General idea: we have n distinct locations $s_1, \dots, s_n$ ($s_i = (\text{lat}, \text{long})$ eg)
the goal is to define a model for the partitioning of s_i into $k_n(t)$ groups

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so we define $\rho_n = \{S_1, \dots, S_{k_n(t)}\}$ the clusters identical, i.e. $s_i \in S_h \iff$ location s_i is in cluster h.
or equivalently let $c_1, \dots, c_n$ index $c_i = h$ iff ecc.

Then the prior for ρ_n is

$$P_{\rho_n}(\rho) = P(\rho = \rho_n) \propto \prod_{h=1}^{k_n} C(S_h, s_h^*)$$

so maybe for us it will become $C(S_h, t)$?

S_h will be like $\{1, \dots, n\}_h$ subset

this is a cohesion function: measures how likely those selected values of S_h would be clustered together a priori

this makes the clustering process location dependent

the set of locations that h to S_h like if $S_h = \{1, 3, 9\}$ then $s_h^* = \{s_1, s_3, s_9\}$

we call this model/law as sPPM (spatial product partition model)

Then the idea is to set to each obs y_i for $i = 1 \to n$ a law/behaviour which is cluster dependent

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$$Y_i = Y(s_i) \mid \theta, c_i \overset{ind}{\sim} f(\theta^*_{c_i}) \quad \text{for } i = 1 \to n$$
$$\theta^*_h \overset{iid}{\sim} G_0 \quad \text{for } h = 1 \to k_n$$

we actually know the ... as above $\rho_n(t)$ as $k_n = |\rho_n(t)|$

$$\{c_i\}_{i=1}^{n} \sim \text{sPPM}$$

for example $\theta^*_{c_i} = (\mu_{c_i}, \sigma^2_{c_i})$ and we set $f(\dots) = N(\dots)$

and we can include covariates with
let $y_i(t)$ the values observed ($y_i(t)$ for us)
let $x_i(t)$ the covariate set there
let $\beta^*_1, \dots, \beta^*_{k_n(t)}$ the cluster specific regression params

the $\theta^*_{c_i}$ of before

(wait, how can we have $k_n(t)$? a varying model for the later inference, i.e. how will the # k_n vary in time? eg the # k_n ... but then the βs are time or "must be the same")

then we model
$$Y_i = Y(s_i) \mid x_i(t), c_i(t), \beta^*, \sigma^2 \overset{ind}{\sim} N(x_i^T \beta^*_{c_i}, \sigma^2)$$

(is all this temporal extension that makes me worry)

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Here there are some more examples of models. note that the step $\{c_i\}_{i=1}^{n} \sim \text{sPPM}$, but that is equivalent to $\rho_n \sim \text{sPPM}$, as for the ρ_n we can translate to get the c_i.

$\rho_n = \{S_1, \dots, S_{k_n}\}$

eg $\begin{pmatrix}2\\3\\7\end{pmatrix}\begin{pmatrix}1\\2\\4\\5\end{pmatrix}\begin{pmatrix}8\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix}$ $k_n = 4$

```
for i in 1:n
| c_i = which(i in S_j for j=1:k_n)
end
```

Ok, now we can understand better the model of our (in thesis) most relevant paper, the paper 4.

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Here they also provide the time extension! at first glance

As they set $w=1 \to m$ the units, ok, they model the partition as $\rho_t = \{S_1(t), \dots, S_{k_t}(t)\}$ ⇒ NICE
or again $c_t = (c_1(t), \dots, c_m(t))$ as before.

So we want to study the $\mathbb{P}$ of getting a certain collection of partitions. So, assuming a MC (Markov) structure, we get our target be as

$$p(\rho_1, \dots, \rho_T) = \underbrace{\mathbb{P}(\rho_1(t) = \rho_1, \dots)}_{\text{marginal prob}} = p(\rho_T | \rho_{T-1}) \cdot \ldots \cdot p(\rho_2|\rho_1) \cdot p(\rho_1)$$

Then they give as prior for the partition structure this

$$\text{“}Pr(\rho|M)\text{”} = p_{\rho_m|M}(\rho|M) = \mathbb{P}(\rho_m = \rho|M) = \frac{M^k}{\underbrace{\prod_{i=1}^m (M+i-1)}_{\text{the constant normalization const}}} \underbrace{\prod_{w=1}^k (|S_w|-1)!}_{\text{the } \Gamma(|S|) \text{ told in the review paper}}$$

we call this law CRP(M).
But that law misses how we want to model the connection among ρ_{t-1} and ρ_t.

$\rho \sim CRP(M)$

So we introduce

$\gamma_{it} = \begin{cases} 1 & \text{unit } i \text{ stays in the same cluster from } t-1 \text{ to } t \\ 0 & \text{otherwise} \end{cases}$

we set here as usual a $\gamma_{wt} \overset{iid}{\sim} Ber(\alpha_t)$ — $\alpha_t = 0: \rho_t \perp \rho_{t-1}$; $\alpha_t = 1: \rho_t = \rho_{t-1}$

Then for better writing everything we introduce, to govern $\gamma_t = (\gamma_1(t), \dots, \gamma_m(t))$ where $\gamma_i(t) \in \{0,1\}$, to say which values we get from the $Ber(\alpha_t)$ for the γ_{it} we introduced above.

We call this model tRPM(α, M), and that is what eventually we will find in the model; about c_t, we eventually have it on the ρ_t.

$$
\begin{aligned}
Y_{it}|\mu_t^\star, \sigma_t^{2\star}, c_t &\overset{\text{ind}}{\sim} N(\mu^\star_{c_{it}t}, \sigma^{2\star}_{c_{it}t}),\ i = 1, \dots, m \text{ and } t = 1, \dots, T, \\
(\mu^\star_{jt}, \sigma^\star_{jt})|\theta_t, \tau_t^2 &\overset{\text{ind}}{\sim} N(\theta_t, \tau_t^2) \times \text{UN}(0, A_\sigma),\ j = 1, \dots, k_t, \\
(\theta_t, \tau_t) &\overset{\text{iid}}{\sim} N(\phi_0, \lambda^2) \times \text{UN}(0, A_\tau),\ t = 1, \dots, T, \\
(\phi_0, \lambda) &\sim N(m_0, s_0^2) \times \text{UN}(0, A_\lambda), \\
\checkmark\ \{c_t, \dots, c_T\} &\sim \text{tRPM}(\alpha, M), \text{ with } \alpha_t \overset{\text{iid}}{\sim} \text{Beta}(a_\alpha, b_\alpha),
\end{aligned} \tag{5}
$$

The effect of this γ_t on the law of ρ is just that it reduces the support, i.e. now

$\mathbb{P}(\rho_t = \lambda | \gamma_t, \rho_{t-1}) = \begin{cases} 0 & \text{if } \rho_{t-1} \text{ and } \rho_t \text{ are not compatible wrt } \gamma_t \\ \dots & \text{other} \end{cases}$

⋮ now it gets more complex, less clear

Anyway, here there was a data generation mechanism so we could use it to make set up our own more models.

$$y(s_i) = y_i = \psi^*_{c_i}(s_i) + x_i^T \beta_{c_i} + \theta(s_i) + x_i^T \beta$$

spatial effect cluster specific + global

covariate effect cluster specific + global

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$$y_i(t) \mid y_i^{(t-1)}, \dots, s_t \overset{\perp}{\sim} \mathcal{N}\Big(\psi^*_{c_{it}}(t) + \underbrace{w_i\, y_i^{(t-1)}}_{\text{to introduce an AR(1) effect}}, \dots\Big)$$

also by knowing just $\overset{\perp}{\sim}$ (i.e. $\overset{iid}{\sim}$) we can use all these models as mixture models

$$y_i(t) \mid \underline{w}_t, f_k \overset{iid}{\sim} f_y(y) = \sum_{k=1}^{K(t)} w_k(t) \cdot f_k(y)$$

where $w_k(t) = \mathbb{P}(c_i(t) = k \mid \underline{w})$

OR

$$y_i(t) \mid s_t, \dots \overset{iid}{\sim} \dots$$

where $s_t \sim \dots$

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