MAXWELL'S EQUATIONS

VARIATIONAL FORMULATION FOR FINITE ELEMENTS METHODS

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Classical Formulation

1865 - "A Dynamical Theory of the Electromagnetic Field"

- Maxwell's Equations in vacuum for the 3D Euclidean space-time

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \cdot \vec{B} = 0 \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{J} \tag{2}$$

- Lorentz's Force

$$\vec{F} = q \cdot \vec{E} + q \cdot \vec{v} \wedge \vec{B}$$

- Continuity Equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Remark

Notice Maxwell's equations constitute a set of eight scalar eqs: six for the components of $\nabla \times \vec{E}$, $\nabla \times \vec{B}$ in (2) and two from the relations in (1). The latter are actually redundant: They can be obtained from (1) imposing homogeneous initial conditions.

Maxwell's Equations

1873 "Treatise on Electricity and Magnetism"

- General form of Maxwell's Equations for the 3D Euclidean space-time

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$$\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \qquad \frac{\partial \vec{D}}{\partial t} - \nabla \times \vec{H} = -\vec{J} \tag{4}$$

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- Continuity Equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \leftrightarrow \qquad \frac{\partial \left(\nabla \cdot \vec{B}\right)}{\partial t} = \frac{\partial \left(\nabla \cdot \vec{D} - \rho\right)}{\partial t} = 0 \tag{5}$$

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- (!)

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0 \qquad \forall\ \vec{A}\left(\vec{x}, t\right) \in \Omega \subseteq \mathbb{E}^3 \times \mathbb{R}^+$$

Time - Harmonic Maxwell's system

We apply the Fourier transform in time to all the involved quantities, considering a radiation with frequency $\omega > 0$, i.e. the fields and the densities assume the form:

$$\vec{A}(\vec{x}, t) = Re\left(e^{-i\omega t}\hat{\vec{A}}(\vec{x})\right) \tag{6}$$

The quantities $\hat{\vec{A}} := \hat{A}(\vec{x})$ are now complex-valued fields.

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- Time - harmonic Maxwell's Equations

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- Redundance of (7.1)

$$\nabla\left(-i\omega\hat{D} - \nabla \times \hat{H}\right) = \nabla\left(-\hat{J}\right) \overset{(!)}{\Longleftrightarrow} -i\omega\nabla \cdot \hat{D} = -\nabla \cdot \hat{J} \overset{(9)}{=} -i\omega\hat{\rho} \Longleftrightarrow (7.1) \tag{10}$$

Integral Formulation

Maxwell' s equations can be viewed also in their integral formulation. It allows also a reinterpretation of the whole theory in terms of differential forms.

Let S be a surface in the euclidean 3D-space, therefore:

$$-i\omega\hat{B} + \nabla \times \hat{E} = 0 \iff i\omega\hat{B} = \nabla \times \hat{E} \tag{11}$$

yields to the equation:

$$i\omega \int_S \hat{B} \cdot \hat{\nu} d\sigma = \int_S \left(\nabla \times \hat{E}\right) \hat{\nu} d\sigma \overset{Stokes' \, Th.}{=} \int_{\partial V} \hat{E} \cdot \hat{\tau} dS \tag{12}$$

Remark

Notice here that we calculate the circuitation of the electric field $\hat{E}$, whereas to for the magnetic one $\hat{B}$ we obtain a surface integral.

Figure: Examples of mesh's elements with flux.

Therefore, it is sensitive to cover the domain making use of rectangular or triangular meshes, whose edges are associated to the electric and faces to the magnetic field.
This is common practice among FEMs. It is also used in the FDTD scheme by Yee, however notice this is a time-domain method, for time-varying fields Link .

Constitutive Laws for linear media

The rules of propagation of waves are dependent on the material we are studying, namely:

- Vacuum or free space:

$$\hat{D} = \epsilon_0 \hat{E} \qquad \hat{B} = \mu_0 \hat{H} \tag{13}$$

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- Conductors:
  The existence of Eddy currents ▸ Link, behaved according to the following equation:

$$\hat{J} = \sigma \hat{E} + \hat{J}_a \tag{15}$$

leads to the second-order Maxwell system:

$$\nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E} \right) - k^2 \epsilon_r \tilde{E} = \tilde{F} \tag{16}$$

$$\nabla \cdot \left(\epsilon_r \tilde{E} \right) = -\frac{1}{k^2} \nabla \cdot \tilde{F} \tag{17}$$

(!) $\tilde{E} := \sqrt{\epsilon_0}\hat{E},\ k := \omega\sqrt{\epsilon_0\mu_0},\ \tilde{F} := ik\sqrt{\mu_0}\hat{J}_a,\ \mu_r := \frac{\mu}{\mu_0},\ \epsilon_r := \frac{1}{\epsilon_0}\left(\epsilon + \frac{i\sigma}{\omega}\right)$ (!) This formulation holds also for dielectrics ($\sigma = 0$).

Interface and boundary conditions

Consider two different materials with different magnetic and electric properties occupying the regions as in the figure:

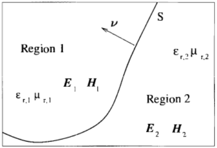


Figure: Geometry for interface boundary conditions

- In order for $\nabla \times \tilde{E}$ to be well-defined "in the least-squares sense"(L^2), we must have:

$$\nu \times \left(\tilde{E}_1 - \tilde{E}_2\right) = 0 \quad on\ S \tag{18}$$

and, similarly, for $\mu_r \tilde{H}$ to have a well-defined divergence ($\nabla \cdot \mu_r \tilde{H} = 0$):

$$\nu \cdot \left(\mu_{r,1}\tilde{H}_1 - \mu_{r,2}\tilde{H}_2\right) = 0 \quad on\ S \tag{19}$$

(!) $\tilde{H}_i := \sqrt{\mu_0}\hat{H}_i \quad \forall i = 1, 2.$

- Usually, the magnetic field has continuous tangential components, whereas the charge density may cause jumps in the normal component of the electric field, i.e.:

$$\nu \times \left(\tilde{H}_1 - \tilde{H}_2\right) = 0 \qquad \nu \cdot \left(\epsilon_{r,1}\tilde{E}_1 - \epsilon_{r,2}\tilde{E}_2\right) = \rho_S \quad \text{on} \ \ S \tag{20}$$

Scattering - 1/2

CASE 1: Scattering from a bounded, inhomogeneous object O occupying a domain D in $\mathbb{R}^3$

The incident field must satisfy the Maxwell's system in absence of the scatterer, i.e.

$$\nabla \times \nabla \hat{E}^i - k^2 \hat{E}^i = \boldsymbol{F} \tag{21}$$

where $\boldsymbol{F}$ is a function describing the current source.

The total field $\hat{E}$ is given by the sum of the incident and the scattered ones:

$$\hat{E} = \hat{E}^i + \hat{E}^s \tag{22}$$

The scattered field is outgoing (Silver-Müller radiation condition):

$$\lim_{\rho \to \infty} \rho \left(\left(\nabla \times \hat{E}^s \right) \times \vec{x} - ik\hat{E}^s \right) = 0 \tag{23}$$

where $\rho := ||\vec{x}||$. This limit is uniform in $\hat{x}$.

Example: [Plane wave]

$$\hat{E}^i = \vec{p} e^{ik\vec{x}\cdot\hat{d}} \tag{24}$$

where $\vec{p}$ is the polarisation, $\hat{d}$ is the direction of propagation versor and $< \vec{p}, \hat{d} >= 0$.

Scattering - 2/2

In order to solve the problem using FEMs, we consider a surface Σ far from the scatterer, and we impose the following absorption condition:

$$\left(\nabla \times \hat{E}\right) \times \nu - ik\hat{E}_T = \left(\nabla \times \hat{E}^i\right) \times \nu - ik\hat{E}^i_T \qquad (25)$$

where $\hat{E}^{(i)}_T := \left(\nu \times \hat{E}^{(i)}|_{\Sigma}\right) \times \nu$

Certainly, this is very different from solving the true scattering problem. However, the difference in the solutions can be made very small, if Σ is sufficiently far from O.

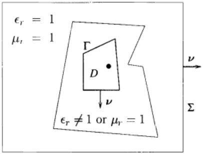


Figure: Scattering problem for FEMs imposing absorbing boundary condition on Σ

CASE 2: Unbounded scatterer in the lower half-plane

Consider w.l.o.g. the background medium to be composed by air and earth, namely at the common boundary we have the usual jumping conditions already discussed before, and the sources of the scattering to be in the air, i.e. $\boldsymbol{F} = 0$ if $x_3 < 0$.
The boundary region is now infinite in extension, thus we must impose an integral radiation condition:

$$\lim_{R \to \infty} \int_{\partial B^{\pm}_R} || \left(\nabla \times \hat{E}^s\right) \times \nu - ik\hat{E}^s||^2 dA = 0 \qquad (26)$$

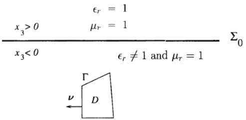


Figure: Scattering problem for FEMs from perfectly conductive objects in a layered medium.

Boundary value problems - 1/3

- **Time-harmonic problem in a cavity**

  Let Ω be a bounded domain with two disjoint connected boundaries Γ and Σ, we search for the solution of the following problem:

$$\begin{cases} \nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E} \right) - k^2 \epsilon_r \tilde{E} = \boldsymbol{F} \quad \text{in } \Omega \\ \nu \times \hat{E} = 0 \quad \text{on } \Gamma \qquad \leftarrow \text{ Perfect conducting boundary} \\ \mu_r^{-1} \left(\nabla \times \hat{E} \right) \times \nu - ik\boldsymbol{\lambda} \hat{E}_T = \boldsymbol{g} \quad \text{on } \Sigma \end{cases} \tag{27}$$

- **Cavity resonator**

  Let Ω be a bounded domain, $\Gamma := \partial\Omega$ Our aim is to find out k scalars and non-trivial electric fields such that:

$$\begin{cases} \nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E} \right) - k^2 \epsilon_r \tilde{E} = \boldsymbol{F} \quad \text{in } \Omega \\ \nu \times \hat{E} = 0 \quad \text{on } \Gamma \\ \nabla \cdot \left(\epsilon_r \hat{E} \right) = 0 \quad \text{in } \Omega \end{cases} \tag{28}$$

- **Scattering from a bounded object**

  The electromagnetic field has an unbounded domain $\mathbb{R}^3 \setminus \bar{D}$, where D is bounded with connected complement and Γ denotes the boundary of the unbounded component of $\mathbb{R}^3 \setminus \bar{D}$. We want to solve the following:

$$\begin{cases} \nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E} \right) - k^2 \epsilon_r \tilde{E} = \boldsymbol{F} \quad \text{in } \Omega \\ \nu \times \hat{E} = 0 \quad \text{on } \Gamma := \partial \left(\mathbb{R}^3 \setminus \ \bar{D} \right) \\ \hat{E} = \hat{E}^i + \hat{E}^s \quad \text{in } \mathbb{R}^3 \setminus \ D \end{cases} \tag{29}$$

Boundary value problems - 2/3

Moreover, the scattered field must satisfy the Silver-Müller condition:

$$\lim_{\rho \to \infty} \rho \left(\left(\nabla \times \hat{E}^s \right) \times \vec{x} - ik\hat{E}^s \right) = 0 \tag{30}$$

where $\rho := ||\vec{x}||$. This limit is uniform in $\hat{x}$.

- Scattering from a buried object

  Consider the following regions:

$$\mathbb{R}^3{}_+ := \left\{ \vec{x} \in \mathbb{R}^3 \mid x_3 > 0 \right\} \qquad \mathbb{R}^3{}_- := \left\{ \vec{x} \in \mathbb{R}^3 \mid x_3 < 0 \right\}$$

We search for solutions of the problem (assume $\nu \times \hat{E} = 0 \quad \text{on } \Gamma := \partial D$):

$$\begin{cases} \nabla \times \nabla \times \tilde{E} - k^2 \tilde{E} = \boldsymbol{F} & \text{in } \mathbb{R}^3{}_+ \\ \nabla \times \nabla \times \tilde{E} - k^2 {\epsilon_r}^e \tilde{E} = 0 & \text{in } \mathbb{R}^3{}_- \setminus \bar{D} \\ \left[\nu \times \hat{E} \right] = 0 = \left[\nu \times \left(\nabla \times \hat{E} \right) \right] & \text{on } \Sigma_0 \ (!) \\ \hat{E} = \hat{E}^i + \hat{E}^s & \text{in } \mathbb{R}^3 \setminus \bar{D} \end{cases} \tag{31}$$

where the incident field satisfy the background Maxwell's system:

$$\begin{cases} \nabla \times \nabla \times \tilde{E}^i - k^2 \tilde{E}^i = \boldsymbol{F} & \text{in } \mathbb{R}^3{}_+ \\ \nabla \times \nabla \times \tilde{E}^i - k^2 {\epsilon_r}^e \tilde{E}^i = 0 & \text{in } \mathbb{R}^3{}_- \\ \left[\nu \times \hat{E}^i \right] = 0 = \left[\nu \times \left(\nabla \times \hat{E}^i \right) \right] & \text{on } \Sigma_0 \ (!) \\ \hat{E} = \hat{E}^i + \hat{E}^s & \text{in } \mathbb{R}^3 \setminus \bar{D} \end{cases} \tag{32}$$

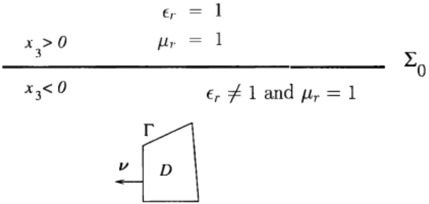
Boundary value problems - 3/3

Finally, we must impose integral radiation conditions on the scattered field $\hat{E}^s$:

$$\lim_{R\to\infty} \int_{\partial B_R^+} \| \left(\nabla \times \hat{E}^s\right) \times \nu - ik\hat{E}^s\|^2 dA = 0 \tag{33}$$

$$\lim_{R\to\infty} \int_{\partial B_R^-} \| \left(\nabla \times \hat{E}^s\right) \times \nu - ik\hat{E}^s\|^2 dA = 0 \tag{34}$$

The simple geometry we have used, is represented in the already - displayed picture:



HILBERT SPACES

Consider $\mathcal{X} \in \boldsymbol{Vct}_{\mathbb{C}}$, then we can define a measure on $\mathcal{X}$, given by the scalar product

$$(.,.)_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \to \mathbb{C} \tag{35}$$

that leads to an induced norm defined as $||\phi||_{\mathcal{X}} = \sqrt{(\phi, \phi)_{\mathcal{X}}} \quad \forall\ \phi \in \mathcal{X}$.

We call the normed space just obtained Hilbert space iff $\mathcal{X}$ is complete with respect to the induced norm. Complete means that every Cauchy sequence in $\mathcal{X}$ admits a limit in $\mathcal{X}$

Example: The space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^3$ open domain, is an Hilbert space with the following measure

$$(\phi, \xi)_{L^2(\Omega)} := \int_{\Omega} \phi \bar{\xi} dV \qquad \forall\ \phi,\ \xi \in L^2(\Omega) \tag{36}$$

The following well-known inequalities hold:

- Cauchy-Schwarz

$$||(u, v)_{\mathcal{X}}|| \leq ||u||_{\mathcal{X}} \cdot ||v||_{\mathcal{X}} \quad \forall u, v \in \mathcal{X} \tag{37}$$

- Arithmetic-Geometric mean

$$\forall u, v \in \mathcal{X}, \exists \delta > 0 : \qquad ||(u, v)_{\mathcal{X}}|| \leq \frac{\delta}{2} ||u||^2{}_{\mathcal{X}} + \frac{1}{2\delta} ||v||^2{}_{\mathcal{X}} \tag{38}$$

Convergence results

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ be a sequence, we have the following notions of convergence:

- Strong convergence

  $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ converges strongly to $u \in \mathcal{X}$ or, shortly, $u_n \underset{n\to\infty}{\longrightarrow} u$, iff

$$||u_n - u||_{\mathcal{X}} \underset{n\to\infty}{\longrightarrow} 0 \tag{39}$$

- Weak convergence

  $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ converges weakly to $u \in \mathcal{X}$ or, shortly, $u_n \underset{n\to\infty}{\rightharpoonup} u$, iff

$$(u_n, \phi)_{\mathcal{X}} \underset{n\to\infty}{\longrightarrow} (u, \phi)_{\mathcal{X}} \quad \forall \phi \in \mathcal{X} \tag{40}$$

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Here is a list of the relevant results:

I) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ be bounded, then there exists a weakly convergent extract sequence.

II) If $U \subset \mathcal{X}$ is a closed subspace and $f \in \mathcal{X}$, then

$$\exists\,!\; g \in U : ||f - g||_{\mathcal{X}} = \inf_{v\in U} ||f - v||_{\mathcal{X}}$$

III) If $U \subset \mathcal{X}$ is a closed subspace and $f \in \mathcal{X}$, then

$$\exists\,!\; u \in U,\ v \in U^{\perp} : f = u + v \Leftrightarrow \mathcal{X} = U \oplus U^{\perp}$$

(!) U closed means it contains all limits of its convergent sequences. In particular, $U^{\perp}$ is closed in $\mathcal{X}$.

Linear Operators: Basics

Consider two Hilbert spaces $\mathcal{X}, \mathcal{Y}$, an operator $A : \mathcal{X} \to \mathcal{Y}$ is said to be:

- linear iff

$$A(\alpha u + \beta v) = \alpha A(u) + \beta(v) \quad \forall \alpha, \beta \in \mathbb{C}, \ \forall u, v \in \mathcal{X} \tag{41}$$

- bounded iff

$$\exists C > 0 : \quad ||A\phi||_{\mathcal{Y}} \leq C \cdot ||\phi||_{\mathcal{X}} \quad \forall \phi \in \mathcal{X} \tag{42}$$

- continous iff

$$\forall \phi \in \mathcal{X} : \{\phi_n\}_{n\in\mathbb{N}} \underset{n\to\infty}{\longrightarrow} \phi \ \text{ in } \mathcal{X} \quad \Rightarrow \quad \{A\phi_n\}_{n\in\mathbb{N}} \underset{n\to\infty}{\longrightarrow} A\phi \ \text{ in } \mathcal{Y} \tag{43}$$

Important result in this setting:

Th: A linear operator is continuous if and only if it is bounded.

We define the operator norm of A as follows:

$$|||A||| := \sup_{\phi\in\mathcal{X}\setminus\{0\}} \frac{||A\phi||_{\mathcal{Y}}}{||\phi||_{\mathcal{X}}} \tag{44}$$

The range and the null-space of A are respectively defined as the subspaces:

$$A(\mathcal{X}) := \{y \in \mathcal{Y} | y = Ax \text{ for some } x \in \mathcal{X}\} \subset \mathcal{Y} \tag{45}$$

$$N(A) := \{x \in \mathcal{X} | Ax = 0\} \subset \mathcal{X} \tag{46}$$

Duality and Representability

Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be a bounded linear operator between Hilbert spaces, there exists a unique operator $A^\dagger \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, called adjoint of A, such that

$$(Ax, y)_{\mathcal{Y}} = \left(x, A^\dagger y\right)_{\mathcal{X}} \quad \forall x \in \mathcal{X},\ \forall y \in \mathcal{Y} \tag{47}$$

Consider f $\in \mathcal{L}(\mathcal{X}, \mathbb{C}) =: \mathcal{X}^*$ a bounded, linear functonal, we define its dual norm as follows:

$$|||f|||_* := \sup_{x \in \mathcal{X} \setminus \{0\}} \frac{||f(x)||_{\mathbb{C}}}{||x||_{\mathcal{X}}} \tag{48}$$

Theorem (Riesz' Representation)

Let $\mathcal{X}$ be a Hilbert space,

$$\forall\, g \in\ \mathcal{X}^*, \exists\, !\, u \in \mathcal{X} : \quad (u, v)_{\mathcal{X}} = g(v) \quad \forall v \in \mathcal{X} \tag{49}$$

Moreover, this isomorphism provides an isometry, i.e. $||u||_{\mathcal{X}} = ||g||_{\mathcal{X}^*}$.

Sesquilinear Forms

Let $\mathcal{X}, \mathcal{Y}$ Hilbert spaces, a mapping $a(.,.) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ defines a sesquilinear form if:

- $a(\alpha_1 u + \alpha_2 v, \phi) = \alpha_1 a(u, \phi) + \alpha_2 a(v, \phi) \quad \forall \alpha_1, \alpha_2 \in \mathbb{C},\ \forall u, v \in \mathcal{X},\ \forall \phi \in \mathcal{Y}$
- $a(u, \beta_1 \phi + \beta_2 \psi) = \overline{\beta_1} a(u, \phi) + \overline{\beta_2} a(v, \psi) \quad \forall \beta_1, \beta_2 \in \mathbb{C},\ \forall u \in \mathcal{X}, \forall \phi, \psi \in \mathcal{Y}$

Example: The scalar product in $L^2(\Omega)$ defines a sesquilinear form.

Given a sesquilinear form $a(.,.) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$, it is said to be:

- bounded if

$$\exists\ C > 0 \text{ ind. of } u \in \mathcal{X} \text{ and } \phi \in \mathcal{Y} : ||a(u, \phi)||_{\mathbb{C}} \leq C \cdot ||u||_{\mathcal{X}} ||\phi||_{\mathcal{Y}} \quad \forall u \in \mathcal{X}, \forall\ \phi \in \mathcal{Y} \tag{50}$$

- coercive if

$$\exists \alpha > 0 \text{ not dependent on } u \in \mathcal{X} : ||a(u, u)||_{\mathbb{C}} \geq \alpha \cdot ||u||_{\mathcal{X}}{}^2 \quad \forall u \in \mathcal{X} \tag{51}$$

(!) Here the sesquilinear form is internal, i.e. $a(.,.) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$

Variational Problems

Given an Hilbert space $\mathcal{X}$ and a bounded coercive sesquilinear form $a(.,.) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, consider the variational problem of finding $u \in \mathcal{X}$ such that

$$a(u, \phi) = f(\phi) \quad \forall \phi \in \mathcal{X} \tag{52}$$

where $f \in \mathcal{X}^*$ is a given functional.
The problem is well-posed (existence, uniqueness, stability). Indeed, the following theorems hold:

Theorem (Lax-Milgram)

Let $a(.,.) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, be a bounded with constant C, coercive with constant α, sesquiinear form, then $\forall f \in \mathcal{X}^*$ there exits a unique solution and, moreover,

$$||u||_{\mathcal{X}} \leq \frac{C}{\alpha} ||f||_{\mathcal{X}^*}$$

Theorem (Generalised Lax-Milgram)

Consider a bounded, sesquilinear form $a(.,.) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ such that:

I) $\exists\, \alpha : \inf\limits_{u \in \mathcal{X}: ||u||_{\mathcal{X}} = 1} \ \sup\limits_{v \in \mathcal{Y}: ||v||_{\mathcal{Y}} \leq 1} \left(||a(u, v)||_{\mathbb{C}} \right) \geq \alpha > 0$ *(Babuška-Brezzi condition)*

II) $\forall v \in \mathcal{Y},\ v \neq 0, \quad \sup\limits_{u \in \mathcal{X}} \left(||a(u, v)||_{\mathbb{C}} \right) > 0$

If $g \in \mathcal{X}^*$, $\exists\, !\ u \in \mathcal{X} : \quad a(u, \phi) = g(\phi) \quad \forall \phi \in \mathcal{Y}$ *and* $\quad ||u||_{\mathcal{X}} \leq \frac{C}{\alpha} ||g||_{\mathcal{Y}^*}$.

SOBOLEV SPACES - 1/2

The variational theory for Maxwell's equations relies on the existence of appropriate Sobolev spaces of scalar and vector functions. Therefore, we need to discuss some basics.

Relevant Sobolev Spaces: Consider a domain $\Omega \subset \mathbb{R}^N$, where $N = 1, 2, 3$

- $C^k(\Omega)$, the space of k-times continuously differentiable functions;
- ${C_0}^k(\Omega)$, the space of k-times continuously differentiable functions with compact support;
- $C^k\left(\overline{\Omega}\right)$, the space of k-times differentiable functions with bounded and uniformly continuous derivatives up to order k;
- $L^p(\Omega) : p \in [1, +\infty)$ the space of functions ϕ such that $\int_\Omega |\phi|^p dV < \infty$
- The space of distributions $T \in ({C_0}^\infty(\Omega))^*$ if $\forall K \subset \Omega$ compact set $\exists$ C, k constants :

$$||T(\phi)||_{\mathbb{C}} \leq C \sum_{||\vec{\alpha}||_1 \leq k} \sup_K \; ||D^{\vec{\alpha}}\phi||_{{C_0}^\infty(\Omega)} \qquad \forall \phi \in {C_0}^\infty(\Omega) \tag{53}$$

The distributional derivative $\partial^{\vec{\alpha}}\phi \in ({C_0}^\infty(\Omega))^*$ is the unique distribution such that

$$\left(\frac{\partial^{\vec{\alpha}}\phi}{\partial x^{\vec{\alpha}}}, \psi\right) = (-1)^{||\vec{\alpha}||_1}\left(\phi, \frac{\partial^{\vec{\alpha}}\psi}{\partial x^{\vec{\alpha}}}\right) \qquad \forall \psi \in {C_0}^\infty(\Omega) \tag{54}$$

(!) If $\phi \in \phi \in C^m(\Omega)$ the strong derivative and the distributional one agree, if $||\vec{\alpha}||_1 \leq m$.

Sobolev Spaces - 2/2

Consider Ω as before, $s \in \mathbb{Z}_+,\ p \in [1, \infty)$, we define the fundamental Sobolev space

$$W^{s,p}(\Omega) := \left\{ \phi \in L^p(\Omega) : \ \partial^{\vec{\alpha}} \phi \in L^p(\Omega) \ \forall \ ||\vec{\alpha}||_1 \leq s \right\} \tag{55}$$

with norm

$$||\phi||_{W^{s,p}(\Omega)} := \left(\sum_{||\vec{\alpha}||_1 \leq s} \int_\Omega |\partial^{\vec{\alpha}} \phi(x)|^p dV(x) \right)^{1/p} \tag{56}$$

and semi-norm

$$||\phi||_{W^{s,p}(\Omega)} := \left(\sum_{||\vec{\alpha}||_1 = s} \int_\Omega |\partial^{\vec{\alpha}} \phi(x)|^p dV(x) \right)^{1/p} \tag{57}$$

(!) For $p = 2$ there is the possibility to define the Sobolev spaces directly via Fourier transform. However, when $N = 2, 3$ it can be shown $H^s\left(\mathbb{R}^N\right) \cong W^{s,2}\left(\mathbb{R}^N\right)$.

For a bounded domain Ω, we define

$$H^s(\Omega) := \left\{ u \in (C_0^\infty(\Omega))^* : u = U|_\Omega \ \textit{for some} \ U \in W^{s,2}\left(\mathbb{R}^N\right) \right\} \tag{58}$$

In order to have a well-defined notion on norm, we set additionally

$$H^s_{\mathbb{R}^N \setminus \overline{\Omega}} := \left\{ u \in W^{s,2}\left(\mathbb{R}^N\right) : supp(u) \subset \mathbb{R}^N \setminus \overline{\Omega} \right\} \subset W^{s,2}\left(\mathbb{R}^N\right) \ \text{(closed sub.)} \tag{59}$$

Thus, there exists a well defined projection $W^{s,2}\left(\mathbb{R}^N\right) \to H^s_{\mathbb{R}^N \setminus \overline{\Omega}}$ and we obtain a measure in $H^s(\Omega)$ as follows:

$$(u, u)_{H^s(\Omega)} := \Big((I - P)\, U, (I - P) \Big)_{H^s(\Omega)} \tag{60}$$

where $u = U|_{|Omega}$, $v = V|_{H^s(\Omega)}$. If $\partial\Omega$ smooth enough, $H^2(\Omega) \cong W^{s,2}(\Omega)$ and $||.||_{H^2(\Omega)} = ||.||_{W^{s,2}(\Omega)}$.

Lipschitz Polyhedra

For Dirichlet boundary conditions, we define the appropriate Sobolev spaces considering the closure of the distribution space in the $W^{s,p}(\Omega)$ norm, i.e.

$$W_0{}^{s,p}(\Omega) := \overline{(C_0{}^{\infty}(\Omega))} \tag{61}$$

Therefore, for $p = 2$, $W_0{}^{s,2}(\Omega) \cong H_0{}^{s}(\Omega)$.
(!) It turns out the functions $u \in H_0{}^{1}(\Omega)$ satisfy $u|_{\partial\Omega} = 0$, i.e. are well-behaved solutions.

Our aim is to study Maxwell's equations on bounded domains (the unbounded can be truncated), however the Sobolev spaces are strongly influenced by the geometry.
A nice class of objects to cover with 2D(3D)-tetrahedra meshes are Lipschitz polyhedra.

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Lipschitz condition and Density

Theorem

Let Ω be bounded, Lipschitz domain in $\mathbb{R}^{\mathbb{N}}$, then:

I) $C^\infty\left(\overline{\Omega}\right)$ *is dense in* $W^{s,p}(\Omega)$, *for* $s \in \mathbb{Z}_+$, $p \in [1, \infty)$;

II) *(Cálderon extension) If* $s \in \mathbb{Z}_+$, $s \geq 1$, $p \in (1, \infty)$ *there exists a continuous, linear extension operator*

$$\Pi : W^{s,p}(\Omega) \to W^{s,p}\left(\mathbb{R}^N\right) \tag{62}$$

such that $(\Pi u)|_\Omega = u \quad \forall u \in W^{s,p}(\Omega)$. *If* $p \geq 2$ *the extension exists for all* $s \geq 0$.

III) $W^{s,2}(\Omega) \cong H^s(\Omega)$, $s \in \mathbb{Z}_+$ *with isometry.*

Lemma

Consider the space $\mathcal{Y} := \left\{ p \in H^1(\Omega) \mid p|_{\partial\Omega} \in H^1(\partial\Omega) \right\}$ *with the graph norm*

$$||p||^2{}_{\mathcal{Y}} = ||p||^2{}_{H^1(\Omega)} + ||p||^2{}_{H^1(\partial\Omega)} \tag{63}$$

then, $C^\infty\left(\overline{\Omega}\right)$ *is dense in* $\mathcal{Y}$.

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Lemma

If Ω *is a bounded and Lipschitz domain, then* $C_0{}^{\infty}(\Omega)$ *is dense in* $L^2(\Omega)$.

Embeddings of General Sobolev spaces - 1/2

A Sobolev space $W^{s,p}(\Omega)$ is said to be embedded in a space X iff

(I) $W^{s,p}(\Omega) \subset X$

(II) the embedding $\mathcal{I} : W^{s,p}(\Omega) \hookrightarrow X$ is continuous,

$$\exists C > 0 : ||\mathcal{I}u||_X \leq C||u||_{W^{s,p}(\Omega)} \qquad \forall u \in W^{s,p}(\Omega) \tag{64}$$

Let Ω^l denote the intersection of an l-dimensional hyperplane with Ω, under which conditions we can have an embedding $\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{m,p}\left(\Omega^l\right)$?

By point (I) of the definition, $u \in W^{m+j,p}(\Omega)$ is a limit of $\{u_n\}_{n\in\mathbb{N}} \in C^\infty\left(\overline{\Omega}\right)$ and these functions are such that $u_n|_{\Omega^l}$ are well-defined. Therefore, the embedding is so $u_n|_{\Omega^l}$ converge to a function in $W^{m,p}\left(\Omega^l\right)$

Theorem (Sobolev's embedding Theorem)

Let Ω be a bounded domain : $\partial\Omega$ is Lipschitz continuous, suppose $m, j \in \mathbb{Z}_+$, $l \in [1, N] \cap \mathbb{N}$, $p \in [1, \infty] \cap \mathbb{R}$, then:

- *If $mp < N$ and $N - mp < l \leq N$*

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}\left(\Omega^l\right) \qquad p \leq q \leq \frac{l \cdot p}{(N - mp)} \tag{65}$$

Embeddings of Sobolev spaces - 2/2

- If $mp = N$, for $l \in [1, N]$ and $q \in [p, \infty)$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}\left(\Omega^l\right) \tag{66}$$

- If $mp > N \geq (m-1)\,p$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow C^j\left(\overline{\Omega}\right) \tag{67}$$

$\mathcal{I}$ defines a compact embedding iff the embedding is a compact operator.

Theorem

Let $\Omega \subset \mathbb{R}^N$ e a bounded, Lipschitz domain and consider a subdomain $\Omega_0 \subseteq \Omega$. Let $j,\ m : m \geq 1,\ j \geq 0$ be integers, $p \in [1, \infty) \cap \mathbb{R}$, the following embeddings are compact:

- *If $mp \leq N$*

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}\left({\Omega_0}^l\right) \qquad 0 < N - mp, l \leq N \text{ and } 1 \leq q < \frac{l \cdot p}{(N - mp)} \tag{68}$$

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(!) For $m = 1,\ p = 2,\ j = 0, l = N, \Omega_0 \equiv \Omega$, the embedding below is compact in dimension $N = 2, 3$

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \tag{71}$$

Embeddings of Fractional Sobolev spaces

Theorem

Let Ω be a bounded Lipschitz domain, if $0 \leq t < s : \; s - \frac{3}{p} = t - \frac{3}{q}$, *we have the embedding*

$$W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega) \tag{72}$$

Furthermore, for $0 \leq t, s < \infty$ *and* $p = q = 2$ *this embedding is compact.*

Let $H^{-1}(\Omega) := \left(H_0{}^1(\Omega)\right)^*$ with usual dual norm.

Trace spaces - 1/3

Let $u \in C^\infty\left(\overline{\Omega}\right)$, we have a well-defined trace operator

$$\gamma_0\left(u\right) := u|_{\partial\Omega} \tag{73}$$

Theorem

Trace theorem If Ω is a bounded Lipschitz domain and $1/p < s \leq 1$, the trace operator has a unique continuous extension as a linear operator

$$W^{s,p}\left(\Omega\right) \twoheadrightarrow W^{s-1/p,p}\left(\partial\Omega\right) \tag{74}$$

Moreover,

$$W_0{}^{1,p} := \left\{u \in W^{1,p} \mid \gamma_0\left(u\right) = 0\right\} \tag{75}$$

(!) This dense space for $p > 1$ consists of function satisfying the homogeneous Dirichlet conditions on the boundary.

When $p > 1$ we can alternative define the above space as

$$W_0{}^{1,p} := \left\{u \in L\left(\Omega\right)^p \mid \nabla u \in \left(L\left(\Omega\right)^p\right)^3, \gamma_0\left(u\right) = 0\right\} \tag{76}$$

where $\nabla : \left(C_0{}^\infty\left(\Omega\right)\right)^* \rightarrow \left(C_0{}^\infty\left(\Omega\right)^*\right)^N$

Trace spaces - 2/3

We describe in details the relevant case $H^{1/2}(\partial\Omega) = W^{1/2,2}(\partial\Omega)$ and its dual $H^{-1/2}(\partial\Omega)$. In particular, for any Lipschitz surface S we have:

$$\langle f, g\rangle_S = \int_S f\bar{g}\,d\Sigma \tag{77}$$

Thus

$$||f||_{H^{-1/2}(\partial\Omega)} = \sup_{g\in H^{+1/2}(\partial\Omega)} \frac{||\langle f, g\rangle||_{\mathbb{C}}}{||g||_{H^{+1/2}(\partial\Omega)}} \tag{78}$$

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For $s > 1$ we define

$$H^s(\partial\Omega) := \left\{u \in L^2(\Omega) \mid u = U|_{\partial\Omega} \text{ for some } U \in H^{s+1/2}(\Omega)\right\} \tag{79}$$

with norm

$$||u||_{H^s(\partial\Omega)} := \inf_{U\in H^{s+1/2}(\Omega): u=U|_{\partial\Omega}} ||u||_{H^{s+1/2}(\Omega)} \tag{80}$$

In particular, $||u||_{H^s(\partial\Omega)} \equiv ||U||_{H^{s+1/2}(\Omega)}$, where $U \in H^{s+1/2}(\Omega)\ :\ U|_{\partial\Omega} = u$ and $(U,\phi)_{H^{s+1/2}(\Omega)} = 0 \quad \forall\, \phi \in H^{s+1/2}(\Omega) \cap {H_0}^1(\Omega)$.

(!) This function exists by Lax-Milgram lemma.

Trace spaces - 3/3

Notice that $H^s(\partial\Omega)$ is complete, since $H^{s+1/2}(\partial\Omega)$ is and therefore it is a Hilbert space.

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- There is no explicit norm that allows us to determine whether a function $g \in \partial\Omega$ is in $H^s(\partial\Omega)$.

Theorem

Let Ω be a bounded Lipschitz polyhedron with boundary-faces $\{\partial\Omega_j\}_{j\in J}$ and $g \in L^2(\partial\Omega)$:

- $g \in H^s(\partial\Omega_j) \quad \forall\, j \in J$ *and* $1 < s < 3/2$*:*
- *If* $\partial\Omega_j \cap \partial\Omega_k = \left\{e_{j,k}\right\}$ *then* $g|_{\partial\Omega_j} \equiv g|_{\partial\Omega_k}$ *on the common edge;*

then, $g \in H^s(\partial\Omega)$*, i.e. there exists an extension to* $H^{s+1/2}(\Omega)$*.*

We define the locally compact $L^2(\Omega)$ space as follows:

$$L_{loc}{}^2(\Omega) := \left\{p \in L^2(\mathcal{O}) \ \forall\, \mathcal{O} \subset \Omega \text{ compact subdomain}\right\} \tag{81}$$

Lemma

If Ω is a bounded, Lipschitz domain and $p \in L_{loc}{}^2(\Omega)$, $\nabla \cdot p \in H^{-1}(\Omega)^3$, then $p \in L^2(\Omega)$.

Well-posedness of the elliptic problems - 1/2

Consider the following mixed elliptic problem on a Lipschitz domain Ω such that $\partial\Omega = \Gamma_N \cup \Gamma_D$, $\Gamma_N \cap \Gamma_D \emptyset$ and ν is the unit outward normal to Γ_N:

$$\begin{cases} -\Delta\phi + c\phi & = f \quad \text{in } \Omega \\ \phi & = \mu_D \quad \text{on } \Gamma_D \\ \dfrac{\partial\phi}{\partial\nu} & = \mu_N \quad \text{on } \Gamma_N \end{cases} \tag{82}$$

$\phi \in H^1(\Omega)$ is a weak solution if

$$(\nabla\phi, \nabla\xi) + c(\phi, \xi) = (f, \xi) + \langle \mu_n, \xi \rangle_{\Gamma_N} \quad \text{and} \quad \phi \equiv \mu_D \text{ on } \Gamma_D \qquad \forall\, \xi \in H^1(\Omega) \; : \; \xi|_{\Gamma_D} = 0 \tag{83}$$

Theorem

Let Ω be a Lipschitz domain, $\mu \in H^{1/2}(\partial\Omega)$ and $f \in H^{-1}(\Omega)$, there exists a unique weak solution $\phi \in H^1(\Omega)$ of

$$-\Delta\phi + \phi = f \text{ in } \Omega \text{ and } \phi = \mu \text{ on } \partial\Omega \tag{84}$$

Furthermore, $\exists\; C > 0 \; : \; ||\phi||_{H^1(\Omega)} \le C\left(||\mu||_{H^{1/2}(\partial\Omega)} + ||f||_{H^{-1}(\Omega)}\right)$.

Lemma (Poincaré inequality)

There exists a positive constant C such that $\forall\; u \in H^1(\Omega)$ the following holds:

$$||u||_{H^1(\Omega)} \le C\left(||\nabla u||_{L^2(\Omega)} = \left|\int_{\partial\Omega} u \; d\Sigma\right|\right) \tag{85}$$

Well-posedness of the elliptic problems - 2/2

Theorem

Let Ω be a Lipschitz domain, let $\mu \in H^{1/2}(\partial\Omega)$, $f \in H^{-1}(\Omega)$, then $\exists\,!$ weak solution $\phi \in H^1(\Omega)$ of

$$-\Delta\phi = f \text{ in } \Omega \quad \text{and} \quad \phi = \mu \text{ on } \partial\Omega \tag{86}$$

Furthermore, $\exists\ C > 0\ :\ ||\phi||_{H^1(\Omega)} \leq C\left(||\mu||_{H^{1/2}(\partial\Omega)} + ||f||_{H^{-1}(\Omega)}\right).$

For Neumann boundary conditions, the following holds:

Theorem

Let Ω be a Lipschitz domain, ν the unit outward normal to the boundary. Let $\mu \in H^{-1/2}(\partial\Omega)$, $f \in \left(H^1(\Omega)\right)^*$, there exists a unique weak solution $\phi \in H^1(\Omega)$ of

$$-\Delta\phi + \phi = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial\phi}{\partial\nu} = \mu \text{ on } \partial\Omega \tag{87}$$

Furthermore, $\exists\ C > 0\ :\ ||\phi||_{H^1(\Omega)} \leq C\left(||\mu||_{H^{-1/2}(\partial\Omega)} + ||f||_{(H^1(\Omega))^*}\right).$

Properties of $H\left(div;\Omega\right)$ - 1/2

We define the following space of functions

$$H\left(div;\Omega\right) := \left\{\vec{u} \in \left(L^2\left(\Omega\right)\right)^3 \mid \nabla \cdot \vec{u} \in L^2\left(\Omega\right)\right\} \tag{88}$$

with norm

$$||\vec{u}||_{H(div;\Omega)} = \left(||\vec{u}||^2{}_{\left(L^2(\Omega)\right)^3} + ||\nabla \cdot \vec{u}||^2{}_{L^2(\Omega)}\right)^{1/2} \tag{89}$$

$H\left(div;\Omega\right)$ is an Hilbert space with the inherited inner product.

Theorem

Let Ω be a bounded 3D-Lipschitz domain, then

$$H\left(div;\Omega\right) = \overline{\left(C^\infty\left(\bar{\Omega}\right)\right)^3} \tag{90}$$

where the closure is taken in the $H\left(div;\Omega\right)$-norm.

(!) This statement holds whenever $\partial\Omega$ is bounded, also if Ω is not. This result will be helpful when dealing with the electromagnetic field at the interface of two different materials.
For it, define the normal trace operator as

$$\gamma_n\left(\vec{v}\right) = \vec{v}|_{\partial\Omega} \cdot \hat{\nu} \qquad \forall \vec{v} \ \in \ \left({C_0}^\infty\left(\Omega\right)^*\right)^3 \tag{91}$$

Properties of $H(div;\Omega)$ - 2/2

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain : ν is the unit outward normal on $\partial\Omega$, then:

I) *γ_n can be extend by continuity to a continuous linear mapping*

$$\gamma_n : H(div;\Omega) \to H^{-1/2}(\partial\Omega) \tag{92}$$

II) *If $\bar{v} \in H(div;\Omega)$, $\phi \in H^1(\Omega)$, then (Green's Th.):*

$$(\vec{v}, \nabla\phi) + (\nabla \cdot \vec{v}, \phi) = \langle \phi, \gamma_n(\vec{v}) \rangle_{\partial\Omega} \tag{93}$$

We set this definition:

$$H_0(div;\Omega) = \overline{(C_0{}^{\infty}(\Omega))^3} \tag{94}$$

where the closure is taken in the $H(div;\Omega)$-norm.

Theorem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, then:

$$H_0(div;\Omega) = \{\vec{v} \in H(div;\Omega) \ : \ \vec{v}|_{\partial\Omega} \cdot \hat{\nu} = 0\} \tag{95}$$

Properties of $H\left(curl;\Omega\right)$ - 1/4

Consider the following space

$$H\left(curl;\Omega\right) := \left\{\vec{v} \in \left(L^2\left(\Omega\right)\right)^3 \mid \nabla \times \vec{v} \in \left(L^2\left(\Omega\right)\right)^3\right\} \tag{96}$$

with norm

$$||\vec{v}||_{H(curl;\Omega)} = \left(||\vec{v}||^2{}_{(L^2(\Omega))^3} + ||\nabla \times \vec{v}||^2{}_{(L^2(\Omega))^3}\right)^{1/2} \tag{97}$$

(!) This space is of particular physical relevance, as it is associated to the finite-energy solutions of the Maxwell's system.

In particular, for $s \geq 0$, set:

$$H^s\left(curl;\Omega\right) := \left\{\vec{v} \in \left(H^s\left(\Omega\right)\right)^3 \mid \nabla \times \vec{v} \in \left(H^s\left(\Omega\right)\right)^3\right\} \tag{98}$$

Moreover, the closure is defined in the usual way, namely:

$$H_0\left(curl;\Omega\right) = \overline{\left(C_0^\infty\left(\bar{\Omega}\right)\right)^3} \tag{99}$$

where the closure is taken in the $H\left(curl;\Omega\right)$-norm.

PROPERTIES OF $H\left(curl;\Omega\right)$ - 2/4

THEOREM

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $\vec{u} \in H\left(curl;\Omega\right)$ be such that $\forall \phi \in \left(C^\infty\left(\bar{\Omega}\right)\right)^3$ such that

$$\left(\nabla \times \vec{u}, \phi\right) - \left(\vec{u}, \nabla \times \phi\right) = 0 \tag{100}$$

then $\vec{u} \in H_0\left(curl;\Omega\right)$

(!) This Green's identities is valid for $\vec{u} \in H_0\left(curl;\Omega\right)$ and $\phi \in H\left(curl;\Omega\right)$, as $\left(C^\infty\left(\bar{\Omega}\right)\right)^3$ is dense in $H\left(curl;\Omega\right)$.

Since we wish to interpret the space $H\left(curl;\Omega\right)$ as the energy, we must verify that functions in it have well-defined tangential trace.
For all $\vec{v} \in \left(C^\infty\left(\bar{\Omega}\right)\right)^3$, $\hat{\nu}$ unit outward normal to Γ, define the following tangential traces:

$$\gamma_t\left(\vec{v}\right) = \hat{\nu} \times \vec{v}|_{\partial\Omega} \tag{101}$$

$$\gamma_T\left(\vec{v}\right) = \hat{\nu} \times \left(\vec{v}|_{\partial\Omega} \times \hat{\nu}\right) \tag{102}$$

THEOREM

Let Ω be a bounded, Lipschitz, domain in $\mathbb{R}^3$, the trace map γ_t can be continuously extended from $H\left(curl;\Omega\right)$ into $\left(H^{-1/2}\left(\partial\Omega\right)\right)^3$.

Fourthermore, $\forall\ \vec{v} \in H\left(curl;\Omega\right)$ and $\phi \in \left(H^1\left(\Omega\right)\right)^3$,

PROPERTIES OF $H\left(curl;\Omega\right)$ - 3/4

(!) The tangential trace $\gamma_t : H\left(curl;\Omega\right) \rightarrow \left(H^{-1/2}\left(\partial\Omega\right)\right)^3$ is not surjective as $\forall\ \vec{v},\ \gamma_t\left(\vec{v}\right)$ is in the tangential space, but $\left(H^{-1/2}\left(\partial\Omega\right)\right)^3$ contains also vectors that are NOT tangential.

Thus, we define:

$$Y\left(\partial\Omega\right) = \left\{\vec{f} \in \left(H^{-1/2}\left(\partial\Omega\right)\right)^3 \ \mid \exists\ \vec{u} \in H\left(curl;\Omega\right) \text{ with } \gamma_t\left(\vec{v}\right) = \vec{f}\right\} \tag{104}$$

with norm

$$||\vec{f}|||_{Y(\partial\Omega)} = \underset{\vec{u}\in H(curl;\Omega):\gamma_t(\vec{v})=\vec{f}}{inf} \quad ||\vec{u}||_{H(curl;\Omega)} \tag{105}$$

The metric space so obtained is indeed a Banach space.

THEOREM

The space $Y\left(\partial\Omega\right)$ is Hilbert. The tangential trace $\gamma_t : H\left(curl;\Omega\right) \rightarrow Y\left(\partial\Omega\right)$ is surjective. The map $\gamma_T : H\left(curl;\Omega\right) \rightarrow Y\left(\partial\Omega\right)^*$ is well-defined and

$$\left(\nabla\times\vec{v},\vec{\phi}\right) - \left(\vec{v},\nabla\vec{\phi}\right) = \left\langle\gamma_t\left(\vec{v}\right),\gamma_T\left(\vec{\phi}\right)\right\rangle_{\partial\Omega} \quad \forall\ \vec{v},\vec{\phi} \in H\left(curl;\Omega\right) \tag{106}$$

PROPERTIES OF $H(curl;\Omega)$ - 4/4

THEOREM

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, then $H_0(curl;\Omega)$ is given by the following expression:

$$\{\vec{v} \in H(curl;\Omega) : \gamma_t(\vec{v}) = 0\} = \left\{\vec{v} \in H(curl;\Omega) : (\nabla \times \vec{v}, \phi) = (\vec{v}, \nabla \times \phi)\ \forall\, \vec{\phi} \in \vec{v} \in \left(C^{\infty}\left(\bar{\Omega}\right)\right)^3\right\} \tag{107}$$

THEOREM

Suppose Ω is a bounded, Lipschitz domain in $\mathbb{R}^3$ such that it is compactly contained as subdomain in $\mathcal{O}$, then there exists a bounded, linear operator

$$E : H(curl;\Omega) \to H\left(curl;\mathbb{R}^3\right) \tag{108}$$

such that

$$E \cdot \vec{v} = \vec{v} \quad \textit{in } \Omega \textit{ and } supp(E\vec{v}) \subset \mathcal{O}. \tag{109}$$

Scalar and Vector Potentials - 1/2

Theorem

Let $\Omega \subset \mathbb{R}^3$, $\vec{u} \in \left(C^1(\Omega)\right)^3$: $\nabla \times \vec{u} = 0$ on Ω, then $\forall\ \mathcal{O} \subset \Omega$ there exists a scalar function $\phi \in C^2(\Omega)$: $\vec{u} = \nabla \cdot \phi$ in $\mathcal{O}$.

(!) If Ω is a simply connected Lipschitz domain, taking unions of parallelepipeds we can show $\vec{u} = \nabla\phi$ in Ω. If we additionally require ϕ to have zero average value, we gain also uniqueness in *phi*.

Theorem

Consider $\Omega \subset \mathbb{R}^3$ is a simply connected Lipschitz domain and suppose $\vec{u} \in \left(L^2(\Omega)\right)^3$,

$$\nabla \times \vec{u} = 0 \text{ in } \Omega \iff \exists\ \phi \in H^1(\Omega) : u = \nabla\phi \tag{110}$$

This ϕ is unique up to an additive constant.

(!) We wish to use a vector potential in order to analyse the regularity of the solutions to the Maxwell's system.

Before doing that, however, it is better to spend some time in studying a convenient geometry for the domain Ω under consideration.

SCALAR AND VECTOR POTENTIALS - 2/2

Let Ω be a bounded, connected, Lipschitz domain, there exists a decomposition of the boundary $\partial\Omega$ into connected components as follows:

$$\partial\Omega = \bigcup \{\partial\Omega_J\}_{j=0}^{k} \quad \text{for some integer } k > 0 \tag{111}$$

where $\partial\Omega_0$ denotes the boundary of the unbounded component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

Let $\Omega_j \subset \mathbb{R}^3 \setminus \overline{\Omega}$ having boundary $\partial\Omega_j$ for $j = 0, \dots, k$. By our definition, Ω_0 is unbounded.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz domain such that $\Omega \subset \mathcal{O}^\circ$

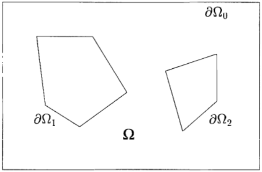

Figure: Simple geometry with labeling of the boundaries in $\mathbb{R}^2$

Finally we obtain the result:

THEOREM

For any function $\vec{u} \in H(div;\Omega) : \nabla\vec{u} = 0$ in Ω and $\langle \vec{u} \cdot \vec{v}, 1\rangle|_{\partial\Omega_j}$ for all $j = 0, \dots, k$, there exists a vector potential

$$\vec{A} \in \left(H^1(\Omega)\right)^3 \ : \ \vec{u} = \nabla \times \vec{A} \text{ in } \Omega \text{ and } \nabla \cdot \vec{A} = 0 \text{ in } \Omega. \tag{112}$$

Application - 1/2

We want here to give a sketch of the variational formulation of the problem, in case of scattering from a buried object (31-34).

Remark

- *For the first two equations in (31) and (32), observe that they are in fact containing derivatives of order two in the distributional sense, as $\nabla \times \nabla$ is a vector having "sum" of derivatives as components and, applying again the curl operator to it, we determine "sums" of second-order partial weak derivatives. Therefore, we are dealing with elliptic equations.*
- *For the equations number three at the interface Σ_0, we are imposing the value of vector functions to be 0, i.e. we have homogeneous Dirichlet conditions;*
- *The last one is just a "conservative law", namely the sum of the incident and scattered fields must be the total field.*
- *The radiation condition here plays a relevant role, as we impose the L^2-norm to vanish when the radius of the sphere approaches infinity. Indeed the field (e.g. plane wave) is outgoing, it is propagating in the radial direction with amplitude decaying as $1/r$ due to sphericity.*

The equations

$$\nabla \times \nabla \times \tilde{E}^i - k^2 \tilde{E}^i = \boldsymbol{F} \qquad \nabla \times \nabla \times \tilde{E}^i - k^2 {\epsilon_r}^e \tilde{E}^i = 0 \tag{113}$$

or equivalently

$$\nabla \times \nabla \times \tilde{E}^i - k^2 \tilde{E}^i - \boldsymbol{F} = 0 \qquad \nabla \times \nabla \times \tilde{E}^i - k^2 {\epsilon_r}^e \tilde{E}^i = 0 \tag{114}$$

Application-2/2

are indeed of the form

$$\nabla\left(\nabla\cdot\tilde{E}\right)-\Delta\tilde{E}-k^2\tilde{E}-\boldsymbol{F}=0 \qquad \nabla\left(\nabla\cdot\tilde{E}\right)-\Delta\tilde{E}-k^2\epsilon_r\tilde{E}=0 \tag{115}$$

as the following equality holds for vector functions:

$$\nabla\times(\nabla\times\phi)=\nabla(\nabla\cdot\phi)-\Delta\phi \tag{116}$$

Therefore, one has to be a little careful about treating the first object. It is the Laplacian plus other second-order term and it is understood that here we need to work properly in the setting of trace spaces, i.e. with multi-trace operators.

A weak solution of the perfect conductor problem is a field $E\in V_0$ such that

$$\int_{B\setminus D} curlE\cdot\overline{curlv}-k_0^2E\cdot\bar{v}dx+\langle ik_0\Lambda\gamma_tE,\gamma_tv\rangle_{\partial B}=\langle ik_0\Lambda\gamma_tE^i-ik_0\gamma_NE^i,\gamma_Tv\rangle_{\partial B} \qquad \forall v\in V_0 \tag{117}$$

where $V_0:=\left\{u\in H\left(curl;B\setminus\bar{D}\right):\gamma_t=0 \text{ on } \partial D\right\}$ and $\gamma_N(v)=\frac{1}{ik}\gamma_t(\nabla\times v)$ is the tangential trace of the magnetic field, whenever v is the electric one.

For more, see the article by F.Hagemann et al.: Link in which the problem displayed above is discussed in its full setting.

!!! THANK YOU FOR YOUR ATTENTION !!!

QUESTIONS ?

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