Laboratory Session : April 30, 2024
Exercises due on : May 13, 2024

Exercise 1: Bayesian Inference for Poisson model

The number of particles emitted by a radioactive source during a fixed interval of time ($\Delta t = 10$ s) follows a Poisson distribution on the parameter μ. The number of particles observed during consecutive time intervals is: 4, 1, 3, 1, 5 and 3.

(a) assuming a *positive uniform* prior distribution for the parameter μ

- determine and draw the posterior distribution for μ, given the data
- evaluate mean, median and variance, both analytically and numerically in R

(b) assuming a *Gamma* prior such that the expected value is $\mu = 3$ with a standard deviation $\sigma = 1$,

- determine and draw the posterior distribution for μ, given the data
- evaluate mean, median and variance, both analytically and numerically in R.

(c) evaluate a 95% credibility interval for the results obtained with different priors. Compare the result with that obtained using a normal approximation for the posterior distribution, with the same mean and standard deviation

Exercise 2: Efficiency using Bayesian approach

A researcher `A` wants to evaluate the efficiency of detector 2 (Det2). For this purpose, he sets up the apparatus shown in the figure 1, where Det2 is sandwiched between Det1 and Det3. Let `n` be the number of signals recorded simultaneously by Det1 and Det3, and `r` be those also recorded by Det2, researcher `A` obtains `n` = 500 and `r` = 312.

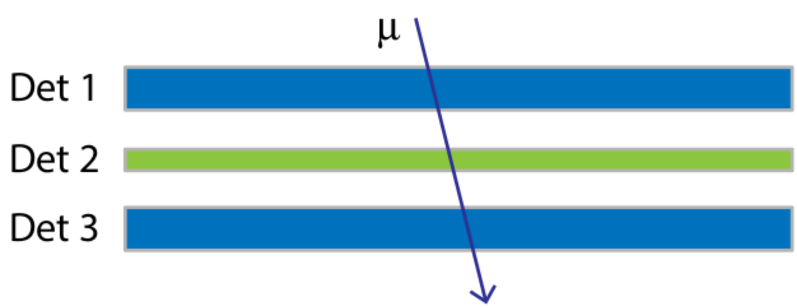

Figure 1:

Assuming a binomial model where `n` is the number of trials and `r` is the number of success out of `n` trials,

a) Evaluate the *mean* and the *variance* using a Bayesian approach under the hypothesis of:

  - uniform prior $\sim \mathcal{U}(0,1)$
  - Jeffrey's prior $\sim$ Beta$(1/2, 1/2)$

b) Plot the posterior distributions for both cases

Taking into account that the same detector has been studied by researcher `B`, who has performed only `n` $= 10$ measurements and has obtained `r` $= 10$ signals,

c) Evaluate the *mean*, the *variance* and the *posterior* distribution using a uniform prior with the results of researcher `B`.

d) Repeat the computation of points a) and b) with the data of researcher `A` using as a prior the posterior obtained from point c).

e) [`Optional`] Compute 95% credible interval using the posterior of the previous point d).

Exercise 3 - Bayesian Inference for Binomial model

- A coin is flipped $n = 30$ times with the following outcomes:

  `T, T, T, T, T, H, T, T, H, H, T, T, H, H, H, T, H, T, H, T, H, H, T, H,`
  `T, H, T, H, H, H`

a) Assuming a flat prior, and a beta prior, plot the likelihood, prior and posterior distributions for the data set.

b) Evaluate the most probable value for the coin probability p and, integrating the posterior probability distribution, give an estimate for a 95% credibility interval.

c) Repeat the same analysis assuming a sequential analysis of the data. Show how the most probable value and the credibility interval change as a function of the number of coin tosses (i.e. from 1 to 30).

d) Do you get a different result, by analysing the data sequentially with respect to a one-step analysis (i.e. considering all the data as a whole) ?

Exercise 4 - Poll

A couple of days before an election in which four parties (A,B,C,D) compete, a poll is taken using a sample of 200 voters who express the following preferences 57, 31,45 and 67 for, respectively, parties A,B,C and D.

Using a Bayesian approach, for all parties

- Calculate the expected percentage of votes and a 68% credibility interval by assuming as prior a
  - uniform prior
  - a prior constructed from the results obtained from another poll conducted the previous week on a sample of 100 voters who expressed the following preferences 32,14,26,28 for, respectively, parties A,B,C and D.
- Sample size to obtain a margin of error less or equal than $\pm 3\%$ for each party