First Meeting Presentation

Project A2

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Politecnico of Milano
Bayesian Statistics course

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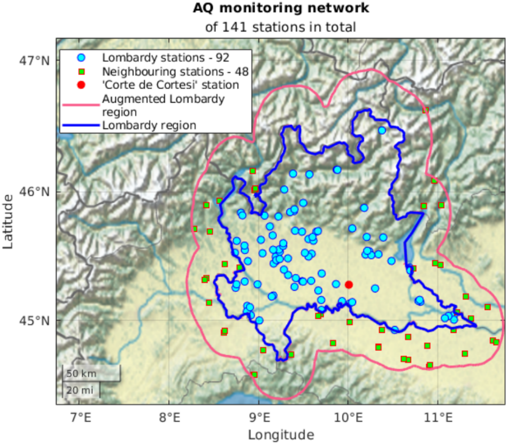
Presentation Flow

The goal of the project

Goal: Clustering weekly data of one year of PM10 (plus covariates)
Dataset: AGRIMONIA project, at
`https://zenodo.org/records/7563265`

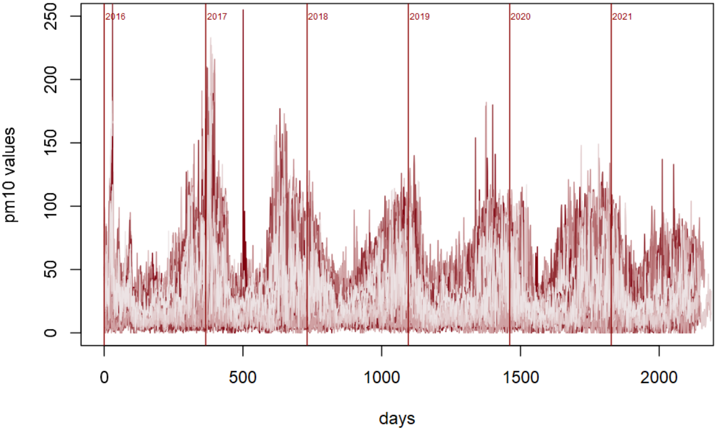
Spatial Exploration

We have 141 stations, which recorded data for 6 years (from 01/01/2016 to 31/12/2021).



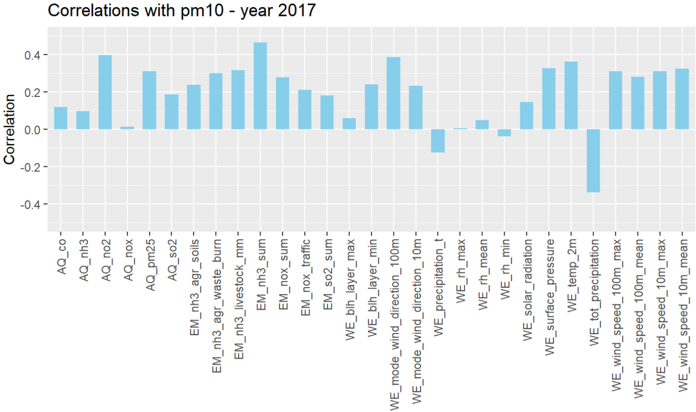
Temporal Exploration

We have 141 stations, which recorded data
for 6 years (from 01/01/2016 to 31/12/2021).



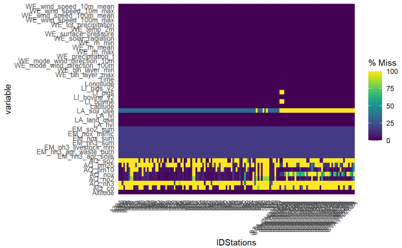
Correlation: PM10 and others

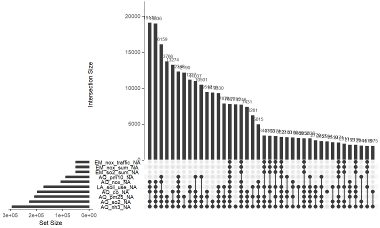
We compute the spearman's correlation index between PM10 and the other covariates, which in the functional framework quantifies with a value in $[-1, 1]$ the tendency of 2 random variables X_t and Y_t to be perfect monotone functions one of each other.



Missing value exploration

Plots to identify a general pattern on missing values.

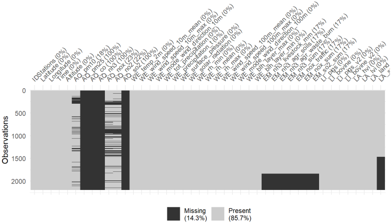


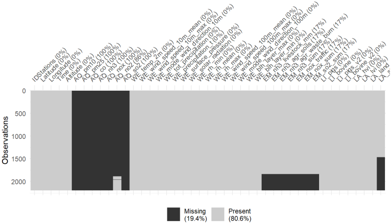


Combinations of variables with the most missing values

Missing value exploration

Plot of missing data divided by station to see particular patterns to help select variables.
Possible to remove some stations that are not measuring PM10 values.
Some columns present missing values in most columns, so we can remove corresponding covariates.
Some missing observations concentrated in specific periods such as during last years.





Missing value exploration

```
 [1] "1274"        "1800"        "1801"        "501"         "504"
 [6] "514"         "529"         "539"         "544"         "545"
[11] "551"         "552"         "573"         "588"         "602"
[16] "603"         "606"         "607"         "626"         "652"
[21] "656"         "657"         "665"         "672"         "682"
[26] "694"         "696"         "698"         "699"         "700"
[31] "702"         "707"         "STA.IT1518A" "STA.IT1751A" "STA.IT1924A"
[36] "STA.IT2282A"
```

First selection to remove non informative stations, then count of missing values for column, removing those above a chosen threshold

```
           IDStations                Latitude
                    0                       0
            Longitude                    Time
                    0                       0
             Altitude                 AQ_pm10
                    0                   12719
              AQ_pm25                   AQ_co
               136792                  136560
               AQ_nh3                  AQ_nox
               217540                   72988
               AQ_no2                  AQ_so2
                21266                  156205
           WE_temp_2m    WE_wind_speed_10m_mean
                    0                       0
  WE_wind_speed_10m_max WE_mode_wind_direction_10m
                    0                       0
  WE_tot_precipitation     WE_precipitation_t
                    0                       0
   WE_surface_pressure     WE_solar_radiation
                    0                       0
            WE_rh_min              WE_rh_mean
                    0                       0
            WE_rh_max   WE_wind_speed_100m_mean
                    0                       0
WE_wind_speed_100m_max WE_mode_wind_direction_100m
                    0                       0
      WE_blh_layer_max        WE_blh_layer_min
                    0                       0
   EM_nh3_livestock_mm       EM_nh3_agr_soils
                38325                   38325
EM_nh3_agr_waste_burn              EM_nh3_sum
                38325                   38325
        EM_nox_traffic              EM_nox_sum
                38325                   38325
            EM_so2_sum                 LI_pigs
                38325                    6576
             LI_bovine              LI_pigs_v2
                 6576                       0
          LI_bovine_v2                  LA_hvi
                    0                       0
                LA_lvi             LA_land_use
                    0                       0
           LA_soil_use
               136656
```

Models: a complex task

Considering the nature of the data, our models should account for different levels of information:

- spatial context
- temporal context
- covariates

which is a not-so-trivial task.
Now we see the general incremental idea to build such models.

Purely spatial model

- We have n distinct locations $\boldsymbol{s}_1, \dots, \boldsymbol{s}_n$, where $\boldsymbol{s}_i = (\text{lat}_i, \text{long}_i)$.
- There we record data y_i and (possibly) covariates $\boldsymbol{x}_i$, for $i = 1, \dots, n$. The goal is to define a model for partitioning them into k groups.
- So we define $\rho = \{S_1, \dots, S_k\}$ the cluster set variable (with $S_h \subseteq \{1, \dots, n\}$ for $h = 1, \dots, k$).
  An equivalent formulation is possible through some cluster indicator variables $c_1, \dots, c_n$; where $c_i = h \iff i \in S_h$ for $i = 1, \dots, n$.
- In general, the law for ρ follows a spatial Product Partition Model (sPPM):

$$p_\rho(\tilde{\rho}) \propto \prod_{h=1}^{k_n} C(\tilde{S}_h, \boldsymbol{s}_h^\star)$$

where $\tilde{\rho} = \{\tilde{S}_1, \dots, \tilde{S}_k\}$, $\boldsymbol{s}_h^\star = \{\boldsymbol{s}_i : i \in \tilde{S}_h\}$ and $C(\tilde{S}_h, \boldsymbol{s}_h^\star)$ is a cohesion function.

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Spatial and temporal model

We have n distinct locations $\boldsymbol{s}_1, \dots, \boldsymbol{s}_n$, where $\boldsymbol{s}_i = (\text{lat}_i, \text{long}_i)$.
There we record data y_i and (possibly) covariates $\boldsymbol{x}_i$, for $i = 1, \dots, n$.
The goal is to define a model for partitioning them into k_t groups, with t spanning over $1, \dots, T$.
So we define $\rho_t = \{S_{1,t}, \dots, S_{k_t,t}\}$ the cluster set variable (with $S_{h,t} \subseteq \{1, \dots, n\}$ for $h = 1, \dots, k_t$).
In general, the law for ρ_t follows a spatial Product Partition Model (sPPM) updated to account for the time relation (stPPM); meaning that we need a formulation of a joint probability model for $\rho_1, \dots, \rho_T$.
This update can be explicited for example by

- supposing a Markov Chain structure, letting ρ_t depend just on ρ_{t-1};
- introducing some cluster reallocation variable $\gamma_{i,t} \in \{0, 1\}$.

Model 1

Garritt L. Page, Fernando A. Quintana, David B. Dahl (2022)
Dependent Modeling of Temporal Sequences of Random Partitions. Journal of Computational and Graphical Statistics, 31:2, 614-627.

$$Y_{it} | \boldsymbol{\mu}_t^\star, \boldsymbol{\sigma}_t^{2\star}, \boldsymbol{c}_t \overset{\text{ind}}{\sim} \mathcal{N}(\mu^\star_{c_{it}t}, \sigma^{2\star}_{c_{it}t}) \quad i = 1, \dots, n \quad \text{and} \quad t = 1, \dots, T$$

$$(\mu_{jt}, \sigma_{jt}) \,|\, \theta_t, \tau_t^2 \overset{\text{ind}}{\sim} \mathcal{N}(\theta_t, \tau_t^2) \times \mathcal{U}(0, A_\sigma) \quad j = 1, \dots, k_t$$

$$(\theta_t, \tau_t) \overset{\text{iid}}{\sim} \mathcal{N}(\phi_0, \lambda^2) \times \mathcal{U}(0, A_\tau) \quad t = 1, \dots, T$$

$$(\phi_0, \lambda) \sim \mathcal{N}(m_0, s_0^2) \times \mathcal{U}(0, A_\lambda)$$

$$\{\boldsymbol{c_t}, \dots, \boldsymbol{c_T}\} \sim \text{tRPM}(\boldsymbol{\alpha}, M) \quad \text{with } \alpha_t \overset{\text{iid}}{\sim} \text{Beta}(a_\alpha, b_\alpha)$$

Adding covariates

Given a partition, now we can easily design models which also account for covariates. For example we can update the previous model into

$$Y_{it}|\beta_t^\star, \boldsymbol{\sigma}_t^{2\star}, \boldsymbol{c}_t \overset{\text{ind}}{\sim} \mathcal{N}(\boldsymbol{x}_{it}^T \beta_{c_{it}t}^\star, \sigma_{c_{it}t}^{2\star})$$
$$(\beta_{jt}, \sigma_{jt}) \,|\, \theta_t, \tau_t^2 \overset{\text{ind}}{\sim} \mathcal{N}(\theta_t, \tau_t^2) \times \mathcal{U}(0, A_\sigma)$$
$$\vdots$$

or we can further characterize the time dependance with some AR$(\cdot)$ model

$$Y_{it}|Y_{it-1}, \beta_t^\star, \boldsymbol{\sigma}_t^{2\star}, \boldsymbol{c}_t \overset{\text{ind}}{\sim} \mathcal{N}(\boldsymbol{x}_{it}^T \beta_{c_{it}t}^\star + Y_{it-1}\eta_i, \sigma_{c_{it}t}^{2\star}(1-\eta_i)^2)$$
$$Y_{i1}|\beta_1^\star, \boldsymbol{\sigma}_1^{2\star}, \boldsymbol{c}_1 \sim \mathcal{N}(\boldsymbol{x}_{i1}^T \beta_{c_{i1}1}^{2\star}, \sigma_{c_{i1}1}^{2\star})$$
$$\vdots$$

Model 2

Mozdzen A., Cremaschi A., Cadonna A., Guglielmi A., Kastner G. (2022)
Bayesian modeling and clustering for spatio-temporal areal data: An application to Italian unemployment. Spatial Statistics 52, 100715.

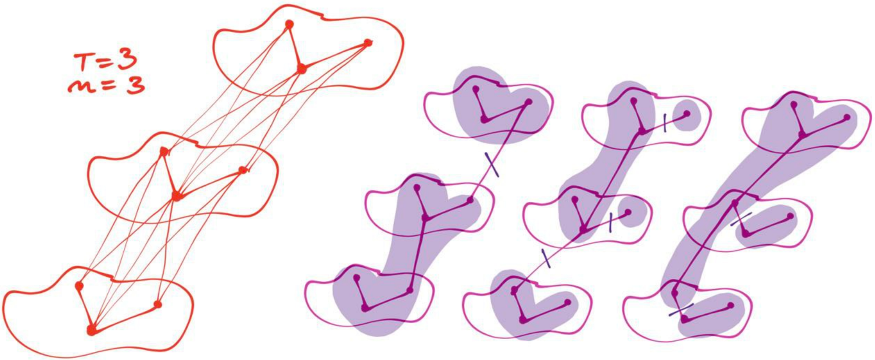
$$
\begin{aligned}
Y_{it} | \boldsymbol{x}_{it}, \beta^\star_{s_i}, w_{it}, \sigma^2, s_i &\overset{\text{ind}}{\sim} \mathcal{N}(\boldsymbol{x}_{it}^T \beta^\star_{s_i} + w_{it}, \sigma^2) \\
\boldsymbol{w}_t | \boldsymbol{w}_{t-1}, \boldsymbol{\xi}^\star_{\boldsymbol{s}}, \boldsymbol{s}, \tau^2, \rho, W &\sim \mathcal{N}_I(\text{diag}(\boldsymbol{\xi}^\star_{\boldsymbol{s}}) \boldsymbol{w}_{t-1}, \tau^2 Q(\rho, W)^{-1}) \\
\boldsymbol{w}_1 | \tau^2, \rho, W &\sim \mathcal{N}_I(\boldsymbol{0}, \tau^2 Q(\rho, W)^{-1}) \\
\sigma^2 &\sim \text{Inv-Gamma}(a_{\sigma^2}, b_{\sigma^2}) \\
\tau^2 &\sim \text{Inv-Gamma}(a_{\tau^2}, b_{\tau^2}) \\
\rho &\sim \text{Beta}(\alpha_\rho, \beta_\rho) \\
\boldsymbol{s} | \alpha &\sim \text{PólyaUrn}(\boldsymbol{s} | \alpha) \\
\alpha &\sim \text{Gamma}(a_\alpha, b_\alpha) \\
\boldsymbol{\phi}^\star_1, \ldots, \boldsymbol{\phi}^\star_{K_I} | \boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}, \alpha_\xi, \beta_\xi &\overset{\text{iid}}{\sim} \text{P}_0, \qquad \boldsymbol{\phi}^\star_j = (\boldsymbol{\beta}^\star_j, \boldsymbol{\xi}^\star_j) \qquad \text{j} = 1, \ldots, \text{K}_I \\
\text{P}_0(d\boldsymbol{\phi}^\star) &= \mathcal{N}_{p+1}(d\boldsymbol{\beta}^\star | \boldsymbol{\mu}_{\boldsymbol{0}}, \Sigma_0) \text{Beta}_{(-1,1)}(d\xi^\star | \alpha_\xi, \beta_\xi)
\end{aligned}
$$

Model 3

Leonardo V. Teixeira, Renato M. Assunção, Rosangela H. Loschi (2019)
Bayesian Space-Time Partitioning by Sampling and Pruning Spanning Trees.
Journal of Machine Learning Research 20, 85, 1–35.

This model works on a graph structure, which incorporates together space and time. That is, from data Y_{jt} for $j = 1, \dots, n$ and $t = 1, \dots, T$, we now move to Y_i for $i \in I = \{1, \dots, nT\}$ by stacking T times the spatial map. So we have a graph $\mathcal{G} = (V, E)$ of nT nodes and edges built according to time and space connections.
The idea is to search a partition $\boldsymbol{\pi} = \{\mathcal{G}_1, \dots, \mathcal{G}_c\}$ of I (with $\mathcal{G}_k$ subgraphs for $\mathcal{G}$), on randomly selected spanning trees $\mathcal{T}$ of $\mathcal{G}$, on which we set cluster-specific parameters $\beta_{\mathcal{G}_1}, \dots, \beta_{\mathcal{G}_c}$.

Model 3



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$$
\begin{aligned}
Y_i | \mathcal{T}, \boldsymbol{\pi}, \beta_{\mathcal{G}_k} &\overset{\text{iid}}{\sim} f\big(Y_i | \beta_{\mathcal{G}_k}; \boldsymbol{x}_i\big) \quad i \in \mathcal{G}_k \\
\beta_{\mathcal{G}_1}, \ldots, \beta_{\mathcal{G}_c} | \mathcal{T}, \boldsymbol{\pi} &\sim \prod_{k=1}^{c} f\big(\beta_{\mathcal{G}_k}\big) \\
p(\boldsymbol{\pi} = \{\tilde{\mathcal{G}}_1, \ldots, \tilde{\mathcal{G}}_c\} | \mathcal{T}) &\sim \prod_{k=1}^{c} \kappa(\tilde{\mathcal{G}}_k) \\
\mathcal{T} &\sim \mathcal{U}(\mathrm{St}(\mathcal{G}))
\end{aligned}
$$

Expected workflow

- PM10 covariates analysis (physical/chemical relation)
- First models implementation and comparison
- Implementation of variations of those simple models, or more complex models from literature
- Gif/Video interactive plots for displaying results

References

- Garritt L. Page, Fernando A. Quintana, David B. Dahl (2022)
  Dependent Modeling of Temporal Sequences of Random Partitions. Journal of Computational and Graphical Statistics, 31:2, 614-627.
- Mozdzen A., Cremaschi A., Cadonna A., Guglielmi A., Kastner G. (2022)
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  Bayesian Space-Time Partitioning by Sampling and Pruning Spanning Trees. Journal of Machine Learning Research 20, 85, 1–35.