STRING ORBIFOLDS AND QUOTIENT STACKS

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This paper provides a brief discussion on orbifolds (string) and quotient stacks.
Motivation: A sigma model on a quotient stack would be a non-singular CFT, even when the quotient is singular.

References: [ES1] Eric Sharpe, String orbifolds and quotient stacks, arXiv: hep-th/0102211; [ES2] Eric Sharpe, Quotient Stacks and String Orbifolds, hep-th/0103008; [nL] https://ncatlab.org/nlab/show/orbifold .

1 Quotient stack $[X/\Gamma]$

Definition 1.1 (Orbifold - nL) An orbifold is a stack presented by an orbifold groupoid.
Roughly speaking, it is a smooth manifold with singularities as fixed points of finite group-actions.

Definition 1.2 (points) For X $\in$ Top,Γ finite, the category $[X/\Gamma]_{pt}$ has:

obj.: Γ-equivariant maps $f : \Gamma \to X : f(g \cdot h) = g \cdot f(h) \quad \forall g, h \in \Gamma$

mor.: $f_1 \xrightarrow{\phi} f_2$, Γ- equivariant bij. $\lambda : \Gamma \to \Gamma : f_2 \circ \lambda = f_1$.

Equivalently, they are orbits of points of X under the action of Γ.

Therefore, there exists a natural projection functor $F_p : [X/\Gamma]_{pt} \longrightarrow (X/\Gamma)_{pt}$ such that $\left([X/\Gamma]_{pt}, F_p\right)$ is a groupoid over $(X/\Gamma)_{pt}$ (discrete category).

What happens at fixed points of Γ? $[X/\Gamma]$ looks like X/Γ, except at singularities, where the quotient stack has extra structure thanks to the natural projection functor $\pi : (X)_{pt} \longrightarrow (X/\Gamma)_{pt}$, $x \to (f_x : \Gamma \to X, f_x(g) = g \cdot x)$.
Observe that $\pi \Rightarrow \pi_g$ $\forall g \in \Gamma$ and, when the action is free, X is a principal Γ-bundle over the quotient.

Definition 1.3 (maps) For X $\in$ Top, Γ finite, the category $[X/\Gamma]_{map}$ has:

obj.: Pairs $\left(E \to Y, E \xrightarrow{f} X\right)$ of a pincipal Γ -bundle and a continuous Γ-equivariant map f, respectively.

mor.: $\left(E_1 \to Y_1, E_1 \xrightarrow{f_1} X\right) \longrightarrow \left(E_2 \to Y_2, E_2 \xrightarrow{f_2} X\right)$ are pairs (ρ, λ) of a continuous function ρ and a bundle morphism λ (with usual constrains on diagrams).

Therefore, there exists a natural projection functor $F_m : [X/\Gamma]_{map} \longrightarrow (X/\Gamma)_{map}$ such that $\left([X/\Gamma]_{map}, F_m\right)$ is a groupoid over $(X/\Gamma)_{map}$, the category of continuous maps into the quotient space.
Moreover, applying Yoneda lemma, we obtain a canonical functor $\pi : (X)_{map} \to [X/\Gamma]_{map}$ such that $\pi \Rightarrow \pi_g$ $\forall g \in \Gamma$, that associates to an object x its class under the action of g and to a continuous map f the pair $\left(f^* \left(X \times \Gamma \xrightarrow{p_1} X\right), f^* (X \times \Gamma) \to X \times \Gamma \xrightarrow{ev} x\right)$.

As usual, we extend notion such as open, surj., inj., local homeo., ... via fiber product, by means of the local property definition.
For example, is $\psi : X \to [X/\Gamma]$ open? Consider $Y \to [Y/\Gamma]$ defined by the pair $\left(E \xrightarrow{\pi} Y, E \xrightarrow{f} X\right)$, then $\pi \cong p_1$ that is always open for every principal Γ-bundle E over any space Y, thus ψ is open. Similarly, Γ discrete $\implies$ π local homeo. Thus ψ, the canonical proj., is also a local homeomorphism.

2 Relation with string orbifolds

A sigma model on the quotient stack is a weighted sum over maps into $[X/\Gamma]$ and continuous maps $\psi : X \to [X/\Gamma]$ correspond to pair $\left(E \xrightarrow{\pi} Y, E \xrightarrow{f} X\right)$ of a p. Γ -bundle over Y and f, Γ-equivariant map. This is exacly the same of having a twisted sector into X.
Considering a Riemannian surface,if we restrict to the largest contractible open set, E is trivialisable, i.e. it admits a local section (it can be extended if the boundary conditions allow it) and this local section is a twisted sector. A map from it to X corresponds to a Γ-equivariant map from the whole bundle into it, i.e. a pair. Thus, twisted sectors are naturally understood in terms of quotient stacks.
Issue: if the action is not free, the functor F_m above does not provide an equivalence of categories, therefore a string orbifold can NOT be thought as a sigma model on the quotient space X/Γ, in general.
However, if X is smooth, the quotient stack is smooth, **regardness of wheter the action of Γ is free or not**.
The following facts hold:

- When the action is free, there exists an equivalence of categories compatible with the proj. map, that respects coverings defining the Grothendieck topology on $[X/\Gamma]$, i.e. $[X/\Gamma] \overset{homeo}{\cong} X/\Gamma$.
- When $\psi : X \to [X/\Gamma]$ defines a principal Γ-bundle,$\psi : X \to [X/\Gamma]$ describes X as total space of a p.Γ-b., regardless of whether or not Γ acts freely on it (sec.6).
  But what is a bundle for generalised spaces? As usual, we use the canonical projection and pullbacks, i.e. ψ is a principal Γ-bundle if $\forall$ map $Y \to [X/\Gamma]$, the projection $Y \times_{[X/\Gamma]} X \to Y$ is a principal Γ-bundle.
- We obtain an orbifold group action on a bundle by putting a Γ-equivariant structure on it

  $$\Gamma\text{- equivariance on X} \longleftrightarrow \text{object of the quotient stack } [X\Gamma]$$

  e.g. Γ- equivariance sheaf $\longleftrightarrow$ sheaf on $[X/\Gamma]$ (see 7.3-7.4), bundles are sheaves, spinors and differential forms are sections of sheaves, a symmetric 2-tensor on the quotient stack (plus ...) gives a Γ- invariant metric on X, etc.. Roughly, whenever a structure is set on the RHS, the geometry naturally "creates" the equivariance on the LHS.
- If $\mathcal{F}$ is a stack with (M,f) as topological atlas the fiber product $Y \times_{\mathcal{F}} M$ is a top. mfd via the first canonical proj, as it is always a local homeo. Therefore, the quotient stack is a top. mfd via the surj.local homeo $f : M \to [X/\Gamma]$, since the canonical map $\psi : X \to [X/\Gamma]$ is always surj. loc. homeo.
- A stack $\mathcal{F}$ is smooth if $\exists$ a smooth manifold M together with a representable surj. local homeo $f : M \to \mathcal{F}$: $M \times_{\mathcal{F}} M \overset{P_{1,2}}{\to} M$ are smooth. (M,f) is said to be an atlas for $\mathcal{F}$. Thus, $[X/\Gamma]$ is a smooth mfd, when X is smooth, Γ acts by diffeomorphisms and the corresponding atlas is (X, ψ). Notice that Γ considered here is always discrete, but its action is not required to be free.
  Moreover, a map of smooth mfds $g : Y \to N$ is smooth iff $Y \times_N M \overset{P_{1,2}}{\to} Y, M$ is smooth.

If $\mathcal{F}$ is the global quotient stack, (Γ acting by diffeo, discrete), a map $\phi : \Sigma \to \mathcal{F}$ is completely determined by a twisted sector and a twisted sector map. The classical action for a sigma model on $\mathcal{F}$ coincides with the action appearing in string orbifolds and the path integral sum over maps ϕ duplicates both the twisted sector sum as well as the sum over maps within each twisted sector, see [ES] sec.9 et seq. for further details.

Main implication:
"[...] String orbifold CFT's do not suffer from any singularities; they behave as if they were sigma models on smooth spaces. This led to the old lore that strings smooth out singularities." However, quotient stacks $[X/\Gamma]$ are always smooth (for X smooth and Γ acting by diffeomorphisms), and so one naturally expects that a sigma model on $[X/\Gamma]$ should always be well-behaved. The old lore "strings smooth out singularities" is merely a consequence of misunderstanding the target space geometry; nothing "stringy" is really involved. [...]" page 7, in [ES2].