UNIVERSITÀ
DELLA CALABRIA

Introduction to coalitional game theory
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Roadmap

1. Coalitional games
2. Characteristic function
3. Superadditivity
4. The Core
5. Shapley Value

Coalitional games

A coalitional game (transferable utility) is a pair $G = (N, v)$, where:

$N = \{1, \ldots, n\}$ is the set of players

$v: 2^N \longrightarrow \mathbb{R}$ is the characteristic function

for each subset of players C, $v(C)$ is the amount that the members of C can earn by working together.

Characteristic function

The characteristic function is a map between each coalition $C \subseteq N$ and its respective value (utility).

Characteristic function (example)

Players and their money

A: $6   **B**: $3   **C**: $3

Ice-cream tubes

500g: $7   **750g**: $9   **1000g**: $11

$v(\emptyset) = v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$
$v(\{A, B\}) = 750; v(\{A, C\}) = 750; v(\{B, C\}) = 0$
$v(\{A, B, C\}) = 1000$

Superadditivity

A characteristic function game $G(N, v)$ is said to be superadditive if it satisfies $v(C_1 \cup C_2) \geq v(C_1) + v(C_2) \; \forall C_1, C_2 \subset N$ such that $C_1 \cap C_2 = \emptyset$.

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$$v(\{A, B\}) \geq v(\{A\}) + v\{(B)\} \Rightarrow 750 \geq 0$$

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$$v(\{A, C\}) \geq v(\{A\}) + v\{(C)\} \implies 750 \geq 0$$

Superaddivity

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$$v(\{B, C\}) \geq v(\{B\}) + v\{(C)\} \Longrightarrow 0 \geq 0$$

The Core

The core is defined as the set of all stable outcomes. That is:

$$\text{Core}(G) = \left\{ \quad \boldsymbol{x} \quad \left| \begin{array}{l} x_i \geq 0 \ \forall i \in N \\[1em] \sum_{i \in N} x_i = v(N) \\[1em] \sum_{i \in C} x_i \geq v(C) \forall C \subseteq N \end{array} \right. \right.$$

Shapley value

The shapley value for a player i is the average marginal contribution of the player i over all possible coalitions.

$$\phi(i,v) = \frac{1}{|N|!} \sum_{\pi \in \Pi_N} v(B(\pi,i) \cup \{i\}) - v(B(\pi,i))$$

Where:

Π_N is the set of all possible permutations of N

$B(\pi,i)$ is the set of predecessors on i in the permutation π

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Computation for player A

Shapley value (example)

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Computation for player A

$$\pi = (A, B, C) \implies v(\{A\}) - v(\{\emptyset\}) = 0 - 0 = 0$$

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$\pi = (B, A, C) \implies v(\{A, B\}) - v(\{B\}) = 750 - 0 = 750$

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Computation for player A

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$\pi = (A, C, B) \implies v(\{A\}) - v(\{\emptyset\}) = 0 - 0 = 0$
$\pi = (B, A, C) \implies v(\{A, B\}) - v(\{B\}) = 750 - 0 = 750$
$\pi = (B, C, A) \implies v(\{A, B, C\}) - v(\{B, C\}) = 1000 - 0 = 1000$

Shapley value (example)

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$$\pi = (A, C, B) \implies v(\{A\}) - v(\{\emptyset\}) = 0 - 0 = 0$$
$$\pi = (B, A, C) \implies v(\{A,B\}) - v(\{B\}) = 750 - 0 = 750$$
$$\pi = (B, C, A) \implies v(\{A,B,C\}) - v(\{B,C\}) = 1000 - 0 = 1000$$
$$\pi = (C, A, B) \implies v(\{A,C\}) - v(\{C\}) = 750 - 0 = 750$$

Shapley value (example)

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Computation for player A

$$\pi = (A, B, C) \implies v(\{A\}) - v(\{\emptyset\}) = 0 - 0 = 0$$
$$\pi = (A, C, B) \implies v(\{A\}) - v(\{\emptyset\}) = 0 - 0 = 0$$
$$\pi = (B, A, C) \implies v(\{A, B\}) - v(\{B\}) = 750 - 0 = 750$$
$$\pi = (B, C, A) \implies v(\{A, B, C\}) - v(\{B, C\}) = 1000 - 0 = 1000$$
$$\pi = (C, A, B) \implies v(\{A, C\}) - v(\{C\}) = 750 - 0 = 750$$
$$\pi = (C, B, A) \implies v(\{A, B, C\}) - v(\{B, C\}) = 1000 - 0 = 1000$$

Shapley value (example)

Computation for player A

$$\pi = (A, B, C) \implies v(\{A\}) - v(\{\emptyset\}) = 0 - 0 = 0$$
$$\pi = (A, C, B) \implies v(\{A\}) - v(\{\emptyset\}) = 0 - 0 = 0$$
$$\pi = (B, A, C) \implies v(\{A, B\}) - v(\{B\}) = 750 - 0 = 750$$
$$\pi = (B, C, A) \implies v(\{A, B, C\}) - v(\{B, C\}) = 1000 - 0 = 1000$$
$$\pi = (C, A, B) \implies v(\{A, C\}) - v(\{C\}) = 750 - 0 = 750$$
$$\pi = (C, B, A) \implies v(\{A, B, C\}) - v(\{B, C\}) = 1000 - 0 = 1000$$

$$\phi(A, v) = \frac{1}{6}(0 + 0 + 750 + 1000 + 750 + 1000) = 583.\overline{33}$$

Shapley value – alternative formulation

The Shapley value can be computed also with this formula

$$\phi(i,v) = \sum_{C \subseteq N} \frac{(|N| - |C|)! \times (|C| - 1)!}{|N|!} (v(C) - v(C \backslash \{i\}))$$