Matheuristic Variants of DSATUR for the Vertex Coloring Problem

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Notation

- $G = (V, E)$ is an undirected graph with vertex set $V = \{v_1, ..., v_n\}$ and edge set E, and denote $I = [\![1; n]\!] = [1; n] \cap \mathbb{Z}$.

- An edge $e = (v_i, v_j) \in E$ with $i < j$ links the underlying vertices (*for VCP there is no sense to consider loop/multiple edges*).

- For $i \in I$, δ_i denotes the set of neighbors of vertex v_i, and $d_i = |\delta_i| = \{j \in I \mid \mathrm{ngb}(v_i, v_j) = 1\}$, where $\mathrm{ngb}(v_i, v_j) = 1$ *iif* $(v_i, v_j) \in E$.[2]

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[2]ngb stands for "neighbor".

A k-coloring of G

- It is an assignment of colors to vertices such that no two adjacent vertices share the same color.

- **VCP** consists in finding a k-coloring of G using the minimum number of colors k (the *chromatic number $\chi(G)$*).

- A valid k-coloring (c) fulfills: $\forall i < j, \left(v_i, v_j\right) \in E \implies c_i \neq c_j$ where $c_i = [\![1; k]\!]$ is the color of v_i.

A partial k-coloring of G

- If a vertex may not be colored, we set $c_i = -1$ $s.t.$ $c_i \in [\![1; k]\!] \cup \{-1\}$

- A partial k-coloring (c) is *feasible* if $\forall i < j, (v_i, v_j) \in E \implies c_i \neq c_j \lor c_i = c_j = -1$.

- Given v_i the saturation table S_i is the set of colors assigned to its colored neighbors: $S_i = \bigcup_{j \in \delta_i}^{n} \{c_j\} \setminus \{-1\}$, and $s_i = |S_i|$ is the saturation degree.

- A total order $\succcurlyeq$ over V is defined as: $v_i \succcurlyeq v_j \iff s_i > s_j \lor (s_i = s_j \land d_i \geq d_j)$

Compact ILP Formulations[3], feasible $iif\ \chi(G) \leq k$

$z_{i,c} \in \{0,1\}$ indicates if vertex v_i is assigned color c.

$y_c \in \{0,1\}$ indicates if color c is used in the coloring.

$$\min \sum_{c=1}^{k} y_c$$

$$s.t.\ \min \sum_{c=1}^{k} z_{i,c} = 1 \quad \forall i \in I$$

$$z_{i,c} + z_{j,c} \leq y_c \quad \forall(v_i, v_j) \in E,$$

$$\forall c \in [\![1; k]\!]$$

The objective minimizes the number of used colors.

The 1st set ensures that each vertex is assigned exactly one color.

The 2nd set ensures that adjacent vertices do not share the same color.

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[3]Efficient formulations: extended column generation by Furini and Malaguti [1], and reduced formulation to MWSSP by Cornaz et al. [2].

Observations

- Having an upper bound of the chromatic number as the initial value k (or simply $k = |V|$) guarantees the optimality of the solution.

- The size of k strongly affects the performance of ILP solvers.

- Symmetries in the model (e.g., colors are permutable) enlarge the search space for Branch-and-Bound algorithms (the same solution can be represented in multiple ways)

Representative ILP Model[4], asymmetric and easily to LP-relax

$x_{i,i'} \in \{0, 1\}$, $\forall i, i' \in V$ s.t. $i \leq i'$, indicates if verticies v_i and $v_{i'}$ share the same color and i is the minimum index of its color class.

$$\min_{z} \sum_{i=1}^{n} x_{i,i}$$

$$s.t. \sum_{i' \leq i} x_{i',i} \geq 1 \qquad \forall i \in I$$

$$x_{j,i} + x_{i,j} \leq x_{j,j} \quad \forall (v_i, v_j) \in E,$$
$$\forall i < j \in I$$

The objective counts the number of representative vertices (i.e., used colors).

The 1st set ensures either $x_{i,i} = 1$ or v_i shares the color with a vertex $v_{i'}$ with $i' < i$.

The 2nd set expresses the color incompatibility between adjacent vertices and $x_{j,i} = 1$ implies that $x_{j,j} = 1$

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[4] A vertex is representative of its color class if it has the minimum index among the vertices sharing the same color.

Standard DSATUR Algorithm

Algorithm 1: Standard DSATUR algorithm

Input: $G = (V, E)$ a non-empty and non-oriented graph

Initialization:

    define partial coloring c with $c_i := -1$ for all $i \in I$

    define saturation table S with $S_i := \emptyset$ for all $i \in I$

    initialize set $U := V$, and color $k := 0$

    while $U \neq \emptyset$

        find $u \in U$, a maximum of $\succcurlyeq$ in U.

        if $|S_u| = k$ **then** $k := k + 1$ // a new color is added

        compute $c_i := \min S_u$ // assign color to u

        remove u from U

        for all $i \in \delta_u \cap U$, $S_i = S_i \cup \{c_i\}$ // update saturation

    end while

return color k and (c) a k-coloring of G

- DSATUR is an adaptive greedy heuristic proposed by Brélaz [3], which colors vertices iteratively.

- Selection of the uncolored verex to color is given with order $\succcurlyeq$, maximizing first the saturation degree and secondly the degree.

- Coloring a new vertex updates saturation, the iteration order of vertices is thus adaptive.

DASTUR Matheuristic Variants

Initialization

Defining an initial partial coloring and computing the saturation table for the uncolored vertices, **before** starting the main DSATUR iterations.

Variants:
1. `maxDeg`: color the vertex with the maximum degree — equivalent to standard DSATUR.
2. `col-`n: cosider n vertices having the maximum degree and color them solving a representative ILP model for the induced subgraph — an exact pre-processing.
3. `clq`: find a maximum clique (heuristically, since it is NP-hard) and color it with different colors — an exact pre-processing
4. `clq-col-`n: combine `clq` and `col-`n.

Thank You!

Bibliography

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[3] D. Brélaz, "New methods to color the vertices of a graph," *Commun. ACM*, vol. 22, no. 4, pp. 251–256, Apr. 1979.