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Non Cooperative Game Theory

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Roadmap

1. Computational limitations of coalitional game theory
2. Strategies to tackle computational limitations
3. Airport Game
4. Montecarlo Approximation
5. Compact representations

Strategic games

- Players compete against each other
- Players are self interested (want to maximise their payoff)
- Players need to decide what action they want to make

Strategic games – Two representations

- Normal form (a.k.a. Matrix form): Lists what payoff each player gets as a function of their actions
  - *As if* player moved simultaneously
- Extensive form: Includes timing of moves and other information
  - Players move sequentially, represented as a tree
    - Chess: white player moves, then black player responds, etc…
  - Keeps track of what players know when they make a move
    - Poker: sequential bets – what can a player see when they bet?

Normal form games

A game in normal form is a tuple $\langle N, A, u \rangle$

- $N = \{1,2,\dots,n\}$ players
- $A = \{A_1, A_2, \dots, A_n\}$ is the set of all actions for all the players
  - A_i is the action set for player i
  - $a = (a_1, a_2, \dots, a_n) \in A = \mathrm{A}_1 \times A_2 \times \cdots \times A_n$ is an action profile
- Utility function (payoff) for player i: $u_i: A \longrightarrow \mathbb{R}$
  - $u = (u_1, u_2, \dots, u_n)$ is a utility function profile

2x2 games as Matrices

We typically consider 2-player games

- Matrix representation: row player and column player

	A	B
C	2, 1	3, 4
D	1, 3	4, 4

2x2 games as Matrices

We typically consider 2-player games

- Matrix representation: row player and column player

	A	B
C	2, 1	3, 4
D	1, 3	4, 4

- One matrix for the row player and one for the column player

	A	B
C	2	3
D	1	4

	A	B
C	1	4
D	3	4

Pure strategy best response

Let's assume player i knew what all the other players would play.

$$a_{-i} = \langle a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$$

The pure strategy best response to a_{-i} is the action which maximises the payoff of player i

$$a_i^* \in \mathrm{BR}(a_{-i}) \Leftrightarrow \forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$$

Pure strategy best response: example

Consider the following game:

	A	B
C	2, 1	3, 4
D	1, 3	4, 4

If column player plays A, the best response for the row player is C

Pure Nash Equilibrium

Generalises the idea of best response for all the players

An action profile is a Pure Nash Equilibrium if each player is playing a best response.

$$a = \langle a_1, a_2, \dots, a_n \rangle \text{ is NE } \Leftrightarrow \forall i \;\; a_i \in \mathrm{BR}(a_{-i})$$

Pure Nash Equilibibiria: Prisoner Dilemma

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Pure Nash Equilibibiria: Pure Coordination

	L	R
L	a, a	0, 0
R	0, 0	a, a

Pure Nash Equilibibiria: Battle of the sexes

	A	C
A	2, 1	0, 0
C	0, 0	1, 2

Pure Nash Equilibibiria: Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Mixed strategies and Nash Equilibria

A strategy s_i for a player i is a probability distribution over A_i

- Pure strategy: only one action is played with probability one
- Mixed strategy: more than one action is played with probability > 0
  - Actions with probability > 0 are called the support of the strategy

- Let S_i be the set of all strategy for player i
- Let $S = S_1 \times S_2 \times \cdots \times S_n$ be the set of all strategy profiles.

Utilities under mixed strategies

- We can no longer read the utilities in the payoff matrix
- We use the expected utility from decision theory

$$u_{i(s)} = \sum_{a \in A} u_{i(a)} \Pr(a|s)$$

$$\Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Utilities under mixed strategies: example

		0.5	0.5
		H	T
0.5	H	1, -1	-1, 1
0.5	T	-1, 1	1, -1

$$u_r\left(\left[\frac{1}{2},\frac{1}{2}\right],\left[\frac{1}{2},\frac{1}{2}\right]\right) = 1 \times 0.25 - 1 \times 0.25 - 1 \times 0.25 + 1 \times 0.25 = 0$$

Best response and mixed Nash equilibrium

Let's assume player i knew what all the other players would play.

$$s_{-i} = \langle s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n \rangle$$

The mixed strategy best response to s_{-i} is the strategy which maximises the payoff of player i

$$s_i^* \in \mathrm{BR}(s_{-i}) \Leftrightarrow \forall s_i \in A_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

Mixed Nash Equilibrium

A strategy profile is a Mixed Nash Equilibrium if each player is playing a best response.

$$s = \langle s_1, s_2, \dots, s_n \rangle \text{ is NE } \Leftrightarrow \forall i \;\; s_i \in \text{BR}(s_{-i})$$

Theorem (Nash, 1950): every finite game has a Nash equilibrium

General condition for a best response

Let A and B be the payoff matrices for the row and the column players respectively.

A strategy s_r^* of the row player is a best response to the column player's strategy s_c if and only if the following condition holds:

$$s_{r,i}^* > 0 \implies (As_c^T) = \max_{k \in A_2}(As_c^T)_k \; \forall i \in A_1$$

Nash Equilibria: Zero-sum games

A 2-player game (A, B) is zero-sum if $A = -B$

Given a strategy x from the row player, the column player can choose a strategy y that limits the payoff of the row player.

Conversely, given a strategy y from the column player, the row player aims at maximising their own payoff.

Nash Equilibria: Zero-sum games

Column player

$$\min_{v,y} v$$
$$\text{s.t.}$$
$$Ay^T \leq \vec{1}v$$
$$y \in S_2$$

Row player

$$\max_{u,x} u$$
$$\text{s.t.}$$
$$Ay^T \geq \vec{1}u$$
$$x \in S_1$$

Nash Equilibria: Zero-sum games

Standard form for the row player

$$\min_{x \in \mathbb{R}^{(m+1)\times 1}} \quad cx$$
$$\text{s.t.}$$
$$M_{ub}x \leq b_{ub}$$
$$M_{eq}x = b_{eq}$$
$$x \geq 0$$

Nash Equilibria: Zero-sum games

$$\min_{x \in \mathbb{R}^{(m+1)\times 1}} \quad cx$$
$$\text{s.t.}$$
$$M_{ub}x \leq b_{ub}$$
$$M_{eq}x = b_{eq}$$
$$x \geq 0$$

$$c = (0, \ldots 0, -1) \in \ \mathbb{R}^{1\times(m+1)}$$
$$M_{ub} = \left(-A^T, \vec{1}\right)$$
$$b_{ub} = (0, \ldots, 0)^T \in \mathbb{R}^n$$
$$M_{eq} = (1, \ldots, 1, 0) \in \mathbb{R}^{1\times(m+1)}$$
$$b_{eq} = 1$$

Mixed Equilibria in 2x2 games

	A	C
A	2, 1	0, 0
C	0, 0	1, 2

- Player 1 plays A with probability p and B with probability $1 - p$
- Player 2 best responds to player 1
  - Player 2 makes player 1 indifferent between their strategies

$$u_1(A) = u_1(C)$$
$$2p + 0(1 - p) = 0 + 1(1 - p)$$
$$p = \frac{1}{3}$$

Mixed Equilibria in 2x2 games

	A	C
A	2, 1	0, 0
C	0, 0	1, 2

- Player 2 plays A with probability q and B with probability $1 - q$
- Player 1 best responds to player 2
  - Player 1 makes player 2 indifferent between their strategies

$$u_2(A) = u_2(C)$$
$$\mathrm{q} + 0(1-p) = 0q + 2(1-p)$$
$$q = \frac{2}{3}$$

Mixed Equilibria in 2x2 games

	A	C
A	2, 1	0, 0
C	0, 0	1, 2

Mixed strategies $\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)$ are a Nash equilibrium