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Coalitional Games and Solution Concepts

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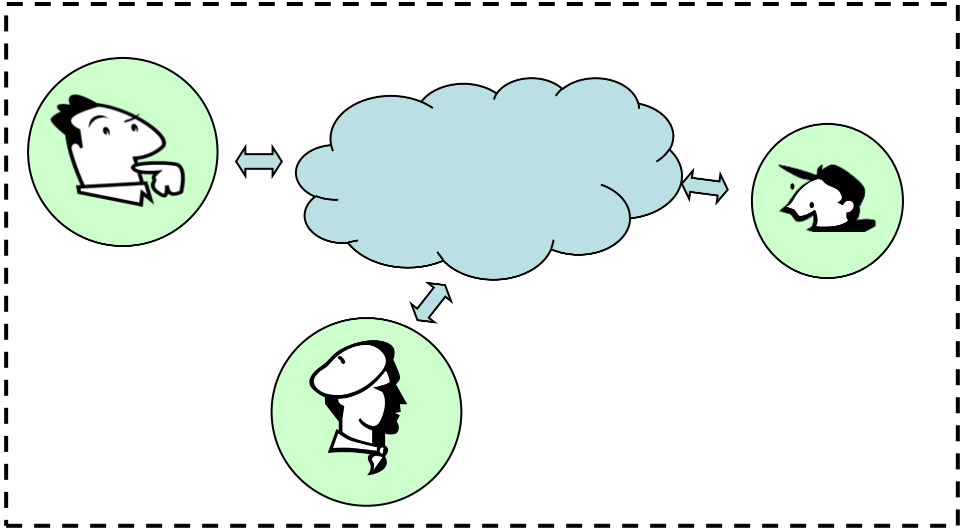
Part 1: Introduction to Coalitional Games

A reference point: Non-Cooperative Games

A **non-cooperative** game is defined by

- a set of agents (players) $N = \{1, \ldots., n\}$
- for each agent $i \in N$, a set of actions S_i
- for each agent $i \in N$, a utility function u_i: $S_1 \times \ldots. \times S_n \rightarrow R$



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Observe that an agent's utility depends not just on her action, but on actions of other agents.

Thus, for agent i finding the best action involves deliberating about what others will do.

Prisoners' dilemma: the rational outcome

- P1's reasoning:
  - if P2 stays quiet, then I should confess
  - if P2 confesses, then I should confess, too
- P2 reasons in the same way

P1 \ P2	Q	C
Q	(-1,-1)	(-4, 0)
C	(0, -4)	(-3, -3)

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Result: both confess and get 3 years in prison

However, if they chose to cooperate and stay quiet, they could get away with 1 year each

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So, why do they not cooperate?

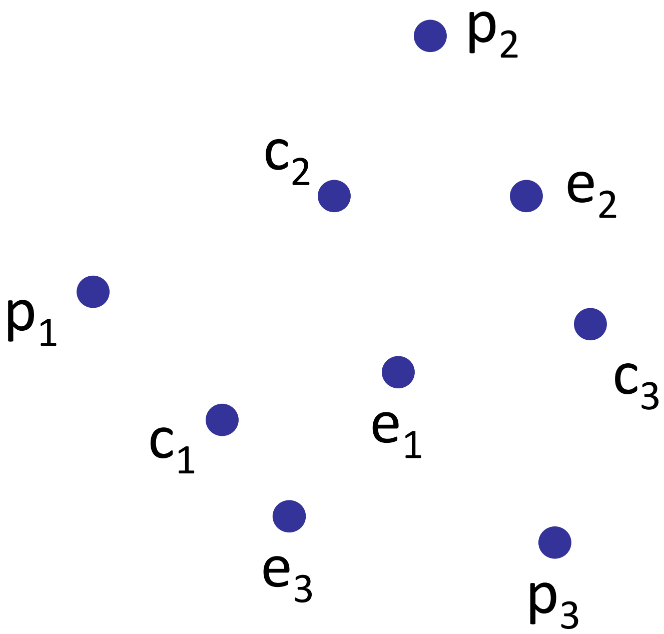
Beyond Non-Cooperative Games

- Cooperation does not occur in prisoners' dilemma, because players *cannot make binding agreements*
- But, what if binding agreements are possible?
- This is exactly the class of scenarios studied by cooperative game theory (and the topic of this lesson)

Coalitions in Cooperative Game Theory

- Task Allocation
- Resource allocation
- Complementary agent expertise

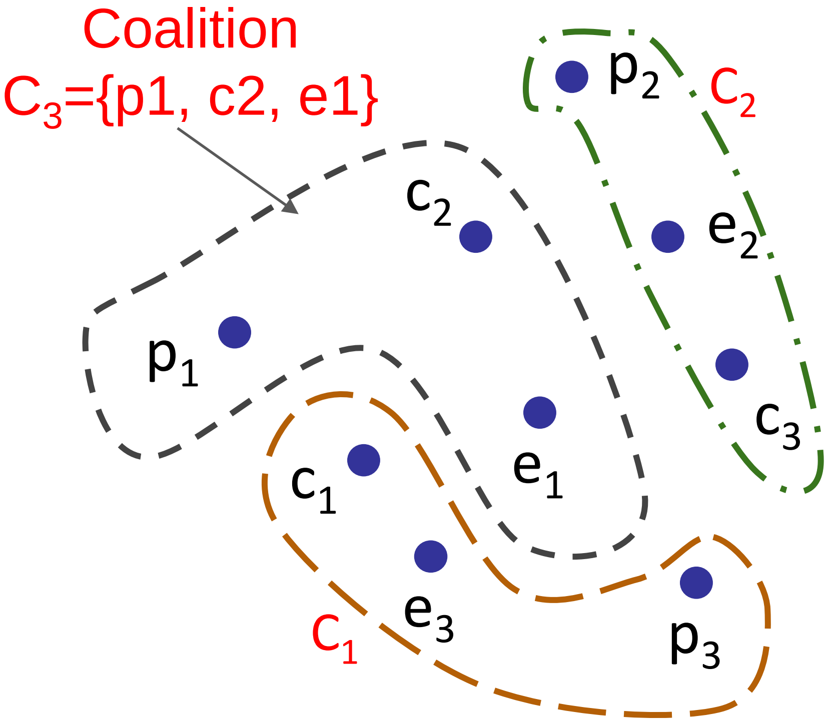
Coalition formation by example



Construction Workers (agents):

- p : plumbers
- c : carpenters
- e : electricians

Coalition formation by example



Agents have to decide:

- who to join

Coalition structure CS=<C_1, C_2, C_3>

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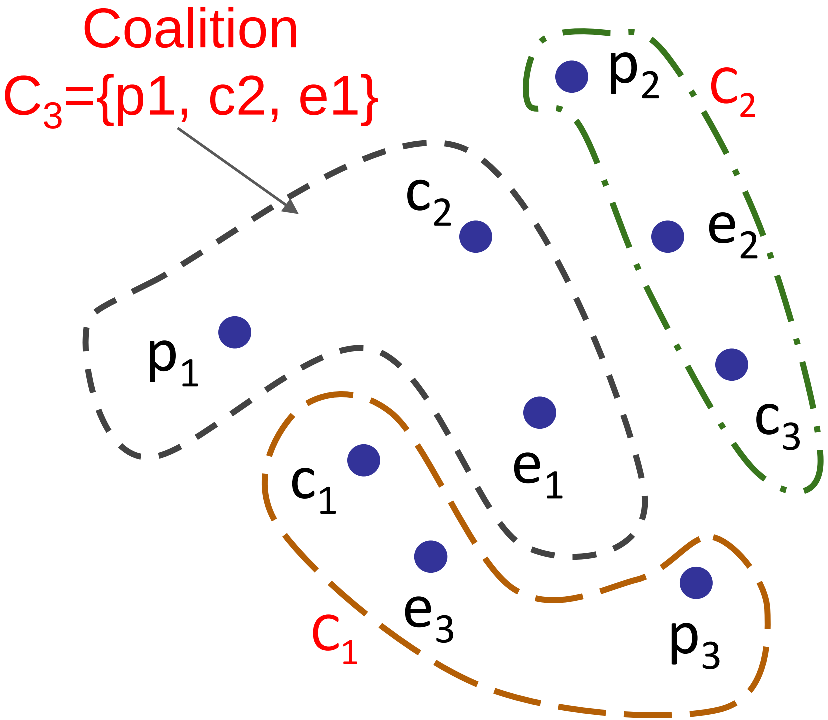


Agents have to decide:

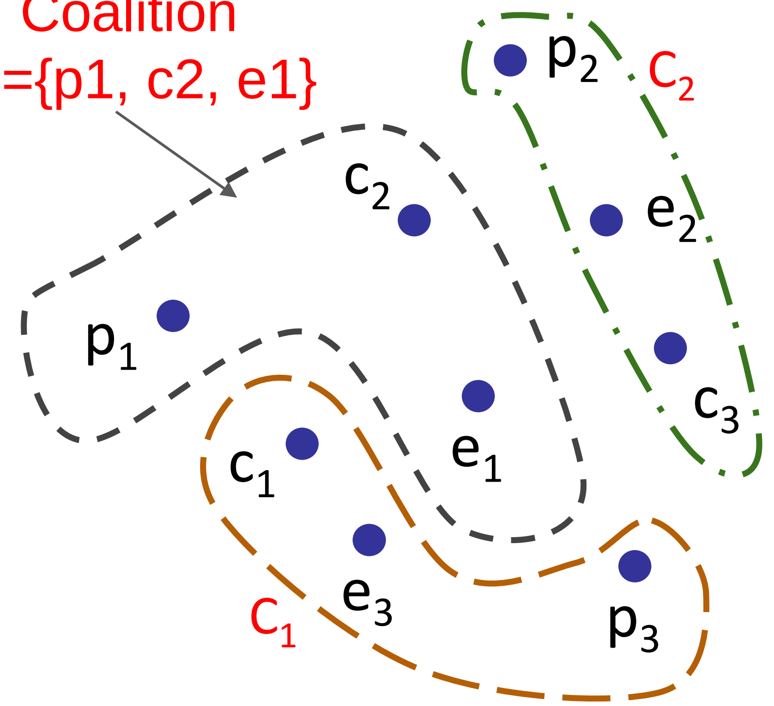
- who to join
- how to act

Coalition structure CS=<C_1, C_2, C_3>

Action vector: $\boldsymbol{a}=\langle a_{C1}, a_{C2}, a_{C3}\rangle$

Coalition formation by example

Coalition
C_3={p1, c2, e1}



$u(C_3 \mid a_{C3})=30$

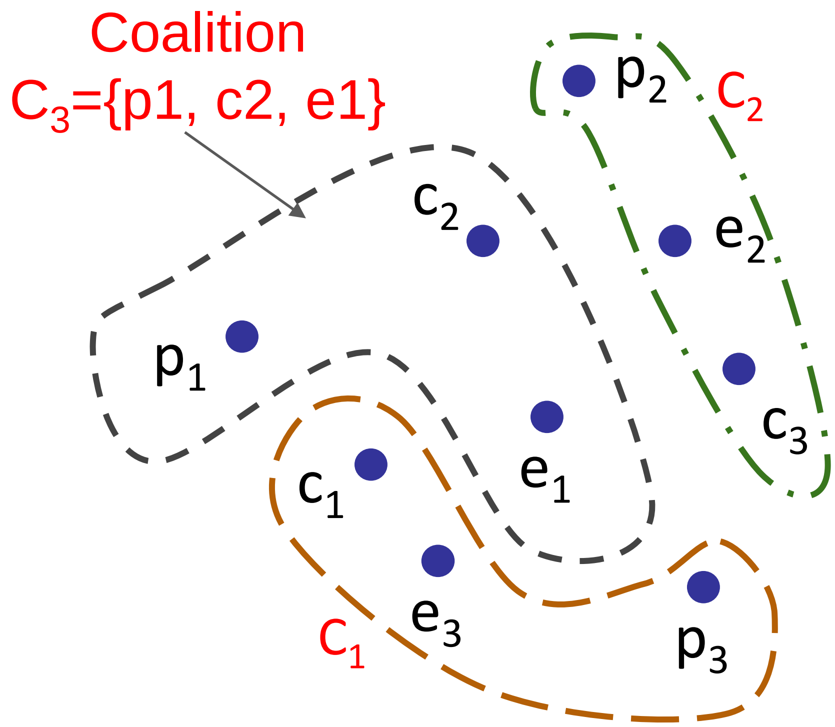
Agents have to decide:

- who to join
- how to act
- how to share profit

Coalition structure CS=<C_1, C_2, C_3>

Action vector: $\boldsymbol{a}=<a_{C1}, a_{C2}, a_{C3}>$

Coalition formation by example



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- how to act
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Coalition structure CS=<C_1, C_2, C_3> ⇓

Action vector: $\boldsymbol{a}=\langle a_{C1}, a_{C2}, a_{C3}\rangle$

$u(C_3\ |a_{C3})=30$ ⇒ Allocation: $\langle p_1=12,\ c_2=3,\ e_1=15\rangle$

Coalitional Games

- Players form *coalitions*
- Each coalition is associated with a *worth*
- A *total worth* has to be distributed

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- Players form *coalitions*
- Each coalition is associated with a *worth*
- A *total worth* has to be distributed

- What can selfish agents expect to get out of joining a coalition?
- What does it mean to have stable coalitions?
- How do coalitions emerge?
- How can coalitional stability be achieved?
- How much does one lose by decentralization?
- How can a designer achieve optimality in task execution by forming necessary coalitions?

A taxonomy of Coalitional Games

Cooperative games model scenarios, where

- agents can benefit from cooperation
- binding agreements are possible
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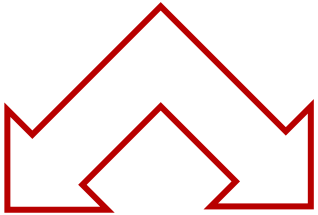

Transferable utility games: payoffs are given to the group and then divided among its members

Non Transferable utility games: group actions result in payoffs to individual group members

NTU Games: Writing Papers

N researchers working at N different universities can form groups to write papers.

- Each group of researchers can work together:
  - The composition of a group determines the quality of the papers they produce.

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- Each author receives a payoff from their own university
  - promotion
  - bonus
  - teaching load reduction

Payoffs are **non-transferable**

TU Games: Buying Ice-Cream

N children, each has some amount of money: the i-th child has b_i dollars

Three types of ice-cream tubs are for sale:

- Type 1 costs \$7, contains 500g
- Type 2 costs \$9, contains 750g
- Type 3 costs \$11, contains 1kg

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Three types of ice-cream tubs are for sale:

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- Children have utility for ice-cream, and do not care about money
- The payoff of each group is the maximum quantity of ice-cream the members of the group can buy by pooling their money
- The ice-cream can be shared arbitrarily within the group

Characteristic Function Games

In general TU games, the payoff obtained by a coalition depends on the actions chosen by other coalitions. These games are also known as partition function games (PFG).

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Characteristic function games (CFG):

- The payoff of each coalition only depends on the action of that coalition
- In such games, each coalition can be identified with the profit it obtains by choosing its best action (**Ice Cream game is a CFG**)

Classes of Cooperative Games

- Any TU game can be represented as an NTU game with a continuum of actions
- *each payoff division outcome in the TU game can be interpreted as an action in the NTU game*

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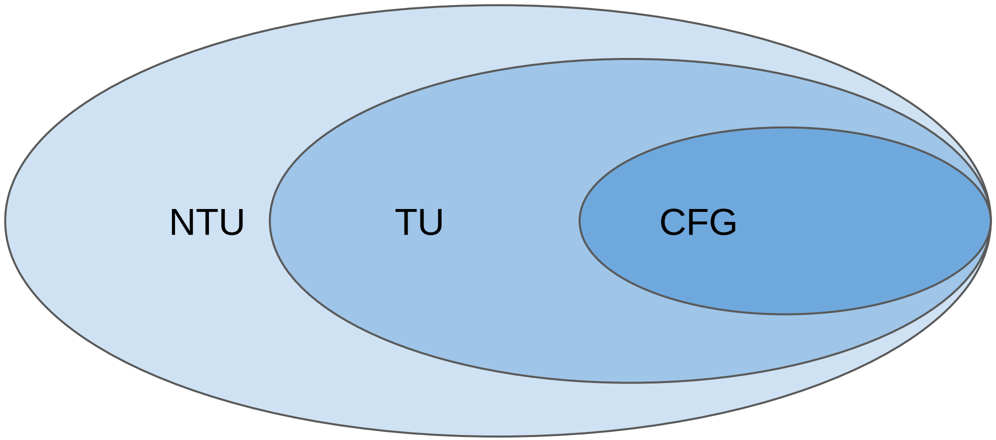

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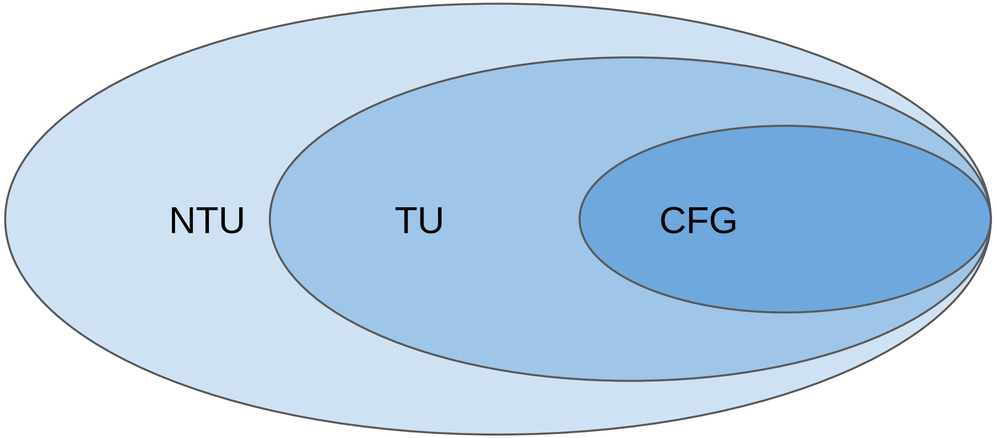

Characteristic function games are often simply called “TU Games”

Formalization of TU Games

A transferable utility game is a pair (N, v), where:

- $N = \{1, \ldots, n\}$ is the set of players (also called grand coalition)
- $v: 2^N \rightarrow \mathbb{R}$ is the characteristic function
  - for each subset of players C, $v(C)$ is the amount that the members of C can earn by working together

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Usually it is assumed that v is

- normalized: $v(\emptyset) = 0$
- non-negative: $v(C) \geq 0$ for any $C \subseteq N$
- monotone: $v(C) \leq v(D)$ for any C, D such that $C \subseteq D$

Ice-cream game: characteristic function

C: €6

M: €4

P: €4

w = 500
p = €7

w = 750
p = €9

w = 1000
p = €11

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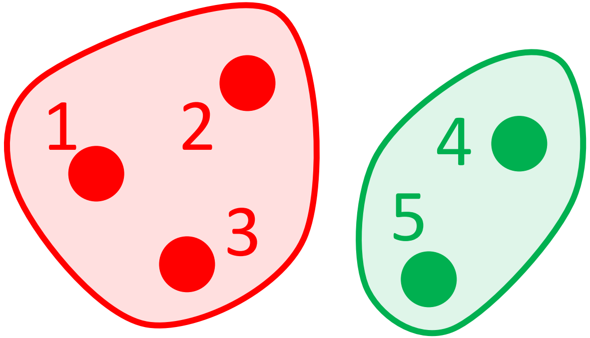
v({C, M}) = 750, v({C, P}) = 750, v({M, P}) = 500

v({C, M, P}) = 1000

Transferable Utility Games: Outcomes

An outcome of a TU game $G = (N, v)$ is a pair $(CS, \underline{\mathbf{x}})$, where:

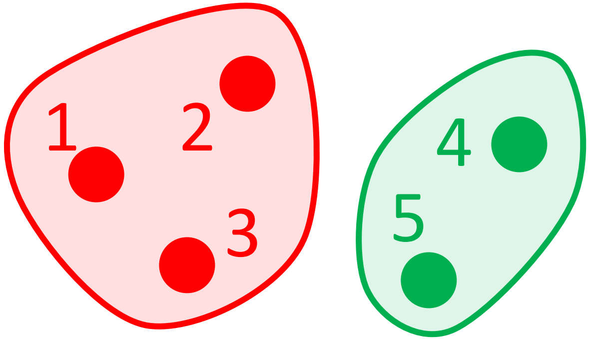
- $CS = (C_1, \ldots, C_k)$ is a coalition structure, i.e., a partition of N:
  - $\cup_i C_i = N$, $C_i \cap C_j = \emptyset$ for $i \neq j$
- $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$ is a payoff vector, which distributes the value of each coalition in CS:
  - $\Sigma_{i \in C}\, x_i = v(C)$ for each C is CS (*Efficiency*)



Outcomes: Example

Suppose v({1, 2, 3}) = **9**, v({4, 5}) = **4**

Then, (({1, 2, 3}, {4, 5}), **(3, 3, 3, 3, 1)**) is an outcome



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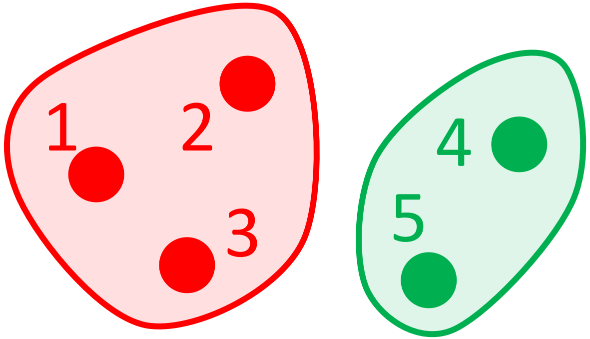
Instead, (({1, 2, 3}, {4, 5}), **(2, 3, 2, 3, 3)**) is **not** an outcome. *Transfers between coalitions are not allowed*



Outcomes: Minimum requirement

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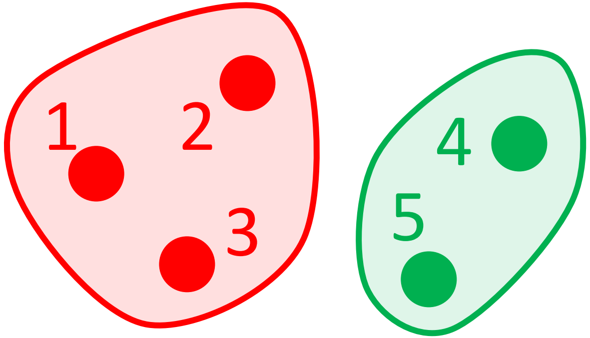
Instead, (({1, 2, 3}, {4, 5}), **(2, 3, 2, 3, 3)**) is **not** an outcome. *Transfers between coalitions are not allowed*

An outcome (CS, **x**) is called an imputation if it satisfies individual rationality:

$$x_i \geq v(\{i\}) \text{ for all } i \in N$$



Superadditive Games

A TU game $G = (N, v)$ is called **superadditive** if $v(C \cup D) \geq v(C) + v(D)$ for any two disjoint coalitions C and D.

Example: $v(C) = |C|^2$: $v(C \cup D) = (|C|+|D|)^2 \geq |C|^2+|D|^2 = v(C) + v(D)$

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In superadditive games, two coalitions can always merge without losing money; hence, we can assume that players form the grand coalition

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$v(\{C, M\}) = 750$, $v(\{C, P\}) = 750$, $v(\{M, P\}) = 500$

$v(\{C, M, P\}) = 1000$

Superadditive Games

Convention: in superadditive games, we identify outcomes with payoff vectors for the grand coalition

► i.e., an outcome is simply a vector $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$ with $\Sigma_{i \in N} x_i = v(N)$

Caution: many papers define outcomes in this way even if the game is not superadditive

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Part 2: Solution Concepts

What is a good outcome?

Consider the ice-cream game with the following characteristic function

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This is a superadditive game: outcomes are payoff vectors (ways to divide 1000). How should the players share the ice-cream?

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This is a superadditive game: outcomes are payoff vectors (ways to divide 1000). How should the players share the ice-cream?

- If they share as (200, 200, 600), Charlie and Marcie can get more ice-cream by buying a 750g tub on their own, and splitting it equally

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This is a superadditive game: outcomes are payoff vectors (ways to divide 1000). How should the players share the ice-cream?

- If they share as (200, 200, 600), Charlie and Marcie can get more ice-cream by buying a 750g tub on their own, and splitting it equally
- The outcome (200, 200, 600) is not **stable**!

Transferable Utility Games: Stability

Definition: the core of a game is the set of all stable outcomes, i.e., outcomes that no coalition wants to deviate from.

$$core(G) = \{(CS, \underline{\mathbf{x}}) \mid \Sigma_{i \in C}\, x_i \geq v(C) \text{ for any } C \subseteq N\}$$

That is, each coalition earns at least as much as it would earn on its own.

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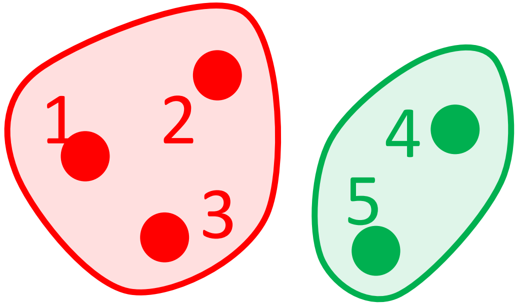
Suppose

- $v(\{1, 2, 3\}) = 9$,
- $v(\{4, 5\}) = 4$,
- $v(\{2, 4\}) = 7$

$((\{1, 2, 3\}, \{4, 5\}), (3, 3, 3, 3, 1))$ is NOT in the core



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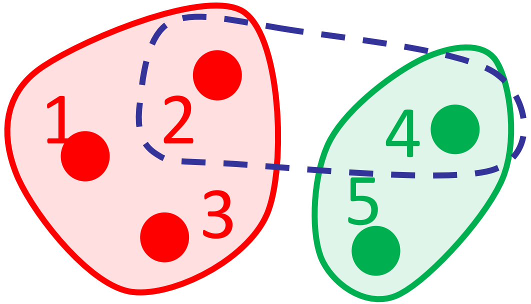
v({2, 4}) = 7



Ice-cream game: Core

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- (200, 200, 600) is not in the core:
  - $v(\{C, M\}) > x_C + x_M$
- (500, 250, 250) is in the core:
  - no subgroup of players can deviate so that each member of the subgroup gets more

Ice-cream game: Core Stability

Consider the ice-cream game with the following characteristic function

$v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$

$v(\{C, M\}) = 750$, $v(\{C, P\}) = 750$, $v(\{M, P\}) = 500$

$v(\{C, M, P\}) = 1000$

$(x_C, x_M, x_P,)$ is in the core if, and only if:

- $x_C + x_M \geq v(\{C, M\})$
- $x_C + x_P \geq v(\{C, P\})$
- $x_P + x_M \geq v(\{P, M\})$

(stability)

- $x_C \geq v(\{C\})$
- $x_P \geq v(\{P\})$
- $x_M \geq v(\{M\})$

(individual rationality)

- $x_C + x_P + x_M = v(\{C, M, P\})$

(efficiency)

Games with empty core

The core is a very attractive solution concept

However, some games have empty cores

$G = (\{1, 2, 3\}, v)$, $v(C) = 1$ if $|C| > 1$ and $v(C) = 0$ otherwise

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Consider an outcome $(CS, \underline{\mathbf{x}})$:

- Assume $CS = (\{1\}, \{2\}, \{3\})$
- Then, the grand coalition can deviate
  - $X_1 + X_2 + X_3 = v(\{1\}) + v(\{2\}) + v(\{3\}) < v(\{1,2,3\})$

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Consider an outcome $(CS, \underline{\mathbf{x}})$:

- Assume $CS = (\{1,2\}, \{3\})$
- Then, either 1 or 2 gets less than 1, so can deviate with 3
  - $X1 + X3 = X1 + 0 < 1 < v(\{1,3\})$

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Consider an outcome $(CS, \underline{\mathbf{x}})$:

- Assume $CS = (\{1,2\}, \{3\})$
- Then, either 1 or 2 gets less than 1, so can deviate with 3
  - $X_1 + X_3 = X_1 + 0 < 1 < v(\{1,3\})$
- Same argument for $CS = (\{1, 3\}, \{2\})$ or $CS = (\{2, 3\}, \{1\})$

Games with empty core

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However, some games have empty cores

$G = (\{1, 2, 3\}, v)$, $v(C) = 1$ if $|C| > 1$ and $v(C) = 0$ otherwise

---

Consider an outcome $(CS, \underline{\mathbf{x}})$:

- Assume $CS = (\{1,2,3\})$
- Then, $x_i > 0$ holds for some i, *(say 3)*
  - so $x(\{1,2\}) < 1$, yet $v(\{1,2\}) = 1$

ε-Core

If the core is empty, then we may want to find approximately stable outcomes

Need to relax the notion of the core:

core: $(CS, \underline{\mathbf{x}})$: $x(C) \geq v(C)$ for all $C \subseteq N$

ε-core: $(CS, \underline{\mathbf{x}})$: $x(C) \geq v(C) - \varepsilon$ for all $C \subseteq N$

It is usually defined for superadditive games only

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It is usually defined for superadditive games only

$G = (\{1, 2, 3\}, v)$, with $v(C) = 1$ if $|C| > 1$, $v(C) = 0$ otherwise

- 1/3-core is non-empty: $(1/3, 1/3, 1/3) \in$ 1/3-core
- ε-core is empty for any $\varepsilon < 1/3$:
  - $x_i \geq 1/3$ for some $i = 1, 2, 3$; so $x(N\setminus\{i\}) \leq 2/3$, $v(N\setminus\{i\}) = 1$

ε-Core and the Least Core

Let $\varepsilon^*(G) = \mathit{inf}\ \{ \varepsilon \mid \varepsilon\text{-core of } G \text{ is not empty}\}$

- it can be shown that $\varepsilon^*(G)$-core is not empty

Definition: $\varepsilon^*(G)$-core is the least core of G

- $\varepsilon^*(G)$ is called the value of the least core

$G = (\{1, 2, 3\}, v)$, with $v(C) = 1$ if $|C| > 1$, $v(C) = 0$ otherwise

- $1/3$-core is non-empty: $(1/3, 1/3, 1/3) \in 1/3$-core
- ε-core is empty for any $\varepsilon < 1/3$:
  - $x_i \geq 1/3$ for some $i = 1, 2, 3$; so $x(N\setminus\{i\}) \leq 2/3$, $v(N\setminus\{i\}) = 1$

Advanced Solution Concepts

There are many solution concepts:

- Nucleolus
- Bargaining set
- Kernel

more sophisticated stability considerations

- Shapley value
- Banzhaf index

based on the concept of fairness

Advanced Solution Concepts

There are many solution concepts:

- **Nucleolus**
- Bargaining set
- Kernel

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The concept of *excess*

How can stability be measured?

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The Nucleolus

Let's arrange excess values in a non-increasing order

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Core imputation

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$$x^* = (1,1,1) \Longrightarrow \theta(x^*) = \{-1,-1,-1,-1,-1,-1\}$$

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The Nucleolus

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<u>Definition</u> [Schmeidler]: The nucleolus $\mathcal{N}(\mathcal{G})$ of a game $\mathcal{G}$ is the set

$$\mathcal{N}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid \nexists y \in X(\mathcal{G}) \text{ s.t. } \theta(y) \prec \theta(x)\}$$

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Objections and counterobjections

An outcome is not in the core if some coalition objects to it; but is the objection itself **plausible**?
Fix an imputation $\underline{\mathbf{x}}$ for a superadditive game $G=(N, v)$

A pair $(\underline{\mathbf{y}}, S)$, where $\underline{\mathbf{y}}$ is an imputation and $S \subseteq N$, is an **objection** of player i against player j to $\mathbf{x}$ if

- $i \in S$, $j \notin S$, $y(S) = v(S)$
- $y_k > x_k$ for all $k \in S$

A pair $(\underline{\mathbf{z}}, T)$, where $\underline{\mathbf{z}}$ is an imputation and $T \subseteq N$, is a **counterobjection** to the objection $(\underline{\mathbf{y}}, S)$ if

- $j \in T$, $i \notin T$, $z(S) = v(S)$, $T \cap S \neq \emptyset$
- $z_k \geq x_k$ for all $k \in T \setminus S$
- $z_k \geq y_k$ for all $k \in T \cap S$

Bargaining Set

An objection is said to be **justified** if it does not admit a counterobjection

Definition: the **bargaining set** of a game G consists of all imputations that do not admit a justified objection

- However, they may admit unjustified objections

The core is the set of all imputations that do not admit an objection.

core ⊆ bargaining set

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Stability vs. Fairness

Consider the game $G = (\{1, 2\}, v)$

- where $v(\varnothing) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$

- $(15, 5)$ is in the core
  - player 2 cannot benefit by deviating

The question is: Is $(15, 5)$ **fair**?

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The question is: Is $(15, 5)$ **fair**?

No! Since 1 and 2 are symmetric

Outcomes in the core may be unfair!

How do we divide payoffs in a **fair** way?

Marginal contribution

A fair outcome would reward each agent according to their contribution.

First attempt:

Given a game $G = (N, v)$, set $x_i = v(\{1, ..., i\text{-}1, i\}) - v(\{1, ..., i\text{-}1\})$

- That is, the payoff to each player is their marginal contribution to the coalition of their predecessors
- We have $x_1 + ... + x_n = v(N)$; $\underline{\mathbf{x}}$ is a payoff vector.

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This does not work, as the payoff to each player depends on the order

- $G = (\{1, 2\}, v)$, with $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$
- $x_1 = v(1) - v(\emptyset) = 5$, $x_2 = v(\{1, 2\}) - v(\{1\}) = 15$

Average Marginal Contribution

Idea: to remove the dependence on ordering, we can average over all possible orderings.

By example:

$G = (\{1, 2\}, v)$, where $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$

- 1, 2: $x_1 = v(1) - v(\emptyset) = 5$, $x_2 = v(\{1, 2\}) - v(\{1\}) = 15$
- 2, 1: $y_2 = v(2) - v(\emptyset) = 5$, $y_1 = v(\{1, 2\}) - v(\{2\}) = 15$

$$z_1 = (x_1 + y_1)/2 = 10, \quad z_2 = (x_2 + y_2)/2 = 10$$

the resulting outcome is fair!

Generalization: Shapley Value

A permutation of $\{1,\dots, n\}$ is a one-to-one mapping from $\{1,\dots, n\}$ to itself

Let $P(N)$ denote the set of all permutations of N

Let $S_\pi(i)$ denote the set of predecessors of i in a permutation $\pi \in P(N)$

For $C \subseteq N$, let $\delta_i(C) = v(C \cup \{i\}) - v(C)$ be the marginal contribution of player i to C

The **Shapley value** of player i in a game $G = (N, v)$ with $|N| = n$ is

$$\varphi_i(G) = 1/n!\ \Sigma_{\pi:\ \pi \in P(N)}\ \delta_i(S_\pi(i))$$

Shapley value: probabilistic interpretation

Suppose that we choose a permutation of players uniformly at random, among all possible permutations of N

Then, φ_i is the expected marginal contribution of player i to the coalition of their predecessors.

Shapley value: properties (1)

Proposition:

in any game G, $\varphi_1 + ... + \varphi_n = v(N)$

Shapley value: properties (2)

Definition:

a player i is a dummy if $v(C) = v(C \cup \{i\})$ for any $C \subseteq N$

Proposition:

if a player i is a dummy, then $\varphi_i = 0$

Shapley value: properties (3)

Definition:

two players i and j are symmetric if $v(C \cup \{i\}) = v(C \cup \{j\})$ for any $C \subseteq N \setminus \{i, j\}$

Proposition:

if i and j are symmetric, then $\varphi_i = \varphi_j$

Shapley value: properties (4)

Definition:

Let $G_1 = (N, u)$ and $G_2 = (N, v)$ be two games with the same set of players. Then $G = G_1 + G_2$ is the game with the set of players N and characteristic function w given by

$$w(C) = u(C) + v(C) \text{ for all } C \subseteq N$$

Proposition:

$$\varphi_i(G_1+G_2) = \varphi_i(G_1) + \varphi_i(G_2)$$

Axiomatic characterization

Consider the following properties:

1. Efficiency: $\varphi_1 + \dots + \varphi_n = v(N)$
2. Dummy: if i is a dummy, $\varphi_i = 0$
3. Symmetry: if i and j are symmetric, $\varphi_i = \varphi_j$
4. Additivity: $\varphi_i(G_1+G_2) = \varphi_i(G_1) + \varphi_i(G_2)$

Theorem: The Shapley value is the only payoff distribution scheme which satisfies the properties 1-4

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Theorem: The Shapely value can be also written as

$$\sum_{C \subseteq N} \frac{(|N|-|C|)!(|C|-1)!}{|N|!}(\varphi(C) - \varphi(C \setminus \{i\}))$$

Slides based on

- Gianluigi Greco, Francesco Lupia, Francesco Scarcello: **The Tractability of the Shapley Value over Bounded Treewidth Matching Games**. IJCAI 2017
- Georgios Chalkiadakis, Gianluigi Greco, Evangelos Markakis: **Characteristic function games with restricted agent interactions: Core-stability and coalition structures**. Artif. Intell. 232: 76-113 (2016)
- Gianluigi Greco, Francesco Lupia, Francesco Scarcello: **Structural Tractability of Shapley and Banzhaf Values in Allocation Games**. IJCAI 2015
- Gianluigi Greco, Enrico Malizia, Luigi Palopoli, Francesco Scarcello: **The Complexity of the Nucleolus in Compact Games**. TOCT 7(1): 3:1-3:52 (2014)
- Gianluigi Greco, Enrico Malizia, Luigi Palopoli, Francesco Scarcello: **On the complexity of core, kernel, and bargaining set**. Artif. Intell. 175(12-13): 1877-1910 (2011)
- Material from tutorials given by Gergios Chalkiadakis, Edith Elkind, Michael Wooldridge and their textbook on «**Computational Aspects of Cooperative Game Theory**»