Machine Learning — Statistical Methods for Machine Learning

Risk analysis for tree predictors

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The risk analysis for ERM over a finite class $\mathcal{H}$ of predictors states that, with probability at least $1-\delta$ with respect the random draw of training set of size m, we have

$$\ell_{\mathcal{D}}(\widehat{h}) \leq \min_{h\in\mathcal{H}} \ell_{\mathcal{D}}(h) + \sqrt{\frac{2}{m}\ln\frac{2|\mathcal{H}|}{\delta}}\ . \tag{1}$$

We can see what happens when applying this result to the class of predictors computed by binary tree classifiers over $\mathcal{X} = \{0,1\}^d$ (i.e., $d \geq 2$ binary attributes). We consider **complete binary trees**: trees whose node have either zero or two children. A full binary tree is a complete binary tree whose leaves (nodes with zero children) are all at the same depth. A complete binary tree with N nodes has always $(N+1)/2$ leaves.

Fact 1. *For each function of the form $h : \{0,1\}^d \to \{-1,1\}$ there exists a binary tree classifier with at most $2^{d+1}-1$ nodes that computes h.*

Proof. Consider a full binary tree with 2^d leaves (which therefore has $2^{d+1}-1$ nodes). The root node implements a binary test on x_1, the 2 nodes at depth 1 implement binary tests on x_2, and so on until the 2^{d-1} nodes at depth $d-1$ which test x_d. Now note that any path from root to a leaf corresponds to a binary sequence in $\{0,1\}^d$. Given any $h : \{0,1\}^d \to \{-1,1\}$, we can assign a label $y_\ell \in \{-1,1\}$ to each leaf ℓ so that if the path to the leaf corresponds to $\boldsymbol{x} \in \{0,1\}^d$, then the label is set to $h(\boldsymbol{x})$. The classifier computed by the tree then corresponds to h. □

Since there are 2^{2^d} binary functions over $\{0,1\}^d$, we can run ERM with a class $\mathcal{H}$ containing 2^{2^d} tree classifiers. The upper bound (1) then becomes

$$\ell_{\mathcal{D}}(\widehat{h}) \leq \min_{h\in\mathcal{H}} \ell_{\mathcal{D}}(h) + \sqrt{\frac{2}{m}\left(2^d \ln 2 + \ln\frac{2}{\delta}\right)}\ .$$

Therefore, in order to make the risk of ERM small, the training set must contain a number m of training examples of the order of 2^d, which is the cardinality of $\mathcal{X} = \{0,1\}^d$. This is a truly extreme case of overfitting.

Limiting the number of nodes. In order to reduce overfitting, we can minimize training error within a smaller class of trees. Consider the set $\mathcal{H}_N$ of all classifiers computed by complete binary tree predictors with exactly N nodes on $\{0,1\}^d$, where $N \ll 2^d$.

Fact 2. $|\mathcal{H}_N| \leq (2de)^N$.

Proof. Note that $|\mathcal{H}_N|$ is smaller than the product of: the number of binary trees with N nodes, the number of ways of assigning binary tests to attributes at the internal nodes, the number of ways of assigning binary labels to the leaves. If we conventionally assign the left child of a node

to the negative result of a test, and the right child to a positive result, a test is uniquely identified just by the index of the tested attribute. Therefore, if the tree has M internal nodes, there are d^M ways of assigning tests to internal nodes. Moreover, since there are $N - M$ leaves, there are 2^{N-M} ways of assigning binary labels to leaves. Therefore, each tree of N nodes can implement up to $d^M 2^{N-M} \leq d^N$ (since $d \geq 2$) classifiers. Finally, the number of complete binary trees with N nodes (N is odd because the tree is complete) is given by the $\frac{N-1}{2}$-th Catalan number

$$C_{\frac{N-1}{2}} = \frac{2}{N+1}\binom{N-1}{(N-1)/2} .$$

Thus, using the standard upper bound $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ derived from Stirling approximation to binomial coefficients, we get

$$|\mathcal{H}_N| \leq \frac{2}{N+1}\left(\frac{2e(N-1)}{N-1}\right)^{\frac{N-1}{2}} d^N < (2ed)^N$$

concluding the proof. □

Hence, if $\widehat{h} = \operatorname{argmin}_{\mathcal{H}_N} \ell_S(h)$ for a given N and training set S, the upper bound (1) becomes

$$\ell_{\mathcal{D}}(\widehat{h}) \leq \min_{h \in \mathcal{H}_N} \ell_{\mathcal{D}}(h) + \sqrt{\frac{2}{m}\left(N\big(1 + \ln(2d)\big) + \ln\frac{2}{\delta}\right)} .$$

From that, we deduce that in this case a training set of size of order $N \ln d$ is enough to control the risk of $\widehat{h} \in \mathcal{H}_N$.

A more refined bound. As it is not clear what N should be used in practice, we now derive a more refined bound. Recall that we control the variance error of ERM in $\mathcal{H}_N$ by making sure that the risk of each predictor in $\mathcal{H}_N$ can exceed its training error by at most ε. We now take a different approach. Namely, we upper bound the risk of a tree predictor h by its training error plus a quantity ε_h that now depends on the size of the tree.

Let $\mathcal{H}$ be the set of all classifiers h_T defined on complete binary trees T with at most $2^{d+1}-1$ nodes. Using our previous calculation, we know that $|\mathcal{H}| = \mathcal{O}\big(d^{2^{d+1}}\big)$. Because of Fact 1, $\mathcal{H}$ implements all binary classifiers $h : \{0,1\}^d \to \{-1,1\}$. Note that the cardinality of $\mathcal{H}$ could be larger than 2^{2^d} because two tree classifiers $h_T, h_{T'}$ may compute the same function $h : \{0,1\}^d \to \{-1,1\}$ even though they are based on distinct complete binary trees T and T'.

We introduce a function $w : \mathcal{H} \to [0,1]$ and call $w(h)$ the weight of tree predictor h. We assume

$$\sum_{h \in \mathcal{H}} w(h) \leq 1 . \tag{2}$$

We can then write the following chain of inequalities, where $\varepsilon_h > 0$ will be chosen later on,

$$\mathbb{P}\left(\exists h \in \mathcal{H} :\ \big|\ell_S(h) - \ell_{\mathcal{D}}(h)\big| > \varepsilon_h\right) \leq \sum_{h \in \mathcal{H}} \mathbb{P}\left(\big|\ell_S(h) - \ell_{\mathcal{D}}(h)\big| > \varepsilon_h\right) \leq \sum_{h \in \mathcal{H}} 2e^{-2m\varepsilon_h^2} .$$

Note that we used Chernoff-Hoeffding bound in the last step. Now, choosing

$$\varepsilon_h = \sqrt{\frac{1}{2m}\left(\ln\frac{1}{w(h)} + \ln\frac{2}{\delta}\right)}$$

we get that

$$\mathbb{P}\left(\exists h \in \mathcal{H} : \left|\ell_S(h) - \ell_{\mathcal{D}}(h)\right| > \varepsilon_h\right) \leq \sum_{h \in \mathcal{H}} \delta w(h) \leq \delta$$

where we used the property (2) of the function w.

A consequence of this analysis is that, with probabilty at least $1 - \delta$ with respect to the training set random draw, we have

$$\ell_{\mathcal{D}}(h) \leq \ell_S(h) + \sqrt{\frac{1}{2m}\left(\ln \frac{1}{w(h)} + \ln \frac{2}{\delta}\right)} \tag{3}$$

simultaneously for every $h \in \mathcal{H}$. This suggests an alternative algorithm to training error minimization: while ERM uses

$$\widehat{h} = \operatorname*{argmin}_{h \in \mathcal{H}_N} \ell_S(h)$$

for a given N, the new approach (which is sometimes called Structural Risk Minimization) leads to the choice

$$\widehat{h} = \operatorname*{argmin}_{h \in \mathcal{H}} \left(\ell_S(h) + \sqrt{\frac{1}{2m}\left(\ln \frac{1}{w(h)} + \ln \frac{2}{\delta}\right)}\right) \ . \tag{4}$$

The function w can be naturally viewed as a complexity measure for the tree predictor h. Note that this analysis offers a different viewpoint on overfitting: $\ell_S(h)$ becomes a good estimate of $\ell_{\mathcal{D}}(h)$ when it is "penalized" by the term

$$\sqrt{\frac{1}{2m}\left(\ln \frac{1}{w(h)} + \ln \frac{2}{\delta}\right)}$$

this accounts for the fact that we used the m training examples to choose a tree predictor h of complexity $w(h)$.

A concrete choice for the function w is obtained as follows. Using coding theoretic techniques, we can encode each tree predictor h with N_h nodes using a binary string $\sigma(h)$ of length $|\sigma(h)| = (N_h + 1)\lceil \log_2(d+3) \rceil + 2\lfloor \log_2 N_h \rfloor + 1 = \mathcal{O}(N_h \log d)$, so that there are no two tree predictors h and h' such that $\sigma(h)$ is a prefix of $\sigma(h')$. Codes of this kind are called *instantaneous* and always satisfy the Kraft inequality

$$\sum_{h \in \mathcal{H}} 2^{-|\sigma(h)|} \leq 1 \ .$$

Thanks to Kraft inequality—which implies property (2)—we can assign weight $w(h) = 2^{-|\sigma(h)|}$ to a classifier h computed by a tree predictor with N_h nodes. Applying bound (3) we get that, with probability at least $1 - \delta$ with respect to the training set random draw,

$$\ell_{\mathcal{D}}(h) \leq \ell_S(h) + \sqrt{\frac{1}{2m}\left(|\sigma(h)| + \ln \frac{2}{\delta}\right)} \qquad \text{(with } |\sigma(h)| = \mathcal{O}(N_h \log d)\text{)}$$

simultaneously for each $h \in \mathcal{H}$. Hence, a learning algorithm for tree predictors can control overfitting by generating predictors $\widehat{h}$ defined by

$$\widehat{h} = \operatorname*{argmin}_{h \in \mathcal{H}} \left(\ell_S(h) + \sqrt{\frac{1}{2m} \left(|\sigma(h)| + \ln \frac{2}{\delta} \right)} \right) .$$

Note that the choice of the weight function w is not determined by the analysis. In particular, we may choose any other w satisfying (2). We should then interpret w as a bias term, giving preference to certain trees as opposed to others. A bias towards smaller trees is an instance of the principle known as *Occam Razor*: if two explanations agree with a set of observations, then the shortest explanation is the one with the biggest predictive power. This is supported by the empirical observation that, given two predictors with the same training error, the "simpler" predictor tends to have smaller risk.