GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

FACULTY FOR MATHEMATICS AND COMPUTER SCIENCE

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

SEMINAR ON ADVANCED FINITE ELEMENTS METHODS

MAXWELL'S EQUATIONS

VARIATIONAL FORMULATION

FOR

FINITE ELEMENTS METHODS

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Contents

Chapter 1

Mathematical Modelling

1.1 Maxwell's classical system

1.1.1 Classical Formulation 1865 "A Dynamical Theory of the Electromagnetic Field"

- Maxwell's Equations in vacuum for the 3D Euclidean space-time

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \cdot \vec{B} = 0 \tag{1.1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{J} \tag{1.2}$$

- Lorentz's Force

$$\vec{F} = q \cdot \vec{E} + q \cdot \vec{v} \wedge \vec{B}$$

- Continuity Equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Remark 1.1.1 Notice Maxwell's equations constitute a set of eight scalar eqs: six for the components of $\nabla \times \vec{E}$, $\nabla \times \vec{B}$ in (2) and two from the relations in (1). The latter are actually redundant: They can be obtained from (1) imposing homogeneous initial conditions.

1.1.2 Maxwell's Equations 1873 "Treatise on Electricity and Magnetism"

- General form of Maxwell's Equations for the 3D Euclidean space-time

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0 \tag{1.3}$$

$$\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \qquad \frac{\partial \vec{D}}{\partial t} - \nabla \times \vec{H} = -\vec{J} \tag{1.4}$$

- Continuity Equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \leftrightarrow \qquad \frac{\partial \left(\nabla \cdot \vec{B}\right)}{\partial t} = \frac{\partial \left(\nabla \cdot \vec{D} - \rho\right)}{\partial t} = 0 \tag{1.5}$$

- **Remark 1.1.2**

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0 \qquad \forall\ \vec{A}\,(\vec{x}, t) \in \Omega \subseteq \mathbb{E}^3 \times \mathbb{R}^+$$

1.2 Maxwell's time-harmonic system

1.2.1 Time - Harmonic Maxwell's system

We apply the Fourier transform in time to all the involved quantities, considering a radiation with frequency $\omega > 0$, i.e. the fields and the densities assume the form:

$$\vec{A}\,(\vec{x}, t) = Re\left(e^{-i\omega t}\hat{\vec{A}}\,(\vec{x})\right) \tag{1.6}$$

The quantities $\hat{\vec{A}} := \hat{A}\,(\vec{x})$ are now complex-valued fields.

- Time - harmonic Maxwell's Equations

$$\nabla \cdot \hat{D} = \hat{\rho} \qquad\qquad \nabla \cdot \hat{B} = 0 \tag{1.7}$$

$$-i\omega\hat{B} + \nabla \times \hat{E} = 0 \qquad\qquad -i\omega\hat{D} - \nabla \times \hat{H} = -\hat{J} \tag{1.8}$$

- Continuity Equation

$$\nabla \cdot \vec{J} - i\omega\hat{\rho} = 0 \tag{1.9}$$

- Redundance of (7.1)

$$\nabla\left(-i\omega\hat{D} - \nabla \times \hat{H}\right) = \nabla\left(-\hat{J}\right) \overset{Rem.}{\iff} -i\omega\nabla \cdot \hat{D} = -\nabla \cdot \hat{J} \overset{(9)}{=} -i\omega\hat{\rho} \iff (7.1) \tag{1.10}$$

1.3 Integral Formulation

Maxwell' s equations can be viewed also in their integral formulation. It allows also a reinterpretation of the whole theory in terms of differential forms.

Let S be a surface in the euclidean 3D-space, therefore:

$$-i\omega\hat{B} + \nabla \times \hat{E} = 0 \iff i\omega\hat{B} = \nabla \times \hat{E} \tag{1.11}$$

yields to the equation:

$$i\omega \int_S \hat{B} \cdot \hat{\nu} d\sigma = \int_S \left(\nabla \times \hat{E}\right) \hat{\nu} d\sigma \overset{Stokes'Th.}{=} \int_{\partial V} \hat{E} \cdot \hat{\tau} dS \tag{1.12}$$

Remark 1.3.1 Notice here that we calculate the circuitation of the electric field $\hat{E}$, whereas to for the magnetic one $\hat{B}$ we obtain a surface integral. Therefore, it is sensitive to cover the domain making use of rectangular or triangular meshes, whose edges are associated to the electric and faces to the magnetic field.

This is common practice among FEMs. It is also used in the FDTD scheme by Yee, however notice this is a time-domain method, for time-varying fields, whereas in FEMs we take the time as fixed (harmonic fields).

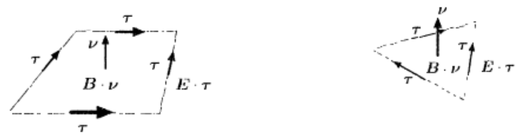


Figure 1.1: Examples of mesh's elements with flux.

1.4 Propagational properties of materials

1.4.1 Constitutive Laws for linear media

The rules of propagation of waves are dependent on the material we are studying, namely:

- Vacuum or free space:

$$\hat{D} = \epsilon_0 \hat{E} \qquad \hat{B} = \mu_0 \hat{H} \tag{1.13}$$

- Inhomogeneous and isotropic material:

$$\hat{D} = \epsilon \hat{E} \qquad \hat{B} = \mu \hat{H} \tag{1.14}$$

- Inhomogeneous and anisotropic material:

  The magnetic and electric properties depend on the direction of the fields.The electric permittivity and the magnetic permeability are replaced by tensors in this case [B] and we may have to use particular techniques to prove the scheme converges.

- Conductors:

  The existence of Eddy currents, behaved according to the following equation:

$$\hat{J} = \sigma \hat{E} + \hat{J}_a \tag{1.15}$$

leads to the second-order Maxwell system:

$$\nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E} \right) - k^2 \epsilon_r \tilde{E} = \tilde{F} \tag{1.16}$$

$$\nabla \cdot \left(\epsilon_r \tilde{E} \right) = -\frac{1}{k^2} \nabla \cdot \tilde{F} \tag{1.17}$$

Remark 1.4.1 $\tilde{E} := \sqrt{\epsilon_0}\hat{E},\ k := \omega\sqrt{\epsilon_0\mu_0},\ \tilde{F} := ik\sqrt{\mu_0}\hat{J}_a,\ \mu_r := \frac{\mu}{\mu_0},\ \epsilon_r := \frac{1}{\epsilon_0}(\epsilon + \frac{i\sigma}{\omega})$

Remark 1.4.2 This formulation holds also for dielectrics ($\sigma = 0$).

1.5 Interface and boundary conditions

Consider two different materials with different magnetic and electric properties occupying the regions as in the figure:

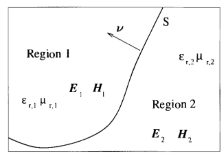


Figure 1.2: Geometry for interface boundary conditions

- In order for $\nabla \times \tilde{E}$ to be well-defined "in the least-squares sense" , we must have:

$$\nu \times \left(\tilde{E}_1 - \tilde{E}_2\right) = 0 \quad on\ S \tag{1.18}$$

and, similarly, for $\mu_r\tilde{H}$ to have a well-defined divergence ($\nabla \cdot \mu_r\tilde{H} = 0$):

$$\nu \cdot \left(\mu_{r,1}\tilde{H}_1 - \mu_{r,2}\tilde{H}_2\right) = 0 \quad on\ S \tag{1.19}$$

Remark 1.5.1 $\tilde{H}_i := \sqrt{\mu_0}\hat{H}_i \quad \forall i = 1, 2.$

- The magnetic field has usually continuous tangential components, whereas the charge density may cause jumps in the normal component of the electric field, i.e.: :

$$\nu \times \left(\tilde{H}_1 - \tilde{H}_2\right) = 0 \qquad \nu \cdot \left(\epsilon_{r,1}\tilde{E}_1 - \epsilon_{r,2}\tilde{E}_2\right) = \rho_S \quad \text{on S} \tag{1.20}$$

The first equation, of course, does not hold when S is a thin conductive layer giving rise to conductive boundary conditions or when singularities of $\boldsymbol{F}$ gives rise to surface currents.

1.6 Scattering problems

1.6.1 Scattering

CASE 1: Scattering from a bounded, inhomogeneous object O occupying a domain D in $\mathbb{R}^3$

The incident field must satisfy the Maxwell's system in absence of the scatterer, i.e.

$$\nabla \times \nabla \hat{E}^i - k^2 \hat{E}^i = \boldsymbol{F} \tag{1.21}$$

where $\boldsymbol{F}$ is a function describing the current source.

The total field $\hat{E}$ is given by the sum of the incident and the scattered ones:

$$\hat{E} = \hat{E}^i + \hat{E}^s \tag{1.22}$$

The scattered field is outgoing (Silver-Müller radiation condition):

$$\lim_{\rho \to \infty} \rho \left(\left(\nabla \times \hat{E}^s \right) \times \vec{x} - ik\hat{E}^s \right) = 0 \tag{1.23}$$

where $\rho := ||\vec{x}||$. This limit is uniform in $\hat{x}$.

Example: [Plane wave]

$$\hat{E}^i = \vec{p} e^{ik\vec{x}\cdot\hat{d}} \tag{1.24}$$

where $\vec{p}$ is the polarisation, $\hat{d}$ is the direction of propagation versor and $< \vec{p}, \hat{d} >= 0$.

In order to solve the problem using FEMs, we consider a surface Σ far from the scatterer, and we impose the following absorption condition:

$$\left(\nabla \times \hat{E} \right) \times \nu - ik\hat{E}_T = \left(\nabla \times \hat{E}^i \right) \times \nu - ik\hat{E}^i_T \tag{1.25}$$

where $\hat{E}^{(i)}_T := \left(\nu \times \hat{E}^{(i)}|_\Sigma \right) \times \nu$

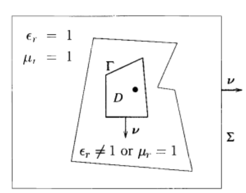


Figure 1.3: Scattering problem for FEMs imposing absorbing boundary condition on Σ

Certainly, this is very different from solving the true scattering problem. However, the difference in the solutions can be made very small, if Σ is sufficiently far from O.

CASE 2: Unbounded scatterer in the lower half-plane

Consider w.l.o.g. the background medium to be composed by air and earth, namely at the common boundary we have the usual jumping conditions already discussed before, and the sources of the scattering to be in the air, i.e. $\boldsymbol{F} = 0$ if $x_3 < 0$. The boundary region is now infinite in extension, thus we must impose an integral radiation condition:

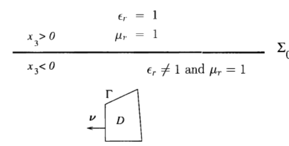


Figure 1.4: Scattering problem for FEMs from perfectly conductive objects in a layered medium.

$$\lim_{R\to\infty} \int_{\partial B_R^{\pm}} || \left(\nabla \times \hat{E}^s\right) \times \nu - ik\hat{E}^s||^2 dA = 0 \tag{1.26}$$

1.7 Boundary value problems

- **Time-harmonic problem in a cavity**

  Let Ω be a bounded domain with two disjoint connected boundaries Γ and Σ, we search for the solution of the following problem:

$$\begin{cases} \nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E}\right) - k^2 \epsilon_r \tilde{E} = \boldsymbol{F} \quad \text{in } \Omega \\ \nu \times \hat{E} = 0 \quad \text{on } \Gamma \\ \mu_r^{-1} \left(\nabla \times \hat{E}\right) \times \nu - ik\boldsymbol{\lambda}\hat{E}_T = \boldsymbol{g} \quad \text{on } \Sigma \end{cases} \tag{1.27}$$

- **Cavity resonator**

  Let Ω be a bounded domain, $\Gamma := \partial\Omega$ Our aim is to find out k scalars and non-trivial electric fields such that:

$$\begin{cases} \nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E}\right) - k^2 \epsilon_r \tilde{E} = \boldsymbol{F} \quad \text{in } \Omega \\ \nu \times \hat{E} = 0 \quad \text{on } \Gamma \\ \nabla \cdot \left(\epsilon_r \hat{E}\right) = 0 \quad \text{in } \Omega \end{cases} \tag{1.28}$$

- **Scattering from a bounded object**

  The electromagnetic field has an unbounded domain $\mathbb{R}^3 \setminus \bar{D}$, where D is bounded with connected complement and Γ denotes the boundary of the unbounded component of $\mathbb{R}^3 \setminus \bar{D}$. We want to solve the following:

$$
\begin{cases}
\nabla \times \left({\mu_r}^{-1} \nabla \times \tilde{E} \right) - k^2 \epsilon_r \tilde{E} = \boldsymbol{F} \quad \text{in } \Omega \\
\nu \times \hat{E} = 0 \quad \text{on } \Gamma := \partial \left(\mathbb{R}^3 \setminus \ \bar{D} \right) \\
\hat{E} = \hat{E}^i + \hat{E}^s \quad \text{in } \mathbb{R}^3 \setminus \ D
\end{cases}
\tag{1.29}
$$

Moreover, the scattered field must satisfy the Silver-Müller condition:

$$
\lim_{\rho \to \infty} \rho \left(\left(\nabla \times \hat{E}^s \right) \times \vec{x} - ik \hat{E}^s \right) = 0 \tag{1.30}
$$

where $\rho := ||\vec{x}||$. This limit is uniform in $\hat{x}$.

- **Scattering from a buried object**

Consider the following regions:

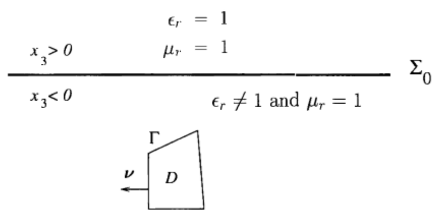
$$
{\mathbb{R}^3}_+ := \left\{ \vec{x} \in \mathbb{R}^3 \mid x_3 > 0 \right\} \qquad {\mathbb{R}^3}_- := \left\{ \vec{x} \in \mathbb{R}^3 \mid x_3 < 0 \right\}
$$

We search for solutions of the problem (assume $\nu \times \hat{E} = 0 \quad \text{on } \Gamma := \partial D$):

$$
\begin{cases}
\nabla \times \nabla \times \tilde{E} - k^2 \tilde{E} = \boldsymbol{F} \quad \text{in } {\mathbb{R}^3}_+ \\
\nabla \times \nabla \times \tilde{E} - k^2 {\epsilon_r}^e \tilde{E} = 0 \quad \text{in } {\mathbb{R}^3}_- \setminus \bar{D} \\
\left[\nu \times \hat{E} \right] = 0 = \left[\nu \times \left(\nabla \times \hat{E} \right) \right] \quad \text{on } \Sigma_0 \\
\hat{E} = \hat{E}^i + \hat{E}^s \quad \text{in } \mathbb{R}^3 \setminus \ \bar{D}
\end{cases}
\tag{1.31}
$$

where the incident field satisfy the background Maxwell's system:

$$
\begin{cases}
\nabla \times \nabla \times \tilde{E}^i - k^2 \tilde{E}^i = \boldsymbol{F} \quad \text{in } {\mathbb{R}^3}_+ \\
\nabla \times \nabla \times \tilde{E}^i - k^2 {\epsilon_r}^e \tilde{E}^i = 0 \quad \text{in } {\mathbb{R}^3}_- \\
\left[\nu \times \hat{E}^i \right] = 0 = \left[\nu \times \left(\nabla \times \hat{E}^i \right) \right] \quad \text{on } \Sigma_0 \\
\hat{E} = \hat{E}^i + \hat{E}^s \quad \text{in } \mathbb{R}^3 \setminus \ \bar{D}
\end{cases}
\tag{1.32}
$$



Finally, we must impose integral radiation conditions on the scattered field $\hat{E}^s$:

$$\lim_{R\to\infty} \int_{\partial B_R^+} || \left(\nabla \times \hat{E}^s\right) \times \nu - ik\hat{E}^s||^2 dA = 0 \tag{1.33}$$

$$\lim_{R\to\infty} \int_{\partial B_R^-} || \left(\nabla \times \hat{E}^s\right) \times \nu - ik\hat{E}^s||^2 dA = 0 \tag{1.34}$$

The simple geometry we have used, is represented in the already - displayed picture:

Chapter 2

Functional Analysis

2.1 Hilbert spaces and Operators' Theory: Basics

2.1.1 Hilbert spaces

Consider $\mathcal{X} \in \boldsymbol{Vct}_{\mathbb{C}}$, then we can define a measure on $\mathcal{X}$, given by the scalar product

$$(.,.)_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \to \mathbb{C} \tag{2.1}$$

that leads to an induced norm defined as $||\phi||_{\mathcal{X}} = \sqrt{(\phi, \phi)_{\mathcal{X}}} \quad \forall\ \phi \in \mathcal{X}$.

We call the normed space just obtained **Hilbert space** iff $\mathcal{X}$ is complete with respect to the induced norm. Complete means that every Cauchy sequence in $\mathcal{X}$ admits a limit in $\mathcal{X}$
Example: The space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^3$ open domain, is an Hilbert space with the following measure

$$(\phi, \xi)_{L^2(\Omega)} := \int_{\Omega} \phi\bar{\xi} dV \qquad \forall\ \phi,\ \xi \in L^2(\Omega) \tag{2.2}$$

The following well-known inequalities hold:

- Cauchy-Schwarz

$$||\ (u, v)_{\mathcal{X}}\ || \leq ||u||_{\mathcal{X}} \cdot ||v||_{\mathcal{X}} \quad \forall u, v \in \mathcal{X} \tag{2.3}$$

- Arithmetic-Geometric mean

$$\forall u, v \in \mathcal{X}, \exists \delta > 0 : \quad || (u, v)_{\mathcal{X}} || \leq \frac{\delta}{2} ||u||^2{}_{\mathcal{X}} + \frac{1}{2\delta} ||v||^2{}_{\mathcal{X}} \tag{2.4}$$

2.1.2 Convergence results

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ be a sequence, we have the following notions of convergence:

- Strong convergence

  $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ **converges strongly** to $u \in \mathcal{X}$ or, shortly, $u_n \underset{n\to\infty}{\longrightarrow} u$, iff

$$||u_n - u||_{\mathcal{X}} \underset{n\to\infty}{\longrightarrow} 0 \tag{2.5}$$

- Weak convergence

  $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ **converges weakly** to $u \in \mathcal{X}$ or, shortly, $u_n \underset{n\to\infty}{\rightharpoonup} u$, iff

$$(u_n, \phi)_{\mathcal{X}} \underset{n\to\infty}{\longrightarrow} (u, \phi)_{\mathcal{X}} \quad \forall \phi \in \mathcal{X} \tag{2.6}$$

Here is a list of the relevant results:

i) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ be bounded, then there exists a weakly convergent extract sequence.

ii) If $U \subset \mathcal{X}$ is a closed subspace and $f \in \mathcal{X}$, then

$$\exists \ ! \ g \in U : ||f - g||_{\mathcal{X}} = \underset{v\in U}{inf} ||f - v||_{\mathcal{X}}$$

iii) If $U \subset \mathcal{X}$ is a closed subspace and $f \in \mathcal{X}$, then

$$\exists \ ! \ u \in U, \ v \in U^{\perp} : f = u + v \Leftrightarrow \mathcal{X} = U \oplus U^{\perp}$$

Remark 2.1.1 U closed means it contains all limits of its convergent sequences. In particular, $U^{\perp}$ is closed in $\mathcal{X}$.

2.1.3 Linear Operators: Basics

Consider two Hilbert spaces $\mathcal{X}, \mathcal{Y}$, an operator $A : \mathcal{X} \to \mathcal{Y}$ is said to be:

- **linear** iff

$$A(\alpha u + \beta v) = \alpha A(u) + \beta(v) \quad \forall \alpha, \beta \in \mathbb{C},\ \forall u, v \in \mathcal{X} \tag{2.7}$$

- **bounded** iff

$$\exists C > 0 : \quad ||A\phi||_{\mathcal{Y}} \leq C \cdot ||\phi||_{\mathcal{X}} \quad \forall \phi \in \mathcal{X} \tag{2.8}$$

- **continous** iff

$$\forall \phi \in \mathcal{X} : \{\phi_n\}_{n\in\mathbb{N}} \underset{n\to\infty}{\longrightarrow} \phi \ \text{ in } \mathcal{X} \quad \Rightarrow \quad \{A\phi_n\}_{n\in\mathbb{N}} \underset{n\to\infty}{\longrightarrow} A\phi \ \text{ in } \mathcal{Y} \tag{2.9}$$

Important result in this setting:

Th: A linear operator is continuous if and only if it is bounded.

We define the **operator norm** of A as follows:

$$|||A||| := \underset{\phi\in\mathcal{X}\setminus\{0\}}{sup} \frac{||A\phi||_{\mathcal{Y}}}{||\phi||_{\mathcal{X}}} \tag{2.10}$$

The **range** and the **null-space** of A are respectively defined as the subspaces:

$$A(\mathcal{X}) := \{y \in \mathcal{Y} | y = Ax \text{ for some } x \in \mathcal{X}\} \subset \mathcal{Y} \tag{2.11}$$

$$N(A) := \{x \in \mathcal{X} | Ax = 0\} \subset \mathcal{X} \tag{2.12}$$

2.1.4 Duality

Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be a bounded linear operator between Hilbert spaces, there exists a unique operator $A^\dagger \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, called **adjoint of A**, such that

$$(Ax, y)_{\mathcal{Y}} = \left(x, A^\dagger y\right)_{\mathcal{X}} \quad \forall x \in \mathcal{X},\ \forall y \in \mathcal{Y} \tag{2.13}$$

Consider f $\in \mathcal{L}(\mathcal{X}, \mathbb{C}) =: \mathcal{X}^*$ a bounded, linear functonal, we define its **dual norm** as follows:

$$|||f|||_* := \underset{x\in\mathcal{X}\setminus\{0\}}{sup} \frac{||f(x)||_{\mathbb{C}}}{||x||_{\mathcal{X}}} \tag{2.14}$$

and the dual paring as:

$$\langle\langle g, u\rangle\rangle_{\mathcal{X}} := g(u) \quad \forall u \in \mathcal{X},\ g \in \mathcal{X}^* \tag{2.15}$$

We therefore obtain a well-posed definition of an operator between dual spaces, namely the so-called **dual operator**, $A^* \in \mathcal{L}(\mathcal{Y}^*, \mathcal{X}^*)$ via the following:

$$\langle\langle Ax, y\rangle\rangle_{\mathcal{Y}} = \langle\langle x, A^* y\rangle\rangle_{\mathcal{X}} \quad x \in \mathcal{C}, y \in \mathcal{Y}^* \tag{2.16}$$

For $\mathcal{V} \subset \mathcal{Y}^*$, the **annihilator** of $\mathcal{V}$ is the subspace:

$$^a\mathcal{V} := \{u \in \mathcal{Y} : \langle\langle, u\rangle\rangle_{\mathcal{Y}} = 0 \ \ \forall g \ \in \mathcal{V}\} \subset \mathcal{Y} \tag{2.17}$$

2.1.5 Density and Representability

Theorem 2.1.1 Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be a bounded, linear operator, then:

$$A(\mathcal{X})^{\perp} = N\left(A^{\dagger}\right) \tag{2.18}$$

$$N\left(A^{\dagger}\right)^{\perp} = \overline{A(\mathcal{X})} = {}^a N(A^*) \tag{2.19}$$

Remark 2.1.2 If $N\left(A^{\dagger}\right) = \{0\}$ or $N(A^*) = \{0\}$, $A(\mathcal{X})$ is dense in $\mathcal{Y}$.

Theorem 2.1.2 (Riesz' Representation) Let $\mathcal{X}$ be a Hilbert space,

$$\forall \ g \in \ \mathcal{X}^*, \exists \ ! \ u \in \mathcal{X} : \quad (u, v)_{\mathcal{X}} = g(v) \quad \forall v \in \mathcal{X} \tag{2.20}$$

Moreover, this isomorphism provides an isometry, i.e. $||u||_{\mathcal{X}} = ||g||_{\mathcal{X}^*}$.

2.2 Lax-Milgram's & Riesz-Fredholm' s Theories for Variations

2.2.1 Sesquilinear Forms

Let $\mathcal{X}, \mathcal{Y}$ Hilbert spaces, a mapping $a(.,.) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ defines a **sesquilinear form** if:

- $a(\alpha_1 u + \alpha_2 v, \phi) = \alpha_1 a(u,\phi) + \alpha_2 a(v,\phi) \quad \forall \alpha_1, \alpha_2 \in \mathbb{C},\ \forall u, v \in \mathcal{X},\ \forall \phi \in \mathcal{Y}$
- $a(u, \beta_1 \phi + \beta_2 \psi) = \overline{\beta_1} a(u,\phi) + \overline{\beta_2} a(v,\psi) \quad \forall \beta_1, \beta_2 \in \mathbb{C},\ \forall u \in \mathcal{X}, \forall \phi, \psi \in \mathcal{Y}$

Example: The scalar product in $L^2(\Omega)$ defines a sesquilinear form.

Given a sesquilinear form $a(.,.) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$, it is said to be:

- **bounded** if

$$\exists\ C > 0 \text{ ind. of } u \in \mathcal{X} \text{ and } \phi \in \mathcal{Y} : ||a(u,\phi)||_{\mathbb{C}} \leq C \cdot ||u||_{\mathcal{X}} ||\phi||_{\mathcal{Y}} \quad \forall u \in \mathcal{X}, \forall\ \phi \in \mathcal{Y} \tag{2.21}$$

- **coercive** if

$$\exists \alpha > 0 \text{ not dependent on } u \in \mathcal{X} : ||a(u,u)||_{\mathbb{C}} \geq \alpha \cdot ||u||_{\mathcal{X}}{}^2 \quad \forall u \in \mathcal{X} \tag{2.22}$$

Remark 2.2.1 Here the sesquilinear form is internal, i.e. $a(.,.) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$

2.3 Variational Problems

Given an Hilbert space $\mathcal{X}$ and a bounded coercive sesquilinear form $a(.,.) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, consider the variational problem of finding $u \in \mathcal{X}$ such that

$$a(u,\phi) = f(\phi) \quad \forall \phi \in \mathcal{X} \tag{2.23}$$

where $f \in \mathcal{X}^*$ is a given functional.

The problem is well-posed (existence, uniqueness, stability). Indeed, the following theorems hold:

Theorem 2.3.1 (Lax-Milgram) Let $a\left(.,.\right) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$, be a bounded with constant C, coercive with constant α, sesquiinear form, then $\forall f \in \mathcal{X}^*$ there exits a unique solution to the variational problem in (57) and, moreover,

$$||u||_{\mathcal{X}} \leq \frac{C}{\alpha}||f||_{\mathcal{X}^*}$$

Theorem 2.3.2 (Generalised Lax-Milgram) Consider a bounded, sesquilinear form $a\left(.,.\right) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{C}$ such that:

i) $\exists\ \alpha : \underset{u\in\mathcal{X}:||u||_{\mathcal{X}}=1}{inf} \quad \underset{v\in\mathcal{Y}:||v||_{\mathcal{Y}}\leq 1}{sup} \Big(||a\left(u,v\right)||_{\mathbb{C}}\Big) \geq \alpha > 0$ (Babuška-Brezzi condition)

ii) $\forall v \in \mathcal{Y},\ v \neq 0, \quad \underset{u\in\mathcal{X}}{sup}\left(||a\left(u,v\right)||_{\mathbb{C}}\right) > 0$

If $g \in \mathcal{X}^*,\ \exists\ !\ u \in \mathcal{X} : \quad a\left(u,\phi\right) = g\left(\phi\right) \quad \forall \phi \in \mathcal{Y}$ and $||u||_{\mathcal{X}} \leq \frac{C}{\alpha}||g||_{\mathcal{Y}^*}$.

2.3.1 Mixed Variational Problems

Sometimes one deals with mixed variational problems, with coercivity only on a restricted portion of the Hilbert spaces under consideration.

Let $\mathcal{X}, \mathcal{S}$ be Hilbert spaces and consider the following two forms

$$a : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C} \qquad b : \mathcal{X} \times \mathcal{S} \rightarrow \mathbb{C} \tag{2.24}$$

such that $\exists C > 0$

$$||a\left(u,\phi\right)||_{\mathbb{C}} \leq C \cdot ||u||_{\mathcal{X}}||\phi||_{\mathcal{X}} \quad \forall u \in \mathcal{X}, \forall\ \phi \in \mathcal{X} \tag{2.25}$$

$$||b\left(u,\xi\right)||_{\mathbb{C}} \leq C \cdot ||u||_{\mathcal{X}}||\xi||_{\mathcal{S}} \quad \forall u \in \mathcal{X}, \forall\ \xi \in \mathcal{S} \tag{2.26}$$

For existence, we assume a to be also $\mathcal{Z}$-coercive, i.e.

$$\exists \alpha > 0 : \quad ||a\left(u,u\right)||_{\mathbb{C}} \geq \alpha \cdot ||u||_{\mathcal{X}}{}^2 \quad \forall u \in \mathcal{Z} \tag{2.27}$$

where $\mathcal{Z} := \{u \in \mathcal{X} | b\left(u,\xi\right) = 0 \ \ \forall \xi \in S\}$

Moreover, b is required to satisfy the Babuška-Brezzi condition, namely:

$$\exists\ \beta > 0 : \forall p \in \mathcal{S} \quad \underset{w\in\mathcal{X}}{sup}\frac{\left(||b\left(w,p\right)||_{\mathbb{C}}\right)}{||w||_{\mathcal{X}}} \geq \beta \cdot ||p||_{\mathcal{S}} \tag{2.28}$$

where β is independent of p.

Theorem 2.3.3 In the previous setting, suppose $f \in \mathcal{X}^*,\ g \in \mathcal{S}$ and consider the problem of finding $u \in \mathcal{X}$ and $p \in \mathcal{S}$:

$$\begin{cases} a\left(u,\phi\right) + b\left(\phi,p\right) = f\left(\phi\right) & \forall \phi \in \mathcal{X} \\ b\left(u,\xi\right) = g\left(\xi\right) & \forall \xi \in \mathcal{S} \end{cases} \tag{2.29}$$

then, there exists a unique solution $(u,p) \in \mathcal{X} \times \mathcal{S}$ to (63) and, additionally, we have:

$$||u||_{\mathcal{X}} + ||p||_{\mathcal{S}} \le C\left(||f||_{\mathcal{X}^*} + ||g||_{\mathcal{S}^*}\right)$$

Remark 2.3.1 The theories we have just discussed do not work for Maxwell's systems of equations, as we need to deal with stability of the basic elliptic problem under perturbations. The right theory to investigate this issue is the one developed by Riesz and Fredholm.

2.4 Riesz-Fredhom Theory

Let $\mathcal{X}$ be an Hilbert space and consider a bounded, linear operator $A \in \mathcal{L}\left(\mathcal{X}\right)$. If $\mathcal{F} \in \mathcal{X}$, we want to solve for $u \in \mathcal{X}$ the following problem:

$$\left(I + A\right) u = \mathcal{F} \tag{2.30}$$

Remember the following relevant result:

Lemma 2.4.1 Let $\mathcal{X}$ be an Hilbert space, $I_{\mathcal{X}}$ is compact $\iff$ $\mathcal{X}$ is finite-dimensional.

Definition 2.4.1 A defines a compact operator if it maps bounded set to relatively compact sets, i.e. sets whose closure is a compact set.

Definition 2.4.2 A subset of an Hilbert space is a **compact subset** if it is sequentially compact, i.e. each of its sequences has an extract convergent in itself.

Theorem 2.4.1 Let B $\in \mathcal{L}\left(\mathcal{X}\right)$ be bounded on an Hilbert space. Suppose $B = A + I$, with A compact and I the identity, then:

- Either $Bu = 0$ has only the trivial solution $u \equiv 0 \in \mathcal{X}$. Therefore, $\forall f \in \mathcal{X}$ the equation $bu = f$ has a unique solution depending continuously on f;
- or $Bu = 0$ has exactly p free solutions, for some positive integer p.

2.4.1 Hilbert-Schmidt Theory of eigenvalues

The second point of the previous lemma can be restated as follows:

$$\exists u \in \mathcal{X} \setminus \{0\} : \; Bu = (I + A)\, u = 0 \Leftrightarrow Au = -u \Leftrightarrow \text{u is an eigenfunction of A} : \lambda_u = -1 \tag{2.31}$$

Remark 2.4.1 Remind that, in general, the existence of eigenvalues is not guaranteed.

Definition 2.4.3 An operator $A \in \mathcal{L}(\mathcal{X})$ is said to be self-adjoint if

$$(Au, v)_{\mathcal{X}} = (u, Av)_{\mathcal{X}} \quad \forall u, v \in \mathcal{X} \tag{2.32}$$

Theorem 2.4.2 If $A \in \mathcal{L}(\mathcal{X})$ is a compact, self-adjoint, linear operator on an Hilbert space $\mathcal{X}$, then there exists a possibly finite sequence of eigenfunctions $\{u_i\}_{i \in I}$ and real eigenvalues $\{\gamma_i\}_{i \in I}$ such that:

1) $Au_j = \gamma_j u_j \quad \forall j \in I : u_j \neq 0$;

2) $(u_k, u_i) = \delta_{k,i} \quad \forall k, i \in I$; i.e. each state is orthogonal to each other;

3) $||\gamma_1||_{\mathbb{R}} \geq ||\gamma_2||_{\mathbb{R}} \geq ||\gamma_3||_{\mathbb{R}} \geq \cdots \geq 0$;

4) If I is infinite, then $\lim_{j \to \infty} \gamma_j = 0$;

5) If I is infinite, $Au = \sum_{j \in I} \gamma_j \left(u, u_j\right)_{\S} u_j$ converges, i.e. there exists a decomposition in terms of the states (physically speaking).

6) If $W :=< u_1, \cdots, u_n >$, then $\mathcal{X} = \overline{W} \oplus N(A)$

Remark 2.4.2 An important case of Maxwell's equations requires A to be self-adjoint.

Chapter 3

Variational Formulation

3.1 General Sobolev spaces

3.1.1 Sobolev Spaces

The variational theory for Maxwell's equations relies on the existence of appropriate Sobolev spaces of scalar and vector functions. Therefore, we need to discuss some basics.

Relevant Sobolev Spaces: Consider a domain $\Omega \subset \mathbb{R}^N$, where $N = 1, 2, 3$

- $C^k(\Omega)$, the space of k-times continuously differentiable functions;
- ${C_0}^k(\Omega)$, the space of k-times continuously differentiable functions with compact support;
- $C^k\left(\overline{\Omega}\right)$, the space of k-times differentiable functions with bounded and uniformly continuous derivatives up to order k;
- $L^p(\Omega) : p \in [1, +\infty)$ the space of functions ϕ such that $\int_\Omega |\phi|^p dV < \infty$
- The space of distributions $T \in ({C_0}^\infty(\Omega))^*$ if $\forall K \subset \Omega$ compact set $\exists\ C, k$ constants :

$$||T(\phi)||_{\mathbb{C}} \leq C \sum_{||\vec{\alpha}||_1 \leq k} \sup_K \; ||D^{\vec{\alpha}}\phi||_{{C_0}^\infty(\Omega)} \quad \forall \phi \in {C_0}^\infty(\Omega) \tag{3.1}$$

Definition 3.1.1 (Weak derivative) The distributional derivative $\partial^{\vec{\alpha}}\phi \in \left(C_0{}^{\infty}\left(\Omega\right)\right)^*$ is the unique distribution such that

$$\left(\frac{\partial^{\vec{\alpha}}\phi}{\partial x^{\vec{\alpha}}}, \psi\right) = (-1)^{||\vec{\alpha}||_1}\left(\phi, \frac{\partial^{\vec{\alpha}}\psi}{\partial x^{\vec{\alpha}}}\right) \quad \forall \psi \in C_0{}^{\infty}\left(\Omega\right) \tag{3.2}$$

Remark 3.1.1 If $\phi \in \phi \in C^m\left(\Omega\right)$ the strong derivative and the distributional one agree, if $||\vec{\alpha}||_1 \leq m$.

Definition 3.1.2 Consider Ω as before, $s \in \mathbb{Z}_+,\ p \in [1, \infty)$, we define the fundamental Sobolev space

$$W^{s,p}\left(\Omega\right) := \left\{\phi \in L^p\left(\Omega\right) :\ \partial^{\vec{\alpha}}\phi \in L^p\left(\Omega\right)\ \forall\ ||\vec{\alpha}||_1 \leq s\right\} \tag{3.3}$$

Definition 3.1.3 (Norm) The norm is defined as

$$||\phi||_{W^{s,p}(\Omega)} := \Big(\sum_{||\vec{\alpha}||_1 \leq s} \int_{\Omega} |\partial^{\vec{\alpha}}\phi\left(x\right)|^p dV\left(x\right)\Big)^{1/p} \tag{3.4}$$

Definition 3.1.4 (Semi-norm) We define the semi-norm on this space to be

$$||\phi||_{W^{s,p}(\Omega)} := \Big(\sum_{||\vec{\alpha}||_1 = s} \int_{\Omega} |\partial^{\vec{\alpha}}\phi\left(x\right)|^p dV\left(x\right)\Big)^{1/p} \tag{3.5}$$

Remark 3.1.2 For $p = 2$ there is the possibility to define the Sobolev spaces directly via Fourier transform. However, when $N = 2, 3$ it can be shown $H^s\left(\mathbb{R}^N\right) \cong W^{s,2}\left(\mathbb{R}^N\right)$.

Definition 3.1.5 For a bounded domain Ω, we define

$$H^s\left(\Omega\right) := \left\{u \in \left(C_0{}^{\infty}\left(\Omega\right)\right)^* : u = U|_{\Omega}\ for\ some\ U \in W^{s,2}\left(\mathbb{R}^N\right)\right\} \tag{3.6}$$

In order to have a well-defined notion on norm, we set additionally the following

Definition 3.1.6

$$H^s_{\mathbb{R}^N \setminus \overline{\Omega}} := \left\{u \in W^{s,2}\left(\mathbb{R}^N\right) : supp(u) \subset \mathbb{R}^N \setminus \overline{\Omega}\right\} \subset W^{s,2}\left(\mathbb{R}^N\right)\ \text{(closed sub.)} \tag{3.7}$$

Thus, there exists a well defined projection $W^{s,2}\left(\mathbb{R}^N\right) \to H^s_{\mathbb{R}^N \setminus \overline{\Omega}}$ and we obtain a measure in $H^s\left(\Omega\right)$ as follows:

Definition 3.1.7

$$(u,u)_{H^s(\Omega)} := \Big((I-P)\,U, (I-P) \Big)_{H^s(\Omega)} \tag{3.8}$$

where $u = U|_{|Omega}, v = V|_{H^s(\Omega)}$. If $\partial\Omega$ smooth enough, $H^2(\Omega) \cong W^{s,2}(\Omega)$ and $||.||_{H^2(\Omega)} = ||.||_{W^{s,2}(\Omega)}$.

3.2 Elementary Geometries

3.2.1 Lipschitz Polyhedra

For Dirichlet boundary conditions, we define the appropriate Sobolev spaces considering the closure of the distribution space in the $W^{s,p}(\Omega)$ norm, i.e.

$$W_0{}^{s,p}(\Omega) := \overline{(C_0{}^{\infty}(\Omega))} \tag{3.9}$$

Therefore, for $p=2$, $W_0{}^{s,2}(\Omega) \cong H_0{}^{s}(\Omega)$.

Remark 3.2.1 It turns out the functions $u \in H_0{}^1(\Omega)$ satisfy $u|_{\partial\Omega} = 0$, i.e. are well-behaved solutions.

Our aim is to study Maxwell's equations on bounded domains (the unbounded can be truncated), however the Sobolev spaces are strongly influenced by the geometry.
A nice class of objects to cover with 2D(3D)-tetrahedra meshes are Lipschitz polyhedra.

Definition 3.2.1 The boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^N$ is said to be Lipschitz continuous iff $\forall\ \hat{x} \in \partial\Omega\ \exists\ \mathcal{O} \in \text{Open}\left(\mathbb{R}^N\right)$, with $\hat{x} \in \mathcal{O}$ and an orthogonal system $\{\zeta_1, \ldots, \zeta_N\} =: \boldsymbol{\zeta}$ such that

- $\exists$ a vector
$$\hat{a} \in \mathbb{R}^N \text{ with } \mathcal{O} = \left\{\boldsymbol{\zeta}' | -a_j < \zeta_j, a_j,\ j \in [1, \ldots, N]\right\} \tag{3.10}$$

- $\exists\ \phi$ a Lipschitz continuous function defined on $\mathcal{O}' = \{\boldsymbol{\zeta} | -a_j < \zeta_j, a_j,\ j \in [1, \ldots, N-1]\}$ with $||\phi\left(\boldsymbol{\zeta}'\right)||_{\mathbb{R}} \leq \frac{a_N}{2}\ \forall \boldsymbol{\zeta}' \in \mathcal{O}'$:

$$\Omega \cap \mathcal{O} = \left\{\boldsymbol{\zeta} \mid \zeta_N < \phi\left(\boldsymbol{\zeta}'\right),\ \boldsymbol{\zeta} \in \mathcal{O}'\right\} \tag{3.11}$$

$$\partial\Omega \cap \mathcal{O} = \left\{\boldsymbol{\zeta} \mid \zeta_N = \phi\left(\boldsymbol{\zeta}'\right),\ \boldsymbol{\zeta} \in \mathcal{O}'\right\} \tag{3.12}$$

Remark 3.2.2 If $\partial\Omega \in C^k, \phi \in C^K(\mathcal{O}')$. For spherical domains, $\partial\Omega \in C^\infty$,i.e. maximal smoothness.

3.3 Density results

3.3.1 Lipschitz condition and Density

Theorem 3.3.1 Let Ω be bounded, Lipschitz domain in $\mathbb{R}^{\mathbb{N}}$, then:

i) $C^\infty\left(\overline{\Omega}\right)$ is dense in $W^{s,p}(\Omega)$, for $s \in \mathbb{Z}_+,\ p \in [1,\infty)$;

ii) (Cálderon extension) If $s \in \mathbb{Z}_+,\ s \geq 1,\ p \in (1,\infty)$ there exists a continuous, linear **extension operator**

$$\Pi : W^{s,p}(\Omega) \to W^{s,p}\left(\mathbb{R}^N\right) \tag{3.13}$$

such that $(\Pi u)|_\Omega = u \quad \forall u \in W^{s,p}(\Omega)$. If $p \geq 2$ the extension exists for all $s \geq 0$.

iii) $W^{s,2}(\Omega) \cong H^s(\Omega),\ s \in \mathbb{Z}_+$ with isometry.

Lemma 3.3.1 Consider the space $\mathcal{Y} := \left\{p \in H^1(\Omega) \mid p|_{\partial\Omega} \in H^1(\partial\Omega)\right\}$ with the graph norm

$$||p||^2{}_{\mathcal{Y}} = ||p||^2{}_{H^1(\Omega)} + ||p||^2{}_{H^1(\partial\Omega)} \tag{3.14}$$

then, $C^\infty\left(\overline{\Omega}\right)$ is dense in $\mathcal{Y}$.

Remark 3.3.1 ${C_0}^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. Therefore, ${W_0}^{k,p}(\Omega) \subset W^{s,p}(\Omega)$ as proper

Lemma 3.3.2 If Ω is a bounded and Lipschitz domain, then ${C_0}^\infty(\Omega)$ is dense in $L^2(\Omega)$.

3.4 Compactness for General Sobolev Spaces

3.4.1 Embeddings of General Sobolev spaces

A Sobolev space $W^{s,p}(\Omega)$ is said to be embedded in a space X iff

(i) $W^{s,p}(\Omega) \subset$ X

(ii) the **embedding** $\mathcal{I} : W^{s,p}(\Omega) \hookrightarrow X$ is continuous,

$$\exists C > 0 : ||\mathcal{I}u||_X \leq C||u||_{W^{s,p}(\Omega)} \quad \forall u \in W^{s,p}(\Omega) \tag{3.15}$$

Let Ω^l denote the intersection of an l-dimensional hyperplane with Ω, under which conditions we can have an embedding $\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{m,p}\left(\Omega^l\right)$?

By point (I) of the definition, $u \in W^{m+j,p}(\Omega)$ is a limit of $\{u_n\}_{n\in\mathbb{N}} \in C^\infty\left(\overline{\Omega}\right)$ and these functions are such that $u_n|_{\Omega^l}$ are well-defined. Therefore, the embedding is so $u_n|_{\Omega^l}$ converge to a function in $W^{m,p}\left(\Omega^l\right)$

Theorem 3.4.1 (Sobolev's embedding Theorem) Let Ω be a bounded domain : $\partial\Omega$ is Lipschitz continuous, suppose $m, j \in \mathbb{Z}_+,\ l \in [1, N] \cap \mathbb{N}$,
$p \in [1, \infty] \cap \mathbb{R}$, then:

- If $mp < N$ and $N - mp < l \leq N$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}\left(\Omega^l\right) \qquad p \leq q \leq \frac{l \cdot p}{(N - mp)} \tag{3.16}$$

- If $mp = N$, for $l \in [1, N]$ and $q \in [p, \infty)$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}\left(\Omega^l\right) \tag{3.17}$$

- If $mp > N \geq (m - 1)\,p$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow C^j\left(\overline{\Omega}\right) \tag{3.18}$$

Definition 3.4.1 $\mathcal{I}$ defines a compact embedding iff the embedding is a compact operator.

Theorem 3.4.2 Let $\Omega \subset \mathbb{R}^N$ e a bounded, Lipschitz domain and consider a subdomain $\Omega_0 \subseteq \Omega$. Let $j,\ m$:
$m \geq 1,\ j \geq 0$ be integers, $p \in [1, \infty) \cap \mathbb{R}$, the following embeddings are compact:

- If $mp \leq N$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}\left({\Omega_0}^l\right) \qquad 0 < N - mp, l \leq N \text{ and } 1 \leq q < \frac{l \cdot p}{(N - mp)} \tag{3.19}$$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}\left({\Omega_0}^l\right) \qquad 0 = N - mp, 1 \leq l \leq N \text{ and } 1 \leq q < \infty \tag{3.20}$$

- If $mp > N$

$$\mathcal{I} : W^{m+j,p}(\Omega) \hookrightarrow C^j\left(\overline{\Omega_0}\right) \tag{3.21}$$

Remark 3.4.1 For $m = 1,\ p = 2,\ j = 0, l = N, \Omega_0 \equiv \Omega$, the embedding below is compact in dimension $N = 2, 3$

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \tag{3.22}$$

3.5 Compactness for Fractional Sobolev Spaces

3.5.1 Sobolev Fractional Spaces

In order to discuss the regularity of solutions to the Maxwell's system, we need to introduce fractional Sobolev spaces.

Definition 3.5.1 Fractional Sobolev space Consider a generic Sobolev Space $W^{s,p}(\Omega)$ and assume $p \in [1, \infty),\ s \in \mathbb{R},\ s \geq 0,\ s = m + \sigma$ for some integer $m \in \mathbb{Z}_+,\ \sigma \in (0,1) \cap \mathbb{R}$, we define a fractional Sobolev space $W^{s,p}(\Omega)$ to be the space of distributions $u \in \left({C_0}^{\infty}(\Omega)\right)^*$ such that

$$\int_\Omega \int_\Omega \frac{{||\partial^{\vec{\alpha}} u(\vec{x}) - \partial^{\vec{\alpha}} u(\vec{y})||_*}^p}{{||\vec{x} - \vec{y}||_\Omega}^{N+\sigma p}}\, dV(\vec{x})\, dV(\vec{y}) < \infty \quad \forall\ \vec{x},\ \vec{y} \in \Omega \text{ and } \forall\ \vec{\alpha} : ||\vec{\alpha}||_1 = m \tag{3.23}$$

Definition 3.5.2 Graph norm We define the graph-norm on this space to be :

$$||u||_{W^{s,p}(\Omega)} = \left\{ {||u||_{W^{m,p}(\Omega)}}^p + \sum_{||\vec{\alpha}||_1 = m} \int_\Omega \int_\Omega \frac{{||\partial^{\vec{\alpha}} u(\vec{x}) - \partial^{\vec{\alpha}} u(\vec{y})||_*}^p}{{||\vec{x} - \vec{y}||_\Omega}^{N+\sigma p}} dV(\vec{x})\, dV(\vec{y}) \right\}^{1/p} \tag{3.24}$$

Remark 3.5.1 In this norm, the space $W^{s,p}(\Omega)\ \forall p \in (1, \infty),\ s \in \mathbb{R} : s \geq 0$ is a separable, reflexive, Banach space.

Definition 3.5.3 We define

$${W_0}^{s,p}(\Omega) := \overline{{C_0}^{\infty}(\Omega)} \tag{3.25}$$

in the $W^{s,p}(\Omega)$-norm.

Notice we still have $H^s(\Omega) \cong W^{s,2}(\Omega),\ \forall s \geq 0$

3.5.2 Embeddings of Fractional Sobolev spaces

Theorem 3.5.1 Let Ω be a bounded Lipschitz domain, if $0 \leq t < s : \;\; s - \dfrac{3}{p} = t - \dfrac{3}{q}$, we have the embedding

$$W^{s,p}\left(\Omega\right) \hookrightarrow W^{t,q}\left(\Omega\right) \tag{3.26}$$

Furthermore, for $0 \leq t, s < \infty$ and $p = q = 2$ this embedding is compact.

Let $H^{-1}\left(\Omega\right) := \left({H_0}^1\left(\Omega\right)\right)^*$ with usual dual norm.

3.6 Trace spaces

Remember that Lipschitz domains $\Omega \subset \mathbb{R}^N$ have the condition $\forall \vec{x} \in \partial\Omega\; \exists$ a Lipschitz continuous map $\phi : \mathcal{O}' \subset \mathbb{R}^{N-1} \to \mathbb{R}$ such that

$$\partial\Omega \cap \mathcal{O} = \left\{\boldsymbol{\zeta} = \left(\boldsymbol{\zeta}', \phi\left(\boldsymbol{\zeta}'\right)\right) \;\mid \boldsymbol{\zeta}' \in \mathcal{O}'\right\}$$

Therefore, locally, $\partial\Omega$ is a $(N-1)$-dimensional hypersurface of $\mathbb{R}^N$. We define

$$\phi\left(\boldsymbol{\zeta}'\right) := \left(\boldsymbol{\zeta}', \phi\left(\boldsymbol{\zeta}'\right)\right)$$

Then ϕ^{-1} exists and it is Lipschitz continuous.

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, a distribution u is in $W^{s,p}\left(\partial\Omega\right)$ for $|s| \leq 1$ iff

$$u \circ \phi \in W^{s,p}\left(\mathcal{O}' \cap \phi^{-1}\left(\partial\Omega \cap \mathcal{O}\right)\right) \quad \forall \mathcal{O},\; \phi \text{ as for Lip. domains} \tag{3.27}$$

Consider $\left(\mathcal{O}_j, \phi_j\right)_{j \in J}$ as an atlas for $\partial\Omega$ satisfying the conditions for Lip. domains, we set the norm to be

$$||u||_{W^{s,p}(\partial\Omega)} := \left(\sum_{j \in J} ||u \circ \phi_j||_{W^{s,p}\left(\mathcal{O}_j' \cap {\phi_j}^{-1}\left(\partial\Omega \cap \mathcal{O}_j'\right)\right)}\right)^{1/p} \tag{3.28}$$

Definition 3.6.1 Graph-norm

For $s \in [0, 1)$ and $\Omega \subset \mathbb{R}^N$ this definition is equivalent to the graph norm

$$||u||_{W^{s,p}(\partial\Omega)} = \left\{ \int_{\partial\Omega} ||u||_C d\sigma + \int_{\partial\Omega} \int_{\partial\Omega} \frac{||u(\vec{x}) - u(\vec{y})||_{\mathbb{C}}^p}{||\vec{x} - \vec{y}||_{\partial\Omega}^{N-1+sp}} d\Sigma(\vec{x})\, d\Sigma(\vec{y}) \right\}^{1/p} \tag{3.29}$$

For $s \in [0,1]$ we have $H^s(\partial\Omega) = W^{s,p}(\partial\Omega)$.

Definition 3.6.2 Let $u \in C^\infty\left(\overline{\Omega}\right)$, we have a well-defined trace operator

$$\gamma_0(u) := u|_{\partial\Omega} \tag{3.30}$$

Theorem 3.6.1 Trace theorem If Ω is a bounded Lipschitz domain and $1/p < s \leq 1$, the trace operator has a unique continuous extension as a linear operator

$$W^{s,p}(\Omega) \twoheadrightarrow W^{s-1/p,p}(\partial\Omega) \tag{3.31}$$

Moreover,

$$W_0{}^{1,p} := \left\{u \in W^{1,p} \mid \gamma_0(u) = 0\right\} \tag{3.32}$$

Remark 3.6.1 This dense space for $p > 1$ consists of function satisfying the homogeneous Dirichlet conditions on the boundary.

Definition 3.6.3 When $p > 1$ we can alternative define the above space as

$$W_0{}^{1,p} := \left\{u \in L(\Omega)^p \mid \nabla \cdot u \in L(\Omega)^{p3}, \gamma_0(u) = 0\right\} \tag{3.33}$$

where $\nabla : (C_0{}^\infty(\Omega))^* \to \left(C_0{}^\infty(\Omega)^*\right)^N$

We describe in details the relevant case $H^{1/2}(\partial\Omega) = W^{1/2,2}(\partial\Omega)$ and its dual $H^{-1/2}(\partial\Omega)$. In particular, for any Lipschitz surface S we have:

$$\langle f, g\rangle_S = \int_S f\bar{g}\, d\Sigma \tag{3.34}$$

Thus

$$||f||_{H^{-1/2}(\partial\Omega)} = \sup_{g\in H^{+1/2}(\partial\Omega)} \frac{||\langle f, g\rangle||_{\mathbb{C}}}{||g||_{H^{+1/2}(\partial\Omega)}} \tag{3.35}$$

Remark 3.6.2 Indeed, $H^{-1/2}(\partial\Omega)$ is the completion of $L^2(\Omega)$ in a suitable norm $:$ we can identify the duality paring with its inner product.

Definition 3.6.4 For $s > 1$ we define

$$H^s(\partial\Omega) := \left\{ u \in L^2(\Omega) \mid u = U|_{\partial\Omega} \text{ for some } U \in H^{s+1/2}(\Omega) \right\} \tag{3.36}$$

with **norm**

$$||u||_{H^s(\partial\Omega)} := \inf_{U \in H^{s+1/2}(\Omega) : u = U|_{\partial\Omega}} ||u||_{H^{s+1/2}(\Omega)} \tag{3.37}$$

In particular, $||u||_{H^s(\partial\Omega)} \equiv ||U||_{H^{s+1/2}(\Omega)}$, where $U \in H^{s+1/2}(\Omega) \ : \ U|_{\partial\Omega} = u$ and $(U, \phi)_{H^{s+1/2}(\Omega)} = 0 \quad \forall\, \phi \in H^{s+1/2}(\Omega) \cap {H_0}^1(\Omega)$.

Remark 3.6.3 This function exists by Lax-Milgram lemma.

Notice that $H^s(\partial\Omega)$ is complete, since $H^{s+1/2}(\partial\Omega)$ is and therefore it is a Hilbert space.

SUMMARY OF OUR ACHIEVEMENTS:

- In this setting any $u \in H^s(\partial\Omega)$ is extendable to $U \in H^{s+1/2}(\Omega)$, when $s > 1$. Therefore these spaces perfectly fit for Dirichlet boundary conditions.

- Scattering problems have usually data arising from traces of a smooth vector field.

- There is no explicit norm that allows us to determine whether a function $g \in \partial\Omega$ is in $H^s(\partial\Omega)$.

Theorem 3.6.2 Let Ω be a bounded Lipschitz polyhedron with boundary-faces $\{\partial\Omega_j\}_{j\in J}$ and $g \in L^2(\partial\Omega)$:

- $g \in H^s(\partial\Omega_j) \quad \forall\, j \in J$ and $1 < s < 3/2$:
- If $\partial\Omega_j \cap \partial\Omega_k = \{e_{j,k}\}$ then $g|_{\partial\Omega_j} \equiv g|_{\partial\Omega_k}$ on the common edge;

then, $g \in H^s(\partial\Omega)$, i.e. there exists an extension to $H^{s+1/2}(\Omega)$.

Definition 3.6.5 Locally compact L^2 space We define the locally compact $L^2(\Omega)$ space as follows:

$${L_{loc}}^2(\Omega) := \left\{ p \in L^2(\mathcal{O}) \ \forall\, \mathcal{O} \subset \Omega \text{ compact subdomain} \right\} \tag{3.38}$$

Lemma 3.6.1 If Ω is a bounded, Lipschitz domain and $p \in {L_{loc}}^2(\Omega)$, $\nabla \cdot p \in H^{-1}(\Omega)^3$, then $p \in L^2(\Omega)$.

3.7 Regularity Results for elliptic equations

3.7.1 Well-posedness of the elliptic problems

Consider the following mixed elliptic problem on a Lipschitz domain Ω such that $\partial\Omega = \Gamma_N \cup \Gamma_D,\ \Gamma_N \cap \Gamma_D \not\emptyset$ and ν is the unit outward normal to Γ_N:

$$\begin{cases} -\Delta\phi + c\phi & = f \quad \text{in } \Omega \\ \phi & = \mu_D \quad \text{on } \Gamma_D \\ \dfrac{\partial\phi}{\partial\nu} & = \mu_N \quad \text{on } \Gamma_N \end{cases} \tag{3.39}$$

$\phi \in H^1(\Omega)$ is a weak solution if

$$(\nabla\phi, \nabla\xi) + c(\phi, \xi) = (f, \xi) + \langle \mu_n, \xi \rangle_{\Gamma_N} \quad \text{and} \quad \phi \equiv \mu_D \text{ on } \Gamma_D \qquad \forall\, \xi \in H^1(\Omega)\ :\ \xi|_{\Gamma_D} = 0 \tag{3.40}$$

Theorem 3.7.1 Let Ω be a Lipschitz domain, $\mu \in H^{1/2}(\partial\Omega)$ and $f \in H^{-1}(\Omega)$, there exists a unique weak solution $\phi \in H^1(\Omega)$ of

$$-\Delta\phi + \phi = f \text{ in } \Omega \text{ and } \phi = \mu \text{ on } \partial\Omega \tag{3.41}$$

Furthermore, $\exists\, C > 0\ :\ ||\phi||_{H^1(\Omega)} \le C\left(||\mu||_{H^{1/2}(\partial\Omega)} + ||f||_{H^{-1}(\Omega)}\right)$.

Lemma 3.7.1 (Poincaré inequality) There exists a positive constant C such that $\forall\, u \in H^1(\Omega)$ the following holds:

$$||u||_{H^1(\Omega)} \le C\Big(||\nabla u||_{L^2(\Omega)} = \left|\int_{\partial\Omega} u \; d\Sigma\right|\Big) \tag{3.42}$$

Theorem 3.7.2 Let Ω be a Lipschitz domain, let $\mu \in H^{1/2}(\partial\Omega),\ f \in H^{-1}(\Omega)$, then $\exists\,!$ weak solution $\phi \in H^1(\Omega)$ of

$$-\Delta\phi = f \text{ in } \Omega \quad \text{and} \quad \phi = \mu \text{ on } \partial\Omega \tag{3.43}$$

Furthermore, $\exists\, C > 0\ :\ ||\phi||_{H^1(\Omega)} \le C\left(||\mu||_{H^{1/2}(\partial\Omega)} + ||f||_{H^{-1}(\Omega)}\right)$.

For Neumann boundary conditions, the following holds:

Theorem 3.7.3 Let Ω be a Lipschitz domain, ν the unit outward normal to the boundary. Let $\mu \in H^{-1/2}(\partial\Omega)$, $f \in \left(H^1(\Omega)\right)^*$, there exists a unique weak solution $\phi \in H^1(\Omega)$ of

$$-\Delta\phi + \phi = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial\phi}{\partial\nu} = \mu \text{ on } \partial\Omega \tag{3.44}$$

Furthermore, $\exists\, C > 0 \;:\; ||\phi||_{H^1(\Omega)} \leq C\left(||\mu||_{H^{-1/2}(\partial\Omega)} + ||f||_{(H^1(\Omega))^*}\right)$.

We introduce the following definition

Definition 3.7.1

$$H^1(\Omega)/\mathbb{R} := \left\{u \in H^1(\Omega) \;\middle|\; \int_{\partial\Omega} u \; d\Sigma = 0\right\} \tag{3.45}$$

Theorem 3.7.4 Let Ω be a Lipschitz domain with unit outward normal ν. Let $\mu \in H^{-1/2}(\partial\Omega)$, $f \in \left(H^1(\Omega)\right)^*$ and suppose

$$\int_{\partial\Omega} \mu \; d\Sigma + \int_{\Omega} f \; dV = 0 \tag{3.46}$$

holds, then there exists a unique weak solution $\phi \in H^1(\Omega)/\mathbb{R}$ of

$$-\Delta\phi = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial\phi}{\partial\nu} = \mu \text{ on } \partial\Omega \tag{3.47}$$

Moreover, $\exists\, C > 0 \;:\; ||\phi||_{H^1(\Omega)} \leq C\left(||\mu||_{H^{-1/2}(\partial\Omega)} + ||f||_{(H^1(\Omega))^*}\right)$.

Theorem 3.7.5 Let Ω be a Lipschitz domain, suppose $\phi \in H^1(\Omega)$ is the weak solution of the problem

$$\Delta\phi = 0 \text{ on } \Omega : \phi|_{\partial\Omega} = \mu \;\in\; H^1(\partial\Omega) \tag{3.48}$$

then

$$\phi \in H^{3/2}(\Omega) \quad \text{and} \quad ||\phi||_{H^{3/2}(\Omega)} \leq C||\mu||_{H^1(\Omega)} \tag{3.49}$$

for some positive constant C.

Theorem 3.7.6 Let Ω be a Lipschitz domain, suppose $\phi \in H^1(\Omega)$ is the weak solution of the problem

$$\Delta\phi = 0 \text{ on } \Omega : \frac{\partial\phi}{\partial\nu} = \mu \;\in\; L^2(\partial\Omega) \quad \text{with } \langle\mu, 1\rangle_{\partial\Omega} = 0 \tag{3.50}$$

then

$$\phi \in H^{3/2}(\Omega) \quad \text{and} \quad ||\phi||_{H^{3/2}(\Omega)} \leq C||\mu||_{L^2(\partial\Omega)} \tag{3.51}$$

for some positive constant C.

Theorem 3.7.7 Let Ω be a Lipschitz polyhedral domain , there exits a $s_\Omega > 0$ such that, if $H^{1+\delta}(\partial\Omega)$, $0 \leq \delta < min(s_\Omega, 1/2)$ and $f \in L^2(\Omega)$, then the weak solution $\phi \in H^1(\Omega)$ of

$$-\Delta\phi = f \text{ in } \Omega \quad \text{and} \quad \phi = \mu \text{ on } \partial\Omega$$

is such that $\phi \in H^{3/2+\delta}(\Omega)$.

3.8 Divergence and Curl Based spaces of functions

3.8.1 Divergence and Curl for distributions

Consider vector functions on $\left(L^2(\Omega)\right)^3$, the scalar product defined component-wise extends as

$$(\vec{u}, \vec{v})_{(L^2(\Omega))^3} = \int_\Omega \sum_{j=1}^{3} u_j \cdot \bar{v}_j \; dV \tag{3.52}$$

For distributions, derivatives are taken in the weak sense and we have the following definitions:

- $\nabla \times \vec{v} : \vec{v} \in \left({C_0}^\infty(\Omega)^*\right)^3$

$$\nabla \times \vec{v} = \Big(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \; \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \; \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big) \tag{3.53}$$

$$(\nabla \times \vec{v}, \phi) = (\vec{v}, \nabla \times \phi) \quad \forall \phi \in \left({C_0}^\infty(\Omega)^*\right)^3 \tag{3.54}$$

- $\nabla \cdot \vec{v} : \vec{v} \in \left({C_0}^\infty(\Omega)^*\right)^3$

$$\nabla \cdot \vec{v} = \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} \tag{3.55}$$

$$(\nabla \cdot \vec{v}, \phi) = -(\vec{v}, \nabla \times \phi) \quad \forall \phi \in \left({C_0}^\infty(\Omega)^*\right)^3 \tag{3.56}$$

- $\nabla \times \nabla \cdot p = 0 \qquad \nabla \cdot \nabla \times \vec{v} = 0 \qquad \forall \vec{v} \in \left({C_0}^{\infty} (\Omega)^* \right)^3, \ \ \forall \ p \ \in {C_0}^{\infty} (\Omega)^*$

3.8.2 Properties of $H\,(div;\Omega)$

We define the following space of functions

Definition 3.8.1

$$H\,(div;\Omega) := \left\{ \vec{u} \in \left(L^2 (\Omega) \right)^3 \ \mid \nabla \cdot \vec{u} \in L^2 (\Omega) \right\} \tag{3.57}$$

Definition 3.8.2 Norm

$$||\vec{u}||_{H(div;\Omega)} = \left(||\vec{u}||^2{}_{(L^2(\Omega))^3} + ||\nabla \cdot \vec{u}||^2{}_{L^2(\Omega)} \right)^{1/2} \tag{3.58}$$

Remark 3.8.1 $H\,(div;\Omega)$ is an Hilbert space with the inherited inner product.

Theorem 3.8.1 Let Ω be a bounded 3D-Lipschitz domain, then

$$H\,(div;\Omega) = \overline{\left(C^{\infty} (\bar{\Omega}) \right)^3} \tag{3.59}$$

where the closure is taken in the $H\,(div;\Omega)$-norm.

Remark 3.8.2 This statement holds whenever $\partial\Omega$ is bounded, also if Ω is not.

This result will be helpful when dealing with the electromagnetic field at the interface of two different materials.

Definition 3.8.3 Normal trace operator We define the normal trace operator as

$$\gamma_n (\bar{v}) = \bar{v}|_{\partial\Omega} \cdot \hat{\nu} \qquad \forall \vec{v} \ \in \ \left({C_0}^{\infty} (\Omega)^* \right)^3 \tag{3.60}$$

3.8.3 Properties of $H\,(div;\Omega)$

Theorem 3.8.2 Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain : ν is the unit outward normal on $\partial\Omega$, then:

i) γ_n can be extend by continuity to a continuous linear mapping

$$\gamma_n : H\,(div;\Omega) \to H^{-1/2} (\partial\Omega) \tag{3.61}$$

ii) If $\bar{v} \in H(div;\Omega)$, $\phi \in H^1(\Omega)$, then (Green's Th.):

$$(\vec{v}, \nabla\phi) + (\nabla \cdot \vec{v}, \phi) = \langle \phi, \gamma_n(\vec{v}) \rangle_{\partial\Omega} \tag{3.62}$$

We set this definition:

$$H_0(div;\Omega) = \overline{(C_0{}^{\infty}(\Omega))^3} \tag{3.63}$$

where the closure is taken in the $H(div;\Omega)$-norm.

Theorem 3.8.3 Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, then:

$$H_0(div;\Omega) = \{\vec{v} \ \in H(div;\Omega) \ : \ \vec{v}|_{\partial\Omega} \cdot \hat{\nu}\} \tag{3.64}$$

3.8.4 Properties of $H(curl;\Omega)$

Consider the following space

$$H(curl;\Omega) := \left\{\vec{v} \in \left(L^2(\Omega)\right)^3 \ | \ \nabla \times \vec{v} \in \left(L^2(\Omega)\right)^3\right\} \tag{3.65}$$

with norm defined as

Definition 3.8.4

$$||\vec{v}||_{H(curl;\Omega)} = \left(||\vec{v}||^2{}_{(L^2(\Omega))^3} + ||\nabla \times \vec{v}||^2{}_{(L^2(\Omega))^3}\right)^{1/2} \tag{3.66}$$

Remark 3.8.3 This space is of particular physical relevance, as it is associated to the finite-energy solutions of the Maxwell's system.

In particular, for $s \geq 0$, set:

$$H^s(curl;\Omega) := \left\{\vec{v} \in (H^s(\Omega))^3 \ | \ \nabla \times \vec{v} \in (H^s(\Omega))^3\right\} \tag{3.67}$$

Moreover, the closure is defined in the usual way, namely:

$$H_0(curl;\Omega) = \overline{\left(C_0^{\infty}(\bar{\Omega})\right)^3} \tag{3.68}$$

where the closure is taken in the $H(curl;\Omega)$-norm.

Theorem 3.8.4 Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $\vec{u} \in H\left(curl;\Omega\right)$ be such that $\forall \phi \in \left(C^{\infty}\left(\bar{\Omega}\right)\right)^3$ such that

$$(\nabla \times \vec{u}, \phi) - (\vec{u}, \nabla \times \phi) = 0 \tag{3.69}$$

then $\vec{u} \in H_0\left(curl;\Omega\right)$

Remark 3.8.4 This Green's identities is valid for $\vec{u} \in H_0\left(curl;\Omega\right)$ and $\phi \in H\left(curl;\Omega\right)$, as $\left(C^{\infty}\left(\bar{\Omega}\right)\right)^3$ is dense in $H\left(curl;\Omega\right)$.

Since we wish to interpret the space $H\left(curl;\Omega\right)$ as the energy, we must verify that functions in it have well-defined tangential trace.

For all $\vec{v} \in \left(C^{\infty}\left(\bar{\Omega}\right)\right)^3$, $\hat{\nu}$ unit outward normal to Γ, define the following tangential traces:

$$\gamma_t\left(\vec{v}\right) = \hat{\nu} \times \vec{v}|_{\partial\Omega} \tag{3.70}$$

$$\gamma_T\left(\vec{v}\right) = \left(\hat{\nu} \times \vec{v}|_{\partial\Omega}\right) \times \hat{\nu} \tag{3.71}$$

Theorem 3.8.5 Let Ω be a bounded, Lipschitz, domain in $\mathbb{R}^3$, the trace map γ_t can be continuously extended from $H\left(curl;\Omega\right)$ into $\left(H^{-1/2}\left(\partial\Omega\right)\right)^3$.
Fourthermore, $\forall\ \vec{v} \in H\left(curl;\Omega\right)$ and $\phi \in \left(H^1\left(\Omega\right)\right)^3$,

$$(\nabla \times \vec{v}, \phi) - (\vec{v}, \nabla \times \phi) = \left\langle \gamma_t\left(\vec{v}\right), \phi\right\rangle \tag{3.72}$$

Remark 3.8.5 The tangential trace $\gamma_t : H\left(curl;\Omega\right) \to \left(H^{-1/2}\left(\partial\Omega\right)\right)^3$ is not surjective as $\forall\ \vec{v},\ \gamma_t\left(\vec{v}\right)$ is in the tangential space, but $\left(H^{-1/2}\left(\partial\Omega\right)\right)^3$ contains also vectors that are NOT tangential.

Thus, we define:

Definition 3.8.5

$$Y\left(\partial\Omega\right) = \left\{\vec{f} \in \left(H^{-1/2}\left(\partial\Omega\right)\right)^3 \ \mid \exists\ \vec{u} \in H\left(curl;\Omega\right) \ \text{with } \gamma_t\left(\vec{v}\right) = \vec{f}\right\} \tag{3.73}$$

Definition 3.8.6 The norm is set as follows:

$$||\vec{f}||_{Y(\partial\Omega)} = \underset{\vec{u}\in H(curl;\Omega):\gamma_t(\vec{v})=\vec{f}}{inf} \quad ||\vec{u}||_{H(curl;\Omega)} \tag{3.74}$$

The metric space so obtained is indeed a Banach space.

Theorem 3.8.6 The space $Y(\partial\Omega)$ is an Hilbert space. The tangential trace $\gamma_t : H(curl;\Omega) \to Y(\partial\Omega)$ is surjective.

The map $\gamma_T : H(curl;\Omega) \to Y(\partial\Omega)^*$ is well-defined and

$$\left(\nabla \times \vec{v}, \vec{\phi}\right) - \left(\vec{v}, \nabla\vec{\phi}\right) = \langle \gamma_t(\vec{v}), \gamma_T\left(\vec{\phi}\right)\rangle_{\partial\Omega} \quad \forall\ \vec{v}, \vec{\phi} \in H(curl;\Omega) \tag{3.75}$$

Theorem 3.8.7 Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, then $H_0(curl;\Omega)$ is given by the following expression:

$$\{\vec{v} \in H(curl;\Omega) : \gamma_t(\vec{v}) = 0\} = \left\{\vec{v} \in H(curl;\Omega) : (\nabla \times \vec{v}, \phi) = (\vec{v}, \nabla \times \phi)\ \ \forall\ \vec{\phi} \in \vec{v} \in \left(C^\infty(\bar{\Omega})\right)^3\right\} \tag{3.76}$$

Theorem 3.8.8 Suppose Ω is a bounded, Lipschitz domain in $\mathbb{R}^3$ such that it is compactly contained as subdomain in $\mathcal{O}$, then there exists a bounded, linear operator

$$E : H(curl;\Omega) \to H\left(curl;\mathbb{R}^3\right) \tag{3.77}$$

such that

$$E \cdot \vec{v} = \vec{v} \quad \text{in } \Omega \text{ and } supp\,(E\vec{v}) \subset \mathcal{O}. \tag{3.78}$$

3.9 Towards the de Rham Theory

3.9.1 Scalar and Vector Potentials

Theorem 3.9.1 Let $\Omega \subset \mathbb{R}^3$, $\vec{u} \in \left(C^1(\Omega)\right)^3 : \nabla \times \vec{u} = 0$ on Ω, then $\forall\ \mathcal{O} \subset \Omega$ there exists a scalar function $\phi \in C^2(\Omega) : \vec{u} = \nabla \cdot \phi$ in $\mathcal{O}$.

Remark 3.9.1 If Ω is a simply connected Lipschitz domain, taking unions of parallelepipeds we can show $\vec{u} = \nabla\phi$ in Ω. If we additionally require ϕ to have zero average value, we gain also uniqueness in *phi*.

Theorem 3.9.2 Consider $\Omega \subset \mathbb{R}^3$ is a simply connected Lipschitz domain and suppose

$\vec{u} \in \left(L^2(\Omega)\right)^3$,

$$\nabla \times \vec{u} = 0 \text{ in } \Omega \iff \exists \phi \in H^1(\Omega) : u = \nabla \phi \tag{3.79}$$

This ϕ is unique up to an additive constant.

Remark 3.9.2 We wish to use a vector potential in order to analyse the regularity of the solutions to the Maxwell's system.

Before doing that, however, it is better to spend some time in studying a convenient geometry for the domain Ω under consideration.

Let Ω be a bounded, connected, Lipschitz domain, there exists a decomposition of the boundary $\partial\Omega$ into connected components as follows:

$$\partial\Omega = \bigcup \{\partial\Omega_J\}_{j=0}^{k} \quad \text{for some integer} \quad k > 0 \tag{3.80}$$

where $\partial\Omega_0$ denotes the boundary of the unbounded component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

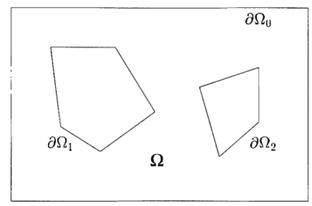


Figure 3.1: Simple geometry with labeling of the boundaries in $\mathbb{R}^2$

Let $\Omega_j \subset \mathbb{R}^3 \setminus \overline{\Omega}$ having boundary $\partial\Omega_j$ for $j = 0, \ldots, k$. By our definition, Ω_0 is unbounded. Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz domain such that $\Omega \subset \mathcal{O}^\circ$

Finally, we obtain the result:

Theorem 3.9.3 For any function $\vec{u} \in H(div; \Omega) : \nabla \vec{u} = 0$ in Ω and $\langle \vec{u} \cdot \vec{v}, 1 \rangle|_{\partial\Omega_j}$ for all $j = 0, \ldots, k$, there exists a vector potential

$$\vec{A} \in \left(H^1(\Omega)\right)^3 \;:\; \vec{u} = \nabla \times \vec{A} \text{ in } \Omega \text{ and } \nabla \cdot \vec{A} = 0 \text{ in } \Omega. \tag{3.81}$$

3.10 An Application: The inverse scattering problem

We want here to give a sketch of the variational formulation of the problem, in case of scattering from a buried object (31-34).

Remark 3.10.1

- For the first two equations in (31) and (32), observe that they are in fact containing derivatives of order two in the distributional sense, as $\nabla \times \nabla$ is a vector having "sum" of derivatives as components and, applying again the curl operator to it, we determine "sums" of second-order partial weak derivatives. Therefore, we are dealing with elliptic equations.
- For the equations number three at the interface Σ_0, we are imposing the value of vector functions to be 0, i.e. we have homogeneous Dirichlet conditions;
- The last one is just a "conservative law", namely the sum of the incident and scattered fields must be the total field.
- The radiation condition here plays a relevant role, as we impose the L^2-norm to vanish when the radius of the sphere approaches infinity. Indeed the field (e.g. plane wave) is outgoing, it is propagating in the radial direction with amplitude decaying as $1/r$ due to sphericity.

The equations

$$\nabla \times \nabla \times \tilde{E}^i - k^2 \tilde{E}^i = \boldsymbol{F} \qquad \nabla \times \nabla \times \tilde{E}^i - k^2 {\epsilon_r}^e \tilde{E}^i = 0 \tag{3.82}$$

or equivalently

$$\nabla \times \nabla \times \tilde{E}^i - k^2 \tilde{E}^i - \boldsymbol{F} = 0 \qquad \nabla \times \nabla \times \tilde{E}^i - k^2 {\epsilon_r}^e \tilde{E}^i = 0 \tag{3.83}$$

are indeed of the form

$$\nabla \left(\nabla \cdot \tilde{E}\right) - \Delta \tilde{E} - k^2 \tilde{E} - \boldsymbol{F} = 0 \qquad \nabla \left(\nabla \cdot \tilde{E}\right) - \Delta \tilde{E} - k^2 \epsilon_r \tilde{E} = 0 \tag{3.84}$$

as the following equality holds for vector functions:

$$\nabla \times (\nabla \times \phi) = \nabla (\nabla \cdot \phi) - \Delta \phi \tag{3.85}$$

Therefore, one has to be a little careful about treating the first object. It is the Laplacian plus other second-order term and it is understood that here we need to work properly in the setting of trace spaces, i.e. with multi-trace operators.

A weak solution of the perfect conductor problem is a field $E \in V_0$ such that

$$\int_{B \setminus D} curlE \cdot \overline{curlv} - k_0^2 E \cdot \bar{v} dx + \langle ik_0 \Lambda \gamma_t E, \gamma_t v \rangle_{\partial B} = \langle ik_0 \Lambda \gamma_t E^i - ik_0 \gamma_N E^i, \gamma_T v \rangle_{\partial B} \qquad \forall v \in V_0 \tag{3.86}$$

where $V_0 := \left\{ u \in H \left(curl; B \setminus \bar{D} \right) : \gamma_t = 0 \text{ on } \partial D \right\}$ and $\gamma_N \left(v \right) = \dfrac{1}{ik} \gamma_t \left(\nabla \times v \right)$ is the tangential trace of the magnetic field, whenever v is the electric one.

For more, see the article by F.Hagemann et al.,[H].

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