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A CO-HOMOLOGICAL APPROACH TO MAXWELL'S ELECTROMAGNETIC FIELD THEORY

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To My Grandfather

Niemand kann dir die Brücke bauen,

auf der gerade du über den Fluß des Lebens schreiten mußt,

niemand außer dir allein.

- F. W. Nietzsche

[No one can construct for you the bridge
upon which precisely you must cross the stream of life,
no one but you yourself alone.]

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Preface

Co-homology is a very rich theory developed by mathematicians, in particular by Georges de Rham, during the XIX century but, as the greater part of the mathematical objects, it seems to be at a first glance a very elegant and murky self-referential construction, of an extremal beauty, but without practical utility. Few people know its strength, especially in applied mathematics and theoretical physics.

In this thesis we will focus on Maxwell's equations and how the co-homology theory plays a key role to give a fully topological proof of the existence of magnetic monopoles, singularities of the electromagnetic field. Moreover, using the general form of the Gauss-Bonnet theorem, we will obtain also a quantization condition, called Schwinger condition, again only by a geometric-topological argument.
We will start with the classical form of these equations, as we already know from basic physics, we will introduce then tensors, using the Special Relativity Theory and its metric, and, finally, in chapter 5, we will come to the topological foundation of the existence of magnetic monopoles, in particular of the Dirac ones, recurring only to the Hopf fibration and the second (co)-homology group of the 3-sphere.

I want express my deep gratitude to Prof. Anna Tozzi, for having initiated me into the fascinating world of algebraic topology and trusted me in this ”experimental” project, and to Prof. Luigi Pilo for having led me into the world of higher physics and its sophisticated theories with enormous patience. Thank you so much.
Lastly, I am really grateful to Prof. Antonio Mecozzi and Prof. Lucio Bedulli for having put at my complete disposal all their knowledge helping me building this work.
Thank you all.

Federica Pasqualone

Chapter 1

The Importance of Geometry in Physics

In what follows we investigate some easy applications of differential geometry on well known topics in physics showing how differential forms are so important to gain a better understanding of their theoretical basis.

1.1 Thermodynamics

Consider a one-component fluid, for which we can write the equation of state as

$$\delta Q = PdV + dU$$

where U is the internal energy and PdV is the work done.
This simplest expression for δQ contains a linear combination of one-forms of a two-dimensional space in the coordinates (V,U) on which the function P(V,U) is defined, so δQ is itself a one-form.
We observe that δQ is not exact, because, otherwise, $\partial P/\partial U$ would vanish everywhere and this would be, clearly, an absurd.

The form $\tilde{\delta} Q$ is, however, a one-form in a two-space and its ideal is therefore closed so, by Frobenius' theorem, there must exist two functions T(V,U) and S(V,U), temperature and entropy of the one-component gas respectively, in terms of which the relation above becomes

$$T\tilde{d}S = P\tilde{d}V + \tilde{d}U$$

Remark 1.1.0.1 Notice that this is a purely mathematical expression for T and S, i.e. it has no relation with the second law of thermodynamics at this step. No mathematical identity of this sort would hold for a multi-component fluid.

1.2 Hamiltonian mechanics

In a dynamical system of equations characterized by the Lagrangian $\mathcal{L}(q, \dot{q})$, we introduce the Hamiltonian

$$H(q,p) = pq - \mathcal{L}(q, \dot{q})$$

where p is the momentum defined by $\dfrac{\partial \mathcal{L}}{\partial q}$.
The Hamiltonian dynamics is then given by

$$\frac{\partial H}{\partial q} = -\frac{dp}{dt} \qquad \frac{\partial H}{\partial p} = \frac{dq}{dt}$$

Consider now the non-degenerate differential two-form

$$\omega := dq \wedge dp$$

defined on the phase space (p,q). It plays a similar role to that of a metric on a Riemannian manifold, namely it provides a bijection between vectors and one-forms: if V is a vector field on M, $\omega(V) := \tilde{V}$ is a one-form field with components $(\tilde{V})_i = \omega_{ij} V^j$; vice-versa, given a one-form field $\tilde{\alpha}$ we define a vector field α as the unique vector such that $\tilde{\alpha} = \omega(\alpha)$.

Definition 1.2.0.1 We say that X is an Hamiltonian vector field if and only if $\tilde{X}$ is an exact one form.

Suppose to have a system with n degrees of freedom and let's extend the theory just developed to its 2n-dimensional phase space.
Define the form ω as follows

$$\omega := \sum_{i=1}^{n} (dq^i \wedge dp_i)$$

Observe that this is actually a symplectic form, thus the phase space becomes a symplectic manifold.

Remark 1.2.0.1 Observe that:

- Thank's to the previous definition, the phase space now it's no more a simple manifold: it has got a natural structure of vector field, so it's fibre bundle is trivial. M is then said to be parallelizable;

- It's possible to show that the momentum p_i for every i is a one-form field on the configuration space M, thus it is a cross-section of the cotangent bundle T*M. It follows that the phase space, whose coordinates are q^i, p_i is T*M and the Hamiltonian is a function on this bundle.
  Moreover, the symplectic form is independent of the coordinates in M and T*M is always orientable, because the volume form $\sigma := \underbrace{\omega \wedge ... \wedge \omega}_{n\ times}$ is always non-zero.

1.3 Cosmology

Einstein's theory of general relativity has given us a complete description of the universe but, at the same time, it has left open the question of its models. We have, at the simplest level, essentially three, evenly possible, sceneries: the "closed", "flat" and "open" universes. This problem arises not at all from Einstein's equations, but from considering the universe approximately homogeneous and isotropic at large scales ($\sim$ 100 Mpc). Being a dynamical theory, it predicts the future and the past of the world, given some initial conditions. Uniformity of the universe is the mean one and it's only using tools of differential geometry, we conclude that only three metric tensor fields are possible.

We give now some definition to treat mathematically concepts as isotropy and homogeneity. The basic idea will be to split the space-time up into a foliation, namely a family of 3-dimensional sub-manifolds filling it up, called hyper-surfaces of constant time (roughly speaking, it's performed just choosing time coordinates at every level) and to restrict the metric tensor $g|$ on each of them. The hyper-surfaces, then, will be space-like if $g|$ is positive definite on the fibre bundle. Uniformity of the cosmology depends on the Killing isometries of these hyper-surfaces.

Definition 1.3.0.1 Let G be the Lie group of isometries of some m-manifold S with metric tensor $g|$, S is said to be isotropic about P if it's isotropy group

$$H_p := \{T \in G \ : \ T(P) = P \ for \ some \ P \in S\}$$

is isomorphic to SO(m). S is isotropic if it is at every point P.

Definition 1.3.0.2 The action of G on S is transitive on S if $\forall P, Q \in S \ \ \exists g \in G :$ $g(P) = Q$.

Definition 1.3.0.3 The manifold S is said to be homogeneous if its isometry group acts transitively on it, i.e. we have the same geometry in a nhood of every point on S.

Definition 1.3.0.4 A cosmological model M is a homogeneous/isotropic cosmology if it has a foliation of space-like hyper-surfaces, each of which is homogeneous/isotropic.

Of course, the properties above are local, but no scientific evidence suggests us we are privileged observers; indeed, the principle of mediocrity holds:
"the properties of the universe we see near us would be seen, on average, by any observer anywhere else in the universe".
We can therefore extend our local homogeneity and isotropy to the whole space. It can be shown that, after some calculations, the metric tensor $g|$ reduces to

$$(h_{i,j}) = \begin{bmatrix} (1-Kr^2)^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \cdot sin(\theta)^2 \end{bmatrix}$$

or better, introducing the curvature K, we have

- $(h_{i,j}) = \dfrac{1}{K} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & sin(\chi)^2 & 0 \\ 0 & 0 & sin(\chi)^2 \cdot sin(\theta)^2 \end{bmatrix}, \quad forK > 0;$

- $(h_{i,j}) = \dfrac{1}{|K|} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & sinh(\chi)^2 & 0 \\ 0 & 0 & sinh(\chi)^2 \cdot sin(\theta)^2 \end{bmatrix}, \quad forK < 0.$

Thus, the geometry really depends only on the sign of K:

- if $K > 0$, the metric is the one of the sphere $\mathbb{S}^3$ of radius $\dfrac{1}{\sqrt{K}}$ and it corresponds to the "closed" model;

- if $K < 0$, we have the unbounded universe or "open";

- if K = 0, we are in $\mathbb{E}^3$, so into a "flat" universe.

Chapter 2

Classical Form of Maxwell's Equations

The almost independent nature of electric and magnetic phenomena disappears when we consider time-dependent problems. Faraday's discovery of induction destroyed this independence and led Maxwell to formulate the existence of displacement currents, which are the reason why light is an electromagnetic wave, basically. The publication of Maxwell's complete work dated back to the 1865 with the paper "A Dynamical Theory of the Electromagnetic Field" and till today it provides the classical description of the electromagnetic field. However, the full import of interconnection between electric and magnetic fields becomes clear within the framework of Special Relativity Theory (STR), originally proposed by Albert Einstein in a paper published 26 September 1905 titled "On the Electrodynamics of Moving Bodies" in what he definitively solved the incongruence of Newtonian mechanics with respect to Maxwell's equations.

Before going further, we warn the reader we will use Lorentz-Heaviside units: a system of units within CGS, named for Hendrik Antoon Lorentz and Oliver Heaviside. They share with CGS-Gaussian units the property that the electric constant ϵ_0 and magnetic constant μ_0 do not appear, having been incorporated implicitly into the unit system and electromagnetic equations. Lorentz-Heaviside units may be regarded as normalizing $\epsilon_0 = 1$ and $\mu_0 = 1$, while at the same time revising Maxwell's equations to use the speed of light c instead.

Lorentz-Heaviside units, like SI units but unlike Gaussian units, are rationalized, meaning that there are no factors of 4π appearing explicitly in Maxwell's equations. The fact that these units are rationalized partly explains their appeal in quantum field theory: the Lagrangian underlying the theory does not have any factors of 4π in these units. Consequently, Lorentz-Heaviside units differ by factors of $\sqrt{4\pi}$ in the definitions of the electric and magnetic fields and of electric charge. They are often used in relativistic theories and in High Energy Physics (particle physics). They are particularly convenient when performing calculations in spatial dimensions greater than three, such as in string theory.

2.1 Formulation using fields

Consider first the classical form of Maxwell's equations in 3D Euclidean space-time and in vacuum. As we already know from basic physics, we have a total of four p.d.e.s for the electric field $\vec{E}$ and the magnetic one $\vec{B}$, namely:

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \cdot \vec{B} = 0 \tag{2.1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{J} \tag{2.2}$$

and, to complete the scenery, remember we have also the Lorentz force equation

$$\vec{F} = q \cdot \vec{E} + q \cdot \vec{v} \wedge \vec{B} \tag{2.3}$$

which governs the dynamics of charges moving with velocity v inside the electromagnetic field, and the continuity one for the current density vector $\vec{J}$

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{2.4}$$

Summarizing the situation, if we fix a point in the space-time, namely $(\vec{x}, t)$, the values of $\vec{E}$ and $\vec{B}$ at that point are completely determined by the Lorentz force exerted on a point test charge q and they are related to the sources ρ and $\vec{J}$ by Maxwell's equations. Sources themselves are not independent with respect to each other, indeed they are strictly related by the continuity equation (2.4).

2.1.1 Substantial features of the electromagnetic field

Let's discuss briefly these equations, doing some relevant considerations on the quantities involved and related properties:

1. asymmetry in (2.1) is a remarkable question and it is related to the fact that, whereas the electric field has point sources, namely charges, at the moment we don't have the proof of the existence of a single magnetic monopole. In nature, positive and negative magnetic poles are always coupled and it's impossible to isolate a single pole. This difference in the intimate structure of the fields $\vec{E}$ and $\vec{B}$ is clearly visible if we consider the related force lines.

   Common features are that the lines never intersect and their number per unit area is proportional to the field strength in the point considered, but, whereas in the case of $\vec{E}$ they start from the positive charge and enter in the negative one in a way to have $\nabla \times \vec{E} = 0$, in the $\vec{B}$ case they are always closed lines starting from N pole and finishing on S pole. Thus, if we take the flux $\Phi_S(\vec{B})$ across an arbitrary surface S we always have zero, accordingly $\nabla \cdot \vec{B} = 0$;

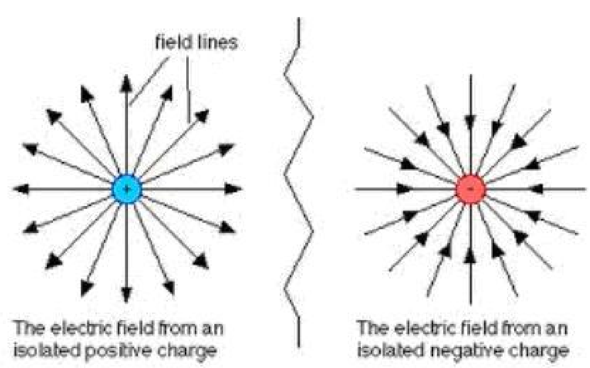


Figure 2.1: Configuration of the electric field in a nhood of a positive or negative isolated charge.

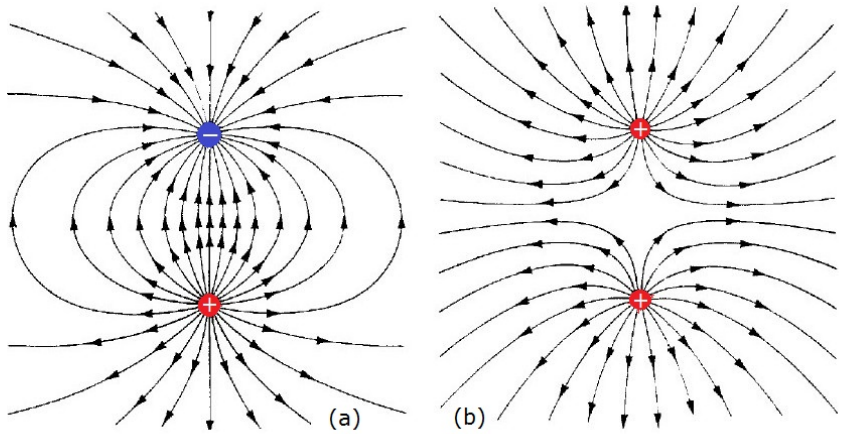


Figure 2.2: Configuration of the electric field lines in a nhood of a dipole in the case of positive-negative charges (a) and in the case two positive charges (b).

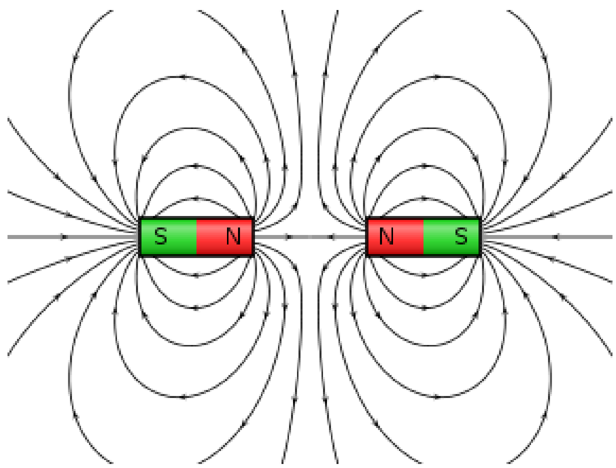


Figure 2.3: Configuration of the magnetic field lines for like poles.

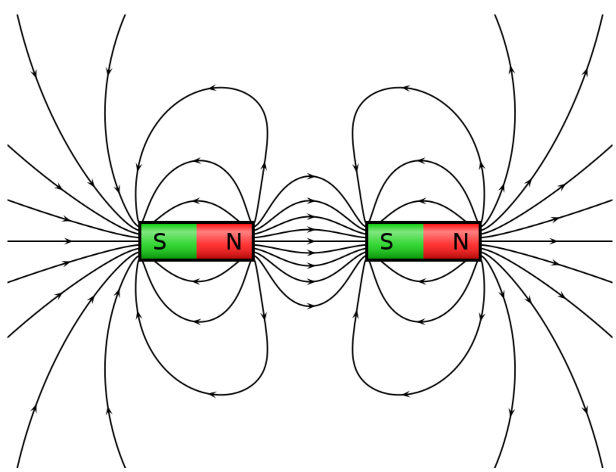


Figure 2.4: Configuration of the magnetic field lines for unlike poles.

2. the nature of the field we are observing depends on the system we are: a charge fixed in certain static reference frame is effectively the source of an electric field but, if we put ourselves on a system that translates with respect to the previous one, we see the same charge moving, i.e. a current, and we can only conclude that it is actually a magnetic field source;

3. in what follows, we consider $\vec{E}$ and $\vec{B}$ as continuous functions of the space-time. We already know this is just an approximation at a macroscopic scale indeed, the correct way to treat the electromagnetic field at a microscopic level is to use quantization, passing to Quantum Electrodynamics (QED);

4. what's more, if we specify the space-time distributions of the sources $\vec{J}$ and ρ, Maxwell's equations, eventually equipped with the relations between $\vec{D}$, $\vec{E}$ and $\vec{H}$, $\vec{B}$, completed by initial conditions and boundary ones, allow us to determine the values of the electromagnetic field in each point of the space-time. Using then (2.3), we get a full description of the dynamics of a particle inside this field;

5. finally, it is important to notice that in (2.1), (2.2) we have eight scalar equations: six from the components of the vectors $\nabla \times \vec{E}$ and $\nabla \times \vec{B}$ and two arising from the scalar equations in (2.1). It's a simple calculation, as we will see below, to show that (2.1) can be easily derived from (2.2), applying ∇ operator and using homogeneous initial conditions, so they are actually redundant.

   From the first one in (2.2), we have indeed

$$\nabla \cdot (\nabla \times \vec{E}) = -\nabla \cdot (\frac{\partial \vec{B}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \cdot \vec{B})$$

$$0 = -\frac{\partial}{\partial t}(\nabla \cdot \vec{B})$$

$$\nabla \cdot \vec{B} = 0$$

From the second one,

$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot (\frac{\partial \vec{E}}{\partial t} + \vec{J}) = \frac{\partial}{\partial t}(\nabla \cdot \vec{E}) + \nabla \cdot \vec{J}$$

and, using the continuity equation for $\vec{J}$ in (2.4),

$$0 = \frac{\partial}{\partial t}(\nabla \cdot \vec{E}) - \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\nabla \vec{E} - \rho)$$

$$\nabla \cdot \vec{E} = \rho$$

Thus, the relevant equations are only the ones in (2.2): Six independent first order coupled p.d.e.s in total, in the six scalar components of the vector fields $\vec{E}$ and $\vec{B}$. Those equations can be solved directly only in few simple cases, this is the reason why, from the next paragraph, we start reducing their number thanks to vector and scalar potentials.

2.1.2 Electromagnetic waves

In the end, the electromagnetic field is a physical entity defined in the space-time regardless of the sources that have generated it, which propagates without the presence of a material medium in the form of waves, called electromagnetic waves. Remind that the electromagnetic field localizes at every point of the space-time a non zero angular momentum, impulse and energy, completely described by the Poynting vector (SI) $\vec{I} := \vec{E} \times \vec{H} = \dfrac{\vec{E} \times \vec{B}}{\mu}$.
In fact, using again the relations in (2.2) with $\vec{J} = 0 = \rho$, we derive wave equations for $\vec{E}$ and $\vec{B}$

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times (-\frac{\partial \vec{B}}{\partial t}) = -\frac{\partial}{\partial t}\,(\nabla \times \vec{B})$$

$$-\nabla^2 \cdot \vec{E} + \nabla \cdot (\nabla \cdot \vec{E}) = -\nabla^2 \cdot \vec{E} = -\frac{1}{c^2} \cdot \frac{\partial}{\partial t}(\frac{\partial \vec{E}}{\partial t}) = -\frac{1}{c^2} \cdot \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{E}_{tt} - c^2 \cdot \Delta \vec{E} = 0 \tag{2.5}$$

Similarly, calculations on $\nabla \times \vec{E}$ lead to

$$\vec{B}_{tt} - c^2 \cdot \Delta \vec{B} = 0 \tag{2.6}$$

To have a more compact form, we can define the d'Alembertian operator $\Box$ in the 3D Euclidean space-time as follows

$$\Box := \Delta - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} \tag{2.7}$$

and we get

$$\begin{cases} \Box \cdot \vec{E} = 0 \\ \Box \cdot \vec{B} = 0 \end{cases} \tag{2.8}$$

Remark 2.1.2.1 We observe that the speed of propagation remains always the light one c, regardless of the frequency waves have and, if $\vec{J} \neq 0$ and $\rho \neq 0$, we simply add in (2.8) first order in time terms that do not influence the principal part of the equations, namely: waves still share the same properties as before.

2.2 Formulation using electrodynamic potentials

As we have already noticed in the previous paragraph, having six independent equations still coupled for the vector fields $\vec{E}$ and $\vec{B}$ is not a suitable situation to solve the problem of determining the values of the electromagnetic field in its generality. What we would like to do in this section is then to reduce these equations to four second order uncoupled p.d.e.s, using some kind of appropriate transformations, called gauges. The benefits of this operation will be not only to have four, instead of six, equations, uncoupled besides, but also to express their relativistic co-variance in terms much more elegant and compact, as we will see in a while.

2.2.1 Scalar and Vector Potentials

Consider the second equation in (2.1), $\nabla \cdot \vec{B} = 0$: It tells us the magnetic field is solenoidal, there exists then a vector potential $\vec{A}$ such that

$$\vec{B} = \nabla \times \vec{A} \tag{2.9}$$

where $\vec{A}$ is a function of space-time, $\vec{A} = \vec{A}(\vec{x}, t) = \vec{A}(x, y, z, t)$ in 3D Euclidean one.

We use then the second homogeneous equation we have, namely the first one in (2.2), and substituting the expression (2.9), we get therefore

$$0 = \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} + \frac{\partial}{\partial t}(\nabla \times \vec{A}) = \nabla \times \vec{E} + \nabla \times \frac{\partial \vec{A}}{\partial t} = \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t})$$

thus

$$\nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0 \tag{2.10}$$

This field shares the property to be curl-free, so, once again, there exists a scalar potential Φ such that

$$-\nabla\Phi = (\vec{E} + \frac{\partial\vec{A}}{\partial t})$$

or better,

$$\vec{E} = -\,(\frac{\partial\vec{A}}{\partial t} + \nabla\Phi) \tag{2.11}$$

Remark 2.2.1.1 The minus sign here is put in order to have the well-known relation for the potential in the electrostatic case, i.e. $\vec{E_0} = -\nabla\cdot V_0 + \text{const.}$

Remark 2.2.1.2 The existence of a vector potential $\vec{A}$ and a scalar potential Φ is subject to the validity of Maxwell's homogeneous equations $\nabla\cdot\vec{B} = 0$ and $\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$. Substituting in them the relations (2.9) and (2.11) we have just found, we obtain indeed the identities

$$\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0$$

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{E} + \frac{\partial\vec{A}}{\partial t}) = -\nabla\times(\nabla\cdot\Phi) = 0$$

At this point, we have only to determine the potentials. To achieve this goal, we make use of the remaining equations in (2.1)-(2.2), called equations for the dynamics, getting

$$\rho = \nabla\cdot\vec{E} = \nabla\cdot(-\nabla\cdot\Phi - \frac{\partial\vec{A}}{\partial t})$$

i.e.

$$\Delta\Phi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\rho \tag{2.12}$$

and

$$\begin{aligned}
\nabla\times\vec{B} &= \frac{1}{c^2}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{J} \\
\nabla\times(\nabla\times\vec{A}) &= \frac{1}{c^2}\cdot\frac{\partial}{\partial t}(-(\frac{\partial\vec{A}}{\partial t} + \nabla\Phi)) + \vec{J} \\
-\nabla^2\cdot\vec{A} + \nabla(\nabla\cdot\vec{A}) &= -\frac{1}{c^2}\cdot(\frac{\partial^2\vec{A}}{\partial t^2} + \nabla\cdot\frac{\partial\Phi}{\partial t}) + \vec{J} \\
-\nabla^2\cdot\vec{A} + \nabla(\nabla\cdot\vec{A} + \frac{1}{c^2}\cdot\frac{\partial\Phi}{\partial t}) &= -\frac{1}{c^2}\cdot(\frac{\partial^2\vec{A}}{\partial t^2}) + \vec{J}
\end{aligned}$$

$$\Delta\vec{A} - \frac{1}{c^2}\cdot\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\cdot(\nabla\cdot\vec{A} + \frac{1}{c^2}\cdot\frac{\partial\Phi}{\partial t}) = -\vec{J} \tag{2.13}$$

i.e.

$$\begin{cases} \Delta\Phi + \dfrac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\rho \\ \\ \Delta\vec{A} - \dfrac{1}{c^2}\cdot\dfrac{\partial^2\vec{A}}{\partial t^2} - \nabla\cdot(\nabla\cdot\vec{A} + \dfrac{1}{c^2}\cdot\dfrac{\partial\Phi}{\partial t}) = -\vec{J} \end{cases} \tag{2.14}$$

Remark 2.2.1.3 In the box above we finally have four independent equations in the unknowns $A_x(\vec{x},t), A_y(\vec{x},t), A_z(\vec{x},t)$ and $\Phi(\vec{x},t)$, that one can solve fixing appropriate boundary and initial conditions. Unfortunately, they are still coupled.

2.2.2 Arbitrary Gauge

As we have observed in the previous remark, we've achieved our first goal to have less equations totally describing the electromagnetic field, and it is a great step forward, but they remain still coupled. The way we will make them independent with respect to each other, is reminding potentials are defined up to gradient of a scalar function, for $\vec{A}$, and a time derivative, for Φ, and that the fields $\vec{E}$ and $\vec{B}$ are invariant for gauges (re-calibrations): transformations of the type

$$\begin{aligned} &\vec{A} \xrightarrow{T_A} \vec{A}' := \vec{A} + \nabla\cdot\Lambda \\ &\Phi \xrightarrow{T_\Phi} \Phi' := \Phi - \frac{\partial\Lambda}{\partial t} \end{aligned} \tag{2.15}$$

where $\Lambda = \Lambda(\vec{x},t)$, at least in $C^{2,2}(\Omega\times\mathbb{R}^+)$, $\Omega\subseteq\mathbb{E}^3$ open set, is called gauge function.

Indeed, from (2.9):

$$\nabla\times\vec{A}' = \nabla\times(\vec{A}+\nabla\cdot\Lambda) = \nabla\times\vec{A} + 0 = \nabla\times\vec{A} = \vec{B}$$

$$\nabla\times\vec{A}' = \vec{B}$$

and, from (2.11):

$$-(\frac{\partial \vec{A}'}{\partial t} + \nabla \cdot \Phi') = -[-\frac{\partial}{\partial t}(\vec{A} + \nabla \cdot \Lambda) + \nabla \cdot (\Phi - \frac{\partial}{\partial t}\Lambda)] =$$

$$= -[\frac{\partial \vec{A}}{\partial t} + \nabla \cdot \frac{\partial \Lambda}{\partial t} + \nabla \cdot \Phi - \frac{\partial}{\partial t}(\nabla \cdot \Lambda)] = -(\frac{\partial \vec{A}}{\partial t} + \nabla \cdot \Phi) = \vec{E}$$

$$-(\frac{\partial \vec{A}'}{\partial t} + \nabla \cdot \Phi') = \vec{E}$$

2.2.3 Lorenz Gauge

In 1867 Lorenz, exploiting the arbitrariness left in the choice of potentials, introduced its own gauge, the so called Lorenz gauge, setting

$$\nabla \cdot \vec{A} + \frac{1}{c^2} \cdot \frac{\partial \Phi}{\partial t} = 0 \tag{2.16}$$

or better, he made an opportune choice on Λ to obtain the above relation (see remark 2.2.3.3). Equation (2.16) is called Lorenz condition.

In place of the dynamics in (2.14), now we have:

$$\begin{cases} \Delta\Phi - \frac{1}{c^2} \cdot \frac{\partial^2 \Phi}{\partial t^2} = -\rho \\ \\ \Delta\vec{A} - \frac{1}{c^2} \cdot \frac{\partial^2 \vec{A}}{\partial t^2} = -\vec{J} \end{cases} \tag{2.17}$$

or, using the d'Alembertian operator,

$$\begin{cases} \Box \cdot \Phi = -\rho \\ \Box \cdot \vec{A} = -\vec{J} \end{cases} \tag{2.18}$$

and, clearly,

$$\begin{cases} B = \nabla \times \vec{A} \\ \vec{E} = -\,(\frac{\partial \vec{A}}{\partial t} + \nabla\Phi) \end{cases} \tag{2.19}$$

Remark 2.2.3.1 If the sources are localized in a bounded domain Ω, we can easily solve (2.18) using delayed (w.r.t. sources) potentials.
Thus, always in L-H units, we have

$$\vec{A}(\vec{x},t) = \int_{\Omega} \frac{\vec{J}(\vec{y},s)}{\Delta\vec{r}} d\tau'(\vec{y})$$

$$\vec{\Phi}(\vec{x},t) = \int_{\Omega} \frac{\rho(\vec{y},s)}{\Delta\vec{r}} d\tau'(\vec{y})$$

where $s := t - \dfrac{\Delta r}{v}$ is the scaled time, v is the signal velocity, depending on the material we consider, and Δr is, as usual, the distance $||\vec{x} - \vec{y}||$.

Remark 2.2.3.2 It's important to notice that the ones in (2.18) are now four second order uncoupled p.d.e.s in the unknown $\vec{A}$ and Φ.

Remark 2.2.3.3 What relation has Λ to satisfy in order for $\vec{A}$ and Φ to fit the Lorenz condition (2.16)? To answer this question, we simply use the arbitrary gauges in (2.15) and do some calculations

$$0 = \nabla \cdot \vec{A}' + \frac{1}{c^2} \cdot \frac{\partial \Phi'}{\partial t} = \nabla \cdot (\vec{A} + \nabla \cdot \Lambda) + \frac{1}{c^2} \cdot \frac{\partial}{\partial t}(\Phi - \frac{\partial \Lambda}{\partial t}) =$$
$$= \nabla \cdot \vec{A} + \Delta \cdot \Lambda + \frac{1}{c^2} \cdot (\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Lambda}{\partial t^2}) = \Delta\Lambda - \frac{1}{c^2} \cdot \frac{\partial^2 \Lambda}{\partial t^2} + \nabla \cdot \vec{A} + \frac{1}{c^2} \cdot \frac{\partial \Phi}{\partial t}$$

Thus, Λ must respect the following equation

$$\Delta\Lambda - \frac{1}{c^2} \cdot \frac{\partial^2 \Lambda}{\partial t^2} = -(\nabla \cdot \vec{A} + \frac{1}{c^2} \cdot \frac{\partial \Phi}{\partial t}) \tag{2.20}$$

that, knowing the term on the right side, has always a solution.

Remark 2.2.3.4 Obviously, $\nabla \cdot \Lambda$ and $\dfrac{\partial \Lambda}{\partial t}$ are not uniquely determined by (2.20), i.e. any $\vec{A}'$, Φ' satisfying (2.15) and Lorenz condition (2.16) is allowed.

Lorenz's work was the first symmetrizing Maxwell's equations after Maxwell himself published his 1865 paper. In 1888, retarded potentials came into general use after Heinrich Rudolf Hertz's experiments on electromagnetic waves and in 1895, a further boost to the theory of retarded potentials came after J. J. Thomson's interpretation of data for electrons.

2.2.4 Coulomb Gauge

Another useful gauge is the Coulomb one, defined by setting

$$\nabla \cdot \vec{A} = 0$$

With this choice, equations in (2.14) simply become

$$\begin{cases} \Delta\Phi = -\rho \\ \Delta\vec{A} - \dfrac{1}{c^2} \cdot \dfrac{\partial^2 \vec{A}}{\partial t^2} = \dfrac{1}{c^2} \cdot \nabla(\dfrac{\partial \Phi}{\partial t}) - \vec{J} \end{cases} \tag{2.21}$$

We trivially notice the first equation is the usual one we meet when dealing with the electrostatic case, although the density ρ is now a function of time $\rho(\vec{x}, t)$, whose solution, as we already know, is given by

$$\vec{\Phi}(\vec{x}, t) = \int_{\Omega} \frac{\rho(\vec{y}, t)}{||\vec{x} - \vec{y}||} d\tau'$$

called Coulomb instantaneous potential (LH units).
This transformation is largely employed when $\rho = 0,\ \vec{J} \equiv 0$, i.e. when

$$\begin{cases} \Delta\Phi = 0 \\ \Delta\vec{A} - \dfrac{1}{c^2} \cdot \dfrac{\partial^2 \vec{A}}{\partial t^2} = 0 \end{cases} \tag{2.22}$$

In this case, it implies indeed $\Phi = 0$ at every point of the space-time, of course imposing homogeneous boundary conditions.
For what concerns the fields, accordingly we have that

$$\begin{cases} B = \nabla \times \vec{A} \\ \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} \end{cases} \tag{2.23}$$

Chapter 3

Relativistic Form of Maxwell's Equations

In this chapter we will go deeper inside the concept of electromagnetic field, studying is relativistic co-variance and using STR to reduce to a single differential form, totally describing the dynamics. At the end of the day, we will have a complete knowledge of the fact electric and magnetic fields are essentially the same physical entity, depending on the way we observe the related phenomena, i.e. the reference frame we are in. We will spend some section just to set up the appropriate mathematical tools to deal with this new theory. As usual, our arguments will be developed in a 3D Euclidean space-time, in vacuum and using LH units.

3.1 Four-Vectors formalism

Definition 3.1.0.1 We set $\underline{x} = (x, y, z, ct) = (x_1, x_2, x_3, x_4)$ as the space-time four-vector with respect to a coordinate system $\Sigma = \{O, x, y, z\}$.

Remark 3.1.0.1 An equivalent definition is to declare the last entry equal to time, but the previous one is more common because of the dimensional equivalence: all the entries are length indeed

Remark 3.1.0.2 Every reference frame we will use in this chapter must be an inertial one. Cases in which there is a frame acceleration are covered in General Relativity Theory. Light essentially bends in presence of a gravitational field, so the dynamics are completely different.

Consider now a second reference frame $\Sigma' = \{O', x', y', z'\}$ moving along $x = x'$ at a constant relative velocity u with respect to Σ, we have then:

$$\underline{x'} = L \cdot \underline{x} \tag{3.1}$$

where the change of variable L is provided by the so-called Lorentz matrix

$$L := \begin{bmatrix} \gamma & 0 & 0 & -\beta\,\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\,\gamma & 0 & 0 & \gamma \end{bmatrix} \tag{3.2}$$

with $\beta := \dfrac{u}{c}$ and $\gamma := \dfrac{1}{\sqrt{1-\beta^2}}$.

Definition 3.1.0.2 Every vector in Σ whose change of variables in a second reference system Σ' is given by equation (3.1) is called four-vector.

It's easy to show that the quantities

$$||\underline{x}||^2 = \; x_1^2 + x_2^2 + x_3^2 - x_4^2$$

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial x_4^2}$$

are relativistically invariant, i.e. invariant under the change of variables (3.1).

3.1.1 The electromagnetic four-potential

Definition 3.1.1.1 We define the current density four-vector as

$$\underline{J} := (\vec{J}, \rho) = (J_1, J_2, J_3, J_4) \tag{3.3}$$

and the electromagnetic four-potential (vector) as

$$\underline{A} := (\vec{A}, \Phi) = (A_x, A_y, A_z, \Phi) \tag{3.4}$$

which combines both electric scalar potential and magnetic vector potential into a single four-vector.

Having performed these definitions, equations in (2.17) simply become

$$\Box \cdot \underline{A} = -\underline{J} \tag{3.5}$$

Remark 3.1.1.1 The quantities involved above, namely $\Box$ and $\underline{J}$, are relativistically invariant, thus $\underline{A}$ also is. Moreover, the electrodynamics, expressed via the $\Box$ operator in compact form in (3.5), are relativistically co-variant: if we find $\underline{A}$ in an inertial reference system, $\underline{A'}$ is completely determined in every other system of this kind just applying the Lorentz matrix on $\underline{A}$. In explicit coordinates

$$\begin{cases} A'_1 = \gamma\ (A_1 - \beta A_4) \Rightarrow A'_x = \gamma\ (A_x - \beta \cdot \Phi) \\ A'_2 = A_2 \Rightarrow A'_y = A_y \\ A'_3 = A_3 \Rightarrow A'_z = A_z \\ A'_4 = \gamma\ (A_4 - \beta A_1) \Rightarrow \Phi = \gamma\ (-\beta \cdot A_x + \Phi) \end{cases}$$

Knowing the potential, $\vec{E}$ and $\vec{B}$ are always obtained from (2.19)

$$\begin{cases} B = \nabla \times \vec{A} \\ \vec{E} = -\ (\dfrac{\partial \vec{A}}{\partial t} + \nabla \Phi) \end{cases}$$

Remark 3.1.1.2 It's possible to have some changes in definitions just performed, depending on the units of measurement used, i.e. SI, CGS, HL.

However, this fact does not influence the essence of the theory.

3.2 Tensor calculus formalism

3.2.1 Metric tensor fields and Minkowski space-time

Definition 3.2.1.1 A metric tensor field $g|$ on a manifold M is a $\binom{0}{2}$ symmetric tensor field, i.e. $g| \in \Sigma^2(\mathbb{E})$, which is invertible at every point on M.

From the point of view of differential geometry, a metric tensor provides an "high level" structure on a manifold: thanks to it we can define notions as distance and curvature, but we add nothing more to the manifold itself; we are simply declaring special a particular tensor.

Definition 3.2.1.2 The signature (p, q, r) of a metric tensor $g|$ is the number (counted with multiplicity) of positive, negative and zero eigenvalues of the real symmetric matrix g_{ab} associated to the metric tensor with respect to a basis.
For a non-degenerate metric tensor (i.e. $r = 0$) an equivalent notation for its signature is $s = p - q$.

Definition 3.2.1.3 Let $\gamma : (-\epsilon, \epsilon) \to M$ be a parametrized curve on the manifold M, for $\epsilon > 0$, if $\gamma'(t) = T$ is its tangent vector, we define length of the infinitesimal arc

$$dl := |g|(T,T)|^{1/2} dt$$

Indeed, $dl^2 = d\gamma(t) \cdot d\gamma(t) = \gamma' dt \cdot \gamma' dt = T dt \cdot T dt = (T \cdot T) dt^2 = g|(T,T) dt^2$.

Remark 3.2.1.1 The reason for the presence of absolute value in the above definition is that, in general, a metric tensor field hasn't got a definite sign. Curves are divided in two classes, using just this sign: if it is positive, dl represent proper distance for space-like curves, while it is time, if negative, and we call time-like this last class of curves. It is zero for null curves.

Definition 3.2.1.4 (Riemannian manifold) A Riemannian manifold (M,g) is a smooth manifold M together with a family of smoothly varying positive definite inner products

$$g = g_x \text{ on } T_x M \qquad \forall x \in M$$

The family $g = \{g_x\}_{x \in M}$ is called a Riemannian metric on M.

Definition 3.2.1.5 Two Riemannian manifolds (M,g), (N,h) are said to be isometric if there exists a smooth diffeomorphism $f : M \longrightarrow N$ such that

$$g_x(X,Y) = h_{f(x)}(f_*X, f_*Y) = h_{f(x)}(df_x \cdot X_x, df_x \cdot Y_x) \quad \forall\ X, Y\ \in\ T_xM,\ \forall\, x \in M$$

Remark 3.2.1.2 Since g_x is a bilinear form on T_xM, it is actually an element of $T_x^*M \otimes T_x^*M$. Thus, "g varies smoothly" simply means g is a smooth section of the bundle $T^*M \otimes T^*M$.

Example 3.2.1.1 On $\mathbb{R}^n$ the Riemannian metric is given by the standard inner product $g_x(v,w) = \langle v, w\rangle\ \ \forall v, w \in\ T_x(\mathbb{R}^n),\ \forall\ x \in \mathbb{R}^n$.
In the end, $(\mathbb{R}^n, g)$ is our commonly used Euclidean space.

Definition 3.2.1.6 (Minkowski space-time) The space $\mathbb{R}^4$, considered as manifold, equipped with a metric of signature +2 [(3,1,0)] is called Minkowski space-time and is the space-time of Special Relativity.
Equivalently, a manifold M with a metric $g|$ is called Minkowski space-time only if there exists a single coordinate system covering the whole M in which $g|$ has components $\eta_{\alpha\beta} = diag(-1, 1, 1, 1)$.

3.2.2 Raising and lowering indices

Given a tensor field on a manifold M, on which is defined a non-singular bilinear form, usually a metric or a symplectic form, we can perform a manipulation on the tensor indices, namely raising or lowering them, to change the tensor rank.

Definition 3.2.2.1 If T is a tensor of rank(p, q), where p+q is its order and p,q denote the upper and lower indices, we

- raise indices to change from a (p,q)-rank to a (p+1,q-1)-rank tensor;
- lower indices to change from a (p,q)-rank to a (p-1,q+1)-rank tensor.

The way we get this result is by multiplying by the co-variant/contra-variant bilinear form and then contracting indices using Einstein notation.
To help understanding, we do some examples on rank one and two tensors but the generalization to rank n is quite immediate; as you will see, it is just a matter of calculations.

Vectors

Consider the simplest case possible: suppose to have a vector $\vec{A} \in \mathbb{R}^n$ and a contra-variant metric tensor g^{ij}, multiplying then by the metric and contracting, we will produce a raise index, as seen in the definition above. Thus:

$$g^{ij}A_j = A^i$$

where i, j: 1,...,n.
Similarly, multiplying by a co-variant metric tensor g_{ij} and contracting, lowers an index, namely:

$$g_{ij}A_j = A_i$$

where i, j: 1,...,n.
Moreover, raising and lowering the same index (or conversely) are inverse operations, in fact:

$$g^{ij}g_{jk} = g_{kj}g^{ji} = \delta^i{}_k = \delta^k{}_i$$

where $\delta_{i,k}$ is the Kronecker delta.

Remark 3.2.2.1 When raising indices of quantities in space-time, it helps to decompose summations into time-like components (where indices are zero) and space-like ones.

Position in the Minkowski space-time

Let $(\mathbb{R}^4, \eta|)$ the Minkowski space-time, where the metric is given by $\eta_{\alpha\beta} = diag(-1,1,1,1)$, consider the position co-variant four-vector, i.e.

$$x_\alpha = (-ct, x, y, z) = (x_0, x_1, x_2, x_3)$$

To raise the index, multiply by the tensor and contract:

$$x^\alpha = \eta^{\alpha\beta} x_\beta = \eta^{\alpha 0} x_0 \; + \; \eta^{\alpha\gamma} x_\gamma$$

where $\alpha, \beta : \; 0, ..., 3, \; \gamma : 1, 2, 3$. The right side of the equation is obtained splitting time-like from space coordinates.
or better

$$\begin{cases} x^0 = \eta^{00} x_0 + \; \eta^{0\gamma} x_\gamma = \eta^{00} x_0 + 0 = \eta^{00} x_0 = (-1) \cdot x_0 = +x_0 = ct \\ x^\alpha = \eta^{\alpha 0} x_0 \; + \; \eta^{\alpha\gamma} x_\gamma = 0 + \delta^{\alpha\gamma} x_\gamma = x_\alpha \end{cases}$$

Thus:

$$x^\alpha = (ct, x, y, z) = (x_0, x_1, x_2, x_3)$$

Rank two tensors

The extension to second order case is simply performed twice multiplying by the contra-variant/co-variant metric, namely:

$$A^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} A_{\gamma\delta} \qquad\qquad A_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} A^{\gamma\delta}$$

3.2.3 Faraday's Co-variant Tensor

Suppose to work with coordinates $\{x_0, x_1, x_2, x_3\} = \{ct, x, y, z\}$ in the Minkowski space-time, where we operate this choice on vectorial indices to make easier the generalization to higher dimensional spaces. Note that the time dependence is now on the first entry of the $\underline{x}$ four-vector, instead of the last one, as in definition 3.1.0.1.

Remark 3.2.3.1 Refer to the appendix for an appropriate discussion on what a tensor is and related topics in mathematics.

Definition 3.2.3.1 We set the Faraday's tensor as the rank two tensor, whose coordinates are given by

$$F_{\mu,\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} \tag{3.6}$$

where $\mu, \nu : 0, ..., 3$, and $\underline{A}, \underline{x}$ as already defined.

This tensor transforms under Lorenz map as

$$F_{\mu,\nu} = \sum_{i,j=0}^{3} L_{\mu,i} L_{\nu,j} F_{i,j} \tag{3.7}$$

and it is actually, by definition, antisymmetric, i.e.

$$F_{\mu,\nu} = -F_{\nu,\mu}$$

where $\mu, \nu : 0, ..., 3$.

Moreover, using the definition of $\underline{A}$ and its relations with the original fields $\vec{E}$ and $\vec{B}$, we reduce to the form

$$\underline{F} := (F_{\mu\nu}) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{3.8}$$

We state, without proof, the following fundamental result

Proposition 3.2.3.1 $F_{[\mu\nu,\gamma]} = 0 \Longleftrightarrow dF = 0 \Longleftrightarrow \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{cases}$

Thus, Maxwell's homogeneous equations are valid if and only if Faraday's tensor derivative is zero.

This fact is not trivial: from now on, we will be able to have a complete knowledge of electromagnetic field dynamics simply operating on a differential form, studying the topology of the space taken in exam and deriving its own co-homology.

The complete set of Maxwell's equations will be converted in a single form in a while, considering the indices contraction given by the metric tensor $g|$.

3.2.4 The Contra-variant Tensor $F^{\mu\nu}$

Definition 3.2.4.1 We define F the antisymmetric $\binom{2}{0}$ tensor with components $F^{\mu\nu} := g^{\mu\alpha} \cdot g^{\nu\beta} \cdot F_{\alpha\beta}$, namely:

$$F^{\mu\nu} := \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{3.9}$$

where, for the sake of simplicity, we have maintained the same notation of definition 3.2.3.1(see next remark for calculations on indices).

In the end, we achieve this powerful result

Proposition 3.2.4.1 $F^{\mu\nu}{}_{,\nu} = \vec{J}^{\mu} \quad \Longleftrightarrow \quad \begin{cases} \nabla \cdot \vec{E} = \rho \\ \nabla \times \vec{B} - \dfrac{\partial \vec{E}}{\partial t} = \vec{J} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{cases}$

where $\vec{J} = (\rho, (\vec{J})_x, (\vec{J})_y, (\vec{J})_z)$ denote only the spatial part of the four-vector current density.

Remark 3.2.4.1 (Calculations on $F_{\mu\nu}$) For completeness, here we do all the calculations needed to reduce to $F^{\mu\nu}$ from $F_{\mu\nu}$, as illustrated in the section about indices manipulation.

Given the co-variant Faraday's tensor $F_{\mu\nu}$ in (3.8),and the Minkowski metric tensor $\eta|$, we have:

$$
\begin{aligned}
F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta} = \eta^{\alpha 0}\eta^{\beta 0}F_{00} + \eta^{\alpha i}\eta^{\beta 0}F_{i0} + \eta^{\alpha 0}\eta^{\beta i}F_{0i} + \eta^{\alpha i}\eta^{\beta j}F_{ij} = \\
= (\eta^{\alpha i}\eta^{\beta 0} - \eta^{\alpha 0}\eta^{\beta i})F_{i0} + \eta^{\alpha i}\eta^{\beta j}F_{ij}
\end{aligned}
$$

remembering $F_{00} = 0, F_{i0} = \ -F_{0i}$.

For $\alpha = 0,\ \beta = k = 1, 2, 3,$

$$
F^{0k} = (\eta^{0i}\eta^{k0} - \eta^{00}\eta^{ki})F_{i0} + \eta^{0i}\eta^{kj}F_{ij} = (0 - \delta_{ki})F_{i0} + 0 = -F_{k0} = F_{0,k}
$$

The case $\beta = 0,\ \alpha = k = 1, 2, 3,$ follows from the previous one, using the fact we will have an anti-symmetric tensor, i.e.:

$$
F^{0k} = -F^{k0}
$$

And, finally, if $\alpha = k = 1, 2, 3$ and $\beta = l = 1, 2, 3,$

$$
F^{kl} = (\eta^{ki}\eta^{l0} - \eta^{k0}\eta^{li})F_{i0} + \eta^{ki}\eta^{lj}F_{ij} = 0 + (\delta_{ki}\delta_{lj})F_{ij} = F_{kl}
$$

In matrix form, it is just (3.9), Q.E.D.

Remark 3.2.4.2 Although we have obtained a more compact form, the tensor just defined is not coordinate-independent, while Faraday's one is. The problem derives from the choice of a particular metric in our definition, the Minkowski one.

Anyway, in next section we will define and use the Hodge star operator to get this obstacle around.

Chapter 4

The Role of the Hodge Star Operator

Our aim in this chapter will be to get the theory a brand reformulation, in order to free from reference frames, i.e. coordinate systems, and bring to light the essence of modern field theory, which is fully related to the topology of spaces we deal with. The relevant feature a space must have to make use of the powerful tool Hodge map provides is to be simply an oriented vector space (or a manifold), endowed with a non degenerate symmetric bilinear form. It is an hypothesis largely satisfied in differential geometry, that's why the Hodge dual plays a relevant role in this kind of mathematics.

4.1 The Hodge Star Operator

4.1.1 Formal definition

The foundation of the Hodge map comes from functional analysis, in particular, from the well known Riesz Representation Theorem, that we state without proof:

Theorem 4.1.1.1 Let $(\mathbb{W}, \langle \cdot, \cdot \rangle)$ a vector space on which is defined an inner product, then

$$\forall\ f \in \mathbb{W}^* \ \exists\ !\ v \ \in \ \mathbb{W} : f(w) = \langle w, v \rangle \quad \forall\ w \ \in \ \mathbb{W}.$$

Moreover, the map

$$\Phi_w : \mathbb{W}^* \longrightarrow \mathbb{W} \qquad f \longrightarrow v : f(w) = \langle w, v \rangle \quad \forall\, w \in \mathbb{W} \tag{4.1}$$

is an isomorphism.

In what follows, we will always work with an oriented n-dimensional inner vector space $\mathbb{V}$ and we refer to the appendix for exterior algebras definition and related constructions.

Remark 4.1.1.1 Let $\{e_1, \ldots, e_n\}$ be a basis for $\mathbb{V}$, remember that

$$\omega \in \Lambda^k(\mathbb{V}),\ \tau \in \Lambda^{n-k}(\mathbb{V}) \Rightarrow \omega \wedge \tau \in \Lambda^n(\mathbb{V})$$

thus, quite obviously:

$$\omega \wedge \tau = \lambda \cdot (e_1 \wedge \cdots \wedge e_n) = \lambda \cdot v \tag{4.2}$$

where λ is an opportune scalar.

Proposition 4.1.1.1 Consider a fixed $\omega \in \Lambda^k(\mathbb{V})$, then there exists a unique linear function $f_\omega \in (\Lambda^{n-k}(\mathbb{V}))^*$ such that

$$\omega \wedge \tau = f_\omega(\tau) \cdot (e_1 \wedge \cdots \wedge e_n) = f_\omega(\tau) \cdot v \ \ \forall\, \tau \in \Lambda^{n-k}(\mathbb{V}) \tag{4.3}$$

Therefore, if $\langle \cdot, \cdot \rangle$ denotes the inner product on $\Lambda^{n-k}(\mathbb{V})$, as a simple application of the Riesz Representation Theorem, the following result holds:

Theorem 4.1.1.2 Given $\omega \in \Lambda^k(\mathbb{V})$, then

$$\exists\,!\ \star\omega \in \Lambda^{n-k}(\mathbb{V}) : f_\omega(\tau) = \langle \tau, \star\omega \rangle$$

where $\star\omega$ is called Hodge dual of ω.

Moreover, the map

$$\begin{aligned} \star : \Lambda^k(\mathbb{V}) &\longrightarrow \Lambda^{n-k}(\mathbb{V}) \\ \omega \longrightarrow \star(\omega) : f_\omega(\tau) &= \langle \tau, \star\omega \rangle \quad \forall\ \tau\ \in\ \Lambda^{n-k}(\mathbb{V}) \end{aligned} \tag{4.4}$$

defines an isomorphism at exterior algebras level, called Hodge isomorphism.

Definition 4.1.1.1 Given $\eta, \zeta \in\ \Lambda^k(\mathbb{V})$, the Hodge dual induces an inner product on that space given by

$$\zeta \wedge \star\eta = \langle \zeta, \eta \rangle v \tag{4.5}$$

where $v :\ v \wedge \star v = v$ is the normalized n-form.

In the case of exterior differential forms on a Riemannian manifold M, v is called the volume form and it is defined as

$$v = \sqrt{|det(g)|} \cdot dx^1 \wedge \dots dx^n \tag{4.6}$$

where g is the matrix associated to the Riemannian metric tensor $g|$.

Viceversa, if an inner product is given on the exterior algebra, the previous equality provides an alternative definition of the Hodge dual.

The Hodge star is thus related to the $\mathbf{L^2}$ inner product on k-forms, by the formula

$$(\zeta, \eta) = \int_M \zeta \wedge \star\eta \qquad \forall\ \zeta, \eta \in \Lambda^k(M) \tag{4.7}$$

Remark 4.1.1.2 We can also see this as an inner product on sections of $\Lambda^k(T^*M)$, commonly denoted as $\Omega^k(M) = \Gamma(\Lambda^k(T^*M))$. Consequently, an interesting application of the Hodge dual on manifold is to define co-differentials in another way than the canonical one, we have already developed in the appendix.

Remark 4.1.1.3 Roughly speaking, with this operator we have created another way to completely describe a tensor. In fact, to assign it we can both:

1. give the numbers of unit components per each combination of the basis vectors, writing down the tensor matrix;

2. find the appropriate scalar, or if you want the correct Hodge dual, which is the unknown factor in the multiplication for the unit volume form.

There is an interesting video on that at https://www.youtube.com/watch?v=CliW7kSxxWU.

4.1.2 Index notation

The Hodge dual can also be defined using index notation, raising and then contracting the indices of a k-form $\eta_{i_1,\dots\, i_k}$ with the n-dimensional Levi-Civita anti-symmetric tensor.

Definition 4.1.2.1 If $\eta \in \Lambda^k(\mathbb{V})$,

$$(\star\eta)_{i_1,\dots,i_{n-k}} = \frac{1}{k!} \cdot \eta^{j_1,\dots,j_k} \sqrt{|det(g)|} \cdot \epsilon_{j_1,\dots,j_k,i_1,\dots,i_{n-k}} \tag{4.8}$$

The absolute value of the determinant of the metric tensor $g|$ is necessary: it may be also negative, i.e. Lorentzian manifolds.
Remember the Levi-Civita symbol expresses in compact form the sign of an index permutation, i.e.:

$$\epsilon_{i_1,\dots,i_n} = \begin{cases} +1 & \text{if } (i_1,\dots,i_n) \text{ is an even permutation of}(1,2,3,\dots,n) \\ -1 & \text{if } (i_1,\dots,i_n) \text{ is an odd permutation of}(1,2,3\dots,n) \\ 0 & \text{otherwise} \end{cases}$$

4.1.3 Relevant examples

Here we explicit the calculations involved in dimension two and four, which are related to analytic functions and Minkowski space-time.

The Euclidean plane

Let's consider $\mathbb{E}^2$, with fixed basis $\{x, y\}$, then we have

$$\star dx = dy \qquad\qquad \star dy = -dx \tag{4.9}$$

$$\star 1 = dx \wedge dy \qquad\qquad \star(dx \wedge dy) = 1 \tag{4.10}$$

We will use (4.7) to show the complex plane has the property of invariance under analytic change of coordinate, a fundamental fact when dealing with complex classes of differentials, or functions in general.

Let f be an holomorphic function $f = u + iv$ of $z = x + iy$, then, using Cauchy-Riemann equations, C-R, we obtain

$$\alpha = pdu + qdv = (p\frac{\partial u}{\partial x} + q\frac{\partial v}{\partial x})dx + (p\frac{\partial u}{\partial y} + q\frac{\partial v}{\partial y})dy = sdx + tdy$$

$$\star\alpha = \star(pdu + qdv) = (pdv - qdu) = (p\frac{\partial v}{\partial x} - q\frac{\partial u}{\partial x})dx + (p\frac{\partial v}{\partial y} - q\frac{\partial u}{\partial y})dy \overset{C-R}{=}$$
$$\overset{C-R}{=} (-p\frac{\partial u}{\partial y} - q\frac{\partial u}{\partial x})dx + (p\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial x})dy = -sdx + tdy$$

Q.E.D.

Minkowski space-time

Let $g|$ the usual Minkowski metric tensor, this time with signature (+ - - -). For the other way around, simply change the sign in the following relations. Note that, in this case, $n = 4$, the Hodge map provides an endomorphism between two forms, indeed

$$\star : \Lambda^2(\mathbb{R}^4) \longrightarrow \Lambda^2(\mathbb{R}^4) \tag{4.11}$$

Fix $\{t, x, y, z\} = \{x_0, x_1, x_2, x_3\}$ as basis in our space-time, it follows that

$$\star dx_0 = dx_1 \wedge dx_2 \wedge dx_3 \qquad \star dx_1 = dx_0 \wedge dx_2 \wedge dx_3$$
$$\star dx_2 = -dx_0 \wedge dx_1 \wedge dx_2 \qquad \star dx_3 = dx_0 \wedge dx_1 \wedge dx_2$$

and

$$\star(dx_0 \wedge dx_1) = -dx_2 \wedge dx_3 \qquad \star(dx_0 \wedge dx_2) = dx_1 \wedge dx_3$$
$$\star(dx_0 \wedge dx_3) = -dx_0 \wedge dx_2 \qquad \star(dx_1 \wedge dx_2) = dx_0 \wedge dx_3$$
$$\star(dx_1 \wedge dx_3) = -dx_0 \wedge dx_2 \qquad \star(dx_2 \wedge dx_3) = dx_0 \wedge dx_1$$

Remark 4.1.3.1 It's important to notice that the sign here depends only on the orientation you have fixed ordering the basis, not on the particular metric you have.

4.2 Remarkable Applications of $\star$

4.2.1 Duality

In the definitions we have just performed, we called $\star\omega$ the Hodge dual of the k-form ω, so we expect to have a concept of some kind of duality associated with this map. In fact, there is: when applied twice, the $\star$ operator provides an identity on the exterior algebra, up to a sign.

Proposition 4.2.1.1 Given a k-form $\eta \in \Lambda^k(\mathbb{V})$,

$$\star \star \eta = (-1)^{k(n-k)} s \cdot \eta \tag{4.12}$$

where s is the metric tensor $g|$ signature sign in a selected basis.

Note that the above result implies the inverse of $\star$ is given by

$$\star^{-1} : \Lambda^k(\mathbb{V}) \longrightarrow \Lambda^{n-k}(\mathbb{V}) \tag{4.13}$$

$$\eta \longrightarrow (-1)^{k(n-k)} s \star \eta \tag{4.14}$$

i.e.

$$\star^{-1} = \begin{cases} s\star & \text{if n is odd} \\ (-1)^k s\star & \text{if n is even} \end{cases} \tag{4.15}$$

4.2.2 Co-differentials

The most important application of the Hodge dual on manifolds is to define the co-differentials, see also section .3 of the appendix.

Definition 4.2.2.1 The co-differential is the map

$$\delta_k : \ \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$$
$$\eta \longrightarrow (-1)^{n(k+1)+1} s \star d \star \eta = (-1)^k \star^{-1} d \star \eta$$

with s=1 in the case of Riemannian manifolds.

Proposition 4.2.2.1 The co-differential is the adjoint of the exterior derivative. If $\zeta \in \Omega^{k+1}(M), \eta \in \Omega^k(M)$, then

$$(\eta, \delta\zeta) = (d\eta, \zeta) \tag{4.16}$$

Proof

It simply follows from Stokes' theorem for smooth forms, when M has empty boundary or when η or $\star\zeta$ have zero boundary values.

$$\int_M d(\eta \wedge \star\zeta) = 0 = \int_M (d\eta \wedge \star\zeta - \eta \wedge star(-1)^{k+1} \star^{-1} d \star \zeta) = (\eta, \delta\zeta) = (d\eta, \zeta)$$

Q.E.D.

Moreover, since $d^2 = 0$, the co-differential has the corresponding property that

$$\delta^2 = s^2 \star d \star \star d \star = (-1)^{k(n-k)} s^3 \star d^2 \star = 0$$

The Laplace-de Rham operator

Another useful operator, that lies at the heart of Hodge theory, is the Laplace-de Rham operator, defined as

$$\Delta = (\delta + d)^2 = \delta d + d\delta$$

It is possible to show it is symmetric and non-negative. Furthermore, it is closed on the class of harmonic forms and it induces an isomorphism of co-homology groups

$$\star : \; H^k_\Delta(M) \longrightarrow \; H^{n-k}_\Delta(M)$$

which, via Poincaré duality, gives an identification of $H^k(M)$ with its dual.

4.2.3 Derivatives in 3D

Now, we use the exterior derivative d and the Hodge star operator $\star$ to generate the classical differential operators on the Euclidean space $\mathbb{E}^3$.

- GRADIENT

  Consider a function, namely a 0-form, $\omega = f(x, y, z)$, then

$$d\omega = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \nabla \cdot f \tag{4.17}$$

- CURL

  Let $\eta = Adx + Bdy + Cdz$ a 1-form, we have

$$d\eta = (\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z})dy \wedge dz + (\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z})dx \wedge dz + (\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})dx \wedge dy$$

$$\star d\eta = (\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z})dx + (\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z})dy + (\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})dz = \nabla \times \eta \tag{4.18}$$

- DIVERGENCE

  With the previous definition of the 1-form η,

$$\star\eta = \star(Adx + Bdy + Cdz) = Ady \wedge dz - Bdx \wedge dz + Cdx \wedge dy$$
$$d \star \eta = (\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z})dx \wedge dy \wedge dz$$

$$\star d \star \eta = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = \nabla \cdot \eta \tag{4.19}$$

- LAPLACIAN

  If $\omega = f(x, y, z)$ is our 0-form, then

$$\Delta\omega = \nabla \cdot (\nabla f) = \star d \star d\omega = (\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}) \tag{4.20}$$

4.3 A Totally Differential Form of Maxwell's equations

At this point, we have all the mathematical tools we need to give Maxwell's equations the most elegant and useful form we can: the differential one.
In the rest of this section, we will make use of rationalized LH units and Minkowski metric, where $v = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. The Euclidean space endowed with the Minskowski metric will be denoted as M.

Definition 4.3.0.1 On the contra-variant tensor already defined in (3.2.4), we set $\star F \ \in \Lambda^2(M)$ to be the contraction

$$(\star F)_{\mu\nu} = \frac{1}{2} \cdot F^{\alpha\beta} v_{\alpha\beta\mu\nu} \tag{4.21}$$

where we have followed the index notation (4.8).

Definition 4.3.0.2 If $\underline{J}$ is the four-vector (3.3), we define the corresponding $J \ \in \Lambda^3(M)$ as:

$$(\star J)_{\mu\nu\gamma} = \frac{1}{2} \cdot J^{\alpha} v_{\alpha\mu\nu\gamma} \tag{4.22}$$

Proposition 4.3.0.1

$$F^{\mu\nu}{}_{,\nu} = J^{\mu} \quad \Longleftrightarrow \quad d \star F = \star J. \tag{4.23}$$

Moreover, $div_v F \underset{(4.19)}{=} \star d \star F = \star(\star J) = J.$

Finally, we get Maxwell's equations a brand new formulation in terms of differential forms, fully coordinate-free, namely:

$$dF = 0 \tag{4.24}$$

$$d \star F = \star J \tag{4.25}$$

Remark 4.3.0.1 Thus, the Hodge map operates on F simply exchanging the role of $\vec{E}$ and $\vec{B}$, roughly speaking. These equations can be used on every manifold you want, i.e. you can solve problems related to steady state in electrostatics, for instance, with any configuration of the conductor surface.

Remark 4.3.0.2 (Magnetic monopoles) Note that $J \in \Lambda^3(M)$ represents the electrical current density. If there were magnetic monopoles, we would have two "current" densities, namely J_e and J_m.
Thus, Maxwell's equations would look like this:

$$dF = \star J_m \tag{4.26}$$

$$d \star F = \star J_e \tag{4.27}$$

becoming, therefore, completely symmetric.

Remark 4.3.0.3 (Conservation of charge) It's a simple exercise to show that, applying the operator d on the equation (4.22), we get $\nabla \cdot \underline{J} = 0$, as we expect to have.

4.3.1 Gauge Conditions

If we apply a gauge transformation on the set of equations defined in (4.24)-(4.25), we derive:

Arbitrary Gauge

$$F = dA \tag{4.28}$$

$$d \star dA = J \tag{4.29}$$

Lorenz Gauge

Analogously, using Lorenz gauge, as we have already defined it in chapter 2, we get:

$$F = dA \tag{4.30}$$

$$\star(- \star d \star d - d \star d \star)A = \star \Box A = J \tag{4.31}$$

where $\Box$ is the so called d'Alembert-Laplace-Beltrami operator on 1-forms on arbitrary Lorentzian space-time.

Remark 4.3.1.1 The previous equations hold if and only if the second real co-homology group of the considered manifold M is 0, i.e. every closed 2-form is exact. We have therefore the topological condition

$$H^2(M) = 0 \tag{4.32}$$

which is valid for every co-homology theory we use, indeed it is always isomorphic to de Rham one.

Chapter 5

Magnetic Monopoles

In the previous chapter we have finally performed a complete transformation of Maxwell's equations into differential forms and we have already observed (see remark 4.3.0.2) that, although we have unified two apparently disjoint forces in nature, namely electric and magnetic one, as different expressions of a common entity, the electromagnetic field, the resulting equations remain still antisymmetric. It's impossible indeed to have an isolated magnetic charge, contrary to what happens for the electric field where point sources are always allowed.
The first attempt to completely symmetrize the couple

$$\begin{cases} dF &= 0 \\ d \star F &= \star J \end{cases} \tag{5.1}$$

was made by Paul Adrien Maurice Dirac in 1931, with the introduction of the Dirac monopole. At first, there was not a topological structure behind this concept, only after the work of 't Hooft(1974), Wu and Yang (1975), Polyakov (1976), Ryder (1978) et alii, it was clear the geometrical and topological frame monopoles fit is the one of fibre bundles.
We start introducing first the Dirac monopole and the related quantization condition without reference to topology, thereafter we will use the theory of co-homology and fibre bundles.

5.1 The Dirac Monopole

Suppose that there exists a magnetic monopole of strength g in the origin, i.e. $\vec{r}=0$, of a fixed reference frame, we have then

$$\nabla \cdot \vec{B} = 4\pi g\ \delta^3(\vec{r}) \tag{5.2}$$

instead of the usual one $\nabla \cdot \vec{B} = 0$.

This equation has the solution

$$\vec{B} = g \cdot \frac{\vec{r}}{r^3} \tag{5.3}$$

simply using the relations

$$\begin{cases} \Delta \cdot \left(\dfrac{1}{r}\right) & = -4\pi\delta^3(\vec{r}) \\ \nabla \cdot \left(\dfrac{1}{r}\right) & = -\dfrac{\vec{r}}{r^3} \end{cases}$$

The total magnetic flux across a sphere S of arbitrary radius R is given therefore by

$$\Phi_s(\vec{B}) = \oint_S \vec{B} \cdot d\vec{S} \overset{D.th.}{=} \int_V \nabla \cdot \vec{B}\ d\tau \overset{(5.2)}{=} \int_V 4\pi g\ \delta^3(\vec{r})\ d\tau = 4\pi g \tag{5.4}$$

which depends only on the strength g.

Consider a particle of electric charge e in the field of this monopole. It is totally described by the wave-function

$$\psi = |\psi| e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r}-Et)} \tag{5.5}$$

In the presence of an electromagnetic field we have the following transformation:

$$\psi \longrightarrow \psi' = \psi \cdot e^{-\frac{ie}{\hbar c}\vec{A}\cdot\vec{r}} \tag{5.6}$$

due to the fact that $\vec{p} \longrightarrow \vec{p} - \dfrac{e}{c} \cdot \vec{A}$ or, in terms of phase, $\alpha \longrightarrow \alpha - \dfrac{e}{\hbar c}\vec{A}\cdot\vec{r}$.

5.1.1 The Dirac string and the quantization condition

If we consider now a loop on the sphere, fixing r and θ, for $\phi \in [0, 2\pi)$, the total change in phase is

$$\Delta\alpha = \frac{e}{\hbar c}\oint \vec{A}\cdot d\vec{s} = \frac{e}{\hbar c}\int \nabla\times\vec{A}\cdot d\vec{S} = \frac{e}{\hbar c}\int \vec{B}\cdot d\vec{S} = \frac{e}{\hbar c}\Phi(r,\theta) \qquad (5.7)$$

where $\Phi(r,\theta)$ is the flux across the portion of sphere bounded by the selected loop (shaded area in the figure).

Thus, we have the following relations:

$$\lim_{\theta\to 0}\Phi(r,\theta) = 0 \qquad\qquad \lim_{\theta\to\pi}\Phi(r,\theta) = 4\pi g \qquad (5.8)$$

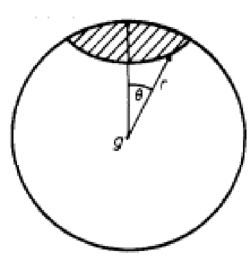


Figure 5.1: Region bounded by the selected loop

because, as θ becomes smaller and smaller, the region bounded by the loop shrinks to a point, so the flux through it is clearly zero. When the angle approaches π, conversely, all the sphere is included, so the flux must coincide with the total one in (5.4). However, for $\theta \to \pi$ the loop shrunk again to a point so the flux must be zero, contradicting the previous result.

On the negative z axis the potential vector $\vec{A}$ has therefore a singularity, indeed we can use the same argument for all possible spheres of every radius. This is known as the Dirac string.

Remark 5.1.1.1 Clearly, it is possible to define the string along any direction, just performing a change in coordinates. In fact, it must be continuous, not necessarily straight.

The singularity in $\vec{A}$ gives rise to the so called Dirac veto: the wave-function should vanish along $\{z < 0\}$. Consequently, there is no necessity that the total change in phase, should be such that

$$\lim_{\theta\to\pi}\Delta\alpha = 0$$

thus, it is indeterminate there. However, to be well defined, we must require the wave-equation ψ to be single-valued, which implies that:

$$\Delta\alpha \underset{(5.7)}{=} \frac{e}{\hbar c} \cdot \lim_{\theta \to \pi} \Phi(r, \theta)$$
$$2\pi n \underset{(5.8)}{=} \frac{e}{\hbar c} \cdot 4\pi g$$

$$eg = \frac{1}{2} \cdot n\hbar c$$

namely,

$$g = \frac{n\hbar c}{2e} \qquad \forall n \in \mathbb{Z} \tag{5.9}$$

called Dirac quantization condition.

Remark 5.1.1.2 Quantum mechanics does not predict that a magnetic monopole must exist but, if it is ever found, the charge must be quantized in terms of e, c and $\hbar$. By the same token, it follows that also the electric charges are quantized and indeed it is. Moreover, the smallest magnetic charge possible is

$$g = \frac{\hbar c}{2|e|} \simeq (\frac{137}{2}) \cdot |e|$$

5.1.2 Wu-Yang potentials

As seen above, we have discovered a singularity in the vector potential $\vec{A}$, the so-called Dirac string. If we consider the pole as end point of a string of magnetic dipoles whose other end is at infinity, as Dirac himself did in 1931, we have (Wentzel 1966)

$$\vec{A} := \left(\frac{-gy}{r(r+z)}, \frac{gx}{r(r+z)}, 0 \right) \tag{5.10}$$

or, in polar coordinates (r, θ, ϕ),

$$\vec{A} := \left(0, 0, \frac{g\ (1 - cos(\theta))}{rsin(\theta)} \right) \tag{5.11}$$

then

$$\nabla \times \vec{A} = \vec{B} = g \cdot \frac{\vec{r}}{r^3} + 4\pi g\ \delta(x)\delta(y)\theta(-z) \tag{5.12}$$

namely there is a singularity that arises along the negative z-axis($\theta = \pi$), as we have required

If we define another vector potential, for instance

$$\vec{A} := \left(\frac{gy}{r(r-z)}, \frac{-gx}{r(r-z)}, 0\right) \tag{5.13}$$

or, in polar coordinates,

$$\vec{A} := \left(0, 0, -\frac{g\ (1+cos(\theta))}{rsin(\theta)}\right) \tag{5.14}$$

this time the Dirac string is along the positive z-axis ($\theta = 0$).

Thus, if we claim that a magnetic monopole exists, the vector potential has a singularity that we cannot eliminate.

How can we solve this problem? In 1975, Wu and Yang proposed essentially to split the surface surrounding the monopole, the sphere, essentially, up into two overlapping regions, define a potential vector on each of them and then use a gauge to have well-defined objects also on the intersection.

Consider thus two overlapping regions on the sphere of figure 5.1:

a) R_a, which excludes the S pole (i.e. the negative z-axis);

b) R_b, the northern hemisphere without the pole (i.e. the positive z axis)

and in each of them define the vector potential as:

a)
$$\vec{A}^a := \left(0, 0, \frac{g\ (1-cos(\theta))}{rsin(\theta)}\right) \tag{5.15}$$

b)
$$\vec{A}^b := \left(0, 0, -\frac{g\ (1+cos(\theta))}{rsin(\theta)}\right) \tag{5.16}$$

where the upper index denotes the region we are considering. By definition, it's easy to see they are both finite in their own domain.

In the intersection $R_a \cap R_b$, on the equator if you want, they are related by a gauge transformation

$$A^b_\mu = A^a_\mu - \frac{i\hbar c}{e} \cdot S \, \frac{\partial S^{-1}}{\partial x^\mu} \tag{5.17}$$

where $S = S_{ab}$ as to be a single-valued function.

If we make then use of (5.15),(5.16), it is clear that

$$S = e^{\frac{2ige}{\hbar c} \cdot \phi} \tag{5.18}$$

satisfies equations (5.17) and the quantization condition in (5.9).

In the end, if we calculate the total flux in $R_a \cup R_b := S^2$, using the Stokes' theorem, we get:

$$\Phi_{S^2} = \int_{S^2} F_{\mu\nu} \cdot dx^{\mu\nu} = \oint \nabla \cdot \vec{A} \cdot d\vec{S} = \int_{R_a} \nabla \times \vec{A}^a \cdot d\vec{S} + \int_{R_b} \nabla \times \vec{A}^b \cdot d\vec{S} =$$

$$= \oint_{\theta=\pi/2} \vec{A}^a \cdot d\vec{l}^a + \oint_{\theta=\pi/2} \vec{A}^b \cdot d\vec{l}^b \underset{orient.}{=} \oint_{\theta=\pi/2} \vec{A}^a \cdot d\vec{l}^a - \oint_{\theta=\pi/2} \vec{A}^b \cdot d\vec{l}^a =$$

$$\underset{(5.17)}{=} \frac{i\hbar c}{e} \oint \frac{d}{d\phi} \ln(S^{-1}) d\phi \underset{(5.18)}{=} 4\pi g$$

in agreement with the equation (5.4).

The Wu-Yang potentials then define a magnetic monopole of strength g.

Remark 5.1.2.1 The equation above can be viewed also as a simple consequence of imposing the gauge general transformation directly in the Schrödinger equation, as it is done in Nakahara's book:

$$\frac{1}{2m} \left(\vec{p} - \frac{e}{c} \cdot \vec{A}\right)^2 \cdot \psi(\vec{r}) = E \cdot \psi(\vec{r})$$

If

$$\vec{A} \longrightarrow \vec{A} + \nabla \cdot \Lambda$$

then the wave-function changes according to

$$\psi \longrightarrow \psi \cdot e^{\dfrac{ie\Lambda}{\hbar c}}$$

In the present case, we have therefore

$$\vec{A}^a - \vec{A}^b = \nabla \cdot (2g\phi) \tag{5.19}$$

that implies directly

$$\Phi_{S^2} = \oint_{\theta=\pi/2} \vec{A}^a \cdot d\vec{l}^a - \oint_{\theta=\pi/2} \vec{A}^b \cdot d\vec{l}^a = \int_0^{2\pi} 2g \cdot d\phi = 4\pi g \tag{5.20}$$

The gauge function Λ chosen in the Wu-Yang treatment of the Dirac monopole is then

$$\Lambda = 2g\phi \tag{5.21}$$

5.1.3 Homology groups and the Hopf map

In our case the space we deal with is in fact a sphere so the curvature K is greater than zero. Till we have an Euclidean space, we can use Stokes' theorem to obtain the above results, as indeed we did, but, to include also curved spaces, we have to generalize the theory just using de Rham co-homology groups.

In the appendix, it is shown that for the spheres the following holds:

$$H^k(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{k=0, n} \\ \{0\} & \text{otherwise} \end{cases}$$

Remark 5.1.3.1 In the appendix we have done it for the continuum case, obtaining in fact

$$H^k(\mathbb{S}^n) = \ \mathbb{R} \text{ if k=0, n,} \quad \{0\}\,\text{otherwise}$$

but here we need the integer co-homology because the quantities involved are always quantized, i.e. the wave-function has to be single-valued.

Thus, in particular, if we consider the second integer co-homology group of the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, which is our case of interest, we find out that it is non-trivial, i.e. on $\mathbb{S}^2$

$$d\omega_2 = 0 \not\Rightarrow \omega_2 = d\omega_1$$

every 2-form is closed but this not implies that it is also exact.

Loosely speaking, the number of generators of the first homology group of X $H_1(X)$ is the number of inequivalent (non-homologous) closed curves in X which are not boundaries of pieces of area in X.

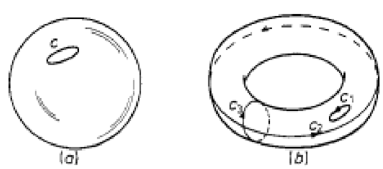


Figure 5.2: Loops on the $\mathbb{S}^2$ and on the torus $\mathbb{T}^2$

If we take a Torus $\mathbb{T}^2$ for instance,

$$H_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$$

in fact, looking at the figure (b) the major and the minor circumferences c_2 and c_3 do not bound areas on the torus, whereas on the sphere $\mathbb{S}^2$ in (a) every closed curve c encloses a piece of area, accordingly

$$H_1(\mathbb{S}^2) = \{0\}$$

If ω is a p-form and c is a (p+1)-chain with boundary δc, we have then the generalized Stokes' theorem

$$\int_{\delta c} \omega = \int_c d\omega$$

The Hopf map

Let the spheres $\mathbb{S}^3 \subset \mathbb{R}^4$ and $\mathbb{S}^2 \subset \mathbb{R}^3$ be parametrized as follows:

$$\mathbb{S}^3 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}$$

$$\mathbb{S}^2 = \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \ : \ \xi_1^2 + \xi_2^2 + \xi_3^2 = 1 \right\}$$

and consider the following change of variables in the complex plain

$$\begin{cases} z_0 & = x_1 + ix_2 \\ z_1 & = x_3 + ix_4 \end{cases} \quad \Longrightarrow \quad \left\{ (z_0, z_1) \ : \ |z_0|^2 + |z_1|^2 = 1 \right\} = \mathbb{S}^3 \tag{5.22}$$

In 1931 Hopf proposed an application that maps $\mathbb{S}^3$ onto $\mathbb{S}^2$ as follows

$$\begin{aligned} & f_H : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \\ & (x_1, x_2, x_3, x_4) \longrightarrow (\xi_1, \xi_2, \xi_3) := \begin{cases} \xi_1 = 2(x_1 x_3 + x_2 x_4) \\ \xi_2 = 2(x_2 x_3 - x_1 x_4) \\ \xi_3 = x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{cases} \end{aligned} \tag{5.23}$$

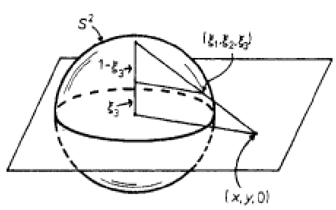


Figure 5.3: Stereographic projection

Alternatively, the 2-sphere can be parametrized using the stereographic projection onto the plane (x, y),as in figure 5.3.

In complex coordinates, z= x + iy, we have therefore that

$$z = \frac{\xi_1 + i\xi_2}{1 - \xi_3} \underset{(5.23)}{=} \frac{x_1 + ix_2}{x_3 + ix_4} \underset{(5.22)}{=} \frac{z_0}{z_1} \tag{5.24}$$

The coordinates z_0 and z_1 are now completely determined by z.

Moreover, if we replace them by $(\lambda z_0, \lambda z_1)$ such that

$$|\lambda z_0|^2 + |\lambda z_1|^2 = |\lambda|^2 = 1 \tag{5.25}$$

z is left unchanged and there exists thus an $\alpha : \ \lambda = e^{i\alpha}$ is the generator of $\mathbb{S}^1 \cong U(1)$.

Locally we have therefore the isomorphism

$$\mathbb{S}^3 \cong \mathbb{S}^2 \times \mathbb{S}^1 \tag{5.26}$$

Remark 5.1.3.2 Globally this is not true at all! Remind that the following holds:

$$H_1(\mathbb{S}^3) = \{0\} \qquad\qquad H_1(\mathbb{S}^2 \times \mathbb{S}^1) = \mathbb{Z} \tag{5.27}$$

From equation (5.25) we see that the Hopf map f_H maps a 1-cycle ($\sim \mathbb{S}^1$) in $\mathbb{S}^3$ onto a point in $\mathbb{S}^2$, as shown in the figure (5.4). Thus, $\mathbb{S}^3$ is a fibre bundle with base space $\mathbb{S}^2$ and fibre $\mathbb{S}^1$.

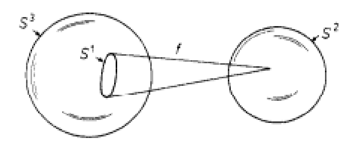


Figure 5.4: Action of the Hopf map (projection)

Therefore the 3-sphere has got, thanks to the continuous surjection f_H, a partition in great circles called Hopf fibration.

Observe that, considering classical linear groups (see Artin M., Algebra, ch.8 for further details), we have $\mathbb{S}^3$ is the group space of the special unitary group $SU(2)$. Since the latter is the group

$$SU(2) := \{P \in M_2(\mathbb{C}) : P^* \cdot P = 1, \det(P) = 1\} \tag{5.28}$$

if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, imposing the conditions $P^{-1} = P^*$ and $\det(P) = 1$, we get respectively:

$$\bar{a} = d,\ \bar{b} = -c \tag{5.29}$$

and

$$\bar{a}a + b\bar{b} = 1 \tag{5.30}$$

that, passing in polar coordinates, give rise to the isomorphism with $\mathbb{S}^3$, as we claim.

Parametrizing the 3-sphere by the Euler angles ψ, θ, ϕ, the $SU(2)$ element corresponding to an arbitrary rotation is

$$U = \begin{bmatrix} cos\left(\frac{\theta}{2}\right) \cdot e^{\dfrac{i(\psi+\phi)}{2}} & i \cdot sin\left(\frac{\theta}{2}\right) \cdot e^{\dfrac{i(\psi-\phi)}{2}} \\ i \cdot sin\left(\frac{\theta}{2}\right) \cdot e^{\dfrac{i(\psi-\phi)}{2}} & cos\left(\frac{\theta}{2}\right) \cdot e^{\dfrac{-i(\psi-\phi)}{2}} \end{bmatrix} = \begin{bmatrix} z_0 & iz_1^* \\ iz_1 & z_0^* \end{bmatrix} \tag{5.31}$$

If we transform now according to

$$\begin{cases} z_0 &= x_1 + ix_2 \\ z_1 &= x_3 + ix_4 \end{cases}$$

we have then

$$x_1 = cos\left(\frac{\theta}{2}\right) \cdot cos\left(\frac{\psi+\phi}{2}\right) \qquad x_2 = cos\left(\frac{\theta}{2}\right) \cdot sin\left(\frac{\psi+\phi}{2}\right) \tag{5.32}$$

$$x_3 = sin\left(\frac{\theta}{2}\right) \cdot cos\left(\frac{\psi-\phi}{2}\right) \qquad x_4 = sin\left(\frac{\theta}{2}\right) \cdot sin\left(\frac{\psi-\phi}{2}\right) \tag{5.33}$$

and, using the Hopf map in (5.23),

$$(\xi_1, \xi_2, \xi_3) := \begin{cases} \xi_1 = cos(\phi) \cdot sin(\theta) \\ \xi_2 = sin(\phi) \cdot sin(\theta) \\ \xi_3 = cos(\theta) \end{cases} \tag{5.34}$$

so that (ϕ, θ) may be identified with the polar angles on $\mathbb{S}^2$ and ψ is the angle of the $\mathbb{S}^1$ fibre.

Finally, if we consider the sections of the 3-sphere corresponding to the great circles

$\psi = \phi,\ \psi = -\phi$, (5.34) gives, respectively:

$$\begin{cases} z_0 & = e^{i\phi} \cdot cos\left(\frac{\theta}{2}\right) \\ z_1 & = sin\left(\frac{\theta}{2}\right) \end{cases} = \begin{cases} z_0 & = \frac{z}{\sqrt{1+|z|^2}} \\ z_1 & = \frac{1}{\sqrt{1+|z|^2}} \end{cases} \tag{5.35}$$

and

$$\begin{cases} z_0 & = cos\left(\frac{\theta}{2}\right) \\ z_1 & = e^{-i\phi} \cdot sin\left(\frac{\theta}{2}\right) \end{cases} = \begin{cases} z_0 & = \frac{1}{\sqrt{1+|w|^2}} \\ z_1 & = \frac{w}{\sqrt{1+|w|^2}} \end{cases} \tag{5.36}$$

where $w = 1/z$.

Equation (5.35) maps $\mathbb{S}^2 \setminus \infty$ (N pole) into $\mathbb{S}^3$, on the other hand, equation (5.36) maps $\mathbb{S}^2 \setminus \{0\}$ (S pole) into $\mathbb{S}^3$. The fact that these maps remain distinct on the overlapping region is another indication that, globally, $\mathbb{S}^3 \ncong \mathbb{S}^2 \times \mathbb{S}^1$.
Moreover, if we define the area 2-form as $\sigma_2 := sin(\theta) d\theta \wedge d\phi$ we then have the area of the unitary 2-sphere is indeed:

$$A(\mathbb{S}^2) = \int_{\mathbb{S}^2} \sigma_2 = 4\pi \tag{5.37}$$

This form is closed but non exact. If there were exact, using Stokes' theorem, we would have:

$$A(\mathbb{S}^2) = \int_{\mathbb{S}^2} \sigma_2 = \int_{\mathbb{S}^2} d\sigma_1 = \int_{\delta\mathbb{S}^2} \sigma_1 = 0 \tag{5.38}$$

since $\delta\mathbb{S}^2 = 0$, and it is clearly an absurd.
However, when viewed in $\mathbb{S}^3$ coordinates it is exact because the second co-homology group, as the homology one, is trivial, i.e.

$$H^2(\mathbb{S}^3) = \{0\} \quad \Rightarrow \quad \sigma_2 = d\sigma_1 \qquad \text{on } \mathbb{S}^3 \tag{5.39}$$

This equation will play a crucial role in what follows.

5.1.4 Geometrical meaning of Wu-Yang potentials and quantization condition

We are now in position to give a topological foundation of magnetic monopoles. The relevant 2-form here, from (5.3)-(5.4), is given by

$$B = g \cdot \sigma_2 = g \cdot sin(\theta) d\theta \wedge d\phi \tag{5.40}$$

where σ_2 denotes the area 2-form as already defined in the previous paragraph.
If we calculate thus the flux, we have

$$\Phi = \int_{\mathbb{S}^2} B = 4\pi g \tag{5.41}$$

since this form is closed but it is not exact. This implies that we cannot define a global vector potential on the 2-sphere such that $\nabla \times \vec{A} = \vec{B}$.
To get this obstacle around, we consider B as a 2-form on the 3- sphere and we obtain its exactness,(5.39), namely there exists a 1-form $A : B = dA$ defined as follows:

$$A := 2g \cdot (x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4) \overset{(5.32/33)}{=} -g \cdot (d\psi + cos(\theta) d\phi) \tag{5.42}$$

whose exterior derivative gives

$$B = dA = 4g \cdot (dx_2 \wedge dx_1 + dx_4 \wedge dx_3) = g \cdot sin(\theta) d\theta \wedge d\phi \tag{5.43}$$

as desired.
Taking now sections 1) $\psi = \phi$ and 2) $\psi = -\phi$, we obtain the following expressions:

1. $A^a = g \cdot (1 - cos(\theta)) d\phi = \frac{g}{r} \cdot \left(\frac{1 - cos(\theta)}{sin(\theta)} \right) \cdot r sin(\theta) d\phi$
2. $A^b = -g \cdot (1 + cos(\theta)) d\phi = \frac{g}{r} \cdot \left(\frac{1 + cos(\theta)}{sin(\theta)} \right) \cdot r sin(\theta) d\phi$

where we have used the 1-form $A = A_\mu dx^\mu$.

In other terms, we get the Wu-Yang potentials already defined in (5.15)-(5.16) as sections in $\mathbb{S}^3$. The construction summarized in figure 5.5.

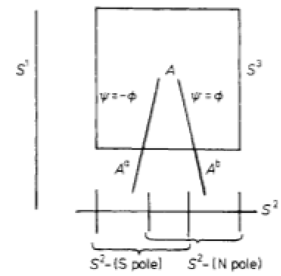


Figure 5.5: Derivation of the Wu-Yang potentials by taking sections $\psi = \pm\phi$ in $\mathbb{S}^3$

It's possible to give also a fully geometrical interpretation of the quantization condition using the Gauss- Bonnet theorem:
The Gaussian curvature K over a closed surface M (2D) is related to the Euler characteristic $\chi(M)$ as follows

$$\frac{1}{2\pi}\int_M KdM = \chi(M) \tag{5.44}$$

We can generalize this statement to the case of vector fields defined on spheres when a connection form is defined.

Affine connection and Riemann tensor

Now, we want define, in some sense, a notion of parallelism on every manifold. Indeed, if we consider a sphere and two great circle passing through the poles, we can transport the same vector V , parallel to the sphere and normal to the circles, and get two anti-parallel resulting vectors. Thus, on a surface as simple as the sphere is impossible to have a global notion of "Euclidean" parallelism. The only thing one can do, in order to perform a kind of parallelism, is to move a vector along a curve without changing its direction, is, therefore, to realize what is called a parallel transport.

Definition 5.1.4.1 An affine connection on a manifold M is a rule for parallel transport.

This tool adds an additional structure to M, it gives shape and curvature remaining completely independent on metric or volume forms.

Definition 5.1.4.2 We say that a vector $\vec{w}$ is parallel-transported along the curve γ iff it's covariant derivative $\nabla_t(\vec{w}) = 0$.

Definition 5.1.4.3 We call geodesic a curve γ that parallel-transports its own tangent vector.

Definition 5.1.4.4 We define Riemann-tensor the commutator of covariant derivatives $[\nabla_\lambda, \nabla_\mu] := \nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda$.

We set as axiom for a flat space the Euclidean axiom of parallel lines, so it's natural to extend this concept using the Riemann-tensor, which measures the curvature of a manifold with a connection.

Definition 5.1.4.5 A space is flat iff the Riemann-tensor vanishes everywhere on it.

Remark 5.1.4.1 It's a remarkable observation that on a flat space M we have a global notion of parallelism, then it's fiber bundle is trivial, i.e. M is parallelizable. Moreover, Minkowski space is just as flat as the Euclidean space.

Coming back to our quantization condition, if we make the transformation

$$\vec{p} \longrightarrow \vec{p} - \frac{e}{c} \cdot \vec{A} \tag{5.45}$$

also the covariant derivative varies as

$$\nabla \longrightarrow \nabla - \frac{ie}{\hbar c} \cdot \vec{A} \tag{5.46}$$

this identifies the connection form as

$$\omega = -\frac{ie}{\hbar c} A \tag{5.47}$$

and the curvature form Ω as

$$\Omega = d\omega - \omega \wedge \omega \overset{(5.42)}{=} d\omega = -\frac{ie}{\hbar c} dA \overset{(5.43)}{=} -\frac{ie}{\hbar c} \cdot g \cdot sin(\theta) d\theta \wedge d\phi \tag{5.48}$$

If we define

$$det\left[1+\left(\frac{1}{2\pi}\right)\cdot\Omega\right]:=1+\omega_1+\omega_2+...+\omega_n$$

and in our case, since Ω is not a matrix, we simply have

$$\omega_1=\left(\frac{1}{2\pi}\right)\cdot\Omega=\frac{eg}{2\pi\hbar c}\cdot sin(\theta)d\theta\wedge d\phi \tag{5.49}$$

the Gauss-Bonnet theorem states that each form ω_i defines a co-homology class whose integral is given by (in our case)

$$\int_{\mathbb{S}^2}\omega_1=\frac{eg}{2\pi\hbar c}\cdot 4\pi=\chi(\mathbb{S}^2)=2 \tag{5.50}$$

(Chern 1972, Baum 1970, Drechsler and Mayer 1977)

i.e.:

$$eg=\hbar c \tag{5.51}$$

which is just the Dirac quantization condition (5.9) for n=2.

Remark 5.1.4.2 The Gauss-Bonnet-Chern theorem has yielded quantization, but the only value allowed is n=2.

In this case, we have achieved the so-called Schwinger condition.

Appendix
The de Rham co-homology theory

.1 The exterior algebra

Tensors

Definition .1.0.1 Let $\mathbb{E}$ be a finite dimensional real vector space and denote by $\mathbb{E}^*$ it's dual, we define tensor of type (p,q) a multi-linear map

$$\text{t: } \underbrace{\mathbb{E}^* \times ... \times \mathbb{E}^*}_{p\ times} \times \underbrace{\mathbb{E} \times ... \times \mathbb{E}}_{q\ times} \longrightarrow \mathbb{R}$$

The symbol $\mathbb{E}_{(p,q)}$ will indicate the real vector space of tensors.

In what follows, we will restrict our attention on tensors of type (0,p), called p-forms and, to simplify the notation, we will set $\mathbb{E}_p := \mathbb{E}_{(0,p)}$.
Being a vector space, we can add and multiply by a scalar every p-form, but we can do more, we can equip it with a product, called tensor product.

Definition .1.0.2 If $\omega \in \mathbb{E}_p$ and $\tau \in \mathbb{E}_q$, the tensor product $\omega \otimes \tau$ belongs to $\mathbb{E}_{p+q}$ and it's defined as

$$\omega \otimes \tau \ := \omega(x_1, ..., x_p)\tau(x_1, ..., x_q).$$

Moreover, if we set $\mathbb{E}_* := \oplus \mathbb{E}_p$ the direct sum of p-forms spaces, this product provides an associative algebra structure and the couple $(\mathbb{E}_*, \otimes)$ is then called tensor algebra.

We are now ready to introduce two special classes of p-forms.

Definition .1.0.3 Let $\Sigma(p)$ be the group of permutations on p elements and $\omega \in \mathbb{E}_p$, we say that:

- ω is a symmetric form and we write $\omega \in \Sigma^p(\mathbb{E})$, if

$$\omega(x_1, ..., x_p) \ = \ \omega(x_{\sigma(1)}, ..., x_{\sigma(p)}) \qquad \forall \sigma \in \Sigma(p);$$

- ω is an exterior form (or alternating tensor) and we write $\omega \in \Lambda^p(\mathbb{E})$, if

$$\omega(x_1, ..., x_p) \; = \; |\sigma| \cdot \omega(x_{\sigma(1)}, ..., x_{\sigma(p)}) \qquad \forall \sigma \in \Sigma(p).$$

Remark .1.0.1 Observe that:

1) $\Lambda^p(\mathbb{E})$ and $\Sigma^p(\mathbb{E})$ are both subspaces of $\mathbb{E}_p$;

2) $\Lambda^0(\mathbb{E}) = \Sigma^0(\mathbb{E}) = \mathbb{R}$;

3) $\Lambda^1(\mathbb{E}) = \Sigma^1(\mathbb{E}) = \mathbb{E}_1 = \mathbb{E}^*$.

Proposition .1.0.1 If $\{\omega_1, ..., \omega_n\}$ is a basis for $\mathbb{E}_1$, then the set of p-forms $\mathcal{E} := \big\{\omega_{i1} \otimes ... \otimes \omega_{ip} : i_1, ..., i_p \in \{1, ..., n\}\big\}$ is a basis of $\mathbb{E}_p$.

Remark .1.0.2 (Determinants) Let $\{e_1, ..., e_n\}$ be a fixed basis for $\mathbb{E}$ and $\{\phi_1, ..., \phi_n\}$ the dual basis, fix indexes $1 \leq i_1 < .... < i_p \leq n$ and define $\omega \in \Lambda^p(\mathbb{E})$ as

$$\omega_{(i_1,...,i_p)}(x_1, ..., x_p) \; := \; \det \phi_{i_j}(x_k).$$

All the forms of this kind are actually exterior ones; indeed, the determinant is multilinear in columns and it change sign after a permutation on them. In particular, they constitute a basis for $\Lambda^p(\mathbb{E})$.
The above argument shows that p-forms are essentially determinants of p x p matrices, therefore p-dimensional volume elements. They are the natural integrands of the multiple oriented integrals.

Wedge product

Having in mind this goal, the search for a primitive in brief, we would like the tensor product to be an internal operation on exterior forms, but this is not true in general. We have then to elaborate a little trick, namely a map, that "alternates" the tensor product in order to have an exterior form.

Definition .1.0.4 We call alternating linear operator the map

$$A : \mathbb{E}_p \to \Lambda^p(\mathbb{E}) \qquad A(\tau)(x_1, ..., x_p) \; := \; \tfrac{1}{p!} \textstyle\sum_{\pi \in \Sigma(p)} |\pi| \cdot \tau(x_{\pi(1)}, ..., x_{\pi(p)})$$

Proposition .1.0.2 Let A be the operator just defined, then:

1) If $\tau \in \mathbb{E}_p$, $\quad A(\tau) \in \Lambda^p(\mathbb{E})$;

2) If $\tau \in \Lambda^p(\mathbb{E})$, $\quad A(\tau) = \tau \;\; and \;\; A^2 = A$.

Definition .1.0.5 We define the exterior or wedge product as the map:

$$\wedge : \; \Lambda^p(\mathbb{E}) \times \Lambda^q(\mathbb{E}) \;\to\; \Lambda^{p+q}(\mathbb{E}) \qquad \wedge(\omega,\tau) := \omega \wedge \tau = \frac{(p+q)!}{p!q!} A(\omega \otimes \tau)$$

This product is associative and distributive so, suitably extended, it defines an algebra structure on $\Lambda^*(\mathbb{E}) := \oplus \Lambda^p(\mathbb{E})$. The couple $(\Lambda^*(\mathbb{E}), \; \wedge)$ is called exterior algebra.

Remark .1.0.3 It's an easy exercise to show that the rational coefficient in the wedge product definition is needed for the unit cube volume to be one.

Proposition .1.0.3 Let $\phi_i \; \in \; (\mathbb{E}_1), \; x_j \in \; \mathbb{E} \; for \; i,j \in \; \{1,...,p\}$, then:

$$\phi_1 \wedge .... \wedge \phi_p(x_1,...,x_p) = \det[\phi_i(x_j)].$$

Moreover, if $\sigma \in \Sigma(p)$, then $\phi_1 \wedge ... \wedge \phi_p = |\sigma| \cdot \phi_{\sigma(1)} \wedge \; ... \; \wedge \phi_{\sigma(p)}$.

Proposition .1.0.4 Let $\{\phi_1,...,\phi_n\}$ be a basis for $\mathbb{E}_1$, then

$$\left\{\phi_{i_1} \;\wedge ... \wedge\; \phi_{i_p} : 1 \leq i_1 < ... < i_p \leq n\right\}$$

is a basis of $\Lambda^p(\mathbb{E})$.
Thus, $\Lambda^p(\mathbb{E})$ has dimension $\binom{n}{p}$ and $\Lambda^p(\mathbb{E}) = \{0\}$, if $p > n$.

Corollary .1.0.1 The algebra $\Lambda^*(\mathbb{E})$ is a graded commutative algebra. Namely, if $\omega \in \Lambda^p(\mathbb{E}), \; \tau \in \Lambda^q(\mathbb{E})$, then

$$\omega \;\wedge\; \tau = (-1)^{pq} \tau \wedge \omega.$$

In particular, the square of an odd degree form is zero.

Operators on p-forms

Let $L : \mathbb{E} \to \mathbb{F}$ be a linear map and consider the transpose of L, i.e.

$$L^* : \mathbb{F}_1 \to \mathbb{E}_1 \quad L^*(\phi)(x) := \phi(Lx).$$

This map extends to a linear map

$$\mathbb{E}_p(L) : \mathbb{F}_p \;\rightarrow\; \mathbb{E}_p$$
$$\mathbb{E}_p(L)(\omega)(x_1, ..., x_p) = \omega(L(x_1), ..., L(x_p))$$

which is closed with respect to the exterior p-forms class, i.e. if $\omega \in \Lambda^p(\mathbb{F})$, then $\mathbb{E}_p(L)(\omega) \in \Lambda^p(\mathbb{E})$.

By restriction, we get therefore a linear map

$$\Lambda^p(L) := \; \mathbb{E}_p(L)|_{\Lambda^p(\mathbb{F})} : \; \Lambda^p(\mathbb{F}) \;\rightarrow\; \Lambda^p(\mathbb{E}),$$

and, by additivity,

$$\Lambda^*(L) : \Lambda^*(\mathbb{F}) \rightarrow \Lambda^*(\mathbb{E}).$$

Proposition .1.0.5 The map L induces a graded algebra homomorphism, in fact $\Lambda^*(\omega \wedge \tau) = \Lambda^*(\omega) \wedge \Lambda^*(\tau)$ and it's a contra-variant functor.

Observe that, thanks to the functorial properties, if L is an isomorphism also L^* is.

Measuring using forms

We want now to equip $\mathbb{E}$ with some structure that will allow us to do measurements on it. Let's construct the theory.

Definition .1.0.6 Let $\mathbb{E}$ be a real vector space with scalar product $\langle \,\cdot\, , \,\cdot\, \rangle : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}$, the isomorphism:

$\flat : \mathbb{E} \rightarrow \mathbb{E}^* \qquad \flat(x)(y) := \langle x, y \rangle$

$\sharp : \mathbb{E}^* \rightarrow \mathbb{E} \qquad \sharp := \flat^{-1}$

are called the musical isomorphism.

Remark .1.0.4 (Orientability) Remind that an orientation on $\mathbb{E}$ is the choice of an equivalence class of the relation $E \sim F \iff |M_{EF}| > 0$, where E and F are two chosen bases for $\mathbb{E}$ and M_{EF} is the basis change matrix.

$\mathbb{E}$ is said to be oriented if such a choice has been made and we call positive the selected basis. Obviously, fixing a basis on $\mathbb{E}$, we induce an orientation also on its dual, simply by declaring positive the dual basis.

Moreover, if E and F are orthonormal and positive, M_{EF} is $\in SO(n)$ i.e. $|M_{EF}| = 1$.

Definition .1.0.7 If $\mathbb{E}$ is an n-dimensional, positive oriented, inner product space and $\{\omega_1, ..., \omega_n\}$ is the dual positive orthonormal basis, the volume form of $\mathbb{E}$ is defined as the n-form

$$v := \omega_1 \wedge ... \wedge \omega_n.$$

Remark .1.0.5 The volume form is well defined, i.e. it does not depend on the choice of the basis (easy exercise).

Definition .1.0.8 Let $\mathbb{E}$ be a n-dimensional oriented inner product space. We define the Hodge operator as follows:

$$\star_p : \Lambda^p(\mathbb{E}) \to \Lambda^{n-p}(\mathbb{E}) \qquad \star_p\,(\eta)(x_1, ..., x_{n-p}) := \langle \eta \wedge \flat(x_1) \wedge ... \wedge \flat(x_{n-p}), v \rangle$$

where v is the volume form.

It may be also defined by extending linearly the map:

$$\star(\omega_{i_1} \wedge ... \wedge \omega_{i_p}) = \omega_{j_1} \wedge ... \wedge \omega_{j_{n-p}},$$

where $\{i_1, ..., i_p, j_1, ..., j_{n-p}\}$ is an even permutation of $\{1, ..., n\}$.

Proposition .1.0.6 $\star$ is a linear isometry and $\star_{n-p} \circ \star_p = (-1)^{p(n-p)} \mathbf{1}_{\Lambda^p(\mathbb{E})}$.

.2 The co-homology groups of an open set

Differential forms

Definition .2.0.1 We define a differential p-form as a smooth map

$$\omega : U \subseteq \mathbb{R}^n \ \rightarrow \ \Lambda^p(\mathbb{R}^n) \cong \mathbb{R}^{\binom{n}{p}}$$

or better, allowing the case of smooth manifolds, as a map which assigns at each point in the open set U a pointed vector in its tangent plane.
Thus, consider the bundle of exterior p-forms

$$\Lambda^p(U) := \ \bigcup_{x \in U} \Lambda^p(T_x U) \cong U \times \Lambda^p(\mathbb{R}^n)$$

and define

$$\tilde{\omega} : U \rightarrow \Lambda^p(U)$$
$$\tilde{\omega}(x) \in \Lambda^p(T_x(U)) \text{ i.e. } \tilde{\omega}(x) := (x,\ \omega(x)) : \omega(x) \in \Lambda^p(\mathbb{R}^n).$$

The set of differential p-forms on U is denoted by $\Omega^p(U)$ and, moreover, it has a obvious structure of module on the space $\mathcal{F}(U)$ of smooth real-valued functions defined on U.

Proposition .2.0.1 Taking $\omega \in \Omega^p(U)$, it induces a $\mathcal{F}(U)$ multi-linear map, denoted by the same symbol,

$$\omega : \mathcal{H}(U) \times ... \times \mathcal{H}(U) \ \rightarrow \ \mathcal{F}(\mathcal{U})$$
$$\omega(X_1, ..., X_p)(x) = \omega(x)(X_1(x), ..., X_p(x))$$

Theorem .2.0.1 (Tensoriality Criterion) An $\mathbb{R}$-multilinear map

$$\omega : \mathcal{H}(U) \times ... \times \mathcal{H}(U) \rightarrow \mathcal{F}(\mathcal{U})$$

is induced by a differential form $\iff$ it is $\mathcal{F}(U)$ multi-linear.

Definition .2.0.2 We define the set of derivation of $\mathcal{F}(\mathcal{U})$ as follows:
$\mathcal{D}(\mathcal{U}) := \{Y : \mathcal{F}(\mathcal{U}) \rightarrow \mathcal{F}(\mathcal{U}) \ linear : \ Y(f \cdot g) = f \cdot Y(g) + g \cdot Y(f) \ \ \forall f, g \in \mathcal{F}(\mathcal{U})\}.$

Definition .2.0.3 Let f $\in \mathcal{F}(U) = \Omega^0(U)$, then the differential of f is the 1-form

$$(\mathrm{df})(\mathrm{x})(\mathrm{X}) := \mathrm{X}(\mathrm{x})(\mathrm{f}), \text{ where } \mathrm{X} \in \mathcal{D}(\mathcal{U})$$

In particular, if we consider the i-th coordinate functions $x_i : U \subseteq \mathbb{R}^n \; open \to \mathbb{R}$, at each point $x \in U$ the differentials $dx_i(x)$ form a basis for $\Lambda^1(U)$.

Proposition .2.0.2 The set $\left\{dx_{i_1}(x) \wedge ... \wedge dx_{i_p}(x) \;\; : 1 \leq i_1 < ... < i_p \leq n\right\}$ is a basis for $\Lambda^p(\mathbb{R}^n)$. Let $\omega \in \Omega^p(U)$, it is written uniquely as

$$\omega = \textstyle\sum_{i_1<..<i_p} \omega_{i_1,...,i_p} dx_{i_1} \wedge ... \wedge dx_{i_p}, \;\; \text{where } \omega_{i_1,...,i_p} \in \mathcal{F}(\mathcal{U}).$$

Finally, if $f \in \mathcal{F}(\mathcal{U})$, we have $df = \sum_{i=1}^n \dfrac{\partial f}{\partial x_i} \cdot dx_i$.

Remark .2.0.1 Observe that $\Omega^p(U)$, as a real vector space, is infinite dimensional but, as an $\mathcal{F}(\mathcal{U}) - module$, is a free module of dimension $\binom{n}{p}$.

Operators on differential forms

Let U $\subseteq \mathbb{R}^n$, V $\subseteq \mathbb{R}^m$ be open sets and $F : U \to V$ a smooth function onto $\mathbb{R}^m$, i.e. $F(x) = (F_1(x), ..., F_m(x))$, then

$$\mathrm{dF(x)}:\mathbb{R}^n \to \mathbb{R}^m$$

is a linear map and we have, as already known from the general theory of forms, an induced map at exterior forms level

$$F^* : \Lambda^p(\mathbb{R}^m) \to \Lambda^p(\mathbb{R}^n).$$

This operator induces a linear map:

$$F^* : \Omega^p(V) \to \Omega^p(U)$$
$$F^*(\omega)(X_1, ..., X_p)(x) := \omega(dF(x)(X_1), .., dF(x)(X_p)).$$

Moreover, if $x_1, ..., x_n, \; y_1, ..., y_m$ are the canonical coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we have

$$F^*(dy_i) = \textstyle\sum_{i=1}^n \dfrac{\partial F_i}{\partial x_j} dx_j,$$

and therefore, if $\omega = \sum_{i_1,...,i_p} \omega_{i_1,...,i_p} dy_{i_1} \wedge ... \wedge dy_{i_p}$, then

$$F^*(\omega)(x) = \textstyle\sum_{i_1,...,i_p} \omega_{i_1,...,i_p}(F(x)) F^*(dy_{i_1}) \wedge ... \wedge F^*(dyi_p)$$

Proposition .2.0.3 The application above is a contra-variant functor. In particular, if F is a diffeomorphism, F* is an isomorphism.

The de Rham co-boundary differential operator

As we have seen in the previous paragraph, a differential can be viewed as a $\mathbb{R}$ -linear operator $d : \Omega^0 \to \Omega^1$. Our proposal is then to define a class of linear operators on $\mathbb{R}$, in order to extend this idea to forms in higher dimensions Ω^p. Thus, we have the following

Theorem .2.0.2 For every $p = \{0, ..., n\}$ there exists a unique family of $\mathbb{R}$ linear operators $d^p : \Omega^p(U) \to \Omega^{p+1}(U)$, such that:

1) $d^0 = d$

2) $d^{p+1} \circ d^p = 0$

3) $If \omega \in \Omega^p(U), \tau \in \Omega^q(U), \quad d^{p+q}\omega \wedge \tau = d^p\omega \wedge \tau + (-1)^p \omega \wedge d^q\tau.$

Moreover, if $F : U \to V$ is a smooth map between open sets and $\omega \in \Omega^p(V)$, then $d^p F^* \omega = F^* d^p \omega$. We call this maps (co-boundary ones) the de Rham differential operators or exterior differentials.

Corollary .2.0.1 d is a local operator,i.e. if $\omega \equiv \tau \in U \Rightarrow d\omega = d\tau \in U.$

We can also define the differentials in a coordinate-independent form, as the following proposition shows:

Proposition .2.0.4 Let $\omega \in \Omega^p(U)$ *and* $X_0, ..., X_p \in \mathcal{H}(\mathcal{U}) \Rightarrow d\omega(X_0, ..., X_p) = \sum_{i=0}^p (-1)^i X_i \cdot \omega(X_0, ..., \hat{X}_i, ..., X_p) + \sum_{i<j} (-i)^{i+j} \omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p)$

Complexes and Co-homology

Definition .2.0.4 We have then constructed a co-chain complex (see A.3), namely a sequence of vector spaces and $\mathbb{R}$ -linear maps

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} ... \xrightarrow{d^{(n-1)}} \Omega^n(U) \longrightarrow 0$$

such that $d^{p+1} \circ d^p = 0$ or, better, $\mathrm{Im}(d^{p-1}) \subseteq \mathrm{Ker}(d^p)$.
This sequence is called the de Rham co-chain complex of U.

Definition .2.0.5 Given the complex above, we define:

- $\mathcal{Z}^p(U) := Ker(d^p)$, the space of p-co-cycles or closed p-forms;
- $\mathcal{B}^p(U) := Im(d^{p-1})$, the space of p- co-boundaries or exact p-forms;
- $\mathcal{H}^p(U) := \dfrac{\mathcal{Z}^p(U)}{\mathcal{B}^p(U)}$ the p-dimensional (de Rham) co-homology of U.

We can also equip the co-homology p-group with some algebraic structure: it can be shown that the wedge product preserves exactness and closedness. In particular, it induces a bilinear map

$$\cup : \mathcal{H}^p(U) \oplus \mathcal{H}^q(U) \to \mathcal{H}^{p+q}(U)$$
$$[\omega] \cup [\tau] := [\omega \wedge \tau]$$

called cup product map.

Remark .2.0.2 The reason we use this notation in place of the wedge one, is that this map is well defined also in other co-homology theories, even if the wedge product has no meaning.

Moreover, suitably extended, it gives $\mathcal{H}^*(U) := \oplus_{p\geq 0}\mathcal{H}^p$ an algebra structure and the couple $(\mathcal{H}^*(U), \cup)$ is called co-homology algebra.

Remark .2.0.3 (Topological invariance of co-homology) As we have already observed, letting F: $U \to V$ be a map between open sets, the functor F* maps closed forms in closed ones and does the same for exact ones, so it's natural to imagine that the de Rham co-homology is a differential topological invariant of U, and so it is.
In fact, F* induces an $\mathbb{R}$ - linear map on the p-groups

$$F^* : \mathcal{H}^p(V) \longrightarrow \mathcal{H}^p(U)$$

and also an algebra homomorphism, denoted by the same symbol,

$$F^* : \mathcal{H}^*(V) \longrightarrow \mathcal{H}^*(U)$$

that are still contra-variant functors. Thus, if F is a diffeomorphism at open sets level, then F* is an algebra isomorphism, proving our thesis about the topological invariance.

.3 A brief introduction to homological algebra

We are now in position to investigate some general properties of co-homology and related structures, namely to say something more about homological algebra. Although the construction is consistent also in the case of modules on commutative rings, we restrict, for the sake of simplicity, only on real vector spaces (not necessarily finite dimensional).

This branch of mathematics analyses sequences of vector spaces and linear maps of the form:

$$\mathcal{E} := \left\{ (\mathbb{E}^p, d^p) \ s.t. \ d^p : \mathbb{E}^p \longrightarrow \mathbb{E}^{p+1} \right\},$$

which will be the objects of our category, and their morphism $\phi : \mathcal{E} \to \mathcal{F}$, where $\mathcal{F}$ is another sequence of the above type. The morphism ϕ is then a sequence of maps $\phi^p : \mathbb{E}^p \to \mathbb{F}^p$ linear, such that the following diagram commutes

$$\begin{array}{ccccc} ... \xrightarrow{d^{p-1}} & \mathbb{E}^p & \xrightarrow{d^p} & \mathbb{E}^{p+1} & \xrightarrow{d^{p+1}} ... \\ & \downarrow \phi^p & & \downarrow \phi^{p+1} & \\ ... \xrightarrow{d^{p-1}} & \mathbb{F}^p & \xrightarrow{d^p} & \mathbb{F}^{p+1} & \xrightarrow{d^{p+1}} ... \end{array}$$

i.e. $\phi^{p+1} \circ d^p = d^p \circ \phi^p$.

The morphism ϕ is an isomorphism iff every ϕ^p is a vector space isomorphism.

Exact sequences and Five Lemma

Using sequences we can characterize maps between whatever type of objects, we can easily see if they are monomorphism, epimorphism or isomorphism. First we need a basic definition:

Definition .3.0.1 A sequence $\mathcal{E} = \{\mathbb{E}^p, d^p\}$ is said to be exact at $\mathbb{E}^p$ iff $Im(d^{p-1}) = Ker(d^p)$. The sequence $\mathcal{E}$ is exact if it happens for every $\mathbb{E}^p$.

Proposition .3.0.1 It's simple to show that:

- A sequence $0 \to \mathbb{E} \xrightarrow{\phi} \mathbb{F}$ is exact at $\mathbb{E}$ $\iff$ ϕ is injective;

- A sequence $\mathbb{E} \xrightarrow{\phi} \mathbb{F} \to 0$ is exact at $\mathbb{F} \iff \phi$ is surjective;
- A sequence $0 \to \mathbb{E} \xrightarrow{\phi} \mathbb{F} \to 0$ is exact $\iff \phi$ is an isomorphism.

Definition .3.0.2 A sequence of the type $0 \to \mathbb{E} \to \mathbb{F} \to \mathbb{G} \to 0$ is called a short sequence.

Short sequences are the building block for long ones, so this definition is of a huge importance. In fact, let

$$... \ \mathbb{E}^{i-1} \xrightarrow{\phi_{i-1}} \mathbb{E}^i \xrightarrow{\phi_i} \mathbb{E}^i \to ...$$

be a sequence and consider the new one

$$0 \to \frac{\mathbb{E}^{i-1}}{Ker(\phi_{i-1})} \xrightarrow{\tilde{\phi}_{i-1}} \mathbb{E}^i \xrightarrow{\tilde{\phi}_i} Im(\phi_i) \to 0$$

where the maps $\tilde{\phi}_{i-1}$ and $\tilde{\phi}_i$ are the induced ones, the above long sequence is exact at $\mathbb{E}^i$ iff the short is exact in $\mathbb{E}^i$.

Proposition .3.0.2 A short exact sequence $0 \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F} \xrightarrow{\psi} \mathbb{G} \longrightarrow 0$ is isomorphic to $0 \longrightarrow \mathbb{E} \xrightarrow{i} \mathbb{E} \oplus \mathbb{F} \xrightarrow{\pi} \mathbb{G} \longrightarrow 0$, where $i(v) = (v, 0)$ and $\pi(v, w) = w$.

Theorem .3.0.1 (Five Lemma) Consider the following diagram

$$\begin{array}{ccccccccc} \mathbb{E}_1 & \xrightarrow{f_1} & \mathbb{E}_2 & \xrightarrow{f_2} & \mathbb{E}_3 & \xrightarrow{f_3} & \mathbb{E}_4 & \xrightarrow{f_4} & \mathbb{E}_5 \\ \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \downarrow \phi_4 & & \downarrow \phi_5 \\ \mathbb{F}_1 & \xrightarrow{g_1} & \mathbb{F}_2 & \xrightarrow{g_2} & \mathbb{F}_3 & \xrightarrow{g_3} & \mathbb{F}_4 & \xrightarrow{g_4} & \mathbb{F}_5 \end{array}$$

If the squares commute, the lines are exact and the ϕ_i are isomorphism for p=1,2,4,5, then ϕ_3 is an isomorphism. [Actually, it's sufficient that only ϕ_2 and ϕ_4 are isomorphism, ϕ_1 surjective and ϕ_5 injective].

Duality Chain/Co-chain Complexes

Definition .3.0.3 A sequence $\mathcal{E} = \{\mathbb{E}^p, d^p\}$ is semi-exact or a co-chain complex if $Im(d^{p-1}) \subseteq Ker(d^p), \ \forall p$. Equivalently, it's a cochain complex if $d^p \circ d^{p-1} = 0$.

Definition .3.0.4 Given a co-chain complex, we set:

- $Z^p(\mathcal{E}) := Ker(d^p)$, the group of p-dimensional co-cycles;
- $B^p(\mathcal{E}) := Im(d^{p-1})$, the group of p-dimensional co-boundaries;
- $H^p(\mathcal{E}) := \dfrac{Z^p(\mathcal{E})}{B^p(\mathcal{E})}$, the p-dimensional co-homology group.

Remark .3.0.1 Clearly, the previous entities are actually vector spaces; we call it groups not to lost the generality of the theory, useful also in Abelian groups or modules context.

Remark .3.0.2 Looking at the definition of a co-homology group as a quotient of co-cycles and co-boundaries, it's easy to see that it gives a measure on complexes: it tell us how much the complex is an exact one.

Remark .3.0.3 If U is an open set of $\mathbb{R}^n$, the de Rham complex

$$... \longrightarrow \Omega^p \xrightarrow{d^p} \Omega^{p+1}(U) \longrightarrow ...$$

is a co-chain complex whose co-homology is the de Rham one, namely $H^p(U)$.

Observe that, if $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism between complexes, it induces linear maps $\phi^{*,p} : H^p(\mathcal{E}) \to H^p(\mathcal{F})$, between co-homology groups, that are actually co-variant functors.
In particular, isomorphic complexes have the "same" co-homology, i.e. isomorphic co-homologies. This is due to the commutativity condition: co-cycles are sent in co-cycles and co-boundaries in co-boundaries.

We want now to investigate the huge link between co-homology groups and homology ones. First, we need some definitions.

Definition .3.0.5 Let's consider a sequence of the type

$$\mathcal{E} := \{(\mathbb{E}_p, \delta_p) \mid \delta_p : \mathbb{E}_p \to \mathbb{E}_{p-1}\}.$$

If it is semi-exact, we call it chain complex.

Definition .3.0.6 For such a complex, define

- $Z_p(\mathcal{E}) := Ker(\delta_p)$, the group of p-dimensional cycles;
- $B_p(\mathcal{E}) := Im(\delta_{p+1})$, the group of p-dimensional boundaries;
- $H_p(\mathcal{E}) := \dfrac{Z_p(\mathcal{E})}{B_p(\mathcal{E})}$, the p-dimensional homology group.

Remark .3.0.4 As in the previous case, given a map $\phi : \mathcal{E} \to \mathcal{F}$, it induces naturally a family of co-variant functors at homology level denoted by the symbol $\phi_{p,*}$ $\forall p$.

Co-homology and homology groups are essentially the same objects, is it possible to find a map between them and what properties will it have?

Definition .3.0.7 Consider a chain complex $\mathcal{E} = \{(\mathbb{E}_p, \delta_p)\ s.t.\ \delta_p : \mathbb{E}_p \to \mathbb{E}_{p-1}\}$, we define the dual complex $\mathcal{E}^* := \{(\mathbb{E}^p, d^p)\}$, where $\mathcal{E}^p = (\mathcal{E}_p)^*$ is the dual space of $\mathbb{E}^p$ and $d^p = \delta^*_{p-1}$ is the transpose of δ_{p-1}.

We claim (simple exercise) that $\mathcal{E}^*$ is actually a co-chain complex, i.e. $d^p \circ d^{p-1} = 0$, thus we can denote by $H_p(\mathcal{E})$ and $H^p(\mathcal{E}^*)$ the homology and co-homology group respectively.

Definition .3.0.8 Let $b : \mathbb{E}^p \times \mathbb{E}_p \to \mathbb{R}$, the bi-linear map defined as $b(\phi, c) := \phi(c)$. Since $(d\phi)(c) = \phi(\delta c)$, we have that if $d\phi = 0$ then $\delta c = 0$. and $b(\phi + d\tau, c + \delta d) = b(\phi, c)$.

Hence, b induces a bi-linear map

$$\tilde{b} : H^p \times H_p \to \mathbb{R}, \tilde{b}([\phi], [c]) := \phi(c)$$

and therefore a linear map

$$K : H^p \to [H_p]^*, K([\phi])([c]) := \phi(c).$$

Theorem .3.0.2 (Universal Coefficient Theorem) The map K is an isomorphism.

Corollary .3.0.1 If a sequence is exact, the dual one is also exact.

Mayer-Vietoris Theorem

Definition .3.0.9 An algebraic homotopy between two morphisms $\phi, \psi : \mathcal{E} \to \mathcal{F}$ of co-chain (chain) complexes is a family of maps $K^p : \mathbb{E}^p \to \mathbb{F}^{p-1}$ ($K_p : \mathbb{E}_p \to \mathbb{F}_{p+1}$) such that $\phi - \psi = d \circ K + K \circ d \quad (\phi - \psi = \delta \circ K + K \circ \delta)$. The two morphisms then are said to be algebraically homotopic.

Proposition .3.0.3 Two algebraically homotopic maps induce the same morphism in co-homology (or homology).

Consider now a short exact sequence of co-chain complexes of the form

$$\{0\} \longrightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow \{0\}$$

The map ϕ is then injective and ψ is surjective at every level, but, in general, ϕ^* and ψ^* are not. However, there is still some relation between the co-homology groups of these complexes.

Theorem .3.0.3 (Mayer-Vietoris Theorem) Given the exact co-chain complexes sequence above, there exist a family of linear maps

$$\Delta^{*,p} : H^p(\mathcal{G}) \to H^{p+1}(\mathcal{E}),$$

called Mayer-Vietoris co-boundaries, such that the sequence

$$... \longrightarrow H^p(\mathcal{E}) \xrightarrow{\phi^*} H^p(\mathcal{F}) \xrightarrow{\psi^*} H^p(\mathcal{G}) \xrightarrow{\Delta^{*,p}} H^{p+1}(\mathcal{E}) \longrightarrow ...$$

is a long exact sequence.

Remark .3.0.5 Clearly, we have the same construction in homology between chain complexes. In this case, the maps Δ_*, p are called the Mayer-Vietoris boundaries.

Remark .3.0.6 Mayer-Vietoris maps are well define at co-homology (homology) level, but not at co-cycles (cycles) one.

Proposition .3.0.4 A morphism between short exact sequence of chain (co-chain) complexes induces a morphism between the associated Mayer-Vietoris exact sequences. In brief, the Mayer-Vietoris operators commute with the induced maps.

Remark .3.0.7 When dealing with Abelian groups or modules over commutative rings instead of vector spaces, we have some attention to pay before extending the general theory just developed.

- The sequence of Abelian groups

$$\{0\} \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}$$

  is a short exact sequence, but is not isomorphic to

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}.$$

  The only way for a sequence to verify this isomorphism is being a split short exact sequence, as the next proposition shows.

  Proposition .3.0.5 A short exact sequence of Abelian groups

$$\{0\} \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \{0\}$$

  splits if and only if there exists a map $r : C \to B$ such that $\psi \circ r = 1_C$.
  This always happens when C is free.

  ;

- If G is an Abelian group, then $G^* := Hom(G, \mathbb{Z})$. We can therefore define the dual of a chain complex of Abelian groups. The map K here is not an isomorphism because the sequence

$$\{0\} \longrightarrow B_{p-1} \longrightarrow \mathbb{Z}_{p-1} \longrightarrow H_{p-1} \longrightarrow \{0\}$$

  does not split.
  If the group H_{p-1} is a free one, the sequence splits and the theorem A.3.2.1 holds as it is stated. The general case, that we won't treat here, is covered by the Universal Coefficient Theorem for Abelian groups.

.4 More on de Rham co-homology

Before starting calculating some co-homology groups, we have to come back for a moment to the general goal of this theory, which is, roughly speaking, the indefinite integration problem: given a form, is it always possible to find a primitive for it? The answer depends on the contest:

- It's always possible to have a local primitive. In fact, given a point $x \in U$ and a form $\beta \in \Omega^{p+1}(U)$, $\exists V \subseteq U$ a nhood of x and a solution $\omega \in \Omega^p(V)$ of the equation $d\omega = \beta|_V \iff d\beta = 0$.
- For the global problem, $d\beta = 0$ is no longer a sufficient condition. The existence of the primitive depends on the topology set on U and on the particular form β taken in exam, as we will see soon.

Definition .4.0.1 Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be two open sets and $F_i : U \to V, i \in \{0, 1\}$ be smooth function, an homotopy is defined as a smooth map

$$H : U \times [0, 1] \subseteq \mathbb{R}^{n+1} \longrightarrow V$$
$$H(x, t) := H_t(x) \quad \forall t \in [0, 1]$$
$$\text{In particular, } H(x, i) := F_i(x) \quad i = 0, 1.$$

Two function are said to be homotopic, writing $F_0 \overset{H}{\sim} F_1$, if there exists such an homotopy between them.

Remark .4.0.1 The smoothness requirement is not needed in general, this kind of homotopy is called smooth homotopy. The reason for adding this hypothesis is that we want a family of integrable function from F_0 to F_1 in order the intermediate integrals to make sense, in brief. Thus, when dealing with forms and differentials is a good property to add.

Definition .4.0.2 We say that U and V are homotopy equivalent if there exist two maps $F : U \to V$ and $G : V \to U$ such that $G \circ F \sim 1_U$ and $F \circ G \sim 1_V$. F and G are said to be the homotopy inverse of each other.

Definition .4.0.3 We say that U is contractible if it is homotopy equivalent to $\mathbb{R}^0$, i.e. to a point.

Example .4.0.1 ($\boldsymbol{H^p}(\mathbb{R}^{\mathbf{0}})$) If we consider $U = \mathbb{R}^0$, the co-homology of U is as follows:

$$H^p(\mathbb{R}^0) \cong \begin{cases} \mathbb{R} & \text{if p=0} \\ \{0\} & \text{if p} \neq 0 \end{cases}$$

Example .4.0.2 (Star-shaped sets) If we consider a convex set, or better, a star shaped set U, it is contractible. In fact, taking $p, q \in U$, the map $H(p,q) = tp + (1-t)q \quad \forall t \in [0,1]$ is a smooth homotopy between 1_U and $F(q) = p$, thus $U \overset{H}{\sim} \mathbb{R}^0$.

Example .4.0.3 (Disjoint unions) Let $U = \dot{\bigcup\limits_{\alpha \in A}} U_\alpha$, where U_α are open sets $\forall\, \alpha \in A$, then $\Omega^p(U) = \underset{\alpha \in A}{\times} \Omega^p(U_\alpha)$ and the differentials preserves decomposition, i.e. if $\omega = \{\omega_\alpha\} \Rightarrow d\omega = \{d\omega_\alpha\}$. Thus,

$$H^p(U) \cong \prod_{\alpha \in A} H^p(U_\alpha)$$

Example .4.0.4 (0-dimensional Co-homology) In this case, the only exact 0-form is the 0-form itself $\Rightarrow$ $H^0(U) \cong$ the closed 0-forms, namely the functions in $\mathcal{F}(U)$ with zero differential. Taking $f \in \mathcal{F}(U) : df = 0$, f is locally constant; in particular, it's constant on the connected components of U. Thus,

$$H^0(U) \cong \underset{c \in C}{\times} H^0(\mathbb{R}^0) = \underset{c \in C}{\times} \mathbb{R}$$

where C is the counter-set of connected components of U.

Homotopy invariance and Poincaré Lemma

An homotopy between two given functions F and G can be viewed as a smooth deformation of one into the other or as a family of curves from F to G. If $H : U \times [0,1] \to V$ is a given homotopy between F and G, there exists a smooth map

$$\bar{H} : U \times \mathbb{R} \to V \qquad \bar{H}(x,i) := F_i(x),\ i \in \{0,1\}\,.$$

In fact, if $\lambda : \mathbb{R} \to [0,1]$ is such that

$$\lambda(t) = \begin{cases} 0 & \text{if t} \leq 0 \\ 1 & \text{if t} \geq 1 \end{cases}$$

it suffices to take $\bar{H}(x,t) := H(x, \lambda(t))$.

Theorem .4.0.1 (Homotopy invariance for co-homology) If $F_i : U \to V, i \in \{0,1\}$ are two homotopic smooth functions, then $F_0^* = F_1^* : H^p(V) \to H^p(U), \ \forall \ p$

Corollary .4.0.1 If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homotopic equivalent open sets, they have isomorphic co-homologies.

Corollary .4.0.2 (Poincaré's Lemma) If U is a contractible open set in $\mathbb{R}^n$ then

$$H^p(U) \cong \begin{cases} \mathbb{R} & \text{if p=0} \\ \{0\} & \text{if p} \neq 0 \end{cases}$$

How to calculate $H^p(U)$

We want now to illustrate a useful procedure that allows us to easy find the co-homology groups. The idea is to split our space U into its smaller components and use the inclusion, projection and Mayer-Vietoris operators to find the unknown groups.

Lemma .4.0.1 Consider the sequence

$$\{0\} \longrightarrow \Omega^p(U_1 \cup U_2) \stackrel{(j_1^*, j_2^*)}{\longrightarrow} \Omega^p(U_1) \oplus \Omega^p(U_2) \stackrel{(k_1^* - k_2^*)}{\longrightarrow} \Omega^p(U_1 \cap U_2) \longrightarrow \{0\}$$

where $j_i : U_i \longrightarrow U_1 \cup U_2$ and $k_i : U_1 \cap U_2 \longrightarrow U_i \quad \forall \ i \in \{1,2\}$ are the canonical inclusions, then it is a short exact sequence of co-chain complexes.

Theorem .4.0.2 (Mayer-Vietoris sequence for de Rham co-homology) Let $V := U_1 \cap U_2, \ U := U_1 \cup U_2$, there exists a family of linear maps $\Delta^* : H^p(V) \longrightarrow H^{p+1}(U)$, called Mayer-Vietoris co-boundaries, such that the sequence below is exact

$$... \longrightarrow H^p(U) \stackrel{(j_1^*, j_2^*)}{\longrightarrow} H^p(U_1) \oplus H^p(U_2) \stackrel{(k_1^* - k_2^*)}{\longrightarrow} H^p(V) \stackrel{\Delta^*}{\longrightarrow} ...$$

Remark .4.0.2 The de Rham co-homology is not the only co-homology theory that exists: every construction which satisfies the Eilenberg-Steenrod axioms below also is a well defined co-homology.

1. Homotopy invariance: Homotopic maps induce the same map in co-homology;

2. Excision: If (X, A) is a pair and U is a subset of X such that the closure of U is contained in the interior of A, then the inclusion map $i : (X \backslash U, A \backslash U) \hookrightarrow (X, A)$ induces an isomorphism in co-homology;

3. Dimension: If M is contractible, then:

$$H^k(M) = \begin{cases} \mathbb{R} & \text{if k=0} \\ \{0\} & \text{otherwise} \end{cases}$$

4. Additivity: If $X = \coprod_\alpha X_\alpha$ is the disjoint union of a family of topological spaces X_α, then $H^k(X) \cong \bigoplus_\alpha H^k(X_\alpha)$;

5. Exactness: Every Mayer-Vietoris sequence in co-homology is exact.

Furthermore, we have developed the theory of de Rham because every other co-homology is indeed isomorphic to this one.

Example .4.0.5 (Spheres)

$$H^k(\mathbb{S}^n) = \begin{cases} \mathbb{R} & \text{k=0, n} \\ \{0\} & \text{otherwise} \end{cases}$$

Example .4.0.6 If X_g is a Riemannian surface with g holes, then:

$$H^k(X_g) = \begin{cases} \mathbb{R} & \text{k=0,2} \\ \mathbb{R}^{2g} & \text{k=1} \\ \{0\} & \text{otherwise} \end{cases}$$

Reduced de Rham co-homology

We introduce now the reduced de Rham co-homology for the sake of having 0 as neutral element for co-homology. In fact, we would like that a contractible space acted like the 0 element.

Definition .4.0.4 Given an open set U, defined the space $\Omega^{-1}(U) := \mathbb{R}$ and the differential $d^{-1} : \Omega^{-1}(U) \to \Omega^0(U)$ as $d^{-1}(a) := a \ \in \Omega^0(U)$, then the sequence

$$\{0\} \longrightarrow \Omega^{-1}(U) \xrightarrow{d^{-1}} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \longrightarrow ...$$

is a co-chain complex called the augmented de Rham complex.

Definition .4.0.5 The reduced de Rham co-homology of U, defined as $\tilde{H}^p(U)$ is then the co-homology of the augmented complex.

Remark .4.0.3 We observe that $\tilde{H}^{-1}(U) = \{0\}$, $H^0(U) \cong \tilde{H}^0(U) \oplus \mathbb{R}$ and $\tilde{H}^p(U) = H^p(U)\ \forall\ p > 0$.

Remark .4.0.4 As anticipated, now the p- dimensional groups of reduced co-homology are all zeros if U is contractible, i.e. $\tilde{H}^p(U) = \{0\}$, $\forall p \geq 0$.

Let $F_1,\ F_2$ two closed subset of $\mathbb{R}^n$ related by an homeomorphism $\phi : F_1 \longrightarrow F_2$, a natural question is: can we find some good relation between the complementary sets? Are they still homeomorphic or homotopic equivalent? The answer, in general, is negative, as the following example shows.

Example .4.0.7 Consider $F_1 = \{x \in \mathbb{R}^2 : |x| = 1\} \cup \{x \in \mathbb{R}^2 : |x| = 2\}$ and $F_2 = \{x \in \mathbb{R}^2 : |x| = 1\} \cup \{x \in \mathbb{R}^2 : |x - (3,0)| = 1\}$. Passing to complementary sets, F_1^c is the union of a circle and two circles, while F_2^c is union of two points and a bouquet of two circles, so their fundamental groups are totally different. We have no homotopic equivalence.

However, there is an interesting relation between complement sets co-homology for closed.

Theorem .4.0.3 (Jordan-Alexander Duality Theorem) Let $F_1,\ F_2 \subseteq \mathbb{R}^n$ be homeomorphic closed sets, then

$$\tilde{H}^k(\mathbb{R}^n \setminus F_1) \cong \tilde{H}^k(\mathbb{R}^n \setminus F_2).$$

Lemma .4.0.2 Let $F \subset \mathbb{R}^n$ be a closed subset, then

$$\tilde{H}^{i+1}(\mathbb{R}^{n+1} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \text{ i} \geq -1.$$

Corollary .4.0.3 If $F \subseteq \mathbb{R}^n$ is a closed set then

$$\tilde{H}^{i+k}(\mathbb{R}^{n+k} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \forall i \geq -k.$$

Lemma .4.0.3 Let $F_i \subseteq \mathbb{R}^n,\ i = 1, 2$ be closed subset and $\phi : F_1 \to F_2$ an homeomorphism, then $\mathbb{R}^{2n} \setminus F \times \{0\}$ is homeomorphic to $\mathbb{R}^{2n} \setminus \{0\} \times F_2$.

Theorem .4.0.4 (Jordan's curve Theorem) Let $\gamma : S^1 \to \mathbb{R}^2$ be (the Jordan's curve) an homeomorphism onto its image, then $\mathbb{R}^2 \setminus \gamma(S^1)$ has exactly two connected components.

Remark .4.0.5 It's clear that the same argument can be extended every time we have a closed hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ and informations on its complement. A good example is the case of compact oriented surface, connected sum of tori and spheres.

.5 Tangent bundle of a differentiable manifold

Our goal is now to extend the theory of differential forms just developed to structures more sophisticated, in some sense, then open sets. In particular, the greater part of objects geometry deals with looks locally like open sets, so the de Rham's constructions fits perfectly in this new context.

Differentiable manifolds

Definition .5.0.1 A differential manifold of dimension n is a set M, together with a family of one-to-one maps $x_\alpha : U_\alpha \longrightarrow M$, where $U_\alpha \subset \mathbb{R}^n$ are open sets, such that:

1) $\bigcup\limits_\alpha x_\alpha(U_\alpha) = M$;
2) For each pair α, β such that $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, we have $x_\alpha^{-1}(W),\ x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and $x_\beta^{-1} \circ x_\alpha,\ x_\alpha^{-1} \circ x_\beta$ are differentiable maps;
3) The family $\{U_\alpha, x_\alpha\}$ is maximal relative to conditions 1) and 2).

Definition .5.0.2 A family $\{U_\alpha, x_\alpha\}$ satisfying conditions 1) and 2) of the previous definition is said to be a differentiable structure on M.

So, roughly speaking, a differentiable manifold is a set M with a differentiable structure defined on.

Tangent Bundle and Parallelizability

Definition .5.0.3 We called tangent bundle to a smooth differentiable manifold M, the sub-manifold of $\mathbb{R}^n \times \mathbb{R}^n$, 2n-dimensional,

$$TM := \dot{\bigcup_{x \in M}} T_xM = \bigcup_{x \in M} \{x\} \times T_xM$$

and we say that $p \in TM \iff p = (x, v) : x \in M, v \in T_xM.$

The tangent bundle is an example of vector bundle, like a covering, and it exists naturally a map $\pi : TM \to M$, defined as $\pi(x, v) = x$.

Definition .5.0.4 A vector field is a map $s : M \to TM \ : \pi \circ s = id_M$, a smooth section of the tangent bundle TM. Roughly, it's a map that associates at every point x in M a vector v tangent to M in that point.

Remark .5.0.1 If we consider a map between two differentiable manifolds $f : M \to M$, it induces an operator on the tangent bundles

$$\tilde{f} : TM \to TN$$
$$(x, v) = (f(x), df_x v)$$

as in the case of open sets.

Locally, a tangent bundle is always defined as a product of a sufficiently small nhood U of a point in M times $\mathbb{R}^n$, but there's no reason to treat it globally as the product $M \times \mathbb{R}^n$. A priori, nothing allows us to identify T_xM and T_yM, if $x \neq y$.

Definition .5.0.5 We say that M is parallelizable if there exists a linear diffeomorphism $t : M \times \mathbb{R}^n \to TM$ such that $t|_{\{x\}\times\mathbb{R}^n} : \{x\} \times \mathbb{R}^n \to T_xM$, called trivialization. Thus, $TM \cong M \times \mathbb{R}^n$.

Proposition .5.0.1 A differentiable manifold M $\subset \mathbb{R}^n$ is parallelizable $\iff$ there exist a family $\{\xi_1, ..., \xi_n\}$ of linearly independent vectors that is a basis for T_xM $\forall x \in M$.

Proposition .5.0.2 If M is parallelizable then $\chi(M) = 0$.

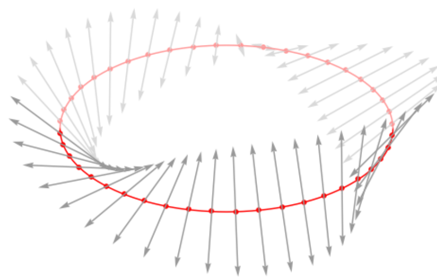
Example .5.0.1 The 2-dimensional sphere S^2 is not parallelizable because $\chi(S^2) = 2$.

Remark .5.0.2 The problem of parallelizability is linked to a huger topic in algebra: the existence in $\mathbb{R}^n$ of a product for which the zero-product property holds. It can be proven that if it's possible to define on $\mathbb{R}^n$ a bi-linear product having the zero-product property, then S^{n-1} is parallelizable.

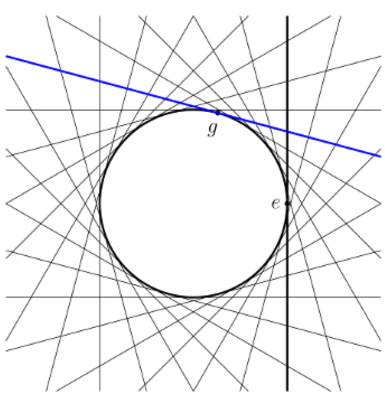
Example .5.0.2 The odd spheres $\left\{S^1,\ S^3,\ S^7\right\}$ are all parallelizable and on $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^6$ we have bi-linear forms giving a domain structure. More explicitly, in the case of S^3, taking $\xi(x) = (-x_2, x_1, -x_4, x_3),\ x \in S^3$, it's easy to show that $\langle \xi(x), x \rangle = 0$.

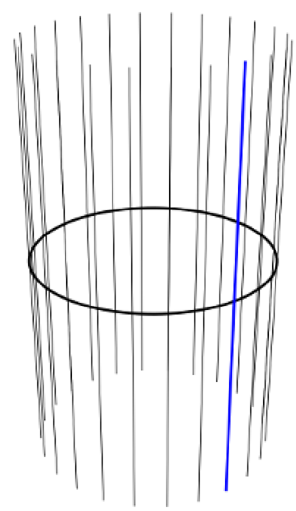
Example .5.0.3 Another beautiful property of the tangent bundle is to be always orientable, also if the manifold is not. It is possible to show it is the case of the Moebius strip, for instance.
The sphere S^1 has got essentially two different tangent bundles:

1. The one below, which is of course non orientable being a Moebius strip;



2. The trivial one, that can be viewed as a kind of "cylinder"





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