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Master's Thesis in Mathematics

PREFACTORISATION ALGEBRAS: INTRODUCTION AND EXAMPLES

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Was sagt dein Gewissen?

- "Du sollst er werden, der du bist."

- Friedrich Wihelm Nietzsche, Die fröhliche Wissenschaft

Simplicity is the ultimate sophistication.

When once you have tasted flight, you will forever walk the earth with your eyes turned skyward, for there you have been, and there you will always long to return.

Learning never exhausts the mind.

Art is never finished, only abandoned.

Painting is poetry that is seen rather than felt, and poetry is painting that is felt rather than seen.

The human foot is a masterpiece of engineering and a work of art.

- Leonardo da Vinci

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Preface

The process of measuring, both in the classical and in the quantum world starts by finding the trajectories the physical system under exam will follow. By methods developped in analytical mechanics, and shortly summarised in the appendix to this script, the first step to take in this direction is to define an action functional for the system under exam and to apply calculus of variations in order to minimise it. The equivalence between the result of this process and the the solutions to the equations of motion, the Euler-Lagrange equations, follows by Hamilton's principle. Such procedure not only applies to classical mechanics, but constitutes the foundation of quantum theories.

Observables are functions defined over the trajectories, as it is apparent that it is possible to make a measurement only on the physical curves that our particles are drawing, roughly. More precisely, on the fields.
To mathematically model the class of observables, we assign to each connected open set of a topological space X a complex vector space

$$\mathbf{Open}_X^c \ni U \longrightarrow \mathrm{Obs}\,(U) \in \mathbf{Vect}_{\mathbb{C}} \tag{1}$$

However, the act of measuring is highly non-trivial in the quantum world. For instance, the double slit experiment shows that an observer collapses the wave function and a beam of electrons starts behaving as a particle, instead of showing the interference pattern of waves. Moreover, it is not possible to combine measurements done during the same time interval. This gives prefactorisation algebras, defined on objects by the assignement above, their peculiar structure map and properties we carefully analyse in this thesis.

In [Costello & Gwilliam [CG1] (2016), §1], a series of two 'quantum measurements' is defined as coupling the system to a first measuring device O during the time period U and to a second apparatus O' during the time period U'. Such open sets $U,\ U'$ are taken to be connected, pairwise disjoint and contained in a bigger set V.
Namely, if we consider connected open sets of the real line, we are doing the experiment during two disjoint time slots (t_0, t_1) and (t_2, t_3) thought as subsets of $\mathbb{R}$ - or of any other real connected open interval (T, S), such that $T \leq \min\{t_0, t_2\}$ and $S \geq \max\{t_1, t_3\}$.

In the above setting, there exists a well-defined map of quantum observables

$$\mathrm{Obs}^{\mathrm{q}}\left(U\right) \otimes \mathrm{Obs}^{\mathrm{q}}\left(U'\right) \rightarrow \mathrm{Obs}^{\mathrm{q}}\left(V\right) \tag{2}$$

where we use the operation of tensor product since these objects live in the category of complex vector spaces $\mathbf{Vect}_{\mathbb{C}}$.

For classical observables, on the contrary, we have a well-defined commutative mapping also on measurements performed during the same interval of time I

$$\mathrm{Obs}^{\mathrm{cl}}\left(I\right) \otimes \mathrm{Obs}^{\mathrm{cl}}\left(I\right) \rightarrow \mathrm{Obs}^{\mathrm{cl}}\left(I\right) \tag{3}$$

Moreover, as standard practice when dealing with quantisation, in the classical limit quantum observables are required to agree with classical ones, i.e.

$$\lim_{\hbar \to 0} \frac{1}{\hbar} \mathrm{Obs}^{\mathrm{q}}\left(U\right) = \mathrm{Obs}^{\mathrm{cl}}\left(U\right) \tag{4}$$

for every $U \in \left(\mathbf{Open}_X^c\right)_0$ connected open set of a topological space $X \in \mathbf{Top}_0$.

Remark 0.0.1 For quantum mechanics, this construction on the real line will recover the Weyl algebra, as expected. The topological space X is usually a manifold, corresponding to the physical space-time.

The factorisation algebras analysed in *op.cit.* have values in co-chain complexes and we can assign a physical meaning to the cohomology classes, for instance, in gauge theories. The 0-th co-homology group represents the physical observables, the 1-st one anomalies to quantise such observables and all the negative ones symmetries of the system. Despite of the fact it is not completely clear their meaning for dimension $n \geq 1$, the expression

$$H^0\left(\mathrm{Obs}^{\mathrm{q}}\left(D_r\right)\right) \ni \int_\gamma \alpha \tag{5}$$

it is well-defined as Wilson operator over the loop γ for $\alpha \in H^1\left(\mathrm{Obs}^{\mathrm{q}}\left(D_r\right)\right)$, where D_r denotes a disk of radius $r > 0$ contained in some manifold M.

Other aspects discussed in [Itzykson & Zuber [IZ] (2005), §3-1] should as well be taken into account when dealing with space-time, aspects that are key in the theory of factorisation algebras. From special realtivity, we know that observables are locally well-defined, experiments must give the same results when performed in inertial reference frames and observables relative to space-like separed regions must commute. In other words, the principles of causality and Lorentz invariance hold and imply the spin to be half-integer and integer for fermions and bosons, respectively, the so-called spin-statistic, as well as the existence for each particle of the corresponding anti-particle, the TCP invariance.

Remark 0.0.2 If we consider a Minkowski space-time with signature $(+,-,-,-)$ and we place two experiments in two different points, we have

$$ds^2 := c^2dt^2 - \left(dx^2 + dy^2 + dz^2\right) = c^2dt^2 - dr^2 \tag{6}$$

Thus, we have three possible scenarios, namely:

$$ds^2 \begin{cases} > 0 \quad \Longleftrightarrow \quad c^2dt^2 > dr^2 \\ = 0 \quad \Longleftrightarrow \quad c^2dt^2 = dr^2 \\ < 0 \quad \Longleftrightarrow \quad c^2 < dr^2 \end{cases} \tag{7}$$

In the first case, the distance covered by a signal travelling at the speed of light is bigger than the physical distance between two event. We say they are time-like separated. In the second case, the events live on the light cone and, in the latter case, in order for the signal to reach the second point starting from the first it is needed to travel faster than the speed on light. This is clearly impossibe, thus we say that they are space-like separated and their observables are uncorrelated.

Since we mentioned correlation of events, we anticipate that is in fact possible to define correlation functions within the theory of factorisation algebras. Indeed, by [Costello & Gwilliam [CG1] (2016), §1.4.4], we have that for a factorisation algebra $\mathcal{F}$ over a manifold M such that $H^0(\mathcal{F}(M)) = \mathbb{R}[[\hbar]]$, ring of formal real power series in $\hbar$, the correlation functions are given as:

$$\langle \bullet \rangle : H^0(\mathcal{F}(U_1)) \times \cdots \times H^0(\mathcal{F}(U_n)) \to H^0(\mathcal{F}(M)) \tag{8}$$

$$(O_1, \ldots, O_n) \to \langle O_1, \ldots, O_n \rangle \tag{9}$$

where $\{U_i\}_{i=1}^n \in \mathbf{Open}_M^c$ is a finite family of connected open sets of M with the additional property of being pairwise disjoint, i.e. $U_i \cap U_j = \emptyset \; \forall \; i \neq j$.

To extend such definition for a field theory over $\mathbb{R}^n$, we first define the vacuum as $\mathbb{R}[[\hbar]]$ - linear map

$$\langle \bullet \rangle_0 : H^0(\mathcal{F}(\mathbb{R}^n)) \to \mathbb{R}[[\hbar]] \tag{10}$$

and then the correlation functions.

This thesis focus on prefactorisation algebras, the mathematical structures that efficiently represent the physical observables, analysing in depth their foundations, properties and the major results this formalism leads to. Large space is also given to the physics that they aim at modelling, in the hope that through these lines the reader will gain some knowledge of the 'big picture' along with the relevant mathematical technicalities.

The algebras introduced in this script are assumed to be unital and over a field. Furthermore, we sometimes modify the letters and notations in the references so the material is as complete and readable as possible. Lastly, when dealing with manifolds, we always assume them to be Hausdorff (T_2), second countable and without boundary, unless otherwise stated.

In the end, let me express my deep gratitude to my supervisors, for assigning me this amazing topic, for the patience and the trust, for every piece of advice given and to the extended group of all the Professors, PostDocs, PhD students, collegues, sponsors, advisors and friends that supported me during these years. I dedicate this to my family and to my grandfather for having made me the person I am, for teaching me, among other things, the value of ancient books and hard work.

Chapter 1

Introduction

1.1 Categorical foundations

As we will frequently make use of them, we collect in this chapter some major definitions and facts from category theory. Additional specific background material, if needed, will be provided in the single sections.
The notations $\mathbf{C}^{0(1)}$ and $\mathbf{C}_{0(1)}$ will equivalently denote the collection of objects (morphisms) of a given category $\mathbf{C}$. This difference is a matter of conventions in the literature and will not affect the theory.

1.1.1 Monoidal Categories

Definition 1.1.1 [Mac Lane [ML] (1978), §VII.1] A **strict monoidal category** consists of a triple $(\mathbf{B}, \square, e)$, formed by a category $\mathbf{B}$, an operation (the product), generally denoted by $\square$, and a multiplicative left and right identity object e.
The product $\square$ is a bifunctor

$$\square : \mathbf{B} \times \mathbf{B} \to \mathbf{B} \tag{1.1}$$

satisfying the following associative (i) and unit (ii) laws

(i)

$$\square\,(\square \times 1) = \square\,(1 \times \square) : \mathbf{B} \times \mathbf{B} \times \mathbf{B} \to \mathbf{B} \tag{1.2}$$

(ii)

$$\square\,(e \times 1) = id_{\mathbf{B}} = \square\,(1 \times e) \tag{1.3}$$

In the former, we further assume

$$(\mathbf{B}\times\mathbf{B})\times\mathbf{B}\cong\mathbf{B}\times(\mathbf{B}\times\mathbf{B}) \tag{1.4}$$

In the latter, we define the arrow $e\times 1$ to be

$$e\times 1:\mathbf{B}\to\mathbf{B}\times\mathbf{B} \tag{1.5}$$

$$b\to(e,b)\quad\forall b\in\mathbf{B}_0 \tag{1.6}$$

and in a similar fashion we proceed with $1\times e$, so the composite arrows in (1.3) are well-defined.

Indeed, we have

$$\begin{tikzcd} \mathbf{B} \arrow[r, "e\times 1"] \arrow[rr, bend right, "id_{\mathbf{B}}"'] & \mathbf{B}\times\mathbf{B} \arrow[r, "\Box"] & \mathbf{B} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{B} \arrow[r, "1\times e"] \arrow[rr, bend right, "id_{\mathbf{B}}"'] & \mathbf{B}\times\mathbf{B} \arrow[r, "\Box"] & \mathbf{B} \end{tikzcd}$$

Remark 1.1.1 Concerning the associativity in (1.2), the compositions of functors on the RHS and on the LHS of the equation are also well-defined. Indeed, for $\Box\circ(\Box\times 1)$, the composite arrow is given by

$$\underbrace{\mathbf{B}\times\mathbf{B}}_{\mathrm{dom}(\Box)}\times\mathbf{B}\xrightarrow{\Box\times 1}\mathbf{B}\times\mathbf{B}\xrightarrow{\Box}\mathbf{B} \tag{1.7}$$

Similarly, for $\Box\circ(1\times\Box)$,

$$\mathbf{B}\times\underbrace{\mathbf{B}\times\mathbf{B}}_{\mathrm{dom}(\Box)}\xrightarrow{1\times\Box}\mathbf{B}\times\mathbf{B}\xrightarrow{\Box}\mathbf{B} \tag{1.8}$$

where dom $(\bullet)$ denotes the domain of the arrow under consideration.

Remark 1.1.2 Notice the symbol $\circ$, denoting composition of arrows, is in general omitted in favour of their simple juxtaposition. For example, in (1.2) we wrote $\Box\,(1\times\Box)$ in place of $\Box\circ(1\times\Box)$.

The product $\Box$ is closed in $\mathbf{B}$, as it takes a pair of objects [morphisms] into an object [morphism] of $\mathbf{B}$, i.e. there are well-defined arrows

(i)

$$\mathbf{B}_0\ni(a,b)\xrightarrow{\Box}a\Box b\in\mathbf{B}_0 \tag{1.9}$$

$\forall\ a,b\in\mathbf{B}_0$

(ii)

$$\mathbf{B}_1 \times \mathbf{B}_1 \ni \{f : a \to a',\ g : b \to b'\} \xrightarrow{\Box} \{f \Box g : a \Box b \to a' \Box b'\} \in \mathbf{B}_1 \tag{1.10}$$

$\forall f, g \in \mathbf{B}_1$

for objects and morphisms, respectively.

Lastly, we require

$$1_a \Box 1_b = 1_{a \Box b} \tag{1.11}$$

$\forall a, b \in \mathbf{B}_0$ and

$$\left(f' \Box g'\right)\left(f \Box g\right) = \left(f' f\right) \Box \left(g' g\right) \tag{1.12}$$

for every pair of sequentially composable arrows (f, f') and (g, g').

Remark 1.1.3 The bifunctor $\Box$ is associative, as in (1.2), both for objects and for morphisms. Analogously, the unit law (1.3) applies to both of them, i.e.

$$e \Box b = b = b \Box e \quad \forall b \in \mathbf{B}_0 \tag{1.13}$$

$$1_e \Box h = h = h \Box 1_e \quad \forall h \in \mathbf{B}_1 \tag{1.14}$$

Example 1.1.1 If **B** is any category, the category of endofunctors $(\mathbf{End}\,(\mathbf{C}), \circ, id_{\mathbf{C}})$ is strict monoidal with bifunctor $\circ$ and unit $id_{\mathbf{C}}$.

Example 1.1.2 Any monoid M [see the relative section] in **Set** with unit $1_{\mathbf{M}}$ gives a strict monoidal category, thought as a discrete category with product given by multiplication of its elements $(\mathbf{M}, \cdot, 1_{\mathbf{M}})$.

When the category is monoidal, but not strict, the above associativity, left and right unit laws hold up to natural isomorphism $\alpha,\ \lambda,\ \rho$ and all the diagrams involving such morphisms must commute.

Definition 1.1.2 [Mac Lane [ML] (1978), ibid.] A **monoidal category** $(\mathbf{B}, \Box, e, \alpha, \lambda, \rho)$ is a category **B**, equipped with a bifunctor $\Box : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$, an object $e \in \mathbf{B}$ and three natural isomorphisms α, called **associator**, λ, called **left unitor** and ρ, the **right unitor**, such that

$$\alpha = \alpha_{b_1, b_2, b_3} : b_1 \Box \left(b_2 \Box b_3\right) \cong \left(b_1 \Box b_2\right) \Box b_3 \tag{1.15}$$

is natural $\forall b_1, b_2, b_3 \in \mathbf{B}_0$ and the pentagonal diagram

$$
\begin{tikzcd}
 & (b_1 \Box b_2) \Box (b_3 \Box b_4) \arrow[dr, "\alpha"] & \\
b_1 \Box (b_2 \Box (b_3 \Box b_4)) \arrow[ur, "\alpha"] \arrow[d, "1\Box\alpha"'] & & ((b_1 \Box b_2) \Box b_3) \Box b_4 \arrow[d, "\alpha\Box 1"] \\
b_1 \Box ((b_2 \Box b_3) \Box b_4) \arrow[rr, "\alpha"] & & (b_1 \Box (b_2 \Box b_3)) \Box b_4
\end{tikzcd}
$$

commutes for every choice of objects in $\mathbf{B}_0$.

Furthermore, the left and right unitor λ and ρ are natural such that

(i)

$$\lambda_b : e \Box b \cong b \qquad\qquad \rho_b : b \Box e \cong b \tag{1.16}$$

for any $b \in \mathbf{B}_0$;

(ii) The triangular diagram

$$
\begin{tikzcd}
b_1 \Box (e \Box b_2) \arrow[rr, "\alpha"] \arrow[dr, "\rho\,\Box 1"'] & & (b_1 \Box e) \Box b_2 \arrow[dl, "1\Box\lambda"] \\
 & b_1 \Box\, b_2 &
\end{tikzcd}
$$

commutes $\forall b_1, b_2 \in \mathbf{B}_0$;

(iii) They coincide on the identity object e, i.e.

$$\lambda_e = \rho_e : e \Box e \to e \tag{1.17}$$

Example 1.1.3 The category of Abelian groups with tensor product and the integers as unit, $(\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \lambda, \rho)$ is monoidal. More generally, for $\mathrm{R} \in \mathbf{CRng}_0$ a commutative ring, $({}_{\mathrm{R}}\mathbf{Mod}, \otimes_{\mathrm{R}}, \mathrm{R})$, the category of left R-modules, is monoidal.

Example 1.1.4 Every category with finite (co)products is monoidal;

Definition 1.1.3 [Mac Lane [ML] (1978), §VII.1, 164] A **(strict) morphism** of monoidal categories

$$T : (\mathbf{B}, \Box, e, \alpha, \lambda, \rho) \to (\mathbf{B}', \Box', e', \alpha', \lambda', \rho') \tag{1.18}$$

is a functor $T : \mathbf{B} \to \mathbf{B}'$ such that, for all objects $a, b \in \mathbf{B}_0$ and morphisms $f, g \in \mathbf{B}_1$

$$T(a \,\Box\, b) = Ta \,\Box\, Tb \qquad T(f \,\Box\, g) = Tf \,\Box\, Tg \qquad Te = e' \tag{1.19}$$

Further, it is compatible with associator, right and left unitor

$$T\alpha_{a,b,c} = \alpha'_{Ta,Tb,Tc} \qquad T\lambda_a = \lambda'_{Ta} \qquad T\rho_a = \rho'_{Ta} \tag{1.20}$$

$\forall a, b, c \in \mathbf{B}_0$.

Remark 1.1.4 Observe we have removed the parentheses indicating the application of the functor T to an object [a morphism] in favour of their juxtaposition. For instance, $T(a)$ is denoted by Ta.

1.1.2 Symmetric monoidal categories

We implement symmetries by equipping a monoidal category with a braiding.

Definition 1.1.4 [Mac Lane [ML], §XI.1, 252 ff., line 20 et seq.] Let $(\mathbf{M}, \Box, e, \alpha, \lambda, \rho)$ be a monoidal category, a **braiding** on it is a family of isomorphisms

$$\gamma_{a,b} : a \,\Box\, b \xrightarrow{\cong} b \,\Box\, a \tag{1.21}$$

natural in $a, b \in \mathbf{M}_0$ which satisfy for e the commutativity

$$\begin{tikzcd}
a \,\Box\, e \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & e \,\Box\, a \arrow[dl, "\lambda"] \\
& a &
\end{tikzcd}$$

and which, with the associator α, make both of the following diagrams commute:

$$\begin{tikzcd}
c \,\Box\, (a \,\Box\, b) \arrow[r, "\alpha"] & (c \,\Box\, a) \,\Box\, b \arrow[r, "\gamma \,\Box\, 1"] & (a \,\Box\, c) \,\Box\, b \\
(a \,\Box\, b) \,\Box\, c \arrow[u, "\gamma"] \arrow[r, "\alpha^{-1}"] & a \,\Box\, (b \,\Box\, c) \arrow[r, "1 \,\Box\, \gamma"] & a \,\Box\, (c \,\Box\, b) \arrow[u, "\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
(b \,\Box\, c) \,\Box\, a \arrow[r, "\alpha^{-1}"] & b \,\Box\, (c \,\Box\, a) \arrow[r, "1 \,\Box\, \gamma"] & b \,\Box\, (a \,\Box\, c) \\
a \,\Box\, (b \,\Box\, c) \arrow[u, "\gamma"] \arrow[r, "\alpha"] & (a \,\Box\, b) \,\Box\, c \arrow[r, "\gamma \,\Box\, 1"] & (b \,\Box\, a) \,\Box\, c \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

Definition 1.1.5 [Mac Lane [ML], §XI.1, 253, line 6 et seq.] A **symmetric monoidal category** is a monoidal category with a braiding γ such that every diagram

$$\begin{tikzcd} a \,\Box\, b \arrow[rr, "\gamma_{a,b}"] \arrow[dr, "1"'] & & b \,\Box\, a \arrow[dl, "\gamma_{b,a}"] \\ & a \,\Box\, b & \end{tikzcd}$$

commutes. In other words,

$$\gamma_{a,b} \circ \gamma_{b,a} = 1 \tag{1.22}$$

For this case, either one of the diagrams above implies the other.

Remark 1.1.5 If two consecutive applications of γ give an isomorphism to the identity and not an equality, the category is said to be braided.

1.1.3 Monoids and Monads

Definition 1.1.6 [Mac Lane [ML] (1978), §VII.3] Given a monoidal category $(\mathbf{B}, \Box, e)$, a **monoid** is an object $c \in \mathbf{B}_0$ together with two arrows $\mu_c : c \,\Box\, c \to c$ (multiplication) and $\eta_c : e \to c$ (unit) such that the diagrams

$$\begin{tikzcd} & (c \,\Box\, c) \,\Box\, c \arrow[dr, "\mu \,\Box\, 1"] & \\ c \,\Box\, (c \,\Box\, c) \arrow[ur, "\alpha"] \arrow[d, "1 \,\Box\, \mu"'] & & c \,\Box\, c \arrow[d, "\mu"] \\ c \,\Box\, c \arrow[rr, "\mu"] & & c \end{tikzcd}$$

$$\begin{tikzcd} e \,\Box\, c \arrow[r, "\eta_c \,\Box\, 1"] \arrow[dr, "\lambda"'] & c \,\Box\, c \arrow[d, "\mu"] & c \,\Box\, e \arrow[l, "1 \,\Box\, \eta_c"'] \arrow[dl, "\rho"] \\ & c & \end{tikzcd}$$

are commutative.

Remark 1.1.6 A monoid, also called semi-group with identity, is a category with one object. It is therefore fully characterised by its class of arrows, a rule for composing them and a unit morphism. Notice morphisms in this context are always sequentially composable.

Definition 1.1.7 [Mac Lane [ML] (1978), ibid.] A **morphism of monoids** $f : (c, \mu, \eta) \to (c', \mu', \eta')$ is an arrow $f : c \to c'$ such that

$$f\mu = \mu' (f \Box f) : c \Box c \to c' \qquad f\eta : \eta' : e \to c' \tag{1.23}$$

Definition 1.1.8 [Mac Lane [ML] (1978), ibid.] Monoids (c, μ_c, η_c) of a monoidal category **B** and arrows between them form a category. We denote it by the symbol $\mathbf{Mon_B}$.

There exists a trivial forgetful functor from the category of monoids in **B** to the category **B** defined by

$$U \ : \mathbf{Mon_B} \longrightarrow \mathbf{B} \tag{1.24}$$

$$(c, \mu_c, \eta_c) \to c \tag{1.25}$$

that has a left adjoint, when the necessary conditions on coproducts [Mac Lane [ML] (1978), §VII.3 , Theorem 2] are satisfied.

Example 1.1.5 Given a set S, we can construct the **free monoid over S**, denoted by the symbol S^*, whose elements are finite sequences of elements of S (also called lists or strings) equipped with concatenation as multiplication map and the empty list $\cdot \to S^*$ as unit.

Example 1.1.6 Among the relevant examples of monoids, we select the following:

- Rings are monoids in the monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$;
- Consider a ring R, R-algebras are monoids in the category of left R-modules $({}_{\mathrm{R}}\mathbf{Mod}, \otimes_{\mathrm{R}}, \mathrm{R})$;
- R-coalgebras are monoids in the opposite category of the left R-modules $\left({}_{\mathrm{R}}\mathbf{Mod}^{\mathrm{op}}, \otimes_{\mathrm{R}}^{\mathrm{op}}, \mathrm{R}\right)$. Indeed, they dualise R-algebras.

Let $(\mathbf{B}, \Box, e)$ be a fixed monoidal category.

Definition 1.1.9 [Mac Lane [ML] (1978), §VII.4] A **left action** of a monoid (c, μ_c, η_c) on an object $a \in \mathbf{B}_0$ is an arrow

$$\nu : c \Box a \to a \tag{1.26}$$

such that the diagram

$$\begin{tikzcd}
c\Box(c \Box a) \arrow[r, "\alpha"] \arrow[d, "1 \Box \nu"'] & (c \Box c) \Box a \arrow[r, "\mu_c \Box 1"] & c \Box a \arrow[d, "\nu"'] & e \Box a \arrow[l, "\eta_c \Box 1"'] \arrow[dl, "\lambda"] \\
c \Box a \arrow[rr, "\nu"'] & & a &
\end{tikzcd}$$

commutes.

Definition 1.1.10 [ibid.] A **morphism of left actions** of c $f : \nu \to \nu'$, is an arrow $f : a \to a'$ in $\mathbf{B}_1$ such that

$$\nu' (1 \,\square\, f) = f\nu : c \,\square\, a \to a' \tag{1.27}$$

The category of left actions of a monoid c is denoted by ${}_c\mathbf{Lact}$.

Example 1.1.7 - The multiplication map is an example of a left-action of c over c, called the **left regular representation** of the object c;

- An action of $R \in \mathbf{CRng}_0$ over $Z \in \mathbf{Ab}$ gives a left R-module. This example extends to algebras over R or dga (differential graded associiaive) algebras and their modules.

Remark 1.1.7 There is a dual notion of right action and left-right actions, defining right and left-right modules (bimodules) respectively.

Example 1.1.8 [MacDonald & Sobral [CF] (2004), §V.1.1] Consider a monoid M and let it act on **Set** as follows:

$$M \times \bullet : \mathbf{Set} \to \mathbf{Set} \tag{1.28}$$

We call such endofunctor T and we observe it comes with a unit [natural transformation]

$$\eta : 1_{\mathbf{Set}} \to T \tag{1.29}$$

$$\eta_X (x) = (e, x) \tag{1.30}$$

and a multiplication [natural transformation]

$$\mu : T^2 \to T \tag{1.31}$$

$$\mu_X (n, n', x) = (nn', x) \tag{1.32}$$

making the following diagram commute

$$\begin{tikzcd}
T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"'] & T \arrow[l, "\eta_T"'] \arrow[d, "T\eta"] \arrow[dl, "1_T"] \\
T^2 \arrow[r, "\mu"'] & T & T^2 \arrow[l, "\mu"]
\end{tikzcd}$$

where $T^i := \underbrace{T \circ \cdots \circ T}_{\text{i times}}$.

(T, η_T, μ_T) is a monoid in the strict monoidal category $(\mathbf{End}\,(\mathbf{Set})\,, \circ, id, 1, 1, 1)$, where the ones denote the trivial associator, left and right unitor, i.e. α, λ, ρ.

The previous example leads to the following general definition

Definition 1.1.11 [MacDonald & Sobral [CF] (2004), §V.1.2] A **monad** on a category $\mathbf{X}$ is a system $T = (T, \eta, \mu)$ consisting of an endofunctor $T : \mathbf{X} \to \mathbf{X}$ and natural transformations $\eta_T : 1_{\mathbf{X}} \to T$ and $\mu_T : T^2 \to T$ making the diagram

$$\begin{tikzcd} T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"'] & T \arrow[l, "\eta_T"'] \arrow[dl, "1_T"] \arrow[d, "T\eta"] \\ T^2 \arrow[r, "\mu"'] & T & T^2 \arrow[l, "\mu"] \end{tikzcd}$$

commute.

Example 1.1.9 [ibid., Examples, (2)] A key example is the **list monad** $\mathcal{L}\,(\mathcal{L}, \mu, \eta)$. The endofunctor $\mathcal{L} : \mathbf{Set} \to \mathbf{Set}$, called list functor, is defined by sending each set X to the set of all finite strings of its elements, denoted by $\mathcal{L}X$. Notice it includes also the empty string. On morphisms,

$$\mathcal{L}f : \mathcal{L}X \longrightarrow \mathcal{L}Y \tag{1.33}$$

$$\mathcal{L}f\,(x_1 \ldots x_n) \mapsto (fx_1 \ldots fx_n) \tag{1.34}$$

The natural unit map $\eta_X : X \to \mathcal{L}X$ takes any set into the string of length one made by itself and the multiplication map $\mu : \mathcal{L}\mathcal{L}X \to \mathcal{L}X$ is defined by sending any string of strings to the string made by the same elements but with parentheses removed.

1.1.4 Elements of Sheaf Theory

Definition 1.1.12 [Bredon [BR] (1997), §I.1, Definition 1.1] Let $X \in \mathbf{Top}_0$ be a topological space. A **presheaf** A (of Abelian groups) on X is a contravariant functor from the category $\mathbf{Open}_X$ of open subsets of X and inclusions to the category of abelian groups $\mathbf{Ab}$

$$\mathbf{Open}_X{}^{\mathrm{op}} \xrightarrow{A} \mathbf{Ab} \tag{1.35}$$

Remark 1.1.8 In general, one may define a presheaf with values in an arbitrary category $\mathbf{C}$.

Definition 1.1.13 [Bredon [BR] (1997), §I.1, Definition 1.2] A **"sheaf"** (of Abelian groups) on $X \in \mathbf{Top}_0$ is a pair $(\mathcal{A}, \pi)$ where:

(i) $\mathcal{A}$ is a topological space (not Hausdorff in general);

(ii) $\pi : \mathcal{A} \longrightarrow X$ is a local homeomorphism onto X;

(iii) each $\mathcal{A}_x = \pi^{-1}(x)$, for $x \in X$, is an Abelian group (and is called the **"stalk"** of $\mathcal{A}$ at x);

(iv) the group operations are continuous.

Remark 1.1.9 Concerning the statement on group operations in (iv): Let $\mathcal{A} \,\Delta\, \mathcal{A}$ be the subspace of $\mathcal{A} \times \mathcal{A}$ consisting of those pairs (α, β) with $\pi(\alpha) = \pi(\beta)$,

$$\mathcal{A} \times \mathcal{A} \supset \mathcal{A} \,\Delta\, \mathcal{A} = \{(\alpha, \beta) : \; \pi(\alpha) = \pi(\beta)\} \tag{1.36}$$

then the function

$$\mathcal{A} \,\Delta\, \mathcal{A} \longrightarrow \mathcal{A} \tag{1.37}$$

$$(\alpha, \beta) \mapsto \alpha - \beta \tag{1.38}$$

is continuous. Equivalently, the two maps

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{A} & \qquad \mathcal{A} \,\Delta\, \mathcal{A} &\longrightarrow \mathcal{A} \\ \alpha &\mapsto -\alpha & (\alpha, \beta) &\mapsto \alpha + \beta \end{aligned}$$

are continuous.

Remark 1.1.10 A sheaf can also be generated by a presheaf by considering the discrete topology over $A(U)$ and generating the stalks as direct limits of the presheaf of sections. For details, see [Bredon [BR] (1997), §I.1.5]. Moreover, [ibid., §1.7, 6 ff, line 28 et seq.] "Sheaves are in one-to-one correspondence with conjunctive monopresheaves. For this reason it is common practice not to distinguish between sheaves and conjunctive monopresheaves." Therefore, we introduce the relevant definitions involved in this statement and we show the equivalence following [Bredon [BR] (1997), §I.1.7]:

Let $\mathcal{A}$ be the sheaf generated by the presheaf A. For any $U \in \mathbf{Open}_X$, collection of open sets of the topological space X, there exists a natural map from the presheaf A to the corresponding sheaf $\mathcal{A}$ defined by

$$\theta_U : A(U) \to \mathcal{A}(U) \tag{1.39}$$

We search for conditions in order for θ_U to be an isomorphism for every chosen U.

Since

$$\mathcal{A}_x = \varinjlim_{x \in U} A(U) \tag{1.40}$$

A section of A over U, $s \in A(U)$, lies in the kernel of the above-defined map θ_U if and only if $\forall x \in U$, $\exists\, V \in \mathcal{I}_x$, a nhood of x contained in U such that

$$s|_V = 0 \tag{1.41}$$

i.e. s is locally trivial. The symbol $\mathcal{I}_x$ denotes the family of nhoods of the point x.

Consider an arbitrary family of open sets $\{U_\alpha\}_{\alpha \in A} \in \mathbf{Open}_X$. The local triviality condition above implies that θ_U is monic for every open set U if and only if the following property, denoted by **(S1)**, holds:

$$\boxed{\text{If } U = \bigcup_{\alpha \in A} U_\alpha \text{ and } \exists\, s, t \in A(U) \ : \ s|_{U_\alpha} = t|_{U_\alpha} \ \forall \alpha \ \Rightarrow \ s = t^2}$$

Definition 1.1.14 A presheaf satisfying **(S1)** is called **monopresheaf**.

If **(S1)** holds, θ_U is surjective on every open set U if and only if the property **(S2)** holds:

$$\boxed{\text{For } U = \bigcup_{\alpha \in A} U_\alpha, \text{ if } \exists\, s_\alpha \in A(U_\alpha),\ s_\beta \in A(U_\beta) : s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta} \ \forall \alpha, \beta \in A \ \Rightarrow \ \exists\, s \in A(U) : s|_{U_\alpha} = s_\alpha \ \forall\, \alpha}$$

In other words, a sheaf satisfes the property **(S2)** if and only if whenever two sections agree on the overlapping of any two opens, there exists a global section that, restricted to the open U_α, is the relative locally defined one s_α, for every choice of the index.

Definition 1.1.15 A presheaf satisfying the property (S2) is called **conjunctive**.

Let $\mathcal{A}$ be the sheaf generated by a conjunctive monopresheaf A, then the sequence [Bredon [BR] (1997), §1.1.7, page 7]

$$0 \to A\left(U\right) \xrightarrow{f} \prod_{\alpha} A\left(U_{\alpha}\right) \xrightarrow{g} \prod_{(\alpha,\beta)} A\left(U_{\alpha,\beta}\right)$$

is exact, where $f\left(s\right) = \prod_{\alpha}\left(s|_{U_{\alpha}}\right)$ and

$$g\left(\prod_{\alpha} s_{\alpha}\right) = \prod_{(\alpha,\beta)}\left(s_{\alpha}|_{U_{\alpha,\beta}} - s_{\beta}|_{U_{\alpha,\beta}}\right)$$

with $U_{\alpha,\beta} := U_{\alpha} \bigcap U_{\beta},\ U = \bigcup_{\alpha} U_{\alpha}$, for a family $\{U_{\alpha}\}_{\alpha\in A} \in \mathbf{Open}_X$.

Remark 1.1.11 The parentheses $(\bullet, \bullet)$ denote pairs and substitute the notation $\langle \bullet, \bullet \rangle$.

Definition 1.1.16 [Bredon [BR] (1997), §V.1, Definition 1.1] A **precosheaf** $\mathfrak{A}$ on $X \in \mathbf{Top}_0$ is a covariant functor from the category of open subsets of X to that of L-modules [where L denotes the base ring being a PID - principal ideal domain]

$$\mathbf{Open}_X \xrightarrow{\mathfrak{A}} \mathbf{Mod}_L \tag{1.42}$$

A precosheaf is a **cosheaf** if the sequence

$$\bigoplus_{(\alpha,\beta)} \mathfrak{A}\left(U_{\alpha} \cap U_{\beta}\right) \xrightarrow{g} \bigoplus_{\alpha} \mathfrak{A}\left(U_{\alpha}\right) \xrightarrow{f} \mathfrak{A}\left(U\right) \to 0$$

is exact for all collections $\{U_{\alpha}\}$ of open sets with $U = \bigcup_{\alpha} U_{\alpha}$, where $g = \sum_{(\alpha,\beta)}\left(\iota^{U_{\alpha}}_{U_{\alpha,\beta}} - \iota^{U_{\beta}}_{U_{\alpha,\beta}}\right)$ and $f = \sum_{\alpha} \iota^{U}_{U_{\alpha}}$.

Remark 1.1.12 The notation ι^U_V denotes the canonical homomorphism $\mathfrak{A}\left(V\right) \to \mathfrak{A}\left(U\right)$ induced by the inclusion $V \subset U$.

1.2 Algebras and vector fields

As it is quite common in the literature to switch between the notions of algebras and vector spaces, it is fundamental to illustrate their interplay and definitions in rigorous categorical terms.
In this section $\mathbb{K}$ denotes a field.

Definition 1.2.1 A **unital associative algebra** over $\mathbb{K}$ is a monoid in the monoidal category of vector spaces $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$.

More explicitly, it is equivalent defined as follows:

Definition 1.2.2 [Loday & Vallette [LV] (2012), §1.1.1] An **associative algebra** over $\mathbb{K}$ is a vector space $\mathcal{A}$ equipped with a multi-linear binary operation, the product,

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \tag{1.43}$$

$$a \times b \to \mu\,(a \otimes b) := a \cdot b \tag{1.44}$$

satisfying the associativity propriety

$$\mu\,(\mu \otimes id_{\mathcal{A}}) = \mu\,(id_{\mathcal{A}} \otimes \mu) \tag{1.45}$$

i.e.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{1.46}$$

$\forall\ a, b, c \in \mathcal{A}$.

Definition 1.2.3 [Loday & Vallette [LV] (2012), §1.1.1, 2] An associative algebra $\mathcal{A}$ over $\mathbb{K}$ is a **unital associative algebra** if there exists further a linear map

$$\eta : \mathbb{K} \to \mathcal{A} \tag{1.47}$$

such that

$$\mu\,(\eta \otimes 1) = \mu\,(1 \otimes \eta) \tag{1.48}$$

In other words, if $1 := \eta\,(1_{\mathbb{K}})$, we have

$$1 \cdot a = a = a \cdot 1 \tag{1.49}$$

$\forall a \in \mathcal{A}$,

In terms of commutative diagrams, associativity and unitality are visualised as

$$\begin{tikzcd}
 & (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \arrow[dr, "\mu\otimes 1"] & \\
\mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \arrow[ur, "\alpha"] \arrow[d, "1\otimes\mu"'] & & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[rr, "\mu"'] & & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K}\otimes\mathcal{A} \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\mu"] & \mathcal{A}\otimes\mathbb{K} \arrow[l, "1\otimes\eta"'] \arrow[dl, "\rho"] \\
& \mathcal{A} &
\end{tikzcd}$$

Remark 1.2.1 In the above diagrams we make use of the notations already introduced in section 1.1.2. Therefore, α is the associator, λ, ρ the left and right unitors, respectively. We removed the subscribt for the unit morphism, i.e. $\eta = \eta_{\mathcal{A}}$, since the meaning is clear from the context.

Definition 1.2.4 [ibid., line 14 et seq.] A **morphism of associative algebras** is a linear map

$$\phi : (\mathcal{A}_1, \mu_1) \to (\mathcal{A}_2, \mu_2) \tag{1.50}$$

compatible with the multiplication maps

$$\phi\left(\mu_1\left(a \otimes b\right)\right) = \mu_2\left(\phi a \otimes \phi b\right) \tag{1.51}$$

$\forall a, b \in \mathcal{A}_1$.

If further, such algebras are unital, we require the additional condition

$$\phi 1_{\mathcal{A}_1} = 1_{\mathcal{A}_2} \tag{1.52}$$

Definition 1.2.5 [ibid., line 14 et seq.] The category of (unital) associative algebras over $\mathbb{K}$ is denoted by **(u)Ass-alg**$_{\mathbb{K}}$.

Remark 1.2.2 The prefix (u)- is usually omitted when unitality is apparent from the context.

Definition 1.2.6 [Loday & Vallette [LV], §1.1.8] A **commutative associative algebra** over $\mathbb{K}$ is a (unital) associative algebra equipped with a **switching map**

$$\gamma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \tag{1.53}$$

$$a \otimes b \to b \otimes a \tag{1.54}$$

such that

$$\mu \circ \gamma = \mu \tag{1.55}$$

Definition 1.2.7 A **morphism of commutative algebras** is a morphism of (unital) associative algebras compatible with the switching map. Namely, with the same notation used in the previous definition of arrows, it reads

$$\phi\left[\mu_1\left(a\otimes b\right)\right]=\phi\left[\mu_1\gamma\left(a\otimes b\right)\right]=\phi\left[\mu_1\left(b\otimes a\right)\right]=\mu_2\left[\phi b\otimes\phi a\right]=\mu_2\gamma\left[\phi b\otimes\phi a\right]=\mu_2\left[\phi a\otimes\phi b\right] \quad (1.56)$$

Definition 1.2.8 [ibid., 7, line 7 et seq.] The category of (unital) commutative algebras over $\mathbb{K}$ is denoted by **(u)Com-alg**$_{\mathbb{K}}$.

Remark 1.2.3 There is also a class of commutative algebra defined without the requirement of associativity, they are called commutative magmatic algebras.

There exist forgetful functors between the categories of commutative algebras, associative algebras and vector spaces over some chosen field $\mathbb{K}$, respectively:

$$\textbf{Com-alg}_{\mathbb{K}}\xrightarrow{F_1}\textbf{Ass-alg}_{\mathbb{K}}\xrightarrow{F_2}\textbf{Vect}_{\mathbb{K}} \quad (1.57)$$

where the former forgets commutativity and the latter multiplication. Such a chain of forgetful functors rigorously formalises the 'abuse of language' and it remains valid in the more general context of modules.

Remark 1.2.4 The left adjoints to such functors, that are briefly presented below, play a relevant role in our theory and enter in the Chevalley-Eilenberg construction for Lie algebras (co-)homology.

Notice the functor $\textbf{Com-alg}_{\mathbb{K}}\xrightarrow{F_1}\textbf{Ass-alg}_{\mathbb{K}}$ always exists as a commutative algebra is also associative [cfr. Loday & Vallette [LV] (2012), §1.1.8, 7, line 9 et seq.].

Concerning free objects in the categories presented above, we state the following definitions:

Definition 1.2.9 [Bourbaki [NBI] (1989), §6, Definition 1.] Let A be a commutative ring, $M\in\textbf{Mod}_A$. The **symmetric algebra of M**, denoted by $S\left(M\right)$, or $\mathrm{Sym}\left(M\right)$, or $S_A\left(M\right)$, is the quotient algebra over A of the tensor algebra $T\left(M\right)$ by the two-sided ideal $\mathfrak{I}'$ - also denoted by $\mathfrak{I}'_M$ - generated by the elements $xy-yx:=x\otimes y-y\otimes x$ of $T\left(M\right)$, where x,y run through M.

In other words, it is the commutative algebra

$$\mathrm{Sym}\left(M\right):=\frac{T\left(M\right)}{\left(xy-yx\right)}\in\textbf{Com-alg}_A^0 \quad (1.58)$$

where the **tensor algebra of M** is defined as

$$T(M) := \bigoplus_{n \geq 0} M^{\otimes^n} \in \textbf{Ass-alg}_A^0 \tag{1.59}$$

The fact that it is indeed commutative derives by the relation we impose on the quotient, namely $xy = yx$.

Remark 1.2.5 - The reader has surely noticed the former is nothing else as the generalisation of the abelianisation procedure in group theory, mutatis mutandis;

- Suppose we are given a vector space $V \in \textbf{Vect}_{\mathbb{K}}$ and a basis $\mathcal{E}$ for it, then the symmetric algebra over V is canonically isomorphic to $\mathbb{K}[\mathcal{E}]$. Similarly, for M a free module, its symmetric algebra is canonically isomorphic to the corresponding polynomial algebra. For more details, see [Bourbaki [NBI] (1997), §6];

- The 0-th piece of the symmetric algebra above is the base ring by convention. This implies we have a commutative unital algebra. In case there is the need to work in a non-unital context, the relevant commutative free algebra will be then the reduced one, where the subscript n starts from one to cut out the base ring. Similarly, the free non-unital associative algebra is the reduced tensor algebra

$$T(M) := \bigoplus_{n \geq 1} M^{\otimes^n} \tag{1.60}$$

In categorical terms, the construction of tensor algebras and symmetric algebras of a vector space V provides left adjoint functors to the functors

$$\textbf{Com-alg}_{\mathbb{K}} \xrightarrow{F_1} \textbf{Vect}_{\mathbb{K}} \tag{1.61}$$

and

$$\textbf{Ass-alg}_{\mathbb{K}} \xrightarrow{F_2} \textbf{Vect}_{\mathbb{K}} \tag{1.62}$$

respectively.

Moreover, the tensor algebra (over a vector space V) is free in the category $\textbf{Ass-alg}_{\mathbb{K}}$ of (unital) associative algebras [Loday & Vallette [LV], §1.1.3, Proposition 1.1.1]. Similarly, the symmetric algebra (over V) is the free commutative unital associative algebra over the vector space V [ibid., §1.1.8, 7, line 13 et seq.].

Remark 1.2.6 Notice that not all the (associative) algebras are free. A free algebra is such that the underlying vector space consists of words from a given alphabet $S = \{s_1, \dots, s_n\}$ and its algebra multiplication is given by concatenation of words. For instance, if $\omega_1, \omega_2 \in S^*$ are two given words from such an alphabet S, then the multiplication map reads $\omega_1\omega_2$. More generally, given a commutative ring $R \in \textbf{CRng}_0$, the free algebra over the alphabet S is given by

$$R\langle S\rangle = \bigoplus_{\omega\in S^*} R\omega \tag{1.63}$$

i.e. direct sum of R-modules over one generator.

Furthermore, there exists a natural adjunction pair

$$\textbf{Ass-alg}_R \leftrightarrows \textbf{Set} \tag{1.64}$$

The left adjoint to the fuctor $\textbf{Ass-alg}_R \to \textbf{Set}$ takes an alphabet into the associative free R-algebra over it.

In particular, given a vector space $V \in \textbf{Vect}_{\mathbb{K}}$, the symmetric algebra over V is obtained as quotient of the tensor algebra $T(V)$ by the commutation relations. Thus, the quotient yields

$$\textbf{Ass-alg}_{\mathbb{K}} \to \textbf{Com-alg}_{\mathbb{K}} \tag{1.65}$$

$$T(V) \to \mathrm{Sym}(V) \tag{1.66}$$

Indeed, given $V \in \textbf{Vect}_{\mathbb{K}}$, there exists a corresponding chain

$$\textbf{Vect}_{\mathbb{K}} \xrightarrow{G_1} \textbf{Ass-alg}_{\mathbb{K}} \xrightarrow{G_2} \textbf{Com-alg}_{\mathbb{K}} \tag{1.67}$$

$$V \longrightarrow T(V) \longrightarrow \mathrm{Sym}(V) \tag{1.68}$$

As composition of adjoints is still an adjoint, [Mac Lane [ML] (1978), §IV.8, 103, Theorem 1.], we have an adjunction between $\textbf{Vect}_{\mathbb{K}}$ and $\textbf{Com-alg}_{\mathbb{K}}$.

Remark 1.2.7 Generalising the notion of vector space, the same applies to an R-module M, i.e. M $\in {}_R\textbf{Mod}$, for R $\in \textbf{CRng}_0$ being a commutative unital ring. In this case, we have the chain of functors

$$\textbf{Mod}_R \xrightarrow{G_1} \textbf{Ass-alg}_R \xrightarrow{G_2} \textbf{Com-alg}_R \tag{1.69}$$

$$M \longrightarrow T(M) \longrightarrow \mathrm{Sym}(M) \tag{1.70}$$

Thus, given a module (vector space), we can always associate to it a commutative algebra: its symmetric algebra.

1.3 Chain complexes and dg-algebras

We recall the definitions of chain and co-chain complexes. In particular, we specialise them to the case of differential graded vector spaces over a field $\mathbb{K}$, whose relative category is denoted by $\textbf{dg-Vect}_{\mathbb{K}}$.

Definition 1.3.1 [Osborne [OS] (2000), §3.2] A **chain complex** $C = \{(C_i, d_i)\}_{i\in\mathbb{Z}}$ consists of a family of Abelian groups C_i and homomorphisms between them $d_i : C_i \to C_{i-1}$, called **differentials** or **boundary maps**

$$\cdots \to C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} C_{i-2} \to \dots \tag{1.71}$$

such that $d_{i-1} \circ d_i = 0$.

Definition 1.3.2 [ibid.] If $C = \left\{\left(C_i, d_i^C\right)\right\}_{i\in\mathbb{Z}}, E = \left\{\left(E_i, d_i^E\right)\right\}_{i\in\mathbb{Z}}$ are two chain complexes, a morphism of chain complexes, called **chain map**, is a family of arrows

$$\psi = \{\psi_i\}_{i\in\mathbb{Z}} : C_i \to E_i \tag{1.72}$$

such that the squares

$$\begin{tikzcd}
\dots \arrow[r] & C_i \arrow[r, "d_i^C"] \arrow[d, "\psi_i"] & C_{i-1} \arrow[r] \arrow[d, "\psi_{i-1}"] & \dots \\
\dots \arrow[r] & E_i \arrow[r, "d_i^E"'] & E_{i-1} \arrow[r] & \dots
\end{tikzcd}$$

commutes.

Remark 1.3.1 Notice we have assumed the complex to be $\mathbb{Z}$ graded. Other gradings are also possible.

Definition 1.3.3 Chain complexes of Abelian groups and chain maps form the **category of chain complexes**. It is denoted by $\mathbf{Ch}\left(\mathbf{Ab}\right)$.

Definition 1.3.4 A **co-chain complex** of Abelian groups is a collection $C = \left\{\left(C^i, \partial_i\right)\right\}_{i\in\mathbb{Z}}$ of pairs made by a family of Abelian groups and homomorphisms between them $\partial_i : C_{i-1} \to C_i$, usually called **co-differentials** or **co-boundary maps**, such that the sequential composition of any two of them, when it is well-defined, gives zero, i.e.

$$\begin{tikzcd}
C_{i-1} \arrow[r, "\partial_i"] \arrow[rr, bend right, "\partial_{i+1}\circ\partial_i=0"'] & C_i \arrow[r, "\partial_{i+1}"] & C_{i+1}
\end{tikzcd}$$

In short, they are square-zero maps.

Remark 1.3.2 In the literature, it is possible as well to find expressions like 'a complex equipped with a differential of co-homological degree +1 (of homological degree −1)'. They all refer to the previous definitions in the obvious way.

Moreover, the notion of chain and co-chain complexes are dual to each other.

Definition 1.3.5 Given a chain complex of Abelian groups $(C_\bullet, d_\bullet)$, there exist canonically defined Abelian groups

$$C^\bullet := (C_\bullet)^* \tag{1.73}$$

and co-boundaries

$$\partial_\bullet := (d_\bullet)^* \tag{1.74}$$

such that $(C^\bullet, \partial_\bullet)$ is a co-chain complex of Abelian groups.

Remark 1.3.3 We use the symbol $\cdot$ to replace the is above, as the reader is now able to define the maps with appropriate subscripts recalling the previous definitions. The symbol $*$ denotes categorical dualisation.

Definition 1.3.6 Co-chain complexes of Abelian groups and co-chain maps are objects and arrows of the category $\mathbf{CoCh}(\mathbf{Ab})$ of co-chain complexes over $\mathbf{Ab}$.

Remark 1.3.4 The symbols d^i may replace ∂_i in the literature.

Remark 1.3.5 (Co-)chain complexes are well-defined also over any Ab-category (preadditive category) $\mathbf{C}$. For example, $\mathbf{C} = \mathbf{Vect}_{\mathbb{K}}$, the category of vector spaces over a chosen field $\mathbb{K}$, or $\mathbf{C} = {}_R\mathbf{Mod}$, the category of left modules over a commutative unital ring R, $R \in \mathbf{CRng}_0$.

Definition 1.3.7 [Mac Lane [ML] (1978), §I.8, 28, line 11 et seq.] A **preadditive category** is a category $\mathbf{A}$ whose morphism sets $\mathbf{A}(a, b)$ are additive Abelian groups for every pair of object $a, b \in \mathbf{A}_0$ and for which the composition

$$\mathbf{A}(b, c) \times \mathbf{A}(a, b) \to \mathbf{A}(a, c) \tag{1.75}$$

$$(g, f) \to g \circ f \tag{1.76}$$

is a bilinear map, i.e.

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f' \tag{1.77}$$

for each parallel pair of sequentially composable arrows $f, f' \in \mathbf{A}(a, b)$, $g, g' \in \mathbf{A}(b, c)$.

Thus, we can restate the latter in terms of the tensor product over $\mathbb{Z}$: The composition is a linear map

$$\mathbf{A}(b,c) \otimes_{\mathbb{Z}} \mathbf{A}(a,b) \to \mathbf{A}(a,c) \tag{1.78}$$

and give an equivalent definition of a preadditive category

Definition 1.3.8 [ibid., 28 ff] An **Ab-category A** consists of the following data:

(i) A set of objects $\mathbf{A}_0$;

(ii) A function $(a,b) \to \mathbf{A}(a,b) \in \mathbf{Ab}_0$ assigning to each ordered pair of objects an Abelian group;

(iii) For all (a,b,c) ordered triple of objects a *composition map*

$$\mathbf{A}(b,c) \otimes_{\mathbb{Z}} \mathbf{A}(a,b) \to \mathbf{A}(a,c) \tag{1.79}$$

$$g \otimes f \to g \circ f \tag{1.80}$$

whenever the arrows involved are sequentially composable;

(iv) For any $a \in \mathbf{A}_0$ a morphism of abelian groups $\mathbb{Z} \to \mathbf{A}(a,a)$ completely determined by the image of the unit of $\mathbb{Z}$ under this map. We further define $\mathrm{Im}(1_{\mathbb{Z}}) =: 1_a$.

These data must satisfying associativity and unit laws as in the general definition of a category.

Definition 1.3.9 [ibid., 29, line 15 et seq.] Let $\mathbf{A}, \mathbf{B}$ be two Ab-categories, a functor between them $T : \mathbf{A} \to \mathbf{B}$ is said to be an **additive functor** if every

$$T : \mathbf{A}(a,a') \to \mathbf{B}(Ta, Ta') \tag{1.81}$$

is an homomorphism of Abelian groups. Namely, T is such that

$$T(f_1 + f_2) = T(f_1) + T(f_2) \tag{1.82}$$

for every $f_1, f_2 \in \mathbf{A}(a,a')$ parallel pair of arrows.

Lastly, we observe the collection of additive functors is closed under composition. Thus, there exists a well-defined category having additive functors as arrows and small Ab-categories as objects denoted by **Ab-cat**.

Definition 1.3.10 [Loday & Vallette (2012), §1..5.4] A **chain complex of vector spaces** over a field $\mathbb{K}$, is a pair (V, d) of a family of graded vector spaces $V := \{V_i\}_{i\in\mathbb{Z}}$ and linear maps $d := \{d_i : V_i \to V_{i-1}\}_{i\in\mathbb{Z}}$ between them, called differentials,

$$\cdots \to V_1 \xrightarrow{d_i} V_0 \xrightarrow{d_0} V_{-1} \to \dots \tag{1.83}$$

such that $d^2 = 0$.

Definition 1.3.11 [ibid., 25] A **degree r morphism** of chain complexes over $\mathbf{Vect}_{\mathbb{K}}$, i.e. an element of

$$\mathbf{Hom}_r\left(\left(V, d_V\right), \left(W, d_W\right)\right) := \prod_{n\in\mathbf{Z}} \mathbf{Hom}\left(V_n, W_{n+r}\right) \tag{1.84}$$

is a graded linear map

$$f : V_\bullet \to W_{\bullet+r} \tag{1.85}$$

such that

$$\partial\left(f\right) = 0 \tag{1.86}$$

where ∂ denotes the below-defined differential.

Definition 1.3.12 [ibid., line 21 et seq.] Given f a degree r morphism of chain complexes, the **differential or boundary** of f is the graded linear mapping of degree r − 1 defined by

$$\partial\left(f\right) = [d, f] := d_W \circ f - (-1)^r f \circ d_V \tag{1.87}$$

Remark 1.3.6 Co-chain complexes of vector spaces over a field $\mathbb{K}$ are defined by dualising the above definition.

Remark 1.3.7 Chain and co-chain complexes over vector spaces are commonly indicated with the name of **differential vector spaces**, as by dualisation it is possibe to switch from chain to co-chain complexes. Roughly, it is just a matter of convention on the directions of the arrows denoting differentials in our sequence of vector spaces.

Consequently, we state the following definition:

Definition 1.3.13 Differential vector spaces over the field $\mathbb{K}$ and degree r morphisms between them define the category $\mathbf{dg\text{-}Vect}_{\mathbb{K}}$ of differential vector spaces over the field $\mathbb{K}$.

Definition 1.3.14 A **differential graded associative algebra**, **dg-algebra** in short, is a monoid object in the category of chain (co-chain) complexes. They are also called chain (co-chain) algebras.

In plain terms, [Loday & Vallette (2012), §1.5.7] a dg-algebra (chain algebra) $(\mathcal{A}, d)$ is a differential graded vector space (chain complex) equipped with an associative unital multiplication map

$$\mu : \mathcal{A}_i \otimes \mathcal{A}_j \longrightarrow \mathcal{A}_{i+j} \tag{1.88}$$

Therefore, a dg-algebra carries a family of differentials of homological degree -1 such that

(i) $d_i \circ d_{i+1} = 0$

(ii) For any homogeneous element α of the graded piece $\mathcal{A}_k$, we have the assignement

$$\mathcal{A}_k \ni \alpha \longrightarrow d\alpha \in \mathcal{A}_{k-1} \tag{1.89}$$

(ii) Each differential satisfies the **graded Leibniz rule**

$$d\,(\alpha \cdot \beta) = d\alpha \cdot \beta + (-1)^k\, \alpha \cdot d\beta \tag{1.90}$$

for $\alpha \cdot \beta := \mu\,(\alpha \otimes \beta)$ product of $\alpha \in \mathcal{A}_k$ and $\beta \in \mathcal{A}_s$ any two arbitrary members.

Remark 1.3.8 The tensor product and the differential on product of chain (co-chain) complexes are well-defined using the tensor product between vector spaces. For instance, in the former case, for $(C_\bullet, d_C), (E_\bullet, d_E) \in$ **dg-Vect**$_{\mathbb{K}}$,

$$(C \otimes E)_\bullet = \sum_{i+j=\bullet} C_i \otimes E_j \tag{1.91}$$

and

$$d_{C\otimes E} = d_C \otimes id_E + id_C \otimes d_E \tag{1.92}$$

The equation holds in the graded sense, i.e.

$$d_{C\otimes E}\,(c \otimes e) = d_C\,(c) \otimes id_E\,(e) + (-1)^i\, id_C\,(c) \otimes d_E\,(e) = d_C\,(c) \otimes e + (-1)^i\, c \otimes d_E\,(e) \tag{1.93}$$

for any $c \otimes e \in C_i \otimes E_j$.

Remark 1.3.9 We use the subscript indices in the definition of chain algebra as it is common practice for homologically graded algebras. On the contrary, when the index appear as superscript the algebra is said to be co-homologically graded.

To better understand this structure, we state an example.

Example 1.3.1 Consider the tensor algebra over a certain vector field $V \in \mathbf{Vect}_{\mathbb{K}}$. We already know this is an associative unital algebra with unit $\mathbb{K}$. The graded pieces are defined as follows:

$$V^0 := \mathbb{K} \qquad\qquad V^i := \underbrace{V \otimes \cdots \otimes V}_{i \text{ times}} = V^{\otimes^i} \tag{1.94}$$

In this way, the tensor algebra is a graded vector space

$$T(V) = \bigoplus_{i \in \mathbb{N}^{\geq 0}} V^i \tag{1.95}$$

or better

$$T(V) = \bigoplus_{i \in \mathbb{N}^{\geq 0}} V^i = \bigoplus_{i \in \mathbb{N}^{\geq 0}} V^{\otimes^i} = \bigoplus_{i \in \mathbb{N}^{\geq 0}} \underbrace{(V \otimes \cdots \otimes V)}_{i \text{ times}} \tag{1.96}$$

equipped with a unital associative multiplication

$$\mu : V^i \otimes V^j \to V^{i+j} \tag{1.97}$$

Indeed,

$$V^i \otimes V^j = \underbrace{(V \otimes \cdots \otimes V)}_{i \text{ times}} \otimes \underbrace{(V \otimes \cdots \otimes V)}_{j \text{ times}} = \underbrace{V \otimes \cdots \otimes V}_{i \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{j \text{ times}} = \underbrace{V \otimes \cdots \otimes V}_{i+j \text{ times}} \tag{1.98}$$

as the tensor product is associative.

Definition 1.3.15 [Loday & Vallette [LV] (2012), §1.5.1, 23, line 17 et seq.] To each element $\alpha = \alpha_1 \dots \alpha_i \in \mathcal{A}^i$ of a graded algebra $\mathcal{A}$, commonly referred to as word, we can assign

(i) Its **weight**, defined as

$$w(\alpha) = i \tag{1.99}$$

(i) Its **degree**, defined as

$$\deg(w) = |\alpha_1| + \cdots + |\alpha_i| \tag{1.100}$$

Remark 1.3.10 The original definition is stated in the reference for graded vector spaces. It extends to graded algebra as, by definition, the latter is a graded vector spaces equipped with a unital multiplication map of degree zero.

We conclude this section with a construction we will frequently deal with in this script.

Definition 1.3.16 [ibid., line 20 et seq.] Given $V = (V_\bullet, d_\bullet)$ a dg-vector space over the field $\mathbb{K}$ and an element r of degree 1 spanning the one-dimensional vector space $\mathbb{K} \cdot r$, the **suspension of V**, also denoted by $V[1]$, is defined as

$$V[1] = \mathbb{K} \cdot r \otimes V = \langle r \rangle_{\mathbb{K}} \otimes V \tag{1.101}$$

where $\langle \bullet \rangle$ denotes the linear span generated by the element $\bullet$ over the subscript field.

Therefore, the suspension of V is a chain complex concentrated in degree one such that

$$(V[1])_i = V_{i-1} \tag{1.102}$$

and

$$d_{V[1]} = -d_V \tag{1.103}$$

1.3.1 Lie algebras and dg Lie algebras

Lie algebras and differential Lie algebras are largely involved in the application of the (pre)factorisation algebra formalism to quantum theories, as they encode symmetries. Therefore, we recall in this section the basic definitions since we will make substantial use of them when dealing with universal enveloping algebras constructed from prefactorisation algebras.

Definition 1.3.17 [Kosmann-Schwarzbach [KS] (2010), §4.1] A **Lie algebra** $\mathfrak{a}$ over [a field] $\mathbb{K}$ is a finite- or infinite-dimensional $\mathbb{K}$-vector space with a $\mathbb{K}$-bilinear, antisymmetric operation [,] satisfying the Jacobi identity

$$\forall\, X, Y, Z \in \mathfrak{a},\ \ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

The bilinear operation [,] is called **Lie bracket** (or simply bracket).

For a finite n-dimensional Lie algebra $\mathfrak{g}$ with basis $\{\gamma_i\}_{i \in I}$, where $I := \{1, \ldots, n\}$, the bracket on the basis elements reads

$$\left[\gamma_i, \gamma_j\right] = \sum_{k=1,\ldots,n} S^k_{ij} \gamma_k \tag{1.104}$$

The $\gamma_{i \in I}$ are referred to as **generators** of the Lie algebra and the complex (or real) coefficients S^k_{ij} as **structural constants**.

Definition 1.3.18 Given two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ over $\mathbb{K}$, a morphism of vector spaces $\phi \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}_1, \mathfrak{g}_2)$ defines a

Lie algebra homomorphism if it is compatible with the Lie brackets. In other words, if

$$\phi\left([X,Y]_{\mathfrak{g}_1}\right) = [\phi(X), \phi(Y)]_{\mathfrak{g}_2} \tag{1.105}$$

The category of Lie algebras and morphisms between them is denoted by $\textbf{Lie-alg}_{\mathbb{K}}$.

Definition 1.3.19 A **differential graded Lie algebra** $\mathfrak{h}$, dg Lie algebra in short, is a chain complex (V, d) equipped with an additional bilinear, graded anti-symmetric internal operation, the Lie bracket, such that

$$d[x,y] = [dx, y] + (-1)^{|x|}[x, dy] \tag{1.106}$$

$\forall\, x, y \in \mathfrak{h}$ and it respects the graded version of the Jacobi identity.

Remark 1.3.11 In other words, we have internalised the structure of Lie algebra in chain complexes.

There is another relevant functor that we need to take into account when switchng between different types of algebras, namely the universal enveloping algebra

$$\mathcal{U}(\cdot) : \textbf{Lie-alg}_{\mathbb{K}} \to \textbf{Ass-alg}_{\mathbb{K}} \tag{1.107}$$

Definition 1.3.20 [Loday & Vallette [LV] (2012), §1.1.1.10] Consider a Lie algebra $\mathfrak{g}$ over some field $\mathbb{K}$ of characteristic zero, the **universal enveloping algebra** is the associative algebra obtained as the quotient

$$\mathcal{U}(\mathfrak{g}) = \frac{\oplus_{n\geq 0}\mathfrak{g}^{\otimes^n}}{(x\otimes y - y\otimes x - [x,y])} \tag{1.108}$$

of the tensor algebra over the vector space $\mathfrak{g}$ by the two-sided ideal generated by arbitrary elements $x, y \in \mathfrak{g}$ such that the relation in parentheses hold.

Moreover, observe there exists a well-defined adjunction [Loday & Vallette [LV] (2012),§1.1.10, 8, Proposition 1.1.3] between the categories of Lie algebras and associative algebras over $\mathbb{K}$ given by

$$\mathcal{U}(\bullet) : \textbf{Lie-alg}_{\mathbb{K}} \rightleftarrows \textbf{Ass-alg}_{\mathbb{K}} : F \tag{1.109}$$

The functor $\mathcal{U}(\cdot)$ takes a Lie algebra to its universal enveloping algebra and F is the forgetful functor inducing a Lie algebra structure on an associative algebra $\mathcal{A}$ by defining the Lie bracket as

$$[x,y]_{\mathfrak{g}_{\mathcal{A}}} := x\otimes y - y\otimes x \tag{1.110}$$

It is a simple calculation to show that such internal operation is anti-symmetric, bilinear and satisfies the Jacobi identity, i.e. the bracket is well-defined.

Remark 1.3.12 Recall the notion of adjunction following [Mac Lane [ML] (1978), §IV.1, 80]: Let $\mathbf{A}$ and $\mathbf{X}$ be categories. An **adjunction** from $\mathbf{X}$ to $\mathbf{A}$ is a triple $\langle F, G, \phi \rangle : \mathbf{X} \rightharpoonup \mathbf{A}$, where F and G are functors

$$F : \mathbf{X} \rightleftarrows \mathbf{A} : G$$

while ϕ is a function which assigns to each pair of objects $x \in \mathbf{X}, a \in \mathbf{A}$ a bijection of sets

$$\phi = \phi_{x,a} : \mathbf{A}\left(Fx, a\right) \cong \mathbf{X}\left(x, Ga\right) \tag{1.111}$$

which is natural in x and a.

Chapter 2

The frame of open sets

Prefactorisation algebras are defined on open sets of a topological space X, therefore we investigate briefly their very nature.

2.1 The category of ordered sets

Definition 2.1.1 Let $X, Y \in$ **Set**, a **relation** R between X and Y, denoted by

$$R : X \nrightarrow Y \tag{2.1}$$

is a subset $R \subset Y \times X$ of the cartesian product.

Remark 2.1.1 The choice of $Y \times X$ in place of $X \times Y$ follows the established conventions in the main reference [Pedicchio & Tholen [CF] (2004)].

Further, a relation R can be represented by the **monic span**

$$Y \xleftarrow{\pi_1} R \subset Y \times X \xrightarrow{\pi_2} X \tag{2.2}$$

where $\pi_1,\ \pi_2$ denote the projection morphisms on the first and the second factor, respectively.

Remark 2.1.2 The name 'monic span' for the above diagram is due to the fact that the inclusion of the relation as subset of the cartesian product $Y \times X$ is a monomorphism.

Definition 2.1.2 Two elements $x \in X,\ y \in Y$ are said to be **R-related**, shortly xRy, iff there exists a relation R between them, i.e. $(x, y) \in R \subset X \times Y$.

Example 2.1.1 Let **Set** be the category of sets and $\subseteq$, denote the relation given by ordering the sets by inclusion. Two sets X, Y are $\subseteq$ - related, $X \subseteq Y$ iff X is contained in Y.

Definition 2.1.3 An **odered set** is a pair (X, R) made of a set X and a relation R satisfying the axioms of reflexivity, i.e. $1_X \subseteq R$, and transitivity, i.e. $RR \subseteq R$. Such a relation is usually denoted by $\leq$.

Remark 2.1.3 In more explicit terms, a relation R on a set X is said to be transitive iff

$$xRx \qquad \forall x \in X \tag{2.3}$$

and transitive iff $\forall\, x, y, z \in X$,

$$xRy \text{ and } yRz \;\Rightarrow\; xRz \tag{2.4}$$

Definition 2.1.4 Taken $(X, R)\,, (Y, R')$ two ordered set, a morphism $f : X \to Y$ is **order-preserving** iff

$$xRy \Rightarrow (fx)\, R'\, (fy) \tag{2.5}$$

We denote by **Ord** the category of order-preserving morphisms and ordered sets.

Furthermore, if we consider a parallel pair of arrows between ordered sets $(X, \leq)\,,\ (Y, \leq) \in \mathbf{Ord}_0$,

$$f, g : (X, \leq) \rightrightarrows (Y, \leq) \tag{2.6}$$

it is possible to impose an order on such morphisms by declaring

$$f \leq g \iff \forall x \in X\ fx \leq gx \text{ in Y} \tag{2.7}$$

i.e. by looking at their pointwise behaviour.

Thus, there exists an endofunctor ϕ between any two arrows $f, g \in \mathbf{Ord}\,(X, Y) \iff f \leq g$

$$\phi : \mathbf{Ord}\,(X, Y) \to \mathbf{Ord}\,(X, Y) \tag{2.8}$$

$$f \mapsto g \tag{2.9}$$

In formulas,

$$(\mathbf{End}\,(\mathbf{Ord}\,(X, Y))\,, \leq) \tag{2.10}$$

is an ordered set.

Consider the following diagram, where T stays for the terminal object in the category **Set**, namely the point-set $\{*\}$. Then the pointwise evaluations of each arrow are represented as maps with common domain T

$$\begin{tikzcd}
T \arrow[d, "x"'] \arrow[dr, "fx\leq gx"] \arrow[drr, dashed, "hfx\leq hgx"] & & \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & Z
\end{tikzcd}$$

If $f \leq g$, we have $fx \leq gx \ \forall x \in X$, by definition, and $\forall h$, sequentially composable arrow, $hf \leq hg$. Indeed, if we define new arrows $hf := F, \ hg := G$, the pointwise evaluations of f and g yield $Fx \leq Gx$ at each point, i.e. $F \leq G$.

Therefore, the composition operation

$$\mathbf{Ord}\,(X, Y) \times \mathbf{Ord}\,(Y, Z) \xrightarrow{\circ} \mathbf{Ord}\,(X, Z) \tag{2.11}$$

$$(f, h) \to h \circ f \tag{2.12}$$

is well-defined and every single member family of objects, that we indicate with notation 1, is mapped as follows

$$1 \to \mathbf{Ord}\,(X, X) \tag{2.13}$$

These arrows give the categorical structure of **Ord**.

Remark 2.1.4 As we have shown in the above diagram, the composition is well-defined and preserves orders. Moreover, there exists a notion of one-object family whose corresponding order exists due to the reflexivity of the order. This implies the existence of the identity arrow on objects, i.e. $1_X \ \forall\, (X, \leq) \in \mathbf{Ord}_0$.

On the direct product of two ordered set, say X and Y, we define a relation as follows

$$(x, y) \leq (v, w) \iff x \leq v \text{ and } y \leq w \tag{2.14}$$

Thus, $(X \times Y, \leq)$ is a product in **Ord**, with order-preserving projection maps.

Thus, the **diagonal functor**

$$\Delta_X : X \to X \times X \tag{2.15}$$

is order-preserving. Indeed, we have the following diagrammatic representation of the arrows involved:

$$\begin{tikzcd}
(X, \leq) \arrow[r, tail, "\Delta_X"] \arrow[dr, "id"'] & (X \times X, \leq) \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"] & (X, \leq) \arrow[dl, "id"] \\
& (X, \leq) &
\end{tikzcd}$$

Moreover, assume there are adjunctions to the diagonal functor defined as follows:

$$\vee \dashv \Delta_X \vdash \wedge \tag{2.16}$$

called **join** $\vee$ (left adj.) and **meet** $\wedge$ (right adj.), respectively.

Definition 2.1.5 The **meet** of two elements x and y in X, an ordered set, is defined as a map

$$(x, y) \mapsto \wedge (x, y) = x \wedge y \tag{2.17}$$

with the property that for all $z \in X$,

$$z \leq x \wedge y \iff z \leq x \text{ and } z \leq y \tag{2.18}$$

Definition 2.1.6 The **join** of two elements x and y in X is defined as a map

$$(x, y) \mapsto \vee (x, y) = x \vee y \tag{2.19}$$

with the property that for all $z \in X$,

$$x \vee y \leq z \iff x \leq z \text{ and } y \leq z \tag{2.20}$$

Remark 2.1.5 The fact that the join and the meet of two elements are defined as maps instead of elements is a subtle point and follows by noticing that the defining properties listed above yield the meet and the join elements as unique up to isomorphism. This reasoning appears in [Wood [CF] (2004). §I.2.4, 19, ex. (5)], where it is clearly shown that the meet exists and it is *the* greatest lower bound of two given elements of a poset. Therefore, we are only allowed to give them as maps once we proved their existence and uniqueness for every ordered pair of elements of an ordered set.

Remark 2.1.6 It is apparent that the notion of meet and join are dual to each other.

Therefore, the diagonal functor has such adjoints iff for each pair (x, y) there exists an element called join of x and y (meet of x and y) such that, for all z, the defining property is verified.

By taking $z = x \wedge y$, it is immediate to verify that $\wedge$ is a greatest lower bound operation, unique up to isomorphism. Dually, the join of any two elements, $w = x \vee y$ is a least upper bound operation.

2.2 Lattices as algebraic objects

Definition 2.2.1 A relation $R : X \nrightarrow X$ is said to be **antisymmetric** iff

$$xRy \text{ and } yRx \Rightarrow x = y \tag{2.21}$$

Definition 2.2.2 A set X on which we define a reflexive, antisymmetric and transitive relation R is called **partially ordered set**.

Remark 2.2.1 In the literature, sets are often called partially ordered when equipped with a transitive, reflexive and antisymmetric relation. Moreover, the term pre-ordered set is used in place of the above-defined ordered set. For (X, R) an ordered set,

$$x \leq y \wedge y \leq x \Rightarrow x \cong y \tag{2.22}$$

We will use such antisymmetry in a while. First, let us give general definition for semilattices and lattices.

Definition 2.2.3 A **meet semilattice** is an ordered set $(X, \leq)$ for which the canonical span

$$1 \leftarrow X \rightarrow X \times X \tag{2.23}$$

has right adjoints. Dually, a **join semilattice** is an ordered set for which the canonical span has left adjoints.

Remark 2.2.2 Duality can be rephrased by saying that X is a meet semilattice iff X^{op} is a join semilattice.

A caracterisation of adjunctions in this setting is given by

Proposition 2.2.1 [Wood [CF], §I.3.2, 21, Proposition] If $f \dashv u : A \to X$, f preserves any bottom elements and joins that exist in X and, dually, u preserves any top elements and meets that exist in A.

Definition 2.2.4 A **bottom element** for an ordered set $(X, \leq)$ is an element

$$0 : 0 \leq x \ \forall x \in X \tag{2.24}$$

The dualisation of the previous definition yields the definition of **top element**.

We prove the equivalence between antisymmetric join semilattices and idempotent structures on sets, following [CF]:

Lemma 2.2.1 [Wood [CF] (2004), §I.3.1, 20, Lemma] An antisymmetric join semilattice is equivalently described as a set X (with no order assumed) together with an idempotent commutative monoid structure.

Proof

Let $(X, \vee, 0)$ be a idempotent, commutative, monoidal structure over a set X. In formulas, a monoidal structure carrying two additional properties:

(i) (*idempotence*) $x \vee x = x$;

(ii) (*commutativity*) $x \vee y = y \vee x$.

Define

$$x \leq y \iff x \vee y = y \tag{2.25}$$

We want to show that this implies that $X \in \mathbf{Set}_0$ is an antisymmetric join semilattice.

For it, firstly we need to prove reflexivity, transitivity and antisymmetry of the relation $\leq$. By using the definition above, we obtain:

(i) $x \vee x = x \Rightarrow x \leq x$;

(ii) $x \leq y \vee y \leq z \Rightarrow x \vee z = (x \wedge y) \vee z = y \vee z = z$;

(iii) $x \leq y \vee y \leq y \Rightarrow x \vee y = y = y \vee x \Rightarrow x = y$.

Secondly, the existence of the bottom element in X is given by the implication:

$$0 \vee x = x \Rightarrow 0 \leq x \ \ \forall x \in X \tag{2.26}$$

Therefore 0, the unit element of the monoidal structure, is the bottom element.

Thirdly, let $x, y \in X$, we have

$$x \vee (x \vee y) = (x \vee x) \vee y = x \vee y \Rightarrow x \leq x \vee y \tag{2.27}$$

$$y \vee (y \vee x) = (y \vee y) \vee x = y \vee x \Rightarrow y \leq y \vee x \tag{2.28}$$

Thus, $x \vee y$ is an upper bound for the set $\{x, y\}$.

Finally, we need to prove $\vee$ defines the operation of join, as in Definition 2.1.6:

Take $x, y, z \in X : x \leq z, y \leq z$, then

$$(x \vee y) \vee z = x \vee y \vee z = x \vee z = z \Rightarrow x \vee y \leq z \tag{2.29}$$

i.e. $x \vee y$ is the join.

Conclusion: The idempotent commutative monoid structure on X implies that X is an antisymmetric join semilattice.

For the reverse implication, suppose $(X, \leq)$ is an antisymmetric join semilattice.

The operation of join, $\vee : X \times X \rightarrow X$, is internal since, by definition, $x \vee y = y \Rightarrow x \leq y$ and the latter is a member of $(X, \leq)$ for every pair of element $x, y \in X$.

The bottom element 0, works as unit of the monoidal structure, since $x \leq 0 \Rightarrow x \vee 0 = x$.

As the relation $\leq$ is reflexive, we have

$$x \leq x \Rightarrow x \vee x = x \tag{2.30}$$

where the implication follows from the definition above.

Due to the antisymmetry property of the relation,

$$(x \leq y) \vee (y \leq x) \Rightarrow (x \vee y = y) \wedge (y \vee x = x) \Rightarrow x = y \Rightarrow x \vee y = y \vee x \tag{2.31}$$

therefore, we have commutativity.

$(X, \vee, 0)$ is a monoid structure over X.

□

Definition 2.2.5 A semilattice that is both meet and join semilattice defines a **lattice**.

Proposition 2.2.2 [Wood [CF] (2004), §I.3.1, 21, Proposition] An antisymmetric lattice is equivalently described as a set X with idempotent commutative monoid structures $(X, \vee, 0)$ and $(X, \wedge, 1)$ related by the 'absorptive laws'

$$x \wedge (x \vee y) = x \text{ and } x \vee (x \wedge y) = x \tag{2.32}$$

Antisimmetric semilattices are purely algebraic objects: They are monoidal categories equipped with a product, join [meet], that is commutative and idempotent, i.e. $x \wedge y = y \wedge x$ and $x \wedge x = x$. Similarly, for $\vee$.

2.3 The Heyting lattice of open sets

Definition 2.3.1 A lattice is said to be **Heyting** iff for each x the order-preserving arrow $x \wedge \bullet$ has a right adjoint $x \Rightarrow \bullet$. If such a Heyting lattice is also antisymmetric, we call it **Heyting algebra**.

Remark 2.3.1 Notice that, if the right adjoint $x \Rightarrow z$ exists, it is a largest element whose meet with x is less or equal than z, for all z.

Definition 2.3.2 The **downsets** of an ordered set $(Y, \leq)$ are the subsets S of Y such that

$$x \leq y \text{ and } y \in S \Rightarrow x \in S \tag{2.33}$$

The collection of all downsets of X with order induced by inclusion of parts is denoted by $\mathbb{D}X$.

Regarding completeness, using the equivalent formulations in [Wood [CF] (2004), §I.5.1, 36, Theorem], we can state the definition as follows:

Definition 2.3.3 An ordered set X is said to be **complete** iff for every subset S of X there exists $\bigvee S$ a least upper bound, or supremum, unique up to isomorphism.

Proposition 2.3.1 [Wood [CF] (2004),§I.4.8, 35, Corollary and §I.5.5, 41, Corollary n.3] Consider a topological space $X \in \mathbf{Top}^0$, denote by $\mathbb{O}(X)$ the lattice of open sets ordered by inclusion, then $\mathbb{O}(X)$ is a complete Heyting lattice.

2.4 Locales and frames

In this section we follow [Picado, Pultr & Tozzi [CF] (2004), §II]. We have shown in the previous section a topological space X comes equipped with a complete lattice of open sets $(\mathbb{O}(X), \subseteq)$, as the union (join) of an arbitrary family of open sets is still open. Moreover, the **infinite distributive law** holds

$$U \wedge \bigvee_{i \in I} V_i = \bigvee (U \wedge V_i) \tag{2.34}$$

Remark 2.4.1 In this context, arbitrary meets are defined as the interior of arbitrary intersections, i.e.

$$\wedge_{i \in I} V_i = \operatorname{int}(\cap_{i \in I} V_i) \tag{2.35}$$

Given a morphism of topological spaces $X \xrightarrow{f} Y$, the associated lattice morphism

$$g : \mathbb{O}(Y) \to \mathbb{O}(X) \tag{2.36}$$

$$U \to g(U) = f^{-1}(U) \tag{2.37}$$

preserves arbitrary joins and finite meets. In the context of sets, these operations correspond to union (join) and intersection (meet).

Thus, we have a contra-variant functor

$$\Phi : \mathbf{Top} \to \mathbf{Frm} \tag{2.38}$$

$$X \mapsto \mathbb{O}(X) \tag{2.39}$$

$$f \mapsto f^{-1} \tag{2.40}$$

where **Frm** denotes the category of frames as defined below.

Definition 2.4.1 The category **Frm** of frames has as objects complete lattices $\mathbb{L}$ such that

$$l \wedge \bigvee P = \bigvee \{l \wedge p \mid p \in P\} \tag{2.41}$$

for every $l \in \mathbb{L},\ P \subseteq \mathbb{L}$. Its class of morphisms, called **frame homomorphisms**, are arrows preserving all joins and finite meets, including their respective identity elements $0, 1$.

Example 2.4.1 This definition generalises the properties of open sets to other contexts such as complete chains, totally ordered sets with a least upper bound (or subsets of posets, depending on the context). An order is said to be total when, in addition to the properties of reflexivity, transitivity and antisymmetry, it is required that, for every two elements $t, s \in (T, \leq)$, either $t \leq s$ or $s \leq t$.

Definition 2.4.2 The category **Loc** of localic maps and locales is defined as

$$(\mathbf{Frm})^{\mathrm{op}} =: \mathbf{Loc} \tag{2.42}$$

With this definition we gain a covariant functor

$$\mathbf{Top} \xrightarrow{\mathrm{Lc}} \mathbf{Loc} \tag{2.43}$$

To each localic map $\phi : X \to Y$ is associated a frame morphism $\phi^* : \mathbb{O}(Y) \to \mathbb{O}(X)$ that has a right adjoint

$$\phi_* : \mathbb{O}(X) \to \mathbb{O}(Y) \tag{2.44}$$

$$U \to \bigvee \{V \mid \phi^*(V) \leq U\} \tag{2.45}$$

Remark 2.4.2 Until this point we made use of the notation $(\mathbb{O}(X), \subseteq)$ for denoting the frame of open sets, since we have not categorified the underlying poset yet. We change it to $\mathbf{Open}_X$, when we aim at indicating the relative category of open sets of a topological space $X \in \mathbf{Top}_0$. Categories are in fact always referred to with boldfaced characters in this script.

Chapter 3

Prefactorisation Algebras

Having largely provide an overview of the basic mahematical tools upon which this algebraic theory is based, we proceed to the definition of prefactorisation algebras. After a first approximative definition, as given in the main reference [CG1], we introduce the notions of multicategory, algebras over operads and poset categories, then we state and discuss the definition given [*ibid.*, §3.1.2] in multicategorical terms. This chapter ends with a new definition, that improves it correcting the issues arised.

We start with a raw definition, following [Costello & Gwilliam [CG1] (2016), §1.2, Definition 2.0.1].

Definition 3.0.1 (First raw definition) A prefactorisation algebra $\mathcal{F}$ on $X \in \mathbf{Top}_0$, taking values in co-chain complexes over a category $\mathbf{C}$, is a rule that assigng a co-chain complex to each open set $U \in \mathbf{Open}_X^0$

$$\mathbf{Open}_X^0 \ni U \longrightarrow \mathcal{F}(U) \in \mathbf{CoCh}(\mathbf{C})_0 \tag{3.1}$$

so that there exist

1. A co-chain map $\mathcal{F}(U) \xrightarrow{\mathcal{F}(\iota_{UV})} \mathcal{F}(V) \in \mathbf{CoCh}(\mathbf{C})_1$ for every inclusion $U \overset{\iota_{UV}}{\hookrightarrow} V$ of opens in $\mathbf{Open}_X^1$;
2. A co-chain map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V) \tag{3.2}$$

   for each finite family of pairwise disjoint open sets contained in V, i.e. $\forall\ \{U_i\}_{i=1}^n \subset \mathbf{Open}_X^0 :\ U_i \cap U_j = \emptyset$ for $i \neq j,\ U_i \in V\ \forall i \in \{1, \ldots, n\}$, induced by the inclusions.

Moreover, compatibility conditions are required. For instance, $\mathcal{F}(\iota_{UV}) \in \mathbf{CoCh}(\mathbf{C})_1$ factorises as in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[rr, "\mathcal{F}(\iota_{UV})"] \arrow[dr, "\mathcal{F}(\iota_{UW})"'] & & \mathcal{F}(V) \\
& \mathcal{F}(W) \arrow[ur, "\mathcal{F}(\iota_{WV})"'] &
\end{tikzcd}$$

whenever the chain of inclusions $U \xhookrightarrow{\iota_{UV}} V \xhookrightarrow{\iota_{VW}} W$ for $U,\ V,\ W \in \mathbf{Open}^0_X$ holds.

This definition can be furter improved by observing we are in fact dealing with multicategories where the family of open sets with additional properties above can be regarded as a generalised poset in the spirit of [Leinster [TL] (2003), Example 2.1.7]. Therefore, we begin by defining and analysing the structures of interest in their generality.

3.1 Multicategories and Operads

Definition 3.1.1 [Leinster [TL] (2003), §2.1, Definition 2.1.1, 35] A **multicategory C** consists of a family of objects, denoted by $\mathbf{C}_0$, and, $\forall n \in \mathbb{N}$, a morphisms class

$$\mathbf{C}\left(a_1, \dots, a_n; a\right) \tag{3.3}$$

taking a finite string of input objects $(a_1, \dots, a_n)$ to a single output object a. An arrow in this class, can be depicted as

$$a_1, \dots, a_n \mapsto a$$

Compositions and identities are defined as:

(i) $\forall n, k_1, \dots, k_n \in \mathbb{N}, a, a_i, a_i^j \in \mathbf{C}_0$

$$\mathbf{C}\left(a_1, \dots, a_n; a\right) \times \mathbf{C}\left(a_1^1, \dots, a_1^{k_1}; a_1\right) \times \cdots \times \mathbf{C}\left(a_1^1, \dots {a_n}^{k_n}; a_n\right)$$

$$\downarrow$$

$$\mathbf{C}\left(a_1^1, \dots, a_1^{k_1}, \dots, a_n^1, \dots, a_n^{k_n}; a\right)$$

$$(\theta, \dots, \theta_n) \mapsto \theta \circ (\theta_1, \dots \theta_n) \tag{3.4}$$

(ii) $\forall\, a \in \mathbf{C}_0, \exists\, 1_a \in \mathbf{C}\left(a; a\right)$

and they satisfy

(a.) Associativity:

$$\theta \circ \left(\theta_1 \circ \left(\theta_1^1, \ldots, \theta_1^{k_1}\right), \ldots, \theta_n \circ \left(\theta_n^1, \ldots, \theta_n^{k_n}\right)\right) = \tag{3.5}$$

$$= (\theta \circ (\theta_1, \ldots, \theta_n)) \circ \left(\theta_1^1, \ldots, \theta_1^{k_1}, \ldots, \theta_n^1, \ldots, \theta_n^{k_n}\right) \tag{3.6}$$

$\forall\, \theta, \theta_i, \theta_i^j$ for which this composition makes sense.

(b.) Identity law:

$$\theta \circ \left(1_{a_1}, \ldots, 1_{a_n}\right) = \theta = 1_a \circ \theta \tag{3.7}$$

for any morphism $\theta : a_1, \ldots, a_n \mapsto a$.

For the sake of visualisation, see this picture from [Leinster [TL] (2003), §2]:

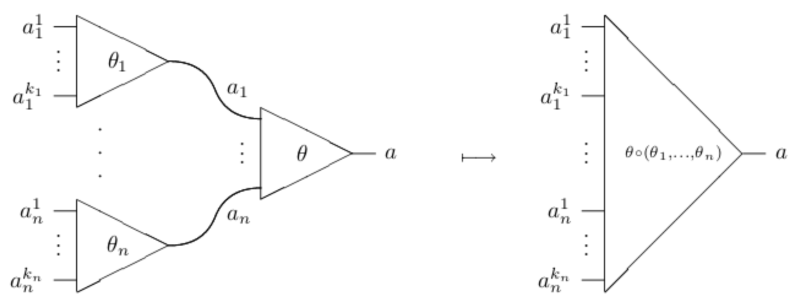

Figure 3.1: Rule of composition

Definition 3.1.2 [Leinster [TL] (2003), §2.1, Definition 2.1.9] Let **C** and **D** be two multicategories, a **map of multicategories**, also called **multifunctor**,

$$\Phi : \mathbf{C} \to \mathbf{D} \tag{3.8}$$

is defined on objects as

$$\Phi : \mathbf{C}_0 \to \mathbf{D}_0 \tag{3.9}$$

$$a \to \Phi(a) \tag{3.10}$$

and on morphisms as

$$\mathbf{C}(a_1, \ldots, a_n; a) \longrightarrow \mathbf{D}(\Phi(a_1), \ldots, \Phi(a_n); \Phi(a)) \tag{3.11}$$

$$\nu \longrightarrow \Phi(\nu) \tag{3.12}$$

$\forall\, a_1, \ldots, a_n, a \in \mathbf{C}_0.$

It respects composition and identities:

(i) If ν_1, ν_2 are any two sequentially composable arrows (in the sense of multicategories), then the functor (assumed covariant) works as follows:

$$\Phi\left(\nu_2 \circ \nu_1\right) = \Phi\left(\nu_2\right) \circ \Phi\left(\nu_1\right) \tag{3.13}$$

(ii) If 1_a is the identity arrow over an arbitrary object $a \in \mathbf{C}_0$,

$$\Phi\left(1_a\right) = 1_{\Phi(a)} \tag{3.14}$$

The small multicategories with functors between them form the category of multicategories, that we will denote by **Multicat**.

Remark 3.1.1 Multicategories are generalisations of categories. Indeed, by considering input strings made of a single objectand mapping them to a single co-domain object, we reconstruct the notion of arrows in a category, i.e. $\alpha \to \beta$. Arrows of this sort, with one object as domain and one as co-domain are said to be **unary**.

Definition 3.1.3 [[Leinster [TL] (2003), §2.2, Definition 2.2.2] A multicategory **C** whose object class consists of a single object is said to be an **operad**.

For **C** being an operad, morphisms classes are of the form

$$P\left(n\right) := \mathbf{C}\big(\underbrace{\clubsuit, \ldots, \clubsuit}_{\text{n}}; \clubsuit\big) \tag{3.15}$$

$\forall\, n \in \mathbb{N}$. This implies

Definition 3.1.4 An operad **C** is a family of n-ary operations

$$\theta := \left\{P\left(n\right)\right\}_{n \in \mathbb{N}} \tag{3.16}$$

whose composition is defined, $\forall n, k_1, \ldots, k_n \in \mathbb{N}$, as

$$P\left(n\right) \times P\left(k_1\right) \times \cdots \times P\left(k_n\right) \to P\left(k_1 + \cdots + k_n\right) \tag{3.17}$$

$$\left(\theta, \theta_1, \ldots, \theta_n\right) \to \theta \circ \left(\theta_1 \circ \cdots \circ \theta_n\right) \tag{3.18}$$

and with identity

$$1 = 1_P \in P(1) \tag{3.19}$$

satisfying associativity and identity laws as in the definiition of a multicategory.

Definition 3.1.5 [*ibid.*, 43] An **arrow of operads** $P \xrightarrow{f} Q$ is a family of morphisms

$$\{f_n : P(n) \to Q(n)\}_{n \in \mathbb{N}} \tag{3.20}$$

preserving composition and identities.

Definition 3.1.6 Operads and arrows between form a category. We denote with the symbol **Operad**.

Remark 3.1.2 This construction can be generalised by taking the hom-sets $P(n)$ to have values in a symmetric monoidal category **C**, instead of being sets. This assumptions gives the notion of **C-operad**.

Remark 3.1.3 A multicategory is sometimes called **coloured operad**, as it is indeed a multi-object operad. Suppose the operad **C** has ♣ as object, then a morphism over n such objects is a map in $\mathbf{C}(n)$

$$g : \clubsuit, \ldots, \clubsuit \to \clubsuit \tag{3.21}$$

Substituting ♣ with different objects [the colours], yields a multicategory, i.e.

$$g' : \clubsuit, \spadesuit, \heartsuit \to \diamondsuit \tag{3.22}$$

Notice in this example, we used a standard French suited deck without jacks, therefore the set of "colours" is made of exactly four elements, the procedure is complete general though.

Example 3.1.1 Vector spaces over $\mathbb{K}$ with multilinear arrows classes $\mathbf{Vect}_{\mathbb{K}}(V_1, \ldots, V_n; W)$ $\forall n \in \mathbb{N}$ define a multicategory. To make the latter an operad, we just modify the length n input string of vector spaces to contain n copies of the same vector space W, the correponding arrow classes are denoted by $\mathbf{Vect}_{\mathbb{K}}(W, \ldots, W; W)$. This multicategory has the additional property to be enriched, as the hom-sets are also vector spaces.

To every (not necessarily strict) monoidal category $(\mathbf{B}, \square, e)$ is possible to associate a multicategory **C**. As from [Leinster [TL] (2003), §2.1, Example 2.1.3], each arrow in **B**

$$\phi : \beta_1 \square \ldots \square \beta_n \longrightarrow \beta \tag{3.23}$$

is mapped to an arrow in **C**

$$\phi' : \beta_1,\ \ldots,\ \beta_n \longrightarrow \beta \tag{3.24}$$

with composition, associativity and identity laws induced by the product $\square$.

Analogously, we can start with a category **A** having finite products and we associate morphisms in a multicategory **C** by

$$\mathbf{C}\,(\alpha_1, \ldots, \alpha_n; \beta) := \alpha_1 \times \cdots \times \alpha_n \to \beta \ \in\ \mathbf{A}_1 \tag{3.25}$$

Example 3.1.2 Let $R \in$ **CRng** be a commutative ring, the monoidal category $(\mathbf{Mod}_R, \otimes, R)$ can be seen as a multicategory by applying the procedure described above: the class of objects stays the same, namely R-modules, and morphisms are R-multilinear maps of the form $\mathbf{Mod}_R\ (M_1, \ldots, M_n; N)\ \forall n \in \mathbb{N},\ \{M_i\}_{i=1}^n, N \in \mathbf{Mod}_R^0$.

Remark 3.1.4 The viceversa it is not necessarily true, as there are multicategories that are not underlying any monoidal category. For instance, consider a sub-collection $\mathbf{D}_0 \subseteq \mathbf{B}_0$ of objects in a monoidal category $(\mathbf{B}, \otimes, e)$, with the property of not being closed under tensor product. On such collection we can define arrows, but we cannot have an underlying monoidal category, since there exists at least one object $b \in \mathbf{B}_0$ that is *not representable* as tensor product of elements in $\mathbf{D}_0$.

The following example is key to understand how an algebra is generated starting from a multicategory **C**.

Example 3.1.3 [Leinster [TL] (2003), §2, example 2.1.16] Any family $\{X\,(a)\}_{a \in S}$ of sets, indexed over S gives a multicategory $\mathbf{End}\,(X)$ with obects-set S, called the **endomorphism multicategory** of X. An arrow $\mathbf{End}\,(X)\,(X\,(a_1)\,, \ldots, X\,(a_n)\,; X\,(a))$ corresponds to

$$\mathbf{Set}\,(X\,(a_1)\,, \ldots, X\,(a_n)\,; X\,(a)) \tag{3.26}$$

i.e.

$$\mathbf{Set}\,(X\,(a_1) \times \cdots \times X\,(a_n)\,; X\,(a)) \tag{3.27}$$

the rules of associativity, identity and composition are as in **Set**.

3.1.1 Symmetric multicategories

Operads equipped with an action of the group Σ of permutations over the objects reserve a special definition. Let Σ_n denote the symmetric group over n elements.

Definition 3.1.7 [Leinster [TL] (2003), §2.2, Definition 2.2.21] A **symmetric multicategory** is a multicategory $\mathbf{C}$ together with a map

$$\bullet \cdot \sigma : \mathbf{C}\left(a_1, \ldots, a_n; b\right) \to \mathbf{C}\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}; b\right) \tag{3.28}$$

$\forall a_1, \ldots, a_n, b \in \mathbf{C}_0$, $\forall \sigma \in \Sigma_n$. such that

$$(\theta \cdot \sigma) \cdot \rho = \theta\left(\sigma \rho\right) \tag{3.29}$$

$$\theta = \theta \cdot 1_{\Sigma_n} \tag{3.30}$$

where $\theta \in \mathbf{C}\left(a_1, \cdot, a_n; b\right)$, $\sigma,\ \rho \in \Sigma_n$. Such a map is therefore a bijection.

Moreover, compatibility with composition reads

$$(\theta \cdot \sigma) \circ \left(\phi_{\sigma(1)} \cdot \pi_{\sigma(1)}, \ldots, \phi_{\sigma(n)} \cdot \pi_{\sigma(n)}\right) = \left(\theta \circ \left(\phi_1, \ldots, \phi_n\right)\right) \cdot \left(\sigma \circ \left(\pi_{\sigma(1)}, \ldots, \pi_{\sigma(n)}\right)\right) \tag{3.31}$$

$\forall \theta, \phi_1, \ldots, \phi_n \in \mathbf{C}^1$ and $\sigma, \pi_1, \ldots, \pi_n \in \Sigma_n$ for which the expression above is well-defined.

In short, a symmetric multicategory comes equipped with an arrow

$$\bullet \cdot \sigma : \mathbf{C}\left(a_1, \ldots, a_n; a\right) \xrightarrow{\sim} \mathbf{C}\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}; a\right) \tag{3.32}$$

$\forall a_1, \ldots, a_n, a \in \mathbf{C}_0$ and $\sigma \in \Sigma_n$.

Definition 3.1.8 [Leinster [TL] (2003), §2.2, Definition 2.2.21] An **arrow of symmetric multicategories** is a map of symmetric multicategories $f : \mathbf{C} \to \mathbf{D}$ such that

$$f\left(\theta \cdot \sigma\right) = f\left(\theta\right) \cdot \sigma \tag{3.33}$$

for any $\theta \in \mathbf{C}^1, \sigma \in \Sigma_\bullet$ for which the above expression is well-defined.

We denote by **SymMulticat** the category of symmetric multicategories.

Definition 3.1.9 A **symmetric operad** is a one-object symmetric multicategory. An operad that is not provided with a symmetric structure is called **non-symmetric operad** or **non-Σ operad**.

Remark 3.1.5 Any *symmetric* monoidal category $(\mathbf{B}, \square, e)$ is naturally converted into a symmetric multicategory defining arrows as we have already discussed and observing, there exists a symmetry map

$$\sigma \cdot \ \bullet : a_{\sigma(1)} \square \cdots \square a_{\sigma(n)} \xrightarrow{\sim} a_1 \square \cdots \square a_n \tag{3.34}$$

This applies in particular to the category of sets **Set**. Another relevant example is the multicategory of vector spaces over a given field $\mathbb{K}$ with multiarrows induced by the tensor product $\otimes$.

To visualise symmetry we refer to the diagrams from [Leinster [TL] (2003), §2.2, 53]:

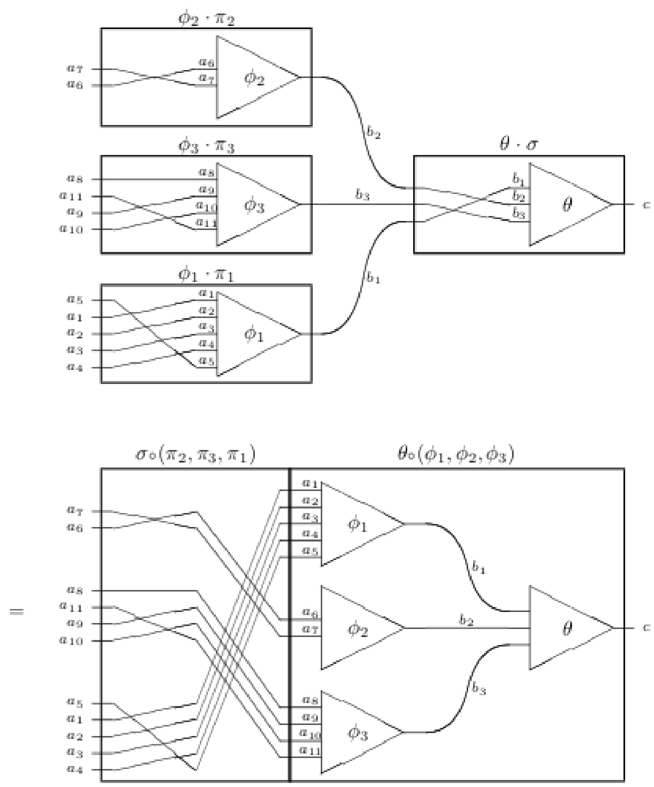

Figure 3.2: Symmetry implementation

3.2 Algebras for an operad

Let **C** denote a multicategory (coloured operad).

Definition 3.2.1 [Leinster [TL] (2003), §2.1, Definition 2.1.12] A **C-algebra**, or algebra for **C**, consists of the following

(i) $\forall a \in \mathbf{C}$, a set $X(a)$;

(ii) For each map $\theta : a_1, \ldots, a_n \to a$ in **C** an arrow

$$\overline{\theta} : X(\theta) : X(a_1) \times \cdots \times X(a_n) \to X(a) \tag{3.35}$$

satisfying associativity and identity axioms as in the general definition of multicategories.

For a better understanding, see the diagrammatic pictures below from [TL].

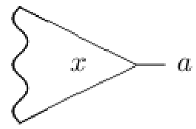

Figure 3.3: Set X(a) whose elements x are labelled by a

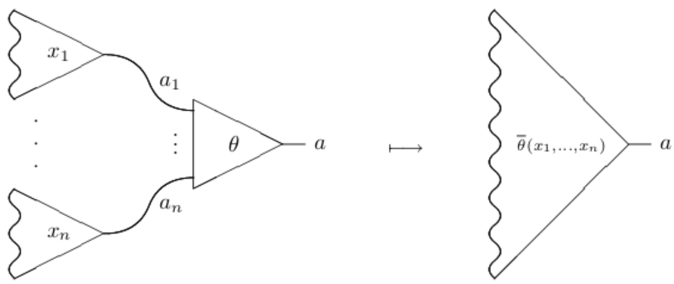

Figure 3.4: Algebra arrow

The definition just provided can be equivalently shortly stated as follows:

Definition 3.2.2 [ibid., Definition 2.1.12] An **algebra for C** is a map from **C** into the multicategory of **Set**.

In particular, notice that such definition extends naturally to the class of symmetric multicategories, since the multicategory of sets is naturally equipped with a symmetry map.

Definition 3.2.3 [Leinster [TL] (2003), §2.2, 54, line 15 et seq.] An **algebra for a symmetric multicategory C** is a map

$$\mathbf{C} \longrightarrow \mathbf{Set} \tag{3.36}$$

of symmetric multicategories.

Remark 3.2.1 In this context, the multicategory of sets we referred to is the one arising from the monoidal structure $(\mathbf{Set}, \times, 1)$.

Definition 3.2.4 [Leinster [TL] (2016), §2.1, 42, line 2 et seq.] A **map of C-algebras** $\alpha : X \to Y$ is a family of functions

$$\left\{X\left(a\right) \overset{\alpha_a}{\to} Y\left(a\right)\right\}_{a\in C_0} \tag{3.37}$$

satisfyig the evident compatibility condition.

The category of **C**-algebras is denoted by $\mathbf{Alg}(\mathbf{C})$.

3.3 The category of Prefactorisation Algebras

We want to apply the formalism just developped in order to obtain a rigorous definition of prefactorisation algebras in the style of multicategories. For it, we need to rediscuss the class of open sets on which the theory is built, as they are required to satisfy additional properties that modify the usual frame structure.

3.3.1 Connected opens as poset category

When dealing with measurements in the quantum world, we need to modify the frame of opens we are working with as we require the supports of the measurements to be disjoint. Therefore, it is useful for our later purposes to define a new collection of open sets of a space X with some additional properties.

We have already analysed order relations in the first chapter. However, we recall the notion of poset in order to categorify it.

Definition 3.3.1 A **poset** $(P, \leq)$ is a set $P \in \mathbf{Set}_0$ together with a **partial order** $\leq$, namely a relation on P that satisfies the properties of reflexivity, transitivity and antisymmetry.

Every poset can be categorified to a poset category, namely:

Definition 3.3.2 A **poset category** consists of elements of a poset P as class of objects $\mathbf{P}_0$ and arrows defined by

$$\exists\, f : x \to y \iff x \leq y \tag{3.38}$$

Otherwise, the hom-set $\mathbf{P}(x, y) = \emptyset$.

Remark 3.3.1 Notice we use the boldface **P** in order to distinguish the categorified poset from the poset.

A first idea would be to categorify the sub-family of $(\mathbb{O}(X), \subseteq)$ consisting of disjoint connected open sets. However, in order to define the structure map of a PFA we require such disjoint open sets to be included in a bigger part, e.g. V. Thus, the objects will be ill-defined, as V is a part cointaining at least another object U, therefore not disjoint from U. As U is completely arbitrary, same reasoning applies if we take any finite string $(U_1, \ldots, U_n)$: the single members, and therefore their disjoint union, are not disjoint from V by construction.

In the main reference of this script, [Costello & Gwilliam [CG1] (2016)], the multicategorical structure on such open sets and the notion of prefactorisation algebra are defined, for a topological space M, as follows:

Definition 3.3.3 [Costello & Gwilliam [CG1] (2016), §3.1.2, Definition 1.2.1] Let $\mathbf{Disj}_M$ denote the following - *symmetric* - multicategory associated to M.

- The objects consist of all *connected* open subsets of M;
- For every (possibly empty) finite collection of open sets $\{U_\alpha\}_{\alpha \in A}$ and open set V, there is a set of maps $\mathbf{Disj}_M \left(\{U\}_{\alpha \in A} | V\right)$.
  If the U_α are pairwise disjoint and all contained in V, then the set of maps is a single point. Otherwise, the set of maps is empty;
- The composition of maps is defined in the obvious way.

Definition 3.3.4 [ibid., §1.2, 40, line 6] A prefactorisation algebra is just an algebra over this - *symmetric* - coloured operad $\mathbf{Disj}_M$.

Remark 3.3.2 We had the term *symmetric* in the original definitions, since the multicategories involved must considered to be symmetric, as explicitly stated in [Costello & Gwilliam [CG1] (2016), §3.1.2, 39]: "Note that we mean the symmetric version of such definitions" (line 4).

Remark 3.3.3 We briefly recall the notions of multicategory and algebra: [Leinster [TL] (2003), §2.1, Definition 2.1.1], a multicategory $\mathbf{C}$ is an arrow from a finite string of input objects in $\mathbf{C}_0$ to a codomain object in the same class, subject to the axioms we stated in the relative section. Moreover, an algebra, in the context of multicategories, is [ibid., Definition 2.1.12] a map from $\mathbf{C}$ into the multicategory of **Set**. Symmetric multicategories are further equipped with a symmetric map involving permutations over the input string of objects.

To overcome the difficulties presented above, notice there exists a better description of the model they propose by using generalised posets as defined in [Leinster [TL] (2003), §2.1, Example 2.1.7]. Indeed, there exists at most one arrow for every hom-set

$$\mathbf{Disj}_M \left(U_1, \ldots, U_n; V\right) \tag{3.39}$$

Example 3.3.1 A familiar example of a generalised poset is the convex hull construction: If we have a finite strings of points in a d-dimensional Euclidean space that we see as singletons, we can give them the structure of a poset by declaring

$$(x_1, \ldots, x_n) \leq y \iff y \text{ is the convex hull of the points} \tag{3.40}$$

where the convex hull is defined as the unique minimal convex set containing the n points. Finally, we categorify it.

Remark 3.3.4 Observe that the join (union) operation defining the monoidal structure $(\mathbf{Open}_X, \vee, 0)$ on the open sets is not internal if we consider a family of *connected disjoint* open sets, as in [Costello & Gwilliam [CG1] (2016), §1.2, Definition 1.2.1]. Indeed,

Proposition 3.3.1 [Willard [SW] (2004), §8.26, Theorem 26.7] If $X = \bigcup X_\alpha$ such that X_α are connected $\forall \alpha$ and $\bigcap_\alpha X_\alpha \neq \emptyset$, then X is connected.

The best description that we can achieve is therefore to consider connected open sets as a poset category, where, if it possible to find an open connected part V of X containing the disjoint union of the input sets, then there exists an arrow. Otherwise, the relative hom-set is empty.
However, we reiterate it, this part V does not coincide with $\dot{\bigcup}_{i=1}^n U_i$, as V it is required to be a connected open set.

Thus, we pose the following definition:

Definition 3.3.5 The sub-family of open sets of a topological space $X \in \mathbf{Top}_0$ with the additional properties of being connected form the collection of objects Open_X^c. The superscript 'c' amounts for "connected".

Concerning the preorder on Open_X^c , we declare

$$U \leq V \iff U \subseteq V \tag{3.41}$$

$$(U_1, \ldots, U_n) \leq V \iff \dot{\cup}_{i=1}^n U_i \subseteq V \tag{3.42}$$

for $U, V, U_i \in \mathrm{Open}_X^c \; \forall i$.

3.3.2 Restating the definition

By virtue of the argument presented in the previous section, where we largely discussed a better rephrasing of the notions involved in the definition of a prefactorisation algebra, we modify [Costello & Gwilliam [CG1] (2016), §3.1.2, Definition 1.2.1] into:

Definition 3.3.6 Let $\left(\mathrm{Open}_X^c, \subseteq\right)$ be the ordered set of connected open parts of a topological space X with set-theoretical inclusion as preorder. The associated *symmetric* poset multicategory $\mathbf{Open}_X^c$ consists of the following:

(i) $\left(\mathbf{Open}_X^c\right)_0$ as objects;

(ii) For any finite string $(U_1, \ldots, U_n) \in \prod^n \left(\mathbf{Open}_X^c\right)_0$ an hom-set $\mathbf{Open}_X^c\,(U_1, \ldots, U_n; V)$, where:

$$\mathbf{Open}_X^c\,(U_1, \ldots, U_n; V) = \begin{cases} \{\emptyset\} & \iff \dot{\cup}_{i=1}^n U_i \nsubseteq V \\ \\ \{f\} & \iff \dot{\cup}_{i=1}^n U_i \subseteq V \;\wedge\; U_i \cap U_j = \emptyset \;\; \forall\, i \neq j \end{cases} \tag{3.43}$$

(iii) An operation of composition: $\forall n, k_1, \ldots, k_n \in \mathbb{N}, V, U_i, U_i^j \in \left(\mathbf{Open}_X^c\right)_0$

$$\mathbf{Open}_X^c \left(U_1, \ldots, U_n; V\right) \times \mathbf{Open}_X^c \left(U_1^1, \ldots, U_1^{k_1}; U_1\right) \times \cdots \times \mathbf{Open}_X^c \left(U_1^1, \ldots U_n{}^{k_n}; U_n\right)$$

$$\downarrow$$

$$\mathbf{Open}_X^c \left(U_1^1, \ldots, U_1^{k_1}, \ldots, U_n^1, \ldots, U_n^{k_n}; V\right)$$

$$(f, \ldots, f_n) \mapsto f \circ (f_1, \ldots f_n) \tag{3.44}$$

whenever the arrows exist and are sequentially composable.

(iv) An identity arrow: $\forall\, U \in \left(\mathbf{Open}_X^c\right)_0, \exists\, 1_U \in \mathbf{Open}_X^c\,(U; U)$

satisfying

(a) Associativity law:

$$f \circ \left(f_1 \circ \left(f_1^1, \ldots, f_1^{k_1}\right), \ldots, f_n \circ \left(f_n^1, \ldots, f_n^{k_n}\right)\right) = \tag{3.45}$$

$$= (f \circ (f_1, \ldots, f_n)) \circ \left(f_1^1, \ldots, f_1^{k_1}, \ldots, f_n^1, \ldots, f_n^{k_n}\right) \tag{3.46}$$

$\forall\, f, f_i, f_i^j$ for which this composition makes sense.

(b.) Identity law:

$$f \circ (1_{U_1}, \ldots, 1_{U_n}) = f = 1_V \circ f \tag{3.47}$$

for any morphism $f : U_1, \ldots, U_n \mapsto V$.

Remark 3.3.5 The single arrow-set $\{f\}$ in (3.43) may be replaced by $\{*\}$ in order to emphasise we have singletons.

Remark 3.3.6 We highlighted the term *symmetric* in the above definition, as it is added following Remark 3.3.2. of the previous section.

Remark 3.3.7 (On the empty set) The empty set $\emptyset$ plays a special role in this poset category. Since $\emptyset$ cannot be written as disjoint union of two proper connected open parts, it cannot be disconnected. Thus, it is an object in the poset category of open connected subsets of X. However, as $\emptyset$ is contained in every connected open and it is the neutral element of the union (join) viewed as monoidal product on the opens, there exists an arrow also when the empty set is an element of the string of input object. In other words, *arrows are invariant under the insertion of the empty set as input object*, being the neutral element of the union. However, we cannot formally declare two input strings of different length to be the same when containing the empty set. It is more convenient to think of the erasing of the empty set as the reduction operation on words of a free algebra. More is actually true: There exists an arrow also when the finite string of input objects is entirely made by copies of $\emptyset$. Back to physics, in the former case, the insertion of an empty time interval does not change the experiment, as it means no time was assigned to the measurement, i.e. there was none, and the latter case is equivalent to the situation in which we do not make any measurement, as the time intervals are all empty. In such a setting, we declare the existence of an arrow in

$$\mathbf{Open}^c_X\,(\emptyset, \ldots, \emptyset; V) \tag{3.48}$$

for V being a proper connected open set, as V is still a part containing $\emptyset$. This is valid for every connected open sets, by definition of topology. Lastly, if not only the input string is completely made out of copies of the empty set, but also the output object is the empty set, we still have an arrow, that we call *trivial arrow*,

$$\emptyset, \ldots, \emptyset \longrightarrow \emptyset \tag{3.49}$$

as the empty set contains the empty set itself. We will see this remark will play a crucial role when it comes to the algebraic properties of prefactorisation algebras.

We redefine the notion of prefactorisation algebra making use of the multicategory of connected open sets of X just introduced as follows:

(i) Replacing [Costello & Gwilliam [CG1] (2016), §3.1.2, 40, lines 6-7]:

Definition 3.3.7 A **prefactorisation algebra** is an algebra over the symmetric poset multicategory $\mathbf{Open}^c_X$. Shortly, it is a $\mathbf{Open}^c_X$-algebra.

(ii) Replacing [Costello & Gwilliam [CG1] (2016), §3.1.2, Definition 1.2.2]:

Definition 3.3.8 Let **C** a symmetric multicategory, a **prefactorisation algebra** with values in **C** is a multi-functor

$$\mathbf{Open}^c_X \xrightarrow{\mathcal{F}} \mathbf{C} \tag{3.50}$$

Since the first definition, following [Leinster [TL] (2003),§2.1, Definition 2.1.12], would give a functor

$$\mathbf{Open}^c_X \to \mathbf{Set}$$

we make use of the second, as it is fully compatible with the theory we develop. In this script a prefactorisation algebra is the below defined functor of multicategories.

Definition 3.3.9 Let **C** a symmetric multicategory, a **prefactorisation algebra** with values in **C** is a multifunctor

$$\mathbf{Open}_X^c \xrightarrow{\mathcal{F}} \mathbf{C} \tag{3.51}$$

Remark 3.3.8 Notice the requirement on the open sets to be connected is not contained in the explicit axiomatic definition presented elsewhere [ibid. §3.1.1]. However it is reasonable to have, and therefore inserted by Costello & Gwilliam as requirement in the multicategorical definition presented in their first volume, since prefactorisation algebras model observables and we cannot take a measure in an interval of time that is not connected. If the interval (of the real line) is disconnected, it is possible to write it as union of two proper parts, but as we ask them to be disjoint, it implies we have an interval decomposable in union of disjoint connected open parts, therefore the structure map should take such parts as two different input objects, as the measurement is performed first in one time interval and then in the other. In other words, the structure map factorises through this decomposition. Connected open sets are thus the "fundamental units" on which the theory is built.

In order to define the category of prefactorisation algebras, we still need to have a clear understanding of the arrows between two of them. Firstly, we substitute the poset multicategory of open sets where needed in the definition contained in the main reference, then we improve it significantly.

Definition 3.3.10 [Costello & Gwilliam [CG1] (2016), §3.1.4, Definition 1.4.1] A morphism of PFAs $\phi : F \to G$ consists of a map

$$\phi_U : F(U) \to G(U) \tag{3.52}$$

for each $U \in \left(\mathbf{Open}_X^c\right)_0$ compatible with the structure maps.

Namely, for any $V \in \left(\mathbf{Open}_X^c\right)_0$, for any finite input string of pairwise disjoint $(U)_i \in \left(\mathbf{Open}_X^c\right)_0$ each contained in V, the following diagram commutes:

$$\begin{tikzcd}
F(U_1) \otimes \cdots \otimes F(U_n) \arrow[r, "\phi_{U_1} \otimes \cdots \otimes \phi_{U_n}"] \arrow[d] & G(U_1) \otimes \cdots \otimes G(U_n) \arrow[d] \\
F(V) \arrow[r, "\phi_V"] & G(V)
\end{tikzcd}$$

Likewise, all the obvious associativity relations are respected.

As we have defined prefactorisation algebras to be functors of multicategories, it is natural to think of morphisms between PFAs as natural transformations. If $\mathcal{F} : \mathbf{Open}^c_X \to \mathbf{C}$ and $\mathcal{G} : \mathbf{Open}^c_X \to \mathbf{C}$ are two prefactorisation algebras taking values in the same symmetric multicategory **C**, an arrow between them is a natural transformation

$$\mathcal{F} \overset{\phi}{\Rightarrow} \mathcal{G} \tag{3.53}$$

In the context of multicategories, a natural transformation is

Definition 3.3.11 [Leinster [TL] (2003), §2.3, Definition 2.3.5] Let $\mathbf{C} \underset{f'}{\overset{f}{\rightrightarrows}} \mathbf{D}$ be a pair of maps between multicategories, a **natural transformation** $\alpha : f \Rightarrow f'$ is a family

$$\left\{ f(a) \xrightarrow{\alpha_a} f'(a) \right\}_{a \in \mathbf{C}_0} \tag{3.54}$$

of unary maps in $\mathbf{D}_1$ such that

$$\alpha_a (f(\theta)) = f'(\theta) \circ (\alpha_{a_1}, \ldots, \alpha_{a_n}) \tag{3.55}$$

for all $\theta : a_1, \ldots, a_n \to a$ in **C**.

For the sake of visualisation, we provide the reader with the diagrams from [ibid., page 61]:

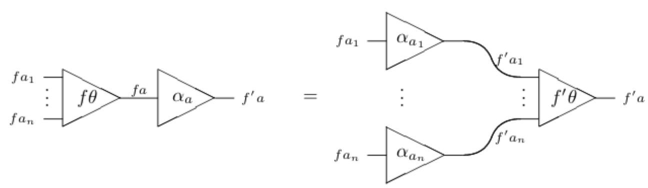

Figure 3.5: Axioms for a natural transformation

We specialise this notion to the case of interest showing we recover the definition in [Costello & Gwilliam [CG1] (2016), Definition 1.4.1].

Definition 3.3.12 Let $\mathcal{F} : \mathbf{Open}^c_X \to \mathbf{C}$, $\mathcal{G} : \mathbf{Open}^c_X \to \mathbf{C}$ be two PFAs taking values in the symmetric multicategory **C**, an **arrow of prefactorisation algebras** is a natural trasformation between them

$$\mathcal{F} \overset{\phi}{\Rightarrow} \mathcal{G} \tag{3.56}$$

is a family of maps

$$\left\{ \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \right\}_{U \in \left(\mathbf{Open}^c_X\right)_0} \tag{3.57}$$

such that

$$\phi_V \circ (\mathcal{F}(f)) = \mathcal{G}(f) \circ (\phi_{U_1}, \dots, \phi_{U_n}) \tag{3.58}$$

for all $f : \mathcal{F}(U_1), \dots, \mathcal{F}(U_n) \to \mathcal{F}(V)$.

In diagrammatic terms: Consider an arrow $f \in \mathbf{Open}^c_X (U_1, \dots, U_n; V)$ in the poset category of connected open sets of a topological space X, the PFA $\mathcal{F}$ induces a map

$$\mathcal{F}(f) \in \mathbf{C}(\mathcal{F}(U_1), \dots, \mathcal{F}(U_n); \mathcal{F}(V)) \tag{3.59}$$

Similarly, for $\mathcal{G}$.

A natural transformation $\mathcal{F} \overset{\phi}{\Rightarrow} \mathcal{G}$ is defined by the property (3.59), i.e. the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(U_1), \dots, \mathcal{F}(U_n) \arrow[r, "\phi_{U_1}, \dots, \phi_{U_n}"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(U_1), \dots, \mathcal{G}(U_n) \arrow[d, "\mathcal{G}(f)"] \\
F(V) \arrow[r, "\phi_V"] & \mathcal{G}(V)
\end{tikzcd}$$

If we represent the multiarrows via the monoidal product, the above coincides exactly with [Costello & Gwilliam [CG1] (2016),§3.1.4, Definition 1.4.1].

Thus, we conclude there exists a well-defined category of prefactorisation algebras defined on open connected sets of a topological spaces X and taking values in a symmetric multicategory $\mathbf{C}$.

Definition 3.3.13 Let X be a topological space, $\mathbf{C}$ be a symmetric multicategory, the **category of prefactorisation algebras over X with values in C** consists of an objects class made out of PFAs and, as morphisms, natural transformation between them. We denote such category by the symbol $\mathbf{PFA}_X(\mathbf{C})$.

Remark 3.3.9 It substitutes the definition in [Costello & Gwilliam [CG1] (2016), §3.1.4, Definition 1.4.2]: "On a space X, we denote the category of prefactorisation algebras on X taking values in the multicategory C by PreFA(X,C)."

Remark 3.3.10 The legs of the natural transformations may be required to satisfy additional properties depending on the codomain category of the PFAs taking into account. For instance, if $\mathbf{C} = \mathbf{Ch}(\mathbf{E})$ denotes chain complexes over a preadditive category $\mathbf{E}$, the ϕ_U are chain maps $\forall\, U \in \left(\mathbf{Open}^c_X\right)_0$.

3.4 Algebraic properties of Prefactorisation Algebras

In [Costello & Gwilliam [CG1] (2016), §3.1.2, 40] it is written:

"Note that if $\mathcal{F}$ is any prefactorisation algebra, then $\mathcal{F}(\emptyset)$ is a commutative algebra object of C" (line 16).

Remark 3.4.1 The latter multicategory is the symmetric multicategory where the PFA takes values and coincides with the symbol **C** in our definition

The expression "commutative algebra object" denotes in the literature a commutative monoid object. We first state the definition of monoid in the context of multicategories, then we discuss if this is compatible with the theory developped by Costello & Gwilliam.

Definition 3.4.1 [Leinster [TL], §2.1, Example 2.1.11] A **monoid in a multicategory C** is an object $c \in \mathbf{C}_0$ equipped with two maps

$$c, c \xrightarrow{\mu} c \qquad\qquad \cdot \xrightarrow{\eta} c \tag{3.60}$$

satisfying associativity and unit laws. The map η goes from the empty string $\cdot$ to the monoid object.

We formalise the above-statement in a lemma and we aim at proving it.

Lemma 3.4.1 If $\mathcal{F}$ is a PFA taking values in a symmetric multicategory **C**, then $\mathcal{F}(\emptyset)$ is a commutative monoid object **C**.

Proof

Step 1. To prove: $\mathcal{F}(\emptyset)$ is a monoid object of **C**.

Firstly, observe $\emptyset \in \mathbf{Open}^c_X$ since it cannot be written as union of two open proper parts, i.e. it cannot be disconnected. Therefore the assignement

$$\left(\mathbf{Open}^c_X\right)_0 \ni \emptyset \xrightarrow{\mathcal{F}} \mathcal{F}(\emptyset) \in \mathbf{C}_0 \tag{3.61}$$

is well-defined.

There exists an arrow in $\mathbf{Open}^c_X(\emptyset, \emptyset; \emptyset)$, as discussed in Remark 3.3.6, that we call trivial 2-arrow, taken by the multifunctor $\mathcal{F} \in (\mathbf{PFA}_X(\mathbf{C}))_0$ to a morphism in

$$\mathbf{C}(\mathcal{F}(\emptyset), \mathcal{F}(\emptyset); \mathcal{F}(\emptyset)) \tag{3.62}$$

Thus, the arrow μ, usually called multiplication,

$$\mu : \mathcal{F}(\emptyset), \mathcal{F}(\emptyset) \longrightarrow \mathcal{F}(\emptyset) \tag{3.63}$$

is well-defined on $\mathcal{F}(\emptyset)$ as required by the definition of a monoid in a multicategory.

Moreover, we have associativity, as the following diagram commutes.

$$\begin{tikzcd}
(\mathcal{F}(\emptyset), \mathcal{F}(\emptyset)), \mathcal{F}(\emptyset) \arrow[rr, "\alpha"] \arrow[d, "{(\mu, 1)}"'] & & \mathcal{F}(\emptyset), (\mathcal{F}(\emptyset), \mathcal{F}(\emptyset)) \arrow[d, "{(1, \mu)}"] \\
\mathcal{F}(\emptyset), \mathcal{F}(\emptyset) \arrow[dr, "\mu"'] & & \mathcal{F}(\emptyset), \mathcal{F}(\emptyset) \arrow[dl, "\mu"] \\
& \mathcal{F}(\emptyset) &
\end{tikzcd}$$

The empty string $\cdot$ is the neutral element of the map above and implies commutativity of the diagram below.

$$\begin{tikzcd}
\cdot, \mathcal{F}(\emptyset) \arrow[r, "{(\eta, 1)}"] \arrow[dr, "="'] & \mathcal{F}(\emptyset), \mathcal{F}(\emptyset) \arrow[d, "\mu"] & \mathcal{F}(\emptyset), \cdot \arrow[l, "{(1, \eta)}"'] \arrow[dl, "="] \\
& \mathcal{F}(\emptyset) &
\end{tikzcd}$$

Therefore $\mathcal{F}(\emptyset)$ is a monoid object in $\mathbf{C}$ with μ as multiplication and empty string as identity object.

Step 2. To show: $\mathcal{F}(\emptyset)$ is a commutative monoid.

Since $\mathbf{C}$ is assumed to be symmetric, we have

$$\mathbf{C}(c_1, \ldots, c_n; c) \xrightarrow{\sim} \mathbf{C}(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c) \tag{3.64}$$

where $\sigma \in \Sigma_n$ is any permutation for which this expression makes sense. This applies to any hom-set, in particular to $\mathbf{C}(\mathcal{F}(\emptyset), \ldots, \mathcal{F}(\emptyset); \mathcal{F}(\emptyset))$. The extended multiplication map, image of an arrow from an input string of n copies of the empty set to the empty set via $\mathcal{F}$,

$$\mathcal{F}(\emptyset), \ldots, \mathcal{F}(\emptyset) \xrightarrow{\mu_n} \mathcal{F}(\emptyset) \tag{3.65}$$

is commutative. Indeed, if we take a trivial arrow on n elements

$$\underbrace{\emptyset, \ldots, \emptyset}_{\text{n copies}} \to \emptyset \tag{3.66}$$

and we permute the input empty sets, the resulting multimorphism agrees with the original one, as nothing changes on elements due to the fact that there are no elements in the $\emptyset$s. The image of these two trivial arrows, as $\mathcal{F}$ is a multifunctor, must coincide. On elements

$$\mathcal{F}(\emptyset), \ldots, \mathcal{F}(\emptyset) \rightarrow \mathcal{F}(\emptyset) \tag{3.67}$$

$$\{x_1, \ldots, x_n = x_{\sigma(1)}, \ldots, x_{\sigma(n)}\} \rightarrow \{x_1 \cdot \ldots \cdot x_n = x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}\} \tag{3.68}$$

Consequently, $\mathcal{F}(\emptyset)$ is a commutative monoid in the multicategory **C**.

□

Example 3.4.1 A commutative monoid in the category of vector spaces over a chosen field $\mathbb{K}$ is an associative commutative algebra over $\mathbb{K}$. Commutative monoid objects in the category of chain complexes are differential graded commutative algebras.

Remark 3.4.2 If we consider an arrow in

$$\mathbf{Open}^c_X\big(\underbrace{\emptyset, \ldots, \emptyset}_{\text{n copies}}; \emptyset\big) \tag{3.69}$$

the family of morphisms in the poset category of open connected parts of the topological space X with domain a string of n empty sets and co-domain the empty set, the prefactorisation algebra takes it into

$$\mathbf{C}\big(\underbrace{\mathcal{F}(\emptyset), \ldots, \mathcal{F}(\emptyset)}_{\text{n copies}}; \mathcal{F}(\emptyset)\big) \tag{3.70}$$

Therefore, there exists a multiplication on n elements induced by the multifunctor in $\mathbf{C}_1$

$$\mu_n : \mathcal{F}(\emptyset), \ldots, \mathcal{F}(\emptyset) \longrightarrow \mathcal{F}(\emptyset) \tag{3.71}$$

extending the above-defined μ.

It is crucial to notice this is *the only case* we can have a natural multiplicative structure over the same object. In fact the axioms we fixed for arrows in the poset category $\mathbf{Open}^c_X$ allow no other multiplication map to arise. For instance, if we take any other connected open U and we try to produce an arrow in

$$\mathbf{Open}^c_X(U, \ldots, U; U) \tag{3.72}$$

the family is empty. Therefore, there is no arrow induced in the target multicategory by the prefactorisation $\mathcal{F}$.

$\mathcal{F}(\emptyset)$ is a very special object, it is the only one that carries an associative multiplication on itself.

Remark 3.4.3 The notion of monoid in a multicategory is presented in [Leinster [TL] (2003)] as a very general construction on strings on objects, following the approach of list monads. However, in [Costello & Gwilliam [CG1] (2016)] they refer to more algebraic aspects of the theory, where a "commutative algebra object" stands for a commutative algebra in the target category where the prefactorisation algebra functor takes its values. Otherwise, it would be easy to build a commutative monoid simply by making use of the empty string. The argument below folows this important remark.

Although we have shown that $\mathcal{F}(\emptyset)$ defines a commutative monoid in every target multicategory, we are interested in unitality in the classical algebraic sense. If the target multicategory is $\mathbf{C} = \mathbf{Vect}_{\mathbb{K}}$, the general theory forces $\mathcal{F}(\emptyset)$ to be commutative associative algebra in the classical sense, as in [Costello & Gwilliam [CG1] (2016), §3.1.1, page 38, line 9], equipped with unit $\eta : \mathbb{K} \to \mathcal{F}(\emptyset)$, as commutative associative unital algebras are monoid objects in $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$. In this context, the multiarrow

$$\mathcal{F}(\emptyset), \ldots, \mathcal{F}(\emptyset) \to \mathcal{F}(\emptyset) \tag{3.73}$$

induced by the prefactorisation algebra functor is further represented in $\mathbf{Vect}_{\mathbb{K}}$ by an arrow having as domain the tensor product of the input objects and target the output one

$$\mathcal{F}(\emptyset) \otimes \cdots \otimes \mathcal{F}(\emptyset) \longrightarrow \mathcal{F}(\emptyset) \tag{3.74}$$

It is apparent the base field carries a unit for such a monoidal product. However, if we chance the target category, $\mathcal{F}(\emptyset)$ may or may not be unital (in the classical algebraic sense).

The unitality of the object $\mathcal{F}(\emptyset)$ plays a fundamental role in the theory.

Definition 3.4.2 [Costello & Gwilliam [CG1] (2016), §3.1.2, Definition 1.2.3] We say that a prefactorisation algebra $\mathcal{F}$ is unital if the commutative algebra $\mathcal{F}(\emptyset)$ is unital.

Remark 3.4.4 If we start by considering both the category of connected open sets and the category in which a PFA takes values as symmetric monoidal categories, $\mathcal{F}$ can be seen as a *strict monoidal functor* sending the identity element of the first category to the second. The open sets, as analysed in chapter two, have a symmetric monoidal structure given by the join (union) as operation and the empty set $\emptyset$ as unit object. The strict monoidal functor defining the PFA is such that $\mathcal{F}(\emptyset) = e$ where e is the unit object of the symmetric monoidal category $(\mathbf{C}, \square, e)$. As these categories are assumed to be symmetric, they have a braiding yielding commutativity ($a \square b \cong b \square a \ \ \forall a, b \in \mathbf{C}_0$). In the case of $\mathcal{F}$ taking values in the symmetric monoidal category $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$ of vector spaces over a field $\mathbb{K}$, the strict monoidal functor $\mathcal{F}$ sends the empty set to the base field $\emptyset \to \mathcal{F}(\emptyset) = \mathbb{K}$. This follows the theory in [Costello & Gwilliam [CG1] (2016),§3.1.3, Definition 1.3.2] and [ibid.,§3.1.1, page 38, line 9 et seq.] where it specialises to vector spaces and symmetric monoidal categories. However, on the connected open parts we consider in this thesis the monoidal structure given by the join with the empty set as neutral element, as already pointed out at the beginning of this remark.

3.4.1 Relation between PFAs and associative algebras on $\mathbb{R}$

In this section we take in exam prefactorisation algebras taking values in vector spaces over the real line

$$\mathbf{Open}^c_{\mathbb{R}} \xrightarrow{\mathcal{F}} \mathbf{Vect}_{\mathbb{R}} \tag{3.75}$$

investigating their relation with associative algebras.

We rephrase the statements in the main reference to adapt them to the new notations introduced in this script and we discuss major results.

Remark 3.4.5 On the symmetric monoidal category of real vector spaces $(\mathbf{Vect}_{\mathbb{R}}, \otimes, \mathbb{R})$ can be induced a multicategorical structure by defining multiarrows using the tensor product operation, as illustrated in the previous chapter. The monoid objects in this category are associative unital algebra. Notice this category is symmetric as there is a well-defined braiding γ on vector spaces inducing the isomorphism

$$V_1 \otimes V_2 \cong V_2 \otimes V_1 \tag{3.76}$$

and such that $\gamma^2 = 1$.

Lemma 3.4.2 [Costello & Gwilliam, §3.1.1, *Example*, 38 ff, line 14 et seq.] Given an associative $\mathcal{A} \in \mathbf{Ass\text{-}alg}_{\mathbb{R}}$, the assignements

$$\mathbf{Open}_{\mathbb{R}}^{c} \ni (\alpha, \beta) \xrightarrow{\mathcal{F}_{\mathcal{A}}} \mathcal{F}_{\mathcal{A}}\left((\alpha, \beta)\right) := \mathcal{A} \in \mathbf{Ass\text{-}alg}_{\mathbb{R}} \tag{3.77}$$

$$\mathbf{Open}_{\mathbb{R}}^{c}\left(\left(\alpha_1, \beta_1\right), \left(\alpha_2, \beta_2\right), \ldots, \left(\alpha_n, \beta_n\right); \mathbb{R}\right) \xrightarrow{\mathcal{F}_{\mathcal{A}}} \mathbf{Vect}_{\mathbb{R}}\left(\mathcal{A}, \ldots, \mathcal{A}; \mathcal{A}\right) \tag{3.78}$$

$$f \mapsto \mathcal{F}_{\mathcal{A}}(f) : \mathcal{A}^{\otimes^n} \to \mathcal{A} \tag{3.79}$$

taking each object to the associative algebra $\mathcal{A}$ over the reals and each morphisms f, when it exists, to a morphism $\mathcal{F}_{\mathcal{A}}(f)$, defines a prefactorisation algebra $\mathcal{F}_{\mathcal{A}} : \mathbf{Open}_{\mathbb{R}}^{c} \to \mathbf{Vect}_{\mathbb{R}}$.

Remark 3.4.6 Notice that the output object can be substituted by any interval containing the disjoint union of the input objects, for instance by a connected interval of the form (α, β) such that $\alpha < \inf_{i \in I} \alpha_i$ and $\beta > \sup_{i \in I} \beta_i$ for $i \in I := \{1, \ldots, n\}$. Furthermore, observe we are working in a symmetric context, therefore the order the input intervals appear in the string is subject to the action of the symmetric group over n elements.

Proof

We need to show the assignements given above on objects and arrows respectively, define an arrow of multicategories. In other words, we have to prove that the above-defined map $\mathcal{F}_{\mathcal{A}} : \mathbf{Open}_{\mathbb{R}}^{c} \to \mathbf{Vect}_{\mathbb{R}}$ respects composition and identites.

1. Identities:

Consider a connected open real interval $J := (\alpha, \beta) \in \left(\mathbf{Open}_{\mathbb{R}}^{c}\right)_0$, the identity arrow over it is defined as

$$1_J : J \to J \tag{3.80}$$

Therefore

$$\mathcal{F}_{\mathcal{A}}(1_J) : \mathcal{F}_{\mathcal{A}}(J) \to \mathcal{F}_{\mathcal{A}\mathcal{A}}(J) \quad = \quad \mathcal{F}_{\mathcal{A}}(1_J) : \mathcal{A} \to \mathcal{A} \tag{3.81}$$

by virtue of the rule on objects in the lemma, i.e.

$$\mathcal{F}_{\mathcal{A}}(1_J) = 1_{\mathcal{F}(J)} = 1_{\mathcal{A}} \tag{3.82}$$

2. Composition:

Take any two sequentially composable arrows in the source multicategory $f_1 \in \mathbf{Open}_{\mathbb{R}}^{c}(I_1, \ldots, I_n; J_0)$ and $f_2 \in$

$\mathbf{Open}^c_{\mathbb{R}}(J_0, J_1, \dots, J_n; J)$, their composition yields a well-defined arrow

$$\mathbf{Open}^c_{\mathbb{R}}(J_0, J_1, \dots, J_n; J) \times \mathbf{Open}^c_{\mathbb{R}}(I_1, \dots, I_n; J_0) \to \mathbf{Open}^c_{\mathbb{R}}(I_1, \dots, I_n; J) \tag{3.83}$$

$$(f_2, f_1) \to f_2 \circ f_1 \tag{3.84}$$

We apply the functor $\mathcal{F}_{\mathcal{A}}$ to such composite arrow.

$$\mathcal{F}_{\mathcal{A}}(f_2 \circ f_1) \in \mathbf{Vect}_{\mathbb{R}}(\mathcal{F}_{\mathcal{A}}(I_1), \dots, \mathcal{F}_{\mathcal{A}}(I_n); \mathcal{F}_{\mathcal{A}}(J)) = \mathbf{Vect}_{\mathbb{R}}(\mathcal{A}, \dots, \mathcal{A}; \mathcal{A}) \tag{3.85}$$

$$\mathcal{F}_{\mathcal{A}}(f_2 \circ f_1) : \mathcal{A}^{\otimes^n} \to \mathcal{A} \tag{3.86}$$

This arrow is completely defined by the rule on objects in the statements of the lemma and represented by the algebra multplication map.

On the other hand, $\mathcal{F}_{\mathcal{A}}(f_2) \circ \mathcal{F}_{\mathcal{A}}(f_1)$ is defined as

$$\mathbf{Vect}_{\mathbb{R}}(\mathcal{F}_{\mathcal{A}}J_0, \mathcal{F}_{\mathcal{A}}J_1, \dots, \mathcal{F}_{\mathcal{A}}J_n; \mathcal{F}_{\mathcal{A}}J) \times \mathbf{Vect}_{\mathbb{R}}(\mathcal{F}_{\mathcal{A}}I_1, \dots, \mathcal{F}_{\mathcal{A}}I_n; \mathcal{F}_{\mathcal{A}}J_0) \to \mathbf{Vect}_{\mathbb{R}}(\mathcal{F}_{\mathcal{A}}I_1, \dots, \mathcal{F}_{\mathcal{A}}I_n; \mathcal{F}_{\mathcal{A}}J)$$

$$(\mathcal{F}_{\mathcal{A}}(f_2), \mathcal{F}_{\mathcal{A}}(f_1)) \to \mathcal{F}_{\mathcal{A}}(f_2) \circ \mathcal{F}_{\mathcal{A}}(f_1) : \underbrace{\mathcal{A} \otimes \ \dots \ \otimes \mathcal{A}}_{\text{n times}} \mapsto \mathcal{A}$$

Again, the assignement on objects and the existence of the algebra multiplication completely determine the arrow.

Remark 3.4.7 We used here the notation $\mathcal{F}_{\mathcal{A}}I$ in place of $\mathcal{F}_{\mathcal{A}}(I)$ on intervals.

It is apparent the two maps coincides

$$\mathcal{F}_{\mathcal{A}}(f_2 \circ f_1) = \mathcal{F}_{\mathcal{A}}(f_2) \circ \mathcal{F}_{\mathcal{A}}(f_1) \tag{3.87}$$

Conclusion

We have just proven the arrow

$$\mathcal{F}_{\mathcal{A}} : \mathbf{Open}^c_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}} \tag{3.88}$$

is a functor of multicategories, i.e. a prefactorisation algebra [Definition 3.3.8].

□

Thus, given an associative algebra over the reals, it is always possible to define a prefactorisation algebra whose structure map is induced by the multiplication of the associative algebra

$$A^{\otimes^n} \ni x_1 \otimes \cdots \otimes x_n \mapsto (\mathcal{F}(f))(x_1 \otimes \cdots \otimes x_n) = x_1 \cdot \ldots \cdot x_n \in A \tag{3.89}$$

such multiplication is associative as in [Definition 1.2.2].

The converse of what we have just shown is not always true, namely we cannot always reconstruct an associative algebra on $\mathbb{R}$, given a prefactorisation algebra. We focus here on the cases in which this is actually possible.

Definition 3.4.3 [based on Costello & Gwilliam [CG1] (2016), §3.2, Definition 2.0.1] Let $\mathcal{F} \in \mathbf{PFA}_{\mathbb{R}}(\mathbf{Vect}_{\mathbb{R}})$ be a prefactorisation algebra over the real line taking values in the multicategory of real vector spaces. We say that $\mathcal{F}$ is **locally constant** if the arrow $\iota_* : \mathcal{F}(U) \to \mathcal{F}(V)$ induced in the target multicategory by the inclusion of open connected parts, is an isomorphism for every inclusion $\left\{U \overset{\iota}{\hookrightarrow} V\right\} \in \left(\mathbf{Open}^c_{\mathbb{R}}\right)_1$.

Since we have introduced the language of multicategories, we rephrase it

Definition 3.4.4 A prefactorisation algebra $\mathcal{F} \in \mathbf{PFA}_{\mathbb{R}}(\mathbf{Vect}_{\mathbb{R}})$ is said to be **locally constant** if, given any multiarrow in the poset category of open connected sets of the topological space X

$$f \in \left(\mathbf{Open}^c_{\mathbb{R}}\right)(I_1, \ldots, I_n; I) \tag{3.90}$$

the corresponding arrow induced by the functor $\mathcal{F}$

$$\mathcal{F}(f) \in \mathbf{Vect}_{\mathbb{R}}(\mathcal{F}(I_1), \ldots, \mathcal{F}(I_n); \mathcal{F}(I)) \tag{3.91}$$

is an isomorphism, meaning that any object $\{\mathcal{F}(I_k)\}_{k \in K}$ of the input string is isomorphic to the output object $\mathcal{F}(V)$. K denotes the set of indices, i.e. $K = \{1, \ldots, n\}$.

Remark 3.4.8 This extension is well-defined from the definition of one single nested interval in [CG1], since we have defined arrows in the posetal category of open connected sets by asking objects in the input string to be pairwise disjoint and all (each one of them) contained in the output object. The isomorphism in [Costello & Gwilliam [CG1] (2016), §3.2, Definition 2.0.1] applies object-wise in the considered string of objects.

It implies the following:

For any arrow $f : I_1, \ldots, I_n \to I$ in the poset category of real connected open sets there exists an induced arrow in $(\mathbf{Vect}_{\mathbb{R}})_1$

$$\mathcal{F}(f) : \mathcal{F}(I_1), \ldots, \mathcal{F}(I_n) \to \mathcal{F}(I) \tag{3.92}$$

Since the prefactorisation algebra $\mathcal{F}$ is locally constant, each inclusion of a connected open part in the output object V yields an isomorphism $\mathcal{F}(I_i) \cong \mathcal{F}(I)$. Therefore, the latter multiarrow is isomorphic to

$$\underbrace{\mathcal{F}(I), \ldots, \mathcal{F}(I)}_{\text{n copies}} \to \mathcal{F}(I) \tag{3.93}$$

Arranging the arrows in a diagram for the sake of visualisation, we obtain:

$$\begin{tikzcd}
I_1 , \arrow[d, "\mathcal{F}"] & \ldots , & I_n \arrow[r, "f"] \arrow[d, "\mathcal{F}"] & V \arrow[d, "\mathcal{F}"] \\
\mathcal{F}(I_1) , \arrow[d, "(\iota_1)_*"] & \ldots , & \mathcal{F}(I_n) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "(\iota_n)_*"] & \mathcal{F}(I) \arrow[d, "1"] \\
\mathcal{F}(I) , & \ldots , & \mathcal{F}(I) \arrow[r, "\mu_n"] & \mathcal{F}(I)
\end{tikzcd}$$

Remark 3.4.9 The bottom arrow is indicated by μ_n as it coincides with

$$\mu_n : \mathcal{F}(I)^{\otimes^n} \to \mathcal{F}(I) \tag{3.94}$$

in the symmetric monoidal category of real vector spaces equipped with monoidal product given by the tensor product and identity object $\mathbb{R}$, the real line. This is indeed the "extended multiplication map" on the tensor product of n vectors.

Thus, we are left with a vector space $\mathcal{F}(I)$ equipped with a multiplication map on n-entries

$$\mu_n : \mathcal{F}(I), \ldots, \mathcal{F}(I) \to \mathcal{F}(I) \tag{3.95}$$

This defines an associative algebra if we prove that the binary operation

$$\mu : \mathcal{F}(I), \mathcal{F}(I) \to \mathcal{F}(I) \tag{3.96}$$

is associative, i.e. $\mu(\mu \otimes id) = \mu(id \otimes \mu)$. In fact, it is the associativity property that allows us the possibility of extending the binary operations to more than two entries. In other words, to compute not taking care of the ordering of the factors.

In particular, the above argument applies if we set the output object I to be the full real line. Moreover, as the multicategory of vector spaces is symmetric, we restate and prove [Costello & Gwilliam [CG1] (2016), §3.2, Lemma 2.0.2].

Lemma 3.4.3 Let $\mathcal{F}$ be a locally constant prefactorisation algebra on $\mathbb{R}$ taking values in the symmetric multicategory of real vector spaces. Let $\mathcal{A} := \mathcal{F}(\mathbb{R})$. Then $\mathcal{A}$ has a natural structure of an associative algebra.

Proof

By definition, the prefactorisation algebra is a functor defined on objects and morphisms as follows:

$$\mathbf{Open}^c_{\mathbb{R}} \ni (\alpha, \beta) \to \mathcal{F}((\alpha, \beta)) \in \mathbf{Vect}_{\mathbb{R}} \tag{3.97}$$

$$\mathbf{Open}^c_{\mathbb{R}}((\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); \mathbb{R}) \to \mathbf{Vect}_{\mathbb{R}}(\mathcal{F}((\alpha_1, \beta_1)), \ldots, \mathcal{F}((\alpha_n, \beta_n)); \mathcal{F}(\mathbb{R})) \tag{3.98}$$

$$f \to \mathcal{F}(f) \tag{3.99}$$

Let $\iota_k : (\alpha_k, \beta_k) \hookrightarrow \mathbb{R}$ for all $k = 1, \ldots, n$, denote the standard inclusions of the k-th interval into the real line.

Remark 3.4.10 In the following we use the notion of nested interval, since we can always deform the "mother" interval to the whole real line, i.e. every open interval is homotopic to the real line. In the case illustrated below, (γ, δ) is thought as the real line, leading to the arrows displayed belo. In fact, in the definition of constant locality extended to the context of multicategories, we have the isomorphism induced by the prefactorisation algebra functor due to the fact the input intervals are all nested in the bigger output one. We could as well have set $\mathcal{F}((\gamma, \delta))$ as $\mathcal{A}$ in the following.

The PFA $\mathcal{F}$ is locally constant, therefore, if we set the image of the real line to be $\mathcal{F}(\mathbb{R}) := \mathcal{A}$, by definition, any open connected real interval gets identified with $\mathcal{A}$. In other words,

$$\mathcal{F}((\alpha_k, \beta_k)) \underset{\psi_k}{\cong} \mathcal{A} \quad \forall k = 1, \ldots, n \tag{3.100}$$

Consequently, for an arbitrary pair of nested (i.e. $\gamma < \alpha_k < \beta_k < \delta$) open intervals

$$(\alpha_k, \beta_k) \underset{\iota_k}{\subset} (\gamma, \delta) \sim \mathbb{R} \tag{3.101}$$

where ι_k denotes the relative inclusion, there exists a commuting cone

$$\begin{tikzcd}
\mathcal{F}((\alpha_k, \beta_k)) \arrow[rr, "(\iota_k)_*"] \arrow[dr, "\psi_k \cong"'] & & \mathcal{F}((\gamma, \delta)) \arrow[dl, "\psi \cong"] \\
& \mathcal{A} &
\end{tikzcd}$$

induced in the category of vector spaces, with $(\iota_k)_* := \mathcal{F}(\iota_k)$, and a corresponding commutative diagram on two input objects

$$\begin{tikzcd}
\mathcal{F}((\alpha_k,\beta_k)) \otimes \mathcal{F}((\alpha_j,\beta_j)) \arrow[r, "\mu"] \arrow[d, "\psi_k\otimes\psi_j \cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}((\gamma,\delta)) \arrow[d, "\psi \cong"] \\
\mathcal{F}(\mathbb{R})^{\otimes^2} =: \mathcal{A}^{\otimes^2} \arrow[r, "\mu_{\mathcal{A}}"] & \mathcal{A} := \mathcal{F}(\mathbb{R})
\end{tikzcd}$$

where the algebra multiplication it is defined as

$$\mu_{\mathcal{A}}(x_k,\ x_j) = \mu_{\mathcal{A}}(x_k \otimes x_j) := x_k \cdot x_j \tag{3.102}$$

and induced by the structure map

$$\mu := (\iota_k)_* \otimes (\iota_j)_* = ((\iota_k)_*, (\iota_j)_*) \tag{3.103}$$

Remark 3.4.11 In the diagram above the multiarrows are represented in the symmetric monoidal category of vector spaces, therefore is displayed the tensor product in place of the usual "," between objects. Similarly in the definition of the horizontal arrows. We implement both the notation, as some reader may want to stick on the more general one using commas.

Let us analyse in details such algebra multiplication:
The ψ_k, respectively ψ, are isomorphisms for all choices of the index k. Therefore, monic and epic arrows.
This implies the following:

$$\forall x_j \in \mathcal{A}\ \exists\ !\ a_k \in \mathcal{F}((\alpha_k,\beta_k)) : \psi_k(a_k) = x_k \tag{3.104}$$

Thus, the multiplication map is defined as:

$$x_k \cdot x_j := \mu_{\mathcal{A}}(x_k \otimes x_j) = \left(\psi \circ \mu \circ (\psi_k \otimes \psi_j)^{-1}\right)(x_k \otimes x_j) \tag{3.105}$$

where μ is the arrow above.

We need to prove $\mathcal{A}$ is an associative unital algebra.

1. Associativity

In order for $\mathcal{A}$ to be an associative algebra, we need to prove

$$\mu_{\mathcal{A}}\left(id, \mu_{\mathcal{A}}\right) = \mu_{\mathcal{A}}\left(\mu_{\mathcal{A}}, id\right) \tag{3.106}$$

In explicit terms, we need to verify the following equation on any three elements:

$$\mu_{\mathcal{A}}\left(\mu_{\mathcal{A}}\left(x_1 \otimes x_2\right) \otimes x_3\right) = \mu_{\mathcal{A}}\left(x_1 \otimes \mu_{\mathcal{A}}\left(x_2 \otimes x_3\right)\right) \tag{3.107}$$

i.e.

$$\left(x_1 \cdot x_2\right) \cdot x_3 = x_1 \cdot \left(x_2 \cdot x_3\right) \tag{3.108}$$

Calculations for the LHS:

$$\begin{aligned}
\mu_{\mathcal{A}}\left(\mu_{\mathcal{A}}\left(x_1 \otimes x_2\right) \otimes x_3\right) &= \mu_{\mathcal{A}}\left(\left[\left(\psi \circ \mu \circ \left(\psi_1 \otimes \psi_2\right)^{-1}\right)\left(x_1 \otimes x_2\right)\right] \otimes x_3\right) = \\
&= \mu_{\mathcal{A}}\left(\left[\left(\psi \circ \mu \circ \left(\psi_2^{-1} \otimes \psi_1^{-1}\right)\right)\left(x_1 \otimes x_2\right)\right] \otimes x_3\right) = \\
&= \mu_{\mathcal{A}}\left(\left[\psi \circ \mu\left(a_1 \otimes a_2\right)\right] \otimes x_3\right) = \mu_{\mathcal{A}}\left(\left[\psi\left(a_{12}\right)\right] \otimes x_3\right) = \\
&= \mu_{\mathcal{A}}\left(x_{12} \otimes x_3\right) = \left(\psi \circ \mu \circ \left(\psi_{12} \otimes \psi_3\right)^{-1}\right)\left(x_{12} \otimes x_3\right) = \\
&= \left(\psi \circ \mu \circ \left(\psi_3^{-1} \otimes \psi_{12}^{-1}\right)\right)\left(x_{12} \otimes x_3\right) = \\
&= \left(\psi \circ \mu\left(a_{12} \otimes a_3\right)\right) = \left[\psi\left(a_{123}\right)\right] = x_{123}
\end{aligned}$$

Calculations for the RHS:

$$\begin{aligned}
\mu_{\mathcal{A}}\left(x_1 \otimes \mu_{\mathcal{A}}\left(x_2 \otimes x_3\right)\right) &= \mu_{\mathcal{A}}\left(x_1 \otimes \left[\left(\psi \circ \mu \circ \left(\psi_2 \otimes \psi_3\right)^{-1}\right)\left(x_2 \otimes x_3\right)\right]\right) = \\
&= \mu_{\mathcal{A}}\left(x_1 \otimes \left[\left(\psi \circ \mu \circ \left(\psi_3^{-1} \otimes \psi_2^{-1}\right)\right)\left(x_2 \otimes x_3\right)\right]\right) = \\
&= \mu_{\mathcal{A}}\left(x_1 \otimes \left[\psi \circ \mu\left(a_2 \otimes a_3\right)\right]\right) = \mu_{\mathcal{A}}\left(x_1 \otimes \left[\psi\left(a_{23}\right)\right]\right) = \\
&= \mu_{\mathcal{A}}\left(x_1 \otimes x_{23}\right) = \left(\psi \circ \mu \circ \left(\psi_1 \otimes \psi_{23}\right)^{-1}\right)\left(x_1 \otimes x_{23}\right) = \\
&= \left(\psi \circ \mu \circ \left(\psi_{23}^{-1} \otimes \psi_1^{-1}\right)\right)\left(x_1 \otimes x_{23}\right) = \\
&= \left(\psi \circ \mu\left(a_1 \otimes a_{23}\right)\right) = \left[\psi\left(a_{123}\right)\right] = x_{123}
\end{aligned}$$

Remark 3.4.12 The (pre)images of the elements under ψ, ψ_k are well-defined as we have already discussed bijectivity of such arrows. Moreover, bijectivity forces uniqueness of the elements involved in the calculations yielding the desired result. The subscript indices for the products are assigned in the obvious way so it easy to recover the factors. Concerning the $a_{ij}s$, by definition of μ,

$$\forall i, j \; \exists \, a_{ij} \in \mathcal{F}(\mathbb{R}) \; : \; \mu\left(a_i, a_j\right) = a_{ij} \tag{3.109}$$

or, in tensorial notation,

$$\forall i, j \; \exists \, a_{ij} \in \mathcal{F}(\mathbb{R}) \; : \; \mu\left(a_i \otimes a_j\right) = a_{ij} \tag{3.110}$$

2. Unitality

Observe $\mathcal{F}$ is required to be unital. By definition, the latter means that $\mathcal{F}(\emptyset)$ is unital, but this object coincides with $\mathcal{A}$, by the constant locality of $\mathcal{F}$. If $\mathcal{F}(\emptyset)$ is unital, so is its isomorphic image $\mathcal{A}$. We have in fact already proven the former is a monoid object in the category of real vector spaces, therefore the isomorphism with $\mathcal{A}$ is an isomorphisms of algebras and the image of the identity element is the identity of $\mathcal{A}$. In plain words, in previous results we obtained $\mathcal{F}(\emptyset)$ is a commutative algebra. In this case, since its unitality is further asked it is equipped with a unit $\mathbb{K} \to \mathcal{F}()$. The isomorphic image carries thus a unit $\mathbb{K} \to \mathcal{A}$ making the latter a unital associative algebra in the sense of [Loday & Vallette [LV] (2012), §1.1.1, 2, line 2 et seq.].

3. Independence of the chosen intervals

Since the intervals we have chosen are completely arbitrary, but such that the relative morphism in $\left(\mathbf{Open}^c_{\mathbb{R}}\right)_1$ exists, the proof is independent of their choice. Indeed, constant locality of the prefactorisation algebra holds on any interval of the real line, i.e. $\mathcal{F}(\alpha, \beta) \cong \mathcal{A}$ for any $(\alpha, \beta) \in \left(\mathbf{Open}^c_{\mathbb{R}}\right)_0$, and we have an associative multiplication map each time the arrow induced by the prefactorisation algebra functor at the level of vector spaces exists. The latter refers back to the existence of an arrow

$$\iota : (\alpha_k, \beta_k), \left(\alpha_j, \beta_j\right) \to (\gamma, \delta) \tag{3.111}$$

in the poset category of open connected parts of the real line.

Conclusion

A unital, locally constant prefactorisation algebra on the real line with values in the symmetric multicategory of vector spaces yields an associative unital algebra structure on the vector space $\mathcal{A}$. In other words, $\mathcal{F}(\mathbb{R}) \in \mathbf{Ass\text{-}alg}_{\mathbb{R}}$. This proves the lemma.

□

Remark 3.4.13 (On symmetries) Notice the poset category $\mathbf{Open}^c_{\mathbb{R}}$ has a well-defined action of the symmetric group yielding equality, as

$$I_1 \dot{\cup} \ldots \dot{\cup} I_n = I_{\sigma(1)} \dot{\cup} \ldots \dot{\cup} I_{\sigma(n)} \tag{3.112}$$

by the property of the union. For vector spaces however, an action of the symmetric group, as required by the definition of symmetric multicategory, results in an isomorphism

$$\mathcal{F}(I_1) \otimes \cdots \otimes \mathcal{F}(I_n) \cong \mathcal{F}(I_{\sigma(1)}) \otimes \cdots \otimes \mathcal{F}(I_{\sigma(n)}) \tag{3.113}$$

In fact, we have already observed the braiding yields $V \otimes W \cong W \otimes V$ for any pair of real vector spaces.

Remark 3.4.14 The extension to a weight n word, reads explicitly as:

$$\mu_{\mathcal{A}}(x_1 \otimes \cdots \otimes x_n) = x_1 \cdot \ldots \cdot x_n \tag{3.114}$$

$$\left(\psi \circ \mu^{(n)} \circ \left(\otimes_{k=1}^{n} \psi_k\right)^{-1}\right)\left(\otimes_{k=1}^{n} x_k\right) = x_1 \cdot \ldots \cdot x_n \tag{3.115}$$

with

$$\mu^{(n)} := ((\iota_1)_* \otimes \cdots \otimes (\iota_n)_*) = ((\iota_1)_*, \ldots, (\iota_n)_*) \tag{3.116}$$

and associativity ensures the multiplication works iteratively as follows, e.g. in $\mathcal{A}^{\otimes^3}$:

$$\mu_{\mathcal{A}}(\mu_{\mathcal{A}}(x_1 \otimes x_2) \otimes x_3) = \mu_{\mathcal{A}}((x_1 \cdot x_2) \otimes x_3) = x_1 \cdot x_2 \cdot x_3 \tag{3.117}$$

the product is well-defined and associative, therefore it is written without further brackets.

Remark 3.4.15 The symbol $\cdot$ will always denote the formal product of the associative algebra associated to a locally constant PFA, unless otherwise specified.

Chapter 4

Examples of Prefactorisation Algebras in Physics

4.1 Prefactorisation Algebras from bimodules

We want to further investigate, as it relates to quantum mechanics, prefactorisation algebras on the real line arising from two different associative unital algebras $\mathcal{A}, \mathcal{B} \in \mathbf{Ass\text{-}alg}_{\mathbb{R}}$ and taking values in vector spaces.

Let $M \in {}_{\mathcal{A}}\mathbf{Mod}_{\mathcal{B}}$ be a left $\mathcal{A}$-module and a right $\mathcal{B}$-module, with usual compatibility relation on multiplication

$$(am)\,b = a\,(mb) \qquad\qquad \forall a \in \mathcal{A}, b \in \mathcal{B}, m \in M \tag{4.1}$$

and, for a fixed point q on the real line, define the following shortcut notations:

$$\mathbb{R}^{<q} := \{y \in \mathbb{R} : y < q\} = (-\infty, q) \tag{4.2}$$

$$\mathbb{R}^{>q} := \{y \in \mathbb{R} : q < y\} = (q, +\infty) \tag{4.3}$$

Remark 4.1.1 Observe that is always possible to define a bimodule over $\mathbf{Ass\text{-}alg}_{\mathbb{R}}$ since an algebra has more structure than a commutative ring. On a ring, there are two internal binary operations, addition and multiplication, with identities. In an associative algebra, being a vector space equipped with multiplication and unit, there is an additional operation coming from scalar multiplication. Thus modules, and bimodules, are well-defined also in this context.

We have already discussed how using associative algebras we can generate prefactorisation algebras. Therefore, we define the prefactorisation algebra $\mathcal{F}_M$ on any real line interval I piece-wise on objects as:

$$\mathcal{F}_M(I) = \begin{cases} \mathcal{A} & \text{if } I \subset \mathrm{Open}^c_{\mathbb{R}<q} \\ \mathcal{B} & \text{if } I \subset \mathrm{Open}^c_{\mathbb{R}>q} \\ M & \text{if } q \in I \subset \mathrm{Open}^c_{\mathbb{R}} \end{cases} \tag{4.4}$$

Remark 4.1.2 The notation $\mathrm{Open}^c_{\bullet}$ is coherent with the one introduced in the previous sections for connected open sets. Notice it is not in boldface, as we are not considering it as a poset multicategory (generalised poset);

For the latter assignement, being M a bimodule and not an associative algebra in itself, we have still to discuss the form of the corresponding prefactorisation algebra. We will see the bimodule structure induces the structure map naturally.

For arrows, consider the following setting:

$$\alpha < \alpha_1 < \beta_1 < \alpha_2 < q < \beta_2 < \alpha_3 < \beta_3 < \beta \tag{4.5}$$

where the point q lies in the central interval.
Applying the prefactorisation functor as defined above corresponds to

$$\begin{tikzcd} (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \arrow[r, "\iota"] \arrow[d, "\mathcal{F}"'] & (\alpha, \beta) \arrow[d, "\mathcal{F}"] \\ \mathcal{F}_M(\alpha_1, \beta_1), \mathcal{F}_M(\alpha_2, \beta_2), \mathcal{F}_M(\alpha_3, \beta_3) \arrow[r, "\iota_*"] & \mathcal{F}_M(\alpha, \beta) \end{tikzcd}$$

Remark 4.1.3 To obtain a lighter notation, we displayed the image objects without double parenthesis in the spirit of Mac Lane, where e.g. $f(a) = fa$ for any arrow f applied to the object a.

Since the prefactorisation algebra we are considering takes values in vector spaces, the latter multiarrow is represented by means of the symmetric monoidal product as

$$\mathcal{A} \otimes M \otimes \mathcal{B} \longrightarrow M \tag{4.6}$$

Thus, the multiplication

$$\mathcal{A} \otimes M \otimes \mathcal{B} \longrightarrow M \tag{4.7}$$

$$a \otimes m \otimes b \to a \cdot m \cdot b \tag{4.8}$$

well-defined by the bimodule structure and its compatibility rule, yields the structure map of the prefactorisation algebra $\mathcal{F}_M$.

Remark 4.1.4 The empty set case needs particular considerations. Suppose we want to study the inclusion of the empty set into (α_2, β_2) containing q as above. The existence of the arrow induced by inclusion $\mathcal{F}_M\emptyset \to M$ implies we have to fix some elements of the bimodule M preserved by the structure map. To achieve this, they will be multiplied on the left by the $1_{\mathcal{A}}$ and on the left by the $1_{\mathcal{B}}$.

Remark 4.1.5 If the intervals considered do not contain the point q of interest, the prefactorisation algebras $\mathcal{F}_M$ simply restrict to the multifunctors $\mathcal{F}_{\mathcal{A}}$ or $\mathcal{F}_{\mathcal{B}}$ depending on the algebra selected, therefore the matter discussed above extends the theory we already presented.

The piece-wise costruction of a prefactorisation algebra just analysed can be implemented in more general contexts. For instance, we fix two points of the real line p and q such that $p < q$ and two bimodules $M \in {}_{\mathcal{A}}\mathbf{Mod}_{\mathcal{B}}$, $N \in {}_{\mathcal{B}}\mathbf{Mod}_{\mathcal{C}}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{Ass\text{-}alg}_{\mathbb{R}}$ are three associative real algebras, and we proceed as follows:

Introduce shortcut notations

$$\mathbb{R}^{<p} := \{y \in \mathbb{R} : y < p\} = (-\infty, q) \tag{4.9}$$

$$\mathbb{R}^{>q} := \{y \in \mathbb{R} : q < y\} = (q, +\infty) \tag{4.10}$$

$$\mathbb{R}^{(p,q)} := \{y \in \mathbb{R} : p < y < q\} = (p, q) \tag{4.11}$$

and define the prefactorisation algebra on objects as

$$\mathcal{F}_{M,N}(\alpha, \beta) = \begin{cases} \mathcal{A} & \text{if } (\alpha, \beta) \subset \text{Open}^c_{\mathbb{R}^{<p}} \\ \mathcal{B} & \text{if } (\alpha, \beta) \subset \text{Open}^c_{\mathbb{R}^{(p,q)}} \\ \mathcal{C} & \text{if } (\alpha, \beta) \subset \text{Open}^c_{\mathbb{R}^{>q}} \\ M & \text{if } (\alpha, \beta) \subset \text{Open}^c_{\mathbb{R}} : \alpha < p < \beta < q \\ N & \text{if } (\alpha, \beta) \subset \text{Open}^c_{\mathbb{R}} : p < \alpha < q < \beta \end{cases} \tag{4.12}$$

Concerning morphisms: each time the connected real intervals in the input string do not contain the points of interests, the functor $\mathcal{F}_{M,N}$ restricts to the prefactorisation algebra generated by the relative associative algebra. As soon as one or both of the points are part of two disjoint intervals in the input string, the prefactorisation algebra functor induces an arrow tensoring with the bimodules.

For instance, consider extrema ordered as follows

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_p < p < \beta_p < \alpha_3 < \beta_3 < \alpha_q < q < \beta_q < \alpha_4 < \beta_4 \tag{4.13}$$

so that $p \in (\alpha_p, \beta_p)$ and $q \in (\alpha_q, \beta_q)$. Set I to be the interval containing both points and all the intervals in the input string, i.e. $I := (A, B) : A < \alpha_1, B > \beta_4$.

The prefactorisation functor produces a multiarrow according to the diagram

$$\begin{tikzcd}
(\alpha_1,\beta_1), (\alpha_2,\beta_2), (\alpha_p,\beta_p), (\alpha_3,\beta_3), (\alpha_q,\beta_q), (\alpha_4,\beta_4) \arrow[r, "\iota"] \arrow[d, "\mathcal{F}"] & I \arrow[d, "\mathcal{F}"] \\
\mathcal{F}_{M,N}(\alpha_1,\beta_1), \mathcal{F}_{M,N}(\alpha_2,\beta_2), \mathcal{F}_{M,N}(\alpha_p,\beta_p), \mathcal{F}_{M,N}(\alpha_3,\beta_3)\, \mathcal{F}_{M,N}(\alpha_q,\beta_q), \mathcal{F}_{M,N}(\alpha_4,\beta_4) \arrow[r, "\iota_*"] & \mathcal{F}_{M,N} I
\end{tikzcd}$$

However, observe that thanks to associativity, this multiarrow factorises through

$$\mathcal{F}_{M,N}(\alpha_1, \beta_p), \mathcal{F}_{M,N}(\alpha_3, \beta_3), \mathcal{F}_{M,N}(\alpha_q, \beta_4) \to \mathcal{F}_{M,N} I \tag{4.14}$$

Thus, we have an arrow

$$M \otimes \mathcal{B} \otimes N \xrightarrow{\mu} \mathcal{F}_{M,N} I \tag{4.15}$$

with the additional property, induced by associativity, that

$$\mu((mb) \otimes n) = \mu(m \otimes (bn)) = \mu(m \otimes b \otimes n) \tag{4.16}$$

since μ has to respect the fact M is a right $\mathcal{B}$-module and N is a left $\mathcal{B}$-module.

In a commutative diagram, for any $m \in M, b \in \mathcal{B}, n \in N$,

$$\begin{tikzcd}
 & m \otimes b \otimes n \arrow[dl] \arrow[dr] \arrow[ddd, "\mu"] & \\
(m \otimes b) \otimes n \arrow[d] & & m \otimes (b \otimes n) \arrow[d] \\
(mb) \otimes n \arrow[dr] & & m \otimes (bn) \arrow[dl] \\
 & mbn &
\end{tikzcd}$$

Consequently, the natural choice for the output object is

$$\mathcal{F}_{M,N} I = M \otimes_{\mathcal{B}} N \tag{4.17}$$

4.1.1 Application to Quantum Mechanics

As a practical application of this method, we illustrate an example from quantum mechanics:

Consider a one-parameter family of unitary operators

$$\{\hat{U}\}_{t \in \mathbb{R}} = \{e^{itH}\}_{t \in \mathbb{R}} \tag{4.18}$$

on a finite-dimensional Hilbert space V whose expectation value during a scattering experiment on a closed time interval $[0, T]$

$$\langle v_{\text{in}} | e^{itH} \hat{O} e^{i(T-t')H} | v_{\text{out}} \rangle \tag{4.19}$$

we aim at calculating. During the experiment, the Hamiltonian is modified by the operator $\hat{O}$ acting in the interval (t, t').

Remark 4.1.6 In the bra-ket notation used in the expression above, $\langle v_{\text{in}}|$ represents the initial state of the system at time $t = 0$, whereas $|v_{\text{out}}\rangle$ denotes its final state at time $t = T$. Moreover, the pairing $\langle v|v'\rangle$ coincides with the inner product of V as Hilbert space $\forall v \in \overline{V}, v' \in V$, for $\overline{V}$ the vector space V equipped with conjugate complex structure.

Remark 4.1.7 The space of states V is finite dimensional, therefore it is possible to identify the unitary operators with a family $\{e^{itH}\}_{t \in \mathbb{R}}$, where H is an Hermitian operator and corresponds to the Hamiltonian of the system determining the its time evolution. For this reason, such operators are usually called **evolution operators**.

Taking the limit for interval of time closer and closer to a fixed time t_0, we think of the average as computed at the time t_0, i.e.

$$\langle v_{\text{in}} | e^{it_0 H} \hat{O} e^{i(T-t_0)H} | v_{\text{out}} \rangle \tag{4.20}$$

Let $A = \mathrm{End}\,(V)$ the space of operators (observables), we have an $*$-structure given by taking the transpose conjugate of the operator yielding the right A-module structure over $\overline{V}$

$$\langle v|M = \langle M^* v| = \langle \overline{M}^T v| \tag{4.21}$$

$\forall\ \langle v| \in \overline{V},\ M \in A,\ |v'\rangle \in V$. Moreover, recall the equality

$$\langle v|M|v'\rangle = \langle M^* v|v'\rangle = \langle v|Mv'\rangle \tag{4.22}$$

Namely M can act both on the right and on the left resulting in the same result.

Define a prefactorisation algebra $\mathcal{F}$ on objects by the assignements:

$$[0, t_0) \xrightarrow{\mathcal{F}} \overline{V} \qquad (t_1, t_2) \xrightarrow{\mathcal{F}} A \qquad (t_3, T] \xrightarrow{\mathcal{F}} V \qquad [0, T] \xrightarrow{\mathcal{F}} \mathbb{C} \tag{4.23}$$

Remark 4.1.8 This follows the original statement in [Costello & Gwilliam, §3.3.2, 47, line 13 et seq.]. However, objects should be open connected interval of the real line. Therefore, although the experiment is conducted in the time interval $[0, T]$, we can refer to larger open intervals containing the initial time, e.g. $(\alpha, t_0) : \alpha < 0$ and the end time, e.g. $(t_3, \beta) : \beta > T$, also if we take trace of the data, in other words we calculate the average, only in the closed time slot $[0, T]$. Similarly we should assign $\mathbb{C}$ to an open connected interval I such that $[0, T] \subset I$, containing all the other open disjoint connected intervals on which the prefactorisation algebra is defined. In this way, the definition is coherent with the theory we have developped so far on open connected sets.

As consequence of the previous remark, we redefine the prefactorisation algebra $\mathcal{F}$ as follows:

$$(\alpha, t_1) \xrightarrow{\mathcal{F}} \overline{V} \qquad (t_2, t_3) \xrightarrow{\mathcal{F}} A \qquad (t_3, \beta) \xrightarrow{\mathcal{F}} V \qquad I \xrightarrow{\mathcal{F}} \mathbb{C} \tag{4.24}$$

By a reasoning analogous to the one just followed for the structure map in case of bimodules and visualised in the last commutative diagram of the previous section, it is reasonable to ask the image via $\mathcal{F}$ of the full time interval $[0, T]$ to be

$$\mathcal{F}\,([0, T]) = \overline{V} \otimes_A V \cong \mathbb{C}. \tag{4.25}$$

or better,

$$\mathcal{F}\,(I) = \overline{V} \otimes_A V \cong \mathbb{C}. \tag{4.26}$$

Remark 4.1.9 The space $\overline{V}$ is the finite-dimensional vector space V equipped with conjugate complex structure and the isomorphism follows from compatibility of the right and left actions with the inner product on the Hilbert space V.

The inclusion of the above disjoint open connected intervals, with extrema given by the ordered times $\alpha < t_0 < t_1 < t_2 < t_3 < \beta$, with $0 \in (\alpha, t_0)$, and $T \in (t_3, \beta)$ and all contained in a bigger open connected interval $I := (t, s)$, induces a multimorphism $\iota_* := \mathcal{F}(\iota)$

$$\begin{tikzcd}
(\alpha, t_0), (t_1, t_2), (t_3, \beta) \arrow[r, "\iota"] \arrow[d, "\mathcal{F}"] & (t, s) \arrow[d, "\mathcal{F}"] \\
\overline{V} \otimes A \otimes V \arrow[r, "\iota_*"] & \mathbb{C}
\end{tikzcd}$$

$$\langle v_{\text{in}}| \otimes \hat{O} \otimes |v_{\text{out}}\rangle \longrightarrow \langle v_{\text{in}}|e^{i(t_1-t_0)H}\hat{O}e^{i(t_3-t_2)H}|v_{\text{out}}\rangle$$

The expected value is calculated by inserting the evolution operators $e^{i\Delta t H}$ in the closed complements of the disjoint intervals of time. In this case, there are no evolution operators corresponding to $(t - \alpha)$ and $(s - \beta)$ since we are considering the system in the initial state at the time t_0 and final state at time t_3.

Remark 4.1.10 The times t_0 and t_3 correspond to the points p and q of the general theory presented in the previous section.

Remark 4.1.11 This is coherent with the underlying physics. Indeed, if $\hat{O} = e^{i(t_2-t_1)H}$, then $\langle v_0|e^{i(t_3-t_0)H}|v_1\rangle$ is the expected value to find the system in $\langle v_0|$ at time t_0 and in $|v_1\rangle$ at time t_3. Furthermore, for $t_3 = t_0$ the decomposition collapses and we recover the inner product on V.

Generalising the construction, we consider the decomposition

$$t < t_0 < t_1 < \cdots < t_{2k-2} < t_{2k-1} < s \tag{4.27}$$

whose relative diagram is

$$\begin{tikzcd}
(t_0, t_1), (t_2, t_3), \ldots, (t_{2k-2}, t_{2k-1}) \arrow[r, "\iota"] \arrow[d, "\mathcal{F}"] & (t, s) \arrow[d, "\mathcal{F}"] \\
\overline{V} \otimes \underbrace{A \otimes \cdots \otimes A}_{\text{k times}} \otimes V \arrow[r, "\iota_*"] & \mathbb{C}
\end{tikzcd}$$

The structure map, for time-ordered operators $\hat{O}_{i=1}^{k}$ reads

$$\langle v_{\text{in}}| \otimes \hat{O}_1 \otimes \cdots \otimes \hat{O}_k \otimes |v_{\text{out}}\rangle \longrightarrow \langle v_{\text{in}}|e^{i(t_0-t)H}\hat{O}_1 e^{i(t_2-t_1)H} \cdots \hat{O}_k e^{i(s-t_{2k-1})H}|v_{\text{out}}\rangle$$

We now focus on the boundaries of the time interval in which the scattering experiment is conducted, namely we take into account the points 0 and T as p and q. We follow the notation in [Costello& Gwilliam [CG1] (2016), §3.3.2, 48, line 9 et seq.] despite the discrepancies pointed out in remark 4.1.8. To solve them, always visualise the closed extrema as limit points of an open connected part containing them.

We present below the structure maps for the different scenarios:

(i) $[0,t) \subset [0,t')$

$$\overline{V} \to \overline{V} \tag{4.28}$$

$$\langle v_0| \to \langle v_0|e^{i(t'-t)H} \tag{4.29}$$

(ii) $(t,T] \subset (t',T]$

$$V \to V \tag{4.30}$$

$$|v_1\rangle \to e^{i(t-t')H}|v_1 \tag{4.31}$$

(iii) $[0,t_0) \dot\cup (t_1,t_2) \subset [0,t')$

$$\overline{V} \otimes A \to \overline{V} \otimes A \tag{4.32}$$

$$\langle v_0| \otimes \hat{O} \to \langle v_0|e^{i(t_1-t_0)H}\hat{O}e^{i(t'-t_2)H} \tag{4.33}$$

(iv) $(t_0,t_1) \subset [0,t') \; : \; t_0 > 0$

$$A \to \overline{V} \otimes A \tag{4.34}$$

$$\hat{O} \to \langle v_{\text{in}}|e^{i(t_0-0)H}\hat{O}e^{i(t'-t_1)H} \tag{4.35}$$

(v) $(t_0, t_1) \subset (t, T]\ :\ t_1 < T$

$$A \to A \otimes V \tag{4.36}$$

$$\hat{O} \to e^{i(t_0 - t_1)H} \hat{O} e^{i(T - t_1)H} |v_{\text{out}}\rangle \tag{4.37}$$

The last two cases are of particular interest due to the fact that they refer to the inclusion of the empty set. Indeed, the scenario in (iv) can be depicted in the form

$$[0, (t_0, t_1), t')$$

It is therefore evident the nested interval does not contain the initial time 0, i.e. the initial state is empty. Dually, for the latter (v)

$$(t, (t_0, t_1), T]$$

where the nested interval clearly does not include the final state at time T.
Therefore, in both cases, we have fixed distinguished elements $e^{i(t'-0)H}$ and $e^{i(T-t)H}$, respectively, and realised the relative initial and final states, $\langle v_{\text{in}}|$ and $|v_{\text{out}}\rangle$, as two idealised states at time 0 and T.

Remark 4.1.12 From the above discussion it is evident the operators are defined on the open conncted set $(0, t)$, whereas the initial and final states belongs to the boundary, i.e. $0, T$, respectively. However, we were able to describe Quantum Mechanics within the formalism of prefactorisation algebras without focusing on the boundary. The initial and final states plays a relevant role in the experiment and it is interesting we need to fix distinguished elements in order to deal with them, when modelling it with prefactorisation algebras.

Remark 4.1.13 We have used the notation $\hat{\bullet}$ for operators. This is common practice in quantum mechanics, where the hat distinguish the operator from the corresponding variable. For instance, x denotes position in some chosen space where the system under exam lives, while $\hat{x}$ the position operator.

Remark 4.1.14 Finally, we point out that if the Hamiltonian is zero, in other words there is no evolution in time for the system, the structure map coincides with the one of a prefactorisation algebra associated to a unital associative algebra.

4.2 Prefactorisation Algebras and Universal Enveloping Algebras

Universal enveloping algebras and, more generally, factorisation envelopes of a sheaf of Lie algebras are essential for the formulation of Noether's theorem in the context of factorisation algebras. Indeed, if the symmetries of a certain quantum field theory on a space X are given by $\mathcal{L}$, sheaf of local homotopy Lie algebras on such a space, the following holds:

Theorem 4.2.1 [Costello & Gwilliam [CG2] (2020), §1.1.5] In this situation, there is a canonical $\hbar$ dependent (shifted) central extension of $\mathcal{L}$ and a map $\mathcal{U}_c(\mathcal{L}) \to \mathrm{Obs}^q$ of factorisation algebras, from the twisted factorisation envelope of $\mathcal{L}$ to the factorisation algebra of observables of the QFT.

Example 4.2.1 Specialising the above result to $\mathcal{L} = \Omega_X^\bullet \otimes \mathfrak{g}$, we recover the so-called **Noether current**, the n-1 form on X valued in Obs^q

$$\Omega_c^1 \otimes \mathfrak{g} \to \mathrm{Obs}^q \tag{4.38}$$

obtained by taking in degree zero the co-chain map $\Omega_c^\bullet(U) \otimes \mathfrak{g}[1] \to \mathrm{Obs}^q(U)$ for every $U \in \mathbf{Open}_X$, induced by the action of the Lie algebra $\mathfrak{g}$ on the theory.

Having understood the importance of factorisation envelops in the theory, we aim at showing it is possible to recover the universal enveloping algebra from a prefactorisation algebra over $\mathbb{R}$. In order to achieve this goal, we begin by introducing the relevant notions and explain in detail the mathematical structures involved.

4.2.1 Motivation: PFAs in Noether's Theorem

Let $\mathfrak{g} \in \mathbf{Lie\text{-}alg}_{\mathbb{R}}^0$ be a Lie algebra over the reals, consider the prefactorisation algebra $\mathfrak{g}^{\mathbb{R}}$ defined on objects as follows

$$\mathbf{Open}_{\mathbb{R}}^c \ni U \underset{\mathfrak{g}^{\mathbb{R}}}{\longrightarrow} \left(\Omega_c^\bullet(U) \otimes \mathfrak{g}, d_{\mathrm{dR}}\right) \in \mathbf{dg\text{-}Lie\text{-}alg}_{\mathbb{R}} \tag{4.39}$$

from the poset category of open connected sets of the real line to the symmetric monoidal category of differential graded Lie algebras with monoidal product given by direct sum on co-chain complexes and de Rham differentials.

Remark 4.2.1 The target monoidal category results in a multicategory by equivalently defining multiarrows, therefore the assignement above gives in fact a multifunctor from $\mathbf{Open}_{\mathbb{R}}^c$ to $\mathbf{dg\text{-}Lie\text{-}alg}_{\mathbb{R}}$, i.e. a prefactorisation algebra according to Definition 3.3.9. It is also a cosheaf of co-chain complexes, but only a *pre*cosheaf of differential graded Lie algebras, since direct sums of the underlying differential graded vector spaces are not co-products in $\mathbf{dg\text{-}Lie\text{-}alg}_{\mathbb{R}}$. Indeed, the coproduct of two dg Lie algebras is obtained going quotient over the direct sum of the underying graded spaces. The natural Lie bracket is induced by the differentials.

Furthermore, recall in Definition 1.1.16 we defined a precosheaf as a covariant functor from open sets to the category of modules over a certain base ring L. However, in the context above the prefactorisation algebra takes values in the category of differential vector spaces over a chosen field $\mathbb{K}$. The definition is still valid as we are in fact dealing with a family of differential vector spaces (modules) equipped with differentials of co-homological degree one squaring to zero. They have more structure, not less.

The PFA $\mathfrak{g}^{\mathbb{R}}$ takes an open connected set U of the real line to the algebra $\Omega_c^{\bullet}(U) \otimes \mathfrak{g}$ given by tensoring the compactly-supported forms over the open U with the Lie algebra $\mathfrak{g}$.

Observe that, from the differential point of view, the only non-trivial piece is the one of forms, since it carries the standard de Rham operator, namely

$$\Omega_c^{\bullet}(U) := \{f \in C^{\infty}(U) : \text{f has compact support}\} \otimes_{\mathbb{R}} \Omega^* \tag{4.40}$$

where $\Omega^* = \oplus_{p=0}^{1} \Omega^p(U)$ is the algebra of forms on the real line with $\{dx\}$ as basis and de Rham differential

$$d_{\mathrm{dR}} : \Omega^p(U) \to \Omega^{p+1}(U) \tag{4.41}$$

Given a Lie algebra $\mathfrak{h}$ it is possible to define its homology by studying the Chevalley-Eilenberg chains $C_{\bullet}(\mathfrak{h})$. This procedure, when applied to the prefactorisation algebra $\mathfrak{g}^{\mathbb{R}}$, is the crucial tool needed to achieve the following major result:

Theorem 4.2.2 [Costello & Gwilliam [CG1] (2016), §3.4, Proposition 4.0.1] Let $\mathcal{H}$ denote the co-homology prefactorisation algebra of $C_*\left(\mathfrak{g}^{\mathbb{R}}\right)$. That is, we take the co-homology of every open and every structure map, so

$$\mathcal{H}(U) = H^*\left(C_*\left(\mathfrak{g}^{\mathbb{R}}(U)\right)\right)$$

for any open U. Then $\mathcal{H}$ is locally constant, and the corresponding associative algebra is isomorphic to $U\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$.

4.2.2 Modules over a Lie algebra and Chevalley-Eilenberg chains

In order to prove the theorem, recall the construction of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ seen in chapter one, section 3.1. This functor takes a Lie algebra to an associative algebra. We are interested in its representations category [Costello & Gwilliam [CG1] (2016), Appendix A.3, 240 et seq.].

Definition 4.2.1 Let **Rep** $(\mathcal{U}(\mathfrak{g}))$ denote the **category of representations of the universal enveloping algebra** of a Lie algebra $\mathfrak{g}$, whose objects and morphisms are respectively left $\mathcal{U}(\mathfrak{g})$- modules and maps between them. In other words, ${}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod} = \mathbf{Rep}(\mathcal{U}(\mathfrak{g}))$.

Remark 4.2.2 In the section concerning prefactorisation algebras from bimodules, we have already analysed the actual possibility of defining modules over an associative algebra or, more generally, an algebra. In fact, the latter has more structure of a commutative ring R, for instance, on which we are used to construct modules. The above category is therefore well-defined.

Define further the category of left $\mathfrak{g}$-module, denoted analogously by **Rep** $(\mathfrak{g})$ or ${}_{\mathfrak{g}}\mathbf{Mod}$, as the category whose objects are representations of the Lie algebra and morphisms arrow between them.

Definition 4.2.2 A **representation** for $\mathfrak{g}$ is a pair (V, ρ) made by a vector space $V \in \mathbf{Vect}_{\mathbb{K}}$ and a bilinear map

$$\rho : \mathfrak{g} \otimes_{\mathbb{K}} V \to V \tag{4.42}$$

$$(\gamma, v) \to \rho(\gamma, v) = \gamma \cdot v = [\gamma, v] \tag{4.43}$$

with the property that, on the bracket,

$$\rho([\gamma_1, \gamma_2] \otimes v) = \rho(\gamma_1 \otimes \rho(\gamma_2 \otimes v)) - \rho(\gamma_2 \otimes \rho(\gamma_1 \otimes v)) \tag{4.44}$$

$\forall\, \gamma_1, \gamma_2 \in \mathfrak{g}, v \in V$.

Moreover, the vector space V is called **support** of the representation.

Remark 4.2.3 In the definition of the representation map (4.43) we have displayed three possible equivalent notations for its image. The second one makes the structure of left $\mathfrak{g}$-module clear and the latter is particularly useful for calculations.

Definition 4.2.3 A **morphism of representations** $\psi \in \mathbf{Rep}(\mathfrak{g})$[1] is a morphism of vector spaces

$$\psi : V \to W \tag{4.45}$$

compatible with the bracket, i.e. such that

$$\psi([\gamma, v]) = [\gamma, \psi(v)] \tag{4.46}$$

$\forall\, \gamma \in \mathfrak{g}, v \in V$.

Definition 4.2.4 The representation of $\mathfrak{g}$ with $V = \mathbb{K}$ is called **trivial representation**.

Remark 4.2.4 The name is due to the fact that, in this case,

$$\rho(\gamma, \lambda) = [\gamma, \lambda] = \gamma \cdot \lambda - \lambda \cdot \gamma = 0 \tag{4.47}$$

$\forall\, \gamma \in \mathfrak{g}, \lambda \in \mathbb{K}$.

Since we have an adjunction between the categories $\mathbf{Lie\text{-}alg}_{\mathbb{K}}$ and $\mathbf{Ass\text{-}alg}_{\mathbb{K}}$ of Lie and associative algebras, respectively,

$$\mathcal{U}(\bullet) : \mathbf{Lie\text{-}alg}_{\mathbb{K}} \rightleftarrows \mathbf{Ass\text{-}alg}_{\mathbb{K}} : F \tag{4.48}$$

we can interchange the two types of left modules, ${}_{\mathfrak{g}}\mathbf{Mod}$ and ${}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod}$ accordingly [Costello & Gwilliam [CG1] (2016), Appendix A.3, 241, line 32 et seq.]. In fact, the commutator on the associative algebra elements, by construction, agrees with the Lie bracket.

Remark 4.2.5 The above-defined adjunction is explained in detail in section 1.3.1 of this thesis.

On the category ${}_{\mathfrak{g}}\mathbf{Mod}$ there exist two natural functors:

Definition 4.2.5 Let $V \in {}_{\mathfrak{g}}\mathbf{Mod}$, the **invariants** of V, left $\mathfrak{g}$-module, are given by the functor

$$\{\bullet\}^{\mathfrak{g}} : {}_{\mathfrak{g}}\mathbf{Mod} \to \mathbf{Vect}_{\mathbb{K}} \tag{4.49}$$

$$V \mapsto V^{\mathfrak{g}} := \{v : [\gamma, v] = 0\ \forall \gamma \in \mathfrak{g}\} \tag{4.50}$$

Definition 4.2.6 Let $V \in {}_{\mathfrak{g}}\mathbf{Mod}$, the **co-invariants** of V, left $\mathfrak{g}$-module, are given by the functor

$$\{\bullet\}_{\mathfrak{g}} : {}_{\mathfrak{g}}\mathbf{Mod} \to \mathbf{Vect}_{\mathbb{K}} \tag{4.51}$$

$$V \mapsto V_{\mathfrak{g}} := \frac{V}{\mathfrak{g}V} \tag{4.52}$$

Remark 4.2.6 (Invariants and co-invariants of the trivial representation) A particular role plays the trivial left $\mathfrak{g}$-module $\mathbb{K}$. Indeed, by taking the quotient of the universal enveloping algebra of $\mathfrak{g}$ by the ideal generated by the latter, $(\mathfrak{g})$, we obtain $\mathbb{K}$.
For it, observe the quotient under exam is given by tensor module over $\mathfrak{g}$

$$\oplus_{n\geq 0}\mathfrak{g}^{\otimes^n} = \mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \ldots \tag{4.53}$$

and relations

$$\begin{cases} [\gamma_1, \gamma_2] = \gamma_1 \otimes \gamma_2 - \gamma_2 \otimes \gamma_1 \\ \\ \gamma_i = 0 \quad \text{for every generator } \gamma_i \text{ of the Lie algebra } \mathfrak{g} \end{cases} \tag{4.54}$$

Thus, the relations forces

$$\frac{\mathcal{U}(\mathfrak{g})}{(\mathfrak{g})} \cong \mathbb{K} \tag{4.55}$$

We conclude the trivial representation is an $\mathcal{U}(\mathfrak{g})$-bimodule, i.e. an object in $\left({}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod}_{\mathcal{U}(\mathfrak{g})}\right)^0$.

Remark 4.2.7 In what follows modules are considered to be objects of the category ${}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod}$. In this way, we recover the Ext and Tor functors.

For any module V we derive below equivalent characterisations of invariants and co-invariants due to the argument presented in the previous remark:

(i)

$$V_{\mathfrak{g}} := \frac{V}{\mathfrak{g}V} = \frac{\mathcal{U}(\mathfrak{g})}{(\mathfrak{g})} \otimes_{\mathcal{U}(\mathfrak{g})} V \cong \mathbb{K} \otimes_{\mathcal{U}(\mathfrak{g})} V \tag{4.56}$$

(ii)

$$V^{\mathfrak{g}} := \{v : [\gamma, v] = 0 \; \forall \gamma \in \mathfrak{g}\} = \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathbb{K}, V) \tag{4.57}$$

In conclusion, by considering both the base field $\mathbb{K}$ and the **support** V of the representation as objects of the category ${}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod}$, we reconstructed the two fundamental operations of homological algebra: formation of homomorphisms groups and taking of tensor products.

$$\begin{tikzcd} {}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod} \arrow[r, shift left, "\mathbb{K}\otimes_{\mathcal{U}(\mathfrak{g})}\bullet"] \arrow[r, shift right, "\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathbb{K}\,,\bullet)"'] & \mathbf{Ab} \end{tikzcd}$$

Remark 4.2.8 The invariants and coinvariants of a module V are defined as functors with values in the category of vector spaces over $\mathbb{K}$. In the above we have indicated **Ab** as target category instead. This is perfectly fine as a vector space is an Abelian group equipped with a further operation given by scalar multiplication. Therefore, there exists an obvious forgetful functor $F : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Ab}$ forgetting such operation. This choice is relevant for the argument to be present below, as we aim at reconstructing the functors Tor and Ext.

The fact that $\mathbb{K}$ is a left and a right $\mathcal{U}(\mathfrak{g})$-module is crucial. The Abelian category ${}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod}$ has enough projectives and injectives. Therefore, the functor

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathbb{K}, \bullet) : {}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod} \longrightarrow \mathbf{Ab} \tag{4.58}$$

is left exact and, if we fix $\mathbb{K}$ as left module, its right derived functor yields $\mathrm{Ext}^{\bullet}_{\mathcal{U}(\mathfrak{g})}(\mathbb{K}, \bullet)$.

On the other hand, $\mathbb{K}$ is also a right $\mathcal{U}(\mathfrak{g})$-module. Thus, fixing it, the functor

$$\mathbb{K} \otimes_{\mathcal{U}(\mathfrak{g})} \bullet : {}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod} \longrightarrow \mathbf{Ab} \tag{4.59}$$

is right exact and its left derived functor is $\mathrm{Tor}^{\mathcal{U}(\mathfrak{g})}_{\bullet}$.

Remark 4.2.9 An object I of a category $\mathbf{B}$ defines an **injective object** if for every monic arrow $\iota : b_1 \rightarrowtail b_2$ and every morphism $f : b_1 \to I$ there exists an arrow $h : b_2 \to I$ making the following diagram commute

$$\begin{tikzcd} & b_2 \arrow[dl, dashed, "h"'] \\ I & b_2 \arrow[l, "f"] \arrow[u, tail, "\iota"'] \end{tikzcd}$$

i.e. $h\iota = f$.

The notion of **projective object** is obtained by dualisation of the former, i.e. inverting arrows in the diagram. Furthermore, a category **B has enough injectives** if for every object $b \in \mathbf{B}_0$ there exists a monic arrow from an injective object I to b, i.e. $g : I \rightarrowtail b$. Analogously, a category **B has enough projectives** if for every object $b \in \mathbf{B}_0$ there exists an epic arrow from the object b to a projective object P.

Finally, for $V \in {}_{\mathfrak{g}}\mathbf{Mod}^0$ a $\mathfrak{g}$-module, we define its Lie algebra homology and co-homology as follows:

Definition 4.2.7 [Costello & Gwilliam [CG1] (2016), Appendix A.3, 241, Definition 3.1.1] The Lie algebra homology groups of V are given by

$$H_{\bullet}(\mathfrak{g}, V) = \mathrm{Tor}^{\mathcal{U}(\mathfrak{g})}_{\bullet}(\mathbb{K}, V) \tag{4.60}$$

whereas its co-homology group are defined by

$$H^{\bullet}(\mathfrak{g}, V) = \mathrm{Ext}^{\bullet}_{\mathcal{U}(\mathfrak{g})}(\mathbb{K}, V) \tag{4.61}$$

Remark 4.2.10 In dimension zero the above assignements give invariants and co-invariants as (co-)homology groups. Indeed,

$$H_0(\mathfrak{g}, V) = \mathbb{K} \otimes_{\mathcal{U}(\mathfrak{g})} V = V_{\mathfrak{g}} \tag{4.62}$$

and

$$H^0(\mathfrak{g}, V) = \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathbb{K}, V) \tag{4.63}$$

In order to define the Chevalley-Eilenberg complexes for Lie algebra (co-)homology, we resolve $\mathbb{K}$, the trivial representation, as $\mathcal{U}(\mathfrak{g})$-module. We showed in fact it can be equivalently written as quotient of the universal enveloping algebra by the ideal generated by $\mathfrak{g}$, i.e.

$$\mathbb{K} \underset{q}{\cong} \frac{\mathcal{U}(\mathfrak{g})}{(\mathfrak{g})} \tag{4.64}$$

Firstly, we produce a free resolution of $\mathbb{K}$

$$\cdots \to \wedge^n \mathfrak{g} \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{g}) \xrightarrow{\phi^n} \cdots \to \mathfrak{g} \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{g}) \xrightarrow{\phi^1} \mathcal{U}(\mathfrak{g}) \xrightarrow{q} \mathbb{K} \to 0 \tag{4.65}$$

where the functions are q, the quotient map, and $\phi^i : \wedge^i \mathfrak{g} \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{g}) \longrightarrow \wedge^{i-1} \mathfrak{g} \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{g})$ given by the expression

$$\begin{gathered}(\gamma_1 \wedge \cdots \wedge \gamma_i) \otimes (x_1 \cdots x_m) \\ \big\downarrow \phi^i \\ \sum_{k=1}^{i} (-1)^{i-k} (\gamma_1 \wedge \cdots \wedge \hat{\gamma}_k \cdots \wedge \gamma_i) \otimes (\gamma_k \cdot x_1 \cdots x_m) - \sum_{1 \le j < k \le i} (-1)^{j+k-1} \left([\gamma_j, \gamma_k] \wedge \gamma_1 \wedge \cdots \wedge \hat{\gamma}_j \wedge \cdots \wedge \hat{\gamma}_k \cdots \wedge \gamma_i \right) \otimes (x_1 \cdots x_m)\end{gathered}$$

$\forall \ \{\gamma_k\}_{k=1}^{i} \in \mathfrak{g}, \ \{x_s\}_{s=1}^{m} \in \mathcal{U}(\mathfrak{g})$.

Remark 4.2.11 The hat indicates removal of the subjacent element.

Secondly, we notice this is a free resolution of $\mathbb{K}$. Therefore, for V a module we are left with

$$\cdots \to \wedge^n \mathfrak{g} \otimes_{\mathbb{K}} V \to \cdots \to \mathfrak{g} \otimes_{\mathbb{K}} V \to V \to 0 \tag{4.66}$$

isomorphic to

$$\mathbb{K} \otimes^{\mathbf{L}}_{\mathcal{U}(\mathfrak{g})} V \tag{4.67}$$

and, by simply dualising the above complex, we gain a second complex

$$0 \cdots \leftarrow \wedge^n \mathfrak{g}^* \otimes_{\mathbb{K}} V \leftarrow \cdots \leftarrow \mathfrak{g}^* \otimes_{\mathbb{K}} V \leftarrow V \tag{4.68}$$

isomorphic to

$$\mathbf{R}\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathbb{K}, V) \tag{4.69}$$

The Tor and Ext derived functors measure the failure of the tensor product and Hom, respectively, to be exact, i.e. to preserve short exact sequences. By [Osborne [OS] (2000), 25, Proposition 2.6] we conclude the two functors above are respectively left and right exact. Therefore the module V is projective and flat [ibid., 28, Definition 2.8].

Remark 4.2.12 Some observations regarding the complexes just defined:

- The second complex has arrows inverted, as dualisation is a contra-variant functor. The notation $*$ refers to the linear dual and the dual bracket reads $d|_{\mathfrak{g}^*} := [\bullet, \bullet] : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$;

- Left and right derived functors, denoted in the above by $\mathbf{L}$ and $\mathbf{R}$, represent a generalisation of Tor and Ext for projective and injective resolutions. For instance, the derived tensor product

$$\bullet \otimes^{\mathbf{L}} \bullet : \mathrm{D}\left(\mathbf{Mod}_{\mathcal{U}(\mathfrak{g})}\right) \times \mathrm{D}\left({}_{\mathcal{U}(\mathfrak{g})}\mathbf{Mod}\right) \to \mathrm{D}\left(\mathbf{Ab}\right) \tag{4.70}$$

has as source categories the derived versions of the Abelian categories we are dealing with. The derived category $\mathrm{D}(\mathbf{B})$ of an Abelian category $\mathbf{B}$ is defined by its universal property of being the localisation of the category of complexes over $\mathbf{B}$ with respect to quasi-isomorphisms.

Finally, by means of the previous constructions, we provide the definitions of Chevalley-Eilenberg chains and co-chains for the Lie algebra (co-)homology of a representation V of $\mathfrak{g} \in \mathbf{Lie\text{-}alg}^0_{\mathbb{K}}$.

Definition 4.2.8 [Costello & Gwilliam [CG1] (2016), 242, Appendix A.3, Definition 3.1.2] The **Chevalley-Eilenberg chains** are defined as

$$C_\bullet(\mathfrak{g}, V) = \left(\mathrm{Sym}_{\mathbb{K}}(\mathfrak{g}[1]) \otimes_{\mathbb{K}} V, d\right) \tag{4.71}$$

with differential

$$d\left(x_1 \wedge \cdots \wedge x_n \otimes v\right) = \sum_{1 \le j < k \le n} (-1)^{j+k} \left[x_j, x_k\right] \wedge x_1 \wedge \ldots \hat{x}_j \wedge \cdots \wedge \hat{x}_k \cdots \wedge x_n \otimes v + \tag{4.72}$$

$$+ \sum_{j=1}^{n} (-1)^{n-j} x_1 \wedge \ldots \hat{x}_j \cdots \wedge x_n \otimes \left[x_j, v\right] \tag{4.73}$$

The **Chevalley-Eilenberg co-chains** are instead given by

$$C^\bullet(\mathfrak{g}, V) = \left(\mathrm{Sym}_{\mathbb{K}}(\mathfrak{g}^*[-1]) \otimes_{\mathbb{K}} V, d\right) \tag{4.74}$$

Consider a basis $\{\gamma_i\}_{i=1}^n$ for the Lie algebra $\mathfrak{g}$. The dual differential $\delta := d|_{\mathfrak{g}^*}$ works on generators of $\mathfrak{g}^*$, denoted by $\left\{\gamma^i\right\}_{i=1}^n \ni \gamma$, as follows:

$$\delta\left(\gamma^k \otimes v\right) = -\sum_{i<j} \gamma^k \left(\left[\gamma_i, \gamma_j\right]\right) \gamma^i \wedge \gamma^j \otimes v + \sum_r \gamma^k \wedge \gamma^r \otimes [\gamma_r, v] \tag{4.75}$$

The extension to any wedge of dual elements tensored with a module one is done by setting the relative differential to be a derivation of co-homological degree one [iteration of Leibniz rule].

Remark 4.2.13 We summarise in this remark some further technicalities:

- For the trivial representation, it is common use to drop the V when writing, e.g. $C_\bullet(\mathfrak{g})$;

- We switched back to $V \in {}_{\mathfrak{g}}\mathbf{Mod}$ using the adjunction pair $F, \mathcal{U}(\bullet)$;

- Chevalley-Eilenberg chain and co-chain yield, by definition, commutative algebras and co-algebras.

4.2.3 Proof of the theorem

After having investigated the category of modules, their properties and resolutions and having defined Chevalley-Eilenberg (co-)chains, we recall and prove the theorem under exam.

Theorem 4.2.3 [Costello & Gwilliam [CG1] (2016), §3.4, Proposition 4.0.1] Let $\mathcal{H}$ denote the co-homology prefactorisation algebra of $C_*\left(\mathfrak{g}^{\mathbb{R}}\right)$. That is, we take the co-homology of every open and every structure map, so

$$\mathcal{H}(U) = H^*\left(C_*\left(\mathfrak{g}^{\mathbb{R}}(U)\right)\right)$$

for any open U. Then $\mathcal{H}$ is locally constant, and the corresponding associative algebra is isomorphic to $U\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$.

Proof

Step I: Constant locality

From the general theory it follows that, if the prefactorisation algebra under exam is proven to be locally constant, there exists an associative algebra arising from it. Therefore, we need to show the constant locality of $\mathcal{H}(U)$: for any two nested intervals $U, V \in \mathbf{Open}^c_{\mathbb{R}} : U \overset{\iota}{\hookrightarrow} V$, the application of the prefactorisation algebra functor $\mathcal{H}(\bullet)$ leads to an isomorphism in co-homology. We prove this for unary arrows, however the procedure can be generalised to any arrow in the poset multicategory $\mathbf{Open}^c_{\mathbb{R}}$.

Consider the inclusion $U \overset{\iota}{\hookrightarrow} V$ and apply the prefactorisation algebra defined on objects in section 4.2.1 as

$$\mathbf{Open}^c_{\mathbb{R}} \ni U \underset{\mathfrak{g}^{\mathbb{R}}}{\longrightarrow} \left(\Omega^\bullet_c(U) \otimes \mathfrak{g}, d_{\mathrm{dR}}\right) \in \mathbf{dg\text{-}Lie\text{-}alg}_{\mathbb{R}} \tag{4.76}$$

The multifunctor $\mathcal{H}$ yields an arrow

$$\Omega^\bullet_c(U) \otimes \mathfrak{g} \overset{\mathfrak{g}^{\mathbb{R}}(\iota)}{\to} \Omega^\bullet_c(V) \otimes \mathfrak{g} \tag{4.77}$$

of compactly supported forms tensored with the Lie algebra $\mathfrak{g}$, where a compact form on U is extended to 0 on the complement $V \setminus U$.

The co-homology prefactorisation algebra gives co-homology groups over the CE (Chevalley-Eilenberg) chains induced by the arrow ι, i.e. $\mathcal{H}\left(\mathfrak{g}^{\mathbb{R}}(\iota)\right)$ reads

$$H^{\bullet}\left(C_{\bullet}\left(\Omega_c^{\bullet}(U)\otimes\mathfrak{g}\right)\right)\to H^{\bullet}\left(C_{\bullet}\left(\Omega_c^{\bullet}(V)\otimes\mathfrak{g}\right)\right) \tag{4.78}$$

By definition of CE chains, the former coincides with an arrow

$$H^{\bullet}\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(U)\otimes\mathfrak{g}\right)[1]\right)\otimes_{\mathbb{R}}\mathbb{R},d\right)\to H^{\bullet}\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(V)\otimes\mathfrak{g}\right)[1]\right)\otimes_{\mathbb{R}}\mathbb{R},d\right) \tag{4.79}$$

As the trivial representation is chosen, we can omit the support resulting in an arrow

$$H^{\bullet}\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(U)\otimes\mathfrak{g}\right)[1]\right),d\right)\to H^{\bullet}\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(V)\otimes\mathfrak{g}\right)[1]\right),d\right) \tag{4.80}$$

We are left to show this is an isomorphism. The latter equation follows by the consecutive application of two functors to $\Omega_c^{\bullet}(U)\otimes\mathfrak{g}\xrightarrow{\mathfrak{g}^{\mathbb{R}}(\iota)}\Omega_c^{\bullet}(V)\otimes\mathfrak{g}$: the Chevalley-Eilenberg functor, from dg Lie algebras to (CE-)chain complexes over them, and the co-homology functor from the category of (CE-)chain complexes to Abelian groups (more extacly commutative algebras over the reals).

Remark 4.2.14 We recover the latter as functor in the category of Abelian group by applying to the commutative algebra created by the CE chains the forgetful functor that forgets both algebra multiplication and scalar multiplication. It results in an Abelian group.

The following diagram helps representing the situation

$$\begin{tikzcd}
\Omega_c^{\bullet}(U)\otimes\mathfrak{g} \arrow[r, "\mathfrak{g}^{\mathbb{R}}(\iota)"] \arrow[d, "C_{\bullet}(\bullet)"'] & \Omega_c^{\bullet}(V)\otimes\mathfrak{g} \arrow[d, "C_{\bullet}(\bullet)"] \\
\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(U)\otimes\mathfrak{g}\right)[1]\right),d\right) \arrow[r, "C_{\bullet}\left(\mathfrak{g}^{\mathbb{R}}(\iota)\right)"] \arrow[d, "H^{\bullet}(\bullet)"'] & \left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(V)\otimes\mathfrak{g}\right)[1]\right),d\right) \arrow[d, "H^{\bullet}(\bullet)"] \\
H^{\bullet}\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(U)\otimes\mathfrak{g}\right)[1]\right),d\right) \arrow[r, "H^{\bullet}\left(C_{\bullet}\left(\mathfrak{g}^{\mathbb{R}}(\iota)\right)\right)"] & H^{\bullet}\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^{\bullet}(V)\otimes\mathfrak{g}\right)[1]\right),d\right)
\end{tikzcd}$$

In terms of categories,

$$\mathbf{Open}_{\mathbb{R}}^{c}\xrightarrow{\mathfrak{g}^{\mathbb{R}}(\bullet)}\mathbf{dgLie\text{-}Alg}_{\mathbb{R}}\xrightarrow{C_{\cdot}(\bullet)}\left(\mathbf{dgAlg}_{\mathrm{sf},1}\right)^{\mathrm{op}}\xrightarrow{H^{\cdot}(\bullet)}\mathbf{Vect}_{\mathbb{R}}\left[\mathbf{Com\text{-}Alg}_{\mathbb{R}}\right] \tag{4.81}$$

Remark 4.2.15 The middle category is the opposite category of the differential graded semifree algebras with generators in degree one. A **semifree dga** is a differential graded algebra whose associated graded algebra is free.

The fact that the co-homology groups in this case reduce to a commutative algebra is proved later.
However, as analysed in chapter one, section 1.2, there is a well-defined forgetful functor

$$\textbf{Com-Alg}_{\mathbb{R}} \to \textbf{Vect}_{\mathbb{R}} \tag{4.82}$$

that forgets all the structures present at the level of CE-chains up to the first line of the diagram, where we have two right $\mathbb{R}$-modules both with basis $\{1, dx\}$, but one with respect to compactly-supported functions in U, $C_c(U)$ and the other one with respect to $C_c(V)$.

Since we are working with compact supported forms over two nested interval of the real line, it is possible to construct a chain map between the realative complexes yielding a chain homotopy equivalence.
Indeed, between the two complexes there is a sequence of commutative squares of the form

$$\begin{tikzcd} \Omega^n_c(U) \arrow[r, "d^n_U"] \arrow[d, "f_n"'] & \Omega^{n+1}_c(U) \arrow[d, "f_{n+1}"] \\ \Omega^n_c(V) \arrow[r, "d^n_V"] & \Omega^{n+1}_c(V) \end{tikzcd}$$

where on the horizontal arrows are displayed the de Rham differentials and on the vertical arrows the elements of the chain map $f := \{f_n\}_{\mathbb{Z}}$.

Using the pair extension/restriction of functions, as $U \subset V$, there exists a chain homotopy equivalence

$$\begin{tikzcd} \left(\Omega^\bullet_c(U), \mathrm{d}_U\right) \arrow[loop left, "h_2"] \arrow[r, shift left, "f"] & \left(\Omega^\bullet_c(V), \mathrm{d}_V\right) \arrow[l, shift left, "g"] \arrow[loop right, "h_1"] \end{tikzcd}$$

i.e. there exists a family of maps $\{g_n\}_{n\in\mathbb{Z}} =: g$ such that

$$1_V - f \circ g = d_V \circ h_1 + h_1 \circ d_V \tag{4.83}$$

$$1_U - g \circ f = d_U \circ h_2 + h_2 \circ d_U \tag{4.84}$$

However, due to homotopy invariance of co-homology, the above-displayed chain homotopy equivalence produces isomorphic co-homology groups in every degree.
We have just proven the bottom row of the diagram below is thus an isomorphism.

$$\begin{tikzcd}
\Omega_c^\bullet(U) = \langle\{1, dx\}\rangle \arrow[r, "id\in\mathbf{Vect}^1_{\mathbb{R}}"] \arrow[d, "H^\bullet(C.(\mathfrak{g}^{\mathbb{R}}U))"'] & \langle\{1, dx\}\rangle = \Omega_c^\bullet(V) \arrow[d, "H^\bullet(C.(\mathfrak{g}^{\mathbb{R}}V))"] \\
H^\bullet\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^\bullet(U)\otimes\mathfrak{g}\right)[1]\right), d\right) \arrow[r, "H^\bullet(C.(\mathfrak{g}^{\mathbb{R}}(\iota)))"] & H^\bullet\left(\mathrm{Sym}_{\mathbb{R}}\left(\left(\Omega_c^\bullet(V)\otimes\mathfrak{g}\right)[1]\right), d\right)
\end{tikzcd}$$

Step II: Construction of the corresponding associative algebra

At this point, having proven the constant locality of the PFA $\mathcal{H}(\cdot)$ on $\mathbb{R}$, we form the associative algebra following the procedure showed in section 3.4.1.

Define

$$\mathcal{H}(\mathbb{R}) =: A_{\mathfrak{g}} \tag{4.85}$$

the induced multiplication map $\mu_{A_{\mathfrak{g}}}$ is therefore

$$\begin{tikzcd}
\mathcal{H}(I_1)\otimes\mathcal{H}(I_2) \arrow[r, "\iota_*"] \arrow[d, "(\psi_1,\,\psi_2)\,\cong"'] & \mathcal{H}(V) \arrow[d, "\psi_V\,\cong"] \\
A_{\mathfrak{g}}\otimes A_{\mathfrak{g}} \arrow[r, "\mu_{A_{\mathfrak{g}}}"] & A_{\mathfrak{g}}
\end{tikzcd}$$

The notation ι_* denotes the arrow induced by the inclusion $\iota : I_1, I_2 \to V$ at the level of co-homology, to be extended in case of inclusion of n intervals in the apparent way.

Remark 4.2.16 The top row arrow exists each time the relative arrow $\iota : I_1, I_2 \to V$ in the poset multicategory $\mathbf{Open}^c_{\mathbb{R}}$ exists. In other words, the open connected intervals $I_1,\ I_2$ are disjoint and contained in V. The latter interval will be considered to be the whole real line, i.e. $V \cong \mathbb{R}$.

In multicategorical notation, for an input string of n objects, the former generalises to a diagram

$$\begin{tikzcd}
\mathbf{Open}^c_{\mathbb{R}}(I_1,\ldots,I_n;\mathbb{R}) \arrow[d] \\
\mathbf{Vect}_{\mathbb{R}}(\mathcal{H}(I_1),\ldots,\mathcal{H}(I_n);\mathcal{H}(\mathbb{R})) \arrow[d, "\cong\ \psi:=(\psi_1,\psi_2,\ldots,\psi_n;\ \psi_{\mathbb{R}})"] \\
\mathbf{Vect}_{\mathbb{R}}\left(A_{\mathfrak{g}},\ldots,A_{\mathfrak{g}};A_{\mathfrak{g}}\right)
\end{tikzcd}$$

Remark 4.2.17 We write the symbol $\cong$ on arrows each time the displayed map is an isomorphism.

Step III: Analysis of the co-homology groups

The complex $\left(\Omega_c^\bullet(\mathbb{R}) \otimes \mathfrak{g}, d_{\mathrm{dR}}\right)$, given by the PFA $\mathfrak{g}^{\mathbb{R}}(\mathbb{R})$, is a differential graded Lie algebra with non-trivial pieces in degree zero and one.
At the level of co-homology, the relative groups read

$$H^\bullet\left(\Omega_c^\bullet(\mathbb{R}) \otimes \mathfrak{g}\right) \cong H_c^\bullet(\mathbb{R}) \otimes \mathfrak{g} \tag{4.86}$$

The application of the Poincaré Lemma for compact co-homology (cf. [Bott & Tu [BT] (1982), §4, 37]), reduces the former family of groups to a single group in dimension one. Indeed, by [ibid., Corollary 4.7.1]:

$$H_c^\bullet(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{in dimension 1} \\ \{0\} & \text{otherwise} \end{cases} \tag{4.87}$$

Thus, we are left with

$$H^\bullet\left(\Omega_c^\bullet(\mathbb{R}) \otimes \mathfrak{g}\right) \cong H_c^\bullet(\mathbb{R}) \otimes \mathfrak{g} \overset{\text{P.L.}}{=} H_c^1(\mathbb{R}) \otimes \mathfrak{g} = \mathbb{R} \otimes \mathfrak{g} = \mathfrak{g}[-1] \tag{4.88}$$

an Abelian Lie algebra.

Definition 4.2.9 [Schottenloher [SM] (2008), §4.4.1] A Lie algebra $\mathfrak{a}$ is an **Abelian Lie algebra** if

$$[X, Y] = 0 \tag{4.89}$$

$\forall\, X, Y \in \mathfrak{a}$.

In fact, for any two elements $f dx \otimes \gamma_1,\ g dx \otimes \gamma_2 \in H_c^1(\mathbb{R}) \otimes \mathfrak{g}$ we have (cf. [ibid., 66, line 3 et seq.])

$$[f dx \otimes \gamma_1, g dx \otimes \gamma_2] = f dx \wedge g dx \otimes [\gamma_1, \gamma_2] = 0 \tag{4.90}$$

as the differential square to zero and the operation we are considering on differential form is the wedge product.

Applying the Chevalley-Eilenberg functor from dg Lie algebras to semifree dg algebras,

$$C_\bullet\left(\Omega_c^\bullet(\mathbb{R}) \otimes \mathfrak{g}\right) \underset{\text{q.i.}}{\simeq} C_\bullet(\mathbb{R} \otimes \mathfrak{g}) \cong C_\bullet(\mathfrak{g}[-1]) \overset{\text{def.}}{=} \mathrm{Sym}_{\mathbb{R}}(\mathfrak{g}[-1][1]) = \mathrm{Sym}_{\mathbb{R}}(\mathfrak{g}) \tag{4.91}$$

we obtain a quasi-isomorphism yielding the following result:

$$A_{\mathfrak{g}} = \mathcal{H}(\mathbb{R}) = H^{\bullet}\left(C_{\bullet}\left(\mathfrak{g}^{\mathbb{R}}(\mathbb{R})\right)\right) \cong \mathrm{Sym}_{\mathbb{R}}(\mathfrak{g}) \tag{4.92}$$

Thus, we conclude there exists an isomorphism in $\mathbf{Vect}_{\mathbb{R}}$ between $A_{\mathfrak{g}}$ and the symmetric algebra of $\mathfrak{g}$ that induces the diagram below

$$\begin{tikzcd}
\mathbf{Open}^{c}_{\mathbb{R}}(I_1, \ldots, I_n; \mathbb{R}) \arrow[d] \\
\mathbf{Vect}_{\mathbb{R}}(\mathcal{H}(I_1), \ldots, \mathcal{H}(I_n); \mathcal{H}(\mathbb{R})) \arrow[d, "\cong"] \\
\mathbf{Vect}_{\mathbb{R}}\left(\mathrm{Sym}_{\mathbb{R}}(\mathfrak{g}), \ldots, \mathrm{Sym}_{\mathbb{R}}(\mathfrak{g}); \mathrm{Sym}_{\mathbb{R}}(\mathfrak{g})\right)
\end{tikzcd}$$

Step IV: Building a splitting central extension

In order to construct the map yielding the isomorphism in [Costello & Gwilliam [CG1] (2016), §3.4, Proposition 4.0.1], we start by analysing central extensions in the category $\mathbf{Lie\text{-}alg}_{\mathbb{R}}$ of Lie algebras over the real line and buiding up the relevant cocycles. Finally, the reasoning will be extended to the differential graded case mutatis mutandis.

Remark 4.2.18 Any vector space V can be made a Lie algebra by equipping it with a trivial Lie bracket, i.e.

$$[v_1, v_2] = 0 \tag{4.93}$$

$\forall\, v_1, v_2 \in V$.

Definition 4.2.10 [Schottenloher [SM] (2008), §4.4.1, Definition 4.1] Let $\mathfrak{a}$ be an Abelian Lie algebra over $\mathbb{K}$ and $\mathfrak{g}$ a Lie algebra over $\mathbb{K}$ [...]. An exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is called a **central extension** of $\mathfrak{g}$ by $\mathfrak{a}$ if $[\mathfrak{a}, \mathfrak{h}] = 0$, that is $[X, Y] = 0$ for all $X \in \mathfrak{a}, Y \in \mathfrak{h}$. Here we identify $\mathfrak{a}$ with the corresponding subalgebra of $\mathfrak{h}$. For such a central extension, the Abelian Lie algebra $\mathfrak{a}$ is realised as an ideal in $\mathfrak{h}$ and $\pi : \mathfrak{h} \to \mathfrak{g}$ serves to identify $\mathfrak{g}$ with $\dfrac{\mathfrak{h}}{\mathfrak{a}}$.

Remark 4.2.19 Some observations on the definition just given:

- "if $[\mathfrak{a}, \mathfrak{h}] = 0$" is equivalent to requiring $\mathfrak{a}$ to be in the center of $\mathfrak{h}$

$$Z(\mathfrak{h}) = \{Y \in \mathfrak{h} \mid \forall\, X \in \mathfrak{g},\ [X, Y] = 0\} \tag{4.94}$$

  This explains the usage of the name "central extension";

- An ideal $\mathfrak{I}$ for a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ closed under bracketing with element of the latter. Namely,

$$[X, Y] \in \mathfrak{I} \tag{4.95}$$

  whenever $X \in \mathfrak{I}, Y \in \mathfrak{g}$.

Applying the former to our matter of concern, we observe the exact sequence

$$0 \longrightarrow H^1_c(\mathbb{R}) \xrightarrow{\text{incl.}} H^1_c(\mathbb{R}) \otimes \mathfrak{g} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

defines a central extension of $\mathfrak{g}$. The first Lie Algebra is Abelian, since the differentials square to zero, and the quotient map reads

$$\mathfrak{g} \cong \frac{H^1_c(\mathbb{R}) \otimes \mathfrak{g}}{H^1_c(\mathbb{R})} \tag{4.96}$$

More precisely, the sequence should be written for the entire complex as

$$0 \longrightarrow H^\bullet_c(\mathbb{R}) \xrightarrow{\text{incl.}} A_{\mathfrak{g}} = H^\bullet\left(C_\bullet\left(\Omega^\bullet_c(\mathbb{R}) \otimes \mathfrak{g}\right)\right) \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

Definition 4.2.11 [Schottenloher [SM] (2008), §4.4.1, Definition 4.2] An exact sequence of Lie algebra homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r] & \mathfrak{h} \arrow[r, "\pi"'] & \mathfrak{g} \arrow[l, bend right, "\beta"'] \arrow[r] & 0 \end{tikzcd}$$

splits if there is a Lie algebra homomorphism $\beta : \mathfrak{g} \to \mathfrak{h}$, called **splitting map**, with $\pi \circ \beta = id_{\mathfrak{g}}$. A central extension which splits is called **trivial extension**, since it is equivalent to the exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a} \oplus \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0$$

[In other words, $\mathfrak{h} \cong \mathfrak{a} \oplus \mathfrak{g}$.]

Remark 4.2.20 The theoretical foundations of the latter isomorphism are discussed below, where another characterisation of the splitting property is provided.

In our case, we will define the splitting map for the following central extension of $\mathfrak{g}$ induced in co-homology

$$0 \longrightarrow H_c^1(\mathbb{R}) \longrightarrow A_{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

that will produce the isomorphism

$$A_{\mathfrak{g}} \cong H_c^1(\mathbb{R}) \oplus \mathfrak{g} \cong H_c^1(\mathbb{R}) \otimes \mathfrak{g} \tag{4.97}$$

Thus, the quotient algebra $\mathfrak{g}$ is forced to be Abelian [commutative]. Indeed,

$$\mathfrak{g} \cong \frac{H_c^1(\mathbb{R}) \otimes \mathfrak{g}}{H_c^1(\mathbb{R})} \cong \frac{H_c^1(\mathbb{R}) \oplus \mathfrak{g}}{H_c^1(\mathbb{R})} \tag{4.98}$$

Before going through the construction and analysis of the co-cycles involved in the proof of the result we are interested in, we recall other relevant aspects of central extensions.

Proposition 4.2.1 [Schottenloher [SM] (2008), §4.4.1, Remark 4.3] For every central extension of Lie algebras

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow[\pi]{} \mathfrak{g} \longrightarrow 0$$

there is a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$ with $\pi \circ \beta = id_{\mathfrak{g}}$ [β not in general a Lie algebra homomorphism].
Let

$$\Theta(X, Y) := [\beta(X), \beta(Y)] - \beta([X, Y]) \tag{4.99}$$

for $X, Y \in \mathfrak{g}$.
Then β is a splitting map if and only if

$$\Theta = 0 \tag{4.100}$$

Further properties of the map Θ [ibid.]:

(i) $\Theta_{(\beta)} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ depends on β, is bilinear and alternating;

(ii) It is compatible with the Jacobi identity, i.e.

$$\Theta\left(X, [Y, Z]\right) + \Theta\left(Y, [Z, X]\right) + \Theta\left(Z, [X, Y]\right) = 0 \tag{4.101}$$

Definition 4.2.12 [Schottenloher [SM] (2008), §4.4.1, Definition 4.4] A map $\Theta : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{a}$ satisfying the properties in (i) and (ii) defines a **two-cocycle** on $\mathfrak{g}$ with values in $\mathfrak{a}$.

The vector spaces isomorphism

$$\mathfrak{h} \cong \mathfrak{g} \oplus \mathfrak{a} \tag{4.102}$$

is provided by the mapping

$$\psi : \mathfrak{g} \times \mathfrak{a} \to \mathfrak{h} \tag{4.103}$$

$$(X, Y) \mapsto \underbrace{\beta\left(X\right)}_{\in\,\mathfrak{h}} + Y =: X \oplus Y \tag{4.104}$$

where $\mathfrak{a} \subset \mathfrak{h}$ as sub-algebra, and becomes a Lie algebra isomorphism with Lie bracket

$$[X \oplus Z,\ Y \oplus Z']_{\mathfrak{h}} := [X, Y]_{\mathfrak{g}} + \underbrace{\Theta\left(X, Y\right)}_{\in\,\mathfrak{a}} \tag{4.105}$$

$\forall\ X, Y \in \mathfrak{g},\ Z, Z' \in \mathfrak{a}$.

Such a bracket can equivalently be defined as

$$[\beta\left(X\right) + Z,\ \beta\left(Y\right) + Z']_{\mathfrak{h}} = \beta\left([X, Y]_{\mathfrak{g}}\right) + \Theta\left(X, Y\right) \tag{4.106}$$

with $\mathfrak{a} \subset \mathfrak{h}$ identified as sub-algebra.

Remark 4.2.21 A sub-algebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{h}$ is defined to be a subspace of the corresponding vector space $V_{\mathfrak{h}}$ on which the latter Lie algebra is defined, closed with respect to the bracket operation in $\mathfrak{h}$, i.e.

$$[X, Y]_{\mathfrak{h}} \in \mathfrak{a} \tag{4.107}$$

for all $X, Y \in \mathfrak{a}$.

Finally, equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are in one-to-one correspondence with the co-homology group $H^2(\mathfrak{g}, \mathfrak{a})$. Consider the notations already introduced for exact sequences and maps, the equivalence is proven in op.cit. the form of a lemma:

Lemma 4.2.1 [Schottenloher [SM] (2008), §4.4.1, Lemma 4.5]

(i) Every central extension $\mathfrak{h}$ of $\mathfrak{g}$ by $\mathfrak{a}$ comes from a 2-cocycle;

(ii) Every cocycle $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ generates a central extension $\mathfrak{h}$ of $\mathfrak{g}$ by $\mathfrak{a}$;

(iii) Such a central extension splits (and this implies that is trivial) if and only if $\exists\ \mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{a})$ with $\Theta(X, Y) = \mu([X, Y])$ for all $X, Y \in \mathfrak{g}$.

By going quotient the splitting characterisation in (iii) on the space of all cocycles, we define the second co-homology group of $\mathfrak{g}$ with values in $\mathfrak{a}$. Such groups are all Abelian. (cf. [Schottenloher [SM] (2008), §4.4.1, Definition 4.6]). In detail, define:

$$\mathrm{Alt}^2(\mathfrak{g}, \mathfrak{a}) := \{\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a} \mid \Theta \text{ satisfies condition (i)}\} \tag{4.108}$$

$$\mathrm{Z}^2(\mathfrak{g}, \mathfrak{a}) := \left\{\Theta \in \mathrm{Alt}^2(\mathfrak{g}, \mathfrak{a}) \mid \Theta \text{ satisfies condition (ii)}\right\} \tag{4.109}$$

$$\mathrm{B}^2(\mathfrak{g}, \mathfrak{a}) := \{\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a} \mid \Theta \text{ satisfies condition (iii)}\} \tag{4.110}$$

Then, the second co-homology group is given by

$$\mathrm{H}^2(\mathfrak{g}, \mathfrak{a}) := \frac{\mathrm{Z}^2(\mathfrak{g}, \mathfrak{a})}{\mathrm{B}^2(\mathfrak{g}, \mathfrak{a})} \tag{4.111}$$

After the brief discussion of the matter involved presented above, we are now in position to apply the new notions to the proof of [Costello & Gwilliam [CG1] (2016), §3.4, Proposition 4.0.1]. We modify the notation in the main reference ([CG1]) to match the one in [SM].

Let $U \in \mathbf{Open}^c_{\mathbb{R}}$ be an open connected interval of the real line, the linear mapping β is defined as

$$\beta : \mathfrak{g} \to A_{\mathfrak{g}} \tag{4.112}$$

$$X \mapsto h \otimes X \tag{4.113}$$

It assigns to an element X of the Lie algebra $\mathfrak{g}$ its tensor product with h, a generator for $H^1_c(U)$. Therefore, it takes values in $H_c^1(U) \otimes \mathfrak{g}$. Moreover, the integral of such a generator is normalised to one, i.e.

$$\int_U dx\ h = 1 \tag{4.114}$$

We want to show β is a splitting map for the central extension

$$0 \longrightarrow H_c^1(\mathbb{R}) \longrightarrow A_{\mathfrak{g}} \underset{\pi}{\overset{\overset{\beta}{\curvearrowleft}}{\longrightarrow}} \mathfrak{g} \longrightarrow 0$$

Namely, we aim at proving β is a Lie algebra morphism such that $\pi \circ \beta = id_{\mathfrak{g}}$. This will imply the Lie algebra $\mathfrak{g}$ is Abelian, then isomorphic to the symmetric algebra of $\mathfrak{g}$ in the category of commutative algebras. In particular, the associative algebra $\mathcal{U}(\mathfrak{g})$ is therefore isomorphic to $A_{\mathfrak{g}}$.

Let $\epsilon > 0$ be a small radius and define the corresponding open interval as $I_0(\epsilon) := (-\epsilon, \epsilon)$. Consider further a function $g_0(x) \in C_c^\infty(I(\epsilon))$, whose integral is normalised to one, and define a new function $g_k := g_0(x-k)$ supported on the deformed interval $I_k(\epsilon) = (k-\epsilon, k+\epsilon)$. The parameter k is varying over the integers.

Thus, using the detailed definition of the generator ϵ introduced above, the morphism β reads on representatives

$$\mathfrak{g} \ni X \overset{\beta}{\longrightarrow} g_0 dx \otimes X \in \Omega_c^1(I_k(\epsilon)) \otimes \mathfrak{g} \tag{4.115}$$

for every k $\in \mathbb{Z}$.

Remark 4.2.22 The choice of g_0 on place of the generic g_k is motivated by the fact that, by construction, the co-cycles $g_0 dx \otimes X$ and $g_k dx \otimes X$ represent the same co-homology class in $\mathbb{R}$, for every $X \in \mathfrak{g}$. In fact, since we have normalised the 1-forms to one ("normalised" meaning their integral over the respective support is one), their difference is exact. For it,

$$0 = 1 - 1 = \int_{I_0(\epsilon)} dx\ g_0 - \int_{I_k(\epsilon)} dx\ g_k = \int_{\mathbb{R}} dx\ \ (g_0 - g_k) = \int_{\mathbb{R}} d\xi \tag{4.116}$$

where $d\xi := dx\ \ (g_0 - g_k)$.

We make use of the associative algebra structure arising from the locally constant PFA $\mathcal{H}$ in order to define the relevant bracket and make calculations. Briefly recalling the construction, let ι_* be the arrow induced by the PFA $\mathcal{H}(\bullet)$ in the multicategory of real vector spaces from a morphism $\iota \in \mathbf{Open}_{\mathbb{R}}^c(I_1, I_2; \mathbb{R})$ in the poset multicategory of open connected parts of the real line, i.e.

$$\iota_* \in \mathbf{Vect}_{\mathbb{R}} \left(\mathcal{H}\left(I_1\right), \mathcal{H}\left(I_2\right); \mathcal{H}\left(\mathbb{R}\right)\right) \tag{4.117}$$

On $A_{\mathfrak{g}} := \mathcal{H}(\mathbb{R})$ there exists an associative algebra structure whose multiplication is given by the commutative diagram below:

$$\begin{tikzcd}
\mathcal{H}\left(I_1\right), \mathcal{H}\left(I_2\right) \arrow[rr, "\iota_*"] \arrow[d, "\cong\ (\psi_1, \psi_2)"'] & & \mathcal{H}\left(\mathbb{R}\right) \arrow[d, "\psi_{\mathbb{R}}"] \\
A_{\mathfrak{g}}, A_{\mathfrak{g}} = \mathcal{H}\left(\mathbb{R}\right), \mathcal{H}\left(\mathbb{R}\right) \arrow[rr, "\mu_{A_{\mathfrak{g}}}"] & & \mathcal{H}\left(\mathbb{R}\right) = A_{\mathfrak{g}}
\end{tikzcd}$$

In explicit terms

$$\mu_{A_{\mathfrak{g}}}\left(x_1, x_2\right) = \left(\psi_{\mathbb{R}} \circ \iota_* \circ \left(\psi_1, \psi_2\right)^{-1}\right)\left(x_1, x_2\right) \tag{4.118}$$

for every $(x_1, x_2) \in (\mathcal{H}(\mathbb{R}), \mathcal{H}(\mathbb{R}))$.
[Or, equivalently, for every $x_1 \otimes x_2 \in \mathcal{H}(\mathbb{R})^{\otimes^2}$. When adopting this notation, $(\psi_1, \psi_2) = (\psi_1 \otimes \psi_2)$].

Remark 4.2.23 The intervals chosen are such that $I_1 < I_2 \subset \mathbb{R}$ therefore they are disjoint and contained in the real line. This guarantees the existence of the induced morphism ι_* used to build the algebra multiplication $\mu_{A_{\mathfrak{g}}}$. Moreover, since we have proven in *Step I* that the prefactorisation algebra is locally constant, the elements $x_1, x_2 \in A_{\mathfrak{g}}$ can be seen as belonging to the factors via the isomorphisms $\mathcal{H}(I_1) \cong \mathcal{H}(I_2) \cong \mathcal{H}(\mathbb{R}) =: A_{\mathfrak{g}}$.

Set Θ to be the following map

$$\Theta : \mathfrak{g} \times \mathfrak{g} \longrightarrow H^1_c\left(\mathbb{R}\right) \tag{4.119}$$

$$\left(X_k, X_j\right) \mapsto \left[\beta\left(X_k\right), \beta\left(X_j\right)\right] - \beta\left(\left[X_k, X_j\right]\right) \tag{4.120}$$

$\forall\ X_k, X_j \in \mathfrak{g}$ with $k, j \in \mathbb{Z}$.
From Proposition 4.2.1 it follows that the central extension under consideration splits if and only if $\Theta = 0$.
We are left to prove it.

Step V: Computations on the cocycles

By definition, $\Theta = 0$ if and only if

$$\left[\beta\left(X_0\right), \beta\left(X_1\right)\right] = \beta\left(\left[X_0, X_1\right]\right) \tag{4.121}$$

for arbitrary elements X_0, X_1 of the Lie algebra $\mathfrak{g}$.

More elegantly, since we have defined β to be a linear morphism $\beta : \mathfrak{g} \to A_{\mathfrak{g}}$, the equation reads

$$[\beta(X_0), \beta(X_1)]_{A_{\mathfrak{g}}} = \beta\left([X_1, X_1]_{\mathfrak{g}}\right) \tag{4.122}$$

We begin by analysing the RHS of the above-displayed equation, whose explicit calculation results, for any $X_0, X_1 \in \mathfrak{g}$, in the following:

$$\beta\left([X_0, X_1]_{\mathfrak{g}}\right) = \beta(X_0 \otimes X_1 - X_1 \otimes X_0) = \beta(X_0 \otimes X_1) - \beta(X_1 \otimes X_0) = \tag{4.123}$$

$$= g_0 dx \otimes (X_0 \otimes X_1) - g_1 dx \otimes (X_1 \otimes X_0) = g_0 dx \otimes X_{01} - g_1 dx \otimes X_{10} \tag{4.124}$$

where the first equality follows by linearity of the morphism $\beta : \mathfrak{g} \to A_{\mathfrak{G}}$ and the latter by observing the tensor product is an internal operation on $\mathfrak{g}$. Further, $g_0 dx$ and $g_1 dx$ are opportune normalised one-forms in $\Omega_c^1(I_k(\epsilon))$, as prescribed by the definition of β. In this case, $k = 0, 1$.

For convenience, the subscript indices in the result are renamed, yielding:

$$\beta\left([X_0, X_1]_{\mathfrak{g}}\right) = g_{01} dx \otimes X_{01} - g_{10} dx \otimes X_{10} \tag{4.125}$$

Remark 4.2.24 The g_{ij}s are compactly-supported smooth functions on the intervals $I_i(\epsilon)$ respectively, for $i, j = 0, 1$ such that the corresponding one-forms are normalised to one. Nothing changes if the indices are set to match the ones of the product Lie algebra elements.

Remark 4.2.25 The order on the (time) intervals is relevant to have a well-defined associative algebra multiplication induced by the PFA on $\mathcal{H}(\mathbb{R})$, but it enters the discussion at a later time.

Concerning the LHS, attention must be paid on the definition of the bracket. Indeed, it does *not* agree with original bracket defined in (4.88), but with the one induced by the associative algebra structure of $A_{\mathfrak{g}}$.

From chapter one, it is known there exist an adjunction pair

$$\mathfrak{U}(\bullet) : \textbf{Lie-alg}_{\mathbb{R}} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \textbf{Ass-alg}_{\mathbb{R}} : F(\bullet)$$

The functor F induces a Lie bracket over a given associative algebra by using its multiplication.

Specialising it to $A_{\mathfrak{g}}$, we obtain the bracket

$$[\alpha_1, \alpha_2]_{FA_{\mathfrak{g}}} := \mu_{A_{\mathfrak{g}}}(\alpha_1, \alpha_2) - \mu_{A_{\mathfrak{g}}}(\alpha_2, \alpha_1) = \alpha_1 \cdot \alpha_2 - \alpha_2 \cdot \alpha_1 \tag{4.126}$$

$\forall\, \alpha_1, \alpha_2 \in A_{\mathfrak{g}}$.

Remark 4.2.26 For the sake of brevity, the notation $[\bullet,\ \bullet]_{A_{\mathfrak{g}}}$ replaces the more correct $[\bullet,\ \bullet]_{FA_{\mathfrak{g}}}$ when the latter is meant.

Regarding multiplication of cocycles, observe that, by taking ϵ small enough, the resulting connected open intervals of the real line are disjoint and ordered as $I_0(\epsilon) < I_1(\epsilon)$. Therefore, the product of the corresponding algebra elements $g_0 dx \otimes X_0 =: \beta(X_0)$ and $g_1 dx \otimes X_1 =: \beta(X_1)$ is well-defined and consists in

$$\beta(X_0) \cdot \beta(X_1) = (g_0 dx \otimes X_0) \cdot (g_1 dx \otimes X_1) \tag{4.127}$$

Thus, to each finite family of connected pairwise-disjoint real intervals (in this case two), there is an associated one-form supported on each of them and, by tensoring with Lie algebra elements, a corresponding representative for a co-homology class in $A_{\mathfrak{g}}$. Moreover, due to the existence of the induced algebra multiplication $\cdot$, denoted by $\mu_{A_{\mathfrak{g}}}$ elsewhere, we are allowed to multiply the elements in $A_{\mathfrak{g}}$.

Remark 4.2.27 Local one-forms can always be extended by zero to the entire real line.

Computations on the LHS of equation (4.119) yield:

$$[\beta(X_0), \beta(X_1)]_{A_{\mathfrak{g}}} = \beta(X_0) \cdot \beta(X_1) - \beta(X_1) \cdot \beta(X_0) = \tag{4.128}$$

$$= (g_0 dx \otimes X_0)(g_1 dx \otimes X_1) - (g_1 dx \otimes X_1)(g_0 dx \otimes X_0) = \tag{4.129}$$

$$= (g_0 dx \otimes g_1 dx) \otimes (X_0 \otimes X_1) - (g_1 dx \otimes g_0 dx) \otimes (X_1 \otimes X_0) = \tag{4.130}$$

$$= (g_{01} dx \otimes X_{01}) - (g_{10} dx \otimes X_{10}) \tag{4.131}$$

Remark 4.2.28 The multiplication symbol $\cdot$ is here removed in favour of simple juxtaposition of the factors.

The calculations just displayed are equivalent to the one obtained by applying the explicit definition of the multiplication map from the commutative diagram presented before. Indeed, the isomorphism $\left(\psi_0^{-1}, \psi_1^{-1}\right)$ applied on the cocycles $(g_0 dx \otimes X_0), (g_1 dx \otimes X_1)$ by construction restricts the forms to the original interval of definition instead of considering them as extended by zero on the full real line. However, this amounts to viewing the one-forms as defined their original support, therefore they are left invariant. The arrow ι_* represented in tensor product notation as

$$\mathcal{H}(I_0) \otimes \mathcal{H}(I_1) \xrightarrow{\iota_*} \mathcal{H}(\mathbb{R}) \tag{4.132}$$

is by definition defined by universal property, since the tensor product is involved. Thus, if we regard the factors as ordered pairs, i.e. at the level of sets, we are in fact multiplying

$$(g_0 dx, X_0) \text{ with } (g_1 dx, X_1) \tag{4.133}$$

therefore the resulting ordered pair $(g_{01} dx, X_{01})$ is well-defined in the target $\mathcal{H}(\mathbb{R})$.

Remark 4.2.29 Furthermore, notice the tensor product of any vector v with itself, namely $v \otimes v$, yields a well-defined vector w in the same vector space. Therefore, if we take two forms on the real line and we tensor them, the resulting vector will be also a multiple of dx, the basis vector. In order to do so, the functions are first extended to zero to the whole real line and then multiplied yielding the compactly-supported functions denoted by g_{01}. Recall, indeed, that the domain of definition of the multiplication of two real compactly-supported smooth functions is the intersection of the respective domains. In formulas, for $f, g \in C_c^\infty(\mathbb{R})$ we have

$$\mathrm{Dom}(f \cdot g) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$$

Analogous reasoning applies to the Lie algebra elements, when seen as vectors. Namely, for every $X_0, X_1 \in \mathfrak{g}$, the internal binary operation of tensor product yields $X_0 \otimes X_1 =: X_{01} \in \mathfrak{g}$. Thus, the above-displayed associative algebra multiplication, coming from the combinations of the internal operations just discussed is internal to $A_\mathfrak{g}$, as wished.

Remark 4.2.30 The reader has noticed we are improperly displaying $\mathcal{H}(\bullet)$ as, at this stage, the argument is given on representatives and not yet on co-homology classes. More properly, it would have been better to write explicitly $\Omega_c^1(\bullet) \otimes \mathfrak{g}$ where needed. The taking of the classes is assumed.

In conclusion, by comparing the RHS and the LHS computations, we obtained the desired equality

$$[\beta(X_0), \beta(X_1)]_{A_\mathfrak{g}} = \beta\left([X_1, X_1]_\mathfrak{g}\right) \tag{4.134}$$

In other words, $\Theta = 0$. Hence, the central extension of Lie algebras

$$\begin{tikzcd} 0 \arrow[r] & H_c^1(\mathbb{R}) \arrow[r] & A_\mathfrak{g} \arrow[r, "\pi"'] & \mathfrak{g} \arrow[l, bend right, "\beta"'] \arrow[r] & 0 \end{tikzcd}$$

splits resulting in the following chain of isomorphisms

$$A_\mathfrak{g} \cong \mathrm{Sym}(\mathfrak{g}) \cong H_c^1(\mathbb{R}) \oplus \mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{g} \tag{4.135}$$

Thus, we recover the statement given at the beginning: such algebra has non-trivial pieces only in degree zero and one.

Furthermore, as the general theory forces $\mathfrak{g}$ to be written as the quotient

$$\mathfrak{g} \cong \frac{H_c^1(R) \oplus \mathfrak{g}}{H_c^1(R)} \tag{4.136}$$

the so-obtained Lie algebra $\mathfrak{g}$ is Abelian.

Finally, by applying the universal envelopping algebra functor, we have then an isomorphism in the category of associative algebras

$$\mathcal{U}(\mathfrak{g}) \cong A_{\mathfrak{g}} \tag{4.137}$$

$\square$

Remark 4.2.31 The proof we have developed is sized on [graded] Lie algebras. However, the prefactorisation algebra functor we are dealing with is in fact taking values in differential graded Lie algebras. Recall the definition on objects:

$$\mathbf{Open}_{\mathbb{R}}^{c} \ni U \xrightarrow{\mathfrak{g}^{\mathbb{R}}} \left(\Omega_c^{\bullet}(U) \otimes \mathfrak{g}, d_{\mathrm{dR}}\right) \in \mathbf{dg\text{-}Lie\text{-}alg}_{\mathbb{R}} \tag{4.138}$$

Observe that the de Rham differential is defined only on the one-forms part, therefore it can be more correctly be written as $\left(d_{\mathrm{dR}}, id_{\mathfrak{g}}\right)$. Therefore, when internalising the Lie algebra structure in $\mathbf{dg\text{-}Vect}_{\mathbb{K}}$, the differential on bracket is always trivial, as one-forms square to zero.

Thus, the above-presented proof is completed by noticing that it holds on co-cycles defined "up to an exact form". In formulas, the map $\beta : \mathfrak{g} \longrightarrow A_{\mathfrak{g}}$ is more precisely to be given on cocycles of the form $\left((g_i dx + d_{dR}\xi) \otimes X_i\right)$ for some $g_i dx \otimes X_i \in \Omega_c^1(\bullet) \otimes \mathfrak{g}$, $\xi \in \Omega_c^1(\bullet)$. Indeed, taking co-homology classes kills the exact form, as it should be from the general theory.

Remark 4.2.32 The map ϕ in the main reference is denoted by β to match the general theory ahead.

Appendix A

Principle of Least Action: Lagrangian and Hamiltonian Mechanics

To motivate the reason beyond the fact physical observables are defined over trajectories, let us start our journey through the matter, by considering a mechanical system whose trajectories we want to determine. One example you may have in mind is the simple pendulum: an inextensible string with a weight at the extremum hanging from some fixed point. Suppose we make the pendulum oscillate by pushing the weight offside. Which kind of trajectories will it draw? How we can calculate them?

The answer is quite straightforward: The system will go along the paths that minimise its energy, roughly. Surely you can imagine the pendulum oscillating back and forth until it realigns with the perpendicular and stops there, in the minimum of its potential. Therefore, *the nature selects some particular tracks among all the possible ones to give rise to motion*. This is just a little - maybe oversimplified -extract from a large discussion that maybe you have already had in an orthodox course in analytical mechanics, but it is sufficient to make the point necessary for our purposes.

To rigorously formalise the above heuristics though, we take some results from [AN] and [IZ]. Readers, however, are strongly encouraged to select their favourite books on the subject.

A.1 Foundations of analytical mechanics

Remark A.1.1 The notation $\mathbf{x}$ denotes in this appendix elements of an n-dimensional vector space, e.g. $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition A.1.1 A **motion** in $\mathbb{R}^n$ is a smooth map $\gamma : J \subset \mathbb{R} \to \mathbb{R}^n$, whose image $\gamma(J)$ define a **trajectory**. Moreover, if $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$ is a motion, its graph $(t, \mathbf{x}(t)) \in \mathbb{R} \times \mathbb{R}^3$ is also a curve, the **world line**. This is relevant, as the space-time of standard usage is $\mathbb{R} \times \mathbb{R}^3$. Lastly, n points moving in an Euclidean space give rise to n world lines with 3n-dimensional configuration space $\mathbb{E}^N \cong \mathbb{E}^{3n}$ - sometimes denoted by the sloppy $\mathbb{R}^N$.

Remark A.1.2 The space of motions constitutes a subspace of the infinite-dimansional space of curves $\mathcal{S}$ connecting any two points of the so-called configuration space.

Remark A.1.3 In a Galileian space, where all the reference frames are inertial, we have among others, the following properties: the space is homogeneous - having the same properties at all points- and isotropic - there is no preferred direction. Further, as time-translations are included in the Galileian group, the laws of motion are time-independent. Thus, for an inertial system, standard Newton's equations

$$\ddot{\mathbf{x}} = F(\mathbf{x}, \dot{\mathbf{x}}, t) \tag{A.1}$$

with $F : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, sufficiently smooth function, reduce to

$$\mathbf{x} = \Phi(\mathbf{x}, \mathbf{x}) \tag{A.2}$$

where $\Phi : \mathbb{R}^{2N} \to \mathbb{R}^N$ smooth enough.
Well-posedness of the defintion stays also when we substitute any subdomain of the real configuration space used and any open time interval on the real line.

A.1.1 Lagrangian systems

Definition A.1.2 A **Lagrangian mechanical system** is given by a smooth manifold M equipped with an action of a one-parameter group of diffeomorphism $\{g^t\}_{t\in\mathbb{R}}$ and a smooth function on its tangent bundle, the **Lagrangian function** $TM \xrightarrow{L} \mathbb{R}$

Definition A.1.3 A curve $\gamma : \mathbb{R} \to M$ is a motion on the configuration manifold of the system, if it is extremal for the functonal

$$\Phi[\gamma] = \int_{t_0}^{t_1} dt\, L(\dot{\gamma}) \tag{A.3}$$

where $\dot{\gamma}(t) \in TM_{\gamma(t)}$ is the velocity vector.

Our goal is to determine extrema of functionals Φ over infinite-dimensional spaces of curves $\mathcal{S}$.

For a Newtonian system, a general Lagrangian is defined as difference between its kinetic energy T and the potential energy V

$$L(\mathbf{q}(t), \mathbf{q}(t), t) := T - V \tag{A.4}$$

Definition A.1.4 An **approximation** for $\gamma = \{(t, \mathbf{x}) : \mathbf{x}(t) = \mathbf{x}, t \in [t_0, t_1]\}$, smooth curve on the space-time, is defined as

$$\gamma'(t) = \{(t, \mathbf{x}) : \mathbf{x} = \mathbf{x}(t) + h(t)\} \tag{A.5}$$

Definition A.1.5 An **increment** for the functional $\Phi[\gamma]$ is the difference

$$\Delta_h(\Phi) := \Phi[\gamma + h] - \Phi[\gamma] \tag{A.6}$$

where h is called **variation** of the curve.

Moreover, we say $\Phi[\gamma]$ to be **differentiable** if the increment is equal to a linear function $F(h)$ in h, called **differential** (or variation of Φ), up to a remainder of order $O\left(h^2\right)$:

$$\Delta_h(\Phi) = F(h) + O\left(h^2\right) \tag{A.7}$$

Remark A.1.4 The above functional $\Phi[\gamma]$ must satisfy some regularity condition in order for us to do calculations. For instance, we require $\Phi \in C^\infty(\Omega \subset \mathcal{S})$ and $h \in C^\infty(\mathbb{R})$, where Ω is some chosen domain.

Definition A.1.6 An **extremal** for the functional $\Phi[\gamma]$ is a curve $\gamma \in C^\infty(\Omega)$ such that $F(\gamma) = 0, \forall h \in C^\infty(\mathbb{R})$.

Example A.1.1 We can consider the problem of finding the extrema of the length functional in an Euclidean space.

In a general Lagrangian setting, where

$$\gamma = \{(t, \mathbf{x}) \,|\mathbf{x} = \mathbf{x}(x), t \in [t_0, t_1]\} \in C^\infty(\mathbb{R} \times \mathbb{R}^n) \tag{A.8}$$

and

$$L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \tag{A.9}$$

Theorem A.1.1 [[AN], §3.12.C, 58] γ is extremal for

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(\mathbf{x}, \mathbf{x}, t)\, dt \tag{A.10}$$

on the space of curves connecting $(t_0, \mathbf{x}_0)$ and $(t_1, \mathbf{x}_1)$ iff the Euler-Lagrange equations are satisfied along γ.

The resulting set of equations is made of n second order differential equations with 2n boundary conditions $\mathbf{x}(t_0) =: \mathbf{x}_0, \mathbf{x}(t_1) =: \mathbf{x}_1$.

Analogous theorem is valid in terms of the so-called **generalised coordinates** and **generalised momenta** q_i and $p_i := \dfrac{\partial L}{\partial \dot{q}_i}$, respectively, i= $1, \ldots, 3$n.

Corollary A.1.1 [[AN], §3.13.A , 60, Corollary] Let $(q_1, \ldots, q_{3n})$ be any coordinates in the configuration space of a system of n mass points. Then the evolution of $\mathbf{q}$ with time is subject to Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{q}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{A.11}$$

Definition A.1.7 We call **action** the following functional

$$S[\gamma] = \int_{t_0}^{t_1} dt\, L(\mathbf{q}, \mathbf{q}, t) \tag{A.12}$$

Definition A.1.8 [[AN], §3.13, 61, Definition] We say that a generalised coordinate q_i is **cyclic** if it does not enter into the Lagrangian, i.e.

$$\frac{\partial L}{\partial q_i} = 0 \tag{A.13}$$

We recall the following relevant result

Theorem A.1.2 [[AN], §3.13, 61, Theorem] The generalised momentum corresponding to a cyclic coordinate is conserved, i.e. $p_i = \text{const}$.

Physical systems are normally subject to constraints. Therefore is natural to introduce holonomic ones in our discussion, i.e. the system is subject to a set of equations in the generalised coordinates (and maybe time).

Definition A.1.9 [[AN], §4.17.B, 77, Definition] Let γ be an m-dimensional surface in 3n-dimensional configuration space of the points $\mathbf{r}_1, \ldots, \mathbf{r}_n$ with masses $\{m_i\}_{i=1}^n$ respectively. Let $\mathbf{q} = (q_1, \ldots, q_m)$ be some coordinates on

$\gamma\ :\ \mathbf{r}_i = \mathbf{r}_i\left(\mathbf{q}\right)$. The system described by the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{q}} - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{A.14}$$

$$L = \frac{1}{2}\sum_{i=1^n} m_i \mathbf{r_i}^2 + U\left(\mathbf{q}\right) \tag{A.15}$$

is called a system of n points with 3n-m **ideal holonomic constraints**. The surface γ is called the **configuration space of the system with constraints**.

If the surface γ is given by $k = 3n - m$ functionally independent equations $f_1\left(\mathbf{r}\right) = 0, \ldots, f_k\left(\mathbf{r}\right) = 0$, then we say that the system is constrained by the relations $f_1 = 0, \ldots, f_k = 0$.

We can enlarge the description of a mechanical system in the case we have a Riemannian manifold M

Definition A.1.10 [[AN], §4.19.B, 77, Definition] Let M be Riemannian, a Lagrangian system is said to be **natural** if

$$L = T - U = \frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle - U \tag{A.16}$$

where U $\in\ C^\infty\left(M; \mathbb{R}\right)$, $\mathbf{v} \in \mathrm{TM}_{\mathbf{x}}$.

Let us state the Noether's theorem for autonomous systems - whose Lagrangian does not depend explicitly on time.

Definition A.1.11 Let $h \in C^\infty\left(M\right)$, the Lagrangian system $\left(M, L\right)$ admits the mapping h if $\forall\ \mathbf{v} \in TM,\ L\left(h_*\mathbf{v}\right) = L\left(dh_{\mathbf{v}}\right) = L\left(\mathbf{v}\right)$.

Theorem A.1.3 [*Noether's Theorem* [[AN], §4.20A, page 88]] If the system $\left(M, L\right)$ admits the one-parameter group of diffeomorphisms $\left\{h^s\right\}_{s\in\mathbb{R}} : M \to M$, then the Lagrangian system of equations corresponding to L has a first integral $I : TM \to \mathbb{R}$.
In local coordinates $\mathbf{q}$ on M the integral I is written in the form

$$I\left(\mathbf{q}, \mathbf{q}\right) = \frac{\partial L}{\partial \mathbf{q}} \left.\frac{dh^s\left(\mathbf{q}\right)}{ds}\right|_{s=0} \tag{A.17}$$

A.1.2 Hamiltonian formalism

The conversion to the Hamiltonian formalism happens by means of a Legendre transformation of the Lagrangian, considered as function of the generalised velocities. Lagrangian mechanics can be therefore seen as a particular case of the Hamiltonian one.

Definition A.1.12 Let $\psi \in C^2(\Omega \subset \mathbb{R}^n; \mathbb{R})$ be such that its Hessian is positive definite in the domain Ω, let $\mathbf{s}$ be another variable. The **Legendre transformation** of ψ with parameter $\mathbf{s}$ is defined as

$$f(\mathbf{s}) := \max_{\mathbf{x}\in\Omega} = \langle \mathbf{s}, \mathbf{x} \rangle - \psi(\mathbf{x}) \tag{A.18}$$

Take Lagrange's equations

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{q}} \quad \text{with } \mathbf{p} = \frac{\partial L}{\partial \mathbf{q}} \tag{A.19}$$

with Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, convex in the second argument $\mathbf{q}$ and smooth enough to make calculations feasible.

Theorem A.1.4 [[AN], §3.15, page 65] The system of Lagrange's equations is equivalent to the system of 2n first-order equations, named Hamilton's equations,

$$\underline{\dot{p}} = -\frac{\partial H}{\partial \mathbf{q}} \quad \mathbf{q} = \frac{\partial H}{\partial \mathbf{p}} \tag{A.20}$$

where

$$H(\mathbf{p}, \mathbf{q}, t) = \mathbf{p} \cdot \mathbf{q} - L(\mathbf{q}, \mathbf{q}, t) \tag{A.21}$$

is the Legendre transformation of $L = L(\mathbf{q})$.

Remark A.1.5 The Legendre transformation is a useful tool in general, as it takes functions on a vector space to functions on its dual and it is involutive, i.e. its square is the identity.

If the system we are studying is a mechanical one, with Lagrangian

$$L = \frac{1}{2} \sum_{i,j} \alpha_{ij}(\mathbf{q}, t) \, \dot{q}_i \dot{q}_j \; + \; U(\mathbf{q}) \tag{A.22}$$

where the kinetic energy is quadratic in $\mathbf{q}$, then

Theorem A.1.5 [[AN], §3.15B, page 66] Under the given assumption, the Hamiltonian H is the total energy $H = T + U$.

Definition A.1.13 The 2n-dimensional space whose variables are $(\mathbf{p}, \mathbf{q})$ is called **phase space**. Additionally considering time and energy as variables, we define an **extended phase space**.

Definition A.1.14 We call **phase flow** the one-parameter group of transformations on the space of solutions of the Hamilton's equations

$$g^t : (\mathbf{p}(0), \mathbf{q}(0)) \to (\mathbf{p}(1), \mathbf{q}(1)) \tag{A.23}$$

for which the following result holds

Theorem A.1.6 [[AN], §3.16B, Liouville's theorem] The phase flow preserves volume.

Moreover, on the phase space, that is the co-tangent bundle of the configuration space, there exists a well-defined Poisson bracket and we can perform strategic change of variables, called canonical transformations, in terms of which Hamiltonian's equations do not change. In addition, the new variables, under certain conditions, may be identified with symmetries of the system.

Definition A.1.15 A **symplectic manifold** $\left(M^{2n}, \omega^2\right)$ is an even-dimensional smooth manifold M equipped with a closed non-degenerate 2-form, i.e.

$$d\omega = 0 \quad \forall\, \xi \in \mathrm{TM}_{\mathbf{x}} \setminus \{0\} \quad \exists \eta \in \mathrm{TM}_{\mathbf{x}} : \omega^2\left(\xi, \eta\right) \neq 0 \tag{A.24}$$

Take $\mathbf{q}$ as local coordinates on a smooth manifold N^n, its co-tangent bundle, of dimension 2n, has coordinates $\{\mathbf{q}, \mathbf{p}\}$ and there exists a well-defined smooth projection

$$\pi : T^*N \longrightarrow N \tag{A.25}$$

$$T_{\mathbf{x}}N \ni \omega \longrightarrow \mathbf{x} \tag{A.26}$$

Thus, on the co-tangent bundle we have a well-defined symplectic structure as follows

Theorem A.1.7 [[AN], §8.37B, page 202] The co-tangent bundle T^*V has a natural symplectic structure. In the local coordinates described above,

$$\omega^2 = d\mathbf{p} \wedge d\mathbf{q} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n \tag{A.27}$$

Definition A.1.16 For all ξ tangent to the symplectic manifold $\left(M^{2n}, \omega^2\right)$ at $\mathbf{x}$, we associate the one-form

$$\omega^1_{\xi}\left(\eta\right) := \omega^2\left(\eta, \xi\right) \quad \forall \eta \in \mathrm{TM}_{\mathbf{x}} \tag{A.28}$$

Such a mapping define an isomorphism $\phi : T^*M_{\mathbf{x}} \to TM_{\mathbf{x}}$.

Definition A.1.17 A **Poisson bracket** is an anti-symmetric, bilinear operation on the phase space, satisfying the Leibniz rule and the Jacobi identity.

Namely, given sufficiently smooth functionals over the phase space F,G,H and scalars λ, μ we have:

1.

$$\{F, G\} = -\{G, F\} \tag{A.29}$$

2.

$$\{\lambda F + \mu G, H\} = \lambda \{F, H\} + \mu \{G, H\} \tag{A.30}$$

3.

$$\{F, \lambda G + \mu H\} = \lambda \{F, G\} + \mu \{F, H\} \tag{A.31}$$

4.

$$\{FG, H\} = \{F, H\} G + F \{G, H\} \tag{A.32}$$

5.

$$\{F, \{G, H, \}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{A.33}$$

Definition A.1.18 Let H be sufficiently smooth function on a symplectic manifold M^{2n}, dH is a differential one-form on M and the corresponding vector field on M ϕdH is called **Hamiltonian vector field** on the configuration space of the system. They form a Lie algebra whose operation is given by Poisson brackets on such a manifold.

Definition A.1.19 The one-parameter family of diffeomorphisms associated with such a vector field as

$$g^t : M^{2n} \to M^{2n} \tag{A.34}$$

$$\left.\frac{d}{dt}\right|_{t=0} g^t \mathbf{x} = \phi\, dH(\mathbf{x}) \tag{A.35}$$

is called **Hamiltonian phase flow**.

Definition A.1.20 The **Poisson bracket** of two functionals over the symplectic manifold M^{2n} is equivalently given as a function on such a manifold, defined by:

$$\{F, H\}(\mathbf{x}) = \left.\frac{d}{dt}\right|_{t=0} F\left(g^t_H(\mathbf{x})\right) \tag{A.36}$$

Definition A.1.21 A **constant of motion** or **first integral** for a system is a function $g = g(\mathbf{p}(t), \mathbf{q}(t), t)$ on the phase space such that $\dot{g} = 0$.

Theorem A.1.8 [*Generalised Noether's Theorem* [[AN], §8.40A]] If a Hamiltonian function H on a symplectic manifold $\left(M^{2n}, \omega^2\right)$ admits a one-parameter group of canonical transformation given by a Hamiltonian F, then F is a first integral of the system with Hamiltonian function H.

Indeed, a constant of motion is given by the following calculation - where the Poisson brackets are computed in local coordinates $\{\mathbf{q}, \mathbf{p}\}$ and the sum over i is omitted for brevity of notation:

$$0 = \dot{g} = \frac{dg}{dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial p_i}\dot{p}_i + \frac{\partial g}{\partial q_i}\dot{q}_i \overset{HE}{=} \frac{\partial g}{\partial t} + \frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial g}{\partial t} + \{g, H\} \tag{A.37}$$

If such a function g is furthermore time-independent, then it constitutes a constant of motion for the system if its bracketing with the Hamiltonian is zero.

Example A.1.2 A time-independent Hamiltonian is by itself a constant of motion:

$$\dot{H} = \{H, H\} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = 0 \tag{A.38}$$

and the described system is indeed conservative, being energy conserved.

A.1.3 Extension to Field Theory: Lagrangian and Hamiltonian densities

This subsections summarise the matter as presented in [IZ].

In order to describe systems with ∞-many d.o.f. we introduce the so-called Lagrangian density $\mathcal{L}(x)$, uniquely defined up to a divergence under gauge, in terms of which the action is

$$S = \int d^4x\ \mathcal{L}(x) \tag{A.39}$$

This density, whose variables we briefly condensed into x, depends on the fields $\phi_i(x)$ and on finitely many of their derivatives, vanishing sufficiently fast at infinity to allow us the integration by parts.
The variation of the action is therefore

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)}\delta\phi_i(x) + \frac{\partial \mathcal{L}(x)}{\partial\left[\partial_\mu \phi_i(x)\right]}\delta\left[\partial_\mu\phi_i(x)\right] \right\} = \tag{A.40}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi_i(x)\right]} \right\} \delta\phi_i(x) \tag{A.41}$$

Therefore, by searching for the extrema of the action, we get the **generalised Euler-Lagrange equations**

$$\frac{\delta S}{\delta \phi_i(x)} = 0 = \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial\left[\partial_\mu \phi_i(x)\right]} \tag{A.42}$$

More precisely, the fields are functions of time, therefore we are searching for the extrema of the action

$$S = \int_{t_1}^{t_2} dt \mathcal{L} [\phi_t] \tag{A.43}$$

under the condition $\phi_{t_1}(x), \phi_{t_2}(x)$ are fixed for some chosen times t_1, t_2 in an open of the real line.

Remark A.1.6 In case of non relativistic theories, the Lagrangian density may involve higher order derivatives.

To switch to the Hamiltonian formalism, we define **conjugate momenta**

$$\pi(x) := \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}} \tag{A.44}$$

and the **Hamiltonian density** by means of a Legendre transformation of the Lagrangian one

$$\mathcal{H} = \pi(x) \dot{\phi}(x) - \mathcal{L}(x) \tag{A.45}$$

Thus, the **Hamiltonian function** is defined as

$$H[\phi, \pi] = \int dx \, \mathcal{H}(\phi, \pi) \tag{A.46}$$

over a subdomain of the phase space and there exists a well-defined Poisson bracket satisying the canonical relations

$$\{\phi(x_1), \pi(x_2)\} = \delta(x_2 - x_1) \tag{A.47}$$

$$\{\pi(x_1), \pi(x_2)\} = \{\phi(x_1), \phi(x_2)\} = 0 \tag{A.48}$$

in terms of which the dinamics is given by

$$\frac{\partial G}{\partial t} = \{G, H\} \tag{A.49}$$

where F and G are functionals over the phase space.

Concerning conservation laws, it is relevant to mention we have results analogous to the previous section, i.e. Noether's Theorems, when extending the theory to fields.

Finally, we quantise such classical picture by switching from Poisson brackets to commutator via

$$\{\cdot, \cdot\} \underset{q}{\longrightarrow} \frac{1}{i\hbar} [\cdot, \cdot] \tag{A.50}$$

with **canonical commutation relations** - immediate consequence of the canonical ones we listed above:

$$[\phi(x_1), \pi(x_2)] = i\hbar\delta(x_2 - x_1) \tag{A.51}$$

$$[\pi(x_1), \pi(x_2)] = [\phi(x_1), \phi(x_2)] = 0 \tag{A.52}$$

This concludes our recap.

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