# Moduli Spaces

# Categorical Formulations, Co-homologies and The Picard Group

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**Incomplete Preliminary Version**

**Abstract**

Solutions to moduli problems provide ways of classifying various geometric objects up to isomorphism. The Picard (Abelian) group, in particular, defined as the set of isomorphism classes of certain invertible sheaves, is the solution to a specific moduli problem: the one of classifying all line bundles up to isomorphism. In fact, given a non-singular curve C of genus g, an invertible sheaf yields a procedure to canonically attach to each such a curve a one-dimensional vector space.

By exploring different categorical formulations of topologies, pioneered by the work of A. Grothendieck, we will move towards the proof of the main result:

**Theorem 0.0.1** [ [M65], p.67 et seq.] The Picard group of the moduli problem $\mathfrak{M}$ is given by the first (étale) co-homology group of $\mathfrak{M}$ with coefficient in the group of units

$$\mathrm{Pic}\left(\mathfrak{M}\right) \simeq \mathrm{H}^1\left(\mathfrak{M}, o^*\right)$$

If time permits, we will also cover Picard schemes, their relations with Hilbert schemes, and the concept of divisors as in [FGA].

## Contents

In this notes $k$ denotes an algebraically closed field with $\mathrm{char}\,(k) = 0$. The reader is assumed to be familiar with the main ideas and categorical constructions of algebraic geometry and homological algebra.

# 1 Introduction

## 1.1 Parameter Spaces

The collections of geometrical objects we are going to analyze in this script arise as a generalization of parameter spaces, therefore as starting point for this notes, we briefly recall their definition and provide an example.

**Definition 1.1.1** [ [H92], Def. 21.1] We say that a variety $\mathcal{H}$, together with a bijection between the points of $\mathcal{H}$ and a collection of varieties $\{X_\alpha\}_{\alpha\in A} \subset \mathbb{P}^n$, for $A$ a set, is a parameter space for the collection $\{X_\alpha\}$ if, for any variety B, the association to each family $\mathcal{V} \subset B \times \mathbb{P}^n$ of varieties belonging to the collection $\{X_\alpha\}_{\alpha\in A}$ of the map $\phi_\mathcal{V} : B \to \mathcal{H}$ induces a bijection between reduced closed families with base $B$, whose fibers are members of the collection $\{X_\alpha\}_{\alpha\in A}$, and regular maps $B \to \mathcal{H}$.

**Example 1.1.1** Let $X = \mathbb{A}^1_k = \mathrm{Spec}\,(k\,[t])$ and $Y = \mathbb{A}^2_k = \mathrm{Spec}\,(k\,[x, y])$, the parameter space $C_\lambda$ of plane cubic curves with a node or a cusp given by polynomials of degree 3 is the image of the map $\phi_\lambda : X \to Y$, whose induced ring homomorphism is

$$\begin{aligned} \phi^\#_\lambda : k\,[x, y] &\to k\,[t] \\ x &\mapsto t^2 - \lambda \\ y &\mapsto t^3 - \lambda t \end{aligned}$$

Let it be denoted by $C_\lambda$, such a space is cut by the paramentric equation $y^2 - x^2\,(x + \lambda) = 0$ and depicted below:

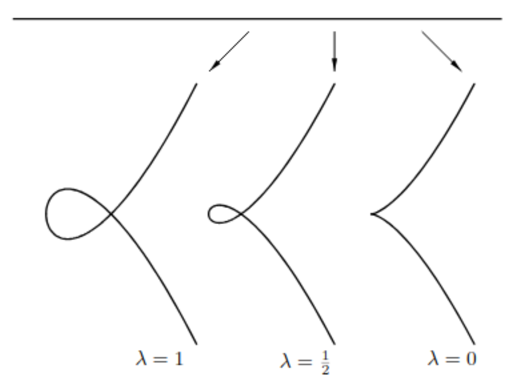


Figure 1: [EH00], p.247

In order to further extend this construction to include the case of isomorphism classes of geometric objects as pre-images of base points, we switch to the functor of points approach to the study of algebraic geometry, and carefully take care of the obstruction to representability of such functor.

The following subsections contain the key technical machinery needed to build up the theory.

## 1.2 Open and Closed Subschemes of an Affine Scheme

**Definition 1.2.1** Let $U$ be an open subset of a scheme $X$, the pair $\left(U, \mathcal{O}_{X|U}\right)$ defines an open subscheme of $X$.

**Remark 1.2.1** The definition above follows by observing that a distinguished open set an affine scheme is again an affine scheme: if $X = \operatorname{Spec}(R)$ and $U = X_f$ is a distinguished open set, then $\left(U, \mathcal{O}_{X|U}\right) = \operatorname{Spec}(R_f)$. Since every open is covered by the distinguished open sets contained in it, $\left(U, \mathcal{O}_{X|U}\right)$ is covered by affine schemes.

**Definition 1.2.2** We define a closed subscheme of *an affine scheme* $X$ to be a scheme $Y$ that is the spectrum of a quotient ring of $R$ by some $I \subset R$.

**Remark 1.2.2** The definition above comes from the observation that if the scheme $X = \operatorname{Spec}(R)$ is an affine scheme and $I \subset R$ an ideal, by identifying a closed subset $V(I) \subset X$ with $Y := \operatorname{Spec}(R/I)$, we obtain an affine scheme. The primes of $R/I$ are in fact the primes of $R$ that contain $I$ taken modulo $I$, thus the topological space $|\operatorname{Spec}(R/I)|$ is canonically homeomorphic to the closed set $V(I) \subset X$.

**Remark 1.2.3** Roughly, the closed subschemes of $X$ are in one to one correspondence with the ideals in $R$. Given $X = \operatorname{Spec}(R)$, the closed subscheme $Y = \operatorname{Spec}(R/I)$ of $X$ contains the closed subscheme $Z = \operatorname{Spec}(R/J)$, if $Z$ is a closed subscheme of $Y$ - if $I \subset J$. In this case $V(J) \subset V(I)$.

## 1.3 Quasi-Coherent Sheaves

We would like to construct a notion of closed subscheme to any scheme, not just the affine one, as presented above. For that, we introduce quasi-coherent sheaves. The first step towards this definition is to substitute a sheaf, called the *ideal sheaf*, for the ideal $I$ associated to a closed subscheme $Y$ of an affine scheme $X = \operatorname{Spec}(R)$.

**Definition 1.3.1** The ideal sheaf $\mathcal{I} = \mathcal{I}_{Y/X}$ of $Y$ in $X$ is defined as the sheaf of ideals of $\mathcal{O}_X$ given on a distinguished open set $V = X_f$ of $X$ by $\mathcal{I}(X_f) = IR_f$.

**Remark 1.3.1** For an affine scheme, the sheaf of ideals $\mathcal{I}$ is the kernel of the restriction map $\mathcal{O}_X \to j_*\mathcal{O}_Y$. The pushforward of the structure scheaf of $Y = \operatorname{Spec}(R/I)$ under the inclusion map $j : |Y| \hookrightarrow |X|$ gets identified with the quotient sheaf $\mathcal{O}_X/\mathcal{I}$.

**Remark 1.3.2** [ [EGAI], 0.4.1.3] More generally. if $(X, \mathcal{A})$ is a ringed space, the sheaf of ideals on $\mathcal{A}$ corresponds to the sub-$\mathcal{A}$-modules.

Not all sheaves of ideals $\mathcal{I}$ arise from ideals of $R$. However, we are interested in exactly those sheaves: The quasi-coherent sheaves of ideals.

**Definition 1.3.2** A quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ on an arbitrary scheme $X$ is a sheaf of ideals $\mathcal{I}$ such that, for every open affine subset $U$ of $X$, the restriction $\mathcal{I}|_U$ is a sheaf of ideals on $U$.

**Remark 1.3.3** A sheaf on $\operatorname{Spec}(N)$ for $N$ a Noetherian ring that corresponds to a finitely generated module has a property called coherence. Thus, it was natural to attach the adjective quasi-coherent to a sheaf of modules, not necessarily finitely generated.

**Definition 1.3.3** A quasi-coherent sheaf $\mathcal{F}$ on a scheme $X$ is a sheaf of $\mathcal{O}_X$ - modules such that for all affine set $U$ and distinguished opens $U_f$ contained in $U$, $U_f \overset{\iota}{\hookrightarrow} U$,

$$\mathcal{F}(U)\left[\frac{1}{f}\right] \xrightarrow{\cong} \mathcal{F}(U_f)$$

is an isomorphism.

**Definition 1.3.4** A quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules over a scheme $X$ is called coherent if all the modules are finitely generated.

## 1.4 The Functor of Points

Let $X$ be a scheme, define a contravariant functor $h_X$ as follows:

$$\mathbf{Schemes}^o \longrightarrow \mathbf{Sets} \tag{1}$$

$$Y \mapsto h_X(Y) = \mathrm{Hom}(Y, X) \tag{2}$$

For $f : Y \to Z$ a morphism of schemes, the corresponding morphism between sets is given by

$$h_X(Z) \to h_X(Y) \tag{3}$$

$$g \mapsto g \circ f \tag{4}$$

**Definition 1.4.1** A functor $F : (\mathbf{Schemes})^o \to \mathbf{Sets}$ is representable if it is of the form $h_X$, for some scheme $X$. Such $h_X$ is called functor of points of $X$.
The name "functor of points" is due to the fact that $h_X(Y)$ is the set of $Y$-valued points of $X$, for $Y$ a given scheme.

**Remark 1.4.1** If such $X$ exists, it is unique. "Representable" is a technical term and refers to the Yoneda Lemma.

Therefore, there exist an equivalence between the category of schemes and a full subcategory of the category of functors

$$\begin{aligned} h : \mathbf{Schemes} &\longrightarrow \mathbf{Fun}\left((\mathbf{Schemes})^o, \mathbf{Sets}\right) \\ X &\to h_X \\ \left\{f : X \to X'\right\} &\mapsto \{h_X(\bullet) \Rightarrow h_{X'}(\bullet)\} \end{aligned}$$

where, for an arbitrary scheme $Y$, the natural transformation sends a morphism to its postcomposition with $f$, i.e. $h_X(Y) \ni g \mapsto f \circ g \in h_{X'}(Y)$.

**Example 1.4.1** Affine Schemes Fact: Let $R \in \mathbf{CRing}$ be a commutative ring, the functor

$$h_X : (\mathbf{R\text{-}Schemes})^o \xrightarrow{\simeq} \mathbf{Fun}\left((\mathbf{R\text{-}Alg}), \mathbf{Sets}\right)$$

is an equivalence, and, $\forall$ R-algebra $A$, the induced functor reads

$$\begin{aligned} h_X^* : \mathbf{CRing} &\to \mathbf{Sets} \\ A &\mapsto h_X(\mathrm{Spec} A) \end{aligned}$$

# 2 Moduli Spaces and Moduli Problems

## 2.1 Flat Families of Schemes

**Definition 2.1.1** A family of schemes is a morphism $\pi : \mathcal{X} \to B$ of schemes. For $b \in B$, the fibers $X_b := \pi^{-1}(b)$ are the schemes in the family and $B$, usually called base, is the parameter space. Roughly, a family of schemes is a scheme defined by parametric equations.

**Definition 2.1.2** A family of closed subschemes of a given scheme $A$ over a base $B$ is a closed subscheme $X \subset B \times A$ together with the restriction to $X$ of the projection map $B \times A \xrightarrow{\pi_1} B$. The fibers $X_b$ are closed subschemes of the fibers $A_b$ of $B \times A$ over $B$.

Our aim is to construct a family of schemes "varying continuously". Naively, we may think of using a family that locally trivializes, i.e. that arises from a genuine projection of a fiber product to one of its factors. In general, this approach is wrong for two main reasons: asking locality in the Zariski topology is too restrictive, and the analytical approach - if $x \in X$ and $b = \pi(x)$, the completion of the local ring $\mathcal{O}_{X,x}$ is a power series ring over the completion of the local ring $\mathcal{O}_{B,b}$ - i.e. requiring smoothness, cuts out interesting families. The solution is asking for flatness.

Let $\mathcal{X}^*$ be the family of subchemes parametrized by $B^* := B \setminus \{0,1\}$ : $\mathcal{X}^* \subset \mathbb{A}^n_{B*} = \mathbb{A}^n_{\mathbb{Z}} \times B^*$ is given by the fibers $X_b$ of $\pi : \mathcal{X}^* \to B^*$. What is the limit of $X_b$ for $b \to 0$, for 0 any closed point?

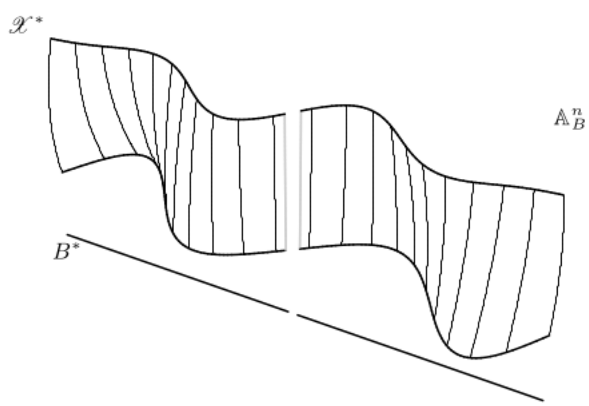


Figure 2: [EH00], p.71

Set $\mathcal{X} := \overline{\mathcal{X}^*}$ the closure of $\mathcal{X}^*$ in $\mathbb{A}^n_B$ and take the

$$\lim_{b \to 0} X_b = X_0 = \overline{\pi_1}^{-1}(0)$$

where $\overline{\pi_1}$ denotes the extension of the projection $\pi_1$ to the closure. Such limits can be calculated making use of Groebner bases.

**Definition 2.1.3** A module $M$ over a ring $R$ is flat iff for every monomorphism of $R$-modules $A \to B$ the induced map

$$M \otimes_R A \to M \otimes_R B$$

is a monomorphism.

**Example 2.1.1** Flat modules include free modules. Moreover, when $R$ is a field, every module is flat. If $R$ is a Dedekind domain, i.e. an integral domain in which every non-zero proper ideal factors into a product of prime ideals, $M$ is flat iff $M$ is torsion-free.

**Definition 2.1.4** A family $\pi : X \to B$ of schemes is flat iff for all point $x \in X$, the local ring $\mathcal{O}_{X,x}$, regarded as $\mathcal{O}_{B,\pi(x)}$-module is flat. In other words, if the morphism induced by $\pi^{\#} : \mathcal{O}_B \to \pi_* \mathcal{O}_X$,

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,\pi(x)}} \mathcal{O}_B \to \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,\pi(x)}} \pi_* \mathcal{O}_X$$

is a monomorphism.

**Remark 2.1.1** Flatness allows us to construct an algebraic analogue to the one of smooth morphisms of compact manifolds, where for $f \in C_c^{\infty}(M, N)$, a smooth morphism of compact manifolds, there exists a dense collection of opens $U$ in $N$ such that $f|_{f^{-1}(U)}$ is a fiber bundle.

## 2.2 Moduli Problems

Suppose we want to identify the set of (isomorphism classes of) non-singular projective curves of genus g over $k$ with the set of closed points of a "moduli scheme" $\mathcal{M}_g$. Using the functor of points approach this amount to find a scheme (if $\exists$) representing the functor

$$\mathcal{M}_g^{\text{fun}} : (k - \textbf{Schemes})^o \to \textbf{Sets}$$
$$B \mapsto \left\{ \mathcal{X} \xrightarrow{\pi} B \right\}$$

that takes an object $B$ into the set of flat morphisms $\pi : \mathcal{X} \to B$, whose fibers are non-singular curves of genus $g$ up to isomorphism, i.e. $\mathcal{X} \cong \mathcal{X}'$ as $B-$schemes.
Such a functor is not necessarily representable. An example elucidating the obstruction to representability is provided in the next subsection.

Moduli spaces are solutions to moduli problems: problems concerning the classification of geometric objects up to some equivalence (usually isomorphism).

## 2.3 The Hilbert Scheme

Subschemes of $\mathbb{P}_k^n$ with Hilbert polynomial $P$ form a scheme, denoted by $\mathcal{H}_P$.

**Definition 2.3.1** The Hilbert functor, the functor of flat families of schemes in $\mathbb{P}_{\mathbb{Z}}^n$ with Hilbert polynomial $P$ is the functor

$$h_P : (\textbf{Schemes})^o \to \textbf{Sets}$$
$$B \mapsto \{\mathcal{X} \subset \mathbb{P}_B^n \ : \ \mathcal{X} \text{ flat subsc. over } B \text{ whose fibers over } b \in B \text{ have Hilbert polynomial } P\}$$

The Hilbert scheme $\mathcal{H}_B$ is the scheme representing $h_P$.

**Remark 2.3.1** For $S = \mathbb{Z}[X_0, \dots X_n]$, $G_S(r, n)$ is an Hilbert scheme: it parametrizes subschemes $X$ of degree 1 and dimension $r$, i.e. those subschemes with Hilbert polynomial $P(m) = \binom{m+r}{r}$ in $\mathbb{P}_S^n$.

## 2.4 Obstructions to Representability

The functor defined in the previous section is not always representable. We will investigate possible obstructions to its representability by studying the example below, involving the classification of non-singular curves of genus one in the complex plane $\mathbb{C}$, up to isomorphism.

**Example 2.4.1** The functor $\mathcal{M}_1^{\text{fun}}$, for the problem of classifying non-singular projective curves of genus one over the complex numbers, is not representable.

In fact, there exists no universal family, i.e. a flat morphism $\mathcal{C} \xrightarrow{\pi} \mathcal{M}_1$ whose fibers are non-singular curves of genus one, such that for every family of non-singular curves of genus one $\mathcal{Y} \xrightarrow{\eta} B$ there are unique morphisms $\phi$ and $\Phi$ forming a fiber product diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\Phi"] \arrow[d, "\eta"'] & \mathcal{C} \arrow[d, "\pi"] \\ B \arrow[r, "\phi"'] & \mathcal{M}_1 \end{tikzcd}$$

It is not even possible to define a tautological family: a morphism $\pi : \mathcal{C} \to \mathcal{M}$ and a bijection between closed points of $\mathcal{M}_1$ and the set of non-singular curves of genus one such that $\forall p \in \mathcal{M}_1$ closed, $\pi^{-1}(p) = [p]$, the isomorphism class over $p$.
This is due to a local and a global obstruction, the latter rooted in the presence of automorphisms.

- *The local obstruction*: Such curves, given parametrically by $y^2 = x(x-2)(x-\lambda)$, for $\lambda \in \mathbb{C} \setminus \{0, 1\}$, are classified by teir j-invariants

$$j(\lambda) = 256 \cdot \frac{\left(\lambda^2 - \lambda + 1\right)^3}{\lambda^2 (\lambda - 1)^2} \tag{5}$$

  In other words, two curves in such parametrized family $C_\lambda$ and $C'_\lambda$ are isomorphic iff $j(\lambda) = j(\lambda')$. For reasons we will not prove here, if such a tautological family would have existed, there should have been one with parameter space the affine line and coordinate $j$. No such a family exists: at the point $j = 0$, the numerator vanish and $j$ vanish with multiplicity three. Moreover, since $j'(-1) = 0$, $j = 1728$ with even multiplicity.

- *The global obstruction*: If a tautological family exists, it could not be universal. Exclude the points corresponding to the two pathological values $\lambda = 0$ and $\lambda = 1728$, a tautological family can in fact be constructed over the line $\mathbb{A}^1_{\mathbb{C}} \setminus \{0, 1728\}$. However, such a family is not universal: Fix $\lambda$, let $B'$ be a variety with fixed-point-free involution $\rho$, and consider the family over $B := B'/\langle\rho\rangle$ obtained from $E \times B$ by the involution

$$((x, y), p) \mapsto ((x, -y), \rho(p))$$

  where $E$ denotes the total space. All of the fibers of this family are isomorphic to $C_\lambda$, therefore it has to come from the constant map $B \to \mathcal{M}_1$. However, it is possible to prove that this family is non-trivial.

The presence of automorphisms causes issues: For a general moduli space two families $\mathcal{X}, \mathcal{X}'$ can be fiber-wise isomorphic, but in the presence of automorphisms on the fibers, there is no way to deduce that those families are isomorphic as $B$-schemes, i.e. $\mathcal{X} \cong \mathcal{X}'$, since those automorphisms are not unique.

For moduli spaces of curves of genus $g \geqslant 4$, it is possible to study automorphism-free moduli spaces (denoted by the 0 as superscipt) of non-singular curves $\mathcal{M}_g^0$. In general, there are two techniques to deal with those obstructions: taking an approximation by a *coarse* moduli space or switching to an algebraic stack. We will pursue the latter, following [M65].

# 3 Étale Co-homology

## 3.1 From Classical Topology to Sites

In classical topology, we define a topology over a space $X$ to be a collection of subsets on $X$ that we call "open" and denote by $\tau$. If we look closely at the relations between those open sets, we realize that they are ordered by set-theoretical inclusion $\subseteq$. In other words, if $U \subseteq V$, there exists an inclusion map

$$f_{U,V} : U \hookrightarrow V$$

In other words, each inclusion of open sets can be described by

$$\mathrm{Hom}\,(U, V) = \begin{cases} f_{U,V} & \text{if } U \subset U \\ \varnothing & \text{otherwise} \end{cases}$$

With this observation, we can define a category $\mathbb{C}$ of open sets, by the following:

1. For $U, V$ open, $\mathrm{Hom}\,(U, V)$ contains at most one element;
2. In $\mathcal{C}$ there exists finite products (corresponding to intersections) and arbitrary sums (unions), and $\mathbb{C}$ has a final object;
3. $V \cap [\bigcup_i U_i] = \bigcup_i (V \cap U_i)$

On such a category $\mathbb{C}$ we are able to define a sheaf $\mathcal{F} : \mathbb{C}^o \to \mathbf{Sets}$. For $U := \bigcup_i U_i$, the sequence

$$\mathcal{F}\,(U) \longrightarrow \prod_i \mathcal{F}\,(U_i) \Longrightarrow \prod_{i,j} \mathcal{F}\,(U_i \cap U_j)$$

is exact and the global sections are given by $\Gamma\,(\mathcal{F}) := |\mathcal{F}\,(X)\,|$.

Co-homology groups of a space $X$, $H^\bullet\,(X; A)$ can be defined with coefficients in groups, Abelian groups $A \in \mathbf{Ab}$. Therefore, in order to use the tools of homological algebra in this context, we require for the sheaf over $\mathbb{C}$ to take values in Abelian groups

$$\mathcal{F} : \mathbb{C}^o \to \mathbf{Ab}$$

**Definition 3.1.1** A sheaf $F$ taking values in $\mathbf{Ab}$ is said to be an Abelian sheaf.

The category of Abelian sheaves $\mathbf{Sh}(\mathbf{Ab})$ is Abelian with enough injectives and the functor

$$\Gamma : \mathbf{Sh}\,(\mathbf{Ab}) \to \mathbf{Ab}$$

is left-exact.

For $\mathcal{F}$ an Abelian sheaf, we define $H^i(\mathcal{F}) : i \geqslant 0$ to be the i-th derived functor.

A further idea of Grothendieck was to drop the assumption that $\operatorname{Hom}(U, V)$ has at most one element by allowing non-trivial automorphisms. This yields the concept of covering

$$\pi_\alpha : U_\alpha \to U$$

**Definition 3.1.2** A site is a category $\mathbb{C}$, whose objects are called "opens", equipped by a Grothendieck topology given by a class of coverings $\operatorname{Cov}(U)$ for every object $U \in \mathbb{C}_0$.

For more about Grothendieck topologies, see my preprint: Gabriel-Ulmer Duality for Topoi, An Introduction, Section 3.1 available at: https://arxiv.org/pdf/2406.04965.

## 3.2 Étale morphisms and Co-homology

In this section schemes are assumed to be seaprated and of finite type. We recall those definitions.

**Definition 3.2.1** Given any map $\alpha : X \to S$ of schemes, define the diagonal subscheme $\Delta \subset X \times_S X$ to be the subscheme defined locally on $X \times_X X$ for each affine open

$$\operatorname{Spec}(A) \subset X \overset{\alpha/\operatorname{Spec} A}{\longrightarrow} \operatorname{Spec}(B) \subset S$$

by the ideal $I := \langle a \otimes 1 - 1 \otimes a \rangle$, generated by $a \otimes 1 - 1 \otimes a \in A \otimes_B A$.

**Definition 3.2.2** $\alpha$ is separated, or $X$ is separated as a scheme over $S$, iff $\Delta$ is a closed subscheme of the product $X \times_X X$.

**Example 3.2.1** Affine schemes are separated.

**Definition 3.2.3** A morphism of schemes $\phi : X \to Y$ is of finite type iff $\forall y \in Y$ there exists $V$ an open sffine $V = \operatorname{Spec}(B) \subset Y$ of $y$ and a finite covering $\phi^{-1}(V) = \bigcup_i U_i$ by affine open sets $U_i \cong \operatorname{Spec}(A_i)$ such that

$$\phi_V^{\#} : B := \mathcal{O}_Y(V) \to \mathcal{O}_X\left(\phi^{-1}V\right) \to \mathcal{O}_X(U_i) =: A_i$$

makes each $A_i$ into a finitely generated algebra over $B$.

**Example 3.2.2** Any subscheme $X$ of $\mathbb{A}_k^n$ or $\mathbb{P}_k^n$ is of finite type over $k$, i.e. the structure morphism $X \to \operatorname{Spec}(k)$ is of finite type.

**Definition 3.2.4** If $f : X \to Y$ is a morphism of schemes, for all closed points $y \in Y$ $f^{-1}(y)$ is a finite set and $\forall x \in f^{-1}(y)$ $\quad f^* : \mathcal{O}_y \to \mathcal{O}_x$ gives rise to an isomorphism

$$\hat{f}^* : \hat{\mathcal{O}}_y \xrightarrow{\cong} \hat{\mathcal{O}}_x$$

then $f$ is étale.

**Remark 3.2.1** Étale morphisms are the analogue of local homeomorphisms in standard topology.

**Definition 3.2.5** Let $X$ be a scheme, the étale topology $X_{\text{ét}}$ of $X$ consists of the following data:

1. A category whose objects $p : U \to X$ are étale morphisms, and morphisms $f$ are commuative triangles

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "p"'] & & V \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

2. Coverings are families $\{\pi_\alpha : U_\alpha \to U\}$ such that $U = \bigcup_\alpha \pi_\alpha (U_\alpha)$.

**Remark 3.2.2** The continuous map between the étale topology $X_{\text{ét}}$ and the Zariski topology$X_{\text{Zar}}$ over $X$, $\sigma : X_{\text{ét}} \to X_{\text{Zar}}$ extend to a coverings and fiber products preserving functor. There exists a third topology that we can put on a scheme, involving holomorphic maps.

**Definition 3.2.6** A curve over $k$ of genus g is a connected, reduced, one-dimensional scheme $X$ such that

$$\dim \left(H^1 (X, \mathcal{O}_X)\right) = g$$

Coming back to the obstructions in finding a solution to the moduli problem, if there exists a universal family for the moduli problem, it is a site on the parameter space $B$.

It is possible to show that if we require the covering in the topology for the moduli topology to be flat or smooth, instead of étale, there are no covering for the final object $M$ because of an infinite group of automorphism. The right topology for the moduli space $\mathfrak{M}$ is the étale topology.

# 4 The Picard Group

**Definition 4.0.1** An invertible sheaf $L$ on the moduli problem consists of the following two data:

1. For all families of non-singular curves $(\pi : \mathcal{X} \to S)$ an invertible sheaf $L(\pi)$ on $S$;

2. For all morphisms

$$\{\pi_1 : \mathcal{X} \to S_1\} \overset{F}{\mapsto} \{\pi_2 : \mathcal{X}_2 \to S_2\}$$

an isomorphism

$$L(\pi_1) \xrightarrow{\cong} F^* (L(\pi_2))$$

satisfying compatibility conditions with respect to pullbacks and composition.

An invertible sheaf is a procedure of attaching a one-dimensional vector space to every non-singular curve of genus g . For $C/k$ a curve and $\pi : C \to \text{Spec}(k)$ the projection, $L(\pi)$ is a one-dimensional vector space attached to $C$. Furthermore, given a family $\{\pi : C \to \text{Spec}(k)\}$, the one-dimensional spaces attached to its fibers form a line bundle and its sections are exactly $L(\pi)$ an invertible sheaf, i.e. this procedure is canonical.

**Definition 4.0.2** The set of isomorphism classes of such invertible sheaves is called the Picard group of the moduli problem $\mathfrak{M}$, denoted by $\text{Pic}(\mathfrak{M})$.

**Remark 4.0.1** The Picard group is an Abelian group with tensor product.

The moduli problems $\mathfrak{M}$ are sites. Let $\pi : \mathcal{X} \to S$ be an open set in the moduli problem equipped with the étale topology (i.e. a modular family of non-singular curves - they have all same dimension and if a curve occurs over the points $s_1$ and $s_2$, the fibers are isomorphic $\pi_1^{-1}(s_1) \cong C \cong \pi_2^{-1}(s_2)$.).

It is possible to show that the assignment $o(\mathcal{X} \to S) = \Gamma(S, o_S)$, defines a sheaf of rings over the moduli problem. Denote by $o^*$ its group of units.

The Picard group can be calculated solely by means of the étale co-homology, once the moduli problem $\mathfrak{M}$ is equipped with the étale topology: opens are modular families of non-singular curves, there exists a final object $M$ and a collection of morphisms onto $M$ define a covering if every curve $C$ occurs exactly in one of the collections.

**Theorem 4.0.1** [ [M65], p.67 et seq.] The Picard group of the moduli problem $\mathfrak{M}$ is given by the first (étale) co-homology group of $\mathfrak{M}$ with coefficient in the group of units

$$\mathrm{Pic}(\mathfrak{M}) \simeq \mathrm{H}^1(\mathfrak{M}, o^*)$$

This result has a beautiful proof using the tecnique of descends, but it will be not covered in this general introduction to advanced tools for moduli problems.

## References

[EGAI] Grothendieck, A., Elements de Geometrie Algebrique I, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 166, 1971, Springer-Verlag Berlin-Heidelberg;

[EH00] Eisenbud, D., Harris, J., The Geometry of Schemes, Graduate Texts in Mathematics n. 197, 2000, Springer-Verlag New York, Inc.;

[FGA] Grothendieck, A., Fondements de la Geometrie Algebrique, Extraits du Seminarie Bourbaki, 1957-1962, Technique de descente et Theoremes d'Existence en Geometrie Algebrique, V. Les schemas de Picard: Theoremes d'Existence et VI. Les schemas de Picard: Proprietes Generales, Fevrier et Mai 1962, 1962, Secretariat Mathematique, 11 rue Pierre Curie, Paris 5e;

[H92] Harris, J., Algebraic Geometry: A First Course, Graduate Texts in Mathematics n. 133, 1992, Springer Science+Business Media, New York;

[HM98] Harris, J., Morrison, I., Moduli of Curves, Graduate Texts in Mathematics n. 187, 1998, Springer-Verlag New York, Inc., New York;

[HS77] Hartshorne, R., Algebraic Geometry, Graduate Texts in Mathematics n.52, 1977, Springer Science+Business Media, New York;

[M65] Mumford, D., Picard groups of Moduli Problems, in Arithmetical Algebraic Geomtry, Proceeding of the Conference Held at Purdue University Dec., 5-7, 1963, Harper's Series in Modern Mathematics, 1965, Harper & Row Publ. Inc., New York.