# Posites: The Foundation of Factorization Homology Notes *preliminary version for 7 min's talk* @CT2025

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# 1 Basics

## 1.1 Quantum Observables

Prefactorization algebras (PFAs) arise naturally as mathematical formalization of quantum observables. In [2] §1 a series of 'quantum measurements' can be described by coupling the system to a first measuring device $O$ during the time interval $U$ and to a second device $O'$ during the time interval $U'$.

**Remark 1.1.1** In the above we are considering time, so this structure has to be defined more precisely on the real line and not on a generalized open set.

Since $U$ and $U'$ are pairwise disjoint and contained in a bigger interval of time $V$, it is reasonable to formalize observables of a quantum theory as a functor from the open sets of the space to a suitable category in a way that there exists a map

$$\mathrm{Obs}^{\mathrm{q}}(U) \otimes \mathrm{Obs}^{\mathrm{q}}\left(U'\right) \to \mathrm{Obs}^{\mathrm{q}}(V)$$

the factorization map.

**Remark 1.1.2** Such map is only defined, again, when the open sets $U$ and $U'$ are disjoint and contained in $V$. This assumption it is not needed for classical observables, e.g. the factorization map can be supported on the interval $I$ alone.

Moreover, as standard practice in quantum mechanics, the quantum observables must agree with the classical ones in the classical limits. In other words,

$$\lim_{\hbar \to 0} \frac{1}{\hbar} \mathrm{Obs}^{\mathrm{q}}(U) = \mathrm{Obs}^{\mathrm{cl}}(U)$$

These simple ideas lies at the core of the whole theory of prefactorization algebras, factorization algebras, and factorization homology.

## 1.2 The frame of open sets

We generalize the first idea in the above section, and define observables on open sets of a topological space, or better on the connected open sets of a topological space, whose frame we will denote by $\mathrm{Open}^c(M)$, for $M \in \mathbf{Mfld}_0^n$ a manifold of dimension $n$.

**Remark 1.2.1** The restriction to connected components is not needed, but useful, since, if the open set is not connected, e.g. $U = U_1 \dot{\bigcup} U_2$, the factorization map will factorize through the connected components again, leading to a further refinement of the previous functor. Therefore, connected open sets are the building blocks of the theory.

Open sets can be seen as a frame, and as a locale via the duality $\mathbf{Fr}^{\mathrm{op}} = \mathbf{Loc}$.

Define the following preorder on the family of connected open sets $\mathrm{Open}^c_M$, where $M$ is the manifold of dimension $n$ under exam [ [6], §3.3.1, eq's 3.41, 3.42]:

$$U \leqslant V \iff U \subseteq V$$
$$(U_1, \ldots, U_n) \leqslant V \iff \dot{\bigcup} U_i \subseteq V$$

where $U, V, U_i \in \mathrm{Open}_M^c \ \forall i$.

The associated symmetric thin (or posetal) multicategory $\mathbf{Open}_M^c$ is defined in [6], Definition 3.3.6., having objects the class $(\mathbf{Open}_M^c)_0$, and as morphisms arrows from a finite string of objects to a target object, with the following property:

$$\mathbf{Open}_M^c (U_1, \ldots, U_n; V) = \begin{cases} \{\varnothing\} & \iff \dot{\bigcup} U_i \nsubseteq V \\ \{*\} & \iff \dot{\bigcup} U_i \subseteq V \wedge U_i \cap U_j = \varnothing \ \forall i \neq j \end{cases} \tag{1}$$

that compose according to the general rules of a classical multicategory, and whose composition respect the associative and identity laws.

# 2 From Prefactorization Algebras to Factorization Homology

## 2.1 Prefactorization Algebras

Prefactorization algebras, and locally constant prefactorization algebras, model the behavior of quantum observables. The definition of prefactorization algebra (in short PFA) is given in [6] Definition 3.3.7 et seq. We recall it below:

**Definition 2.1.1** A prefactorization algebra $\mathcal{F}$ is an algebra over the symmetric posetal multicategory $\mathbf{Open}_M^c$. In other words, it is a $\mathbf{Open}_M^c$-algebra with values in a suitable category $\mathbf{C}$: A multifunctor

$$\mathcal{F} : \mathbf{Open}_M^c \to \mathbf{C}$$

Prefactorization algebras assemble in a category, denoted by $\mathbf{PFA}_M(\mathbf{C})$, whose arrows are natural transformations between such algebras, see [6], Definitions 3.3.10 - 3.3.13.

**Remark 2.1.1** [ [6], Remark 3.3.10] The legs of the natural transformations may have additional properties, e.g. they become chain maps, if the target category is $\mathbf{Ch}(\mathbf{E})$, for some preadditive category $\mathbf{E}$.

Let us specialize to the case of a prefactorization algebra defined on the real line with values in vector spaces.

**Definition 2.1.2** [ [6], Definition 3.4.4] A prefactorization algebra $\mathcal{F} \in \mathbf{PFA}_{\mathbb{R}}(\mathbf{Vect}_{\mathbb{R}})$ is locally constant iff given any multiarrow

$$f \in \mathbf{Open}_M^c (I_1, \ldots, I_n : I)$$

the corresponding image under $\mathcal{F}$

$$\mathcal{F}(f) \in \mathbf{Vect}_{\mathbb{R}} (\mathcal{F}(I_1), \ldots, \mathcal{F}(I_n); \mathcal{F}(I))$$

is an isomorphism, i.e. $\mathcal{F}(I_j) \cong \mathcal{F}(I) \ \forall j : 1, \ldots, n$.

**Remark 2.1.2** Locally constant prefactorization algebras on the real line are associative algebras. In fact, the isomorphism just described reads (in the category of vector spaces over the real line) as an assignment of a map

$$\mu : \mathcal{F}(I)^{\otimes_n} \to \mathcal{F}(I)$$

that can be proven to be the multiplication map of an associative algebra structure.

## 2.2 Factorization Algebras

We would like to extend the theory including local-to-global data. This can be done by defining the notion of Weiss covering.

**Definition 2.2.1** [ [2], Definition 1.2.1] Let $U$ be an open set, a collection of open sets $\mathfrak{U} := \{U_i \mid i \in I\}$ is a Weiss cover for $U$ iff for all finite collections of points $\{x_1, \ldots, x_k\}$ in U, there is an open set $U_i \in \mathfrak{U}$ such that $\{x_1, \ldots, x_k\} \subset U_i$.

Such a cover defines a topology on the space, called Weiss topology.

**Remark 2.2.1** [ [2], Example page 171] A Weiss cover equips the manifold with way more open sets, it is a sort of 'exponentiation' operation.
If $(M, g)$ is a smooth manifold equipped with a Riemannian metric $g$, given a collection of points $\{q_n\}_{n\in\mathbb{N}}$ of $M$, taking all finite, pairwise disjoint unions of disks from the class

$$\mathfrak{D} := \left\{D_{1/m}(q_n) \mid m \in \mathbb{N}, n \in \mathbb{N}\right\}$$

for $D_{1/m}(q_n) := \{q \in M : d(q_n, q) < 1/m\}$ defines a countable Weiss cover.

**Definition 2.2.2** [ [2], Definition 4.0.1] A prefactorization algebra $\mathcal{F}$ is a (strict) factorization algebra iff it is a cosheaf with respect to the Weiss topology (co-descent condition). Moreover, it is multiplicative iff the structure map is an isomorphism

$$\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \dot{\cup} V) \tag{2}$$

for every pair of disjoint opens $U, V$.

**Definition 2.2.3** [ [2], Definition 4.0.1] A factorization algebra $\mathcal{F}$ on a n-manifold $M$ is locally constant if for each inclusion of open disks $D \subset D'$, the map $\mathcal{F}(D) \to \mathcal{F}(D')$ is a quasi-isomorphism.

**Remark 2.2.2** [ [2], Theorem 4.0.2 et seq.] There is an equivalence of $(\infty, 1)$-categories between $E_n$ algebras and locally constant factorization algebras on $\mathbb{R}^n$. Finally, we observe that $E_n$-algebras form a full subcategory of the category of factorization algebras on $\mathbb{R}^n$.

We rephrase the definition in topos-theoretical terms.

**Definition 2.2.4** [ [1], Definition 2.20] In the Weiss Grothendieck topology on the category $\mathbf{Mfld}^n$, a sieve $\mathfrak{U} \subset \mathbf{Mfld}^n_{/M}$ is a covering sieve if, for each finite subset $S \subset M$, there is an object $\left(U \xhookrightarrow{e} M\right) \in \mathfrak{U}$ for which $S \subset e(U)$.

The $\infty$-category of Weiss sheaves on $\mathbf{Mfld}^n$ is the full $\infty$-subcategory

$$\mathbf{Shv}^W(\mathbf{Mfld}^n) \subset \mathbf{PShv}(\mathbf{Mfld}^n)$$

consisting of those presheaves $\mathcal{G} : \mathbf{Mfld}^{\mathrm{op}} \to \mathbf{Spaces}$ for which the canonical functor

$$(\mathfrak{U}^{\mathrm{op}})^{\triangleleft} \simeq (\mathfrak{U}^{\triangleright})^{\mathrm{op}} \longrightarrow \left(\mathbf{Mfld}^n_{/M}\right)^{\mathrm{op}} \longrightarrow (\mathbf{Mfld}^n)^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathbf{Spaces}$$

is a limit diagram for each covering sieve as defined above.

**Proposition 1** [ [1], Proposition 2.22] Restriction along $\mathcal{D}\mathrm{isk}^n \hookrightarrow \mathcal{M}\mathrm{fld}^n$ defines an equivalence between $\infty$- categories

$$\mathbf{Shv}^W (\mathcal{M}\mathrm{fld}^n) \xrightarrow{\simeq} \mathbf{PShv} (\mathcal{D}\mathrm{isk}^n)$$

**Remark 2.2.3** $\mathcal{M}\mathrm{fld}^n$ denotes the category of n-manifolds equipped with the $C^\infty$ compact-open topology, while the category $\mathbf{Mfld}^n$ carries the discrete topology. Similarly for the dubcategory of disks. Such categories of disks are symmetric monoidal and consist of disjoint unions of n-dimensional Euclidean spaces.

**Definition 2.2.5** [ [1], Definition 3.12] If $\mathcal{V}$ is a symmetrci monoidal $\infty$-cat, $M$ an n-manifold, the $\infty$-category of $\mathcal{V}$-valued factorization algebras on $M$ is the full $\infty$-subcategory of algebras in $\mathcal{V}$ over the multicategory $\mathbf{Open}^c (M)$ as in the pullback among $\infty$-categories

$$\begin{tikzcd}
\mathbf{Alg}_M (\mathcal{V}) \arrow[r] \arrow[d] & \mathbf{Alg}_{\mathbf{Open}^c_M} (\mathcal{V}) \arrow[d] \\
\mathbf{cShv}^W_{\mathcal{V}} (M) \arrow[r] & \mathbf{Fun} (\mathbf{Open}^c (M), \mathcal{V})
\end{tikzcd}$$

## 2.3 Factorization Homology

All of the above can be viewed in terms of factorization homology, a $\mathcal{V}$-valued functor from two inputs, one geometric, the other algebraic (an n-manifold, and an n-disk algebra (or a stack over it))

$$(M, A) \mapsto \int_M A$$

In fact, factorization algebras in the work of Costello and Gwilliam can be recovered by the assignment

$$A \mapsto \left\{ U \mapsto \int_{U \subseteq M} A \right\} \tag{3}$$

where $\int A : \mathbf{Open}^c (M) \to \mathcal{V}$, is an $\mathcal{V}$-valued algebra on the posetal category of open connected sets.

# 3 Posites as Foundation: Coming soon ...

In the setting of the theory, we are naturally dealing with a posite as in the work of Johnstone [ [4], Part C1.1, Examples 1.1.16 (e)], a site whose underlying category is posetal (thin).

Observations:

- Subterminal objects in $[P^{\mathrm{op}}, \mathbf{Set}]$ are functors $F$ such that $F(p) = \varnothing$ or $F(p) = \{*\}$, i.e. they are the lower subsets of $P$, up to iso. Sounds familiar?
- For a sheaf topos over **Top** the subterminal obj's are the open subsets of a topological space.

- An order structure can be given on a space without topology nor metric: The convex hull is defined only in terms of linear spaces!
- Thierry Coquand(1996) and Ayberk Tosum (2020) developped a formal system to deal with topology over a poset (also in univalent foundations), Here you are the original axioms:
  Given a Post system $(X, \mathrm{Cov})$ consisting of a set $X$ and a class of subsets $\mathrm{Cov} \subset \mathcal{P}(X)$. Let $U \in \mathcal{P}(X)$, define inductively $u \triangleleft U$ via

a. $u \triangleleft U$ if $u \in U$;

b. $u \triangleleft U$ if $S \triangleleft U$ for one $S \in \mathrm{Cov}_u$.

If we represent a derivation of $u \triangleleft U$ as tree, then its branching is determined by the size of the element in the cover. Sounds familiar?

## References

[1] Ayala, D., Francis, J., *A Factorization Homology Primer*, in Handbook of Homotopy Theory, Ch. 2, Chapman and Hall/CRC, 2020, https://doi.org/10.1201/9781351251624;

[2] Costello, K., Gwilliam, O., *Factorization Algebras in Quantum Field Theory*, Vol. 1, Cambridge University Press, Cambridge, 2016, https://doi.org/10.1017/9781316678626;

[3] Johnstone, P., T., *Stone Spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, England, 1982;

[4] Johnstone, P., T., *Sketches of An Elephant: A Topos Theory Compendium*, Oxford Logic Guides, vol. 43-44. Oxford University Press, Oxford, 2002;

[5] Mac Lane S., Moerdijk, I., *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Universitext, Springer-Verlag New York, Inc., 1994, https://doi.org/10.1007/978-1-4612-0927-0;

[6] Pasqualone, F., *Prefactorization Algebras: Introduction and Examples*, Master's Thesis in Mathematics, 2023, available online at https://github.com/Federica-Rike/Theses_Mathematics/blob/main/Federica_Pasqualone_FMaster_Thesis.pdf;

[7] Picado, J., Pultr, A., *Frames and Locales: Topology Without Points*, Frontiers in Mathematics, Birkhaeuser/Springer Basel AG, Basel, 2012.