{i\beta\dot{\alpha}}\gamma_{i\alpha\dot{\beta}} = 2\delta_{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}} \tag{3.36}$$

Proposition 3.2.5 If (3.28) and (3.29) are verified, the above-defined map γ defines a normed triality. Viceversa, given any normed triality on that space, its components in a suitable basis must verify (3.28) and (3.29).

Proof

See [ES] pag 100. ♠

Hence, we have just established the following 1:1 correspondence:
SYMT d = 3, 4, 6, 10 $\longleftrightarrow$ normed trialities for n = 1, 2, 4, 8 $\longleftrightarrow$ division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

The group of outer automorphisms T, defined in section (3.2.1) is in this case, considering γ as triality, exactly G = SO(d - 2) $\times$ I.
G is the invariance group of the light-cone algebra:
Any $g \in G$ is represented by $\lambda,\ \sigma_+,\ \sigma_- \in M_n(\mathbb{R})$ acting on $P_i \in V$, $Q_\alpha \in S_+$ and $Q_{\dot{\alpha}} \in S_-$ respectively. The following possibilities hold:

- g is a rotation in SO(d -2) and the matrices are vector and spinor representations;
- g is an internal symmetry with associated λ= 1 and $\sigma_\pm$ in I;
- g is the product of such things.

Hence λ is orthogonal and the invariance of the first and third parentheses in

$$\Big\{Q_\alpha,\ Q_\beta\Big\} = 2\ \delta_{\alpha\beta}P_+ \qquad \Big\{Q_\alpha,\ Q_{\dot{\beta}}\Big\} = 2\ \gamma_{i\alpha\dot{\beta}}P_i \qquad \Big\{Q_{\dot{\alpha}},\ Q_{\dot{\beta}}\Big\} = 2\ \delta_{\dot{\alpha}\dot{\beta}}P_- \tag{3.37}$$

implies that also $\sigma_\pm$ are orthogonal.
The second relation above, gives $(\lambda,\ \sigma_+\ \sigma_-)$ is a triple for the triality γ.
Conversely, given any triple, it leaves the superalgebra above unchanged and hence it is an element of G.

The trivial representations $(\mu,\ \nu\ \rho) \to \mu$, $(\mu,\ \nu\ \rho) \to \nu$ and $(\mu,\ \nu\ \rho) \to \rho$ permuted by Σ_3 at the end of the previous section, are in this case the representations carried by V, $S_\pm$.

A more explicit description of the group G in each of the case of interest is given in [ES], page 103.

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