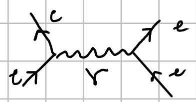
THE JONES POLYNOMIAL AND QFT

- PART I -

FEYNMAN DIAGRAMS, AN ORIENTED STATE MODEL FOR $V_K(t)$

§1. Motivation from physics - Feynman diagrams (1948)

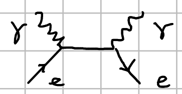
Quantum Electrodynamics (QED) simplest of the dynamical theories



γ photon (mediator)

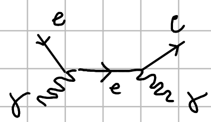
Scattering Møller
$e^+ + e^+ = e^+ + e^+$

Pair annihilation



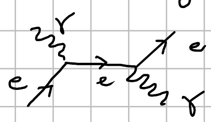
$e^- + e^+ = \gamma + \gamma$

Pair production



$\gamma + \gamma = e^+ + e^-$

Scattering Compton



$e^+ + \gamma = \gamma + e^+$

Any lepton does the same as they don't carry colours!

Physicists are interested in these three values : (processes in nature)
- cross-section (1)
- decay rates times (2)
- bounded states (3)

This answers the question: why are these diagrams useful?

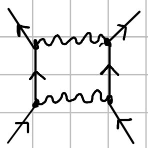
(1) scattering - when particles collide

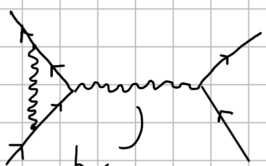
(2) when a particle naturally loses energy and or changes during time (*) see end of this talk notes

(3) Quantum states of a particle subject to a potential such that the particle has the tendency to remain localised in one or more regions of space.

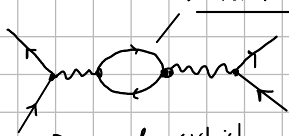
One can increase vertices --- for example 4

virtual particles (not observed particles lie inside loops) — they do not lie on their mass shell

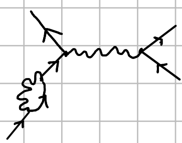




boson (〰 stays for γ one)



real particle



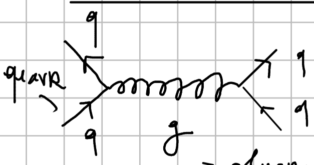
- There are ∞ many Feynman diagrams for a given process, BUT as each vertex of QED contributes by a factor of $\alpha_e = \frac{e^2}{\hbar c}$ (fine structure constant)

⇒ diagrams with more & more vertices contribute less & less to the final result and can be therefore safely ignored (usually $o(V^4)$ in QED).

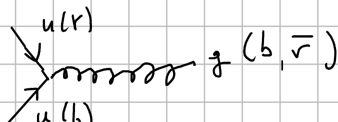
- There are conservation laws ∀ vertex (mass, energy, spin) to be imposed.

• What if we have colours? QCD (Quantum Chromodynamics)

Colour (like charge) is conserved



but



8 gluons (bicoloured)

r = red   b = blue   g = green

as quarks carry colour

Here is possible to have glueballs too: bound states of interacting gluons

However, observe that now $\alpha_g > 1$ (not $\alpha_e < 1$ as above)

& α_g is not constant, but depends on distance between particles! ASYMPTOTIC FREEDOM

→ Diagrams with more vertices will contribute more & more.

In QED vacuum is a dielectric, the vacuum polarisation partially screens the charge and reduces its field.

In QCD we have both quark polarisation & gluon polarisation diagrams.

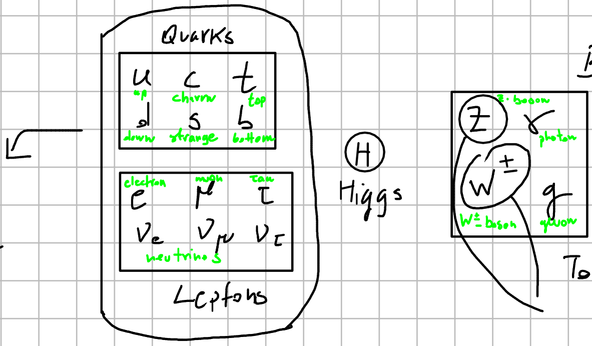
drives $\alpha_s \nearrow$ at short distances — drives it $\swarrow$

Let $f :=$ # flavours & $n :=$ # colours $\Rightarrow a = 2f - 11n$.

For the standard model, $f = 6$, $n = 3$ $\Rightarrow a = -21$.

The standard model

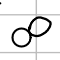
Quarks and leptons acquire masses (amount depends on the coupling strength) interacting with the H field.



Fermions (Dirac eq.)

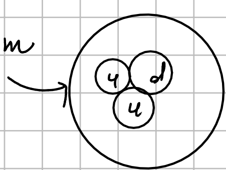
To explain the non-zero masses of Z and W, bosons & fermions masses are generated by the Higgs mechanism

→ QED vs QCD: electric charge is a true feature of particles, no colour is carried by elementary ones.

( mesons) ( baryons) $W^\pm$, Z weak interactions (bosons)

Bound states of quarks

hydrogen atom (nucleus)

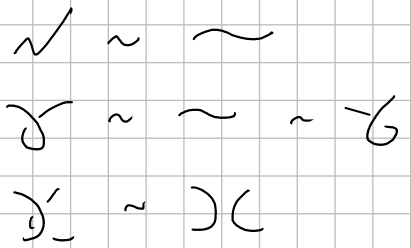


proton p

CONSERVATION LAWS: charge, colours, baryon & lepton spin numbers (mass & momenta).

§2. Back to knots: what we already know.

Reidemeister moves
∼ ambient isotopy

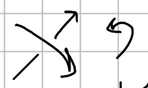
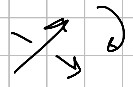


0.

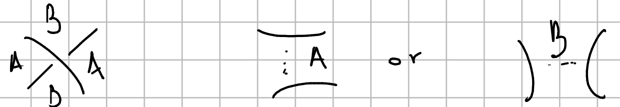
1.

2.

3. Just sliding of the top string ✓

→ Remember also the parity  $+1$  -1

→ Invariants emerge by summing / integrating over the set of states for a given diagram (evolution of states ~ fund. group).

→ Splicing  ⟹ for n splicings, we have 2^n configurations!

→ If σ is a state of K, $\langle K|\sigma\rangle$ is the commutative product of labels attached to σ & $\|\sigma\| = \#$ of loops in σ -1

→ BRACKET POLYNOMIAL $\quad \langle K\rangle = \langle K\rangle(A,B,d) = \sum_\sigma \langle K|\sigma\rangle \underbrace{\langle\sigma\rangle}_{= d^{\|\sigma\|}}$

This is not a topological invariant: The bracket with $B = A^{-1}$ & $d = -(A^2 + 1/A^2)$ is invariant under type 2 & 3 Reidemeister moves, but the invariance under 1. is gained via normalisation!

Let K be an oriented link $\qquad w(K) = \sum_{p\in C(K)} \varepsilon(p)$

(1) $w(\ \ ) = 1 + \underbrace{w(\ \ )}_{=0}$ & $w(\ \ ) = -1 + \underbrace{w(\ \ )}_{=0}$ (2)

NORMALISED BRACKET $\qquad \mathcal{L}_K := (-A^3)^{-w(K)} \langle K\rangle$

(1*) $\langle\ \ \rangle = (-A^3)\langle \sim \rangle$ & $\langle\ \ \rangle = (-A)^{-3}\langle\sim\rangle$ (2*)

Property: $\mathcal{L}_{K^*}(A) = \mathcal{L}_K(A^{-1})$

⟹ if $\mathcal{L}_K(A) \neq \mathcal{L}_K(A^{-1})$, $K \not\sim K^*$ ⟹ $\mathcal{L}_K$ DETECTS CHIRALITY ✓

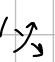
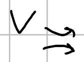
Property: If T is a knot, $w(T)$ does not depend on the orientation

⟹ $\mathcal{L}_T = (-A^3)^{-w(T)}\langle T\rangle \longrightarrow \mathcal{L}_L = \langle L\rangle$ for a linked diagram ✓

Theorem: $\mathcal{L}_K(1/\sqrt[4]{t}) = V_K(t)$ ← Jones polynomial

normalised bracket

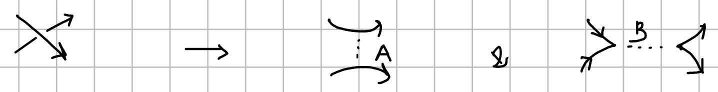
Def. $V_K(t)$ is a Laurent polynomial in $\sqrt{t}$ assigned to an oriented K:

1. $K \sim K' \Rightarrow V_K(t) = V_{K'}(t)$;
2. $V_{\bigcirc} = 1$;
3. $\frac{1}{t} V$  $- t\, V$  $= (\sqrt{t} - 1/\sqrt{t})\, V$ .

§3. A new perspective [why section 1. makes sense]

Jones polynomial $\quad V_K(t) = (-t^{3/4})^{\omega(K)} \langle K \rangle \left(\frac{1}{\sqrt[4]{t}}\right)$

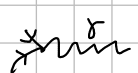
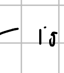
– What about splicing a Feynman diagram?

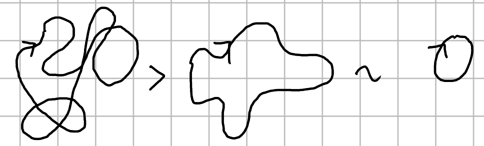


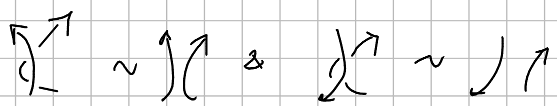
interaction ✓   creation & annihilation ✓

⚠ Time here is local & coincides with the arrows on the diagram!

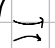
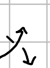
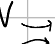
Observe that: 

– topological diagrams do not know about field effect  – is ignored ()

– 

• We impose (cross) channel unitarity, i.e. 
(interaction of particles)

& triangle invariance

$\Rightarrow V^{+}$  $= A\,V$  $+ B\,V$  & V^{-}  $= A'\,V$  $+ B'\,V$ 

$V_{\bigcirc K} = \delta V_K$ (loop value)

We want to determine A', B', δ

Remark: It is not true that  (otherwise is trivial)

and  However  ✓

§ 3.1 Calculations

1. Channel unitarity (& cross-channel unitarity)

$$V = AV + BV = A\{A'V + B'V\} + B\{A'V + B'V\} = AA'V + (AB' + BB'\delta + BA')V \overset{\text{unitarity}}{\Longrightarrow} \begin{cases} AA' = 1 \\ AB' + BB'\delta + BA' = 0 \end{cases}$$

From cross-channel unitarity (left as exercise), we obtain an analogous set of eq.s. Combining both of them, we get:

$$\left.\begin{array}{l} AA' = 1 = BB' \\ \delta = -\left(\frac{A}{B} + \frac{A'}{B'}\right) = -\left(\frac{B'}{A'} + \frac{B}{A}\right) \end{array}\right\} \quad A' = A^{-1}, \quad B' = B^{-1}, \quad \delta = -\left(\frac{A}{B} + \frac{B}{A}\right) \quad (1)$$

✓

2. Annihilation & crossing (this implies triangle invariance).

$$V = A'V + B'V = A'\{A'V + B'V\} + B'\{A'V + B'V\} =$$

$$= \big(\underbrace{(A')^2 + (B')^2 + \delta A'B'}_{=0}\big)V + A'B'V \overset{(1)}{=} \frac{1}{AB} V \quad \left(\text{the particle just pass by}\right)$$

Similarly, $V = AB\,V$ and also $V = AB\,V$ (a)

$V = \frac{1}{AB} V$ (b) and $V = AB\,V$ (c)

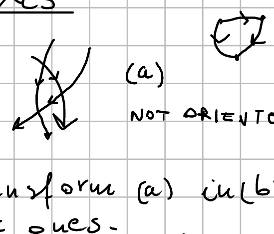
$\therefore$ $V = V$ ✓ (creation-annihilation version of the triangle move) :)

It implies triangle invariance. Indeed:

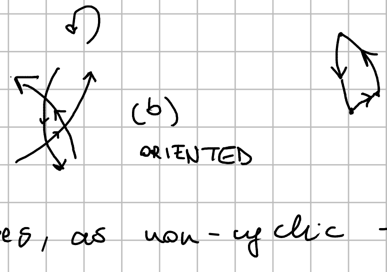
$$V = AV + BV = AV + BV = V$$

3. Oriented triangles

Def.



(a) NOT ORIENTED



(b) ORIENTED

However, we can transform (a) in (b) via R. moves, as non-cyclic triangle moves can generate cyclic ones.

$\Rightarrow$ We found:
$$V = A V + B V$$
$$V = \frac{1}{A} V + \frac{1}{B}$$

4. Type I. invariance (invariance under R I)

$$V = A V_{\circ} + B V = (A\delta + B)\, V$$

$$A\delta + B \overset{(1)}{=} A\left(-\frac{A}{B} - \frac{B}{A}\right) + B = -A^2/B$$

$$\Rightarrow V = V \iff A\delta + B = 1 = -A^2/B \iff B = -A^2 \ \checkmark$$

§3.2 The Jones polynomial

We have just concluded $B = -A^2$. Let $A = -\sqrt{t} \Rightarrow B = -t$, this implies

$$\begin{cases} V = -\sqrt{t}\, V - t\, V \\ V = -\frac{1}{\sqrt{t}} V - \frac{1}{t} V \end{cases}$$

$$t^{-1} V + (-t) V = \left(-\frac{\sqrt{t}}{t} + \frac{t}{\sqrt{t}}\right) V = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V$$

$\Rightarrow V_K(t)$ is the 1-variable Jones polynomial.

Lemma:
$$V = t\sqrt{t}\, V \qquad V = \frac{1}{t\sqrt{t}} V$$
$$V = t\sqrt{t}\, V \qquad V = \frac{1}{t\sqrt{t}} V$$

$$V = -\sqrt{t}\, V - t\, V = \left(-\sqrt{t} - t\left(-\sqrt{t} - \frac{1}{\sqrt{t}}\right)\right) V = t\sqrt{t}\, V \ \checkmark$$

Analogously for the other cases.

- Let's call $\hat{V}$ the renormalised Jones poly, then:

$$t\hat{V} - \frac{1}{t}\hat{V} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\hat{V}$$

$$\hat{\delta} = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) \text{ the loop variable}$$

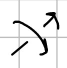
Physical interpretation: For corresponding state expansion, we have:

$$\hat{V}_K = \sum_{\vec{s}} \langle K|\vec{s}\rangle \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{\|\vec{s}\|}$$

where $\langle K|\vec{s}\rangle$ is the product of the vertex weights

$\|\vec{s}\| :=$ # of loops in the oriented state

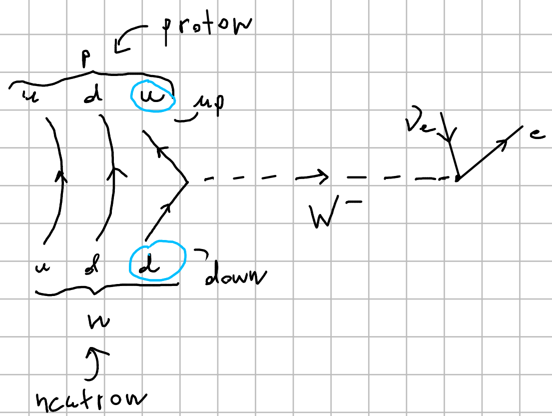
$\forall \vec{s}$ the sign is $(-1)^{\pi}$ where π is the parity of the number of creation - annihilation splices in the state $\vec{s}$

 $+1$ ↺ (positive oriented)

 -1 ↻ (negative oriented)

✓

* On the decays (some examples):
  - γ-decay: does not change the structure or composition of an atom. Some unstable nuclei dissipate excess of energy via electromagnetic radiation. Orin
  - β-decay: $n = p + e + \bar{\nu}_e$ ← antineutrino



In this process the quark changes its flavour. A neutron transforms into a proton by the emission of an electron and an antineutrino.

THE JONES POLYNOMIAL AND QFT

- PART II -

STRINGS & BRAIDED CATEGORIES: THE JONES POLYNOMIAL REVEALED

§1. A primer in category theory

Def. A directed graph is a set of objects O, a set of morphisms (arrows) A and two functions

$$\begin{tikzcd} A \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] & O \end{tikzcd}$$

ex. $f : a \longrightarrow b$, $f \in A$

$$\left.\begin{aligned} \mathrm{dom}(f) &= a \\ \mathrm{cod}(f) &= b \end{aligned}\right\} \in O$$

In this graph, the set of composable arrows is the set:

$$A \times_O A = \{ \langle g, f \rangle : g, f \in A \ \& \ \mathrm{dom}\, g = \mathrm{cod}\, f \}$$

called the product over O.

Def. A category is a graph with two additional functions:

(identity) $O \xrightarrow{id} A$, $a \longrightarrow id_a$ $\qquad$ $A \times_O A \xrightarrow{\circ} A$, $\langle g, f \rangle \longrightarrow g \circ f$ (composition)

such that $\mathrm{dom}(id_a) = a = \mathrm{cod}(id_a)$ $\quad \forall a \in O$

$\mathrm{dom}(g \circ f) = \mathrm{dom}\, f$

$\mathrm{cod}(g \circ f) = \mathrm{cod}\, g$ $\quad \forall \langle g, f \rangle \in A \times_O A$

and - associativity: for $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d$, $k \circ (g \circ f) = (k \circ g) \circ f$

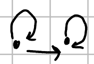
- unit law: $\forall\ f : a \to b,\ g : b \to c$ $\quad 1_b \circ f = f$ & $g \circ 1_b = g$

Remark (notation): If C is a category, the set of objects is denoted also by C^0, the set of morphisms by C^1 or $C^1(a,b)$ when one wants to stress dom & cod. We also write this last one has

$$\mathrm{hom}(a,b) = \{ f \mid f \in C^1,\ \mathrm{dom}\, f = a,\ \mathrm{cod}\, f = b \}$$

Examples: $\mathbb{0}$ $\emptyset$ category, i.e. $O^0 = O^1 = \emptyset$

$\mathbb{1}$  one obj. and one (identity) arrow.

$\mathbb{2}$  $\qquad$ $\mathbb{3}$  and so on...

$\downarrow\downarrow$ two obj & two nontrivial arrows (e.g. $g, f : a \rightrightarrows b$) called parallel arrows

<u>Monoids</u> : A monoid is a category with one object. It can be described as a set M with a binary operation $M \times M \to M$ associative and unital. It is a semigroup with identity.

For any category $\mathcal{C}$ and $a \in \mathcal{C}^0$, $\mathrm{Hom}(a,a)$ is a monoid.

Set, Set$_*$, Top, Top$_*$, Toph (morphisms are homotopy classes of maps)
Ab, Grp, R-Mod, Ring, ...

Def. A morphism of categories is called functor, e.g. $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$.

$\forall\, c_1 \in \mathcal{C}_1^0 \quad \Phi(c_1) \in \mathcal{C}_2^0$

$\forall\, f \in \mathcal{C}_1^1(a,b) \quad \Phi(f) \in \mathcal{C}_2^1(\Phi(a), \Phi(b))$

in such a way that $\Phi(1_c) = 1_{\Phi(c)} \quad \forall\, c \in \mathcal{C}_1^0$

if covariant $\Phi(g \circ f) = \Phi(g) \circ \Phi(f) \quad \forall\, \langle g, f\rangle \in \mathcal{C}_1^1$

(if contra-variant) (or $\Phi(g \circ f) = \Phi(f) \circ \Phi(g)$)

<u>Examples</u>: $\mathrm{Top}_* \xrightarrow{\pi_n} \mathrm{Grp} \qquad (X, x_0) \to \pi_n(X, x_0)$

$\mathrm{Grp} \xrightarrow{\phi} \mathrm{Ab} \qquad G \to G/[G,G]$

forgetful $\mathrm{Grp} \xrightarrow{U} \mathrm{Set} \qquad G \to U(G)$ forgetting group structure

$\mathrm{Set} \xrightarrow{\mathcal{P}} \mathrm{Set} \qquad X \longrightarrow \mathcal{P}(X)$

$\{X \xrightarrow{f} Y\} \longrightarrow \{\mathcal{P}(X) \xrightarrow{\mathcal{P}f} \mathcal{P}(Y)\}$

what if one uses the inverse image? f^{-1}

$\mathcal{P}(1_X) = 1_{\mathcal{P}X}$
$\mathcal{P}(fg) = \mathcal{P}f \circ \mathcal{P}(g)$
covariance

Def. An arrow $m : a \to b$ is monic in $\mathcal{C}$ iff $\forall\, f_1, f_2 : d \rightrightarrows a$, $mf_1 = mf_2 \Rightarrow f_1 = f_2$ (left cancellable).

monic in Set? inj ✓

Def An arrow $h : a \to b$ is epic (epi) in $\mathcal{C}$ iff $\forall\, g_1, g_2 : b \rightrightarrows c$ $g_1 h = g_2 h \Rightarrow g_1 = g_2$ (right cancellable)

epi in Set? surj ✓

Def. Two objects $a, b \in \mathcal{C}^0$ are isomorphic ($a \cong b$) iff $\exists$ an isomorphism $e : a \to b$, i.e. iff $\exists$ an arrow $e : a \to b$ invertible (i.e. $\exists\, e^{-1} : b \to a \mid e^{-1}e = 1_a$ & $ee^{-1} = 1_b$).

iso in Set? bijections ✓

§2 Monoidal categories* (a functor of functors is a natural transfo.)

Roughly, a monoidal category, is a category equipped with a product.

<u>Def.</u> A <u>strictly</u> monoidal category $\langle B, \square, e\rangle$ is a category B with a bifunctor (functor in both variables) $\square : B\times B \to B$ which is

- associative $\square(\square\times 1) \overset{(1)}{=} \square(1\times\square) : B\times B\times B \longrightarrow B$
- has a unit, i.e. an object e : $\square(e\times 1) \overset{(2)}{=} id_B = \square(1\times e)$

where $(e\times 1)(c) \longrightarrow \langle e, c\rangle$ $\qquad e\,\square\, c = c = c\,\square\, e \quad \forall c\in B^0$

$B \longrightarrow B\times B$ $\qquad 1_e \,\square\, f = f = f\,\square\, 1_e \quad \forall f\in B^1$

How $\square$ works?

On objects: $(a,b)\in(B^0)^2 \longrightarrow a\,\square\, b\in B^0$

On morphisms: $(f,g)\in(B^1)^2 \longrightarrow f\,\square\, g : a\,\square\, b\to a'\,\square\, b'$ for $f: a\to a'$, $g: b\to b'$

such that

$$1_a\,\square\, 1_b = 1_{a\square b}$$

$$(f'\square g')(f\square g) = f'f\,\square\, g'g$$

<u>Def.</u> A monoidal category $B = \langle B, \square, e, \alpha, \lambda, \rho\rangle$ is a category B with a bifunctor $\square : B\times B\to B$, $e\in B$ as above and three <u>natural isomorphisms</u> α, λ, ρ as follows:

(1) <u>associativity</u> $\qquad \alpha = \alpha_{a,b,c} : a\,\square(b\,\square\, c) \cong (a\,\square\, b)\,\square\, c \qquad \forall a,b,c\in B^0$

$$\begin{tikzcd}
a\square(b\square(c\square d)) \arrow[r, "\alpha"] \arrow[d, "1\square\alpha"'] & (a\square b)\square(c\square d) \arrow[r, "\alpha"] \arrow[d, phantom, "\circlearrowright"] & ((a\square b)\square c)\square d \\
a\square((b\square c)\square d) \arrow[rr, "\alpha"'] & & (a\square(b\square c))\square d \arrow[u, "\alpha\square 1"']
\end{tikzcd}$$

(2) <u>unit law (left & right)</u> $\qquad \lambda_a : e\,\square\, a \cong a \qquad \rho_a : a\,\square\, e\cong a \qquad \forall a\in B^0$

$$\begin{tikzcd}
a\square(e\square c) \arrow[rr, "\alpha"] \arrow[dr, "1\square\lambda"'] & \circlearrowright & (a\square e)\square c \arrow[dl, "\rho\square 1"] \\
& a\square c &
\end{tikzcd}$$

& $\lambda_e = \rho_e : e\,\square\, e\to e$ ✓

Examples: $(Set, \times)$ $(Ring, \otimes)$

They are usually called <u>tensor categories</u> rather than monoidal in some textbook. $(Vect_{\mathbb{F}}, \otimes, 1\text{-dim sub.})$

<u>Def.</u> A <u>monoid</u> $m \in B$ (monoidal) is an object together with two arrows $\mu : m \,\square\, m \to m$ $\quad \eta : e \to m$ such that

$$\begin{tikzcd} m \,\square\, (m \,\square\, m) \arrow[r, "\alpha"'] \arrow[d, "1 \square \mu"'] & (m \,\square\, m) \,\square\, m \arrow[r, "\mu \square 1"'] & m \,\square\, m \arrow[d, "\mu"] \\ m \,\square\, m \arrow[rr, "\mu"'] & & m \end{tikzcd} \qquad (1)$$

$$\begin{tikzcd} e \,\square\, m \arrow[r, "\eta \square 1"] \arrow[dr, "\lambda"'] & m \,\square\, m \arrow[d, "\mu"] & m \,\square\, e \arrow[l, "1 \square \eta"'] \arrow[dl, "\rho"] \\ & m & \end{tikzcd} \qquad (2)$$

- An <u>arrow between monoids</u> $f : \langle m, \mu, \eta \rangle \to \langle m', \mu', \eta' \rangle$ is a morphism $f : m \to m'$ such that

$$f\mu = \mu'(f \,\square\, f) : m \,\square\, m \to m'$$
$$f\eta = \eta' : e \to m'$$

$\to$ The monoids in B form therefore a category denoted by Mon_B and $\exists$ a forgetful functor $U : Mon_B \to B$
$$\langle m, \mu, \eta \rangle \to m$$

that just forgets the multiplication μ.

<u>Ex</u>amples: $\langle R\text{-mod}, \otimes, R \rangle \longleftrightarrow R\text{-algebras}$

B $\quad \langle Ab, \otimes, \mathbb{Z} \rangle \longleftrightarrow \mathbb{Z}\text{-modules}$ $\quad Mon_B$

<u>To</u> see: The simplicial category as <u>UNIVERSAL MONOID</u>.

$$\Delta \to Top$$

* The coherence theorem part is skept for time reasons (please see Mac Lane).

§3. Braiding (Ch. XI Mac Lane)

Def. (3.1) A monoidal category $\mathcal{B}$ is called <u>symmetric</u> if the commutativity property holds up to a natural isomorphism $\gamma : a \square b \cong b \square a$. (see def 3.3)

- Is it always possible to strictify, i.e. $\gamma^4 = 1$? <u>NO</u>

  $\exists$ examples of categories with a twist, i.e. $\gamma^2 \neq 1$ — braided cat. !!

Def. (3.2) A braiding for a monoidal category M consists of a family of isos $\gamma_{a,b} : a \square b \cong b \square a$ natural in $a, b \in M^{\circ}$:

$$\begin{tikzcd} a \square e \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & \arrow[d, phantom, "\circlearrowleft"] & e \square a \arrow[dl, "\lambda"] \\ & a & \end{tikzcd}$$

and*

$$\begin{tikzcd} (ab)c \arrow[r, "\gamma"] \arrow[d, "\alpha^{-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & c(ab) \arrow[d, "\alpha"] \\ a(bc) \arrow[d, "1 \square \gamma"'] & (ca)b \arrow[d, "\gamma \square 1"] \\ a(cb) \arrow[r, "\alpha"'] & (ac)b \end{tikzcd} \qquad \begin{tikzcd} a(bc) \arrow[r, "\gamma"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & (bc)a \arrow[d, "\alpha^{-1}"] \\ (ab)c \arrow[d, "\gamma \square 1"'] & b(ca) \arrow[d, "1 \square \gamma"] \\ (ba)c \arrow[r, "\alpha"'] & b(ac) \end{tikzcd}$$

* dropping $\square$

<u>Remark</u>: If γ is a braiding on M, γ^{-1} also is.

<u>Def.</u> (3.3) A symmetric monoidal category, is a category M with a braiding γ:

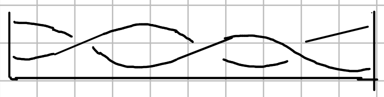
$$\begin{tikzcd} ab \arrow[r, "\gamma_{a,b}"] \arrow[dr, "id_{ab}"'] & ba \arrow[d, "\gamma_{b,a}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & ab \end{tikzcd}$$

Remark: For symmetric monoidal cat. all formal diagrams involving α & γ commute (coherence).

For braided m-cat. $\exists$ ∞ number of canonical automorphisms $1, \gamma^2, \gamma^4, \dots, \gamma^{-2}, \gamma^{-4}, \dots : a \square b \longrightarrow a \square b$.

§4. The Artin braid groups B_n and the braid category

A braid is formed by strings fixed at the extremes of the diagram as follows:



The one in the example is an element of B_3, the Artin braid group on 3 strings.

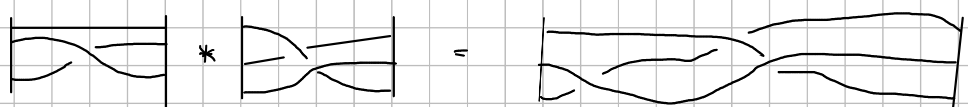
Def. The n-th Artin braid group is defined as

$$B_n = \{ \text{braids on } n \text{ strings} \} / \sim$$

where $b_1 \sim b_2$ in B_n iff b_1 can be continuously deformed into b_2 without crossing or cutting the component strings.

- On B_n we define the composition operation simply composing the single paths (strings) in the braids (juxtapposition).

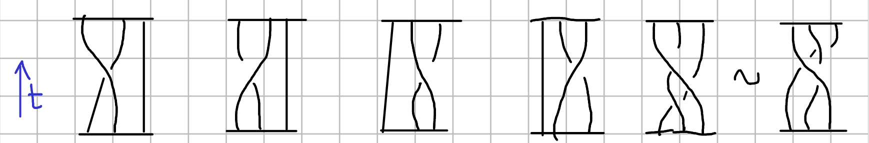
Example: on B_3



$b_1 \quad * \quad b_2 \quad = \quad b_1 * b_2$

§4.1 Generators of B_n & S_n

On B_3, consider the following



$\sigma_1 \quad \sigma_1^{-1} \quad \sigma_2 \quad \sigma_2^{-1} \quad \sigma_1\sigma_2\sigma_1 \sim \sigma_2\sigma_1\sigma_2$

σ_i := i-th string under the $(i+1)$-th. (*)

$\sigma_2\sigma_1$ | Composition reads way

$\Rightarrow \quad B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ as presentation.

It is quite natural to think about the existence of relation between the n-th Artin group B_n and the group S_n of permutations over n elements $B_n \to S_n$.

can also be omitted, as it is a group.

$$\boxed{B_n = \langle \sigma_1, \dots, \sigma_{n-1} : \underbrace{\sigma_i\sigma_i^{-1} = 1}, \ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1, \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle}$$

σ_i as in (*) above.

- Little observation: B_n can be also seen as fundamental group of T_n, where we fix $P = \mathbb{E}^2$ and define $T_n = \{ (x_1, \dots, x_n) \mid x_i \in P \ \forall i \}$

$\Rightarrow B_n$ = fundamental group of T_n. (to become clear later)

Remember: every permutation $\sigma \in S_n$ can be written as a finite product of transpositions.

Ex. $(123) \to (321) = (13)$ ✓

$(12345) \to (54312) = (12)(24)(15)$ ✓

The permutation group S_n is presented as follows:

$$\boxed{S_n = \langle \tau_i := (i, i+1) \mid \tau_i^2 = 1,\ \tau_i \tau_j = \tau_j \tau_i \text{ for } |i-j| \neq 1,\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \rangle}$$

$\Rightarrow$ It is apparent that $\exists$ an homomorphism $B_n \to S_n$, $\gamma_i \to \tau_i$ $\quad \forall i = 1, \dots, n-1$

§ 4.2 The braid category $\mathcal{B}$

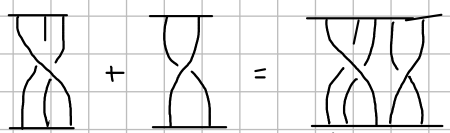
All the braid groups B_n can be assembled into a category $\mathcal{B}$, called the braid category, whose:

- objects are the natural numbers $\mathbb{N}^{\geq 0}$
- arrows are the braids $n \to n$ ($\nexists$ arrow $n \to m$ for $n \neq m$ and only the identity arrow $\emptyset \to \emptyset$)

$\Rightarrow$ This defines a monoidal category $\mathcal{B}$ with product $+ : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$.

The addition operation $+$:

- on objects, it is just addition of natural numbers ✓
- On morphisms,

ex: 

$\in B_3 \qquad \in B_2 \qquad \in B_5$

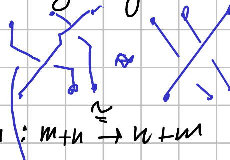
$(3 \to 3) + (2 \to 2) = (5 \to 3+2 = 5)$

as defined for standard $\square$ operation ✓

In general, $(n \to n) + (m \to m) = (n+m) \to (n+m)$

This operation can be shown to be associative and with a unit given by the empty braid $\emptyset \to \emptyset$ $\Rightarrow$ $\langle \mathcal{B}, +, \emptyset \to \emptyset \rangle$ is a strict mon. category.

$\to$ What about commutativity? $\gamma^2 \neq 1 \Rightarrow$ not symmetric.



The addition is commutative and one can define a natural trafo $\gamma_{m,n} : m+n \xrightarrow{\sim} n+m$ just crossing m strings over n strings. $\to$ we have a braiding !!

§5. Braid groups & The Jones Polynomial

§5.1 Von Neumann algebras

Def. Given a commutative ring R, an <u>R-algebra</u> is an R-module with an associative R-bilinear operation which also contains a multiplicative identity. It can be viewed as a monoid in $\langle R\text{-mod}, \otimes, R\rangle$.

Def. A <u>free R-algebra on a set E</u> is an algebra of polynomials with coefficients in R and non commuting indeterminates taken from the set E.

- Consider a sequence $\{A_n\}_{n=2}^{\infty}$ of free additive algebras on multiplicative generators $\{e_i\}_{i=1}^{n-1}$ (for each A_n), i.e. modules over the ring $\mathbb{C}[\tau, \tau^{-1}]$, where τ as in the relations:

von Neumann algebras

1. $e_i^2 = e_i$
2. $e_i e_{i\pm1} e_i = \tau e_i$
3. $e_i e_j = e_j e_i \quad |i-j| > 2$

and the following tower generated by inclusions of algebras as follows:

$M_0 := N$, $M_1 := M$, $M_2 := \langle M_1, e_1\rangle$ where $e_1 : M_1 \to M_0$ is the projection & $\langle M_1, e_1\rangle$ denotes the algebra generated by M_1 plus e_1 (adding 1 gen.).

$\Rightarrow \quad M_0 \subset M_1 \subset M_2 \subset M_3 \subset \dots \subset M_n \subset M_{n+1} \subset \dots$

$e_i : M_i \to M_{i-1} \ \forall i$ & $M_{i+1} := \langle M_i, e_i\rangle$.

§5.2 Jones algebras

- Jones constructed such a tower of algebras with $\{e_i\}$:

$e_i e_{i\pm1} e_i = \tau e_i \ \forall i$ & $e_i e_j = e_j e_i \ \forall i,j : |i-j| > 1$

where $\tau := [M_1 : M_0]$ (index).

Moreover, he defined a trace $tr : M_n \to \mathbb{C}$ satisfying the <u>Markov Property</u>: $tr(w e_i) = \tau \cdot tr(w) \quad \forall w \in \langle M_0, e_1, \dots, e_{i-1}\rangle$ ← "span" symbol

Def. A function $f: \mathcal{A} \to \mathbb{C}$, where $\mathcal{A}$ is an algebra, is called <u>trace</u> iff $tr(ab) = tr(ba) \quad \forall a, b \in \mathcal{A}$

Example: The ordinary trace of matrices is indeed a trace.

So far we have then the following:

$e_i^2 = e_i$ $e_i e_{i\pm 1} e_i = \tau e_i$ $e_i e_j = e_j e_i \quad \lvert i-j \rvert > 1$ gen.: $e_1, \dots, e_{n-1}$ Jones Algebra $\mathcal{A}_n$	$\sigma_i \sigma_i^{-1} = 1$ (it's a group) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $\sigma_i \sigma_j = \sigma_j \sigma_i$ gen.: $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$ n-th cylindrical braid group B_n

§ 5.3 Jones construction of a representation of B_n

Def. Given a group G, a <u>representation of G</u> is a finite dimensional complex vector space E^n along with a morphism $\rho: G \to GL_n(E)$, $GL_n(E) = \{\mathbb{C}\text{-linear maps } \Phi: E\,\circlearrowleft\}$. It is denoted by (E, ρ), E is called support of the representation.

→ In particular, if $G \leq GL_n(E)$ ⇒ the representation induced by $i: G \hookrightarrow GL_n(E)$ is called <u>fundamental representation</u>.

Consider the following representation $\rho_n(\sigma_i) \to a e_i + b$, $\rho_n: B_n \to \mathcal{A}_n$ for some a, b appropriately chosen. We already defined $tr: \mathcal{A}_n \to \mathbb{C}[t, t^{-1}]$ a trace on $\mathcal{A}_n$ ⇒ It is natural to compose them, i.e.:

$$tr \circ \rho_n : B_n \longrightarrow \mathbb{C}[t, t^{-1}]$$

We want to prove this gives indeed the Jones polynomial $V_K(t)$ upon normalisation.

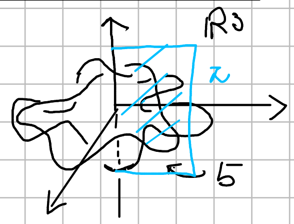
Remark: We know it is ambient isotopy invariant for oriented links.

Thanks to theorems by Markov & Alexander, we will show $V_K(t)$ is not only well-defined on braids, but for arbitrary knots and links.

§ 5.4 Alexander's & Markov's Theorems

Def. Let $b \in B_n$ for some $n \in \mathbb{N}^{\geq 0}$ a braid, then its closure $\bar{b}$ is obtaining by glueing together the initial and terminal points of b in order.

ALEXANDER'S THEOREM : Each link in 3d-space is ambient isotopic to a link in the form of a closed braid.



It is always possible to arrange $\bar{b}$ in such a way that $\exists$ an axis through the center as in the figure on the left.
If we take a plane π, it is rotated ↻ or ↺ wrt the vertical axis, as we follow the orientation on $\bar{b}$.

It is also apparent that $\forall \pi$ $|\pi \cap \bar{b}| = \#$ of strings $= n$.

Now follow along the knot or line and when the strand starts circulating incorrectly, just throw it over the axis. The knot / line will be a braid $\bar{b}$ after this procedure. □

• How many different ways are there to represent a link as closed braid?

i.e. to modify $b, \tilde{b}$ in such a way that their closures are ambient isotopic links?

(1.) Using the Markov move: add a generator $B_n \to B_{n+1}$ and close b^+

$$b \to b^+$$

More specifically, if $\beta \in B_n \Rightarrow \beta, \beta\sigma_n, \beta\sigma_n^{-1}$ are such that $\bar{\beta} \sim \overline{\beta\sigma_n} \sim \overline{\beta\sigma_n^{-1}}$

where $\sim$ means ambient isotopic. ✓

(2.) Compose the conjugate braid $g\beta g^{-1}$ for some $g \in B_n$ and take the closure $\overline{g\beta g^{-1}} \Rightarrow \overline{g\beta g^{-1}} \sim \bar{\beta}$. ✓

MARKOV'S THEOREM: Let $\beta_n \in B_n$ and $\beta'_m \in B_m$ $\Rightarrow L := \overline{\beta_n}, L' := \overline{\beta'_m}$ are ambient isotopic $\Leftrightarrow$ β'_m is obtained from β_n, via a series of:

a) equivalences b) conjugations c) Markov moves

Using this theorem, we can use the presentations of the braid group to extract topological information about knots and links.

↳ Let R be a com. ring and $\{J_n : B_n \to R\}_{n\geq 2}$ family of functions $\Rightarrow$ by Markov's theorem, they can be used to construct link invariants if the following holds:

(i) $b \sim b' \Rightarrow J_n(b) = J_n(b') \quad \forall\, b, b' \in B_n$

(ii) $g, b \in B_n \Rightarrow J_n(b) = J_n(gbg^{-1})$

(iii) $b \in B_n \Rightarrow \exists\, \alpha \in R$ ($\perp\!\!\!\perp$ of n) : $\quad J_{n+1}(b\sigma_n) = \alpha^{+1} J_n(b)$

$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b)$$

Remark: for $\overline{b'} = \overline{b\sigma_n}$, the Markov move $b \to b' \underset{\text{eq.}}{\Leftrightarrow}$ RI on $\overline{b}$.

($\overline{b\sigma_n}$ RI. positive, $\overline{b\sigma_n^{-1}}$ RI. negative)

Def. Let $b = \sigma_{i_1}^{a_1} \cdots \sigma_{i_k}^{a_k}$ be a braid, then we define its writhe as

$$\sum_{s=1}^{k} a_k =: w(b)$$

It is apparent that $w(b) = w(\overline{b})$.

Def. Let $\{J_n\}_{n\geq 2}$ be the family of functions above, we call it a Markov trace on $\{B_n\}$.

Def. Let $L \sim \overline{b}$, $b \in B_n$ for any link L (existence of $\overline{b}$ follows by Alexander's theorem)

$\Rightarrow$ we define the link invariant $J(L)$ for the Mar. trace $\{J_n\}$ the following:

$$\boxed{R \ni J(L) = \alpha^{-w(b)} J_n(b)}$$

Notice that the function J_n is fixed by the choice of the representative $b \in B_n$ for the link.

Proposition: If $L \sim L' \Rightarrow J(L) = J(L')$

Let $L \sim \overline{b}$, $L' \sim \overline{b'}$ $b, b' \in B_n$ (via Alexander's theorem as usual)

as $L \cup L'$ by hypothesis, $\overline{b} \sim \overline{b'}$ via a sequence of moves of type a), b), c) in Markov's theorem $\Rightarrow w(\overline{b}) = w(\overline{b'}) \Rightarrow J(L) = J(L')$ ⊠

§6. Bracket for Braids

Def. Let $\langle\,\rangle : B_n \to \mathbb{Z}[A, A^{-1}]$ be defined as $\langle b \rangle = \langle \bar{b} \rangle$ the bracket for braids or equivalently, if $J_n : B_n \to \mathbb{Z}[A, A^{-1}]$, then $J_n(b) = \langle \bar{b} \rangle$.

$\to$ J_n is a Markov trace with $\boxed{\alpha = -A^3}$.

§6.1 States of a braid

Remember $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$ & generalise

$\hookrightarrow$ $\langle || \dots | \times | \dots || \rangle = A \langle || \dots | \rangle + A^{-1} \langle || \dots | \asymp | \dots | \rangle$

ith positions
(i+1)th

$\Rightarrow$ $\langle \sigma_i \rangle = A \langle 1_n \rangle + A^{-1} \langle U_i \rangle$

U_i is a new element in the braid, a composition of cup $\cup$ and cap $\cap$ at the i-th & (i+1)-th strand

$\Rightarrow$ Each state of $\bar{b}$ can be written as the closure of products of U_i

Define $\sigma_i := A + A^{-1} U_i$, $\sigma_i^{-1} := A^{-1} + A U_i$ $\Rightarrow$ $b = \mathcal{U}(b)$

sum of products of U_i

Just inserting for each generator σ_i of the braid word, the new definitions in terms of the U_i ✓

$\Rightarrow \langle b \rangle = \langle \mathcal{U}(b) \rangle = \langle \mathcal{U}(\bar{b}) \rangle = \langle \bar{b} \rangle = \delta^{\|\mathcal{U}\|}$

$\|\mathcal{U}\| = \#$ loops in $\bar{\mathcal{U}} - 1$

$\delta = -\left(\frac{1}{A^2} + A^2\right)$

$\Rightarrow$ $\mathcal{U}(b) = \sum_s \langle b | s \rangle \mathcal{U}_s$

where s indexes all terms in the products and $\langle b|s \rangle$ is as usual the product of the labels A, A^{-1}.

$\Rightarrow \langle \mathcal{U}(b) \rangle = \langle \Sigma_s \langle b/s \rangle \mathcal{U}_s \rangle = \Sigma_s \langle b|s \rangle \langle \mathcal{U}_s \rangle = \Sigma_s \langle b|s \rangle\, \delta^{\|s\|}$

§6.2 Temperley-Lieb algebra $A_n = TL_n(\delta)$

statistical mechanics

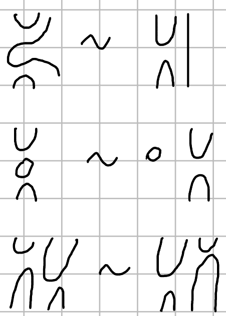
In terms of U_i, we have a free multiplicative algebra on $U_1, \dots, U_{n-1}$ multiplicative generators and relations, called Temperley Lieb algebra A_n.

A_n is a $\mathbb{Z}[A, A^{-1}]$-module with $\delta = -\left(\frac{1}{A^2} + A^2\right) \in \mathbb{Z}[A, A^{-1}]$ loop value.

The relation for A_n are

$$[A] \quad \begin{cases} U_i U_{i\pm 1} U_i = U_i \\ U_i^2 = \delta U_i \\ U_i U_j = U_j U_i & |i-j|>1 \end{cases}$$

Visualisation:



$U_1 = U_1 U_2 U_1$

$U_1^2 = \delta U_1$

$U_1 U_3 = U_3 U_1$

$\Rightarrow$ Remember Jones algebra? Now define $\boxed{e_i := \frac{U_i}{\delta} \ \forall i}$

$$\Rightarrow \begin{cases} e_i^2 = e_i \\ e_i e_{i\pm 1} e_i = \tau e_i & \text{with } \tau = 1/\delta^2 \end{cases}$$

from the bracket for braids we recovered the Jones algebra! ☺

§ 6.3 $TL_n(\delta)$ as diagram monoid on $2n$ points $\mathcal{D}_n$

Def. Let $\mathcal{D}_n$ denote the collection of all topological equivalence classes of diagrams obtained by connecting pairs of points in two parallel rows of n points.

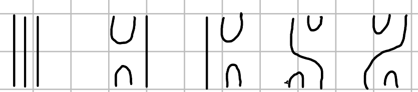
The arc connecting these points must be such that:

(a) All arcs are drawn between the two rows;

(b) No two arcs cross one another;

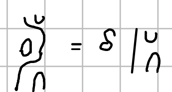
(c) $a, b \in \mathcal{D}_n$ are equivalent if $\exists$ a planar isotopy between them.

$\mathcal{D}_n$ is called diagram monoid on $2n$ points.

Example:  elements of $\mathcal{D}_3$ (6 points)

$1_3 \quad U_1 \quad U_2 \quad r \quad s$

$(\mathcal{D}_n, \cdot)$ comes equipped with a multiplication. If $a, b \in \mathcal{D}_n$ $a \cdot b = \delta^k c$ for some $c \in \mathcal{D}_n$, k # of loops in the $a \cdot b$.

Example: $\xrightarrow{rs}$  $= \delta \, U_2$ ✓

Proposition: The elements $\{1, U_1, \dots, U_{n-1}\}$ generate $\mathcal{D}_n$. If $\mathcal{D}_n \ni x \sim P := \prod U_i$ $\sim Q := \prod U_j$ $\Rightarrow P$ is obtained by Q applying the relations in $[A]$.

§7. The Jones Polynomial revealed

Finally we have all the elements in order to investigate the true nature of $V_K(t)$.

- Define a map $\rho_n : B_n \longrightarrow TL_n(\delta)$ via $\begin{cases} \rho(\sigma_i) = A + A^{-1} U_i \\ \rho(\sigma_i^{-1}) = A^{-1} + A U_i \end{cases}$ (n fixed by choice of B_n)

We know already that for $b \in B_n$, $\langle \bar{b} \rangle = \Sigma_s \langle b|s \rangle \langle \bar{U}_s \rangle = \langle b \rangle$ and

$\rho(b) = \Sigma_s \langle b|s \rangle \langle U_s \rangle$ as $\rho(xy) = \rho(x) \cdot \rho(y)$ $\forall x, y \in B_n$.

$\langle \bar{U}_s \rangle := \#$ loops in $\bar{U}_s - 1$

- Define $tr : TL_n(\delta) \rightarrow \mathbb{Z}[A, A^{-1}]$
$U \rightarrow tr(U) = \langle \bar{U} \rangle$

realisation of Jones trace on the von Neumann algebra A_n (by loop counts)

for $U \in D_n$ and extend it by linearity to $TL_n(\delta)$.

$$\Rightarrow \boxed{\langle b \rangle = tr(\rho(b))}$$

bracket

Jones Polynomial

Conclusion: $V_K(t)$, the Jones polynomial in one variable is a trace on a representation of the braid group B_n to the Temperley Lieb algebra $TL_n(\delta)$.

To prove (exercise): The map $\rho : B_n \rightarrow TL_n(\delta)$ defined above is indeed a representation, i.e. is a morphism:

(i) $\rho(\sigma_i \sigma_i^{-1}) = \rho(\sigma_i)\rho(\sigma_i^{-1}) = \rho(\sigma_i)\rho(\sigma_i)^{-1} = id$ $\forall i$

(ii) $\rho(\sigma_i \sigma_j) = \rho(\sigma_j \sigma_i)$ $\forall i, j : |i - j| > 1$

(iii) $\rho(\sigma_i \sigma_{i+1} \sigma_i) = \rho(\sigma_{i+1} \sigma_i \sigma_{i+1})$ $\forall i$