THE JONES POLYNOMIAL AND QFT

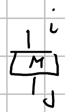
– PART III –

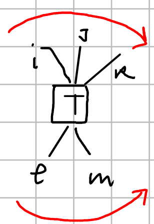
FROM VACUUM-VACUUM EXPECTATIONS TO THE DUAL UNIVERSAL ENVELOPING ALGEBRA OF $\mathfrak{sl}_2(\mathbb{C})$: THE YANG-BAXTER EQUATION

§ 1. Abstract tensors

We want to be able to model a scattering process of the form illustrated in the first talk by Feynman diagrams. For such objects, if we ignore the internal lines, we observe there are input and output particle. It is sensitive therefore to introduce the following notation:

diagrams

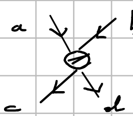

$\leftrightarrow$ $M = [m^i_j]$ — where i, j are labels for the input & output particle respectively.

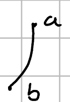

$\leftrightarrow$ $T^{ijk}_{\ell m}$


Orientation convention for indices

← tensors

Looks familiar?

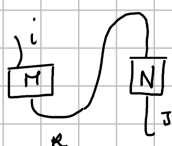
$\Rightarrow$

$\longleftrightarrow$ S^{ab}_{cd}


$\longleftrightarrow$ $\delta^a_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$

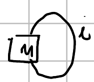
Warning:  $\leftrightarrow$ M_{ab} later

Since objects with this sort of indices are tensors in maths, we call them abstract tensors here. (generalisation of matrices ~ matrix algebra with indices)

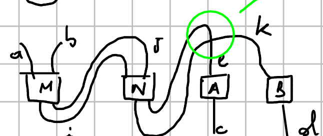
- What about "matrix multiplication"?

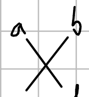
$\Sigma_k M^i_k N^k_j \leftrightarrow M^i_K N^K_j \leftrightarrow$  $=$  chain of boxes

Einstein's summation convention

$tr(M) = \Sigma_i M^i_i \leftrightarrow M^i_i \longleftrightarrow$  loop

general rule:

Example: $M^{ab}_{ij} \cdot N^{ij}_{k\ell} \cdot A^{\ell}_{c} \cdot B^{k}_{d} \longleftrightarrow$ 

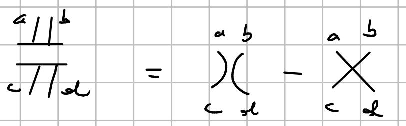

$P^{ab}_{cd} = \delta^a_d \, \delta^b_c$

permutation

Application: S_3 in terms of abstract tensors

$I = \{1, 2\}$ index set $\qquad \varepsilon_{ij} := \begin{cases} 1 & i<j \\ -1 & i>j \\ 0 & i=j \end{cases} \qquad \varepsilon^{ij} = \varepsilon_{ij}$

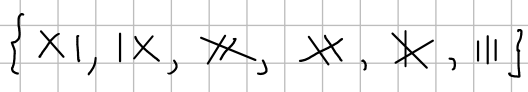
let (diagrams) $\sqcap \leftrightarrow \varepsilon_{ij}$, $\sqcup \leftrightarrow \varepsilon^{ij}$ (tensors) $\Rightarrow$



$\leftrightarrow \quad \varepsilon^{ab}\varepsilon_{cd} = \delta^a_c\delta^b_d - \underbrace{\delta^a_d\delta^b_c}_{P^{ab}_{cd}}$

The matrix P represents a transposition!

Generalizing it for three strands, we have S_3:

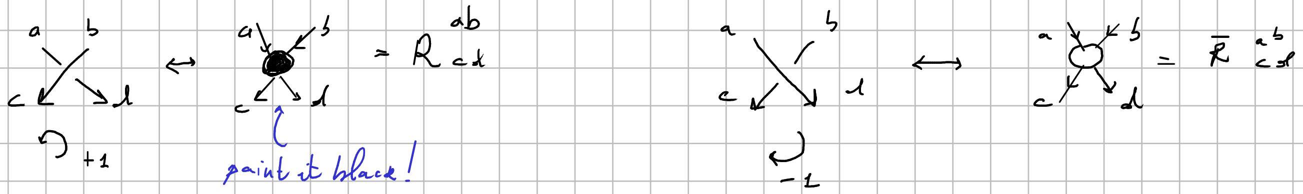


$S_3 = \{ X|,\ |X,\ \ldots,\ |||\}$

(12) (23) (132) (123) (13) id_3

§2. Towards the R-matrix

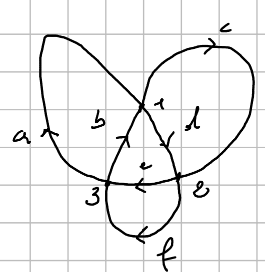
How to associate this structures to knots/links? Consider the crossings



Proceeding in this way $\forall$ crossings, we have established the following correspondence:

$$\boxed{\text{oriented link diagram } K \longleftrightarrow \text{contracted abstract tensor } T(K)}$$

Example:



$(*)\quad T(K) = \sum_{a,\ldots,f} R^{ab}_{cd} R^{dc}_{ef} R^{ef}_{ba} = \sum_{\sigma} \langle K | \sigma \rangle$

Let R be a commutative ring and $I = \{a, b, \ldots, f\}$ the index set $\Rightarrow R^{ab}_{cd} \in R \quad \forall$ 4-tuple of indices

$\Rightarrow$ We have just discovered the following correspondence:

A choice of labels from I for the edges $\longleftrightarrow$ state of K

$\{\sigma : \underset{\text{edges}}{E(K)} \to I\} \xleftrightarrow[\text{colouring}]{\hspace{4em}} \sigma$

- $T(K)$ is identical to the shadow of K with LABELLED NODES (black/white). Such a diagram

(*) is a formal product of R:: and $\bar{R}$:: ✓

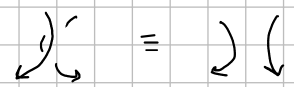
§3. The Yang-Baxter equation for R (and $\bar{R}$).

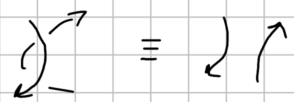
Now, in the same spirit of talk 1 (for Feynman diagrams), we want to impose channel & cross-channel unitarity and triangle invariance ($R\,\mathrm{II}$, $R\,\mathrm{III}$).

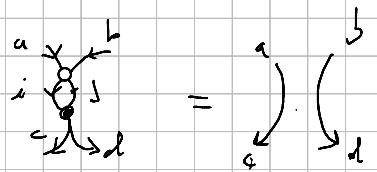
1. Channel (cross-channel) unitarity

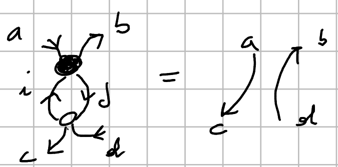
$$\bar{R}^{ab}_{ij} R^{ij}_{cd} = \delta^a_c \delta^b_d \qquad \& \qquad R^{ia}_{jb} \bar{R}^{jd}_{ic} = \delta^a_c \delta^d_b$$

($R\,\mathrm{II}\,A$) ($R\,\mathrm{II}\,B$)



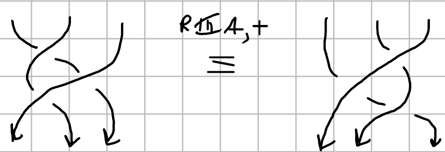


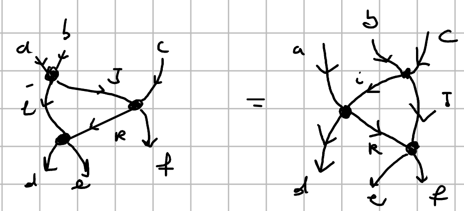




2. Triangle invariance

$$R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef}$$





Imposing this on $T(K)$ we have the Yang-Baxter equations:

$$(\mathrm{III}\,A,+): \quad \sum_{i,j,k \in I} R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = \sum_{i,j,k \in I} R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef} \qquad \text{for } R$$

$$(\mathrm{III}\,A,-): \quad \sum_{i,j,k \in I} \bar{R}^{ab}_{ij} \bar{R}^{jc}_{kf} \bar{R}^{ik}_{de} = \sum_{i,j,k \in I} \bar{R}^{bc}_{ij} \bar{R}^{ai}_{dk} \bar{R}^{kj}_{ef} \qquad \text{for } \bar{R}$$

Theorem: If the matrices R and $\bar{R}$ satisfy i) channel unitarity, ii) cross-channel unitarity and iii) Yang-Baxter equation, then:

$T(K)$ is a regular isotopy invariant for oriented diagrams K.

§ 3.1 Physical interpretation

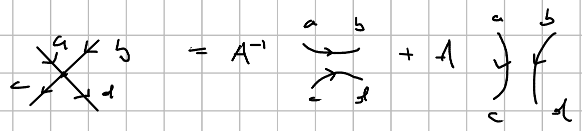
This equation in reality involves extra parameters momenta, spin, charge ... $\Rightarrow R^{ab}_{cd}$ can be seen as the probability amplitude for this particular combination of the parameters in and out $(ab \to cd)$.

For now, consider only spins $\Rightarrow$ the conservation of spin (or charge) reads $a+b = c+d$ for $R^{ab}_{cd} \neq 0$.

This must be taken into account when searching for a solution of YBE.

§ 3.2 A solution for the Yang-Baxter equation: the R-matrix

Example :



$$\Rightarrow \quad R^{ab}_{cd} = A\,\delta^a_c\,\delta^b_d + A^{-1}\,\delta^{ab}\,\delta_{cd}$$

just to visualise they arise from horizontal strands

where $\langle O \rangle = \delta^a_a = \boxed{n = -\left(\frac{1}{A^2} + A^2\right)}$ i.e.

For the index set $I = \{1, 2, 3, \dots, n\}$

A is chosen such that $\boxed{nA^2 + A^4 + 1 = 0}$

This solution arises from the bracket for the Jones Polynomial.

Remember: $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

$\langle O \rangle = \delta = -(A^2 + 1/A^2)$

This suggested the matrix model in the example above, where $\langle O \rangle = \delta^a_a = n$ the loop value is the trace of a Kronecker delta, for $I = \{1, \dots, n\}$.

Remark: Be careful about the time arrow you choose.

- If the index set $I = \{1, \dots, n\}$, then the "tensor contraction" $T(K)$ will satisfy:

  i) $T(\times) = A\,T(\asymp) + B\,T()()$

  ii) $T(OK) = n \cdot T(K)$

  iii) $T(O) = n$

$\Rightarrow T(K) = \langle K \rangle$ generalised bracket with loop value the integer n.

Observe that if $B := A^{-1}$, $n := -\left(A^2 + \frac{1}{A^2}\right)$ will not guarantee that R satisfies the YBE.

§ 3.3 Checking the R-matrix

Let $R^{ab}_{cd} = A \; {}^{a}_{c}\big)\big({}^{b}_{d} + B \; {}^{a \smile b}_{c \frown d} \quad \longleftrightarrow$ (crossing) $= A$)~($+ B$ (pair)

and expand the triangles (for invariance) $\Rightarrow$ we have:

$$[\;] - [\;] = A^2 B \,[\;\cdot\;] + AB^2 \,[\;\cdot\;] + B^3 \,[\;\cdot\;] =$$

$$= (A^2 B + n B^2 A + B^3)\,[\;\cdot\;] \qquad (*)$$

$\Rightarrow$ The YBE is verified $\Leftrightarrow$ $(*) = 0$ $\Leftrightarrow$ $B(A^2 + AB\cdot n + B^2) = 0$ $\Leftrightarrow$

$$\begin{cases} B = 0 \Rightarrow \text{trivial solution } \checkmark \\ A^2 + nAB + B^2 = 0 \Rightarrow (A \neq 0 \neq B) \quad n = -\left(\frac{A}{B} + \frac{B}{A}\right) \end{cases}$$

Looks familiar?

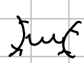
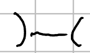
What about channel unitarity?

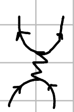
$$\langle \;\rangle = AB \langle)(\rangle + (A^2 + B^2 + ABn) \langle \asymp \rangle = AB \langle)(\rangle$$

$\Rightarrow$ even for $B \neq \frac{1}{A}$, this model is invariant for RII. ✓

Conclusion: We have directly verified that $R^{ab}_{cd} = A\delta^a_c \delta^b_d + A^{-1} \delta^{ab} \delta_{cd}$ with $n = -(A^2 + 1/A^2)$ is a solution for the YBE. ✓

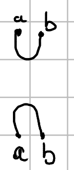
Remark on notation for Feynman diagrams above:

$t \uparrow$  spin-preserving interaction  above

$t \uparrow$  annihilation - creation pair  above

§4. The vacuum - vacuum expectation

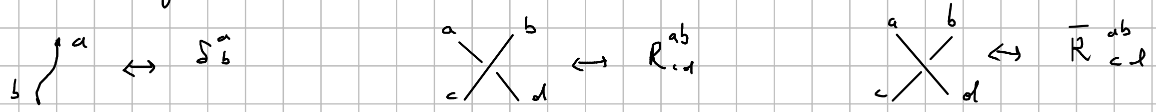
Introduce some new notations for the formal Feynman diagrams:



$t\uparrow$ $\longleftrightarrow$ M^{ab} creation of spins a & b from vacuum (cup)

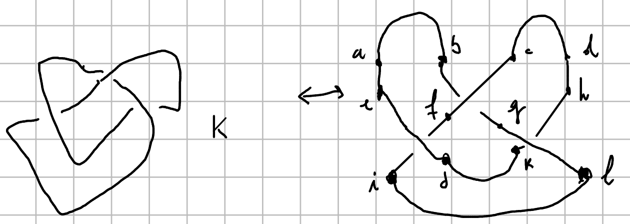
$t\uparrow$ $\longleftrightarrow$ M_{ab} annihilation ~ to vacuum (cap)

The rest stays the same, as in the previous section:



$\longleftrightarrow \delta^a_b$ $\qquad \longleftrightarrow R^{ab}_{cd}$ $\qquad \longleftrightarrow \bar{R}^{ab}_{cd}$

Example of calculations:



$t\uparrow$ $\longleftrightarrow$ $\longleftrightarrow$ $\tau(K) = M_{ab} M_{cd}\, \delta^a_e\, \delta^d_h\, R^{bc}_{fg}\, \bar{R}^{ef}_{ij}\, M^{il} M^{jk}$

If the tensor are valued in a commutative ring $\mathcal{R}$, $\tau(\mathcal{R})$ is the vacuum-vacuum expectation (Kaufmann name for a quasi-physical model - vacuum expectation).

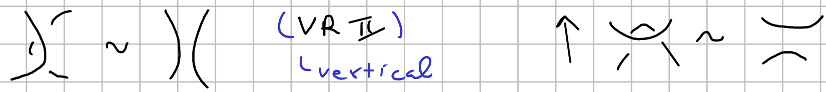
Also here,

$\sum_i P_{ai} Q_{ib} = (PQ)_{ab}$ ✓

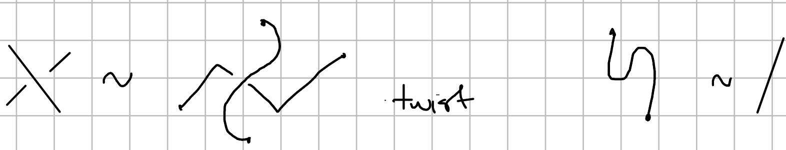
as in QM (Quantum Mechs) the prob. amplitude for the concatenation of processes is obtained by summing the products of the amplitudes of the intermediate configurations in the process over all possible configurations.

> Claim: We would like this to be a TQFT (Top. Quantum Field Theory).
> i.e. $\tau(K)$ should be invariant under regular isotopy on K.

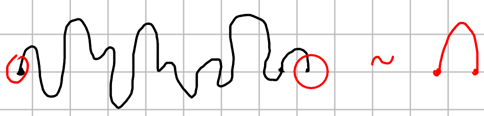
- Reidemeister moves relative to the time's arrow:



$\uparrow$ ~ (VR II) └vertical $\qquad \uparrow$ ~ (HR II) horisontal



~ twist $\qquad$ ~ topological move



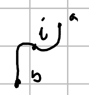
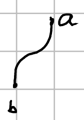
+ Remark: ~ angles rule ✓

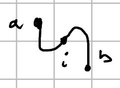
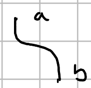
A given level intersects the diagr. in isolated points that are min, max or transv. int

⇒ Two link diagrams arranged transversal to a given time direction (height function) are regularly isotopic ⇔ one can be obtained from the other by a sequence of moves of the types

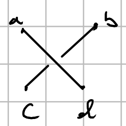
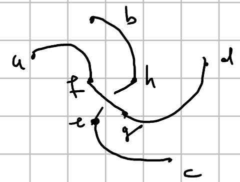
a) top. moves
b) twist
c) VRII
d) VRIII with all crossing of same type relative to the time's arrow.

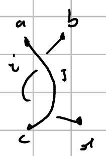
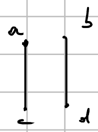
Let's see how this translates in our model:

a) top. move  ↔ $M_{bi}M^{ia}$ ≈  ↔ δ^a_b ⇔ $M_{bi}M^{ia} = \delta^a_b$

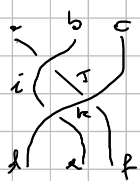
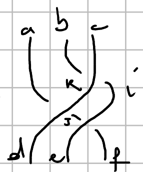
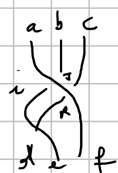
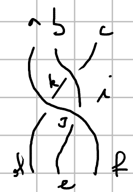
 ↔ $M^{ai}M_{ib}$ ≈  ↔ δ^a_b ⇔ $M^{ai}M_{ib} = \delta^a_b$

⇒ $(M_{ab})^{-1} = M^{ab}$ and viceversa.

b) twist  ≈  $\bar R^{ab}_{cd} = M_{ce} R^{ef}_{gh} M^{bh} \delta^a_f \delta^g_d$

c) (VR II)  ≈  ↔ $\bar R^{ab}_{ij} R^{ij}_{cd} = \delta^a_c \delta^b_d$

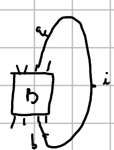
⇒ $\bar R R = I = R \bar R$ ✓

d) (VR III)  ≈  ,  ≈  Yang - Baxter equations for R and $\bar R$

Theorem: If R and $\bar R$ satisfy the YBE plus $\bar R^{ab}_{cd} = M_{ce} R^{ef}_{gh} M^{bh} \delta^a_f \delta^g_d$ and if M_{ab} and M^{ab} are inverse matrices ⇒ $\tau(R)$ is an invariant of regular isotopy. (⇒ we have a TQFT) ✓

§ 4.1 Braids, brackets and R-matrices

Consider $K = \bar B$ for some braid $B \in B_n$ ⇒ each closure strand has one max and one min

 ↔ $M_{ai}M^{bi} =: \eta^b_a$

if $\rho(B)$ is the product of R-matrices, called interaction tensor, then:

$$\boxed{\tau(K) = \mathrm{Tr}\left(\eta^{\otimes n}\rho(B)\right)} \qquad \text{if } B \in B_n \qquad \text{(Markov trace)}$$

If $\tau(K)$ must satisfy the bracket eq. $\tau(\times) = A\,\tau(\asymp) + A^{-1}\,\tau(\,)(\,)$

$\Rightarrow$ ${}^{a}_{c}\!\times^{b}_{d} = \boxed{R^{ab}_{cd}} = A \; {}^{a}\!\cup^{b}_{\;c\,\cap\,d} + A^{-1}\; {}^{a}_{c}\supset\subset^{b}_{d} = \boxed{A\, M_{cd}\, M^{ab} + A^{-1}\,\delta^a_c\,\delta^b_d}$

Suppose that ${}^{a}\bigcirc^{b} = \sum_{a,b} M_{ab} M^{ab} = d = -(A^2 + 1/A^2)$

If $U = M^{ab} M_{cd}$, $\begin{cases} U^2 = d\,U \\ R = AU + A^{-1} I \end{cases}$ and R satisfies the YBE ✓

Conclusion: To have a model for the bracket and a solution to YBE, we simply need a pair of matrices (M_{ab}, M^{ab}) : $d = -(A^2 + 1/A^2)$.

Example: [A simple model] Assume $M_{ab} = M^{ab}$, i.e. $M^2 = I$, i.e.

$\Rightarrow d = \sum_{a,b} (M_{ab})^2$.

Let $M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $\Rightarrow$ $M^2 = 1_2$ & $d = i^2 + (-i)^2 = -1 + (-1) = -2$

$\Rightarrow A = A^{-1} = 1$ ✓

Deform M to $\tilde{M} = \begin{pmatrix} 0 & i\cdot A \\ -iA^{-1} & 0 \end{pmatrix}$ $\Rightarrow$ $M^2 = id_2$ & $d = -\left(A^2 + \frac{1}{A^2}\right)$ as desired.

For the sake of calculations:

$\tilde{M}$ =

	1	2
1	0	iA
2	$-iA^{-1}$	0

$\Rightarrow U = \tilde{M} \otimes \tilde{M}$ =

	11	12	21	22
11	0	0	0	0
12	0	$-A^2$	1	0
21	0	1	$-A^{-2}$	0
22	0	0	0	0

$\Rightarrow R = AM \otimes M + A^{-1} I \otimes I$ =

A^{-1}	0	0	0
0	$-A^3 + A^{-1}$	A	0
0	A	0	0
0	0	0	A^{-1}

§5. Penrose binors, spin-network & SL(2)

We want to study more in detail the case $M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ already mentioned.

Observe the following:

$$M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \cdot \varepsilon \quad \text{where we define}$$

$$[\varepsilon_{ab}] = \begin{cases} 1 & a<b \\ -1 & a>b \\ 0 & \text{otherwise} \end{cases} \quad \text{with } a,b \in \{1,2\} \quad \& \text{ remember } \quad \varepsilon_{ab} = \underset{a\;\;b}{\sqcap}, \quad \varepsilon^{ab} = \overset{a\;\;b}{\sqcup}$$

Lemma: If P is a matrix of commuting associative scalars (e.g. $P \in M_2(\mathbb{C})$)

$$\Rightarrow \quad P\varepsilon P^T = |P| \cdot \varepsilon$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} ab - ab & -bc + ad \\ -da + cb & -dc + cd \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & ad - bc \\ -ad + bc & 0 \end{pmatrix} = (ad - bc) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = |P| \cdot \varepsilon \qquad \square$$

Def. Let $R \in \mathrm{CRing}$ (commutative, associative, unital ring), then

$$SL_2(R) := \{ P \in M_2(R) \mid P\varepsilon P^T = \varepsilon \} =: SL(2)$$

→ The identity $\times = A \asymp + A^{-1})($ for the R-matrix, becomes now an $SL(2)$-invariant tensor identity

$$\varepsilon^{ab}\varepsilon_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c$$

for $A = \pm 1$.

Remark: In Penrose calculus, there is no distinction between over and undercrossing.

Penrose binors arise for $A = -1 \Rightarrow \times + \asymp +)(= 0$.

Here each crossing receives a $-$ sign and $\cup \sim \cap$. ✓

The Penrose binors are a special case of the bracket ⇒ a special case of $V_K(t)$.

The general theory developed by Roger Penrose is called theory of spin-networks.

§5.1 Penrose binors

We investigate a little more the idea behind this form of calculus for $SL(2)$-invariant tensors called Penrose calculus.

Start with spinors ψ^A (2-vectors over $\mathbb{C}$): $A = \{1,2\}$ and impose that $\psi\psi^*$ is $SL_2(\mathbb{C})$-invariant, i.e. this inner product is real. Let $\psi^*_A = \varepsilon_{AB}\psi^B$, so that

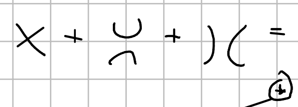
$\psi\psi^* = \psi^A\psi^*_A = \psi^A\varepsilon_{AB}\psi^B$ and take $(U^A_B) \in SL_2\,\mathbb{C} \Rightarrow (U\psi)(U\psi^*) = (U^A_I\psi^I) \cdot$

$\cdot\, \varepsilon_{AB} U^B_J \psi^J = (U^A_I U^B_J \varepsilon_{AB})(\psi^I\psi^J) = \varepsilon_{IJ}\psi^I\psi^J = \psi\psi^*$ ✓

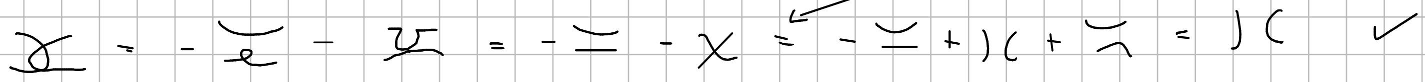
Diagrams :

but this destroys topological invariance.

$\Rightarrow$ Penrose solved the pb associating a minus sign to each crossing AND to each minimum. The loop value is now $d = 2$ and



Example of calculations :



§ 5.2 The quantum group for SL(2)

If we deform ε to $\tilde{\varepsilon} := \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix}$ we get a sensitive model for the bracket & a solution for YBE. Now the question is : For what algebraic structure is $\tilde{\varepsilon}$ a basic invariant ?

Consider $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $[P_{ij}]$ are elem. of some associative BUT NOT COMMUTATIVE algebra and impose $\begin{cases} P^T \tilde{\varepsilon} P = \tilde{\varepsilon} \\ P \tilde{\varepsilon} P^T = \tilde{\varepsilon} \end{cases}$ (*), then :

Proposition (9.5) : (*) $\Leftrightarrow$

$$\boxed{\begin{array}{ll} ba = q\,ab & dc = q\,cd \\ ca = q\,ac & db = q\,bd \\ bc = cb & \\ ad - da = (q^{-1} - q)\,bc & \\ ad - q^{-1}(bc) = 1 & \end{array}} \quad (\ast\ast)$$

with $q := A^2$

Note that $ad - q^{-1}(bc) = 1 = \det(P)$ for $q = 1$ & we restore commutativity !

> However, proposition 9.5 does not give rise to a group $SL(2)_q$ leaving $\tilde{\varepsilon}$ invariant. Indeed $(PQ)^T \neq Q^T P^T$ in the non-commutative case.

The universal associative algebra defined by the relations **, is called $SL(2)_q$, "the quantum group $SL(2)_q$".

it is NOT a group ! ^^

This "group" is quite interesting, as it has some relation with $\mathfrak{sl}_2$ and comes equipped with a nice comultiplication $\Delta : SL(2)_q \to SL(2)_q \otimes SL(2)_q$.

§6. The universal enveloping algebra $U(\mathfrak{sl}_2)$

Given the relations $PEP^T = E$ and $P^TEP = E$, let $\mathcal{A}(E)$ denote the resulting universal algebra.

$\Rightarrow \mathcal{A}(\tilde{\varepsilon}) = SL(2)_q$, where $q^2 = A$. Define $\mathcal{A}(E) \xrightarrow{\Delta} \mathcal{A}(E) \otimes \mathcal{A}(E)$ by the formula:

$$\Delta(P^i_j) = \sum_{k \in I} P^i_k \otimes P^k_j$$

and extend linearly over the ground ring (the ground ring is $\mathbb{C}[A, A^{-1}]$ for $SL(2)_q$).

Proposition: $\Delta : \mathcal{A}(E) \to \mathcal{A}(E) \otimes \mathcal{A}(E)$ is a map of algebras.

- $\Delta(\mathcal{A}(E)) \subset \mathcal{A}(E) \otimes \mathcal{A}(E)$ is an algebra leaving E invariant.

In our case, $\mathcal{A}(\tilde{\varepsilon})$ has generator $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and relations:

$$\boxed{\begin{array}{ll} ba = qab & dc = qcd \\ ca = qac & db = qbd \\ bc = cb & \\ ad - da = (q^{-1} - q)bc & \\ ad - q^{-1}bc = 1 & \end{array}}$$

with:
$$\Delta(a) = a \otimes a + b \otimes c$$
$$\Delta(b) = a \otimes b + b \otimes d$$
$$\Delta(c) = c \otimes a + d \otimes c$$
$$\Delta(d) = c \otimes b + d \otimes d$$

$\Delta : \mathcal{A}(\tilde{\varepsilon}) \to \mathcal{A}(\tilde{\varepsilon})^{\otimes 2}$ comultiplication

- This algebra is an example of Hopf algebra (see next talk).

- The antipode is given by the map $\gamma : \mathcal{A}(\tilde{\varepsilon}) \circlearrowleft$ such that $\gamma(a) = d$, $\gamma(d) = a$, $\gamma(b) = -qb$, $\gamma(c) = -\frac{1}{q}c$. Extend linearly on the sums and define it on products so that $\gamma(xy) = \gamma(y)\gamma(x)$ (anti-homomorphism).

§6.1 Deforming the algebra of functions.

The algebra $\mathcal{A}(\tilde{\varepsilon})$ is a deformation of the algebra of functions on the group $SL_2(\mathbb{C})$. Let's explain this construction for G arbitrary group.

Let $F(G) := \{ f : G \to \mathbb{C} \text{ functions} \}$, define $m : F(G) \otimes F(G) \to F(G)$ by $m(f \otimes g)(x) = f(x)g(x)$ and $\Delta : F(G) \to F(G) \otimes F(G)$ by $\Delta(f)(x \otimes y) = f(xy)$.

- Let $\gamma : F(G) \to F(G)$ be defined by $\gamma(f)(x) = f(x^{-1})$ the antipode

and $\varepsilon : F(G) \to \mathbb{C}$, $\varepsilon(f) = f(1_G)$ & $\eta : \mathbb{C} \to F(G)$, $\eta(z)(g) = z \cdot f(g)$ $\forall z \in G$ unit & counit.

- This gives $F(G)$ the structure of an Hopf algebra.

$\to$ In our case, $F(G) = F(SL_2(\mathbb{C}))$. We expect that $F_q(SL(2)) := SL(2)_q$ is related to functions on a deformation of the Lie algebra of $SL(2)$.

Def. A Lie algebra $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$, equipped with a bracket operation $[\,,\,] : \mathfrak{g}^2 \to \mathfrak{g}$ that is bilinear, alternative $[x, x] = 0\ \forall x \in \mathfrak{g}$ & satisfies the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ $\forall x, y, z \in \mathfrak{g}$.

§ 6.2 Deforming the Lie algebra of $SL_2(\mathbb{C})$

The Lie algebra of $SL_2(\mathbb{C})$, denoted by $\mathfrak{sl}_2(\mathbb{C})$, is defined as follows

$$\mathfrak{sl}_2(\mathbb{C}) = \{ M \in \mathfrak{gl}_2(\mathbb{C}) \mid Tr(M) = 0 \}$$ as it is apparent that

$$\det(e^M) = e^{tr(M)} = 1 \iff tr(M) = 0 \quad \checkmark$$

Remember that the exponential matrix gives the way to go from the Lie group G to the corresponding Lie algebra of G.

It is generated by the three matrices $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

i.e. $\quad \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} r & s \\ t & -r \end{pmatrix} = rH + sX^+ + tX^- \right\}$

with bracket given by:

$$[H, X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [X^+, X^-] = H. \quad \checkmark$$

- Define a deformation of this Lie algebra, called <u>the quantum universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, denoted $U_q(\mathfrak{sl}_2)$</u>, as follows:

$$[H, X^+] = 2X^+, \; [H, X^-] = -2X^- \text{ and } [X^+, X^-] = \sinh\left(\left(\tfrac{\hbar}{2}\right)H\right) / \sinh\left(\tfrac{\hbar}{2}\right)$$

where $H, X^\pm$ are the generators and $q := e^{\hbar}$. Planck constant

Observe that for $\hbar \to 0$ (classic limit), we obtain the brackets above. ✓

Denote $\mathcal{U}(\mathfrak{sl}(2))_q := U_q$, the comultiplication is given by the formulas

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X^{\pm 1}) = X^\pm \otimes e^{(\hbar/4)H} + e^{(-\hbar/4)H} \otimes X^\pm$$

→ this is an Hopf algebra (one has to define γ, ϵ, η as well).

§ 6.3 The dual universal enveloping algebra

In order to clearly see the link, we need to <u>dualise</u>: U_q^* inherits a multiplication from the co-multiplication Δ on U_q and a co-multiplication from the multiplication. Let $\rho, \rho' : \mathcal{U} \to \mathbb{C}[\hbar, \hbar^{-1}]$, define $\rho\rho'(\alpha) = \rho \otimes \rho'(\Delta(\alpha))$.

Define $\Delta(\rho) : \mathcal{U} \otimes \mathcal{U} \to \mathbb{C}[\hbar, \hbar^{-1}]$ via $\Delta(\rho)(\alpha \otimes \beta) = \rho(\alpha \cdot \beta)$ where $\alpha\beta$ product in $\mathcal{U}$.

Consider now the representation $\tilde{\rho} : U_q \to \mathfrak{sl}_2$ of U_q given by:

$$\tilde{\rho}(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \tilde{\rho}(X^+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \rho(X^-) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{(trivial)}$$

and note that $\sinh\left(\tfrac{\hbar}{2}\tilde{\rho}(H)\right) = \sinh\left(\tfrac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} \sinh(\hbar/2) & 0 \\ 0 & -\sinh(\hbar/2) \end{pmatrix}$.

Hence, $\dfrac{\sinh((\hbar/2)\,\tilde{\rho}(H))}{\sinh(\hbar/2)} = \tilde{\rho}(H).$

→ It is easy to prove $\tilde{\rho}$ gives indeed a representation for $\mathcal{U}_q$.

⇒ we can write $\tilde{\rho} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a matrix of functions on $\mathcal{U}_q$, with

$$\begin{array}{lll|}
a(H) = 1 & a(X^+) = 0 = a(X^-) & \\
b(H) = 0 & b(X^+) = 1 & b(X^-) = 0 \\
c(H) = 0 & c(X^+) = 0 & c(X^-) = 1 \\
d(H) = -1 & d(X^+) = 0 = d(X^-) &
\end{array}$$

Lemma: The Hopf algebra generated by a, b, c, d as above is isomorphic to $\mathcal{A}(\check{\epsilon}) = F_q(SL(2))$. ⇒ $F_q(SL(2)) \cong \mathcal{U}_q^*$ with $q = e^{\hbar/2}$.

§ 6.4 Final remarks

Looking at $SL(2)_q^*$ as $\mathcal{U}_q(\mathfrak{sl}_2)$: $\begin{cases} [H, X^+] = 2X^+ \\ [H, X^-] = -2X^- \\ [X^+, X^-] = \sinh(\hbar/2\, H)/\sinh(\hbar/2) \\ \text{and } \Delta \text{ as above.} \end{cases}$

allows us to establish this algebra as deformation of the Lie algebra of $SL(2)$. One can study the representation of these quantum groups. The universal R-matrix emerges from the Lie algebra formalism. Specific solutions are defined changing the representation $\tilde{\rho}$. In the fundamental representation $\tilde{\rho}$ above, we recover the R-matrix for the bracket.

Conclusion: We moved from the bracket as vacuum-vacuum expectation to the dual universal enveloping algebra $\mathcal{U}_q^*(\mathfrak{sl}_2)$. This algebraic structure arises therefore from $\langle \asymp \rangle = A \langle \frown\smile \rangle + B \langle)(\rangle$, topological invariance and QM ideas. (+ DTQFT & generalisation of classical symmetry).
└ discrete top. quan. field theory

THE JONES POLYNOMIAL AND QFT

– PART IV –

HOPF ALGEBRAS: QUANTUM GROUPS FROM A CATEGORICAL VIEWPOINT

§1. Introduction

The theory of quantum groups (algebraists) or quantum topology (topologists) developed from the works of Jones (1984) and Drinfel'd/Jimbo (1985).

Quantum groups are 1-par. deformations of semi-simple complex Lie algebras, arising therefore from the same procedure we did in the last talk.

Thanks to new ideas coming from conformal field theory by E. Witten (1988), the Jones polynomial, seen as path integral, was related to the construction of a 2-dim. modular functor. From then on TQFT is one of the most active field of research (Atiyah, Segal, Kauffmann, Turaev, Seiberg, ...).

- We search for invariants of knots and 3-mfd from algebraic objects that formalise the properties of modules over quantum groups at a root of unit.

§ 1.1 Modular categories & modular functors

we already discussed these arise from the study of commutativity of $\otimes$

Def. A modular category is a tensor category* with some braiding/twisting that is semisimple: its objects can be decomposed into simple objects (like irreducible modules in representation theory) and has the property of finiteness: such a decomposition is finite.

Def. A modular functor (n-dim.) is a functor $J : \Sigma \in \mathrm{mfd}^n_{cl} \to J(\Sigma) \in \mathrm{Proj}(K\text{-mod})$ from a closed n manifold(1) (with some additional structure) to a projective K-module(2) called module of states of Σ. (sounds familiar?

For arrows, it takes homeomorphisms of manifolds to isomorphisms of modules. Moreover, $J(\Sigma \sqcup \Sigma') = J(\Sigma) \otimes_K J(\Sigma')$ & $J(\emptyset) = K \in \mathrm{CRing}$.

Def.(1) A closed manifold is a manifold without boundary and compact.

Def.(2) A module P is projective iff $\exists h'$: the following commute

$$\begin{tikzcd} & N \arrow[d, two heads, "h"] \\ P \arrow[ur, dashed, "\exists h'"] \arrow[r, "g"'] & M \end{tikzcd}$$

in K-mod

i.e. every arrow $g: P \to M$ factors through every epic $h: N \to M$. (see p. 118 ML)

* $\langle \mathrm{Ab}, \otimes, \mathbb{Z} \rangle \leftrightarrow$ Rings

$\langle K\text{-Mod}, \otimes_K, K \rangle \leftrightarrow$ K-algebras

$\langle K\text{-Mod}^{op}, \otimes^{op}_K, K \rangle \leftrightarrow$ K-coalg.

monoidal cat. monoids

Now extend the modular functor to a TQFT, that associates

$\{$homs of modules of states$\}$ $\longrightarrow$ $\{$cobordisms "spacetimes"$\}$

Def. An $(n+1)$-dim. cobordism is a compact manifold M^n whose boundary can be decomposed as $\partial M = \partial_- M \mathbin{\dot\cup} \partial_+ M$, disjoint union of two closed manifolds called top and bottom base.

Def. An $(n+1)$-dim TQFT is formed by an n-dim. modular functor J and an operator invariant of $(n+1)$-cobordisms τ.

This operator invariant τ assigns to a cobordism M the homomorphism $\tau(M): J(\partial_- M) \to J(\partial_+ M)$. It should be invariant under homs of cobordisms and multiplicative wrt the disjoint union of cobordisms. Moreover, it should be compatible with gluings of the cobordisms along their common base $\partial_+(M_1) = \partial_-(M_2) \Rightarrow \tau(M) = k\,\tau(M_2)\circ\tau(M_1): J(\partial_- M_1) \to J(\partial_+ M_2)$. This k is called anomaly of the gluing. The most interesting TQFTs are anomaly free, i.e. $k=1$.

Remark: To speak of a TQFT, we need to specify the class of spaces and cobordisms subject to the application of J and τ.

Here is the link $\quad \{\text{modular category}\} \xrightarrow{1-1} \left\{3\text{-dim. TQFT}\right\}$ (anomaly-free)

§1.2. How to create these modular categories (sensitive for physics)?

$$\boxed{\text{Quantum groups at a root of unity} \longrightarrow \text{modular categories}}$$

The modular categories leading to meaningful TQFTs arise from the theory of representation of quantum groups at a root of unity.

- $U_q(\mathfrak{g})$ is an Hopf algebra over $\mathbb{C}$ obtained by a 1-parameter deformation of the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

The Jones polynomial (we know already) is obtained for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

- To achieve finitness we take deformation parameter q to be a complex root of unity. However, we loose semisimplicity $\Rightarrow$ we go quotient to regain it.

GOAL ✓ ACHIEVED

$\rightarrow$ The fin. dim. modules over $U_q(\mathfrak{g})$ form a semisimple tensor category with braiding & twisting!

§2. A primer in Lie algebras

§2.1 Basic definitions

Def. A <u>Lie algebra</u> $\mathfrak{g}$ over $\mathbb{K}$ is a (finite or infinite dimensional) $\mathbb{K}$-vector space with a $\mathbb{K}$-bilinear, antisymmetric operation $[\,,]$ (called Lie bracket): $\mathfrak{g}^2 \to \mathfrak{g}$ satisfying Jacobi identity. (see previous talk; bilinearity + $[x,x] \overset{\forall x \in \mathfrak{g}}{=} 0 \Rightarrow$ anti-commutativity)

Example: $(\mathbb{R}^3, \times)$ real 3-d. Lie algebra vector product as operation.

Example: $E \in \mathrm{Vect}_{\mathbb{K}}$ $u, v \in \mathcal{L}(E)$ $[u,v] = u \circ v - v \circ u$ commutator

if $E = Gl_n(\mathbb{K})$: $[X,Y] = XY - YX$ $\forall X, Y \in Gl_n(\mathbb{K})$, this defines a n^2 dimensional Lie algebra over $\mathbb{K}$, called $\mathfrak{gl}_n(\mathbb{K})$.

Def. A <u>Lie subalgebra</u> $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is a vector subspace over $\mathbb{K}$ closed under the bracket defined on $\mathfrak{g}$.

→ Each vector subspace of $\mathfrak{gl}_n(\mathbb{K})$ closed under the commutator gives rise to a Lie algebra over $\mathbb{K}$.

Def. A Lie algebra is called <u>abelian</u> iff $[\,,] = 0$.

Def. An <u>ideal</u> of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$: $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. $\{0\}$ and $\mathfrak{g}$ are the trivial ideals.

Def. The <u>center of $\mathfrak{g}$</u> is the abelian ideal of $\mathfrak{g}$

$$Z(\mathfrak{g}) := \{ X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g},\ [X,Y] = 0 \}$$

Def. A Lie algebra $\mathfrak{g}$ is called <u>simple</u> if $\mathfrak{g}$ has no nontrivial ideals and is not of dimension 0 or 1.

Def. A Lie algebra $\mathfrak{g}$ is called <u>semi-simple</u> if $\mathfrak{g}$ has no nontrivial abelian ideals.

Def. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, a <u>representation of $\mathfrak{g}$</u> is a $\mathbb{K}$-linear mapping $R: \mathfrak{g} \to \mathfrak{gl}(E)$, where E is a finite-dim. $\mathbb{C}$-vector space & $\forall X, Y \in \mathfrak{g}$ $[R(X), R(Y)]_{\mathfrak{gl}(E)} = R[X,Y]_{\mathfrak{g}}$. It is denoted by (E, R).

Def. (E,R) is called <u>irreducible representation</u> if the only R-invariant subspaces are E and $\{0\}$.

§ 2.2 Root systems & Cartan matrices

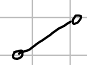
This section refers to: J. Bosshardt, The classification of simple complex Lie Algebras. (for more also Fulton, Harris, Representation Theory, 1991)

Def. A root system R is a collection of roots in E^n:

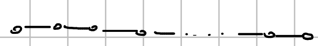
1. R finite and spans E^n;
2. If α is a root $\Rightarrow$ its only multiple roots are $\pm\alpha$.
3. If α, β are roots $\Rightarrow W_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is a root (W-invariance).
4. If α, β roots $\Rightarrow n_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is a root.

Def. A root system R is reducible if it can be partitioned into two subsets S and S' of R: each root in S is orthogonal to every root in S'. Otherwise, R is irreducible.

Proposition: The root system of a simple Lie algebra $\mathfrak{g}$ is irreducible.

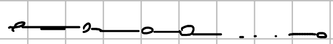
Def. The Dynkin diagram of R is obtained by drawing a $\circ$ for each simple root and connecting them via an edge  at an angle $\hat{\alpha}$ representing the actual angle between them.
The number of lines joining the nodes α and β is $-n_{\beta\alpha}$.

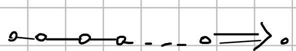
Example:

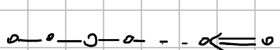
$\mathfrak{sl}_{n+1}(\mathbb{C})$ 

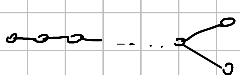
Fact: If the system is irreducible, the D. diagram is connected.

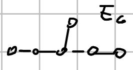
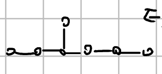
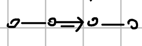
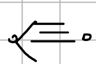
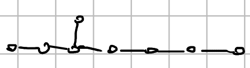
Theorem: The connected diagrams are:

 A_n $n \geq 1$ $\mathfrak{sl}_{n+1}(\mathbb{C})$

 B_n $n \geq 2$ $\mathfrak{so}_{2n+1}(\mathbb{C})$

 C_n $n \geq 3$ $\mathfrak{sp}_{2n}(\mathbb{C})$

 D_n $n \geq 4$ $\mathfrak{so}_{2n}(\mathbb{C})$

 E_6  E_7 F_4   G_2  E_8

For $\mathfrak{sl}_2(\mathbb{C})$ $\longleftrightarrow$ $L_2 - L_1 = \alpha$ A_1 $\{\pm\alpha\}$

Def. The <u>Cartan matrix</u> of a simple Lie algebra $\mathfrak{g}$ is the matrix $A_{\mathfrak{g}} = [a_{ij}]$ with $a_{ij} = \dfrac{2(r_i, r_j)}{(r_j, r_j)}$ where r_i are the simple* roots of the Lie algebra $\mathfrak{g}$

Def.* A root r in R root system of $\mathfrak{g}$ is called <u>simple</u> if it cannot be decomposed as sum of roots in R.

<u>Properties of Cartan matrices</u> :

1) $a_{ii} = 2 \quad \forall i$
2) $a_{ij} \leq 0$
3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
4) $A = \underbrace{D}_{\text{diagonal}} \cdot \underbrace{S}_{\text{symmetric}}$

§3. Hopf algebras

K-Algebras are monoids of $\langle K\text{-Mod}, \otimes_K, K \rangle$

Def. Let A be an algebra over $K \in CRing$ equipped with <u>a co-multiplication</u> $\Delta: A \to A^{\otimes 2}$: $\Delta(1_A) = 1_A \otimes 1_A$, a <u>co-unit</u> $\varepsilon: A \to K : \varepsilon(1_A) = 1_K = 1$ and a K-linear antihomomorphism $s: A \circlearrowleft$ called <u>antipode</u>, $(A, \Delta, \varepsilon, s)$ defines an <u>Hopf algebra</u> if these homomorphisms together with the multiplication $m: A \otimes A \to A$, satisfy the following identities:

$$\begin{cases} (id_A \otimes \Delta)\Delta = (\Delta \otimes id_A)\Delta & 1.1a \\ m(s \otimes id_A)\Delta = m(id_A \otimes s)\Delta = \varepsilon \cdot 1_A & 1.1b \\ (\varepsilon \otimes id_A)\Delta = (id_A \otimes \varepsilon)\Delta = id_A & 1.1c \end{cases}$$

(of course $(A \otimes A) \otimes A = A \otimes (A \otimes A)$ & $K \otimes A = A \otimes K = A$ are also true)

$\to$ s is antihomomorphism for both <u>algebra</u> and <u>co-algebra</u>, indeed:

1) $m(s \otimes s) = s \circ m \circ P_A : A^{\otimes 2} \to A$ where P_A is responsible for the flip $a \otimes b \to b \otimes a$

2) $P_A(s \otimes s)\Delta = \Delta \circ s : A \to A^{\otimes 2}$ and $s(1_A) = 1_A$, $\varepsilon \circ s = \varepsilon : A \to K$.

§ 3.1 Dual Hopf algebra

- If an Hopf algebra A, viewed as K-module, is projective of finite type, we can define an Hopf algebra structure on $A^* = \mathrm{Hom}(A, K)$. The multiplication, co-multiplication and antipode on A^* are dual to the respective ones in A.

  unit / counit in A^* $\longleftrightarrow$ counit / unit in A

  y homomorphism $\longleftrightarrow$ $y(1_A) : A^* \longrightarrow K$

§ 3.3 Examples

(1) $K[G]$ group ring of G; Δ, ε, S are defined on the additive generators of the group G by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$ $\forall g \in G$.

(2) Let G be finite, as K-module this algebra A is freely generated by $\{\delta_g\}_{g\in G}$ where $\delta_g \cdot \delta_h = \begin{cases} 0 & g \neq h \\ \delta_g & g = h \end{cases}$

$$\Delta(\delta_g) = \sum_{h\in G} \delta_h \otimes \delta_{h^{-1}g}, \quad S(\delta_g) := \delta_{g^{-1}} \quad \& \text{ the counit } \varepsilon : A \to K$$

is defined as $\varepsilon(\delta_g) = \begin{cases} 1 & g = 1_G \\ 0 & \text{otherwise} \end{cases}$

$\sum_{g\in G} \delta_g$ is the unit of A (prove 1.1a → 1.1.c)

<u>Fact</u>: A is dual to $K[G]$.

(3) $U(\mathfrak{g})$ for $\mathfrak{g}$ a Lie algebra, is called <u>universal enveloping algebra of $\mathfrak{g}$</u> and it is an Hopf algebra equipped with the following morphisms:

for $g \in \mathfrak{g}$ $\quad \Delta(g) := g \otimes 1 + 1 \otimes g \quad \varepsilon(g) := 0 \quad S(g) := -g$

§ 4. The category of representations and R-matrices (Hints)

We already discussed that a particular form of the R-matrix is achieved by choosing a representation in talk 3. We want to motivate it a little more here (please refer to Turaev / Mac Lane as this section may be not self-contained).

Def We call M an <u>A-module of finite rank</u> if it is a left A-module whose underlying K-module is projective of finite type.

Def. The <u>category of representations</u> of an algebra A over K is denoted by $\mathrm{Rep}(A)$ and consists of the following data:

$$\mathrm{Rep}(A)^0 = \{A\text{-modules of finite rank}\}$$

$$\mathrm{Rep}(A)^1 = \{A\text{-linear homomorphisms between } A\text{-mod. of finite rank}\}$$

$$\boxed{\mathrm{Rep}(A) \text{ is a monoidal } \underline{\text{Ab-category}} \text{ with duality}}$$

We look for <u>ribbon categories</u> in the class $\mathrm{Rep}(A)$, where A is an Hopf algebra, as:

$\exists$ a braiding in $\mathrm{Rep}(A)$ $\longrightarrow$ A is <u>quasi-triangular</u> for some universal R-matrix $R \in A^{\otimes 2}$

- A quasi-triangular Hopf algebra is the dual concept to a braided category. Here the role of braiding is played by $R \in A^{\otimes 2}$ and this invertible element is called a universal R-matrix of A.
- There is a general method to produce QT Hopf algebras called Drinfel'd double construction: If A is an Hopf algebra over a field $\Rightarrow$ $\exists$ a quasi-triangular structure on $A \otimes A^*$.

Def. A <u>ribbon</u> Hopf algebra is a triple (A, R, v) consisting of a QTHA A with R universal R-matrix of A and an element $v \in Z(A)$ invertible: $\Delta(v) = P_A(R)R(v \otimes v)$ & $s(v) = v$.

Such a v is called <u>universal twist of A</u>.

<u>Lemma</u>: Take $V \in \mathrm{Rep}(A)$, $\vartheta_V : V \to V$ denotes the multiplication by $v \in A$.

Let (A, R, v) be a ribbon HA $\Rightarrow$ $\{\vartheta_V\}_V$ is a twist in the braided monoidal category $\mathrm{Rep}(A)$, compatible with duality.

<u>Theorem</u>: If A ribbon HA $\Rightarrow$ $\mathrm{Rep}(A)$ ribbon Ab-category.

§5. Quantum groups at a root of unity

- Quantum groups are Hopf algebras obtained by a 1-par. deformation of the $\mathcal{U}(\mathfrak{g})$ for $\mathfrak{g}$ simple complex Lie algebra.
- We restrict ourselves to quantum groups corresponding to simple Lie algebras $\mathfrak{g}$ of types A, D, E with Cartan matrix $(a_{ij})_{1\le i,j\le m}$ $m \geq 1$.
- Fix $q \in \mathbb{C} \setminus \{0, \pm 1\}$, $U_q(\mathfrak{g})$ is the algebra over $\mathbb{C}$ defined by $4m$ generators E_i, F_i, K_i, K_i^{-1} for $i = 1, \dots, m$ and relations:

a) $\begin{cases} K_i K_j = K_j K_i \\ K_i K_i^{-1} = K_i^{-1} K_i = 1 \end{cases}$ b) $\begin{cases} K_i E_j = q^{a_{ij}} E_j K_i \\ K_i F_j = q^{-a_{ij}} F_j K_i \end{cases}$

c) $E_i F_j - F_j E_i = \delta_{ij} \left(\dfrac{K_i - K_i^{-1}}{q - q^{-1}} \right)$

d) $E_i E_j = E_j E_i$ & $F_i F_j = F_j F_i$ for $a_{ij} = 0$

e) $\begin{cases} E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 & \text{for } a_{ij} = -1 \\ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 & \text{for } a_{ij} = 1 \end{cases}$

$i, j = 1, \dots, m$

Remember this is called quantum group (not a group) !!

- We want to equip $U_q(\mathfrak{g})$ with the structure of an HA:

→ co-multiplication:
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \quad \forall i$$
$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \quad \forall i$$
$$\Delta(K_i) = K_i \otimes K_i \quad \forall i$$

→ antipode:
$$S(K_i) = K_i^{-1} \quad \forall i$$
$$S(E_i) = -E_i / K_i \quad \forall i$$
$$S(F_i) = -F_i \cdot K_i \quad \forall i$$

→ counit:
$$\varepsilon(K_i) = 1 \quad \forall i$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0 \quad \forall i$$

and then extend by linearity. ✓

Little remark if you know about it otherwise skip

- For $K_i := \exp\left(-\hbar H_i /_2\right)$ $q := e^{-\hbar/2}$ and $\hbar \to 0$ we recover the standard relations among the Chevalley generators of $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

§ 5.1 Conclusion

$\mathcal{U}_q(\mathfrak{g})$

$\rightarrow$ $\boxed{\text{If } q \text{ is a root of unity of even order, } \Rightarrow \mathcal{U}_q(\mathfrak{g}) \text{ is a modular category.}}$

$\rightarrow$ $\boxed{\text{If } q \text{ generic, } \mathcal{U}_q(\mathfrak{g}) \text{ is a semisimple category with an } \infty \text{ number of simple objects.}}$

! THANK YOU FOR THE ATTENTION !