UNIVERSITÀ DELLA CALABRIA

Introduction to utility theory and decision making

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Roadmap

1. Basic concepts (alternatives, preferences, …)
2. Representing ordinal preferences (ordinal utility)
3. Beyond ordinal preferences (cardinal utility, VNM utility, …)
4. Attitude towards risk
5. Applications and introduction to decision making

Basic Concepts: Alternatives

An agent chooses between a set X of alternatives

Alternatives are

- Mutually exclusive
- Exhaustive

Basic Concepts: Alternatives

Example:

Options = {Deep Learning, Algorithmic Game Theory}

X = {

DL = Deep Learning,

AGT = Algorithmic Game Theory,

DLAGT = Deep Learning and Algorithmic Game Theory,

N = None

}

Basic Concepts: Preferences

Preferences are a relation $\succcurlyeq$ on X, which is a subset of $X \times X$.

$\succcurlyeq$ is complete iff $\forall\, x,y \in X$, $x \succcurlyeq y$ or $y \succcurlyeq x$

$\succcurlyeq$ is transitive iff $\forall\, x,y,z \in X$, $[x \succcurlyeq y$ and $y \succcurlyeq z] \Rightarrow x \succcurlyeq z$

Basic Concepts: Preference Relation

A preference is a preference relation iff it is complete and transitive.

- Strict preference:

$$x \succ y \Leftrightarrow [x \succcurlyeq y \text{ and } y \not\succcurlyeq x]$$

- Indifference:

$$x \sim y \Leftrightarrow [x \succcurlyeq y \text{ and } y \succcurlyeq x]$$

Representing preferences as utilities

A preference relation can be represented by a utility function $u : X \to \mathbb{R}$ in the following sense:

$$x \succcurlyeq y \Leftrightarrow u(x) \geq u(y) \quad \forall x, y \in X$$

If a player finds x at least as good as y then $u(x)$ must be at least as high as $u(y)$.

In this sense, a player acts *as if* they are trying to maximise the value of $u(\cdot)$.

Theorem: Ordinal Representation

Theorem 1: Let X be finite. A preference can be represented by a utility function if and only if it is complete and transitive. Moreover, if $u : X \to \mathbb{R}$ represents $\succcurlyeq$, and if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $f \circ u$ also represents $\succcurlyeq$.

By the last statement of the theorem, such utilities are called ordinal, i.e. only the order information is relevant.

Necessary conditions

Transitivity is a necessary condition

Let $X = \{a, b, c\}$, suppose $a \succ b \succ c \succ a \Rightarrow u(a) > u(b) > u(c) > u(a)$. This is absurd.

Completeness is a necessary condition

If we have incomplete preferences, at most we can construct an order for a subset of X.

Proof of Theorem 1

Transitivity and completeness are necessary and sufficient

Let $X = \{X_1, \ldots, X_n\}$, we can partition the elements of X into k *indifference classes* $C_1, \ldots, C_k$ such that $C_1 \succ C_2 \succ \ldots \succ C_k$.

Thus, we can define u so that:

$$u(x) = k \qquad \forall x \in C_1,$$

$$u(x) = k\text{-}1 \qquad \forall x \in C_2,$$

$$\ldots,$$

$$u(x) = 1 \qquad \forall x \in C_k$$

Ordinal utilities: examples

X = $\{a, b, c\}$

Preferences: $a \succcurlyeq b \succcurlyeq c$

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Preferences: $a \succcurlyeq b \succcurlyeq c$

$u(a) = -1$	$u(a) = 30$	$u(a) = 90$
$u(b) = -2$	$u(b) = 20$	$u(b) = 2$
$u(c) = -3$	$u(c) = 10$	$u(b) = 1$

The numerical value of ordinal utility is not interpretable: ordinal utility does not tell us the magnitude with which a player prefers a to b.

Beyond ordinal utilities: lotteries

A simple lottery is a tuple $L = (p_1, x_1; p_2, x_2; \ldots; p_n, x_n)$

- Monetary prizes $x_1, x_2, \ldots, x_n \in X \subseteq \mathbb{R}$
- Probability distribution $(p_1, p_2, \ldots, p_n)$

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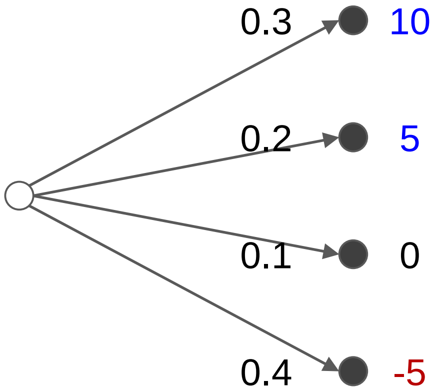
Example L = (0.3, 10; 0.2, 5; 0.1, 0; 0.4, -5)



Expected value

Consider the lottery L = (0.3, 10; 0.2, 5; 0.1, 0; 0.4, -5) we can compute the expected value to know its worth, as follows:

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In our case we have:

$$\mathbb{E}(L) = 0.3 \times 10 + 0.2 \times 5 - 0.4 \times 5 = 2$$

The expected value of L is 2. This means that, each time we play such a lottery, we expect to win a value of 2.

Expected utility

Given a preference relation $\succcurlyeq$ on $\mathcal{L}$, a utility function $U : \mathcal{L} \to \mathbb{R}$ is an expected utility function if it can written as:

$$U(L) = \sum_{i=1}^{n} p_i u(x_i)$$

for some function $u : \mathbb{R} \to \mathbb{R}$.

The function u is called a Bernoulli utility function.

Von Neumann and Morgenstern Utility

Von Neumann and Morgenstern (VNM) provided the conditions by which a preference relation $\succcurlyeq$ on $\mathcal{L}$ can be represented by a utility function.

Axiom 1: (Preference order)

$\succcurlyeq$ is complete and transitive

Axiom 2: (Continuity)

if $L \succ M \succ N$, there exists $p \in [0,1]$ such that $pL + (1\text{-}p)N \sim M$

Axiom 3: (Independence)

for any lottery N and $p \in [0,1]$, $L \succcurlyeq M \Leftrightarrow pL + (1\text{-}p)N \succcurlyeq pM + (1\text{-}p)N$

VNM Theorem

Theorem 2: (VNM) A binary relation $\succ$ over $\mathcal{L}$ has an expected utility representation if and only if it satisfies axioms 1–3. Moreover, if U and V are expected utility representations of $\succ$, then there exist constants $a, b \in \mathbb{R}$, $a > 0$, such that $U(\,\cdot\,) = aV(\,\cdot\,) + b$.

The last statement tells us that the VNM utility representation is unique up to a affine transformations.

Proof of VNM Theorem

We divide the proof in two parts:

- Part 1: we show that $U(L) = \sum_{i=1}^{n} p_i u(x_i)$

- Part 2: we show that $L \succ M \Leftrightarrow U(L) > U(M), \quad L,M \in \mathcal{L}$

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It follows that, the utility of a lottery $M = \sum_i p_i o_i$ is the expectation of u

$$u(M) = u\left(\sum_i p_i o_i\right) = \sum_i p_i u(o_i) = \sum_i p_i q_i$$

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However,

$$M' = \left(\sum_i p_i q_i\right) \cdot o_n + \left(\sum_i p_i (1 - q_i)\right) \cdot o_1 = U(M) \cdot o_n + (1 - U(M)) \cdot o_1$$

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Suppose $L \succ M$, we can define L' and M' as follows

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Since $L' \succ M' \Rightarrow U(L) > U(M)$. Hence, $L \succ M \Leftrightarrow U(L) > U(M)$.

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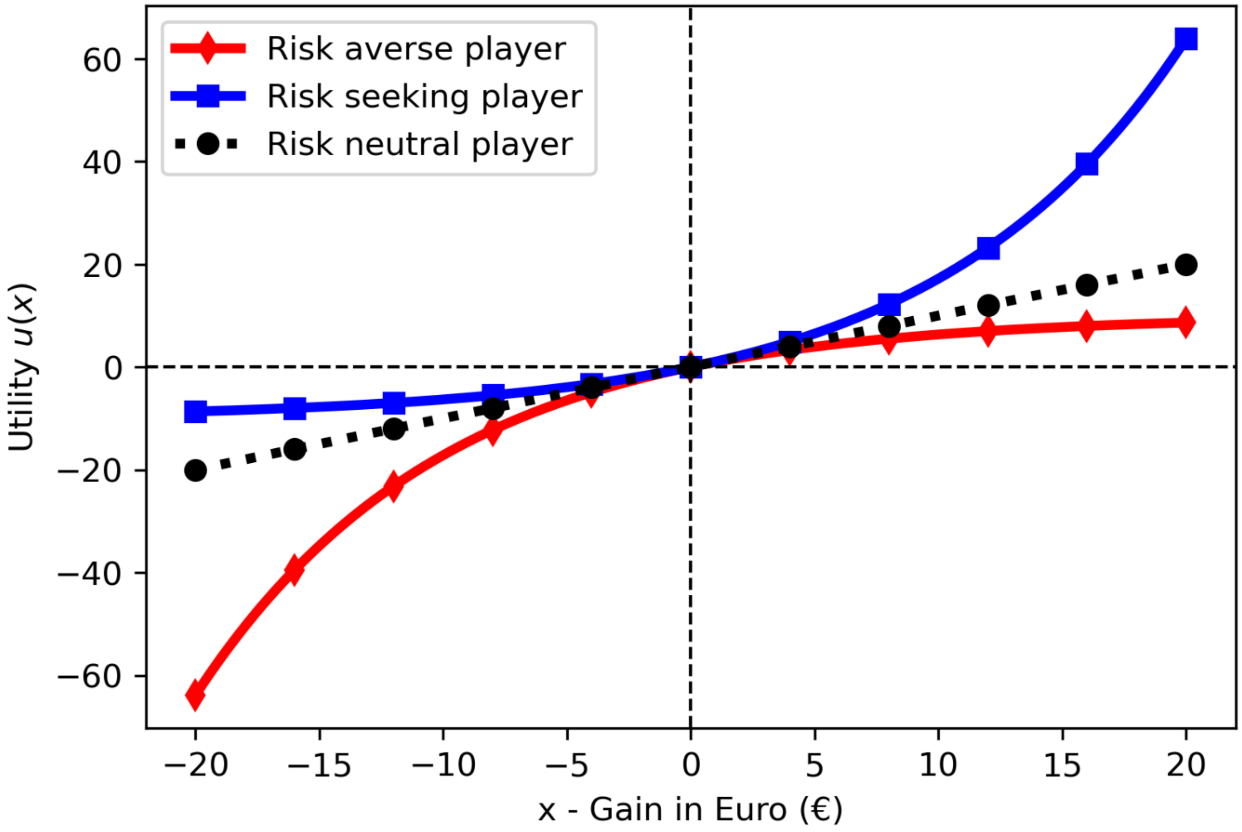

Applications: risk sharing

Suppose we have two risk averse players with $u(x) = \sqrt{x}$ and two risky assets $A_1, A_2 = (0.5, 100; 0.5, 0)$. Suppose A_1 and A_2 are independent.

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$$u(A_m) = 0.25 \times \sqrt{100} + 0.5 \times \sqrt{50} \approx 6$$

Applications: Insurance

We have a risk averse player with $u(x) = \sqrt{x}$ and an asset $A = (0.5, 100; 0.5, 0)$. We have a risk neutral insurance company with lots of money.

What premium P would the player pay to insure their asset?

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$$u(100 - P) \geq 0.5 \times u(100) + 0.5 \times u(0) \Rightarrow \sqrt{100 - P} \geq 5 \Rightarrow$$
$$\Rightarrow 100 - P \geq 25 \Rightarrow -P \geq -100 + 25 \Rightarrow P \leq 75$$

Applications: Insurance

What premium would the insurance company require to ensure the player's asset?

$$P \geq 0.5 \times 100 + 0.5 \times 0 \Rightarrow P \geq 50$$

Hence, both parties would gain if the company insures the player's asset for a premium $P \in [50, 75]$.

Recap on VNM utility theory

- VNM utility theory provides a way to construct cardinal utility functions
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  - Expected utility maximisation
  - Highly used in financial and economic applications
- VNM utility theory does not describe human behaviour
  - An alternative is not preferred because it is associated with a higher utility. Rather, it is associated with a higher utility because it is the preferred alternative.